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Abstract

Let L_1, L_2 be compact special Lagrangian submanifolds of a Calabi–Yau manifold, and suppose L_1, L_2 intersect transversally at a point p. One can construct another special Lagrangian submanifold M by gluing a Lawlor neck [8] into $L_1 \cup L_2$ at p; see Butscher [2], D. Lee [9], Y. Lee [10], and Joyce [6]. By construction, M is close to the Lawlor neck near p and to $L_1 \cup L_2$ away from p. The main result of this paper is a uniqueness theorem for special Lagrangian submanifolds which are close to the Lawlor neck near p and to $L_1 \cup L_2$ away from p; see Theorem 1.1.

1. Introduction

Let (W, Ω) be a Calabi–Yau manifold of complex dimension m, i.e., a Kähler manifold W with a holomorphic m-form Ω such that $|\Omega| = 2^{m/2}$. We call Ω a complex volume form. Let M be an oriented submanifold of W. We call M a special Lagrangian submanifold of (W, Ω) if $\Omega|_M$ is the volume form of M. If M is a special Lagrangian submanifold, then M is minimal and Lagrangian; see Harvey and Lawson [4, Corollary 1.11, Chapter III].

Let $(z^1, \ldots, z^m)$ be the complex coordinates on $\mathbb{C}^m$, and set

$$\Omega' = dz^1 \wedge \cdots \wedge dz^m.$$

Ω' is a complex volume form on $(\mathbb{C}^m, g')$. Set

$$L_1' = \mathbb{R}^m = \{(r_1, \ldots, r_m) \in \mathbb{C}^m | r_1, \ldots, r_m \in \mathbb{R}\}.$$

L_1' is a special Lagrangian submanifold with respect to Ω'. Let $\theta_1, \ldots, \theta_m \in (0, \pi)$ with $\theta_1 + \cdots + \theta_m = \pi$, and set

$$L_2' = \{(r_1 e^{i\theta_1}, \ldots, r_m e^{i\theta_m}) \in \mathbb{C}^m | r_1, \ldots, r_m \in \mathbb{R}\}.$$

L_2' is a special Lagrangian submanifold with respect to Ω'. Let K be a Lawlor neck [8], i.e., a special Lagrangian submanifold of $(\mathbb{C}^m, \Omega')$ which is asymptotic to $L_1' \cup L_2'$ at ∞ and diffeomorphic to $\mathbb{R} \times S^{m-1}$, where S^{m-1} is the sphere of dimension $m - 1$.

Let L_1, L_2 be compact special Lagrangian submanifolds of (W, Ω), and suppose L_1, L_2 intersect only at a point p. Let J be the complex structure of the Kähler manifold W, and g the Kähler metric of W. Let $\{J_s\}_{s>0}$ be a smooth family of complex structures on W converging to J as $s \to +0$. Let g_s be a smooth family of Kähler metrics with respect to J_s converging to g as $s \to +0$, and Ω_s a smooth family of complex volume forms with respect to g_s converging to Ω as $s \to +0$. Let $0 < a_s < b_s$, and suppose

$$a_s = Rs,\ b_s = O(s^\beta) \text{ for some } R > 0,\ 0 < \beta < 1, \tag{1.1}$$

where $O(s^\beta)$ is an infinitesimal of order s^β. Let ω_s be the symplectic form $g_s(J_s\bullet,\bullet)$, and ω' the standard symplectic form on $\mathbb{C}^m$. Let f_s be a smooth family of Darboux charts centered at p in (W,ω_s), i.e.,

$$p \in U \subset W,\ f_s : U \to \mathbb{C}^m,\ f_s(p) = 0,\ f_s^*\omega' = \omega_s.$$

Suppose df_s maps $T_pL_1, T_pL_2 \subset T_pW$ onto $L_1', L_2' \subset \mathbb{C}^m$ respectively. Suppose F_s is a smooth family of diffeomorphisms of W such that F_s converges to the identity as $s \to +0$, and $F_s^*\omega_s = \omega$, where $\omega = g(J\bullet,\bullet)$. Consider a compact special Lagrangian submanifold M_s of (W,Ω_s) satisfying:

(B1) there exist $a_s' > a_s$ and a normal vector field u_s on $M_s \cap B(a_s)$ in (W,g_s) such that $M_s \cap B(a_s)$ is contained in the graph of u_s on $f_s^{-1}(sK) \cap B(a_s')$, and $\|u_s\|_{C^1} = o(s)$;

(B2) there exist $b_s' < b_s$ and a normal vector field v_s on $F_s(L_1 \cup L_2) \setminus B(b_s')$ in (W,g_s) such that $M_s \setminus \overline{B(b_s)}$ is contained in the graph of v_s on $F_s(L_1 \cup L_2) \setminus B(b_s')$, such that $\|v_s\|_{C^1} = o(s^\beta)$, and such that $v_s \lrcorner \omega_s$ is an exact 1-form on $F_s(L_1 \cup L_2) \setminus B(b_s')$;

here $B(r)$ is the metric ball of radius $r > 0$ centered at p in (W,g_s), and $o(s), o(s^\beta)$ are infinitesimals of order higher than s, s^β respectively. For the gluing construction of M_s, see D. Lee [9, Theorem 1] or Joyce [6, Theorem 9.10]. The main result of this paper is a uniqueness theorem for M_s. Suppose $m > 2$.

Theorem 1.1 (The Main Result)**.** *There exists at most one compact special Lagrangian submanifold M_s of (W,Ω_s) satisfying* (B1) *and* (B2) *whenever $s > 0$ is sufficiently small and $R > 0$ is sufficiently large.*

Here, $R > 0$ is as in (1.1).

(B1) and (B2) are assumptions on $M \cap B(a_s)$ and $M \setminus \overline{B(b_s)}$ respectively. We do not make any assumption on

$$M_s \cap (B(b_s) \setminus \overline{B(a_s)}). \tag{1.2}$$

We shall give the idea of the proof of the main result of this paper. Let M_s be as in Theorem 1.1. We prove that (1.2) is close to

$$(T_pL_1 \cup T_pL_2) \cap (B(b_s) \setminus \overline{B(a_s)}). \tag{1.3}$$

Once this has been done, we can prove Theorem 1.1 by the maximum principle as in Thomas and Yau [15, Lemma 4.2].

We shall explain how we prove that (1.2) is close to (1.3). We do it in a way similar to the proof of Simon's theorem [13, Theorem 5, p563]. It is a uniqueness theorem for smooth tangent cones of minimal submanifolds with isolated singular points. Consider a minimal submanifold Y with an isolated singular point. It is important in the proof of Simon's theorem that Y satisfies a monotonicity formula on balls centered at the singular point. On the other hand, (1.2) does not satisfy the same monotonicity formula as Y since (1.2) is not contained in any ball centered at p. We prove a different monotonicity formula for (1.2). Suppose for simplicity that M is a special Lagrangian submanifold of $(\mathbb{C}^m,\Omega')$, and M is a closed subset of $B(b) \setminus \overline{B(a)}$, where $B(b), B(a)$ are the balls of radii $b > a$ centered at $0 \in \mathbb{C}^m$. We prove that

$$\int_{M\cap\partial B(c)} r^{1-m}\partial_r \lrcorner \Omega' \le \int_{M\cap\partial B(d)} r^{1-m}\partial_r \lrcorner \Omega', \tag{1.4}$$

for almost every c, d with $a < c < d < b$, where r is the Euclidean distance from 0, and ∂_r is the vector field $\partial/\partial r$. This is a higher-dimensional analogue of Hofer's energy estimate for pseudo-holomorphic curves in symplectizations of contact manifolds [5, pp534–539]. Actually, (1.4) holds only for the Euclidean metric. For a general metric, we prove a monotonicity formula with an error term.

In Simon's theorem, it is assumed that the minimal submanifold Y has a smooth tangent cone

$$[0,\infty) \times X/\{0\} \times X, \tag{1.5}$$

where X is a compact smooth manifold. It is important in the proof of Simon's theorem that the distance of Y from (1.5) satisfies an a-priori C^1-estimate. We replace (1.5) by $(a_s, b_s) \times X$ since we consider (1.3). We prove that the distance of (1.2) from (1.3) satisfies a similar a-priori C^1-estimate.

Using the monotonicity formula and the a-priori estimate, we prove that (1.2) is close to (1.3). This is the key step to the proof of Theorem 1.1.

We begin with the statement of the key step to the proof of Theorem 1.1; see Section 2. In Section 3 we prove the monotonicity formula for special Lagrangian submanifolds of annuli. In Section 4 we prove the a-priori estimate similar to that of Simon. In Section 5 we complete the proof of the main result of this paper.

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2. Statement of the Key Step

In this section we state the key step to the proof of the main result of this paper; see Theorem 2.2.

We begin with a review of calibrated geometry [4]. Let W be a Riemannian manifold. An m-form ϕ on W is said to be of comass ≤ 1 if $\phi(v_1, \dots, v_m) \leq 1$ for every orthonormal vector fields $v_1, \dots, v_m$ on W. A closed m-form of comass ≤ 1 on W is called a calibration of degree m on W. Let ϕ be a calibration of degree m on W. Let M be an oriented submanifold of W. We call M a ϕ-submanifold of W if $\phi|_M$ is the volume form of M. By a theorem of Harvey and Lawson [4], ϕ-submanifolds of W are minimal submanifolds of W.

We shall set up the notation which we use in the statement of Theorem 2.2 below. Let g' be the Euclidean metric on $\mathbb{R}^n$, i.e.,

$$g' = dy^1 \otimes dy^1 + \cdots + dy^n \otimes dy^n$$

in the coordinates $(y^1, \dots, y^n)$ on $\mathbb{R}^n$. Let ϕ' be a calibration of degree m on $(\mathbb{R}^n, g')$. Suppose ϕ' is parallel, i.e.,

$$\phi' = \phi'_{i_1 \dots i_m} dy^{i_1} \wedge \cdots \wedge dy^{i_m}$$

for some $\phi'_{i_1 \dots i_m} \in \mathbb{R}$. Let r be the radial coordinate $|\bullet|$ on $(\mathbb{R}^n \setminus \{0\}, g')$. Set

$$\psi' = (\partial_r \lrcorner \phi')\,|_{S^{n-1}}, \tag{2.1}$$

where ∂_r is the vector field $\partial/\partial r$, $\lrcorner$ is the interior product of vector fields with differential forms, and S^{n-1} is the unit sphere of $(\mathbb{R}^n, g')$. For every orthonormal vector fields $v_1, \dots, v_{m-1}$ on S^{n-1}, we have

$$\psi'(v_1, \dots, v_{m-1}) = \phi'(\partial_r, v_1, \dots, v_{m-1}) \leq 1 \tag{2.2}$$

since $\partial_r, v_1, \ldots, v_{m-1}$ are orthonormal. Therefore, ψ' is an $(m-1)$-form of comass ≤ 1 on S^{n-1}. Let X be a oriented submanifold of S^{n-1}. We call X a ψ'-submanifold if $\psi'|_X$ is the volume form of X.

Proposition 2.1. *ψ'-submanifolds of S^{n-1} are minimal submanifolds of S^{n-1}.*

Proof. Let X be a ψ'-submanifold of S^{n-1}. Set

$$CX = \{rx \in \mathbb{R}^n | r \in (0, \infty), x \in X\}.$$

Then, by (2.2), CX is a ϕ'-submanifold of $(\mathbb{R}^n, g')$. Therefore, CX is a minimal submanifold of $(\mathbb{R}^n, g')$. Therefore, X is a minimal submanifold of S^{n-1}. □

Let I be an open interval of $(0, \infty)$, and X a submanifold of S^{n-1}. We embed $I \times S^{n-1}$ into $\mathbb{R}^n$ by $(r, y) \mapsto ry$. Let ν be a normal vector field on $I \times X$ in $(I \times S^{n-1}, g')$. Set

$$\|\nu\|_{C^0_{\rm cyl}} = \sup_{I \times X} |\nu|/r, \ \|\nu\|_{C^1_{\rm cyl}} = \sup_{I \times X} \big(|\nu|/r + |D\nu|\big),$$

where $D\nu$ is the covariant derivative of ν. These are induced by the cylindrical metric g'/r^2 on $(0, \infty) \times S^{n-1}$. Set

$$G_{\rm cyl}(\nu) = \Big\{ \frac{r}{\sqrt{r^2 + |\nu(rx)|^2}} \big(rx + \nu(rx)\big) \ \Big| \ r \in I, x \in X \Big\}.$$

The key step to the proof of the main result of this paper is the following

Theorem 2.2. *Let ϕ' be a parallel calibration of degree m on the Euclidean space $(\mathbb{R}^n, g')$, and ψ' the $(m-1)$-form (2.1) on the unit sphere S^{n-1} of $(\mathbb{R}^n, g')$. Let X be a compact ψ'-submanifold of S^{n-1}. Let $0 < l < 1$. Then, there exist $\epsilon_0, \eta_0, C_0, c_0 > 0$ depending only on l, m, n, X, ϕ' such that if:*

(A0) *$0 < \epsilon < \epsilon_0$;*
(A1) *$0 < a_0 < b_0 < a_1 < b_1$, $a_0/b_0 = a_1/b_1 = l$;*
(A2) *g is a Riemannian metric on $B^n(b_1)$ with*

$$\|g - g'\|_{C^1(B^n(b_1))} \leq \epsilon, \ \|g - g'\|_{C^2(B^n(b_1))} \leq 1$$

with respect to g', and $B^n(b_1)$ is the ball of radius b_1 centered at 0 in $(\mathbb{R}^n, g')$;
(A3) *ϕ is a calibration on $(B^n(b_1), g)$ with*

$$(1 + \log \frac{b_1}{a_0}) \sup_{B^n(b_1)} |\phi - \phi'| \leq \epsilon,$$

where $|\bullet|$ is with respect to g', and $B^n(b_1)$ is the ball of radius b_1 centered at 0 in $(\mathbb{R}^n, g')$;
(A4) *M is a closed subset of $(a_0, b_1) \times S^{n-1}$, and M is a ϕ-submanifold with respect to g;*
(A5) *there exists a normal vector field ν_i on $(a_i, b_i) \times X$ in $((a_i, b_i) \times S^{n-1}, g'/r^2)$, where $i = 0, 1$, such that*

$$M \cap ((a_i, b_i) \times S^{n-1}) = G_{\rm cyl}(\nu_i) \ \text{with} \ \|\nu_i\|_{C^1_{\rm cyl}} \leq \epsilon,$$

then there exists a normal vector field ν on $(a_0, b_1) \times X$ in $((a_0, b_1) \times S^{n-1}, g'/r^2)$ such that

$$M = G_{\rm cyl}(\nu) \ \text{with} \ \|\nu\|_{C^1_{\rm cyl}} \leq C_0 \epsilon^{c_0}. \tag{2.3}$$

We prove Theorem 2.2 in Section 5.

3. A Monotonicity Formula

In this section we prove a monotonicity formula for calibrated submanifolds of annuli; see Proposition 3.4. This is a higher-dimensional analogue of an energy estimate of Hofer [5, pp534–539] for pseudo-holomorphic curves in symplectizations of contact manifolds.

Let g be a Riemannian metric on $\mathbb{R}^n$, and ϕ a calibration of degree m on $(\mathbb{R}^n, g)$.

Proposition 3.1. *Let M be a ϕ-submanifold of $(\mathbb{R}^n, g)$. If ν is a normal vector field on M in $(\mathbb{R}^n, g)$, then we have*

$$(\nu \lrcorner \phi)|_M = 0.$$

Proof. It suffices to prove that for every point $p \in M$ and orthonormal vectors $v_1, \dots, v_{m-1} \in T_pM$, we have

$$\phi_p(\nu_p, v_1, \dots, v_{m-1}) = 0. \tag{3.1}$$

Choose $v \in T_pM$ so that $\phi_p(v, v_1, \dots, v_{m-1}) = 1$. Consider

$$t \mapsto \phi_p((\sin t)\nu_p + (\cos t)v, v_1, \dots, v_{m-1}).$$

By the definition of calibration, this attains maximum 1 at $t = 0$. Differentiating it at $t = 0$, we have (3.1). □

Let g' be the Euclidean metric on $\mathbb{R}^n$. Let r be the radial coordinate on the Euclidean space $(\mathbb{R}^n, g')$, and ∂_r the vector field ∂/∂_r. In the same way as Harvey and Lawson [4, Lemma 5.11, II.5], we shall prove the following

Proposition 3.2. *Let M be a ϕ-submanifold of $(\mathbb{R}^n, g)$. Then, we have*

$$\langle \overrightarrow{TM}, \partial_r \lrcorner dr \wedge \phi \rangle = |\mathrm{pr}_{TM^\perp} \partial_r|^2, \tag{3.2}$$

where $\langle \bullet, \bullet \rangle$ is the canonical pairing of poly-vector fields and differential forms, $\overrightarrow{TM}$ is the m-vector field on M dual to $\phi|_M$, $r = |\bullet|$ is with respect to the Euclidean metric g', and $\mathrm{pr}_{TM^\perp}$ is the projection of $\mathbb{R}^n$ onto the normal bundle of M in $(\mathbb{R}^n, g)$.

Proof. By Proposition 3.1, we have

$$\langle \nu \wedge \overrightarrow{TM}, dr \wedge \phi \rangle = \langle \nu, dr \rangle \langle \overrightarrow{TM}, \phi \rangle, \text{ where } \nu = \mathrm{pr}_{TM^\perp} \partial_r.$$

This proves (3.2). □

Set

$$\psi = \frac{m}{r^m} \int_0^r (\partial_r \lrcorner \phi) dr. \tag{3.3}$$

Proposition 3.3. *ψ is an $(m-1)$-form on $\mathbb{R}^n \setminus \{0\}$ such that*

$$\phi = d\left(\frac{r^m}{m}\psi\right). \tag{3.4}$$

Proof. Set $\chi = \partial_r \lrcorner \phi$, and $\omega = \partial_r \lrcorner dr \wedge \phi$. Then, we have

$$\phi = dr \wedge \chi + \omega. \tag{3.5}$$

Since $\partial_r \lrcorner \chi = \partial_r \lrcorner \omega = 0$, we may regard χ and ω as smooth families of differential forms on S^{n-1}. By the definition of calibration, $d\phi = 0$. Therefore, we have

$$d_{S^{n-1}} \chi = \partial_r \omega, \tag{3.6}$$

where $d_{S^{n-1}}$ is the exterior differentiation on S^{n-1}. By (3.5) and (3.6), we have

$$\phi = d\left(\int_0^r \chi dr\right) = d\left(\int_0^r (\partial_r \lrcorner \phi) dr\right).$$

By (3.3), this proves (3.4). □

Let ϕ' a parallel calibration of degree m on the Euclidean space $(\mathbb{R}^n, g')$, Set

$$\psi' = r^{1-m}\partial_r \lrcorner \phi'. \tag{3.7}$$

Then, (3.3) holds with ϕ', ψ' in place of ϕ, ψ respectively.

We shall prove a monotonicity formula with an error term. When $\phi = \phi'$, it has no error term.

Proposition 3.4. *There exists $C_{m,n} > 0$ depending only on m, n such that*

$$\left|m^{-1} d\psi - r^{-m}\partial_r \lrcorner dr \wedge \phi\right|_{\mathrm{cyl}} \leq C_{m,n} \sup |\phi - \phi'|, \tag{3.8}$$

where $|\bullet|_{\mathrm{cyl}}$ is with respect to the metric g'/r^2.

Proof. By (3.4) and (3.7), we have

$$m^{-1} d\psi - r^{-m}\partial_r \lrcorner dr \wedge \phi = dr/r \wedge (r^{1-m}\partial_r \lrcorner \phi - r^{1-m}\partial_r \lrcorner \phi' + \psi' - \psi). \tag{3.9}$$

By (3.3) and (3.7), we have

$$|r^{1-m}\partial_r \lrcorner \phi - r^{1-m}\partial_r \lrcorner \phi'|_{\mathrm{cyl}} \leq c \sup |\phi - \phi'|,$$
$$|\psi - \psi'|_{\mathrm{cyl}} \leq c \sup |\phi - \phi'|$$

for some $c > 0$ depending only on m, n. Therefore, by (3.9), we have (3.8). □

We shall prove a proposition which we use in the proof of Lemma 3.6 below. We also use it in the key step to proof of the main result of this paper.

Proposition 3.5. *Let M be a ϕ-submanifold of $(\mathbb{R}^n, g)$, and suppose M is a closed subset of $(a, b) \times S^{n-1}$, where $(a, b) \times S^{n-1}$ is embedded into $\mathbb{R}^n$ by $(r, y) \mapsto ry$. There exist $\epsilon_{m,n}, C'_{m,n} > 0$ depending only on m, n such that if*

$$(1 + m \log \frac{b}{a}) \sup_{(a,b)\times S^{n-1}} |\phi - \phi'| \leq \epsilon_{m,n}, \quad \sup_{(a,b)\times S^{n-1}} |g - g'| \leq 1, \tag{3.10}$$

then we have

$$\begin{aligned} \mathrm{Vol}(M, g/r^2) \leq{}& C'_{m,n} \log \frac{b}{a} \limsup_{r\to b} \left| \int_{M\cap\{r\}\times S^{n-1}} \psi \right| \\ &+ C'_{m,n}(1 + m \log \frac{b}{a}) \int_M |\mathrm{pr}_{TM^\perp}\partial_r|^2 \,\mathrm{dVol}(M, g/r^2). \end{aligned} \tag{3.11}$$

Proof. By (3.4), we have

$$\mathrm{Vol}(M, g/r^2) = \int_M \phi/r^m = \int_M (dr/r) \wedge \psi + m^{-1}\int_M d\psi.$$

By (3.8), we have

$$m^{-1}\int_M d\psi \leq \int_M |\mathrm{pr}_{TM^\perp}\partial_r|^2 \,\mathrm{dVol}(M, g/r^2) + C_{m,n} \sup |\phi - \phi'| \,\mathrm{Vol}(M, g'/r^2).$$

By (3.8) and (3.2), we have

$$\begin{aligned}
&\int_M (dr/r)\wedge\psi \le \log\frac{b}{a}\limsup_{r\to b}\left|\int_{M\cap\{r\}\times S^{n-1}}\psi + \int_{M\cap([a,r]\times S^{n-1})} d\psi\right| \\
&\le m\log\frac{b}{a}\limsup_{r\to b}\left|\int_{M\cap\{r\}\times S^{n-1}} m^{-1}\psi\right| + \int_M |\mathrm{pr}_{TM^\perp}\partial_r|^2\,\mathrm{dVol}(M,g'/r^2) \\
&+ mC_{m,n}\log\frac{b}{a}\sup|\phi-\phi'|\,\mathrm{Vol}(M,g'/r^2).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathrm{Vol}(M,g/r^2) \\
&\le m\log\frac{b}{a}\limsup_{r\to b}\left|\int_{M\cap\{r\}\times S^{n-1}}\psi\right| + (1+m\log\frac{b}{a})\int_M |\mathrm{pr}_{TM^\perp}\partial_r|^2\,\mathrm{dVol}(M,g/r^2) \\
&+ C_{m,n}(1+m\log\frac{b}{a})\sup|\phi-\phi'|\,\mathrm{Vol}(M,g'/r^2).
\end{aligned}$$

By (3.10), we have

$$C_{m,n}(1+m\log\frac{b}{a})\sup|\phi-\phi'|\,\mathrm{Vol}(M,g'/r^2)\le(1/2)\,\mathrm{Vol}(M,g/r^2).$$

Thus, we have (3.11). □

We shall prove a lemma which we use in the key step to the proof of the main result of this paper. It is similar to a lemma of Simon [13, Lemma 3, p561]. We however use the monotonicity formula for ϕ-submanifolds of annuli.

Lemma 3.6. *Let ϕ' be a parallel calibration of degree m on the Euclidean space $(\mathbb{R}^n, g')$, and let ψ' be as in (3.7). Let X be a compact ψ'-submanifold of S^{n-1}. Let $\epsilon>0$, and $0<\lambda<\lambda''<\lambda'<1$. Then, there exists $\delta>0$ such that if:*

- (P1) *g is a Riemannian metric on $B^n(1)$ with $\|g-g'\|_{C^1(B^n(1))}\le\delta$, where $\|\bullet\|_{C^1}$ is with respect to g', and $B^n(1)$ is the unit ball of $(\mathbb{R}^n,g')$;*
- (P2) *ϕ is a calibration on $(B^n(1),g)$ with $\sup_{B^n(1)}|\phi-\phi'|\le\delta$, where $|\bullet|$ is with respect to g', and $B^n(1)$ is the unit ball of $(\mathbb{R}^n,g')$;*
- (P3) *M is a ϕ-submanifold of $(\mathbb{R}^n,g)$, and M is a closed subset of $(\lambda,1)\times S^{n-1}$, where $(\lambda,1)\times S^{n-1}$ is embedded into $\mathbb{R}^n$ by $(r,y)\mapsto ry$;*
- (P4) *there exists a normal vector field ν on $(\lambda',1)\times X$ in $((\lambda',1)\times S^{n-1}, g'/r^2)$ such that*

  $$M\cap((\lambda',1)\times S^{n-1}) = G_{\mathrm{cyl}}(\nu) \text{ with } \|\nu\|_{C^1_{\mathrm{cyl}}}\le\delta$$

  in the notation of Section 2;
- (P5) $\int_M |\mathrm{pr}_{TM^\perp}\partial_r|^2\,\mathrm{dVol}(M,g/r^2)\le\delta$,

then there exists a normal vector field ν' on $(\lambda,1)\times S^{n-1}$ in $((\lambda,1)\times S^{n-1}, g'/r^2)$ such that

$$M = G_{\mathrm{cyl}}(\nu') \text{ with } \|\nu'|_{(\lambda'',\lambda')\times S^{n-1}}\|_{C^{1,1/2}_{\mathrm{cyl}}}\le\epsilon,$$

where $C^{1,1/2}_{\mathrm{cyl}}$ is the Hölder space with respect to the metric g'/r^2 on $(\lambda,1)\times S^{n-1}$.

Proof. Suppose there does not exist such δ. Then, for every $j=2,3,4,\dots$, there exist g_j,ϕ_j,M_j such that (P1), (P2), (P3), (P4) and (P5) hold with $\delta=1/j$, and the following holds:

(P6) there does not exist any normal vector field ν'_j on $(\lambda'',\lambda')\times X$ in $((\lambda'',\lambda')\times S^{n-1}, g'/r^2)$ such that

$$M_j = G_{\mathrm{cyl}}(\nu'_j) \text{ with } \|\nu'_j\|_{C^{1,1/2}_{\mathrm{cyl}}} \le \epsilon.$$

By (P1), (P2) and (P3), we may apply Proposition 3.5. Therefore, by (3.11), (P4) and (P5), we have

$$\sup_{j=2,3,4,\dots} \mathrm{Vol}(M_j, g_j/r^2) < \infty. \tag{3.12}$$

Therefore, by (P1), we have

$$\sup_{j=2,3,4,\dots} \mathrm{Vol}(M_j, g') < \infty.$$

By (P1) and (P3), we have

$$\lim_{j\to\infty} \big(\text{the mean curvature of } M_j \text{ in } ((\lambda,1)\times S^{n-1}, g')\big) = 0 \tag{3.13}$$

in the C^0-topology. Thus, by Allard's compactness theorem [1, Theorem 5.6], there exists a subsequence M_{j_k} converging as varifolds to some rectifiable varifold M_∞ in $((\lambda,1)\times S^{n-1}, g')$.

Let $\|M_\infty\|$ be the Radon measure on $((\lambda,1)\times S^{n-1}, g')$ induced by M_∞. We shall prove

$$a^m \|M_\infty\|(a^{-1}E) = \|M_\infty\|(E) \tag{3.14}$$

for every $a > 0, E \subset (\lambda,1)\times S^{n-1}$ with $aE \subset (\lambda,1)\times S^{n-1}$. It suffices to prove

$$\frac{d}{da} a^m \int_{(\lambda,1)\times S^{n-1}} f(ar) h d\|M_\infty\| = 0 \tag{3.15}$$

for every smooth functions $h : S^{n-1} \to [0,\infty)$ and $f : (\lambda,1) \to [0,\infty)$ with $a(\mathrm{supp} f) \subset (\lambda,1)$. By (3.8), (P2), (P5) and (3.12), we have

$$\lim_{j\to\infty} \int_{M_j} d\psi_j \to 0,$$

where ψ_j is as in (3.3) with ϕ_j in place of ϕ. Therefore, by (P3) and (3.4), we have

$$\begin{aligned}
\text{the left-hand side of (3.15)} &= \frac{d}{da} \lim_{k\to\infty} a^m \int_{M_{j_k}} f(ar) h d\Big(\frac{r^m}{m}\psi_{j_k}\Big) \\
&= \lim_{k\to\infty} \int_{M_{j_k}} \frac{d}{da}((ar)^m f(ar)) \frac{dr}{r} \wedge h\psi_{j_k} \\
&= \lim_{k\to\infty} \int_{M_{j_k}} a^{-1} \frac{d}{dr}((ar)^m f(ar)) dr \wedge h\psi_{j_k} \\
&= \lim_{k\to\infty} -\int_{M_{j_k}} a^{-1} (ar)^m f(ar) dh \wedge \psi_{j_k}.
\end{aligned}$$

Therefore, by (3.7), (P2) and (3.12), we have

$$\text{the left-hand side of (3.15)} = \lim_{k\to\infty} -\int_{M_{j_k}} a^{-1}(ar)^m f(ar) r^{1-m} \partial_r \lrcorner (dh \wedge \phi_{j_k}).$$

By Proposition 3.1, we have

$$\int_{M_{j_k}} r^{-m} \partial_r \lrcorner (dh\wedge\phi_{j_k}) = \int_{M_{j_k}} \langle \mathrm{pr}_{TM_{j_k}^\perp} \partial_r, dh\rangle \, \mathrm{dVol}(M_{j_k}, g_{j_k}/r^2).$$

This converges to 0 by (P5) and (3.12). Thus, we have (3.15). This proves (3.14).

By (P4), the restriction of M_∞ to $(\lambda',1)\times S^{n-1}$ is equal to $(\lambda',1)\times X$ as varifolds in $((\lambda',1)\times S^{n-1},g')$. Therefore, by (3.14), we have

$$M_\infty=(\lambda,1)\times X \text{ as varifolds in } ((\lambda,1)\times S^{n-1},g').$$

Therefore, M_{j_k} converges to $(\lambda,1)\times X$ as varifolds in $((\lambda,1)\times S^{n-1},g')$. Therefore, by (3.13) and Allard's regularity theorem [1, Theorem 8.19], M_{j_k} converges to $(\lambda,1)\times X$ in the local $C^{1,1/2}$-topology in $(\lambda,1)\times S^{n-1}$. This contradicts (P6), which completes the proof of Lemma 3.6. □

4. A-Priori Estimate

In this section we prove an a-priori estimate similar to that of Simon [13] for an evolution equation (4.5) below.

Let X be a compact smooth Riemannian manifold, V a smooth real vector bundle on X with a fibre metric and a metric connection. Let C^∞_x be the space of smooth sections of $V\to X$. Let $E: C^\infty_x\to\mathbb{R}$ satisfy

$$Ev=\int_X F(x,v,D_xv)dx \tag{4.1}$$

for every $v\in C^\infty_x$, where D_xv is the covariant derivative of v, and $F=F(x,v,p)$ is a $\mathbb{R}$-valued smooth function of $x\in X$, $v\in V|_x$, $p\in T^*_xX\otimes V|_x$. Suppose F satisfies the following conditions:

(C1) $(v,p)\mapsto F(x,v,p)$ is a real-analytic function on the vector space $V|_x\oplus(T^*_xX\otimes V|_x)$ for every $x\in X$;

(C2) there exists $c>0$ such that for every $x\in X,\xi\in T^*_xX, v\in V|_x$,

$$\frac{d^2}{dh^2}F(x,0,h^2\xi\otimes v)\Big|_{h=0}>c|\xi|^2|v|^2.$$

By (C1), one can use the Łojasiewicz estimate [11]. This is important in the proof of a result of Simon; for the statement, see Proposition 4.1 below. (C2) is called the Legendre–Hadamard condition. Let $-\operatorname{grad}E: C^\infty_x\to C^\infty_x$ be the Euler–Lagrange operator of E, i.e.,

$$\big(\operatorname{grad}E(v),v'\big)_{L^2_x}=\frac{d}{dh}E(v+hv')\Big|_{h=0}$$

for every $v,v'\in C^\infty_x$, where

$$(v'',v')_{L^2_x}=\int_X\big(v''(x),v'(x)\big)dx; \tag{4.2}$$

here $\big(v''(x),v'(x)\big)$ is the inner product on the fibre $V|_x$ at $x\in X$. Suppose

$$\operatorname{grad}E(0)=0,\ \text{ where } 0\in C^\infty_x. \tag{4.3}$$

Let $t_0<t_\infty$. Let $C^\infty_{t,x}(t_0,t_\infty)$ be the space of all smooth sections $u=u(t,x)$ with $u(t,x)\in V|_x$ for every $(t,x)\in(t_0,t_\infty)\times X$. Let $C^{k,\mu}_{t,x}(t_0,t_\infty)$ be the Hölder spaces with respect to the product metric on $(t_0,t_\infty)\times X$. Set $u(t)=u(t,\bullet)\in C^\infty_x$ for every $u=u(t,x)\in C^\infty_{t,x}(t_0,t_\infty)$.

We shall state a result of Simon which we use in the proof of Lemma 4.3 below.

Proposition 4.1 (Simon [13, Lemma 1, p542])**.** *There exist $\delta_0, \theta > 0$ depending only on X, V, E such that if $t_0 < t_3 < t_4 < t_\infty, u \in C^\infty_{t,x}(t_0, t_\infty), \delta > 0$ and if*

$$\begin{aligned}
&\|u\|_{C^{2,1/2}_{t,x}(t_3,t_4)} \le \delta_0, \\
&\sup_{t\in[t_3,t_4]} \big(E(0) - E\big(u(t)\big)\big) \le \delta, \qquad (4.4)\\
&\|\partial_t u(t) + \operatorname{grad} E\big(u(t)\big)\|_{L^2_x} \le (3/4)\|\partial_t u(t)\|_{L^2_x} \text{ for every } t \in [t_3, t_4],
\end{aligned}$$

then we have

$$\int_{t_3}^{t_4} \|\partial_t u(t)\|_{L^2_x} dt \le (4/\theta)\big(\big|E\big(u(t_3)\big) - E(0)\big|^\theta + \delta^\theta\big).$$

Here, $\|\bullet\|_{L^2_x}$ is with respect to (4.2).

Consider $u = u(t, x) \in C^\infty_{t,x}(t_0, t_\infty)$ satisfying

$$\partial_t^2 u - \partial_t u - \operatorname{grad} E(u) + R(u, \partial_t u, \partial_t^2 u) = f \tag{4.5}$$

as in Simon [13], where $f \in C^\infty_{t,x}(t_0, t_\infty)$ satisfies

$$\|\partial_t^k f(t)\|_{C^2_x} \le C_f e^{-2(t-t_0)} \text{ for every } t \in (t_0, t_\infty), k = 0, 1, 2 \tag{4.6}$$

for some $C_f > 0$, and $R : C^\infty_x \times C^\infty_x \times C^\infty_x \to C^\infty_x$ satisfies

$$\begin{aligned}
R(v, v^{(1)}, v^{(2)}) =& A(x, v, D_x v, v^{(1)}) \cdot D_x^2 v \otimes v^{(1)} \\
&+ \sum_{(k,l)=(0,1),(1,1),(0,2)} B_{kl}(x, v, D_x v, v^{(1)}) \cdot D_x^l v^{(k)}
\end{aligned} \tag{4.7}$$

for every $v, v^{(1)}, v^{(2)} \in C^\infty_x$, where $A = A(x, v, p, q)$, $B_{kl} = B_{kl}(x, v, p, q)$ are smooth functions of $x \in X$, $v \in V|_x$, $p \in T^*_x X \otimes V|_x$, $q \in V|_x$ with $A(x, v, p, q) \in \operatorname{Hom}(\bigotimes^2 T^*_x X \otimes V|_x \otimes V|_x, V|_x)$, $B_{kl}(x, v, p, q) \in \operatorname{Hom}(\bigotimes^l T^*_x X \otimes V|_x, V|_x)$ and $B_{kl}(x, 0, 0, 0) = 0$ for every $x \in X$, $(k, l) = (1, 0), (1, 1), (2, 0)$. Then, for every $C_2' > 0$, there exists $\delta_4 = \delta_4(X, V, E, R, C_2') > 0$ such that if $\|u\|_{C^{1,1/2}_{t,x}(t_0,t_\infty)} \le \delta_4$, then we have

$$|R(u(t), \partial_t u(t), \partial_t^2 u(t))| \le C_2'(|\partial_t u(t)| + |D_x \partial_t u(t)| + |\partial_t^2 u(t)|). \tag{4.8}$$

Let $H : C^\infty_x \to C^\infty_x$ be the linearized operator of $\operatorname{grad} E$ at $0 \in C^\infty_x$. Then, (4.5) is of the form

$$\partial_t^2 u - \partial_t u - Hu = \sum_{0\le k+l\le 2} a_{kl}(x, u, D_x u, \partial_t u) \cdot D_x^l \partial_t^k u + f, \tag{4.9}$$

where $a_{kl} = a_{kl}(x, u, p, q)$ are smooth functions of $x \in X$, $v \in V|_x$, $p \in T^*_x X \otimes V|_x$, $q \in V|_x$ with $a_{kl}(x, v, p, q) \in \operatorname{Hom}(\bigotimes^l T^*_x X \otimes V|_x, V|_x)$, $a_{kl}(x, 0, 0, 0) = 0$ for every $x \in X, 0 \le k + l \le 2$. Therefore, there exists $\delta_2 = \delta_2(X, V, E, R) > 0$ such that if $u \in C^\infty_{t,x}(t_0, t_\infty)$ with $\|u\|_{C^{1,1/2}_{t,x}(t_0,t_\infty)} \le \delta_2$, then we have

$$\max_{0\le k+l\le 2} \|a_{kl}(x, u, D_x u, \partial_t u)\|_{C^{0,1/2}_{t,x}(t_0,t_\infty)} \le \delta_1, \tag{4.10}$$

where $\delta_1 = \delta_1(X, V, E) > 0$ is given below. By the Legendre–Hadamard condition (C2), $\partial_t^2 - \partial_t - H$ is elliptic on $C^\infty_{t,x}(t_0, t_\infty)$. Therefore, there exists $\delta_1 = \delta_1(X, V, E) > 0$ such that if $T > 0$, if $w, g \in C^\infty_{t,x}(-T/3, T/3)$ and if

$$\partial_t^2 w - \partial_t w - Hw = \sum_{0\le k+l\le 2} b_{kl}(t, x) \cdot D_x^l \partial_t^k w + g \tag{4.11}$$

with $\max_{0\le k+l\le 2}\|b_{kl}\|_{C^{0,1/2}_{t,x}(-T/3,T/3)}\le\delta_1$, then we have

$$\|w\|_{C^{2,1/2}_{t,x}(-T/5,T/5)}\le C_1\|w\|_{L^2_{t,x}(-T/4,T/4)}+C_1\|g\|_{C^{0,1/2}_{t,x}(-T/4,T/4)} \tag{4.12}$$

for some $C_1=C_1(X,V,E;T)>0$; here $L^2_{t,x}(t',t'')$ is with respect to the product metric on $(t',t'')\times X$. (4.12) is a Schauder estimate for elliptic systems; see Douglis–Nirenberg [3] and Morrey [12].

We shall state a proposition which we use in the proof of Lemma 4.3 below. One can prove it in the same way as a result of Simon; see [13, Lemma 2, p549] or [14, Lemma 3.3, Part II].

Proposition 4.2. *There exist $h,T_3,\delta_3>0$ depending only on X, V, E such that if $T>T_3$, if $w,g\in C^\infty_{t,x}(0,3T)$ satisfy* (4.11) *with $\|b_{kl}\|_{C^0_{t,x}(0,3T)}\le\delta_3$, and if*

$$\|g\|_{L^2_{t,x}(0,3T)}\le{\delta_3}^{1/3}\|w\|_{L^2_{t,x}(T,2T)}\ \text{with}\ \|w\|_{L^2_{t,x}(0,3T)}<\infty,$$

then we have

$$\begin{aligned}
&\|w\|_{L^2_{t,x}(2T,3T)}\le e^{-hT}\|w\|_{L^2_{t,x}(T,2T)}\Longrightarrow\|w\|_{L^2_{t,x}(T,2T)}\le e^{-hT}\|w\|_{L^2_{t,x}(0,T)},\\
&\|w\|_{L^2_{t,x}(T,2T)}\ge e^{hT}\|w\|_{L^2_{t,x}(0,T)}\Longrightarrow\|w\|_{L^2_{t,x}(2T,3T)}\ge e^{hT}\|w\|_{L^2_{t,x}(T,2T)},\\
&\|w\|_{L^2_{t,x}(T,2T)}\ge e^{-hT}\|w\|_{L^2_{t,x}(0,T)}\ \text{and}\ \|w\|_{L^2_{t,x}(2T,3T)}\le e^{hT}\|w\|_{L^2_{t,x}(T,2T)}\\
&\Longrightarrow\|w(t)\|_{L^2_x}\le(3/2)\|w(t')\|_{L^2_x}\ \text{for every}\ t,t'\in(T,2T)\\
&\text{and}\ \|\partial_t w(t)\|_{L^2_x}\le(1/2)\|w(t)\|_{L^2_x}\ \text{for every}\ t\in(T,2T).
\end{aligned}$$

We shall prove a lemma which we use in the key step to the main result of this paper. It is similar to a result of Simon [13, Theorem 1, p534]. Simon's result is an a-priori estimate on $(0,\infty)\times X$. We however consider $(t_0,t_\infty)\times X$ with (t_0,t_∞) bounded. We prove the lemma for completeness.

Lemma 4.3. *Let X,V,E,R be as above. Let $t_0<t_\infty$, and $f\in C^\infty_{t,x}(t_0,t_\infty)$ with* (4.6) *for some $C_f>0$. Then, there exist $\theta,\delta_*,C_*>0$ depending only on X,V,E,R,C_f such that if $t_*\in(t_0,t_\infty)$, if $u\in C^\infty_{t,x}(t_0,t_*)$ satisfies* (4.5) *and if*

$$\|u\|_{C^{1,1/2}_{t,x}(t_0,t_*)}\le\delta_*, \tag{4.13}$$

$$\limsup_{t\to t_0}\|u(t)\|_{L^2_x}\le\delta, \tag{4.14}$$

$$\sup_{t\in(t_0,t_*)}\Big(E(0)-E\big(u(t)\big)\Big)\le\delta, \tag{4.15}$$

$$\|\partial_t u\|_{L^2_{t,x}(t_0,t_*)}\le\sqrt{\delta} \tag{4.16}$$

for some $0<\delta<\min\{1,\delta_*\}$, then we have

$$\sup_{t\in(t_0,t_*)}\|u(t)\|_{L^2_x}<C_*\delta^\theta. \tag{4.17}$$

Proof. By (4.14), it suffices to prove

$$\int_{t_0}^{t_*}\|\partial_t u(t)\|_{L^2_x}dt<C_*\delta^\theta. \tag{4.18}$$

By the Schwartz inequality and (4.16), for every $(t',t'')\subset(t_0,t_*)$, we have

$$\int_{t'}^{t''}\|\partial_t u(t)\|_{L^2_x}dt\le\sqrt{t''-t'}\|\partial_t u\|_{L^2_{t,x}(t',t'')}\le\sqrt{(t''-t')\delta}. \tag{4.19}$$

Let $T > 0$ be a sufficiently large constant; in the proof of Lemma 4.3 a constant means a real number depending only on X, V, E, R, C_f. If $t_* - t_0 < 8T$, then by (4.19), we have (4.18); we may therefore assume $t_* - t_0 \geq 8T$. Choose $t_1, t_6 \in (t_0, t_*)$ so that $T \leq t_1 - t_0 \leq 2T$, $T \leq t_* - t_6 \leq 2T$ and $t_6 - t_1 = jT$ for some integer $j \geq 4$. Then, by (4.19), we have

$$\int_{t_0}^{t_1} \|\partial_t u(t)\|_{L^2_{t,x}} \leq \sqrt{T\delta}, \quad \int_{t_6}^{t_*} \|\partial_t u(t)\|_{L^2_{t,x}} \leq \sqrt{T\delta}. \tag{4.20}$$

By (4.13), u satisfies (4.9) with (4.10). Therefore, u satisfies the Schauder estimate (4.12). Therefore, by (4.13) and (4.6), we have

$$\|u\|_{C^{2,1/2}_{t,x}(t_1,t_6)} \leq C_1'\delta_* + C_1''e^{-2T}$$

for some constants $C_1', C_1'' > 0$. We may therefore assume that

$$\|u\|_{C^{2,1/2}_{t,x}(t_1,t_6)} \text{ is sufficiently small.} \tag{4.21}$$

Differentiating (4.5) with respect to t and using (4.21), we have:

$$w = \partial_t u, g = \partial_t f \text{ satisfy (4.11) with } \|b_{kl}\|_{C^{0,1/2}_{t,x}(t_1,t_6)} \text{ sufficiently small.} \tag{4.22}$$

We may therefore apply Proposition 4.2 to $\partial_t u$ repeatedly on (t_1, t_6) since $t_6 - t_1 \geq 4T$ is assumed to be sufficiently large. Therefore, there exist constants $h, \delta_3, c_3 > 0$ and integers i_1, i_2 with $1 \leq i_1 \leq i_2 \leq j-1$ such that: if $1 < i_1$, then we have

$$\begin{aligned} &\text{either } \|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \leq e^{-hT}\|\partial_t u\|_{L^2_{t,x}(t_1+(i-1)T, t_1+iT)} \\ &\text{or } {\delta_3}^{1/3}\|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \leq \|C_f e^{-2(t-t_0)}\|_{L^2_{t,x}(t_1+(i-1)T,\infty)} \end{aligned} \tag{4.23}$$

for every $i \in \{1, \ldots, i_1 - 1\}$; if $i_1 < i_2$, then we have

$$c_3 e^{-2(t-t_0)} \leq \|\partial_t u(t)\|_{L^2_x} \leq (3/2)\|\partial_t u(t')\|_{L^2_x} \tag{4.24}$$

for every $t, t' \in (t_1 + i_1 T, t_1 + i_2 T)$ with $|t' - t| \leq T$ and we have

$$\|\partial_t^2 u(t)\|_{L^2_x} \leq (1/2)\|\partial_t u(t)\|_{L^2_x} \tag{4.25}$$

for every $t \in (t_1 + i_1 T, t_1 + i_2 T)$; if $i_2 < j - 1$, then we have

$$\|\partial_t u\|_{L^2_{t,x}(t_1+(i-1)T, t_1+iT)} \leq e^{-hT}\|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \tag{4.26}$$

for every $i \in \{i_2 + 1, \ldots, j - 1\}$. Set $t_5 = t_1 + i_2 T$. Then, by (4.26) and (4.19), we have

$$\begin{aligned} \int_{t_5}^{t_6} \|\partial_t u(t)\|_{L^2_x} dt &\leq \sum_{i=i_2}^{j} \sqrt{T}\|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \\ &\leq \sqrt{T}(1 - e^{-hT})^{-1}\sqrt{\delta}. \end{aligned} \tag{4.27}$$

In a similar way, by (4.23), there exists a constant $C_{T,h} > 0$ such that

$$\int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2_x} dt \leq C_{T,h}\sqrt{\delta}. \tag{4.28}$$

If $i_1 = i_2$, then by (4.20) and (4.28), we have (4.18); we may therefore assume $i_1 < i_2$. Set $t_3 = t_2 + T/3$, $t_4 = t_5 - T/3$. Then, by (4.19), we have

$$\int_{t_2}^{t_3} \|\partial_t u(t)\|_{L_x^2} dt \leq \sqrt{(T/3)\delta}, \int_{t_4}^{t_5} \|\partial_t u(t)\|_{L_x^2} dt \leq \sqrt{(T/3)\delta}, \tag{4.29}$$

$$\int_{t_2}^{t_3+T/4} \|\partial_t u(t)\|_{L_x^2} dt \leq \sqrt{(7T/12)\delta}. \tag{4.30}$$

By (4.22), we may apply the Schauder estimate (4.12) to $w = \partial_t u, g = \partial_t f$. Therefore, by (4.6) and (4.24), there exists a constant $C_2 > 0$ such that for every $t \in [t_3, t_4]$, we have

$$\|D_x \partial_t u(t)\|_{L_x^2} \leq C_2 \|\partial_t u(t)\|_{L_x^2}. \tag{4.31}$$

By (4.21), u satisfies (4.5) with R satisfying (4.8). Therefore, by (4.25) and (4.31), for every $t \in [t_3, t_4]$, we have

$$\|\partial_t u(t) + \operatorname{grad} E\big(u(t)\big)\|_{L_x^2} = \|\partial_t^2 u(t) + R\|_{L_x^2} \leq (3/4)\|\partial_t u(t)\|_{L_x^2}.$$

Therefore, by (4.21) and (4.15), we have (4.4). Therefore, by Proposition 4.1, we have

$$\int_{t_3}^{t_4} \|\partial_t u(t)\|_{L_x^2} dt \leq (4/\theta)\left(\Big|E\big(u(t_3)\big) - E(0)\Big|^\theta + \delta^\theta\right) \tag{4.32}$$

for some constant $\theta > 0$. Since E satisfies (4.1) with (4.3) and u satisfies the Schauder estimate (4.12), there exist constants $C_3', C_3 > 0$ such that

$$\begin{aligned} \big|E\big(u(t_3)\big) - E(0)\big| &\leq C_3' \|u(t_3)\|_{C_x^1}^2 \\ &\leq C_3 \left(\sup_{t \in (t_3 - T/4, t_3 + T/4)} \|u(t)\|_{L_x^2} + e^{-2(t_3 - t_0)} \right)^2. \end{aligned} \tag{4.33}$$

By (4.14), (4.20), (4.28) and (4.30), there exists a constant $C_4 > 0$ such that

$$\sup_{t \in (t_3 - T/4, t_3 + T/4)} \|u(t)\|_{L_x^2} \leq \limsup_{t \to t_0} \|u(t)\|_{L_x^2} + \int_{t_0}^{t_3 + T/4} \|\partial_t u(t)\|_{L_x^2} dt \leq C_4 \sqrt{\delta}.$$

By (4.24) and (4.19), there exists a constant $C_5 > 0$ such that

$$e^{-2(t_3 - t_0)} \leq C_5 \sqrt{\delta}.$$

Thus, (4.33) is bounded by $C_6 \delta^\theta$ for some constant $C_6 > 0$. Therefore, (4.32) is bounded by $C_7 \delta^\theta$ for some constant $C_7 > 0$. Therefore, by (4.20), (4.27), (4.28) and (4.29), we have (4.18). This completes the proof of Lemma 4.3. □

5. Completion of the Proof

In this section we complete the proof of the main result of this paper.

We shall first prove Theorem 2.2.

Proof of Theorem 2.2. Let ϕ' be a parallel calibration of degree m on the Euclidean space $(\mathbb{R}^n, g')$, and let ψ' be as in (2.1) in Section 2, or equivalently as in (3.7) in Section 3. Let X be a compact ψ'-submanifold of S^{n-1}. Let $0 < l < 1$. Suppose:

(S0) $\epsilon > 0$ is sufficiently small;

(S1) $0 < a_0 < b_0 < a_1 < b_1$, $a_0/b_0 = a_1/b_1 = l$;

(S2) g is a Riemannian metric on $B^n(b_1)$ with

$$\|g - g'\|_{C^1} \leq \epsilon, \|g - g'\|_{C^2} \leq 1$$

with respect to g', and $B^n(b_1)$ is the ball of radius b_1 centered at 0 in $(\mathbb{R}^n, g')$;

(S3) ϕ is a calibration of degree m on $(B^n(b_1), g)$ with

$$(1 + \log \frac{b_1}{a_0}) \sup_{B^n(b_1)} |\phi - \phi'| \leq \epsilon$$

where $|\bullet|$ is with respect to g', and $B^n(b_1)$ is the ball of radius b_1 centered at 0 in $(\mathbb{R}^n, g')$;

(S4) M is a ϕ-submanifold of $(\mathbb{R}^n, g)$, and M is a closed subset of $(a_0, b_1) \times S^{n-1}$, where $(a_0, b_1) \times S^{n-1}$ is embedded into $\mathbb{R}^n$ by $(r, y) \mapsto ry$;

(S5) there exists a normal vector field ν_i on $(a_i, b_i) \times X$ in $((a_i, b_i) \times S^{n-1}, g')$, where $i = 0, 1$, such that

$$M \cap ((a_i, b_i) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu_i) \text{ with } \|\nu_i\|_{C^1_{\mathrm{cyl}}} \leq \epsilon$$

in the notation of Section 2.

Let ψ be as in (3.3) in Section 3. Then, by (S3) and (S5), we have

$$\sup_{r \in (a_0, b_0) \cup (a_1, b_1)} \left| \int_{M \cap \{r\} \times S^{n-1}} \psi - \mathrm{Vol}(X) \right| \leq C\epsilon \tag{5.1}$$

for some constant $C > 0$; in the proof of Theorem 2.2 a constant means a real number depending only on l, m, n, X and ϕ'.

By Proposition 3.4, the Stokes Theorem and (5.1), we have

$$\int_M |\mathrm{pr}_{TM^\perp} \partial_r|^2 \,\mathrm{dVol}(M, g/r^2) \leq (2C/m)\epsilon + C_{m,n} \sup |\phi - \phi'| \,\mathrm{Vol}(M, g'/r^2). \tag{5.2}$$

Therefore, by Proposition 3.5 and (5.1), we have

$$\mathrm{Vol}(M, g'/r^2) \leq C' \log \frac{b_1}{a_0} + C'(1 + m \log \frac{b_1}{a_0}) \left(\epsilon + \sup |\phi - \phi'| \,\mathrm{Vol}(M, g'/r^2)\right)$$

for some constant $C' > 0$. Therefore, by (S3), we have

$$\mathrm{Vol}(M, g'/r^2) \leq C'' \log \frac{b_1}{a_0} \tag{5.3}$$

for some constant $C'' > 0$. Therefore, by (5.2), we have

$$\int_M |\mathrm{pr}_{TM^\perp} \partial_r|^2 \,\mathrm{dVol}(M, g/r^2) \leq C''' \epsilon \tag{5.4}$$

for some constant $C''' > 0$. Choose a constant $\epsilon_* > 0$ so that if I is an open interval of $(0, \infty)$, and if ν is a normal vector field on $I \times X$ in $(I \times S^{n-1}, g')$ with $\|\nu\|_{C^0_{\mathrm{cyl}}} \leq \epsilon_*$, then $G_{\mathrm{cyl}}(\nu)$ is contained in a tubular neighbourhood of $I \times X$ in $(I \times S^{n-1}, g')$. Here, $G_{\mathrm{cyl}}(\nu)$ is as in Section 2. If ϵ_* is sufficiently small, then we have

$$|r\partial_r(\nu/r)|^2 \leq 2|\mathrm{pr}_{TM^\perp} \partial_r|^2,$$

as in [13, (7.13), p561] or [14, 3.2, Part I]. Therefore, by (5.4), we have

$$\int_{M \cap (I \times S^{n-1})} |r\partial_r(\nu/r)|^2 \,\mathrm{dVol}(M, g/r^2) \leq 2C''' \epsilon. \tag{5.5}$$

Choose $0 < \lambda < \lambda'' < \lambda' < 1$ so that $l\lambda' < \lambda < \lambda'' < l < \lambda'$. By (5.4), we may apply Lemma 3.6 to $M \cap ((\lambda b_1, b_1)) \times S^{n-1}$. Therefore, there exists a normal vector field ν on $(\lambda b_1, b_1) \times X$ in $((\lambda b_1, b_1) \times S^{n-1}, g'/r^2)$ such that

$$M \cap ((\lambda b_1, b_1) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu) \text{ with } \|\nu|_{(\lambda'' b_1, \lambda' b_1) \times X}\|_{C^{1,1/2}_{\mathrm{cyl}}} \leq \epsilon_*. \tag{5.6}$$

Let S_* be the set of all $b_* \in [\lambda' a_0/\lambda, a_1)$ such that there exists a normal vector field ν on $(b_*, a_1) \times X$ in $((b_*, a_1) \times S^{n-1}, g'/r^2)$ such that

$$M \cap ((b_*, b_1) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu) \text{ with } \|\nu|_{(b_*, a_1) \times X}\|_{C^{1,1/2}_{\mathrm{cyl}}} \leq \epsilon_*. \tag{5.7}$$

S_* is non-empty since $\lambda'' b_1 \in S_*$ by (5.6).

Proposition 5.1. *Suppose $b_* \in S_* \cap [\lambda a_0/\lambda', b_1)$, and let ν be as in* (5.7). *Then, there exist constants $c_{10}, C_{10} > 0$ such that*

$$\|\nu|_{(b_*, b_1) \times X}\|_{C^1_{\mathrm{cyl}}} \leq C_{10} \epsilon^{c_{10}}.$$

Proof. By Proposition 2.1, X is a minimal submanifold of S^{n-1}. By (S4), M is a minimal submanifold of $((a_0, b_1) \times S^{n-1}, g)$ with $\|g - g'\|_{C^2(B(b_1))} \leq 1$ as in (S2). Set

$$u(t, x) = e^{t/m} \nu(e^{-t/m} x), \ t_0 = -m \log b_1, t_* = -m \log b_*.$$

Then, by a result of Simon [14, Remark 3.3, Part I], u satisfies (4.5) for some E, R, f depending only on m, n, X. We shall apply Lemma 4.3 to u. By (5.7), we have

$$\|\nu|_{(b_*, a_1) \times X}\|_{C^{1,1/2}_{\mathrm{cyl}}} = \|u\|_{C^{1,1/2}_{t,x}(-m \log a_1, t_*)} \leq \epsilon_*. \tag{5.8}$$

Therefore, we have (4.13). By (S5), we have (4.14). By (5.5), (5.8) and (S2), there exists a constant $C_{11} > 0$ such that

$$\|\partial_t u\|^2_{L^2_{t,x}(t_0, t_*)} = \int_{(b_*, b_1) \times X} |r \partial_r (\nu/r)|^2 dr/r \, \mathrm{dVol}(X) \leq C_{11} \epsilon.$$

Therefore, we have (4.16). It suffices therefore to prove (4.15). In a way similar to (5.1), by (5.7), we have

$$\begin{aligned} &\sup_{b \in (b_*, b_1)} \mathrm{Vol}(X) - \mathrm{Vol}\left(M \cap \{b\} \times S^{n-1}, g'/r^2\right) \\ &\leq \sup_{b \in (b_*, b_1), b' \in (a_1, b_1)} \int_{M \cap \{b'\} \times S^{n-1}} \psi - \int_{M \cap \{b\} \times S^{n-1}} \psi + C_{12} \epsilon \end{aligned}$$

for some constant $C_{12} > 0$. By Proposition 3.4, (5.4), (5.3) and (S3), we have

$$\sup_{b \in (b_*, b_1), b' \in (a_1, b_1)} \int_{M \cap \{b'\} \times S^{n-1}} \psi - \int_{M \cap \{b\} \times S^{n-1}} \psi \leq C_{13} \epsilon$$

for some constant $C_{13} > 0$. Thus, there exists a constant $C_{14} > 0$ such that

$$\sup_{b \in (b_*, b_1)} \mathrm{Vol}(X) - \mathrm{Vol}\left(M \cap \{b\} \times S^{n-1}, g'/r^2\right) \leq C_{14} \epsilon.$$

Therefore, we have (4.15). We may now apply Lemma 4.3 to u. Therefore, as in (4.17), we have

$$\sup_{t \in (t_0, t_*)} \|u\|_{L^2_x} \leq C_{15} \epsilon^{c_{15}}$$

for some constants $c_{15}, C_{15} > 0$. Therefore, by interpolation and (5.8), we have

$$\|\nu|_{(b_*, a_1) \times X}\|_{C^1_{\mathrm{cyl}}} = \|u\|_{C^1_{t,x}(-m \log a_1, t_*)} \leq C_{10} \epsilon^{c_{10}}$$

for some constants $C_{10}, c_{10} > 0$. By (S5), this proves Proposition 5.1. $\square$

Suppose $b_* \in S_*$. Then, by Proposition 5.1, we may apply Lemma 3.6 to $M \cap ((\lambda b_*/\lambda', b_*/\lambda') \times S^{n-1})$. Therefore, $\lambda'' b_*/\lambda' \in S_*$. Therefore, b_* is an interior point in S_*. S_* is thus an open subset of $[\lambda' a_0/\lambda, a_1)$.

By definition, S_* is a closed subset of $[\lambda' a_0/\lambda, a_1)$. S_* is thus a non-empty open closed subset of $[\lambda' a_0/\lambda, a_1)$. Therefore, $S_* = [\lambda' a_0/\lambda, a_1)$. Therefore, $\lambda' a_0/\lambda \in S_*$. Therefore, by (5.7) and Proposition 5.1, we have

$$M \cap ((\lambda' a_0/\lambda, b_1) \times S^{n-1}) = G_{\mathrm{cyl}}(\nu) \text{ with } \|\nu|_{(\lambda' a_0/\lambda, b_1)\times X}\|_{C^1_{\mathrm{cyl}}} \le C_{10}\epsilon^{c_{10}}.$$

Therefore, by (S5), we have (2.3). This completes the proof of Theorem 2.2. $\square$

We shall prove the main result of this paper.

Proof of Theorem 1.1. Let W be a Calabi–Yau manifold of complex dimension m with Kähler metric g, and $f_s : (U, \omega_s) \to (\mathbb{C}^m, \omega')$ a smooth family of Darboux charts centered at p as in Section 1. Choose a normal chart $f' : U \to \mathbb{C}^m$ centered at p in (W, g) with $df' = df_s$ at p. For every $s > 0$ sufficiently small, choose a_0, b_0, a_1, b_1 so that

$$(a_0, b_0) \times S^{2m-1} \subset f'(B(a_s) \setminus \overline{B(a_s/2)}),\ (a_1, b_1) \times S^{2m-1} \subset f'(B(2b_s) \setminus \overline{B(b_s)}),$$

where $(a, b) \times S^{2m-1}$ are embedded into $\mathbb{C}^m$ by $(r, y) \mapsto ry$. Suppose there exists a compact special Lagrangian submanifold M_s satisfying (B1) and (B2) in Section 1. Set

$$M = f'(M_s \cap U) \cap ((a_0, b_1) \times S^{2m-1}). \tag{5.9}$$

We shall prove (A1), (A2), (A3), (A4) and (A5) of Theorem 2.2. It is easy to see (A1) and (A4). We have (A2) since f' is a normal chart. We have (A3) since $a_s = O(s), b_s = O(s^\beta)$ by assumption. It suffices therefore to prove (A5). Let K be a Lawlor neck as in Section 1. As in Butscher [2, Theorem 6], K satisfies:

$$f' \circ f_s^{-1}(sK) \cap ((a_0, b_0) \times S^{2m-1}) \text{ is sufficiently close to } (a_0, b_0) \times X$$

in the C^1_{cyl}-topology whenever $s > 0$ is sufficiently small and R is sufficiently large. Here, C^1_{cyl} is as in Section 2. Since M_s satisfies (B1) as in Section 1, we have:

$$f' \circ f_s^{-1}(sK) \cap ((a_0, b_0) \times S^{2m-1}) \text{ is sufficiently close to } M \cap ((a_0, b_0) \times S^{2m-1})$$

in the C^1_{cyl}-topology whenever $s > 0$ is sufficiently small. Thus, we have:

$$M \cap ((a_0, b_0) \times S^{2m-1}) \text{ is sufficiently close to } (a_0, b_0) \times X \tag{5.10}$$

in the C^1_{cyl}-topology. Since M_s satisfies (B2) as in Section 1, we have:

$$M \cap ((a_1, b_1) \times S^{2m-1}) \text{ is sufficiently close to } (a_1, b_1) \times X \tag{5.11}$$

in the C^1_{cyl}-topology whenever $s > 0$ is sufficiently small. By (5.10) and (5.11), we have (A5). We may now apply Theorem 2.2 to M. Therefore, as in (2.3), M is the graph of some normal vector field on $(a_0, b_1) \times X$ in $(a_0, b_1) \times S^{2m-1}$ with C^1_{cyl}-norm converging to 0 as $s \to +0$.

By using a partition of unity, choose a compact Lagrangian submanifold N_s of (W, ω_s) such that

$$N_s \cap B(a_s) = f_s^{-1}(sK) \cap B(a_s),\ N_s \setminus \overline{B(b_s)} = F_s(L_1 \cup L_2),$$

and $f'(N_s \cap U) \cap ((a_0, b_1) \times S^{2m-1})$ is sufficiently close to $(a_0, b_1) \times X$ in the C^1_{cyl}-topology whenever $s > 0$ is sufficiently small; see Y. Lee [10, 3. Approximate submanifolds] or Joyce [7, Definition 6.2]. Then, for every $s > 0$ sufficiently small, there exists $c_0 \in (a_0, b_0)$ such that $M \cap ((b_0, a_1) \times S^{2m-1})$ is contained in the graph of some normal vector field on $f'(N_s \cap U) \cap (c_0, a_1) \times S^{2m-1}$ in $((c_0, a_1) \times S^{2m-1}, f'_* g_s)$ with C^1-norm converging to 0 as $s \to +0$. Therefore, since M_s satisfies (B1) and (B2) as in Section 1, M_s is the graph of some normal vector field x on N_s in (W, g_s) whenever $s > 0$ is sufficiently small.

$x \lrcorner \omega_s$ is a 1-form on M_s since M_s is Lagrangian with respect to ω_s. Since K is diffeomorphic to $\mathbb{R} \times S^{m-1}$ with $m > 2$, the restriction of $x \lrcorner \omega_s$ to $M_s \cap B(a_s)$ is an exact 1-form. By assumption, the restriction of $x \lrcorner \omega_s$ to $M_s \setminus \overline{B(b_s)}$ is an exact 1-form; see (B2) in Section 1. By assumption, $F_s(L_1) \setminus \overline{B(b_s)}$ and $F_s(L_2) \setminus \overline{B(b_s)}$ do not intersect. Thus, $x \lrcorner \omega_s = dh$ for some smooth function $h : N_s \to \mathbb{R}$.

M_s is thus the graph of dh on N_s. In the same way, if M'_s is another special Lagrangian submanifold as in Theorem 1.1, M'_s is the graph of dh' on N_s for some smooth function $h' : N_s \to \mathbb{R}$. Therefore, M'_s is a time-independent Hamiltonian deformation of M_s. Therefore, by a result of Thomas and Yau [15, Lemma 4.2], $M'_s = M_s$. This completes the proof of Theorem 1.1, the main result of this paper. □

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