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BRODY CURVES AND MEAN DIMENSION

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Abstract. We study the mean dimensions of the spaces of Brody curves. In particular we give the formula of the mean dimension of the space of Brody curves in the Riemann sphere. A key notion is a non-degeneracy of Brody curves introduced by Yosida (1934). We develop a deformation theory of non-degenerate Brody curves and apply it to the calculation of the mean dimension. Moreover we show that there are sufficiently many non-degenerate Brody curves.

1. Introduction

1.1. **Main results.** Let $z = x + y\sqrt{-1} \in \mathbb{C}$ be the standard coordinate in the complex plane $\mathbb{C}$. Let $f = [f_0 : f_1 : \cdots : f_N] : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic map (f_i: holomorphic function). We define $|df|(z) \geq 0$ by

$$|df|^2(z) := \frac{1}{4\pi}\Delta \log(|f_0|^2 + |f_1|^2 + \cdots + |f_N|^2) \quad \left(\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

$|df|(z)$ is classically called a spherical derivative. It evaluates the dilatation of the map f with respect to the Euclidean metric on $\mathbb{C}$ and the Fubini-Study metric on $\mathbb{C}P^N$. (See the equation (6) in Section 4.2.)

A holomorphic map $f : \mathbb{C} \to \mathbb{C}P^N$ is called a Brody curve ([3]) if it satisfies $|df|(z) \leq 1$ for all $z \in \mathbb{C}$. Let $\mathcal{M}(\mathbb{C}P^N)$ be the space of Brody curves in $\mathbb{C}P^N$. It is endowed with the compact-open topology (the topology of uniform convergence on compact subsets): A sequence of Brody curves $\{f_n\} \subset \mathcal{M}(\mathbb{C}P^N)$ converges to $f \in \mathcal{M}(\mathbb{C}P^N)$ if and only if for any compact subset $K \subset \mathbb{C}$ we have $\sup_{z\in K} d(f_n(z), f(z)) \to 0$ as $n \to \infty$. ($d(\cdot,\cdot)$ is the distance on $\mathbb{C}P^N$ with respect to the Fubini-Study metric.) $\mathcal{M}(\mathbb{C}P^N)$ is an infinite dimensional compact metrizable space, and it admits the following continuous $\mathbb{C}$-action.

$$\mathcal{M}(\mathbb{C}P^N) \times \mathbb{C} \to \mathcal{M}(\mathbb{C}P^N), \quad (f(z), a) \mapsto f(z + a).$$

The main purpose of the paper is to study the mean dimension $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$ of this infinite dimensional dynamical system. (Mean dimension is an invariant of topological dynamical systems introduced by Gromov [12]. We review its definition in Section 2.1.)

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Let $f:\mathbb{C}\to\mathbb{C}P^N$ be a Brody curve. We define an energy density $\rho(f)$ by setting

$$\rho(f) := \lim_{R\to\infty}\frac{1}{\pi R^2}\left(\sup_{a\in\mathbb{C}}\int_{|z-a|<R}|df|^2dxdy\right). \tag{1}$$

(This limit always exists. See Section 2.2.) We define the Nevanlinna-Shimizu-Ahlfors characteristic function $T(r,f)$ $(r\geq 1)$ by

$$T(r,f) := \int_1^r\left(\int_{|z|<t}|df|^2dxdy\right)\frac{dt}{t}.$$

From the Brody condition $|df|\leq 1$, we have $T(r,f)\leq \pi r^2/2$. We define $\rho_{\mathrm{NSA}}(f)$ by

$$\rho_{\mathrm{NSA}}(f) := \limsup_{r\to\infty}\frac{2}{\pi r^2}T(r,f).$$

It is easy to see $\rho_{\mathrm{NSA}}(f)\leq\rho(f)$.

Let $\rho(\mathbb{C}P^N)$ be the supremum of $\rho(f)$ over $f\in\mathcal{M}(\mathbb{C}P^N)$, and let $\rho_{\mathrm{NSA}}(\mathbb{C}P^N)$ be the supremum of $\rho_{\mathrm{NSA}}(f)$ over $f\in\mathcal{M}(\mathbb{C}P^N)$. We know (see Section 2.2)

$$0<\rho_{\mathrm{NSA}}(\mathbb{C}P^N)\leq\rho(\mathbb{C}P^N)<1.$$

The main result of this paper is the following:

Theorem 1.1.

$$2(N+1)\rho(\mathbb{C}P^N)\leq\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\leq 4N\rho_{\mathrm{NSA}}(\mathbb{C}P^N).$$

Corollary 1.2.

$$\dim(\mathcal{M}(\mathbb{C}P^1):\mathbb{C}) = 4\rho(\mathbb{C}P^1) = 4\rho_{\mathrm{NSA}}(\mathbb{C}P^1).$$

From Theorem 1.1, $4\rho(\mathbb{C}P^1)\leq\dim(\mathcal{M}(\mathbb{C}P^1):\mathbb{C})\leq 4\rho_{\mathrm{NSA}}(\mathbb{C}P^1)$. Since $\rho_{\mathrm{NSA}}(\mathbb{C}P^1)\leq\rho(\mathbb{C}P^1)$, we get the corollary.

The formula $\dim(\mathcal{M}(\mathbb{C}P^1):\mathbb{C}) = 4\rho_{\mathrm{NSA}}(\mathbb{C}P^1)$ was conjectured in [22, p. 1643, (4)]. This formula is very surprising (at least for the authors) because the definitions of the left-hand-side and the right-hand-side are totally different.

The upper bound $\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\leq 4N\rho_{\mathrm{NSA}}(\mathbb{C}P^N)$ was already proved in [19, Theorem 1.5] by using the Nevanlinna theory. (Remark: We used the notation $e(f)$ for $\rho_{\mathrm{NSA}}(f)$ in [19].) The purpose of the present paper is to prove the lower bound $\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\geq 2(N+1)\rho(\mathbb{C}P^N)$.

1.2. **Non-degenerate Brody curves.** For $a\in\mathbb{C}$ and $r>0$ we set $D_r(a):=\{z\in\mathbb{C}|\,|z-a|\leq r\}$. The following is a key-notion of the paper. This notion was first introduced by Yosida [23]. (Gromov [12, p. 399] also discussed it in a more general situation. See also Eremenko [5, Section 4] and Remark 1.4 below.)

Definition-Lemma 1.3. Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. Then the following two conditions are equivalent.
(i) Any constant curve does not belong to the closure of the $\mathbb{C}$-orbit of f. (In other words, for any sequence of complex numbers $\{a_n\}_{n\geq 1}$, the sequence of Brody curves $\{f(z+a_n)\}_{n\geq 1}$ does not converge to a constant curve.)
(ii) There exist $\delta > 0$ and $R > 0$ such that for all $a \in \mathbb{C}$ we have $\|df\|_{L^\infty(D_R(a))} \geq \delta$.

f is said to be non-degenerate if it satisfies one of (and hence both) the above conditions.

Proof. The following argument is given in [23]. Suppose that the condition (ii) fails. Then for any $n \geq 1$ there is $a_n \in \mathbb{C}$ such that $\|df\|_{L^\infty(D_1(a_n))} \leq 1/n$. Taking a subsequence, we can assume that the sequence $\{f(z + a_n)\}_{n\geq 1}$ converges to a Brody curve $g(z)$. Then $\|dg\|_{L^\infty(D_1(0))} = 0$. This implies that g is a constant curve.

Suppose the condition (ii) holds. Let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers. If $\{f(z + a_n)\}_{n\geq 1}$ converges to $g(z)$, then $\|dg\|_{L^\infty(D_R(0))} \geq \delta$. Hence $g(z)$ is not a constant curve. This proves the condition (i). □

Remark 1.4. The above argument also proves that the conditions in Definition-Lemma 1.3 are equivalent to the following:
(ii') For any $R > 0$ there exists $\delta > 0$ such that for all $a \in \mathbb{C}$ we have $\|df\|_{L^\infty(D_R(a))} \geq \delta$.

Yosida [23, Theorem 4] proved (i) $\Leftrightarrow$ (ii') for the case of $N = 1$. In [23] Brody curves $f : \mathbb{C} \to \mathbb{C}P^1$ satisfying (i) are called meromorphic functions of 1st category. In Eremenko [5, Section 4] Brody curves $f : \mathbb{C} \to \mathbb{C}P^N$ satisfying (i) are called binormal curves. Gromov [12, p. 399] used the terminology "uniformly nondegenerate".

Example 1.5. $f(z) = e^z \in \mathcal{M}(\mathbb{C}P^1)$ is a degenerate (i.e. not non-degenerate) Brody curve. A non-constant elliptic function $f(z) \in \mathcal{M}(\mathbb{C}P^1)$ is a non-degenerate Brody curve.

In our viewpoint, non-degenerate Brody curves are "non-singular points" of the space $\mathcal{M}(\mathbb{C}P^N)$, and they behave very nicely for the calculation of the mean dimension:

Theorem 1.6. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve with $\|df\|_{L^\infty(\mathbb{C})} < 1$. Then*

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \geq 2(N + 1)\rho(f).$$

The following theorem means that there are "sufficiently many" non-degenerate Brody curves:

Theorem 1.7. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic map with $\|df\|_{L^\infty(\mathbb{C})} < 1$. Then for any $\varepsilon > 0$ there exists a non-degenerate Brody curve $g : \mathbb{C} \to \mathbb{C}P^N$ satisfying $\|dg\|_{L^\infty(\mathbb{C})} < 1$ and $\rho(g) \geq \rho(f) - \varepsilon$.*

Proof of Theorem 1.1, assuming Theorems 1.6 and 1.7. The upper bound $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \leq 4N\rho_{\mathrm{NSA}}(\mathbb{C}P^N)$ was already proved in [19, Theorem 1.5]. Here we prove the lower bound. Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. Let $0 < c < 1$ and set $f_c(z) = f(cz)$. Then $|df_c|(z) = c|df|(cz)$ and $\rho(f_c) = c^2\rho(f)$. Since $\|df_c\|_{L^\infty(\mathbb{C})} \leq c < 1$, we can apply Theorem 1.7 to f_c. Then for any $\varepsilon > 0$ there exists a non-degenerate Brody curve $g : \mathbb{C} \to \mathbb{C}P^N$ satisfying $\|dg\|_{L^\infty(\mathbb{C})} < 1$ and $\rho(g) \geq \rho(f_c) - \varepsilon = c^2\rho(f) - \varepsilon$. By Theorem 1.6

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \geq 2(N+1)\rho(g) \geq 2(N+1)(c^2\rho(f) - \varepsilon).$$

Let $\varepsilon \to 0$ and $c \to 1$. We get $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \geq 2(N+1)\rho(f)$. Taking the supremum over $f \in \mathcal{M}(\mathbb{C}P^N)$, we get $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \geq 2(N+1)\rho(\mathbb{C}P^N)$. $\square$

2. Some preliminaries

2.1. **Review of mean dimension.** In this subsection we review the definition of mean dimension. For the detail, see Gromov [12] and Lindenstrauss-Weiss [14]. (For some related works, see also Lindenstrauss [13] and Gournay [7, 8, 9, 10].)

Let (X, d) be a compact metric space, and let Y be a topological space. Let $\varepsilon > 0$. A continuous map $f : X \to Y$ is called an ε-embedding if $\mathrm{Diam} f^{-1}(y) \leq \varepsilon$ for all $y \in Y$. Here $\mathrm{Diam} f^{-1}(y)$ is the supremum of $d(x_1, x_2)$ over $x_1, x_2 \in f^{-1}(y)$. We define $\mathrm{Widim}_\varepsilon(X, d)$ as the minimum integer $n \geq 0$ such that there are an n-dimensional polyhedron P and an ε-embedding $f : X \to P$.

For example, let $X = [0, 1] \times [0, \varepsilon]$ with the Euclidean distance. Then the projection $\pi : X \to [0, 1]$ is an ε-embedding, and we have $\mathrm{Widim}_\varepsilon(X, \mathrm{Euclid}) = 1$. The following example is very important in the later argument. This was given by Gromov [12, p. 333]. (For the detailed proof, see Gournay [8, Lemma 2.5] or Tsukamoto [22, Appendix].)

Example 2.1. Let V be a finite dimensional Banach space over $\mathbb{R}$, and set $B_r(V) := \{x \in V | \, \|x\| \leq r\}$ for $r > 0$. For $0 < \varepsilon < r$,

$$\mathrm{Widim}_\varepsilon(B_r(V), \|\cdot\|) = \dim V.$$

Here we consider the norm distance on $B_r(V)$.

For a subset $\Omega \subset \mathbb{C}$ and $r > 0$, we define $\partial_r\Omega$ as the set of $a \in \mathbb{C}$ satisfying $D_r(a) \cap \Omega \neq \emptyset$ and $D_r(a) \cap (\mathbb{C} \setminus \Omega) \neq \emptyset$. Let Ω_n $(n \geq 1)$ be a sequence of bounded Borel subsets of $\mathbb{C}$. It is called a Følner sequence if for all $r > 0$

$$\frac{\mathrm{Area}(\partial_r\Omega_n)}{\mathrm{Area}(\Omega_n)} \to 0 \quad (n \to \infty).$$

For example, the sequence $\Omega_n := D_n(0)$ is a Følner sequence. The sequence $\Omega_n := [0, n] \times [0, n]$ is also Følner. We need the following "Ornstein-Weiss lemma". (For the proof, see Gromov [12, pp. 336-338].)

Lemma 2.2. *Let $h : \{\text{bounded Borel subsets of } \mathbb{C}\} \to \mathbb{R}_{\geq 0}$ be a map satisfying the following three conditions.*
(i) If $\Omega_1 \subset \Omega_2$, then $h(\Omega_1) \leq h(\Omega_2)$.

(ii) $h(\Omega_1 \cup \Omega_2) \le h(\Omega_1) + h(\Omega_2)$.
(iii) For any $a \in \mathbb{C}$ *and any bounded Borel subset* $\Omega \subset \mathbb{C}$, *we have* $h(a+\Omega) = h(\Omega)$ *where* $a + \Omega := \{a + z \in \mathbb{C} | z \in \Omega\}$.

Then for any Følner sequence Ω_n $(n \ge 1)$ *in* $\mathbb{C}$, *the limit of the sequence*

$$\frac{h(\Omega_n)}{\mathrm{Area}(\Omega_n)} \quad (n \ge 1)$$

exists, and its value is independent of the choice of a Følner sequence.

Suppose that the Lie group $\mathbb{C}$ continuously acts on a compact metric space X. Here we don't assume that the distance is invariant under the group action. For a subset $\Omega \subset \mathbb{C}$, we define a new distance d_Ω on X by

$$d_\Omega(x, y) := \sup_{a \in \Omega} d(a.x, a.y).$$

It is easy to see that the map $\Omega \mapsto \mathrm{Widim}_\varepsilon(X, d_\Omega)$ satisfies the three conditions in Lemma 2.2 for each $\varepsilon > 0$. So we define a mean dimension $\dim(X : \mathbb{C})$ by

$$\dim(X : \mathbb{C}) := \lim_{\varepsilon \to +0} \left(\lim_{n \to \infty} \frac{\mathrm{Widim}_\varepsilon(X, d_{\Omega_n})}{\mathrm{Area}(\Omega_n)} \right)$$

where Ω_n $(n \ge 1)$ is a Følner sequence in $\mathbb{C}$. The value of the mean dimension $\dim(X : \mathbb{C})$ is independent of the choice of a Følner sequence, and it is a topological invariant. (That is, it is independent of the choice of a distance on X compatible with the topology.) For example, we have

$$\begin{aligned} \dim(X : \mathbb{C}) &= \lim_{\varepsilon \to +0} \left(\lim_{R \to \infty} \frac{\mathrm{Widim}_\varepsilon(X, d_{D_R(0)})}{\pi R^2} \right) \\ &= \lim_{\varepsilon \to +0} \left(\lim_{R \to \infty} \frac{\mathrm{Widim}_\varepsilon(X, d_{[0,R] \times [0,R]})}{R^2} \right). \end{aligned} \tag{2}$$

2.2. **Energy density.** Here we explain some basic properties of the energy density $\rho(f)$ introduced in (1). Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. Then the map

$$\Omega \mapsto \sup_{a \in \mathbb{C}} \int_{a + \Omega} |df|^2 dxdy$$

clearly satisfies the three conditions in Lemma 2.2, where $\Omega \subset \mathbb{C}$ is a bounded Borel subset. Therefore we can define the energy density $\rho(f)$ by

$$\rho(f) := \lim_{n \to \infty} \frac{1}{\mathrm{Area}(\Omega_n)} \left(\sup_{a \in \mathbb{C}} \int_{a + \Omega_n} |df|^2 dxdy \right),$$

where Ω_n $(n \ge 1)$ is a Følner sequence in $\mathbb{C}$. In particular, we have

$$\begin{aligned} \rho(f) &= \lim_{R \to \infty} \frac{1}{\pi R^2} \left(\sup_{a \in \mathbb{C}} \int_{|z-a| < R} |df|^2 dxdy \right) \\ &= \lim_{R \to \infty} \frac{1}{R^2} \left(\sup_{a, b \in \mathbb{R}} \int_{[a, a+R] \times [b, b+R]} |df|^2 dxdy \right). \end{aligned} \tag{3}$$

From this we get

$$\rho(f) \geq \limsup_{R\to\infty} \frac{1}{\pi R^2}\int_{|z|<R} |df|^2 dxdy \geq \limsup_{R\to\infty} \frac{2}{\pi R^2} T(R,f) =: \rho_{\mathrm{NSA}}(f).$$

If f is elliptic (i.e. there is a lattice $\Lambda \subset \mathbb{C}$ such that $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda$), then

$$\rho(f) = \limsup_{R\to\infty} \frac{1}{\pi R^2}\int_{|z|<R} |df|^2 dxdy = \rho_{\mathrm{NSA}}(f) = \frac{1}{\mathrm{Area}(\mathbb{C}/\Lambda)}\int_{\mathbb{C}/\Lambda} |df|^2 dxdy.$$

In the paper [20] we studied the quantity

$$\limsup_{R\to\infty} \frac{1}{\pi R^2}\int_{|z|<R} |df|^2 dxdy.$$

Some methods and results in [20] can be also applied to $\rho(f)$. For example, from [20, Proposition 2.6, Proposition 3.1] (Proposition 3.1 in [20] follows from a result of Calabi [4, Thoerem 8],), there exists $0 < c(N) < 1$ such that for all Brody curves $f : \mathbb{C} \to \mathbb{C}P^N$ and all $a, b \in \mathbb{R}$

$$\int_{[a,a+1]\times[b,b+1]} |df|^2 dxdy \leq c(N).$$

Hence

$$\rho(\mathbb{C}P^N) = \sup_{f\in\mathcal{M}(\mathbb{C}P^N)} \rho(f) \leq c(N) < 1.$$

Moreover, from [20, Proposition 5.10], there exists $r > 0$ such that for all Brody curves $f : \mathbb{C} \to \mathbb{C}P^1$ and all $a, b \in \mathbb{R}$

$$\frac{1}{r^2}\int_{[a,a+r]\times[b,b+r]} |df|^2 dxdy \leq 1 - 10^{-100}.$$

Hence we get an explicit (but very rough) bound:

$$\rho(\mathbb{C}P^1) \leq 1 - 10^{-100}.$$

In the paper [22, Section 1.2] we constructed an elliptic function $f : \mathbb{C} \to \mathbb{C}P^1$ such that f is a Brody curve and

$$\rho(f) = \rho_{\mathrm{NSA}}(f) = \frac{2\pi}{\sqrt{3}}\left(\int_1^\infty \frac{dx}{\sqrt{x^3-1}}\right)^{-2} = 0.6150198678198\ldots.$$

Hence

$$\frac{2\pi}{\sqrt{3}}\left(\int_1^\infty \frac{dx}{\sqrt{x^3-1}}\right)^{-2} \leq \rho(\mathbb{C}P^1) \leq 1 - 10^{-100}.$$

The authors think that it is very wonderful if the first inequality is an equality.

It is very difficult to determine the value of $\rho(\mathbb{C}P^N)$, but we have the following clear result on its asymptotic behavior: The sequence $\rho(\mathbb{C}P^N)$ $(N \geq 1)$ is a non-decreasing sequence, and from [20, Theorem 1.5], we have

$$\lim_{N\to\infty} \rho(\mathbb{C}P^N) = 1.$$

Moreover the proof of [20, Theorem 1.5] also shows

$$\lim_{N\to\infty} \rho_{\mathrm{NSA}}(\mathbb{C}P^N) = 1.$$

3. Proof of Theorem 1.6

In this section we prove Theorem 1.6 assuming Propositions 3.1 and 3.2 below. Theorem 1.7 will be proved in Section 6. Let $T\mathbb{C}P^N$ be the tangent bundle of $\mathbb{C}P^N$. It naturally admits a structure of a holomorphic vector bundle. We consider the Fubini-Study metric on it. Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve, and let $f^*T\mathbb{C}P^N$ be the pull-back of $T\mathbb{C}P^N$ by f. $f^*T\mathbb{C}P^N$ is a holomorphic vector bundle over the complex plane $\mathbb{C}$, and its Hermitian metric is given by the pull-back of the Fubini-Study metric. Let H_f be the space of holomorphic sections $u : \mathbb{C} \to f^*T\mathbb{C}P^N$ satisfying $\|u\|_{L^\infty(\mathbb{C})} < +\infty$. $(H_f, \|\cdot\|_{L^\infty(\mathbb{C})})$ is a complex Banach space (possibly infinite dimensional). We set $B_r(H_f) := \{u \in H_f | \, \|u\|_{L^\infty(\mathbb{C})} \leq r\}$ for $r \geq 0$.

Proposition 3.1. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve with $\|df\|_{L^\infty(\mathbb{C})} < 1$. Then there exist $\delta > 0$ and a map*

$$B_\delta(H_f) \to \mathcal{M}(\mathbb{C}P^N), \quad u \mapsto f_u,$$

satisfying the following two conditions:
(i) $f_0 = f$.
(ii) For all $u, v \in B_\delta(H_f)$ and $z \in \mathbb{C}$

$$|d(f_u(z), f_v(z)) - |u(z) - v(z)|| \leq \frac{1}{8} \|u - v\|_{L^\infty(\mathbb{C})}.$$

Here $d(\cdot,\cdot)$ is the distance on $\mathbb{C}P^N$ defined by the Fubini-Study metric, and $|u(z) - v(z)|$ is the fiberwise norm of $f^*T\mathbb{C}P^N$.

Let $R > 0$ and $\Lambda \subset \mathbb{C}$. Λ is said to be an R-square if $\Lambda = [a, a+R] \times [b, b+R]$ for some $a, b \in \mathbb{R}$.

Proposition 3.2. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve. Then for any R-square $\Lambda \subset \mathbb{C}$ with $R > 2$ there exists a finite dimensional complex subspace $V \subset H_f$ satisfying the following two conditions:*
(i)

$$\dim_{\mathbb{C}} V \geq (N+1) \int_\Lambda |df|^2 dxdy - C_f R.$$

Here C_f is a positive constant depending only on f (and independent of R, Λ).
(ii) For all $u \in V$ we have $\|u\|_{L^\infty(\mathbb{C})} \leq 2 \|u\|_{L^\infty(\Lambda)}$.

Propositions 3.1 and 3.2 will be proved later (Sections 4 and 5.) Here we prove Theorem 1.6, assuming them.

Proof of Theorem 1.6. We define a distance on $\mathcal{M}(\mathbb{C}P^N)$ by

$$\mathrm{dist}(g,h) := \sum_{n=0}^{\infty} \frac{1}{10^n} \sup_{|z|\le n} d(g(z),h(z)), \quad (g,h \in \mathcal{M}(\mathbb{C}P^N)).$$

Then $|\mathrm{dist}(g,h) - d(g(0),h(0))| \le (1/9)\sup_{z\in\mathbb{C}} d(g(z),h(z))$. Hence for $\Omega \subset \mathbb{C}$

$$|\mathrm{dist}_\Omega(g,h) - \sup_{z\in\Omega} d(g(z),h(z))| \le \frac{1}{9}\sup_{z\in\mathbb{C}} d(g(z),h(z)). \tag{4}$$

Let $\delta > 0$ be the positive constant introduced in Proposition 3.1. Let $\Lambda \subset \mathbb{C}$ be an R-square ($R > 2$). By Proposition 3.2, there exists $V = V_\Lambda \subset H_f$ satisfying the conditions (i) and (ii) in Proposition 3.2. We investigate the map $B_\delta(H_f) \to \mathcal{M}(\mathbb{C}P^N)$, $u \mapsto f_u$, (given by Proposition 3.1) and its restriction to $B_\delta(V) := V \cap B_\delta(H_f)$.

From the condition (ii) of Proposition 3.1, for $u, v \in B_\delta(H_f)$, we have $\sup_{z\in\mathbb{C}} d(f_u(z), f_v(z)) \le (9/8)\|u-v\|_{L^\infty(\mathbb{C})}$. Hence $(B_\delta(H_f), \|\cdot\|_{L^\infty(\mathbb{C})}) \to \mathcal{M}(\mathbb{C}P^N)$ is continuous. For $u, v \in B_\delta(H_f)$

$$\begin{aligned}
&\left|\mathrm{dist}_\Lambda(f_u,f_v) - \sup_{z\in\Lambda}|u(z)-v(z)|\right| \\
&\le \left|\mathrm{dist}_\Lambda(f_u,f_v) - \sup_{z\in\Lambda} d(f_u(z),f_v(z))\right| + \left|\sup_{z\in\Lambda} d(f_u(z),f_v(z)) - \sup_{z\in\Lambda}|u(z)-v(z)|\right| \\
&\le \frac{1}{9}\sup_{z\in\mathbb{C}} d(f_u(z),f_v(z)) + \frac{1}{8}\|u-v\|_{L^\infty(\mathbb{C})} \quad \text{(by Proposition 3.1 (ii) and (4))} \\
&\le \frac{1}{4}\|u-v\|_{L^\infty(\mathbb{C})}.
\end{aligned}$$

Thus

$$\|u-v\|_{L^\infty(\Lambda)} \le \mathrm{dist}_\Lambda(f_u,f_v) + \frac{1}{4}\|u-v\|_{L^\infty(\mathbb{C})}.$$

For $u, v \in B_\delta(V) = V \cap B_\delta(H_f)$, we have $\|u-v\|_{L^\infty(\mathbb{C})} \le 2\|u-v\|_{L^\infty(\Lambda)}$ (Proposition 3.2 (ii)). Hence

$$\|u-v\|_{L^\infty(\mathbb{C})} \le 4\,\mathrm{dist}_\Lambda(f_u,f_v), \quad (u,v \in B_\delta(V)).$$

Hence for $\varepsilon < \delta/4$,

$$\begin{aligned}
\mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N), \mathrm{dist}_\Lambda) &\ge \mathrm{Widim}_{4\varepsilon}(B_\delta(V), \|\cdot\|_{L^\infty(\mathbb{C})}) \\
&= \dim_{\mathbb{R}} V \quad \text{(by Example 2.1)} \\
&\ge 2(N+1)\int_\Lambda |df|^2 dxdy - 2C_f R \quad \text{(by Proposition 3.2 (i))}.
\end{aligned}$$

Since $\mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N), \mathrm{dist}_\Lambda) = \mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N), \mathrm{dist}_{[0,R]\times[0,R]})$, for $\varepsilon < \delta/4$, the quantity $\mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N), \mathrm{dist}_{[0,R]\times[0,R]})$ is bounded from below by

$$2(N+1)\left(\sup_\Lambda \int_\Lambda |df|^2 dxdy\right) - 2C_f R.$$

Here Λ runs over all R-squares. Dividing this by R^2 and letting $R \to \infty$, we get

$$\lim_{R\to\infty}\left(\frac{1}{R^2}\mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N), \mathrm{dist}_{[0,R]\times[0,R]})\right) \geq 2(N+1)\rho(f).$$

Here we have used (3). Let $\varepsilon \to 0$. Then $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \geq 2(N+1)\rho(f)$ by (2). $\square$

Remark 3.3. The above argument also gives the lower bound on the local mean dimension $\dim_f(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$. (Local mean dimension is a notion introduced in [16].) The readers can skip this remark.

Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve with $\|df\|_{L^\infty(\mathbb{C})} < 1$. Let $B_r(f)_\mathbb{C} \subset \mathcal{M}(\mathbb{C}P^N)$ $(r > 0)$ be the set of $g \in \mathcal{M}(\mathbb{C}P^N)$ satisfying $\mathrm{dist}_\mathbb{C}(f, g) \leq r$. Since $f_0 = f$, if $(4/5)r \leq \delta$ then $u \in B_{(4/5)r}(H_f)$ satisfies $f_u \in B_r(f)_\mathbb{C}$. Let $\Lambda \subset \mathbb{C}$ be an R-square $(R > 2)$. As in the above proof, for $4\varepsilon < (4/5)r \leq \delta$, we get

$$\mathrm{Widim}_\varepsilon(B_r(f)_\mathbb{C}, \mathrm{dist}_\Lambda) \geq 2(N+1)\int_\Lambda |df|^2 dxdy - 2C_f R.$$

Hence

$$\dim_f(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) := \lim_{r\to+0}\left\{\lim_{\varepsilon\to+0}\left(\lim_{R\to\infty}\frac{1}{R^2}\sup_{\Lambda: R\text{-square}}\mathrm{Widim}_\varepsilon(B_r(f)_\mathbb{C}, \mathrm{dist}_\Lambda)\right)\right\}$$
$$\geq 2(N+1)\rho(f).$$

Then $\dim_{loc}(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) := \sup_{f\in\mathcal{M}(\mathbb{C}P^N)} \dim_f(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$ satisfies

$$2(N+1)\rho(\mathbb{C}P^N) \leq \dim_{loc}(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \leq \dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \leq 4N\rho_{\mathrm{NSA}}(\mathbb{C}P^N).$$

The proof is the same as the proof of Theorem 1.1. In particular we get

$$\dim_{loc}(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) = \dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}).$$

4. Proof of Proposition 3.1

In this section we prove Proposition 3.1.

4.1. **Analytic preliminaries.** Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. As in Section 3, let $T\mathbb{C}P^N$ be the tangent bundle of $\mathbb{C}P^N$ with the natural holomorphic vector bundle structure, and let $E := f^*T\mathbb{C}P^N$ be the pull-back of $T\mathbb{C}P^N$. E is a holomorphic vector bundle over the complex plane $\mathbb{C}$. Its Hermitian metric h is given by the pull-back of the Fubini-Study metric. E is equipped with the unitary connection ∇ defined by the holomorphic structure and the metric h.

Let $1 < p < \infty$ be a real number, and $k \geq 0$ be an integer. Let $a \in L^p_{k,loc}(\Lambda^{0,i}(E))$ $(i = 0, 1)$ be a locally L^p_k-section of $\Lambda^{0,i}(E)$ (the $\mathcal{C}^\infty$-vector bundle of $(0, i)$-forms valued in E). For a subset $\Omega \subset \mathbb{C}$, we set

$$\|a\|_{L^p_k(\Omega)} := \left(\sum_{n=0}^{k}\int_\Omega |\nabla^n a|^p dxdy\right)^{1/p}.$$

We define the $\ell^\infty L^p_k$-norm $\|a\|_{\ell^\infty L^p_k}$ by

$$\|a\|_{\ell^\infty L^p_k} := \sup_{z\in\mathbb{C}} \|a\|_{L^p_k(D_1(z))}.$$

Let $\ell^\infty L^p_k(\Lambda^{0,i}(E))$ be the Banach space of all $a \in L^p_{k,loc}(\Lambda^{0,i}(E))$ satisfying $\|a\|_{\ell^\infty L^p_k} < +\infty$.

Lemma 4.1. *(i) For $a \in L^2_{2,loc}(\Lambda^{0,i}(E))$,*

$$\|a\|_{L^\infty(\mathbb{C})} \le \text{const}\, \|a\|_{\ell^\infty L^2_2}.$$

(Precisely speaking, if the right-hand-side is finite then the left-hand-side is also finite and satisfies the inequality.)
(ii) If $a \in L^p_{2,loc}(\Lambda^{0,i}(E))$ with $p > 2$, then

$$\|a\|_{L^\infty(\mathbb{C})} + \|\nabla a\|_{L^\infty(\mathbb{C})} \le \text{const}_p\, \|a\|_{\ell^\infty L^p_2}.$$

Proof. Since $\mathcal{M}(\mathbb{C}P^N)$ is compact, there are $\delta > 0$ and $\text{const}_k > 0$ $(k \ge 0)$ such that for every $z \in \mathbb{C}$ there is a trivialization u of the holomorphic vector bundle E over a neighborhood of $D_\delta(z)$ such that $u_* h = (h_{\alpha\bar\beta})_{\alpha\beta}$ (the Hermitian matrix representing h under the trivialization u) satisfies $\|h_{\alpha\bar\beta}\|_{\mathcal{C}^k(D_\delta(z))}, \left\|h^{\alpha\bar\beta}\right\|_{\mathcal{C}^k(D_\delta(z))} \le \text{const}_k$. (Here $(h^{\alpha\bar\beta}) = (h_{\alpha\bar\beta})^{-1}$.) Then the norms $\|a\|_{L^p_k(D_\delta(z))}$ and $\|a\|_{L^\infty(D_\delta(z))}$ are equivalent to $\|u\circ a\|_{L^p_k(D_\delta(z))}$ and $\|u\circ a\|_{L^\infty(D_\delta(z))}$ uniformly in $z \in \mathbb{C}$ respectively. (We consider $u \circ a$ as a $\mathbb{C}^N$-valued $(0,i)$-form in $D_\delta(z)$.) Hence the Sobolev embedding theorem (Gilbarg-Trudinger [6, Chapter 7.7]) implies

$$\|a\|_{L^\infty(D_\delta(z))} \le \text{const}\, \|a\|_{L^2_2(D_\delta(z))}.$$

Here the important point is that const is independent of $z \in \mathbb{C}$. Thus $\|a\|_{L^\infty(\mathbb{C})} \le \text{const}\, \|a\|_{\ell^\infty L^2_2}$. (ii) can be proved in the same way. □

Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function satisfying $\|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty$ for all $k \ge 0$. We set $\bar\partial^*_\varphi(a) := e^{-\varphi}\bar\partial^*(e^\varphi a)$ for $a \in \Omega^{0,1}(E)$. Here $\bar\partial^*$ is the formal adjoint of the Dolbeault operator $\bar\partial : \Omega^0(E) \to \Omega^{0,1}(E)$ with respect to the Hermitian metric h. $\bar\partial^*_\varphi$ is the formal adjoint of $\bar\partial$ with respect to the metric $e^\varphi h$. We define the operator $\Box_\varphi : \Omega^{0,i}(E) \to \Omega^{0,i}(E)$ by setting

$$\Box_\varphi a := \bar\partial^*_\varphi \bar\partial a \quad (i = 0), \quad \Box_\varphi a := \bar\partial\bar\partial^*_\varphi a \quad (i = 1).$$

Lemma 4.2. *For $a \in \ell^\infty L^p_{k+2}(\Lambda^{0,i}(E))$,*

$$\|a\|_{\ell^\infty L^p_{k+2}} \le \text{const}_{p,k,\varphi}\left(\|a\|_{\ell^\infty L^p} + \|\Box_\varphi a\|_{\ell^\infty L^p_k}\right).$$

More precisely, if $a \in L^p_{k+2,loc}(\Lambda^{0,1}(E))$ and the right hand side of the above is finite then $a \in \ell^\infty L^p_{k+2}$ and satisfies the above inequality.

Proof. We use the trivialization u of E introduced in the proof of Lemma 4.1. Since $\|\varphi\|_{\mathcal{C}^l(\mathbb{C})} < +\infty$ for all $l \geq 0$, under the trivialization u, the operator $\Box_\varphi$ is represented as

$$\Box_\varphi = (-1/2)\Delta + A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C$$

over a neighborhood of $D_\delta(z)$ where the $\mathcal{C}^l$-norms ($l \geq 0$) of the matrices A, B, C over $D_\delta(z)$ are bounded uniformly in $z \in \mathbb{C}$. Then from the L^p-estimate (Gilbarg-Trudinger [6, Chapter 9.5])

$$\|a\|_{L^p_{k+2}(D_{\delta/2}(z))} \leq \mathrm{const}_{p,k,\varphi}\left(\|a\|_{L^p(D_\delta(z))} + \|\Box_\varphi a\|_{L^p_k(D_\delta(z))}\right).$$

The desired estimate follows from this. $\Box$

4.2. **Perturbation of the Hermitian metric.** Here we develop a perturbation technique of a Hermitian metric (Lemma 4.5 below). Gromov also discussed it in [12, p. 399]. Tsukamoto [22, Section 4.3] studied an easier situation.

Lemma 4.3. *Let $g : \mathbb{C} \to \mathbb{R}_{\geq 0}$ be a non-negative smooth function with $\|g\|_{\mathcal{C}^k(\mathbb{C})} < +\infty$ for all $k \geq 0$. We suppose that the following non-degeneracy condition holds: There exist $\delta > 0$ and $R > 0$ such that for all $p \in \mathbb{C}$ we have $\|g\|_{L^\infty(D_R(p))} \geq \delta$. Then there exists a smooth function $\varphi : \mathbb{C} \to \mathbb{R}$ satisfying*

$$(-\Delta + 1)\varphi = -g, \quad \|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty \quad (\forall k \geq 0), \quad \sup_{z\in\mathbb{C}} \varphi(z) < 0.$$

Here $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Proof. We need the following sublemma.

Sublemma 4.4. *Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ ($\varphi \in \mathcal{C}^2_{loc}$). Suppose that the norms $\|\varphi\|_{L^\infty(\mathbb{C})}$ and $\|(-\Delta+1)\varphi\|_{L^\infty(\mathbb{C})}$ are both finite. Then*

$$\|\varphi\|_{L^\infty(\mathbb{C})} \leq 4\,\|(-\Delta+1)\varphi\|_{L^\infty(\mathbb{C})}.$$

Proof. Take $z_0 \in \mathbb{C}$ such that $|\varphi(z_0)| \geq \|\varphi\|_{L^\infty(\mathbb{C})}/2$. For simplicity, we suppose $z_0 = 0$. Moreover we suppose $\varphi(0) \geq 0$. (If $\varphi(0) < 0$ then we apply the following argument to $-\varphi$.) We define $w : \mathbb{C} \to \mathbb{R}$ by

$$w(z) := \frac{1}{2\pi}\int_0^{2\pi} e^{(x\cos\theta + y\sin\theta)/\sqrt{2}}d\theta.$$

w satisfies

$$(-\Delta + 1/2)w = 0, \quad \min_{z\in\mathbb{C}} w(z) = w(0) = 1, \quad w(z) \to +\infty \quad (|z| \to +\infty).$$

Then $(-\Delta+1)w = w/2 \geq 1/2$. For $\varepsilon > 0$, set $M := 2\,\|(-\Delta+1)\varphi\|_{L^\infty(\mathbb{C})} + \varepsilon > 0$.

$$(-\Delta+1)\left(Mw - \varphi\right) \geq M/2 - (-\Delta+1)\varphi \geq \|(-\Delta+1)\varphi\|_{L^\infty(\mathbb{C})} + \varepsilon/2 - (-\Delta+1)\varphi \geq \varepsilon/2.$$

Since the function $Mw - \varphi$ is positive for $|z| \gg 1$, the weak minimum principle (Gilbarg-Trudinger [6, Chapter 3.1, Corollary 3.2]) implies that this function is non-negative everywhere. Hence

$$\|\varphi\|_{L^\infty(\mathbb{C})}/2 \leq \varphi(0) \leq Mw(0) = M = 2\,\|(-\Delta+1)\varphi\|_{L^\infty(\mathbb{C})} + \varepsilon.$$

Let $\varepsilon \to 0$. We get

$$\|\varphi\|_{L^\infty(\mathbb{C})} \leq 4\,\|(-\Delta+1)\varphi\|_{L^\infty(\mathbb{C})}.$$

□

Let $\phi_n : \mathbb{C} \to [0,1]$ $(n \geq 1)$ be a cut-off function such that $\phi_n = 1$ over $D_n(0)$ and $\mathrm{supp}(\phi_n) \subset D_{n+1}(0)$. We want to solve the equation $(-\Delta+1)\varphi = -\phi_n g$. The following is a standard L^2-argument.

Let $L^2_1(\mathbb{C})$ be the space of L^2-functions $\varphi : \mathbb{C} \to \mathbb{R}$ satisfying $\partial\varphi/\partial x, \partial\varphi/\partial y \in L^2$ with the inner product $\langle \varphi, \varphi' \rangle_{L^2_1} := \langle \varphi, \varphi' \rangle_{L^2} + \langle \partial\varphi/\partial x, \partial\varphi'/\partial x \rangle_{L^2} + \langle \partial\varphi/\partial y, \partial\varphi'/\partial y \rangle_{L^2}$. Consider the bounded linear functional:

$$L^2_1(\mathbb{C}) \to \mathbb{R}, \quad \varphi \mapsto -\langle \varphi, \phi_n g \rangle_{L^2}.$$

From the Riesz representation theorem, there uniquely exists $\varphi_n \in L^2_1(\mathbb{C})$ satisfying $\langle \varphi, \varphi_n \rangle_{L^2_1} = -\langle \varphi, \phi_n g \rangle_{L^2}$ for all $\varphi \in L^2_1(\mathbb{C})$. This implies $(-\Delta+1)\varphi_n = -\phi_n g$ as a distribution. From the local elliptic regularity, φ_n is smooth and $\|\varphi_n\|_{L^\infty(\mathbb{C})} < +\infty$. Then we can apply Sublemma 4.4 to φ_n and get

$$\|\varphi_n\|_{L^\infty(\mathbb{C})} \leq 4\,\|\phi_n g\|_{L^\infty(\mathbb{C})} \leq 4\,\|g\|_{L^\infty(\mathbb{C})} < +\infty.$$

By the local elliptic regularity, for every compact subset $K \subset \mathbb{C}$ and $k \geq 0$, the sequence $\|\varphi_n\|_{\mathcal{C}^k(K)}$ $(n \geq 1)$ is bounded. Then we can choose a subsequence $n_1 < n_2 < n_3 < \dots$ such that φ_{n_k} converges to some φ in $\mathcal{C}^\infty$ over every compact subset of $\mathbb{C}$. φ satisfies $(-\Delta+1)\varphi = -g$ and $\|\varphi\|_{L^\infty(\mathbb{C})} \leq 4\,\|g\|_{L^\infty(\mathbb{C})}$. By the elliptic regularity, $\|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty$ for all $k \geq 0$.

Note that we have not used the non-degeneracy condition of the function g so far. We need it for the proof of the condition $\sup_{z\in\mathbb{C}} \varphi(z) < 0$.

Set $M := \sup_{z\in\mathbb{C}} \varphi(z)$. There are $z_n \in \mathbb{C}$ $(n \geq 1)$ such that $\varphi(z_n) \to M$. Set $\varphi_n(z) := \varphi(z + z_n)$ and $g_n(z) := g(z + z_n)$. Then

$$(-\Delta+1)\varphi_n = -g_n.$$

The sequences $\|\varphi_n\|_{\mathcal{C}^k(\mathbb{C})}$ and $\|g_n\|_{\mathcal{C}^k(\mathbb{C})}$ $(n \geq 1)$ are bounded for every $k \geq 0$. Hence by choosing a subsequence (denoted also by φ_n and g_n), we can assume that φ_n and g_n converge to φ_∞ and g_∞ respectively in $\mathcal{C}^\infty$ over every compact subset of $\mathbb{C}$. They satisfy

$$g_\infty \geq 0, \quad (-\Delta+1)\varphi_\infty = -g_\infty \leq 0, \quad \varphi_\infty(z) \leq \varphi_\infty(0) = M.$$

From the non-degeneracy condition of g, the function g_∞ is not zero. Hence if φ_∞ is a constant, then $\varphi_\infty = -g_\infty$ is a negative constant function and $M < 0$. If φ_∞ is not a constant, then the strong maximum principle [6, Chapter 3.2, Theorem 3.5] implies that φ_∞ cannot achieve a non-negative maximum value. Hence $M = \varphi_\infty(0) = \max_{z\in\mathbb{C}} \varphi_\infty(z) < 0$. $\square$

Recall that $f : \mathbb{C} \to \mathbb{C}P^N$ is a Brody curve and $E = f^*T\mathbb{C}P^N$. For $a \in \Omega^{0,1}(E)$ we have the Weintzenböck formula:

$$\bar\partial\bar\partial^* a = \frac{1}{2}\nabla^*\nabla a + \Theta a, \tag{5}$$

where $\Theta := [\nabla_{\partial/\partial z}, \nabla_{\partial/\partial \bar z}]$ is the curvature operator. The crucial fact for the analysis of this paper is that the holomorphic bisectional curvature of the Fubini-Study metric is positive. From this, there exists a positive constant c such that

$$h(\Theta a, a) \geq c|df|^2|a|^2.$$

This means that the curvature operator is positive where $|df|$ is positive. The non-degeneracy condition of the map f enters into the argument through this point. (See the condition (ii) of Definition-Lemma 1.3.) In the next lemma we will prove that if f is non-degenerate then we can perturb the Hermitian metric h so that the curvature is uniformly positive:

Lemma 4.5. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve. There is a smooth function $\varphi : \mathbb{C} \to \mathbb{R}$ with $\|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty$ $(\forall k \geq 0)$ satisfying the following. Let Θ_φ be the curvature of the Hermitian metric $h_\varphi := e^\varphi h$. Then there is $c' > 0$ such that*

$$h_\varphi(\Theta_\varphi a, a) \geq c'|a|^2_{h_\varphi}$$

for all $a \in \Omega^{0,1}(E)$.

Proof. We have $\Theta_\varphi a = \frac{-\Delta\varphi}{4}a + \Theta a$ for $a \in \Omega^{0,1}(E)$, and hence

$$h_\varphi(\Theta_\varphi a, a) = e^\varphi\left(\frac{-\Delta\varphi}{4}|a|_h^2 + h(\Theta a, a)\right) \geq e^\varphi\left(\frac{-\Delta\varphi}{4} + c|df|^2\right)|a|_h^2.$$

By the non-degeneracy of f and Lemma 4.3, there is a smooth function $\varphi : \mathbb{C} \to \mathbb{R}$ satisfying

$$(-\Delta + 1)\varphi = -4c|df|^2, \quad \|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty \quad (\forall k \geq 0), \quad \sup_{z\in\mathbb{C}} \varphi(z) < 0.$$

Then

$$h_\varphi(\Theta_\varphi a, a) \geq e^\varphi(-\varphi/4)|a|_h^2 = (-\varphi/4)|a|^2_{h_\varphi} \geq (-\sup_{z\in\mathbb{C}}\varphi(z)/4)|a|^2_{h_\varphi}.$$

Hence $c' := -\sup_{z\in\mathbb{C}}\varphi(z)/4 > 0$ satisfies the statement. $\square$

In our convention, the Fubini-Study metric $g_{i\bar j}$ on $\mathbb{C}P^N$ is given by

$$g_{i\bar j} = \frac{1}{2\pi}\frac{\partial^2}{\partial z_i \partial \bar z_j}\log(1 + |z_1|^2 + \cdots + |z_N|^2)$$

over $\{[1 : z_1 : \cdots : z_N]\} \subset \mathbb{C}P^N$. The spherical derivative $|df|(z)$ for a holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$ satisfies

$$f^*\left(\sqrt{-1}\sum g_{i\bar{j}} dz_i d\bar{z}_j\right) = |df|^2 dxdy. \tag{6}$$

The Fubini-Study metric $g_{i\bar{j}}$ satisfies the Kähler-Einstein equation

$$\mathrm{Ric}_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log(\det(g_{k\bar{l}})) = 2\pi(N+1)g_{i\bar{j}}.$$

From this, the curvature operator $\Theta = [\nabla_{\partial/\partial z}, \nabla_{\partial/\partial \bar{z}}]$ in (5) satisfies

$$\frac{\sqrt{-1}}{2\pi}\mathrm{tr}(\Theta) dz d\bar{z} = (N+1)|df|^2 dxdy \tag{7}$$

since $\mathrm{tr}(\Theta) dz d\bar{z} = f^*(\sum \mathrm{Ric}_{i\bar{j}} dz_i d\bar{z}_j)$. The equation (7) will be used in the proof of Proposition 5.1. Note that the form $(\sqrt{-1}/2\pi)\mathrm{tr}(\Theta) dz d\bar{z}$ is the Chern form representing $c_1(E)$ although we have $c_1(E) = 0$ because $H^2(\mathbb{C}; \mathbb{Z}) = 0$.

4.3. **L^∞-estimate.** Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve, and let $\varphi : \mathbb{C} \to \mathbb{R}$ be a smooth function introduced in Lemma 4.5. Propositions 4.6 and 4.7 below essentially use the positivity of the curvature Θ_φ.

The following L^∞-estimate was proved in [22, Proposition 4.2].

Proposition 4.6. *Let $a \in \Omega^{0,1}(E)$ be an E-valued $(0,1)$-form of class $\mathcal{C}^2$ ($a \in \mathcal{C}^2_{loc}$). Set $b := \Box_\varphi a$. If $\|a\|_{L^\infty(\mathbb{C})}, \|b\|_{L^\infty(\mathbb{C})} < +\infty$, then*

$$\|a\|_{L^\infty(\mathbb{C})} \leq \mathrm{const}_{f,\varphi} \|b\|_{L^\infty(\mathbb{C})}.$$

Proof. The proof is similar to the proof of Sublemma 4.4. For the detail, see [22, pp. 1648-1649]. □

Proposition 4.7. *Let $b \in L^2_{2,loc}(\Lambda^{0,1}(E))$ and suppose $\|b\|_{L^\infty(\mathbb{C})} < +\infty$. Then there uniquely exists $a \in L^2_{4,loc}(\Lambda^{0,1}(E))$ satisfying*

$$\Box_\varphi a = b, \quad \|a\|_{L^\infty(\mathbb{C})} < +\infty.$$

Moreover $\|a\|_{L^\infty(\mathbb{C})} + \|\nabla a\|_{L^\infty(\mathbb{C})} \leq \mathrm{const}_{f,\varphi} \|b\|_{L^\infty(\mathbb{C})}$.

Proof. The uniqueness follows from Proposition 4.6. (Note the Sobolev embedding $L^2_{4,loc} \hookrightarrow \mathcal{C}^2_{loc}$ in $\mathbb{R}^2$.) So the problem is the existence. We have the Weinzenböck formula: for $a \in \Omega^{0,1}(E)$

$$\Box_\varphi a = \frac{1}{2}\nabla_\varphi^* \nabla_\varphi a + \Theta_\varphi a,$$

where ∇_φ is the unitary connection on E with respect to the metric $h_\varphi = e^\varphi h$. Θ_φ satisfies the positivity condition in Lemma 4.5.

Let $\phi_n : \mathbb{C} \to [0,1]$ be a cut-off function such that $\phi_n = 1$ over $D_n(0)$ and $\mathrm{supp}(\phi_n) \subset D_{n+1}(0)$. From the positivity of the curvature, as in the proof of Lemma 4.3, a standard L^2-argument shows that there is $a_n \in L^2_1(\Lambda^{0,1}(E))$ (the space of L^2-sections a of $\Lambda^{0,1}(E)$

satisfying $\nabla_\varphi a \in L^2$) satisfying $\Box_\varphi a_n = \phi_n b$ as a distribution. (For the detail, see [22, Lemma 5.3].) The local elliptic regularity implies $a_n \in L^2_{4,loc}$. By Lemmas 4.1 (i) and 4.2,

$$\begin{aligned}\|a_n\|_{L^\infty(\mathbb{C})} &\le \mathrm{const}\, \|a_n\|_{\ell^\infty L^2_2} \le \mathrm{const}_\varphi \left(\|a_n\|_{\ell^\infty L^2} + \|\Box_\varphi a_n\|_{\ell^\infty L^2}\right)\\ &\le \mathrm{const}_\varphi \left(\|a_n\|_{L^2} + \|\phi_n b\|_{L^\infty(\mathbb{C})}\right) < +\infty.\end{aligned}$$

By Proposition 4.6 we have $\|a_n\|_{L^\infty(\mathbb{C})} \le \mathrm{const}_{f,\varphi} \|b_n\|_{L^\infty(\mathbb{C})} \le \mathrm{const}_{f,\varphi} \|b\|_{L^\infty(\mathbb{C})}$. Then for any compact set $K \subset \mathbb{C}$ the sequence $\|a_n\|_{L^2_2(K)}$ ($n \ge 1$) is bounded. By choosing a subsequence $n_1 < n_2 < n_3 < \dots$, the sequence a_{n_k} converges to some a weakly in $L^2_2(D_R(0))$ (and hence strongly in $L^\infty(D_R(0))$) for every $R > 0$. a satisfies $\Box_\varphi a = b$, and $\|a\|_{L^\infty(\mathbb{C})} \le \sup_{n\ge 1} \|a_n\|_{L^\infty(\mathbb{C})} \le \mathrm{const}_{f,\varphi} \|b\|_{L^\infty(\mathbb{C})}$. By the local elliptic regularity $a \in L^2_{4,loc}$. By Lemmas 4.1 (ii) and 4.2

$$\|a\|_{L^\infty(\mathbb{C})} + \|\nabla a\|_{L^\infty(\mathbb{C})} \le \mathrm{const}\, \|a\|_{\ell^\infty L^2_3} \le \mathrm{const}_\varphi \left(\|a\|_{\ell^\infty L^3} + \|b\|_{\ell^\infty L^3}\right) \le \mathrm{const}_{f,\varphi} \|b\|_{L^\infty(\mathbb{C})}.$$

□

4.4. **Deformation theory.** Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve with $\|df\|_{L^\infty(\mathbb{C})} < 1$. In this subsection we study a deformation of f and prove Proposition 3.1. Gromov [12, pp. 399-400, Projective interpolation theorem] studied a different kind of deformation theory. Our argument is a generalization of the deformation theory of elliptic Brody curves developed in [22].

Consider the following map (see McDuff-Salamon [17, p. 40]):

$$\Phi : \ell^\infty L^2_3(E) \to \ell^\infty L^2_2(\Lambda^{0,1}(E)), \quad u \mapsto P_u(\bar\partial \exp u) \otimes d\bar z.$$

Here $\exp u = \exp_{f(z)} u(z)$ is defined by the exponential map of the Fubini-Study metric, and

$$\bar\partial \exp u := \frac{1}{2}\left(\frac{\partial}{\partial x}\exp u + J\frac{\partial}{\partial y}\exp u\right) \quad (J\text{: complex structure of } \mathbb{C}P^N).$$

$P_{u(z)} : T_{\exp_{f(z)} u(z)}\mathbb{C}P^N \to T_{f(z)}\mathbb{C}P^N$ is the parallel translation along the geodesic $\exp_{f(z)}(tu(z))$ ($0 \le t \le 1$).

Φ is a smooth map between the Banach spaces. $\Phi(0) = 0$ and the derivative of Φ at the origin is equal to the Dolbeault operator:

$$d\Phi_0 = \bar\partial : \ell^\infty L^2_3(E) \to \ell^\infty L^2_2(\Lambda^{0,1}(E)).$$

Proposition 4.8. *There is a bounded linear operator $Q : \ell^\infty L^2_2(\Lambda^{0,1}(E)) \to \ell^\infty L^2_3(E)$ satisfying $\bar\partial \circ Q = 1$.*

Proof. We will prove that the map

$$\Box_\varphi = \bar\partial\bar\partial^*_\varphi : \ell^\infty L^2_4(\Lambda^{0,1}(E)) \to \ell^\infty L^2_2(\Lambda^{0,1}(E)) \tag{8}$$

is an isomorphism. ($\varphi : \mathbb{C} \to \mathbb{R}$ is a smooth function introduced in Lemma 4.5.) Then $Q := \bar{\partial}_\varphi^* \Box_\varphi^{-1} : \ell^\infty L_2^2(\Lambda^{0,1}(E)) \to \ell^\infty L_3^2(\Lambda^{0,1}(E))$ becomes a right inverse of $\bar{\partial}$. The injectivity of the map (8) directly follows from the L^∞-estimate in Proposition 4.6.

On the other hand, by Proposition 4.7, for every $b \in \ell^\infty L_2^2(\Lambda^{0,1}(E))$ there is $a \in L^\infty \cap L_{4,loc}^2(\Lambda^{0,1}(E))$ satisfying $\Box_\varphi a = b$. By Lemma 4.2, $a \in \ell^\infty L_4^2$. Thus the map (8) is surjective. $\Box$

Let H_f be the Banach space of all L^∞-holomorphic sections of E introduced in Section 3. H_f is equal to the kernel of the map $\bar{\partial} : \ell^\infty L_3^2(E) \to \ell^\infty L_2^2(\Lambda^{0,1}(E))$ by Lemmas 4.1 and 4.2. Moreover the norms $\|\cdot\|_{\ell^\infty L_k^2}$ ($k \geq 0$) are all equivalent to the norm $\|\cdot\|_{L^\infty(\mathbb{C})}$ over H_f.

From Proposition 4.8 and the implicit function theorem, there are $r > 0$ and a smooth map $\alpha : \{u \in H_f |\, \|u\|_{L^\infty(\mathbb{C})} < r\} \to \mathrm{Im}Q$ ($\mathrm{Im}Q \subset \ell^\infty L_3^2(E)$ is a closed subspace) such that

$$\Phi(u + \alpha(u)) = 0, \quad \alpha(0) = 0, \quad d\alpha_0 = 0.$$

The first and second conditions imply that $f_u := \exp_f(u + \alpha(u))$ becomes a holomorphic curve with $f_0 = f$. The third condition implies that for any $\varepsilon > 0$ there exists $0 < \delta < r$ such that if $u, v \in H_f$ satisfies $\|u\|_{L^\infty(\mathbb{C})}, \|v\|_{L^\infty(\mathbb{C})} \leq \delta$ then $\|\alpha(u) - \alpha(v)\|_{L^\infty(\mathbb{C})} \leq \varepsilon \|u - v\|_{L^\infty(\mathbb{C})}$.

Proof of Proposition 3.1. Since $\|df\|_{L^\infty(\mathbb{C})} < 1$, if $\delta \ll 1$, the holomorphic curves f_u ($u \in B_\delta(H_f)$) satisfy $\|df_u\|_{L^\infty(\mathbb{C})} \leq 1$. We will prove that if $0 < \delta < r$ is sufficiently small then the map

$$B_\delta(H_f) \ni u \mapsto f_u \in \mathcal{M}(\mathbb{C}P^N)$$

satisfies the conditions in Proposition 3.1. The condition (i) ($f_0 = f$) is OK. So we want to prove the condition (ii).

We choose $0 < \delta < r$ sufficiently small so that all $u, v \in B_\delta(H_f)$ satisfy

$$\|\alpha(u) - \alpha(v)\|_{L^\infty(\mathbb{C})} \leq (1/20) \|u - v\|_{L^\infty(\mathbb{C})},$$

and that if $v_1, v_2 \in T_p\mathbb{C}P^N$ are two tangent vectors satisfying $|v_1|, |v_2| \leq 2\delta$ then

$$|d(\exp(v_1), \exp(v_2)) - |v_1 - v_2|| \leq (1/20)|v_1 - v_2|.$$

The former condition comes from $d\alpha_0 = 0$, and the latter is just a standard property of the exponential map. Then all $u, v \in B_\delta(H_f)$ satisfy

$$\begin{aligned} |d(\exp(u + \alpha(u)), \exp(v + \alpha(v))) - |u + \alpha(u) - v - \alpha(v)|| &\leq (1/20)\, |u + \alpha(u) - v - \alpha(v)| \\ \leq (1/20) \|u - v\|_{L^\infty(\mathbb{C})} + (1/20) \|\alpha(u) - \alpha(v)\|_{L^\infty(\mathbb{C})} &\leq (1/20 + 1/400) \|u - v\|_{L^\infty(\mathbb{C})}, \end{aligned}$$

and

$$||u + \alpha(u) - v - \alpha(v)| - |u - v|| \leq |\alpha(u) - \alpha(v)| \leq (1/20) \|u - v\|_{L^\infty(\mathbb{C})}.$$

These inequalities imply the condition (ii):

$$|d(\exp(u + \alpha(v)), \exp(v + \alpha(v))) - |u - v|| \leq (1/8) \|u - v\|_{L^\infty(\mathbb{C})}.$$

□

5. Study of H_f: proof of Proposition 3.2

In this section we prove Proposition 3.2. Let $R > 0$, and let $\Lambda = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{C}$ be an R-square (i.e. $b_1 = a_1 + R$ and $b_2 = a_2 + R$). For $0 < r < R/2$, we set

$$\partial_r \Lambda = \{([a_1, a_1 + r) \cup (b_1 - r, b_1]) \times [a_2, b_2]\} \cup \{[a_1, b_1] \times ([a_2, a_2 + r) \cup (b_2 - r, b_2])\}.$$

(This notation is used only in this section. It conflicts with the notation $\partial_r \Omega$ introduced in Section 2.1.) The following is a preliminary version of Proposition 3.2.

Proposition 5.1. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. Let $\varepsilon > 0$, and let $\Lambda \subset \mathbb{C}$ be an R-square with $R > 2$. Then there exists a finite dimensional complex subspace $W \subset \Omega^0(E)$ (the space of $\mathcal{C}^\infty$-sections of $E = f^* T\mathbb{C}P^N$) satisfying the following three conditions.*
(i)

$$\dim_{\mathbb{C}} W \geq (N+1) \int_{\Lambda} |df|^2 dxdy - C_\varepsilon R,$$

where C_ε is a constant depending only on ε. (The important point is that it is independent of R.)
(ii) All $u \in W$ satisfy $u = 0$ outside of Λ.
(iii) All $u \in W$ satisfy $\left\|\bar{\partial} u\right\|_{L^\infty(\mathbb{C})} \leq \varepsilon \left\|u\right\|_{L^\infty(\mathbb{C})}$.

Proof. Set $\Lambda = [a_1, b_1] \times [a_2, b_2]$. Let $\varphi_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ be smooth functions such that $0 \leq \varphi_i' \leq 1$, $\varphi_i(x) = x$ over $[a_i + 1/2, b_i - 1/2]$, $\varphi(x) = \varphi(a_i + 1/4)$ over $x \leq a_i + 1/4$ and $\varphi_i(x) = \varphi(b_i - 1/4)$ over $x \geq b_i - 1/4$. Moreover we assume that, for $k \geq 1$, $|\varphi_i^{(k)}| \leq \mathrm{const}_k$ (depending only on $k \geq 1$).

We define a $\mathcal{C}^\infty$-map $\tilde{f} : \mathbb{C} \to \mathbb{C}P^N$ by $\tilde{f}(x + \sqrt{-1}y) := f(\varphi_1(x) + \sqrt{-1}\varphi_2(y))$. We have $|d\tilde{f}|(z) := \max_{u \in T_z\mathbb{C},\, |u|=1} |d\tilde{f}(u)| \leq 1$ for all $z \in \mathbb{C}$. Let $\tilde{E} := \tilde{f}^* T\mathbb{C}P^N$ be the pull-back of $T\mathbb{C}P^N$ by $\tilde{f}$. $\tilde{E}$ is a complex vector bundle over $\mathbb{C}$ with the Hermitian metric $\tilde{h}$ (the pull-back of the Fubini-Study metric) and the unitary connection $\tilde{\nabla}$ (the pull-back of the Levi-Civita connection on $T\mathbb{C}P^N$). From the definition of $\tilde{f}$, the connection $\tilde{\nabla}$ is flat over $\partial_{1/4}\Lambda$. Flat connections over $\partial_{1/4}\Lambda$ are classified by their holonomy maps $\pi_1(\partial_{1/4}\Lambda) \to U(N)$. Hence there is a bundle trivialization (as a Hermitian vector bundle) g of $\tilde{E}$ over $\partial_{1/4}\Lambda$ such that $g(\tilde{\nabla}) = d + A$ (A: connection matrix) satisfies

$$\|A\|_{\mathcal{C}^k(\partial_{1/4}\Lambda)} \leq \mathrm{const}_k \quad (k \geq 0).$$

Here const_k are universal constants depending only on k. (The important point is that they are independent of R.) Let $\psi : \Lambda \to [0, 1]$ be a cut-off function such that $\psi = 1$ over $\Lambda \setminus \partial_{1/5}\Lambda$, $\psi = 0$ over $\partial_{1/6}\Lambda$, and $\|\psi\|_{\mathcal{C}^k(\Lambda)} \leq \mathrm{const}_k$. We define a unitary connection ∇' on $\tilde{E}$ over Λ by $\nabla' := g^{-1}(d + \psi A)$. ($\nabla' = \tilde{\nabla}$ over $\Lambda \setminus \partial_{1/5}\Lambda$.) Under the trivialization g, the metric $\tilde{h}$ and the connection ∇' are equal to the standard metric and the product connection of $\partial_{1/6}\Lambda \times \mathbb{C}^N$ over $\partial_{1/6}\Lambda$.

Consider an elliptic curve $\mathbb{T} := \mathbb{C}/(R\mathbb{Z} + R\sqrt{-1}\mathbb{Z})$, and let $\pi : \mathbb{C} \to \mathbb{T}$ be the natural projection. We define a complex vector bundle E' over $\mathbb{T}$ as follows. $E' = \tilde{E}$ over $\pi(\Lambda \setminus \partial_{1/5}\Lambda) \cong \Lambda \setminus \partial_{1/5}\Lambda$, and $E'|_{\pi(\partial_{1/4}\Lambda)}$ is equal to the product bundle $\pi(\partial_{1/4}\Lambda) \times \mathbb{C}^N$. We glue these by the map g. The metric $\tilde{h}$ and the connection ∇' naturally descend to the metric and connection on E' (also denoted by $\tilde{h}$ and ∇').

Let $\Theta' := [\nabla'_{\partial/\partial z}, \nabla'_{\partial/\partial \bar{z}}]$ be the curvature of ∇'. From the definition, $\Theta' = [\nabla_{\partial/\partial z}, \nabla_{\partial/\partial \bar{z}}]$ over $\pi(\Lambda \setminus \partial_{1/2}\Lambda) \cong \Lambda \setminus \partial_{1/2}\Lambda$, and $|\Theta'| \le \mathrm{const}$ (a universal constant) all over $\mathbb{T}$. Then by (7)

$$\int_{\mathbb{T}} c_1(E') = \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{T}} \mathrm{tr}(\Theta')dzd\bar{z} \ge (N+1)\int_\Lambda |df|^2 dxdy - \mathrm{const}\cdot R. \tag{9}$$

Let $\bar{\partial}_{\nabla'} : \Omega^0(E') \to \Omega^{0,1}(E')$ be the Dolbeault operator over $\mathbb{T}$ twisted by the unitary connection ∇' (i.e. the $(0,1)$-part of the covariant derivative $\nabla' : \Omega^0(E) \to \Omega^1(E)$). Let $H^0_{\nabla'}$ be the space of $u \in \Omega^0(E')$ satisfying $\bar{\partial}_{\nabla'}u = 0$. From the Riemann-Roch formula and the above (9)

$$\dim_{\mathbb{C}} H^0_{\nabla'} \ge \int_{\mathbb{T}} c_1(E') \ge (N+1)\int_\Lambda |df|^2 dxdy - \mathrm{const}\cdot R. \tag{10}$$

Lemma 5.2. *For all $u \in H^0_{\nabla'}$,*

$$\|\nabla' u\|_{L^\infty(\mathbb{T})} \le K\,\|u\|_{L^\infty(\mathbb{T})}.$$

Here K is a universal constant (independent of f, R, Λ).

Proof. The connection ∇' has the following property: There is a universal constant $r > 0$ such that for every $p \in \mathbb{T}$ there is a bundle trivialization v of a Hermitian vector bundle E' over $D_r(p)$ satisfying $v(\nabla') = d + A'$ with

$$\|A'\|_{\mathcal{C}^k(D_r(p))} \le \mathrm{const}_k \quad (k \ge 0).$$

Then the result follows from the elliptic regularity. □

Let $\tau = \tau(\varepsilon) > 0$ be a small number which will be fixed later. We take points $p_1, \dots, p_M \in \pi(\partial_1\Lambda)$ with $M \le \mathrm{const}_\tau \cdot R$ such that for every $p \in \pi(\partial_1\Lambda)$ there is p_i satisfying $d(p, p_i) \le \tau$. We define $V \subset H^0_{\nabla'}$ as the space of $u \in H^0_{\nabla'}$ satisfying $u(p_i) = 0$ for all $i = 1, \dots, M$. From (10),

$$\dim_{\mathbb{C}} V \ge \dim_{\mathbb{C}} H^0_{\nabla'} - \dim_{\mathbb{C}}\left(\bigoplus_{i=1}^M E'_{p_i}\right) \ge (N+1)\int_\Lambda |df|^2 dxdy - C_\varepsilon R. \tag{11}$$

Let $u \in V$ and $p \in \pi(\partial_1\Lambda)$. Take p_i satisfying $d(p, p_i) \le \tau$. From $u(p_i) = 0$ and Lemma 5.2,

$$|u(p)| \le \tau\,\|\nabla' u\|_{L^\infty(\mathbb{T})} \le \tau K\,\|u\|_{L^\infty(\mathbb{T})}.$$

We choose $\tau > 0$ so that $\tau K < 1$. Then the maximum of $|u|$ is attained in $\mathbb{T} \setminus \pi(\partial_1\Lambda)$.

Let $\phi : \mathbb{C} \to \mathbb{R}$ be a cut-off such that $\phi = 1$ over $\Lambda \setminus \partial_1\Lambda$, $\mathrm{supp}(\phi)$ is contained in the interior of $\Lambda \setminus \partial_{1/2}\Lambda$, and $|d\phi| \leq 10$. For $u \in V$, we set $u' := \phi u$. Here we identify the region $\Lambda \setminus \partial_{1/2}\Lambda$ with $\pi(\Lambda \setminus \partial_{1/2}\Lambda)$ where we have $E' = E$, and we consider u' as a section of E over the plane $\mathbb{C}$. Set $W := \{u' | u \in V\}$. We have $\|u'\|_{L^\infty(\mathbb{C})} = \|u\|_{L^\infty(\mathbb{T})}$. Hence, by (11), we get the condition (i):

$$\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} V \geq (N+1) \int_\Lambda |df|^2 dxdy - C_\varepsilon R.$$

The condition (ii) is obviously satisfied. $\bar{\partial} u' = \bar{\partial}\phi \otimes u$ is supported in $\partial_1\Lambda$.

$$\left\|\bar{\partial} u'\right\|_{L^\infty(\mathbb{C})} \leq 10 \left\|u\right\|_{L^\infty(\pi(\partial_1\Lambda))} \leq 10\tau K \left\|u\right\|_{L^\infty(\mathbb{T})} = 10\tau K \left\|u'\right\|_{L^\infty(\mathbb{C})}.$$

We choose $\tau > 0$ so that $10\tau K \leq \varepsilon$. Then the condition (iii) is satisfied. □

Proof of Proposition 3.2. Let $\varepsilon > 0$ be a small number which will be fixed later. By Proposition 5.1, for this ε and any R-square Λ ($R > 2$), there is a finite dimensional complex subspace $W \subset \Omega^0(E)$ satisfying the conditions (i), (ii), (iii) in Proposition 5.1. By Proposition 4.7, there is a linear map

$$W \to \Omega^{0,1}(E), \quad u \mapsto a,$$

such that

$$\bar{\partial}\bar{\partial}_\varphi^* a = \bar{\partial} u, \quad \left\|\bar{\partial}_\varphi^* a\right\|_{L^\infty(\mathbb{C})} \leq C'_f \left\|\bar{\partial} u\right\|_{L^\infty(\mathbb{C})} \leq C'_f \cdot \varepsilon \left\|u\right\|_{L^\infty(\mathbb{C})}.$$

Set $u' := u - \bar{\partial}_\varphi^* a$. Then $\bar{\partial} u' = 0$ and $\|u'\|_{L^\infty(\mathbb{C})} \geq (1 - C'_f\varepsilon) \|u\|_{L^\infty(\mathbb{C})}$. We choose $\varepsilon > 0$ so that $1 - C'_f\varepsilon > 0$. We set $V := \{u' | u \in W\}$. Then $V \subset H_f$ and

$$\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W \geq (N+1) \int_\Lambda |df|^2 dxdy - C_\varepsilon R.$$

For $u \in W$ (recall $\mathrm{supp}(u) \subset \Lambda$)

$$\|u'\|_{L^\infty(\mathbb{C})} \leq (1 + C'_f\varepsilon) \|u\|_{L^\infty(\mathbb{C})} = (1 + C'_f\varepsilon) \|u\|_{L^\infty(\Lambda)},$$

$$\|u'\|_{L^\infty(\Lambda)} \geq (1 - C'_f\varepsilon) \|u\|_{L^\infty(\Lambda)}.$$

Hence

$$\|u'\|_{L^\infty(\mathbb{C})} \leq \frac{1 + C'_f\varepsilon}{1 - C'_f\varepsilon} \|u'\|_{L^\infty(\Lambda)}.$$

We choose $\varepsilon > 0$ so small that

$$\frac{1 + C'_f\varepsilon}{1 - C'_f\varepsilon} \leq 2.$$

□

6. Infinite gluing: proof of Theorem 1.7

We prove Theorem 1.7 in this section. Our method is gluing: We glue infinitely many rational curves to a (possibly degenerate) Brody curve $f : \mathbb{C} \to \mathbb{C}P^N$, and construct a non-degenerate one.

A kind of "infinite gluing construction" is classically used for the proof of Mittag-Leffler's theorem. Probably another origin of infinite gluing construction is the shadowing lemma in dynamical system theory (for example, see Bowen [2, Chapter 3]). Angenent [1] developed a shadowing lemma for an elliptic PDE. Gromov [12, p. 403] suggested an idea of gluing infinitely many rational curves to a (pseudo-)holomorphic curve. Macrì-Nolasco-Ricciardi [15] developed gluing infinitely many selfdual vortices. Gournay [7, 11] studied an infinite gluing method for pseudo-holomorphic curves. Tsukamoto [18, 21] studied gluing infinitely many Yang-Mills instantons.

First we establish a result on gluing one rational curve:

Proposition 6.1. *There are $\delta_0 > 0$, $R_0 > 0$ and $K > 0$ satisfying the following statement. Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. If f satisfies $\|df\|_{L^\infty(D_R(p))} < \delta_0$ for some $p \in \mathbb{C}$ and $R \geq R_0 + 1$, then there exists a holomorphic curve $g : \mathbb{C} \to \mathbb{C}P^N$ satisfying the following three conditions.*
(i) $\delta_0 \leq \|dg\|_{L^\infty(D_R(p))} \leq 2/3$.
(ii) $||dg|(z) - |df|(z)| \leq K/|z-p|^3$ over $|z-p| > R$.
(iii) $d(f(z), g(z)) \leq K/|z-p|^3$ for $z \neq p$.

Proof. The proof is just a calculation. It may be helpful for some readers to consider the case of $N = 1$ by themselves. Let $\varepsilon > 0$ be a sufficiently small number. δ_0, R_0, K and ε will be fixed later. Several conditions will be imposed on them through the argument, but basically they need to satisfy

$$\delta_0 \ll \frac{\varepsilon}{R_0}, \quad R_0 \gg 1, \quad \varepsilon \ll \frac{1}{R_0^4}.$$

Fix $a > 0$ so that the curve $q : \mathbb{C} \to \mathbb{C}P^N$ defined by $q(z) := [1 : a/z^3 : \cdots : a/z^3]$ satisfies $\|dq\|_{L^\infty(\mathbb{C})} = 1/12$. Here

$$|dq|(z) = \frac{3a\sqrt{N}r^2}{\sqrt{\pi}(r^6 + Na^2)} \quad (r = |z|).$$

We can suppose $\|dq\|_{L^\infty(D_{R_0}(0))} = 1/12$ since we choose $R_0 \gg 1$.

From the symmetry we can assume $p = 0$ and $f(0) = [1 : 0 : \cdots : 0]$. Let $f(z) = [1 : f_1(z) : \cdots : f_N(z)]$ where $f_i(z)$ are meromorphic functions in $\mathbb{C}$. Since $|df| \leq \delta_0$ over $|z| \leq R$ with $R \geq R_0 + 1$, if we choose δ_0 sufficiently small ($\delta_0 \ll \varepsilon/R_0$), we have

$$|f_i(z)| \leq \varepsilon, \quad |f_i'(z)| \leq \varepsilon \quad (|z| \leq R_0). \tag{12}$$

Set $g_i(z) := f_i(z) + a/z^3$, and we define $g : \mathbb{C} \to \mathbb{C}P^N$ by $g(z) := [1 : g_1(z) : \cdots : g_N(z)]$. We will prove that this map g satisfies the conditions (i), (ii), (iii).

First we study the condition (iii). The Fubini-Study metric is given by

$$ds^2 = \frac{\sum_{i=1}^N |dz_i|^2 + \sum_{1\leq i<j\leq N} |z_j dz_i - z_i dz_j|^2}{\pi(1+\sum |z_i|^2)^2} \quad \text{on } \{[1 : z_1 : \cdots : z_N]\}.$$

$$ds^2 \leq \frac{\sum |dz_i|^2 + 2(\sum |z_i|^2)(\sum |dz_i|^2)}{\pi(1+\sum |z_i|^2)^2} \leq \frac{2(1+\sum |z_i|^2)\sum |dz_i|^2}{\pi(1+\sum |z_i|^2)^2} \leq \frac{2}{\pi}\sum |dz_i|^2$$

Hence $ds \leq \sqrt{2/\pi}\sqrt{\sum_{i=1}^N |dz_i|^2}$. Thus for $f(z) = [1 : f_1(z) : \cdots : f_N(z)]$ and $g(z) = [1 : f_1(z) + a/z^3 : \cdots : f_N(z) + a/z^3]$ we get

$$d(f(z), g(z)) \leq \sqrt{2/\pi}\sqrt{\sum_{i=1}^N |a/z^3|^2} = \frac{a\sqrt{2N/\pi}}{|z|^3}. \tag{13}$$

Next we study the conditions (i) and (ii). We have

$$|df|(z) = \frac{\sqrt{\sum |f_i'(z)|^2 + \sum_{i<j} |f_i'(z)f_j(z) - f_i(z)f_j'(z)|^2}}{\sqrt{\pi}(1+\sum |f_i(z)|^2)},$$

$$|dg|(z) = \frac{\sqrt{\sum |g_i'(z)|^2 + \sum_{i<j} |g_i'(z)g_j(z) - g_i(z)g_j'(z)|^2}}{\sqrt{\pi}(1+\sum |g_i(z)|^2)},$$

where

$$g_i' = f_i' - \frac{3a}{z^4}, \quad g_i'g_j - g_ig_j' = (f_i'f_j - f_if_j') + \frac{3a}{z^4}(f_i - f_j) + \frac{a}{z^3}(f_i' - f_j').$$

Case 1: Suppose $r := |z| \leq R_0$. We will prove $\delta_0 \leq \|dg\|_{L^\infty(D_{R_0}(0))} \leq 2/3$. From (12),

$$|g_i(z)| \leq \varepsilon + \frac{a}{r^3} \leq \frac{2a}{r^3}, \quad |g_i'(z)| \geq \frac{3a}{r^4} - \varepsilon \geq \frac{3a}{2r^4}.$$

Here we have supposed $\varepsilon \leq \min(a/R_0^3, 3a/(2R_0^4))$. Then

$$|dg|(z) \geq \frac{\sqrt{N}(3a/(2r^4))}{\sqrt{\pi}(1+4Na^2/r^6)} = \frac{3a\sqrt{N}r^2}{2\sqrt{\pi}(r^6+4Na^2)} \geq \frac{3a\sqrt{N}r^2}{8\sqrt{\pi}(r^6+Na^2)} = \frac{|dq|(z)}{8}.$$

Hence $\|dg\|_{L^\infty(D_{R_0}(0))} \geq (1/8)\|dq\|_{L^\infty(D_{R_0}(0))} = 1/96 \geq \delta_0$. (Here we have supposed $\delta_0 \leq 1/96$.) On the other hand,

$$|dg|(z) = \frac{\sqrt{\sum |3az^2 - z^6 f_i'|^2 + \sum_{i<j} |z^6(f_i'f_j - f_j'f_i) + 3az^2(f_i - f_j) + az^3(f_i' - f_j')|^2}}{\sqrt{\pi}(r^6 + \sum |a + z^3 f_i|^2)}.$$

From (12),

$$|a + z^3 f_i| \geq a - \varepsilon R_0^3 \geq \frac{a}{2}, \quad (\text{here we suppose } \varepsilon R_0^3 \leq a/2).$$

$$r^6 + \sum |a + z^3 f_i|^2 \geq r^6 + \frac{Na^2}{4} \geq \frac{r^6 + Na^2}{4}.$$

$$|3az^2 - z^6 f_i'| \leq 3ar^2 + r^6\varepsilon \leq r^2(3a + R_0^4\varepsilon) \leq 4ar^2, \quad (\text{we suppose } R_0^4\varepsilon \leq a).$$

$$|z^6(f_i'f_j - f_j'f_i) + 3az^2(f_i - f_j) + az^3(f_i' - f_j')| \leq r^2(2\varepsilon^2 R_0^4 + 6a\varepsilon + 2a\varepsilon R_0) \leq \frac{ar^2}{\sqrt{\binom{N}{2}}}.$$

Here we have supposed $2\varepsilon^2 R_0^4 + 6a\varepsilon + 2a\varepsilon R_0 \leq a/\sqrt{\binom{N}{2}}$. Then

$$|dg|(z) \leq \frac{4ar^2\sqrt{16N+1}}{\sqrt{\pi}(r^6 + Na^2)} \leq \frac{24ar^2\sqrt{N}}{\sqrt{\pi}(r^6 + Na^2)} = 8|dq|(z) \leq \frac{2}{3}, \quad (|dq| \leq 1/12).$$

Thus we get $\delta_0 \leq \|dg\|_{L^\infty(D_{R_0}(0))} \leq 2/3$.

Case 2: Suppose $|z| \geq R_0$. We will prove $||df|(z) - |dg|(z)| \leq K/r^3$ for an appropriate $K > 0$. We have

$$\big||f_i|^2 - |g_i|^2\big| \leq (|f_i| + |g_i|) \cdot |f_i - g_i| \leq (2|f_i| + a/r^3)(a/r^3) \leq (2|f_i| + a/R_0^3)(a/r^3),$$

$$\sum \big||f_i|^2 - |g_i|^2\big| \leq \frac{a}{r^3}\left(2\sum |f_i| + \frac{Na}{R_0^3}\right) \leq \frac{2a}{r^3}\left(1 + \sum |f_i|\right) \quad (\text{we suppose } \frac{Na}{R_0^3} \leq 2).$$

If $|f_i| \geq a/r^3$, then

$$|g_i|^2 \geq \left(|f_i| - a/r^3\right)^2 \geq \frac{|f_i|^2}{2} - \frac{a^2}{r^6} \geq \frac{|f_i|^2}{2} - \frac{a^2}{R_0^6}, \quad ((x-y)^2 \geq \frac{x^2}{2} - y^2).$$

If $|f_i| < a/r^3$, then

$$|g_i|^2 \geq 0 > \frac{|f_i|^2}{2} - \frac{a^2}{r^6} \geq \frac{|f_i|^2}{2} - \frac{a^2}{R_0^6}.$$

Therefore we always have $|g_i|^2 \geq |f_i|^2/2 - a^2/R_0^6$.

$$1 + \sum |g_i|^2 \geq \left(1 - \frac{Na^2}{R_0^6}\right) + \frac{1}{2}\sum |f_i|^2 \geq \frac{1}{2}\left(1 + \sum |f_i|^2\right) \quad (\text{we suppose } \frac{Na^2}{R_0^6} \leq \frac{1}{2}).$$

Hence

$$\begin{aligned} \left|\frac{1}{1+\sum |g_i|^2} - \frac{1}{1+\sum |f_i|^2}\right| &\leq \frac{\frac{4a}{r^3}\left(1+\sum |f_i|\right)}{\left(1+\sum |f_i|^2\right)^2} \leq \frac{4a\sqrt{N+1}\sqrt{1+\sum |f_i|^2}}{r^3\left(1+\sum |f_i|^2\right)^2} \\ &= \frac{4a\sqrt{N+1}}{r^3\left(1+\sum |f_i|^2\right)^{3/2}} \leq \frac{4a\sqrt{N+1}}{r^3(1+\sum |f_i|^2)}. \end{aligned} \tag{14}$$

Then, from $g_i' = f_i' - 3a/z^4$ and the above (14),

$$\begin{aligned} \left|\frac{|g_i'|}{1+\sum |g_k|^2} - \frac{|f_i'|}{1+\sum |f_k|^2}\right| &\leq \left|\frac{|g_i'|}{1+\sum |g_k|^2} - \frac{|g_i'|}{1+\sum |f_k|^2}\right| + \left|\frac{|g_i'|}{1+\sum |f_k|^2} - \frac{|f_i'|}{1+\sum |f_k|^2}\right| \\ &\leq \frac{4a\sqrt{N+1}(|f_i'| + 3a/r^4)}{r^3(1+\sum |f_k|^2)} + \frac{3a}{r^4(1+\sum |f_k|^2)}. \end{aligned}$$

From $|df| \leq 1$, we have $|f_i'|/(1+\sum |f_k|^2) \leq \sqrt{\pi}$. Hence the above is bounded by

$$\frac{4a\sqrt{N+1}}{r^3}(\sqrt{\pi} + 3a/r^4) + 3a/r^4 \leq \frac{4a\sqrt{N+1}}{r^3}(\sqrt{\pi} + 3a) + \frac{3a}{r^3}.$$

Here we have supposed $r \geq R_0 \geq 1$. Set $K_a := 4a\sqrt{N+1}(\sqrt{\pi}+3a)+3a$. Then

$$\left|\frac{|g_i'|}{1+\sum|g_k|^2} - \frac{|f_i'|}{1+\sum|f_k|^2}\right| \leq \frac{K_a}{r^3}. \tag{15}$$

From (14), for $i<j$,

$$\begin{aligned}\left|\frac{|g_i'g_j-g_j'g_i|}{1+\sum|g_k|^2} - \frac{|f_i'f_j-f_j'f_i|}{1+\sum|f_k|^2}\right| &\leq \left|\frac{|g_i'g_j-g_j'g_i|}{1+\sum|g_k|^2} - \frac{|g_i'g_j-g_j'g_i|}{1+\sum|f_k|^2}\right| + \left|\frac{|g_i'g_j-g_j'g_i|}{1+\sum|f_k|^2} - \frac{|f_i'f_j-f_j'f_i|}{1+\sum|f_k|^2}\right| \\ &\leq \frac{4a\sqrt{N+1}|g_i'g_j-g_j'g_i|}{r^3(1+\sum|f_k|^2)} + \frac{|(g_i'g_j-g_j'g_i)-(f_i'f_j-f_j'f_i)|}{1+\sum|f_k|^2}\end{aligned}$$

From $g_i'g_j-g_j'g_i = (f_i'f_j-f_j'f_i)+(3a/z^4)(f_i-f_j)+(a/z^3)(f_i'-f_j')$, this is bounded by

$$\begin{aligned}&\frac{4a\sqrt{N+1}}{r^3}\left(\frac{|f_i'f_j-f_j'f_i|}{1+\sum|f_k|^2}+\frac{3a(|f_i|+|f_j|)}{r^4(1+\sum|f_k|^2)}+\frac{a(|f_i'|+|f_j'|)}{r^3(1+\sum|f_k|^2)}\right)\\ &\quad+\frac{3a(|f_i|+|f_j|)}{r^4(1+\sum|f_k|^2)}+\frac{a(|f_i'|+|f_j'|)}{r^3(1+\sum|f_k|^2)}.\end{aligned} \tag{16}$$

From $|df|\leq 1$,

$$\frac{|f_i'f_j-f_j'f_i|}{1+\sum|f_k|^2} \leq \sqrt{\pi}, \quad \frac{|f_i'|+|f_j'|}{1+\sum|f_k|^2} \leq 2\sqrt{\pi}.$$

Since $i<j$,

$$\frac{|f_i|+|f_j|}{1+\sum|f_k|^2} \leq \frac{\sqrt{2}\sqrt{|f_i|^2+|f_j|^2}}{1+\sum|f_k|^2} \leq \sqrt{2}.$$

Hence the above (16) is bounded by

$$\begin{aligned}&\frac{4a\sqrt{N+1}}{r^3}\left(\sqrt{\pi}+\frac{3a\sqrt{2}}{r^4}+\frac{2a\sqrt{\pi}}{r^3}\right)+\frac{3a\sqrt{2}}{r^4}+\frac{2a\sqrt{\pi}}{r^3}\\ &\leq \frac{4a\sqrt{N+1}}{r^3}(\sqrt{\pi}+3a\sqrt{2}+2a\sqrt{\pi})+\frac{3a\sqrt{2}}{r^3}+\frac{2a\sqrt{\pi}}{r^3}.\end{aligned}$$

Here $r\geq R_0\geq 1$. Set $K_a' := 4a\sqrt{N+1}(\sqrt{\pi}+3a\sqrt{2}+2a\sqrt{\pi})+3a\sqrt{2}+2a\sqrt{\pi}$. Then

$$\left|\frac{|g_i'g_j-g_j'g_i|}{1+\sum|g_k|^2} - \frac{|f_i'f_j-f_j'f_i|}{1+\sum|f_k|^2}\right| \leq \frac{K_a'}{r^3}.$$

From this and (15),

$$\left||dg|(z)-|df|(z)\right| \leq (1/\sqrt{\pi})\sqrt{N(K_a/r^3)^2+\binom{N}{2}(K_a'/r^3)^2} = \frac{\sqrt{NK_a^2+\binom{N}{2}(K_a')^2}}{\sqrt{\pi}r^3}.$$

Here we have used the inequality

$$\left|\sqrt{x_1^2+\cdots+x_l^2}-\sqrt{y_1^2+\cdots+y_l^2}\right| \leq \sqrt{(x_1-y_1)^2+\cdots+(x_l-y_l)^2}.$$

Set

$$K := \max\left(a\sqrt{2N/\pi}, \sqrt{NK_a^2+\binom{N}{2}(K_a')^2}/\sqrt{\pi}\right).$$

(This K satisfies the condition (iii) by (13).) Then

$$||df|(z) - |dg|(z)| \leq \frac{K}{r^3} \quad (r \geq R_0).$$

Thus we have proved the condition (ii).

For $R_0 \leq |z| \leq R$,

$$|dg|(z) \leq \|df\|_{L^\infty(D_R(0))} + \frac{K}{R_0^3} \leq \delta_0 + \frac{1}{2} \leq \frac{2}{3},$$

where we have chosen R_0 and δ_0 so that $K/R_0^3 \leq 1/2$ and $\delta_0 \leq 1/6$. In Case 1, we proved $\delta_0 \leq \|dg\|_{L^\infty(D_{R_0}(0))} \leq 2/3$. Thus we get the condition (i):

$$\delta_0 \leq \|dg\|_{L^\infty(D_R(0))} \leq 2/3.$$

□

Proof of Theorem 1.7. Let $\|df\|_{L^\infty(\mathbb{C})} \leq 1-\tau$, $(0 < \tau \leq 1)$. Let δ_0, R_0, K be the positive numbers introduced in Proposition 6.1. For $\varepsilon > 0$, we set $\delta := \min(\delta_0, \sqrt{\varepsilon})$. Let $R = R(\varepsilon, \tau) \geq R_0 + 1$ be a large positive number which will be fixed later.

We index the elements of $\mathbb{Z}^2$ by natural numbers: $\mathbb{Z}^2 = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3), \dots\}$. For $n \geq 1$, we set $p_n := 2R(\alpha_n + \sqrt{-1}\beta_n)$ and $\Lambda_n := \{x + y\sqrt{-1} \in \mathbb{C} | \, |x - 2R\alpha_n| \leq R, |y - 2R\beta_n| \leq R\}$. The squares Λ_n $(n \geq 1)$ give a tiling of the plane $\mathbb{C}$.

We inductively define the sequence of Brody curves $f_n : \mathbb{C} \to \mathbb{C}P^N$ $(n \geq 0)$ as follows. We set $f_0 := f$. Suppose we have defined f_n.

(1) If $\|df\|_{L^\infty(\Lambda_{n+1})} \geq \delta$, then we set $f_{n+1} := f_n$.
(2) If $\|df\|_{L^\infty(\Lambda_{n+1})} < \delta$ and $\|df_n\|_{L^\infty(\Lambda_{n+1})} \geq \delta_0$, then we set $f_{n+1} := f_n$.
(3) If $\|df\|_{L^\infty(\Lambda_{n+1})} < \delta$ and $\|df_n\|_{L^\infty(\Lambda_{n+1})} < \delta_0$, then we apply Proposition 6.1 to f_n and p_{n+1} (note $D_R(p_{n+1}) \subset \Lambda_{n+1}$) and get a holomorphic map $f_{n+1} : \mathbb{C} \to \mathbb{C}P^N$ satisfying the following (i), (ii), (iii).
   (i) $\delta_0 \leq \|df_{n+1}\|_{L^\infty(D_R(p_{n+1}))} \leq 2/3$.
   (ii) $||df_{n+1}|(z) - |df_n|(z)| \leq K/|z - p_{n+1}|^3$ over $|z - p_{n+1}| > R$.
   (iii) $d(f_n(z), f_{n+1}(z)) \leq K/|z - p_{n+1}|^3$ for $z \neq p_{n+1}$.

For every $n \geq 1$, by (i) and (ii)

$$|df_n|(z) \leq \max(1-\tau, 2/3) + \sum_{k:|z-p_k|>R} \frac{K}{|z-p_k|^3} \leq \max(1-\tau, 2/3) + \frac{\mathrm{const} \cdot K}{R^3}.$$

Here const is a positive constant independent of n. We choose R so large that the right hand side is bounded by $\max(1-\tau/2, 3/4) < 1$. Then all $f_n : \mathbb{C} \to \mathbb{C}P^N$ become Brody curves, and we can continue the above inductive construction infinitely many times. Moreover, for all $n \geq 1$,

$$\|df_n\|_{L^\infty(\mathbb{C})} \leq \max(1-\tau/2, 3/4). \tag{17}$$

For any compact set $\Omega \subset \mathbb{C}$, by the condition (iii), there exists $n(\Omega) \geq 1$ such that

$$\sum_{n \geq n(\Omega)} \sup_{z \in \Omega} d(f_n(z), f_{n+1}(z)) \leq \sum_{k: d(p_k, \Omega) \geq 1} \frac{K}{d(p_k, \Omega)^3} < +\infty.$$

Hence the sequence f_n converges to a holomorphic curve $g : \mathbb{C} \to \mathbb{C}P^N$ uniformly over every compact subset of $\mathbb{C}$. From (17) we have $\|dg\|_{L^\infty(\mathbb{C})} \leq \max(1 - \tau/2, 3/4) < 1$. We will prove that g is non-degenerate and $\rho(g) \geq \rho(f) - \varepsilon$.

For proving the non-degeneracy of g, it is enough to show $\|dg\|_{L^\infty(\Lambda_n)} \geq \delta/2$ for all $n \geq 1$. (See the condition (ii) of Definition-Lemma 1.3.)

Case 1: If $|df|(z) \geq \delta$ for some $z \in \Lambda_n$, then

$$|dg|(z) \geq \delta - \sum_{k: k \neq n} \frac{K}{|z - p_k|^3} \geq \delta - \frac{\text{const} \cdot K}{R^3}.$$

We can choose R so large that $\|dg\|_{L^\infty(\Lambda_n)} \geq \delta/2$.

Case 2: If $|df|(z) < \delta$ for all $z \in \Lambda_n$, then for some $k \in \{n-1, n\}$ and $w \in \Lambda_n$ we have $|df_k|(w) \geq \delta_0$. Hence

$$|dg|(w) \geq \delta_0 - \sum_{l: l \neq n} \frac{K}{|w - p_l|^3} \geq \delta - \frac{\text{const} \cdot K}{R^3}.$$

We can choose R so large that $\|dg\|_{L^\infty(\Lambda_n)} \geq \delta/2$.

We have proved that g is non-degenerate. Next we will prove $\rho(g) \geq \rho(f) - \varepsilon$. For this sake, it is enough to prove that for every $n \geq 1$

$$\frac{1}{(2R)^2} \int_{\Lambda_n} |dg|^2 dxdy \geq \frac{1}{(2R)^2} \int_{\Lambda_n} |df|^2 dxdy - \varepsilon. \tag{18}$$

Case 1: If $\|df\|_{L^\infty(\Lambda_n)} \geq \delta$, then for all $z \in \Lambda_n$

$$\left| |dg|^2(z) - |df|^2(z) \right| \leq 2 \left| |dg|(z) - |df|(z) \right| \leq \sum_{k: k \neq n} \frac{2K}{|z - p_k|^3} \leq \frac{\text{const} \cdot K}{R^3} \leq \varepsilon$$

for sufficiently large R. Hence (18) holds if we choose R sufficiently large.

Case 2: If $\|df\|_{L^\infty(\Lambda_n)} < \delta$, then (recall $\delta = \min(\delta_0, \sqrt{\varepsilon})$)

$$\frac{1}{(2R)^2} \int_{\Lambda_n} |df|^2 dxdy \leq \delta^2 \leq \varepsilon.$$

Hence (18) holds trivially.

Thus we have proved $\rho(g) \geq \rho(f) - \varepsilon$. □

References

[1] S. Angenent, The shadowing lemma for elliptic PDE. In: Dynamics of infinite dimensional systems, Nato Adv. Sci. Inst. Ser. F Comput. Systems Sci., **37** (1987) 7-22

[2] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Second revised edition, with a preface by David Ruelle, edited by Jean-René Chazottes, Lecture Notes in Mathematics, **470** Springer-Verlag, Berlin (2008)

[3] R. Brody, Compact manifolds and hyperbolicity, Trans. Amer. Math. Soc. **235** (1978) 213-219
[4] E. Calabi, Isometric imbedding of complex manifolds, Ann. of Math. **58** (1953) 1-23
[5] A. Eremenko, Normal holomorphic curves from parabolic regions to projective spaces, preprint, Purdue university (1998), arXiv: 0710.1281
[6] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin (2001)
[7] A. Gournay, Dimension moyenne et espaces d'applications pseudo-holomorphes, thesis, Département de Mathématiques d'Orsay (2008)
[8] A. Gournay, Widths of l^p balls, arXiv:0711.3081, to appear in Houston J. of Math.
[9] A. Gournay, On a Hölder covariant version of mean dimension, C. R. Acad. Sci. Paris **347** (2009) 1389-1392
[10] A. Gournay, A metric approach to von Neumann dimension, Discrete Contin. Dyn. Syst. **26** (2010) 967-987
[11] A. Gournay, Complex surfaces and interpolation on pseudo-holomorphic cylinder, arXiv: 1006.1775
[12] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps: I, Math. Phys. Anal. Geom. **2** (1999) 323-415
[13] E. Lindenstrauss, Mean dimension, small entropy factors and an embedding theorem, Inst. Hautes Études Sci. Publ. Math. **89** (1999) 227-262
[14] E. Lindenstrauss, B. Weiss, Mean topological dimension, Israel J. Math. **115** (2000) 1-24
[15] M. Macrí, M. Nolasco, T. Ricciardi, Asymptotics for selfdual vortices on the torus and on the plane: a gluing technique, SIAM J. Math. Anal. **37** (2005) 1-16
[16] S. Matsuo, M. Tsukamoto, Instanton approximation, periodic ASD connections, and mean dimension, J. Funct. Anal. **260** (2011) 1369-1427
[17] D. McDuff, D. Salamon, J-holomorphic curves and symplectic topology, American Mathematical Society Colloquium Publications, **52**, American Mathematical Society, Providence (2004)
[18] M. Tsukamoto, Gluing an infinite number of instantons, Nagoya Math. J. **188** (2007) 107-131
[19] M. Tsukamoto, Moduli spaces of Brody curves, energy and mean dimension, Nagoya Math. J. **192** (2008) 27-58
[20] M. Tsukamoto, A packing problem for holomorphic curves, Nagoya Math. J. **194** (2009) 33-68
[21] M. Tsukamoto, Gauge theory on infinite connected sum and mean dimension, Math. Phys. Anal. Geom. **12** (2009) 325-380
[22] M. Tsukamoto, Deformation of Brody curves and mean dimension, Ergod. Th. & Dynam. Sys. **29** (2009) 1641-1657
[23] K. Yosida, On a class of meromorphic functions, Proc. Phys.-Math. Soc. Japan **16** (1934) 227-235

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