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ON THE ACC FOR LENGTHS OF EXTREMAL RAYS

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Abstract. We discuss the ascending chain condition for lengths of extremal rays. We prove that the lengths of extremal rays of n-dimensional $\mathbb{Q}$-factorial toric Fano varieties with Picard number one, which are sometimes called fake weighted projective spaces, satisfy the ascending chain condition.

Contents

1. Introduction

We discuss the ascending chain condition (ACC, for short) for (minimal) lengths of extremal rays.

First, let us recall the definition of $\mathbb{Q}$-*Fano* varieties. Note that our definition is more restrictive than the usual one.

Definition 1.1 ($\mathbb{Q}$-Fano varieties)**.** Let X be an n-dimensional normal projective variety with only log canonical singularities. Assume that X is $\mathbb{Q}$-factorial, $-K_X$ is ample, and $\rho(X) = 1$. In this case, we call X a $\mathbb{Q}$-*Fano* variety.

From now on, we want to discuss the following conjecture. It seems to be the first time that the ascending chain condition for lengths of extremal rays of $\mathbb{Q}$-Fano varieties is discussed in the literature.

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Conjecture 1.2 (ACC for lengths of extremal rays)**.** *We put*

$$\mathcal{L}_n := \{l(X) \mid X \textit{ is an } n\textit{-dimensional } \mathbb{Q}\textit{-Fano variety}\}$$

such that

$$l(X) := \min_C(-K_X \cdot C)$$

where C is an integral curve on X. For every n, the set $\mathcal{L}_n$ satisfies the ascending chain condition.

We note that $l(X) \leq 2 \dim X$ (see, for example, [Fj3, Theorem 18.2]). Although, for inductive treatments, it may be better to consider the ascending chain condition for lengths of extremal rays of *log Fano* pairs (X, D) such that the coefficients of D are contained in a set satisfying the descending chain condition, we only discuss the case when $D = 0$ for simplicity. In this paper, we are mainly interested in toric $\mathbb{Q}$-Fano varieties. So, we define

$$\mathcal{L}_n^{\mathrm{toric}} := \{l(X) \mid X \text{ is an } n\text{-dimensional toric } \mathbb{Q}\text{-Fano variety}\}.$$

In the literature, the toric $\mathbb{Q}$-Fano varieties are sometimes called *fake weighted projective spaces.*

Let X be an n-dimensional fake weighted projective space. Then we have $l(X) \leq n + 1$. Furthermore, $l(X) \leq n$ if $X \not\simeq \mathbb{P}^n$ (cf. [Fj1, Proposition 2.9]). We can easily check that $X \simeq \mathbb{P}(1, 1, 2, \cdots, 2)$ if and only if $l(X) = n$ (cf. [Fj1, Section 2], [Fj2, Proposition 2.1], and [Fj4]).

The following result is the main theorem of this paper, which supports Conjecture 1.2.

Theorem 1.3 (Main theorem)**.** *For every n, $\mathcal{L}_n^{\mathrm{toric}}$ satisfies the ascending chain condition.*

In 2003, Professor Vyacheslav Shokurov explained his ideas on minimal log discrepancies, log canonical thresholds, and lengths of extremal rays to the first author at his office. He pointed out some analogies among them and asked the ascending chain condition for lengths of extremal rays. It is a starting point of this paper.

We close this section with examples. Example 1.4 shows that the $\mathcal{L}_n^{\mathrm{toric}}$ does not satisfy the descending chain condition. Example 1.5 implies that the ascending chain condition does not necessarily hold for minimal lengths of extremal rays of birational type.

Example 1.4. We consider $X_k = \mathbb{P}(1, k-1, k)$ with $k \geq 2$. Then

$$l(X_k) = \frac{2}{k-1}.$$

Therefore, $l(X_k) \to 0$ when $k \to \infty$.

Example 1.5. We fix $N = \mathbb{Z}^2$ and let $\{e_1, e_2\}$ be the standard basis of N. We consider the cone $\sigma = \langle e_1, e_2 \rangle$ in $N' = N + \mathbb{Z}e_3$, where $e_3 = \frac{1}{b}(1, a)$. Here, a and b are positive integers such that $\gcd(a, b) = 1$. We put $Y = X(\sigma)$ is the associated affine toric surface which has only one singular point P. We take a weighted blow-up of Y at P with the weight $\frac{1}{b}(1, a)$. This means that we divide σ by e_3 and obtain a fan Δ of $N'_{\mathbb{R}}$. We define $X = X(\Delta)$. It is obvious that X is $\mathbb{Q}$-factorial and $\rho(X/Y) = 1$. We can easily obtain

$$K_X = f^* K_Y + \left(\frac{1+a}{b} - 1 \right) E,$$

where $E = V(e_3) \simeq \mathbb{P}^1$ is the exceptional curve of f, and

$$-K_X \cdot E = 1 - \frac{b-1}{a}.$$

We note that

$$-K_X \cdot E = \min_C (-K_X \cdot C)$$

where C is a curve on X such that $f(C)$ is a point because $\overline{NE}(X/Y) = NE(X/Y)$ is spanned by E. In the above construction, we put $a = k^2$ and $b = mk + 1$ for any positive integers k, m. Then it is obvious that $\gcd(a, b) = 1$. Thus we obtain

$$-K_X \cdot E = 1 - \frac{m}{k}.$$

Therefore, the minimal lengths of K_X-negative extremal rays do not satisfy the ascending chain condition in this local setting. More precisely, the minimal lengths of K_X-negative extremal rays can take any values in $\mathbb{Q} \cap (0, 1)$ in this example.

We note that the minimal length of a K_X-negative extremal toric *birational* contraction morphism $f : X \to Y$ is bounded by $\dim X - 1$ (cf. [Fj4]).

For estimates of lengths of extremal rays of toric varieties and related topics, see [Fj1], [Fj2], and [Fj4].

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2. Preliminaries

In this section, we prepare various definitions and notation. We recommend the reader to see [Fj1, Section 2] for basic calculations.

2.1. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n. A toric variety $X(\Delta)$ is associated to a *fan* Δ, a collection of convex cones $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying:

(i) Each convex cone σ is a rational polyhedral in the sense there are finitely many $v_1, \cdots, v_s \in N \subset N_{\mathbb{R}}$ such that

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s;\ r_i \geq 0\} =: \langle v_1, \cdots, v_s \rangle,$$

and it is strongly convex in the sense

$$\sigma \cap -\sigma = \{0\}.$$

(ii) Each face τ of a convex cone $\sigma \in \Delta$ is again an element in Δ.

(iii) The intersection of two cones in Δ is a face of each.

Definition 2.2. The *dimension* $\dim \sigma$ of σ is the dimension of the linear space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$ spanned by σ.

We define the sublattice N_σ of N generated (as a subgroup) by $\sigma \cap N$ as follows:

$$N_\sigma := \sigma \cap N + (-\sigma \cap N).$$

If σ is a k-dimensional simplicial cone, and $v_1, \cdots, v_k$ are the first lattice points along the edges of σ, the *multiplicity* of σ is defined to be the *index* of the lattice generated by the $\{v_i\}$ in the lattice N_σ;

$$\operatorname{mult}(\sigma) := [N_\sigma : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k].$$

We note that $X(\sigma)$ is non-singular if and only if $\operatorname{mult}(\sigma) = 1$.

Let us recall a well-known fact. See, for example, [M, Lemma 14-1-1].

Lemma 2.3. *A toric variety $X(\Delta)$ is $\mathbb{Q}$-factorial if and only if each cone $\sigma \in \Delta$ is simplicial.*

2.4. The *star* of a cone τ can be defined abstractly as the set of cones σ in Δ that contain τ as a face. Such cones σ are determined by their images in $N(\tau) := N/N_\tau$, that is, by

$$\overline{\sigma} = \sigma + (N_\tau)_{\mathbb{R}}/(N_\tau)_{\mathbb{R}} \subset N(\tau)_{\mathbb{R}}.$$

These cones $\{\overline{\sigma}; \tau \prec \sigma\}$ form a fan in $N(\tau)$, and we denote this fan by $\operatorname{Star}(\tau)$. We set $V(\tau) = X(\operatorname{Star}(\tau))$. It is well-known that $V(\tau)$ is an $(n-k)$-dimensional closed toric subvariety of $X(\Delta)$, where $\dim \tau = k$. If $\dim V(\tau) = 1$ (resp. $n-1$), then we call $V(\tau)$ a *torus invariant curve* (resp. *torus invariant divisor*). For the details about the correspondence between τ and $V(\tau)$, see [Fl, 3.1 Orbits].

2.5 (Intersection Theory)**.** Assume that Δ is simplicial. If $\sigma, \tau \in \Delta$ span γ with $\dim \gamma = \dim \sigma + \dim \tau$, then

$$V(\sigma) \cdot V(\tau) = \frac{\mathrm{mult}(\sigma) \cdot \mathrm{mult}(\tau)}{\mathrm{mult}(\gamma)} V(\gamma)$$

in the *Chow group* $A^*(X)_{\mathbb{Q}}$. For the details, see [Fl, 5.1 Chow groups]. If σ and τ are contained in no cone of Δ, then $V(\sigma) \cdot V(\tau) = 0$.

2.6 (Toric $\mathbb{Q}$-Fano varieties)**.** Now we fix $N \simeq \mathbb{Z}^n$. Let $\{v_1, \cdots, v_{n+1}\}$ be a set of primitive vectors such that $N_{\mathbb{R}} = \sum_i \mathbb{R}_{\geq 0} v_i$. We define n-dimensional cones

$$\sigma_i := \langle v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{n+1} \rangle$$

for $1 \leq i \leq n+1$. Let Δ be the complete fan generated by n-dimensional cones σ_i and their faces for every i. Then we obtain a complete toric variety $X = X(\Delta)$ with Picard number $\rho(X) = 1$. We call it a *$\mathbb{Q}$-factorial toric Fano variety with Picard number one* or simply a *toric $\mathbb{Q}$-Fano variety*. It is sometimes called a *fake weighted projective space*. We define $(n-1)$-dimensional cones $\mu_{i,j} = \sigma_i \cap \sigma_j$ for $i \neq j$. We can write $\sum_i a_i v_i = 0$, where $a_i \in \mathbb{Z}_{>0}$ for every i and $\gcd(a_1, \cdots, a_{n+1}) = 1$. Then we obtain

$$0 < V(v_l) \cdot V(\mu_{k,l}) = \frac{\mathrm{mult}(\mu_{k,l})}{\mathrm{mult}(\sigma_k)},$$

$$V(v_i) \cdot V(\mu_{k,l}) = \frac{a_i}{a_l} \cdot \frac{\mathrm{mult}(\mu_{k,l})}{\mathrm{mult}(\sigma_k)},$$

and

$$\begin{aligned} -K_X \cdot V(\mu_{k,l}) &= \sum_{i=1}^{n+1} V(v_i) \cdot V(\mu_{k,l}) \\ &= \frac{1}{a_l} \Big(\sum_{i=1}^{n+1} a_i\Big) \frac{\mathrm{mult}(\mu_{k,l})}{\mathrm{mult}(\sigma_k)}. \end{aligned}$$

For the procedure to compute intersection numbers, see 2.5 or [Fl, p.100].

Let us recall the following easy lemma, which will play crucial roles in the proof of our main theorem: Theorem 1.3. The proof of Lemma 2.7 is obvious by the description in 2.6.

Lemma 2.7. *We use the notations in* 2.6. *We consider the sublattice N' of N spanned by $\{v_1, \cdots, v_{n+1}\}$. Then the natural inclusion $N' \to N$ induces a finite toric morphism $f : X' \to X$ from a weighted* *projective space X' such that f is étale in codimension one. In particular, $X(\Delta)$ is a weighted projective space if and only if $\{v_1, \cdots, v_{n+1}\}$ generates N.*

For a toric description of *weighted projective spaces*, see [Fj1, Section 2].

2.8. In Lemma 2.7, we consider $C = V(\mu_{k,l}) \simeq \mathbb{P}^1 \subset X$ and the unique torus invariant curve $C' \subset X'$ such that $f(C') = C$. We put

$$m_{k,l} := \deg(f|_{C'} : C' \to C) \in \mathbb{Z}_{>0}$$

for every (k, l). Then we can check that

$$m_{k,l} = |N(\mu_{k,l})/N'(\mu_{k,l})|$$

by definitions, where $N'(\mu_{k,l}) = N'/N'_{\mu_{k,l}}$ and $N(\mu_{k,l}) = N/N_{\mu_{k,l}}$. Let D be a Cartier divisor on X. Then we obtain

$$C \cdot D = \frac{1}{m_{k,l}}(C' \cdot f^*D)$$

by the projection formula. Therefore, we have

$$\begin{aligned} C \cdot V(v_k) &= V(\mu_{k,l}) \cdot V(v_k) \\ &= \frac{\operatorname{mult}(\mu_{k,l})}{\operatorname{mult}(\sigma_l)} = \frac{\gcd(a_k, a_l)}{m_{k,l} a_l}. \end{aligned}$$

2.9 (Lemma on the ACC)**.** We close this section with an easy lemma for the ascending chain condition.

Lemma 2.10. *We have the following elementary properties.*

(1) *If A satisfies the ascending chain condition, then any subset B of A satisfies the ascending chain condition.*

(2) *If A and B satisfy the ascending chain condition, then so does*

$$A + B = \{a + b \,|\, a \in A, b \in B\}.$$

(3) *If there exists a real number t_0 such that*

$$A \subset \{x \in \mathbb{R} \,|\, x \geq t_0\}$$

and $A \cap \{x \in \mathbb{R} \,|\, x > t\}$ is a finite set for any $t > t_0$, then A satisfies the ascending chain condition.

All the statements in Lemma 2.10 directly follow from definitions.

3. Proof of the main theorem

In this section, we prove the main theorem of this paper: Theorem 1.3. We will freely use the notation in Section 2.

Proof of Theorem 1.3. Without loss of generality, we may assume that

$$\frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} \leq \frac{\mathrm{mult}(\mu_{k,l})}{a_k\mathrm{mult}(\sigma_l)}$$

for every (k,l). We note that

$$\frac{\mathrm{mult}(\mu_{k,l})}{a_k\mathrm{mult}(\sigma_l)} = \frac{\mathrm{mult}(\mu_{k,l})}{a_l\mathrm{mult}(\sigma_k)}$$

for every $k \neq l$. In our notation,

$$l(X) = \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} \sum_{i=1}^{n+1} a_i.$$

Therefore, we can write

$$\mathcal{L}_n^{\mathrm{toric}} = \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} \sum_{i=1}^{n+1} a_i \,\middle|\, \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} \leq \frac{\mathrm{mult}(\mu_{k,l})}{a_k\mathrm{mult}(\sigma_l)} \text{ for every } (k,l) \right\}.$$

It is sufficient to prove that

$$\mathcal{M}_i = \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} a_i \,\middle|\, \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} \leq \frac{\mathrm{mult}(\mu_{k,l})}{a_k\mathrm{mult}(\sigma_l)} \text{ for every } (k,l) \right\}$$

satisfies the ascending chain condition. It is because $\mathcal{L}_n^{\mathrm{toric}}$ is contained in

$$\left\{ \frac{\mathrm{mult}(\mu_{1,2})}{\mathrm{mult}(\sigma_2)} \right\} + \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{\mathrm{mult}(\sigma_1)} \right\} + \mathcal{M}_3 + \cdots + \mathcal{M}_{n+1}.$$

We note that

$$\left\{ \frac{\mathrm{mult}(\mu_{1,2})}{\mathrm{mult}(\sigma_2)} \right\}, \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{\mathrm{mult}(\sigma_1)} \right\} \subset \left\{ \frac{1}{m} \,\middle|\, m \in \mathbb{Z}_{>0} \right\}.$$

Therefore, it is sufficient to prove the following proposition by Lemma 2.10.

Proposition 3.1. *For $3 \leq i \leq n+1$, $\mathcal{M}_i \cap \{x \in \mathbb{R} \,|\, x > \varepsilon\}$ is a finite set for every $\varepsilon > 0$.*

From now on, we fix i with $3 \leq i \leq n+1$. Since

$$\mathcal{M}_i = \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} a_i \,\middle|\, \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} \leq \frac{\mathrm{mult}(\mu_{k,l})}{a_k\mathrm{mult}(\sigma_l)} \text{ for every } (k,l) \right\},$$

we have

$$\begin{aligned}\varepsilon &< \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)}a_i \\ &= \frac{\mathrm{mult}(\mu_{1,2})}{a_1\mathrm{mult}(\sigma_2)} \cdot \frac{a_i\mathrm{mult}(\sigma_j)}{\mathrm{mult}(\mu_{i,j})} \cdot \frac{\mathrm{mult}(\mu_{i,j})}{\mathrm{mult}(\sigma_j)} \\ &\le \frac{\mathrm{mult}(\mu_{i,j})}{\mathrm{mult}(\sigma_j)}\end{aligned}$$

for every $1 \le j \ne i \le n+1$. Therefore, we obtain

$$\frac{\mathrm{mult}(\sigma_j)}{\mathrm{mult}(\mu_{i,j})} \le \llcorner \varepsilon^{-1} \lrcorner$$

for every $1 \le j \ne i \le n+1$, where $\llcorner \varepsilon^{-1} \lrcorner$ is the integer satisfying $\varepsilon^{-1} - 1 < \llcorner \varepsilon^{-1} \lrcorner \le \varepsilon^{-1}$. We put

$$\mathbb{Z}(i,j) = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{i-1} + \mathbb{Z}v_{i+1} + \cdots + \mathbb{Z}v_{j-1} + \mathbb{Z}v_{j+1} + \cdots + \mathbb{Z}v_{n+1}$$

for $j \ne i$ and

$$\mathbb{Z}(j) = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{j-1} + \mathbb{Z}v_{j+1} + \cdots + \mathbb{Z}v_{n+1}.$$

We consider the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}(i,j) \arrow[r] \arrow[d] & \mathbb{Z}(j) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_{\mu_{i,j}} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\pi"] & N/N_{\mu_{i,j}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_{\mu_{i,j}}/\mathbb{Z}(i,j) \arrow[r] \arrow[d] & N/\mathbb{Z}(j) \arrow[r, "p"] \arrow[d] & \mathcal{A}^j_{(i,j)} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We note that

$$\left|\mathcal{A}^j_{(i,j)}\right| = \frac{\mathrm{mult}(\sigma_j)}{\mathrm{mult}(\mu_{i,j})} \le \llcorner \varepsilon^{-1} \lrcorner.$$

Therefore, for any $v \in N$,

$$p \circ \pi \left((\llcorner \varepsilon^{-1} \lrcorner)!v\right) = 0$$

in $\mathcal{A}^j_{(i,j)}$. Thus,

$$\pi \left((\llcorner \varepsilon^{-1} \lrcorner)!v\right) \in N_{\mu_{i,j}}/\mathbb{Z}(i,j).$$

This holds for every $1 \leq j \neq i \leq n+1$. So we have

$$(\llcorner \varepsilon^{-1} \lrcorner)! v \in \mathbb{Z}(i) = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{i-1} + \mathbb{Z}v_{i+1} + \cdots + \mathbb{Z}v_{n+1}.$$

Note that

$$\bigcap_{j \neq i} \big(N_{\mu_{i,j}} / \mathbb{Z}(i,j) \big) = \{0\} \subset N/\mathbb{Z}(i).$$

Therefore, we obtain

$$1 \leq m_{1,2} \leq (\llcorner \varepsilon^{-1} \lrcorner)!.$$

Moreover,

$$\varepsilon < \frac{\mathrm{mult}(\mu_{1,i})}{\mathrm{mult}(\sigma_1)} = \frac{\gcd(a_1, a_i)}{m_{1,i} a_1} \leq \frac{\gcd(a_1, a_i)}{a_1}.$$

By the same way, we obtain

$$\varepsilon < \frac{\gcd(a_2, a_i)}{a_2}.$$

We note the following obvious inequality

$$\frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i \leq \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \cdot \frac{a_i \mathrm{mult}(\sigma_2)}{\mathrm{mult}(\mu_{2,i})} \leq 1.$$

Lemma 3.2.

$$\gcd(l, a_i) = \frac{\gcd(a_1, a_i) \cdot \gcd(a_2, a_i)}{\gcd(d, a_i)}$$

where $d := \gcd(a_1, a_2)$ *and*

$$\frac{a_1 a_2}{d} = \mathrm{lcm}(a_1, a_2) =: l.$$

Proof of Lemma 3.2. It can be checked easily by direct calculations. □

Therefore, we obtain

$$\frac{\gcd(l, a_i)}{l} = \frac{\gcd(a_1, a_i)}{a_1} \cdot \frac{\gcd(a_2, a_i)}{a_2} \cdot \frac{d}{\gcd(d, a_i)} > \varepsilon^2 \frac{d}{\gcd(d, a_i)} \geq \varepsilon^2.$$

This means that

$$\frac{l}{\gcd(l, a_i)} \leq \varepsilon^{-2}.$$

Thus, we have

$$\begin{aligned} 1 &\geq \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i = \frac{a_i}{m_{1,2} l} = \frac{\gcd(l, a_i)}{l} \cdot \frac{a_i}{m_{1,2} \gcd(l, a_i)} \\ &\geq \varepsilon^2 \frac{a_i}{m_{1,2} \gcd(l, a_i)}. \end{aligned}$$

So, we obtain

$$\frac{a_i}{\gcd(l, a_i)} < \varepsilon^{-2} m_{1,2} \leq \varepsilon^{-2}(\llcorner \varepsilon^{-1} \lrcorner)!.$$

On the other hand,

$$\frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i = \frac{a_i}{m_{1,2} l}.$$

We note that

$$\frac{a_i}{m_{1,2} l} = \frac{\frac{a_i}{\gcd(l, a_i)}}{m_{1,2} \frac{l}{\gcd(l, a_i)}}.$$

This implies that $\mathcal{M}_i \cap \{x \in \mathbb{R} \,|\, x > \varepsilon\}$ is a finite set. It is because

$$\frac{a_i}{\gcd(l, a_i)}, \ \frac{l}{\gcd(l, a_i)}, \ \text{and} \ \ m_{1,2}$$

are positive integers and

$$\frac{a_i}{\gcd(l, a_i)} \leq \varepsilon^{-2}(\llcorner \varepsilon^{-1} \lrcorner)!, \ \frac{l}{\gcd(l, a_i)} \leq \varepsilon^{-2}, \ \text{and} \ m_{1,2} \leq (\llcorner \varepsilon^{-1} \lrcorner)!.$$

Therefore, $\mathcal{L}_n^{\mathrm{toric}}$ satisfies the ascending chain condition. □

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