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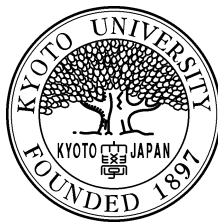
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Minimal stratifications for line arrangements and positive homogeneous presentations for fundamental groups

by

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Abstract

The complement of a complex hyperplane arrangement is known to be homotopic to a minimal CW complex. There are several approaches to the minimality. In this paper, we restrict our attention to real two dimensional cases, and introduce the “dual” objects so called minimal stratifications. The strata are explicitly described as semialgebraic sets. The stratification induces a partition of the complement into a disjoint union of contractible spaces, which is minimal in the sense that the number of codimension k pieces equals the k -th Betti number.

We also discuss presentations for the fundamental group associated to the minimal stratification. In particular, we show that the fundamental groups of complements of a real arrangements have positive homogeneous presentations.

1 Introduction

In 1980s Randell found an algorithm for presenting the fundamental group of the complement $M(\mathcal{A})$ of arrangement \mathcal{A} of complexified lines in \mathbb{C}^2 ([R1, F]). Various algorithms for doing this were found subsequently ([A, CS, MT]). It was observed that these presentations are minimal in the sense that the numbers of generators and relations are equal to $b_1(\pi_1)$ and $b_2(\pi_1)$, respectively, (c.f. $b_i(M) = b_i(\pi_1(M(\mathcal{A})))$ for $i \leq 2$ [R2]) and several presentations are homotopic to $M(\mathcal{A})$. (It is not clear to the author that whether or not every minimal presentation is homotopic to $M(\mathcal{A})$, which is true for braid-monodromy presentation [Li].)

These works have been partially generalized to higher dimensional cases. Let \mathcal{A} be an arrangement of hyperplanes in \mathbb{C}^ℓ . The complement $M(\mathcal{A}) =$

$\mathbb{C}^\ell \setminus \mathcal{A}$ is proved to be homotopic to a minimal CW complex, that is, a finite CW complex in which the number of p -cells equals the p -th Betti number [PS, DP, R3]. The minimality is expected to have applications to topological problems of arrangements. In order to apply, we need to make explicit how cells in the minimal CW complex are attached. There are two approaches to describe the minimal structure of $M(\mathcal{A})$, one is based on classical Morse theoretic study of Lefschetz's theorem on hyperplane section [Y1], the other is based on discrete Morse theory of Salvetti complex [SS, D]. There are also some applications to computations of local system (co-)homology groups [GS, Y2, Y3].

The purpose of this paper is to describe the “dual” object to the minimal CW complex for $\ell = 2$. We introduce the minimal stratification $M(\mathcal{A}) = X_0 \supset X_1 \supset X_2$ for the complement $M(\mathcal{A})$ such that

- $X_0 \setminus X_1 = U$ is a contractible 4-manifolds,
- $X_1 \setminus X_2 = \bigsqcup_{i=1}^{b_1(M)} S_i^\circ$ is a disjoint union of contractible 3-manifolds, such that the number of pieces is equal to the 1st Betti number $b_1(M)$, and
- $X_0 = \bigsqcup_{\lambda=1}^{b_2(M)} C_\lambda$ is a disjoint union of contractible 2-manifolds (chambers), such that the number of pieces is equal to the 2nd Betti number $b_2(M)$.

(see Theorem 4.2 for details). We describe explicitly the strata as semialgebraic sets. For such stratification, we can take generators and relations of $\pi_1(M)$ which are dual to the strata. By analyzing the incidence relation of strata, we obtain a presentation for π_1 which is not rely on braid monodromy or Zariski-van Kampen method. The resulting presentation has only positive homogeneous relations.

This paper is organized as follows. In §2, as a motivating example, we compare the minimal stratification with Morse theoretic description of minimal CW complex for a very simple example: two points $\{0, 1\}$ in \mathbb{R} . In §3 we recall basic facts and introduce the sail $S(\alpha, \beta)$ bound to lines. The sail is a 3-dimensional semialgebraic submanifold of $M(\mathcal{A})$ which will be used to define the minimal stratification. §4 contains the main result. The proof will be given in §7. In §5 we discuss the presentation for $\pi_1(M(\mathcal{A}))$ associated to the minimal stratification. The generators are taken as transversal loops to the strata. In §6 we take meridian generators for the fundamental group. By computing relations in the previous section with respect to the new generators, we reach the positive homogeneous presentation.

2 A one-dimensional example

Example 2.1. Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\varphi(z) := \frac{(z+1)^2}{\sqrt{z(z-1)}}$. We consider $|\varphi| : M \rightarrow \mathbb{R}$ as a Morse function which has three critical points $z = -1, \frac{5-\sqrt{17}}{4}, \frac{5+\sqrt{17}}{4}$ with index $0, 1, 1$ respectively. Note that all critical points are real and $0 < \frac{5-\sqrt{17}}{4} < 1 < \frac{5+\sqrt{17}}{4}$. The unstable manifolds present a one-dimensional CW complex which is homotopic to M . Since $|\varphi(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$, the unstable cells are as in Figure 1. It is not easy to describe the unstable manifolds explicitly even for one-dimensional cases. Nevertheless, the stable manifolds can be explicitly described: two open segments $(0, 1), (1, \infty)$ and the remainder $U = M \setminus ((0, 1) \cup (1, \infty))$.

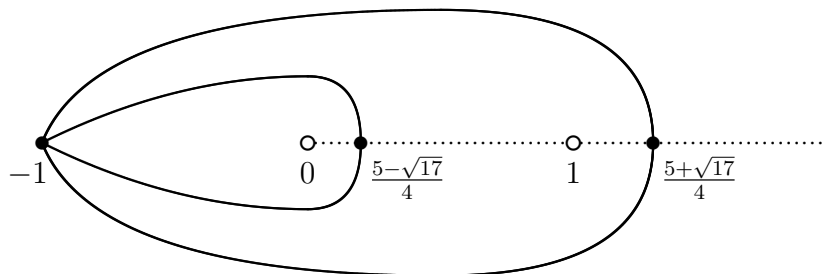


Figure 1: Unstable and stable manifolds (thick and dotted line, respectively).

We have a partition $U \sqcup (0, 1) \sqcup (1, \infty)$ of M by contractible pieces, and note that the number of codimension zero piece is equal to $b_0(M) = 1$ and that of codimension one is $b_1(M) = 2$. Also note that codimension one pieces $(0, 1)$ and $(1, \infty)$ are nothing but chambers of the real hyperplane arrangement $\{0, 1\}$. These pieces are expressed in terms of defining linear forms as follows,

$$\begin{aligned} (0, 1) &= \left\{ z \in M \mid \frac{z-1}{z} \in \mathbb{R}_{<0} \right\}, \\ (1, \infty) &= \left\{ z \in M \mid \frac{-1}{z-1} \in \mathbb{R}_{<0} \right\}, \end{aligned} \tag{1}$$

where $\mathbb{R}_{<0}$ is the set of negative real numbers.

The homotopy types of the unstable cells for higher dimensional cases are discussed in [Y1]. The unstable cell itself is highly transcendental. On the other hand, the stable manifolds are semi-algebraic sets (open segments and their complements). The purpose of this paper is to generalize the “dual” description for two-dimensional real arrangements. We will give an explicit stratification for the complement of complexified real lines in \mathbb{C}^2 such that

- each disjoint piece is contractible, and
- the number of codimension k pieces is equal to k -th Betti number.

We will see that formulae similar to (1) define one-codimensional pieces.

3 Basic notation

3.1 Setting

A real arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ is a finite set of affine lines in the affine plane \mathbb{R}^2 . Each line is defined by some affine linear form

$$\alpha_H(x_1, x_2) = ax_1 + bx_2 + c = 0, \quad (2)$$

with $a, b, c \in \mathbb{R}$ and $(a, b) \neq (0, 0)$. A connected component of $\mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ is called a chamber. The set of all chambers is denoted by $\text{ch}(\mathcal{A})$. The affine linear equation (2) defines a complex line $\{(z_1, z_2) \in \mathbb{C}^2 \mid az_1 + bz_2 + c = 0\}$ in \mathbb{C}^2 . We denote the set of complexified lines by $\mathcal{A}_{\mathbb{C}} = \{H_{\mathbb{C}} = H \otimes \mathbb{C} \mid H \in \mathcal{A}\}$. The object of our interest is the complexified complement $M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$.

3.2 Generic flags and numbering of lines

Let \mathcal{F} be a generic flag in \mathbb{R}^2

$$\mathcal{F} : \emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 = \mathbb{R}^2,$$

where \mathcal{F}^k is a generic k -dimensional affine subspace.

Definition 3.1. For $k = 0, 1, 2$, define the subset $\text{ch}_k^{\mathcal{F}}(\mathcal{A}) \subset \text{ch}(\mathcal{A})$ by

$$\text{ch}_k^{\mathcal{F}}(\mathcal{A}) := \{C \in \text{ch}(\mathcal{A}) \mid C \cap \mathcal{F}^k \neq \emptyset, C \cap \mathcal{F}^{k-1} = \emptyset\}.$$

The set of chambers decomposes into a disjoint union, $\text{ch}(\mathcal{A}) = \text{ch}_0^{\mathcal{F}}(\mathcal{A}) \sqcup \text{ch}_1^{\mathcal{F}}(\mathcal{A}) \sqcup \text{ch}_2^{\mathcal{F}}(\mathcal{A})$. The cardinality of $\text{ch}_k^{\mathcal{F}}(\mathcal{A})$ is given as follows, which is an application of Zaslavski's formula [Z].

Proposition 3.2.

$$\begin{aligned} \#\text{ch}_0^{\mathcal{F}}(\mathcal{A}) &= b_0(M(\mathcal{A})) = 1, \\ \#\text{ch}_1^{\mathcal{F}}(\mathcal{A}) &= b_1(M(\mathcal{A})) = n, \\ \#\text{ch}_2^{\mathcal{F}}(\mathcal{A}) &= b_2(M(\mathcal{A})). \end{aligned}$$

3.3 Assumptions on generic flag and numbering

Throughout this paper, we assume that the generic flag \mathcal{F} satisfies the following conditions:

- \mathcal{F}^1 does not separate intersections of \mathcal{A} ,
- \mathcal{F}^0 does not separate n -points $\mathcal{A} \cap \mathcal{F}^1$.

Then we can choose coordinates x_1, x_2 so that \mathcal{F}^0 is the origin, \mathcal{F}^1 is given by $x_2 = 0$, all intersections of \mathcal{A} are contained in the upper-half plane $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ and $\mathcal{A} \cap \mathcal{F}^1$ is contained in the half line $\{(x_1, 0) \mid x_1 > 0\}$.

We set $H_i \cap \mathcal{F}^1$ has coordinates $(a_i, 0)$. By changing the numbering of lines and signs of the defining equation α_i of $H_i \in \mathcal{A}$ we may assume

- $0 < a_n < a_{n-1} < \dots < a_1$, and
- the origin \mathcal{F}^0 is contained in the negative half plane $H_i^- = \{\alpha_i < 0\}$.

Remark 3.3. Sometimes it is convenient to consider 0-th line H_0 to be the line at infinity H_0 with defining equation $\alpha_0 = -1$ and $a_0 = +\infty$.

We also put $\text{ch}_0^{\mathcal{F}}(\mathcal{A}) = \{C_0\}$ and $\text{ch}_1^{\mathcal{F}}(\mathcal{A}) = \{C_1, \dots, C_n\}$ so that $C_k \cap \mathcal{F}^1$ is equal to the interval (a_k, a_{k-1}) . (We use the convention $a_0 = +\infty$.) It is easily seen that the chambers C_0 and C_k ($k = 1, \dots, n$) have the following expression.

$$\begin{aligned} C_0 &= \bigcap_{i=1}^n \{\alpha_i < 0\}, \\ C_k &= \bigcap_{i=0}^{k-1} \{\alpha_i < 0\} \cap \bigcap_{i=k}^n \{\alpha_i > 0\}, \quad (k = 1, \dots, n). \end{aligned} \tag{3}$$

(We consider $\alpha_0 < 0$ whole \mathbb{R}^2 .) The notations introduced in this section are illustrated in Figure 2.

3.4 Sails bound to lines

Let $\alpha, \beta \in \mathbb{C}[z_1, z_2]$ be polynomials of $\deg \leq 1$. We assume that $\alpha \neq 0, \beta \neq 0$ and they are linearly independent over \mathbb{C} . (Note that we allow the situation that one of α or β is equal to a non-zero constant.)

Definition 3.4. For α and β as above, we define the *sail bound to α and β* by

$$S(\alpha, \beta) = \left\{ z = (z_1, z_2) \in \mathbb{C}^2 \mid \alpha(z)\beta(z) \neq 0, \frac{\alpha(z)}{\beta(z)} \in \mathbb{R}_{<0} \right\}.$$

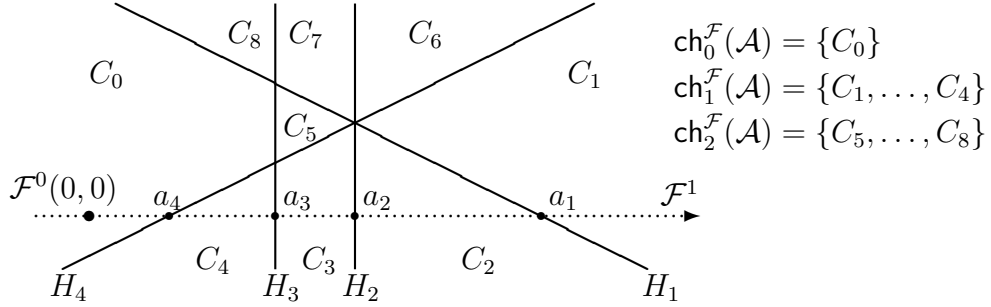


Figure 2: Numbering of lines and chambers.

The sail $S(\alpha, \beta)$ is a closed subset of $\mathbb{C}^2 \setminus \{\alpha\beta = 0\}$. Furthermore we have:

Lemma 3.5. $S(\alpha, \beta)$ is an orientable 3-dimensional manifold. More precisely,

- (1) if α and β determine intersecting lines, then $S(\alpha, \beta)$ is diffeomorphic to $\mathbb{C}^* \times \mathbb{R}_{<0}$.
- (2) else, (i.e., either α and β determine parallel lines or one of α and β is a nonzero constant), then $S(\alpha, \beta)$ is diffeomorphic to $\mathbb{C} \times \mathbb{R}_{<0}$.

Proof. Case (1): Suppose that $\deg \alpha = \deg \beta = 1$ and two lines intersect. Then the map

$$\begin{aligned} (\alpha, \beta) : \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto (\alpha(z), \beta(z)) \end{aligned}$$

is isomorphic. The image of the sail $S(\alpha, \beta)$ by the map (α, β) is

$$\left\{ (s, t) \in \mathbb{C}^2 \mid s \cdot t \neq 0, \frac{s}{t} \in \mathbb{R}_{<0} \right\},$$

where s, t are coordinates of the target \mathbb{C}^2 . The image is isomorphic to $\mathbb{C}^* \times \mathbb{R}_{<0}$ by the isomorphism $(s, t) \mapsto (t, s/t)$ of $(\mathbb{C}^*)^2$.

Case (2): Suppose that $\deg \alpha = \deg \beta = 1$ and two lines are parallel. In this case we may assume that $\beta = p\alpha + q$ with $p, q \in \mathbb{C}^*$. Choose another linear equation γ such that lines $\alpha = 0$ and $\gamma = 0$ are intersecting. Then

$$\begin{aligned} (\alpha, \gamma) : \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto (\alpha(z), \gamma(z)) \end{aligned}$$

is isomorphic. The image of $S(\alpha, \beta)$ is expressed as

$$\left\{ (s, t) \in \mathbb{C}^2 \mid s \cdot t \neq 0, \frac{s}{ps + q} \in \mathbb{R}_{<0} \right\}.$$

It is easily checked that the set

$$\left\{ s \in \mathbb{C} \left| \frac{s}{p(s + \frac{q}{p})} \in \mathbb{R}_{<0} \right. \right\}$$

is an open arc connecting 0 and $-\frac{q}{p} \in \mathbb{C}$. Thus $S(\alpha, \beta)$ is isomorphic to the product of the open arc and \mathbb{C} .

Case (3): The proof for the case $\deg \alpha = 1$ and $\deg \beta = 0$ is similar to the case (2). \square

3.5 Orientations

For the purpose of obtaining a presentation for the fundamental group of $M(\mathcal{A})$, intersection numbers of loops and sails play crucial roles. It is necessary to specify the orientation of the sail $S(\alpha, \beta)$.

We first recall that the orientation of \mathbb{C}^2 is given by the identification

$$\mathbb{C}^2 \xrightarrow{\sim} \mathbb{R}^4$$

$$(z_1, z_2) \longmapsto (x_1, y_1, x_2, y_2),$$

where $z_i = x_i + \sqrt{-1}y_i$. Consider the map $\varphi = \frac{\alpha}{\beta} : \mathbb{C}^2 \setminus \{\alpha\beta = 0\} \rightarrow \mathbb{C}$. Since $S(\alpha, \beta)$ is connected, it is enough to specify an orientation of $T_p S(\alpha, \beta)$ for a point $p \in S(\alpha, \beta)$. The following two ordered direct sums determine an orientation of $S(\alpha, \beta)$:

$$\begin{aligned} T_p S(\alpha, \beta) \oplus N_p(S(\alpha, \beta), \mathbb{C}^2) &= T_p \mathbb{C}^2 \\ T_{\varphi(p)} \mathbb{R}_{<0} \oplus \varphi_* N_p(S(\alpha, \beta), \mathbb{C}^2) &= T_{\varphi(p)} \mathbb{C}, \end{aligned}$$

where $N_p(S, \mathbb{C}^2)$ is a normal bundle. Note that we consider the orientation of $\mathbb{R}_{<0}$ induced from the inclusion $\mathbb{R}_{<0} \subset \mathbb{R}$.

Remark 3.6. $S(\alpha, \beta)$ and $S(\beta, \alpha)$ are the same as manifolds, but orientations are different.

The above definition is equivalent to saying as follows. Let $c : (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}^2 \setminus \{\alpha\beta = 0\}$ be a differentiable map transversal to $S(\alpha, \beta)$. Assume that $c^{-1}(S(\alpha, \beta)) = \{0\}$. Then c intersects $S(\alpha, \beta)$ positively (denoted by $I_{c(0)}(S(\alpha, \beta), c) = +1$) if and only if

$$\varphi_*(\dot{c}(0)) \in T_{\varphi(c(0))} \mathbb{C} \simeq \mathbb{C}$$

has positive imaginary part (Figure 3).

Let us look at an example showing how the intersection numbers are computed.

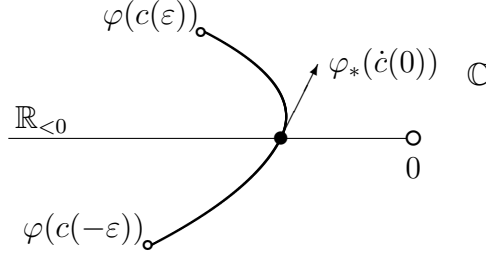


Figure 3: $\varphi \circ c : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{C}$.

Example 3.7. Let $\varphi(z_2, z_1) = \frac{z_2}{z_1}$ and

$$S := S(z_2, z_1) = \{(z_1, z_2) \in (\mathbb{C}^*)^2 \mid \varphi(z_2, z_1) \in \mathbb{R}_{<0}\}.$$

Fix positive real numbers $r, \varepsilon > 0$ and an argument $0 \leq \theta_0 < 2\pi$. Consider the continuous map

$$\begin{aligned} \gamma : \mathbb{R}/2\pi\mathbb{Z} &\longrightarrow (\mathbb{C}^*)^2 \\ t &\longmapsto r(\cos \theta_0, \sin \theta_0) + \sqrt{-1}\varepsilon(\cos t, \sin t). \end{aligned}$$

Then $\gamma(t) \in S$ if and only if $\varphi(\gamma(t)) = \frac{r \sin \theta_0 + \sqrt{-1}\varepsilon \sin t}{r \cos \theta_0 + \sqrt{-1}\varepsilon \cos t}$ is a negative real number. Since

$$\frac{r \sin \theta_0 + \sqrt{-1}\varepsilon \sin t}{r \cos \theta_0 + \sqrt{-1}\varepsilon \cos t} = \frac{r^2 \sin \theta_0 \cos \theta_0 + \varepsilon^2 \sin t \cos t + \sqrt{-1} \cdot r \cdot \varepsilon \sin(t - \theta_0)}{r^2 \cos^2 \theta_0 + \varepsilon^2 \cos^2 t},$$

it is contained in $\mathbb{R}_{<0}$ if and only if $t = \theta_0, \theta_0 + \pi$ and $\sin \theta_0 \cdot \cos \theta_0 < 0$ (equivalently either $\frac{\pi}{2} < \theta_0 < \pi$ or $\frac{3\pi}{2} < \theta_0 < 2\pi$). In such cases it is easily seen that $\Im \varphi_*(\dot{\gamma}(\theta_0)) > 0$ and $\Im \varphi_*(\dot{\gamma}(\theta_0 + \pi)) < 0$. Hence we have

$$I_{\gamma(\theta_0)}(S, \gamma) = +1, \text{ and } I_{\gamma(\theta_0 + \pi)}(S, \gamma) = -1.$$

4 Minimal Stratification

4.1 Main result

In this section we shall give an explicit stratification of the complement $M(\mathcal{A})$ by using chambers and sails. We keep the notations as in §3.3. First recall that the sail defined by α_i and α_{i-1} is

$$S(\alpha_{i-1}, \alpha_i) = \left\{ z \in \mathbb{C}^2 \mid \alpha_{i-1}(z) \cdot \alpha_i(z) \neq 0, \frac{\alpha_{i-1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\}.$$

(we use the convention $\alpha_0 = -1$). Then

$$S_i := S(\alpha_{i-1}, \alpha_i) \cap M(\mathcal{A})$$

is an oriented 3-dimensional closed submanifold of $M(\mathcal{A})$ for $i = 1, \dots, n$. These S_i 's stratify the complement $M(\mathcal{A})$.

Proposition 4.1. Let $C \in \text{ch}(\mathcal{A})$ and $i = 1, \dots, n$. The following are equivalent.

- (a) $C \subset S_i$.
- (b) $C \cap S_i \neq \emptyset$.
- (c) $\alpha_i(C) \cdot \alpha_{i-1}(C) < 0$. (We use the convention $\alpha_0 = -1$.)

Now we state the main result.

Theorem 4.2. *The closed submanifolds $S_1, \dots, S_n \subset M(\mathcal{A})$ satisfy the following.*

- (i) S_i and S_j ($i \neq j$) intersect transversely, and $S_i \cap S_j = \bigsqcup C$, where C runs all chambers satisfying $\alpha_i(C)\alpha_{i-1}(C) < 0$ and $\alpha_j(C)\alpha_{j-1}(C) < 0$.
- (ii) $S_i^\circ := S(\alpha_i, \alpha_{i-1}) \setminus \bigcup_{C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})} C$ is a contractible 3-manifold.
- (iii) $U := M(\mathcal{A}) \setminus \bigcup_{i=1}^n S_i$ is a contractible 4-manifold.

The proof will be given in §7.

Remark 4.3. Theorem 4.2 gives rise to a partition of $M(\mathcal{A})$ into disjoint union of contractible manifolds $M(\mathcal{A}) = U \sqcup \bigsqcup_{i=1}^n S_i \sqcup \bigsqcup_{C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})} C$. Such partitions are obtained in [IY] for any dimension. However, the partition $M(\mathcal{A}) = \bigsqcup S_\lambda$ in [IY] is not given by a stratification, in other words, it does not satisfy the following property: $\overline{S_\lambda} \setminus S_\lambda$ is a union of other pieces of smaller dimensions. We do not know explicit minimal stratification for dimension ≥ 3 .

5 Dual presentation for the fundamental group

Using Theorem 4.2, we give a presentation for the fundamental group $\pi_1(M(\mathcal{A}))$. The idea is that we take the base point in U and transversal loop to each S_i as a generator, then relations are generated by loops around chambers $C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})$.

5.1 Transversal generators

Fix a base point $* \in U$ and a point $p_i \in S_i^\circ$. There exists a continuous curve $\eta_i : [0, 1] \rightarrow M(\mathcal{A})$ such that

- $\eta_i(0) = \eta_i(1) = *$,
- $\eta_i(1/2) = p_i$ and $\eta_i^{-1}(S_i) = \{1/2\}$,
- η_i intersects S_i° transversely and positively, that is, $I_{p_i}(S_i^\circ, \eta_i) = 1$, and it does not intersect S_j for $j \neq i$.

Since U and S_i° are contractible, the homotopy type of η_i is independent of the choice of η_i .

Let $\eta : [0, 1] \rightarrow M(\mathcal{A})$ be a continuous map with $\eta(0), \eta(1) \in U$ (not necessarily $\eta(0) = \eta(1) = *$). Since U is contractible, there exist paths c_1 from the base point $*$ to $\eta(0)$ and c_2 from $\eta(1)$ to $*$. Then $c_1\eta c_2$ is a loop which homotopy class $[c_1\eta c_2] \in \pi_1(M(\mathcal{A}), *)$ is uniquely determined by η . We denote the class by $[\eta] \in \pi_1(M(\mathcal{A}), *)$ for simplicity.

Lemma 5.1. *With the notation above, $[\eta_1], \dots, [\eta_n]$ generate $\pi_1(M(\mathcal{A}), *)$.*

Proof. Let $\eta : [0, 1] \rightarrow M(\mathcal{A})$ be a continuous map such that $\eta(0) = \eta(1) = *$. By the transversality homotopy theorem (e.g., [GP, Chap 2]), we can perturb η into a new loop such that the following hold:

- The image of η is disjoint from $\bigsqcup_{C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})} C$.
- The image of η intersects $\bigsqcup_{i=1}^n S_i^\circ$ transversely.

Suppose that $\eta^{-1}(\bigsqcup_{i=1}^n S_i^\circ) = \{t_1, \dots, t_N\}$ with $0 < t_1 < \dots < t_N < 1$ and $\eta(t_k) \in S_{m_k}^\circ$. From the transversality, the intersection number $\varepsilon_k := I_{\eta(t_k)}(S_{m_k}^\circ, \eta)$ is either $+1$ or -1 because of transversality. The class $[\eta] \in \pi_1(M(\mathcal{A}), *)$ is expressed as

$$[\eta] = [\eta_{m_1}]^{\varepsilon_1} [\eta_{m_2}]^{\varepsilon_2} \dots [\eta_{m_N}]^{\varepsilon_N}.$$

Thus any $[\eta] \in \pi_1(M)$ is generated by $[\eta_1], \dots, [\eta_n]$. □

Remark 5.2. If we fix the base point in $\mathcal{F}_\mathbb{C}^1 = \mathcal{F}^1 \otimes \mathbb{C}$, then we may choose transversal generators as in Figure 4.

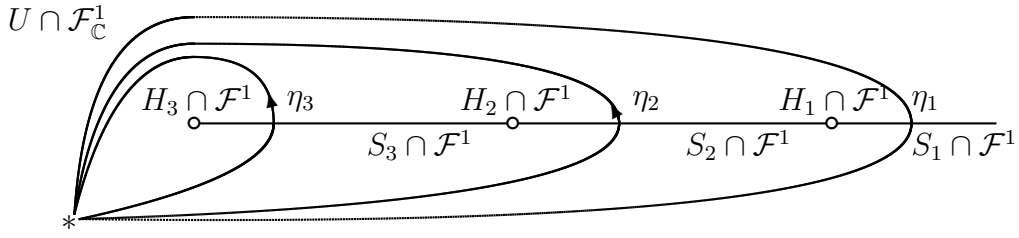


Figure 4: Transversal generators η_1, η_2, η_3 .

5.2 Chamber relations

As we have seen in the previous section, the transversal generators determine a surjective homomorphism

$$G : F\langle \eta_1, \dots, \eta_n \rangle \longrightarrow \pi_1(M(\mathcal{A}), *),$$

from the free group generated by η_1, \dots, η_n to $\pi_1(M(\mathcal{A}), *)$. We will prove that the kernel of the above map is generated by conjugacy classes of meridian loops around chambers $C \subset M(\mathcal{A})$, $C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})$.

Let $\eta : [0, 1] \rightarrow M(\mathcal{A})$ be a loop with $\eta(0) = \eta(1) = *$. Suppose that η represents an element of $\text{Ker } G$. Then η is null-homotopic in $M(\mathcal{A})$, and hence there is a homotopy $\sigma : [0, 1]^2 \rightarrow M(\mathcal{A})$ such that $\sigma(t, 0) = \eta(t)$, $\sigma(t, 1) = \sigma(0, s) = \sigma(1, s) = *$. We can perturb σ in such a way that

- $\sigma(\partial[0, 1]^2) \cap \bigsqcup_{C \in \text{ch}_2} C = \emptyset$.
- σ intersects $\bigsqcup_{C \in \text{ch}_2} C$ transversely.

Let $\sigma^{-1}(\bigsqcup_{C \in \text{ch}_2} C) = \{q_1, \dots, q_L\}$. We choose a meridian loop v_i in $[0, 1]^2$ around each point q_i with the base point $(0, 0)$. Let $\alpha : [0, 1] \rightarrow \partial([0, 1]^2)$ be the loop with the base point $(0, 0)$ that goes along the boundary in the counter clockwise direction. Then α is homotopically equivalent to a product of meridians v_i . Since η is homotopically equivalent to $\sigma \circ \alpha$, it is also homotopically equivalent to the product of meridian loops $\sigma \circ v_i$ that are meridian loops of chambers. (Figure 5.)

We will describe the relations more explicitly in §5.3.

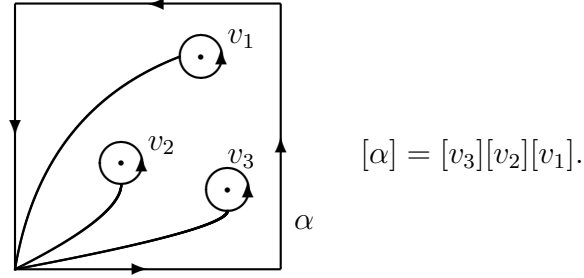


Figure 5: Inverse images of chambers.

5.3 Dual presentation

Let $i = 1, \dots, n$ and $C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})$. We define the i -th degree $d_i(C) \in \{-1, 0, +1\}$ by

$$d_i(C) = \begin{cases} -1 & \text{if } \alpha_{i-1}(C) < 0 < \alpha_i(C), \\ +1 & \text{if } \alpha_{i-1}(C) > 0 > \alpha_i(C), \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

(Here we use the convention $\alpha_0 = -1$, in particular, $\alpha_0(C) < 0$ for any chamber C . See §5.6 for examples.)

We will prove (in §5.5) that the meridian loop of $C \subset M(\mathcal{A})$ ($C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})$) is conjugate to the word

$$E(C) := \eta_n^{d_n(C)} \eta_{n-1}^{d_{n-1}(C)} \dots \eta_1^{d_1(C)} \cdot \eta_n^{-d_n(C)} \eta_{n-1}^{-d_{n-1}(C)} \dots \eta_1^{-d_1(C)}. \quad (5)$$

Theorem 5.3. *With notation as above, the fundamental group $\pi_1(M(\mathcal{A}), *)$ is isomorphic to the group defined by the presentation*

$$\langle \eta_1, \dots, \eta_n \mid E(C), C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A}) \rangle.$$

Remark 5.4. The information about homotopy type of $M(\mathcal{A})$ is encoded in the degree map $d_i : \text{ch}_2^{\mathcal{F}}(\mathcal{A}) \rightarrow \{0, \pm 1\}$. Indeed, it plays a role when we present cellular chain complex with coefficients in a local system (see §5.7).

Before proving Theorem 5.3 we introduce some terminology.

5.4 Pivotal argument

Let us denote the argument of the line H_i by θ_i , that is the angle of two positive half lines of \mathcal{F}^1 and H_i (see Figure 6). By the assumption on generic flag, arguments $\theta_1, \dots, \theta_n$ satisfy

$$0 < \theta_n \leq \theta_{n-1} \leq \dots \leq \theta_1 < \pi. \quad (6)$$

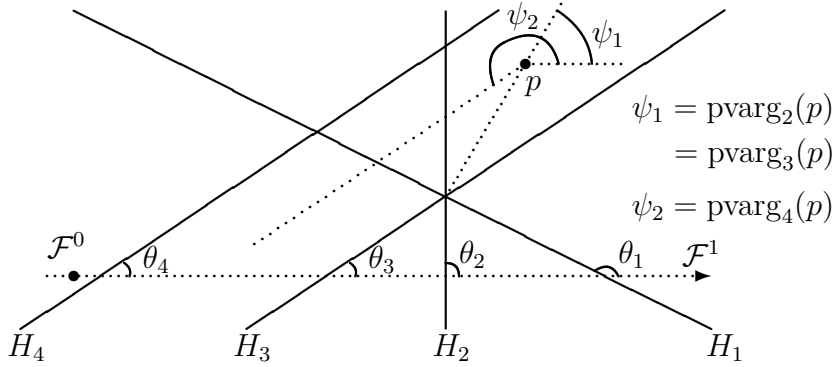


Figure 6: Pivotal arguments.

Remark 5.5. Sometimes it is convenient to define $\theta_0 := \theta_1$.

Definition 5.6. Let $p = (x_1, x_2) \in \mathbb{R}^2$ be a point different from $H_i \cap H_{i-1}$. For $i = 1, \dots, n$, define the i -th pivotal argument $\text{pvarg}_i(p) \in [0, 2\pi)$ by

$$\text{pvarg}_i(p) = \begin{cases} \arg(\vec{qp}), & \text{if } i > 1 \text{ and } H_i \cap H_{i-1} (\neq \emptyset) = \{q\}, \\ \theta_i + \pi, & \text{if } i = 1 \text{ or } i > 1, H_i \text{ is parallel to } H_{i-1}. \end{cases}$$

And also

$$|\text{pvarg}_i(p)| = \begin{cases} \text{pvarg}_i(p), & \text{if } 0 \leq \text{pvarg}_i(p) < \pi, \\ \text{pvarg}_i(p) - \pi, & \text{if } \pi \leq \text{pvarg}_i(p) < 2\pi. \end{cases}$$

We have the following.

Proposition 5.7. Let $p \in \mathbb{R}^2$. Suppose $\alpha_i(p) \cdot \alpha_{i-1}(p) < 0$ ($i > 1$).

- If H_{i-1} and H_i intersects, then $\theta_i < |\text{pvarg}_i(p)| < \theta_{i-1}$.
- If H_{i-1} and H_i are parallel, then $\theta_i = |\text{pvarg}_i(p)| = \theta_{i-1}$.

Using pivotal arguments, we can describe the intersection number of the sail $S_i = S(\alpha_{i-1}, \alpha_i) \cap M(\mathcal{A})$ and a curve, which is a generalization of Example 3.7.

Example 5.8. Let $p(x_1, x_2) \in \mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ and $\varepsilon > 0$. Consider the loop

$$\begin{aligned} \gamma: \mathbb{R}/2\pi\mathbb{Z} &\longrightarrow M(\mathcal{A}) \\ t &\longmapsto (x_1, x_2) + \sqrt{-1}\varepsilon(\cos t, \sin t). \end{aligned}$$

If $\alpha_i(p) \cdot \alpha_{i-1}(p) > 0$, then γ does not intersect S_i . If $\alpha_i(p) \cdot \alpha_{i-1}(p) < 0$, then $\gamma^{-1}(S_i) = \{\text{pvar}_i(p), \text{pvar}_i(p) + \pi\}$. We have

$$\begin{aligned} I_{\gamma(\text{pvar}_i(p))}(S_i, \gamma) &= 1, \text{ and} \\ I_{\gamma(\text{pvar}_i(p)+\pi)}(S_i, \gamma) &= -1. \end{aligned}$$

Combining this with the degree d_i , we have the following.

Proposition 5.9. Let $p(x_1, x_2) \in \mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ and the loop γ be as in Example 5.8. Let us denote by C the chamber which contains p . We have

$$\begin{aligned} I_{\gamma(|\text{pvar}_i(p)|)}(S_i, \gamma) &= d_i(C), \text{ and} \\ I_{\gamma(|\text{pvar}_i(p)|+\pi)}(S_i, \gamma) &= -d_i(C). \end{aligned}$$

5.5 Proof of Theorem 5.3

Now we prove Theorem 5.3. Let $C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})$ and $p \in C$. We take a meridian loop $\gamma : \mathbb{R}/2\pi\mathbb{Z} \rightarrow M(\mathcal{A}), t \mapsto \gamma(t)$ as in Example 5.8. Then γ intersects S_i at $t = |\text{pvar}_i(p)|$ and $t = |\text{pvar}_i(p)| + \pi$ with intersection numbers $d_i(C)$ and $-d_i(C)$, respectively. (This logically includes that γ does not intersect C if and only if $d_i(C) = 0$.) In particular, from Proposition 5.7, $\theta_{i-1} \leq |\text{pvar}_i(p)| \leq \theta_i$ provided $d_i(C) \neq 0$. From Eq. (6), the loop γ intersects $S_n, S_{n-1}, \dots, S_1, S_n, S_{n-1}, \dots, S_1$ in this order with intersection numbers $d_n(C), d_{n-1}(C), \dots, d_1(C), -d_n(C), -d_{n-1}(C), \dots, -d_1(C)$. Hence the loop γ is homotopic to the word $E(C)$ in Eq. (5).

5.6 Examples

Example 5.10. Let $\mathcal{A} = \{H_1, \dots, H_5\}$ be a line arrangement and \mathcal{F} be a flag pictured in Figure 7. Then $\text{ch}_2^{\mathcal{F}}(\mathcal{A}) = \{C_6, C_7, \dots, C_{12}\}$ consists of 7 chambers. The degrees can be computed as follows.

	d_1	d_2	d_3	d_4	d_5
C_6	0	0	-1	1	-1
C_7	0	-1	0	1	-1
C_8	0	-1	1	0	-1
C_9	0	-1	1	0	0
C_{10}	-1	0	1	0	0
C_{11}	-1	0	0	1	0
C_{12}	-1	0	0	1	-1

The fundamental group $\pi_1(M(\mathcal{A}), *)$ has the following presentation.

$$\begin{aligned} \pi_1(M(\mathcal{A}), *) = \langle \eta_1, \dots, \eta_5 \mid & E(C_6) : \eta_5^{-1} \eta_4 \eta_3^{-1} \eta_5 \eta_4^{-1} \eta_3 \\ & E(C_7) : \eta_5^{-1} \eta_4 \eta_2^{-1} \eta_5 \eta_4^{-1} \eta_2 \\ & E(C_8) : \eta_5^{-1} \eta_3 \eta_2^{-1} \eta_5 \eta_3^{-1} \eta_2 \\ & E(C_9) : \eta_3 \eta_2^{-1} \eta_3^{-1} \eta_2 \\ & E(C_{10}) : \eta_3 \eta_1^{-1} \eta_3^{-1} \eta_1 \\ & E(C_{11}) : \eta_4 \eta_1^{-1} \eta_4^{-1} \eta_1 \\ & E(C_{12}) : \eta_5^{-1} \eta_4 \eta_1^{-1} \eta_5 \eta_4^{-1} \eta_1 \rangle. \end{aligned}$$

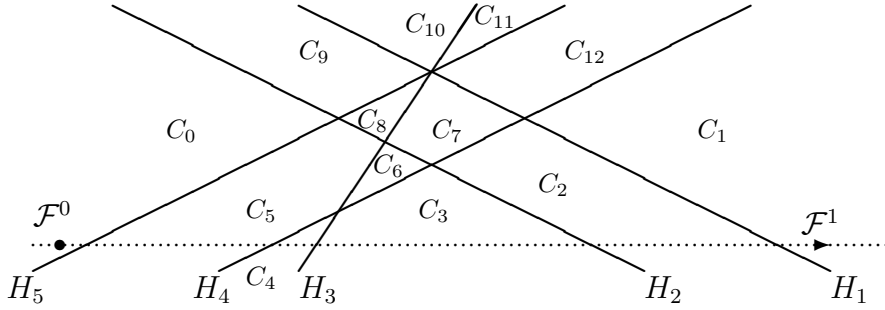


Figure 7: Example 5.10 and 6.7.

Example 5.11. Let $\mathcal{A} = \{H_1, \dots, H_5, H_6\}$ be a line arrangement and \mathcal{F} be a flag pictured in Figure 8. Then $\text{ch}_2^{\mathcal{F}}(\mathcal{A}) = \{C_7, C_8, \dots, C_{17}\}$ consists of 11 chambers. The degrees can be computed as follows.

	d_1	d_2	d_3	d_4	d_5	d_6
C_7	-1	1	0	-1	0	0
C_8	-1	1	0	0	-1	0
C_9	-1	1	0	0	-1	1
C_{10}	-1	0	1	0	-1	1
C_{11}	-1	0	1	-1	0	1
C_{12}	-1	0	1	-1	0	0
C_{13}	-1	1	0	0	0	0
C_{14}	-1	0	1	0	0	0
C_{15}	-1	0	0	1	0	0
C_{16}	-1	0	0	0	1	0
C_{17}	-1	0	0	0	0	1

The fundamental group $\pi_1(M(\mathcal{A}), *)$ has the following presentation.

$$\begin{aligned} \pi_1(M(\mathcal{A}), *) = \langle \eta_1, \dots, \eta_6 \mid & E(C_7) : \eta_4^{-1} \eta_2 \eta_1^{-1} \eta_4 \eta_2^{-1} \eta_1 \\ & E(C_8) : \eta_5^{-1} \eta_2 \eta_1^{-1} \eta_5 \eta_2^{-1} \eta_1 \\ & E(C_9) : \eta_6 \eta_5^{-1} \eta_2 \eta_1^{-1} \eta_6^{-1} \eta_5 \eta_2^{-1} \eta_1 \\ & E(C_{10}) : \eta_6 \eta_5^{-1} \eta_3 \eta_1^{-1} \eta_6^{-1} \eta_5 \eta_3^{-1} \eta_1 \\ & E(C_{11}) : \eta_6 \eta_4^{-1} \eta_3 \eta_1^{-1} \eta_6^{-1} \eta_4 \eta_3^{-1} \eta_1 \\ & E(C_{12}) : \eta_4^{-1} \eta_3 \eta_1^{-1} \eta_4 \eta_3^{-1} \eta_1 \\ & E(C_{13}) : \eta_2 \eta_1^{-1} \eta_2^{-1} \eta_1 \\ & E(C_{14}) : \eta_3 \eta_1^{-1} \eta_3^{-1} \eta_1 \\ & E(C_{15}) : \eta_4 \eta_1^{-1} \eta_4^{-1} \eta_1 \\ & E(C_{16}) : \eta_5 \eta_1^{-1} \eta_5^{-1} \eta_1 \\ & E(C_{17}) : \eta_6 \eta_1^{-1} \eta_6^{-1} \eta_1 \rangle \end{aligned}$$

The relations $E(C_{13}), \dots, E(C_{17})$, indicate that the large loop η_1 is contained in the center of the group.

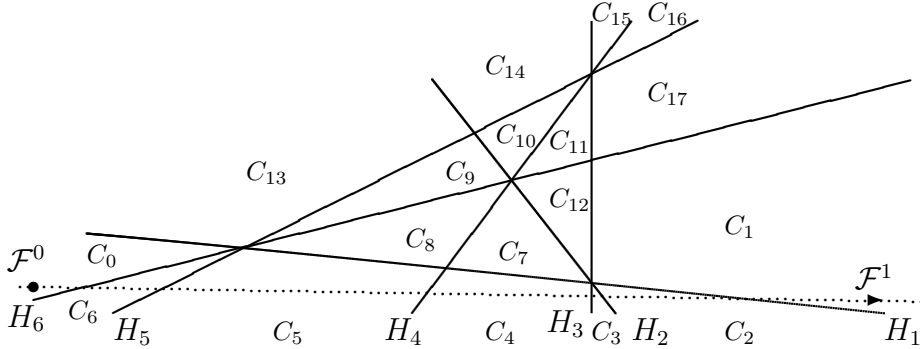


Figure 8: Example 5.11 and 6.8.

Remark 5.12. Example 5.11 gives a presentation for the pure braid group with 4-strands. See also Example 6.8.

5.7 Twisted minimal chain complex

Let \mathcal{L} be a complex rank one local system on $M(\mathcal{A})$. \mathcal{L} is determined by nonzero complex numbers (monodromy around H_i) $q_i \in \mathbb{C}^*$, $i = 1, \dots, n$. Fix a square root $q_i^{1/2} \in \mathbb{C}^*$ for each i . For given chambers C, C' , let us define

$$\Delta(C, C') := \prod_{H_i \in \text{Sep}(C, C')} q_i^{1/2} - \prod_{H_i \in \text{Sep}(C, C')} q_i^{-1/2},$$

where $H_i \in \text{Sep}(C, C')$ runs over all hyperplanes which separate C and C' . With these notation, we can describe a chain complex which computes homology groups with coefficients in \mathcal{L} .

Theorem 5.13. Denote by $\mathbb{C}[\text{ch}_k^{\mathcal{F}}(\mathcal{A})] := \bigoplus_{C \in \text{ch}_k^{\mathcal{F}}(\mathcal{A})} \mathbb{C} \cdot [C]$ the vector space spanned by $\text{ch}_k^{\mathcal{F}}(\mathcal{A})$. Recall that $\text{ch}_1^{\mathcal{F}}(\mathcal{A}) = \{C_1, C_2, \dots, C_n\}$ and $\text{ch}_0^{\mathcal{F}}(\mathcal{A}) = \{C_0\}$. Then the linear maps

$$\begin{aligned} \nabla : \text{ch}_2^{\mathcal{F}}(\mathcal{A}) &\longrightarrow \text{ch}_1^{\mathcal{F}}(\mathcal{A}), [C] \longmapsto \sum_{i=1}^n d_i(C) \Delta(C, C_i) [C_i], \\ \nabla : \text{ch}_1^{\mathcal{F}}(\mathcal{A}) &\longrightarrow \text{ch}_0^{\mathcal{F}}(\mathcal{A}), [C_i] \longmapsto \Delta(C_0, C_i) [C_0], \end{aligned}$$

determines a chain complex $(\mathbb{C}[\text{ch}_{\bullet}^{\mathcal{F}}(\mathcal{A})], \nabla)$ which homology group is isomorphic to

$$H_k(\mathbb{C}[\text{ch}_{\bullet}^{\mathcal{F}}(\mathcal{A})], \nabla) \simeq H_k(M(\mathcal{A}), \mathcal{L}).$$

See [Y2, Y3] for details and applications.

6 Positive homogeneous presentations

6.1 Left and right lines

In this section, we give an alternative presentation for the fundamental group $\pi_1(M(\mathcal{A}))$. It is presented with positive homogeneous relations as:

$$\begin{aligned} \text{Generators} &: \gamma_1, \gamma_2, \dots, \gamma_n, \\ \text{Relations, } R(C) &: \gamma_1 \gamma_2 \dots \gamma_n = \gamma_{i_1(C)} \gamma_{i_2(C)} \dots \gamma_{i_n(C)}, \end{aligned}$$

where C runs over all $\text{ch}_2^{\mathcal{F}}(\mathcal{A})$ and $(i_1(C), \dots, i_n(C))$ is a permutation of $(1, \dots, n)$ associated to C .

Definition 6.1. Let $C \in \text{ch}(\mathcal{A})$ be a chamber. The line $H_i \in \mathcal{A}$ is said to be *passing the left side of C* if $C \subset \{\alpha_i > 0\}$. Similarly, The line $H_i \in \mathcal{A}$ is said to be *passing the right side of C* if $C \subset \{\alpha_i < 0\}$.

Remark 6.2. Sometimes it is convenient to consider 0-th line H_0 is passing the right side of C for any chamber C . (Recall that $\alpha_0(C) = -1$ by our convention.)

Definition 6.3. For a chamber $C \in \text{ch}(\mathcal{A})$, define the decomposition $\{1, \dots, n\} = I_R(C) \sqcup I_L(C)$ as follows.

$$\begin{aligned} I_R(C) &= \{i \mid H_i \text{ passes the right side of } C\}, \\ I_L(C) &= \{i \mid H_i \text{ passes the left side of } C\}. \end{aligned}$$

The notion right/left is related to the map d_i . The proof of the next proposition is straightforward.

Proposition 6.4. Let $C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})$.

- If H_{i-1} is passing right side of C and H_i is passing left side of C , then $d_i(C) = -1$.
- If H_{i-1} is passing left side of C and H_i is passing right side of C , then $d_i(C) = 1$.
- Otherwise, $d_i(C) = 0$.

6.2 Positive homogeneous relations

For a chamber $C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})$, arranging the right/left indices increasingly as

$$\begin{aligned} I_R(C) &= \{i_1(C) < i_2(C) < \cdots < i_k(C)\}, \\ I_L(C) &= \{i_{k+1}(C) < i_{k+2}(C) < \cdots < i_n(C)\}. \end{aligned}$$

Then we introduce the following homogeneous relation.

$$\Gamma(C) : \gamma_1 \gamma_2 \cdots \gamma_n = \gamma_{i_1(C)} \gamma_{i_2(C)} \cdots \gamma_{i_n(C)}. \quad (7)$$

Theorem 6.5. *With notation as above, the fundamental group $\pi_1(M(\mathcal{A}), *)$ is isomorphic to the group defined by the presentation*

$$\langle \gamma_1, \dots, \gamma_n \mid \Gamma(C), C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A}) \rangle.$$

Remark 6.6. Note that all relations in the above presentation are positive homogeneous. It is similar to the “conjugation-free geometric presentation” introduced in [EGT1, EGT2]. However they require stronger properties on relations. Indeed they prove that the fundamental group of Ceva arrangement (Figure 8) does not have conjugation-free geometric presentation.

Example 6.7. Let $\mathcal{A} = \{H_1, \dots, H_5\}$ be a line arrangement and \mathcal{F} be a flag pictured in Figure 7.

	$I_R(C)$	$I_L(C)$
C_6	124	35
C_7	14	235
C_8	134	25
C_9	1345	2
C_{10}	345	12
C_{11}	45	123
C_{12}	4	1235

Hence the fundamental group has the following presentation.

$$\begin{aligned} \pi_1(M(\mathcal{A}), *) &\simeq \langle \gamma_1, \dots, \gamma_5 \mid 12345 = 12435 = 14235 = 13425 \\ &= 13452 = 34512 = 45123 = 41235 \rangle. \end{aligned}$$

Here we denote 12345 instead of $\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$ for simplicity.

Example 6.8. Let $\mathcal{A} = \{H_1, \dots, H_6\}$ be a line arrangement and \mathcal{F} be a flag pictured in Figure 8.

	$I_R(C)$	$I_L(C)$
C_7	23	1456
C_8	234	156
C_9	2346	15
C_{10}	346	125
C_{11}	36	1245
C_{12}	3	12456
C_{13}	23456	1
C_{14}	3456	12
C_{15}	456	123
C_{16}	56	1234
C_{17}	6	12345

Hence the fundamental group has the following presentation.

$$\begin{aligned} \pi_1(M(\mathcal{A}), *) &\simeq \langle \gamma_1, \dots, \gamma_5 \mid 123456 \\ &= 231456 = 234156 = 234615 = 346125 = 361245 = 312456 \\ &= 234561 = 345612 = 456123 = 561234 = 612345 \rangle. \end{aligned}$$

6.3 Proof of Theorem 6.5

The new presentation in Theorem 6.5 is obtained by changing generators as $\eta_i = \gamma_i\gamma_{i+1} \dots \gamma_n$, or equivalently,

$$\begin{aligned} \gamma_1 &= \eta_1\eta_2^{-1} \\ \gamma_2 &= \eta_2\eta_3^{-1} \\ &\dots \\ \gamma_{n-1} &= \eta_{n-1}\eta_n^{-1} \\ \gamma_n &= \eta_n. \end{aligned} \tag{8}$$

Remark 6.9. If we fix the base point in $\mathcal{F}_\mathbb{C}^1 = \mathcal{F}^1 \otimes \mathbb{C}$, then we may choose meridian generators $\gamma_1, \dots, \gamma_n$ as in Figure 9. (Compare Figure 4.)

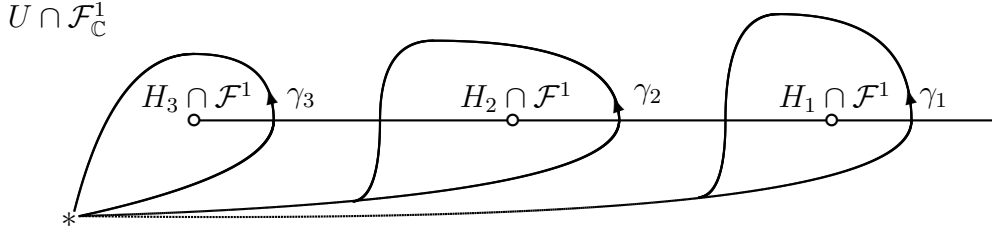


Figure 9: Meridian generators $\gamma_1, \gamma_2, \gamma_3$.

Proposition 6.10. By the change (8), the relation $E(C) = 1$ (Eq. (5)) is equivalent to $\Gamma(C)$ (Eq. (7)).

Proof. We distinguish four cases according to H_1 and H_n are passing right/left of C .

Case (1). Both H_1 and H_n are passing right side of C .

Case (2). H_1 is passing right and H_n is passing left side of C .

Case (3). Both H_1 and H_n are passing left side of C .

Case (4). H_1 is passing left and H_n is passing right side of C .

Case (1). We may take $1 < i_1 < \dots < i_{2k} < n$ in such a way that

$$\overbrace{1, 2, \dots}^{\text{right}}, \overbrace{i_1, i_1 + 1, \dots}^{\text{left}}, \overbrace{i_2, i_2 + 1, \dots}^{\text{right}}, \dots, \overbrace{i_{2k}, i_{2k} + 1, \dots, n}^{\text{right}}.$$

In this case we have

$$I_R(C) = \{i_0 = 1, 2, \dots, i_1 - 1, i_2, i_2 + 1, \dots, i_3 - 1, \dots, i_{2k}, i_{2k} + 1, \dots, n\},$$

$$I_L(C) = \{i_1, i_1 + 1, \dots, i_2 - 1, i_3, \dots, i_4 - 1, \dots, i_{2k-1}, i_{2k-1} + 1, \dots, i_{2k} - 1\}.$$

Then by Proposition 6.4, $d_{i_{2g-1}}(C) = -1, d_{i_{2g}}(C) = 1$ ($g = 1, \dots, k$) and otherwise, $d_i(C) = 0$. Hence the word $E(C)$ is equal to

$$E(C) = \eta_{i_1}^{-1} \eta_{i_2}^1 \eta_{i_3}^{-1} \dots \eta_{i_{2k}}^1 \cdot \eta_{i_1}^1 \eta_{i_2}^{-1} \eta_{i_3}^1 \dots \eta_{i_{2k}}^{-1}.$$

Using (8), we have

$$\begin{aligned} E(C) &= \eta_{i_1}^{-1} (\eta_{i_2} \eta_{i_3}^{-1}) \dots (\eta_{i_{2k-2}} \eta_{i_{2k-1}}^{-1}) \eta_{i_{2k}} \cdot (\eta_{i_1} \eta_{i_2}^{-1}) \dots (\eta_{i_{2k-1}} \eta_{i_{2k}}^{-1}) \\ &= \eta_{i_1}^{-1} \cdot (\gamma_{i_2} \dots \gamma_{i_3-1}) \dots (\gamma_{i_{2k-2}} \dots \gamma_{i_{2k-1}-1}) \cdot (\gamma_{i_{2k}} \dots \gamma_n) \\ &\quad \cdot (\gamma_{i_1} \dots \gamma_{i_2-1}) \dots (\gamma_{i_{2k-1}} \dots \gamma_{i_{2k}-1}). \end{aligned}$$

Since the equality $E(C) = e$ holds, by multiplying $\gamma_1 \gamma_2 \dots \gamma_n$ from the left, we have (note that $\gamma_1 \gamma_2 \dots \gamma_n \eta_{i_1}^{-1} = \gamma_1 \gamma_2 \dots \gamma_{i_1-1}$)

$$\begin{aligned} \gamma_1 \gamma_2 \dots \gamma_n &= (\gamma_1 \gamma_2 \dots \gamma_{i_1-1}) (\gamma_{i_2} \dots \gamma_{i_3-1}) \dots (\gamma_{i_{2k}} \dots \gamma_n) \\ &\quad \cdot (\gamma_{i_1} \gamma_{i_1+1} \dots \gamma_{i_2-1}) (\gamma_{i_3} \dots \gamma_{i_4-1}) \dots (\gamma_{i_{2k-1}} \dots \gamma_{i_{2k}-1}), \end{aligned}$$

which is equal to the relation $\Gamma(C)$.

The remaining cases (2), (3) and (4) are handled in the same way. \square

7 Proofs of main results

In this section, we prove Theorem 4.2. For this purposes, it is convenient to describe $M(\mathcal{A})$ in terms of tangent bundle of \mathbb{R}^2 .

7.1 Tangent bundle description

We identify \mathbb{C}^2 with the total space $T\mathbb{R}^2$ of the tangent bundle of \mathbb{R}^2 via

$$\begin{aligned} T\mathbb{R}^2 &\longrightarrow \mathbb{C}^2 \\ (\mathbf{x}, \mathbf{y}) &\longmapsto \mathbf{x} + \sqrt{-1}\mathbf{y}, \end{aligned}$$

where $\mathbf{y} \in T_{\mathbf{x}}\mathbb{R}^2$ is a tangent vector of \mathbb{R}^2 at $\mathbf{x} \in \mathbb{R}^2$. Let $H \subset \mathbb{R}^2$ be a line and $H_{\mathbb{C}} \subset \mathbb{C}^2$ be its complexification. Then $H_{\mathbb{C}}$ is identified by the above map with

$$H_{\mathbb{C}} \simeq \{(\mathbf{y} \in T_{\mathbf{x}}\mathbb{R}^2) \mid \mathbf{x} \in H, \mathbf{y} \in T_{\mathbf{x}}H\}. \quad (9)$$

For $\mathbf{x} \in \mathbb{R}^2$, write $\mathcal{A}_{\mathbf{x}}$ the set of lines passing through \mathbf{x} . Then we have the following (see [Y1, §3.1].):

$$M(\mathcal{A}) \simeq \{(\mathbf{y} \in T_{\mathbf{x}}\mathbb{R}^2) \mid \mathbf{x} \in \mathbb{R}^2, \mathbf{y} \notin T_{\mathbf{x}}H, \text{ for } H \in \mathcal{A}_{\mathbf{x}}\}.$$

It is straightforward to check the following from (9).

Lemma 7.1. *If $\mathbf{x} + \sqrt{-1}\mathbf{y} \in M(\mathcal{A})$, then $(\mathbf{x} + t\mathbf{y}) + \sqrt{-1}\mathbf{y} \in M(\mathcal{A})$ for any $t \in \mathbb{R}$.*

Thus lines and the complement $M(\mathcal{A})$ are preserved under the linear uniform motion. The next lemma shows that the sail $S(\alpha, \beta)$ is also preserved under the linear uniform motion. The next lemma will be used repeatedly to construct deformation retractions for certain subsets of $M(\mathcal{A})$.

Lemma 7.2. *Let α, β be linear forms (as in Definition 3.4). Suppose $\mathbf{x} + \sqrt{-1}\mathbf{y} \in S(\alpha, \beta)$. Then $(\mathbf{x} + t\mathbf{y}) + \sqrt{-1}\mathbf{y} \in S(\alpha, \beta)$ for any $t \in \mathbb{R}$. Conversely, if $\mathbf{x} + \sqrt{-1}\mathbf{y} \notin S(\alpha, \beta)$, then $(\mathbf{x} + t\mathbf{y}) + \sqrt{-1}\mathbf{y} \notin S(\alpha, \beta)$ for any $t \in \mathbb{R}$.*

Proof. Set $\alpha(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b$ and $\beta(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} + d$, where $\mathbf{a}, \mathbf{c} \in (\mathbb{R}^2)^*$ and $b, d \in \mathbb{R}$. By assumption,

$$\frac{\alpha(\mathbf{x} + \sqrt{-1}\mathbf{y})}{\beta(\mathbf{x} + \sqrt{-1}\mathbf{y})} = \frac{\mathbf{a} \cdot \mathbf{x} + \sqrt{-1}\mathbf{a} \cdot \mathbf{y} + b}{\mathbf{c} \cdot \mathbf{x} + \sqrt{-1}\mathbf{c} \cdot \mathbf{y} + d} = \frac{\alpha(\mathbf{x}) + \sqrt{-1}\mathbf{a} \cdot \mathbf{y}}{\beta(\mathbf{x}) + \sqrt{-1}\mathbf{c} \cdot \mathbf{y}} = r \in \mathbb{R}_{<0}.$$

Hence

$$\alpha(\mathbf{x}) = r\beta(\mathbf{x}) \text{ and } \mathbf{a} \cdot \mathbf{y} = r\mathbf{c} \cdot \mathbf{y}. \quad (10)$$

The assertion follows from

$$\frac{\alpha(\mathbf{x} + t\mathbf{y} + \sqrt{-1}\mathbf{y})}{\beta(\mathbf{x} + t\mathbf{y} + \sqrt{-1}\mathbf{y})} = \frac{\alpha(\mathbf{x}) + t\mathbf{a} \cdot \mathbf{y} + \sqrt{-1}\mathbf{a} \cdot \mathbf{y}}{\beta(\mathbf{x}) + t\mathbf{c} \cdot \mathbf{y} + \sqrt{-1}\mathbf{c} \cdot \mathbf{y}} = r. \quad (11)$$

The second part follows immediately from the first part. \square

Suppose that $\mathbf{x} + \sqrt{-1}\mathbf{y} \in S(\alpha, \beta)$ and $\mathbf{a} \cdot \mathbf{y} \neq 0$. Set $t = -\frac{\alpha(\mathbf{x})}{\mathbf{a} \cdot \mathbf{y}}$. Then by (11) above, $\alpha(\mathbf{x}) + t\mathbf{a} \cdot \mathbf{y} = \alpha(\mathbf{x} + t\mathbf{y}) = 0$ and $\beta(\mathbf{x}) + t\mathbf{c} \cdot \mathbf{y} = \beta(\mathbf{x} + t\mathbf{y}) = 0$, which implies that the line $\mathbf{x} + \mathbb{R} \cdot \mathbf{y}$ is passing through the intersection $H_\alpha \cap H_\beta$ of two lines $H_\alpha = \{\alpha = 0\}$ and $H_\beta = \{\beta = 0\}$. We obtain the following description of the sail.

Proposition 7.3. Let α and β be as in Lemma 7.2.

- (i) Suppose $H_\alpha = \{\alpha = 0\}$ and $H_\beta = \{\beta = 0\}$ are not parallel. Then $\mathbf{x} + \sqrt{-1}\mathbf{y} \in S(\alpha, \beta)$ if and only if either
 - $\alpha(\mathbf{x})\beta(\mathbf{x}) < 0$ and \mathbf{y} is tangent to the line $\overline{\mathbf{x} \cdot (H_\alpha \cap H_\beta)}$ passing through \mathbf{x} and the intersection $H_\alpha \cap H_\beta$, or
 - $\alpha(\mathbf{x}) = \beta(\mathbf{x}) = 0$ (i.e., $\{\mathbf{x}\} = H_\alpha \cap H_\beta$) and $\mathbf{y} \neq 0$ such that the line $\mathbf{x} + \mathbb{R} \cdot \mathbf{y}$ is passing through the domain $\{\mathbf{x} \in \mathbb{R}^2 \mid \alpha(\mathbf{x})\beta(\mathbf{x}) < 0\}$.
- (ii) Suppose H_α and H_β are parallel. Then $\mathbf{x} + \sqrt{-1}\mathbf{y} \in S(\alpha, \beta)$ if and only if $\alpha(\mathbf{x})\beta(\mathbf{x}) < 0$ and \mathbf{y} is either zero or parallel to H_α .
- (iii) Suppose α is a nonzero constant. (In this case, β should be degree one.) Then $\mathbf{x} + \sqrt{-1}\mathbf{y} \in S(\alpha, \beta)$ if and only if $\alpha(\mathbf{x})\beta(\mathbf{x}) < 0$ and \mathbf{y} is either zero or parallel to H_β .

(See Figure 10.)

Define

$$|\arg(\mathbf{y})| := \begin{cases} \arg(\mathbf{y}), & \text{if } 0 \leq \arg(\mathbf{y}) < \pi \\ \arg(\mathbf{y}) - \pi, & \text{if } \pi \leq \arg(\mathbf{y}) < 2\pi. \end{cases}$$

Using the above and Proposition 5.7, we have

Proposition 7.4. Let $\mathbf{x} + \sqrt{-1}\mathbf{y} \in S_i = S(\alpha_{i-1}, \alpha_i) \cap M(\mathcal{A})$.

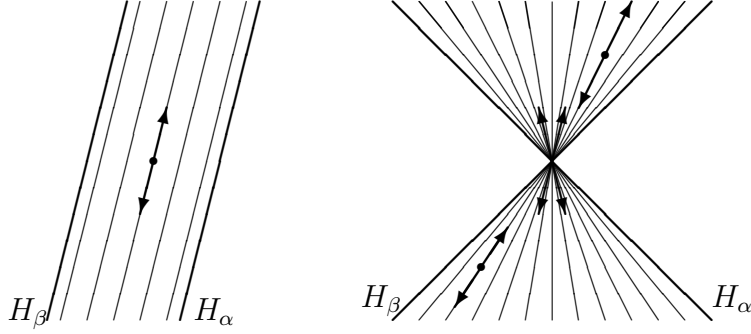


Figure 10: Sails $S(\alpha, \beta)$.

- If \mathbf{x} is the intersection $H_{i-1} \cap H_i$, then $\mathbf{y} \neq \mathbf{0}$ and $\theta_{i-1} < |\arg(\mathbf{y})| < \theta_i$.
- If \mathbf{x} is not the intersection $H_{i-1} \cap H_i$, then $\alpha_{i-1}(\mathbf{x})\alpha_i(\mathbf{x}) < 0$ and $\mathbf{y} \neq \mathbf{0}$ with $|\arg(\mathbf{y})| = |\text{pvarg}_i(\mathbf{x})|$ or $\mathbf{y} = \mathbf{0}$.

Now we prove Theorem 4.2 (i):

$$S_i \cap S_j = \bigsqcup C,$$

where C runs all chambers satisfying $\alpha_{i-1}(C)\alpha_i(C) < 0, \alpha_{j-1}(C)\alpha_j(C) < 0$ ($1 \leq i < j \leq n$). Suppose that $\mathbf{x} + \sqrt{-1}\mathbf{y} \in S_i \cap S_j$ and $\mathbf{y} \neq \mathbf{0}$. Then by Proposition 7.4 and Proposition 5.7, we have

$$\theta_j \leq |\arg(\mathbf{y})| \leq \theta_{j-1}, \text{ and } \theta_i \leq |\arg(\mathbf{y})| \leq \theta_{i-1}.$$

This happens only when $\theta_{i-1} = \theta_i = \theta_{j-1} = \theta_j$, which means that H_{i-1}, H_i, H_{j-1} and H_j are parallel. However, since $\{\mathbf{x} \in \mathbb{R}^2 \mid \alpha_{i-1}\alpha_i < 0\}$ and $\{\mathbf{x} \in \mathbb{R}^2 \mid \alpha_{j-1}\alpha_j < 0\}$ are parallel strips, which do not intersect. This is a contradiction. Hence we have $\mathbf{y} = \mathbf{0}$, and $S_i \cap S_j$ is a union of chambers. (See Figure 11.)

7.2 Contractibility of S_i°

Now we prove that $S_i^\circ = S_i \setminus \bigcup_{C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})} C$ is contractible. Let us denote $A_i := S(\alpha_{i-1}, \alpha_i) \cap \mathcal{F}^1$. Since A_i is obviously contractible, therefore it suffices to construct a deformation retract onto A_i , that is, a family of continuous map $f_t : S_i^\circ \rightarrow S_i^\circ$ which satisfies $f_0 = \text{id}_{S_i^\circ}$, $f_1(S_i^\circ) = A_i$ and $f_t|_{A_i} = \text{id}_{A_i}$.

Define a continuous map $\rho : S_i^\circ \rightarrow A_i$, $\mathbf{x} + \sqrt{-1}\mathbf{y} \mapsto \rho(\mathbf{x} + \sqrt{-1}\mathbf{y})$ by

- (1) if $\mathbf{y} \neq \mathbf{0}$, then $\rho(\mathbf{x} + \sqrt{-1}\mathbf{y}) = A_i \cap (\mathbf{x} + \mathbb{R} \cdot \mathbf{y})$,

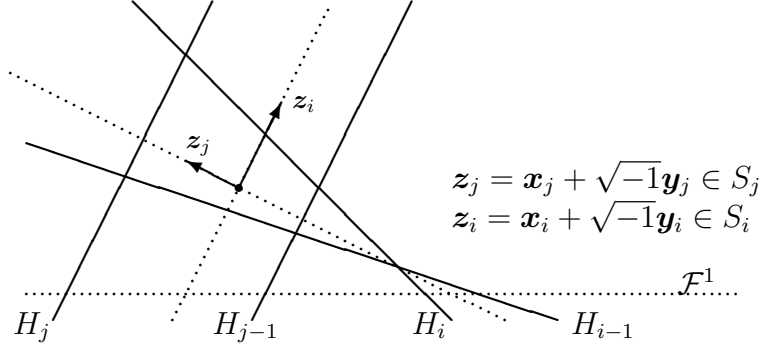


Figure 11: S_i and S_j intersect transversely.

- (2) if $H_i \cap H_{i-1} \neq \emptyset$ and $\mathbf{y} = \mathbf{0}$, then $\rho(\mathbf{x} + \sqrt{-1}\mathbf{y}) = A_i \cap \overline{(\mathbf{x} \cdot H_i \cap H_{i-1})}$, where $(\mathbf{x} \cdot H_i \cap H_{i-1})$ is the line passing through \mathbf{x} and the intersection $H_i \cap H_{i-1}$,
- (3) if $H_i \cap H_{i-1} = \emptyset$ and $\mathbf{y} = \mathbf{0}$, then $\rho(\mathbf{x} + \sqrt{-1}\mathbf{y}) = A_i \cap L_{\mathbf{x}}$, where $L_{\mathbf{x}}$ is the line passing through \mathbf{x} and parallel to H_i .

By Proposition 7.4, ρ is a well-defined continuous map. Note that $\rho|_{A_i} = \text{id}_{A_i}$.

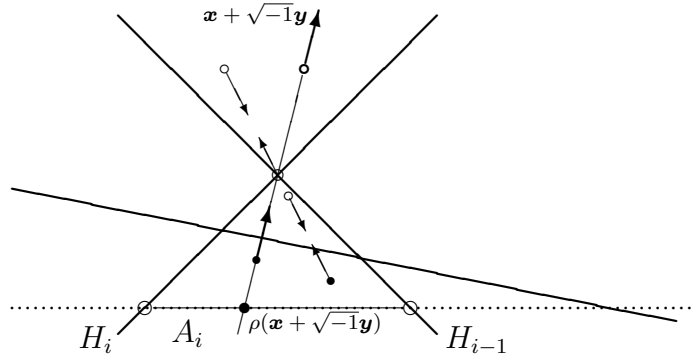


Figure 12: Deformation retract $\rho(\mathbf{x} + \sqrt{-1}\mathbf{y})$.

Define

$$f_t(\mathbf{x} + \sqrt{-1}\mathbf{y}) = ((1-t)\mathbf{x} + t\rho(\mathbf{x} + \sqrt{-1}\mathbf{y})) + \sqrt{-1}(1-t)\mathbf{y}.$$

If $\mathbf{y} \neq \mathbf{0}$, then the real part $((1-t)\mathbf{x} + t\rho(\mathbf{x} + \sqrt{-1}\mathbf{y}))$ is on the line $\mathbf{x} + \mathbb{R} \cdot \mathbf{y}$ and the imaginary part is nonzero provided $t \neq 1$. Hence $f_t(\mathbf{x} + \sqrt{-1}\mathbf{y}) \in S_i^\circ$ (see also Lemma 7.2). If $\mathbf{y} = \mathbf{0}$, then \mathbf{x} is contained in the chamber C_i . (Otherwise, \mathbf{x} is contained in some chamber $C \in \text{ch}_2^{\mathcal{F}}(\mathcal{A})$ which does not intersect \mathcal{F}^1 .) Hence $f_t(\mathbf{x}) \in C_i \subset S_i^\circ$. The map f_t determines a deformation contraction of S_i° onto A_i . (See Figure 12.)

7.3 Contractibility of U

We break the proof of the contractibility of $U = M(\mathcal{A}) \setminus \bigcup_{i=1}^n S_i$ up into n steps.

7.3.1 Filtration U_k

Definition 7.5. Define $U_0 = U$ and

$$U_k := \{\mathbf{z} = \mathbf{x} + \sqrt{-1}\mathbf{y} \in U \mid \alpha_1(\mathbf{x}) \leq 0, \dots, \alpha_k(\mathbf{x}) \leq 0\},$$

for $k = 1, \dots, n$.

Obviously

$$U = U_0 \supset U_1 \supset \dots \supset U_n,$$

and

Proposition 7.6. U_n is star-shaped. In particular, U_n is contractible.

Therefore it is enough to construct a deformation retract $\rho_k : U_k \rightarrow U_{k+1}$ for $k = 0, \dots, n-1$.

7.3.2 The case $k = 0$

First we construct a deformation retraction $\rho_0 : U = U_0 \rightarrow U_1 = \{\mathbf{z} = \mathbf{x} + \sqrt{-1}\mathbf{y} \mid \alpha_1(\mathbf{x}) \leq 0\}$.

Let $\mathbf{z} = \mathbf{x} + \sqrt{-1}\mathbf{y} \in U_0 \setminus U_1$. Then, by definition, $\alpha_1(\mathbf{x}) > 0$. Recall that

$$S_1 = \{\mathbf{x} + \sqrt{-1}\mathbf{y} \mid \alpha(\mathbf{x}) > 0 \text{ and } \mathbf{y} \text{ is either zero or parallel to } H_1\}.$$

Therefore $\mathbf{z} \notin S_1$ implies that the affine line $\mathbf{x} + \mathbb{R} \cdot \mathbf{y} \subset \mathbb{R}^2$ is not parallel to H_1 , hence intersects H_1 . Denote by $\tau(\mathbf{z}) \in \mathbb{R}$ the unique real number satisfying $\mathbf{x} + \tau(\mathbf{z})\mathbf{y} \in H_1$. Define the family of continuous map $f_t : U_0 \rightarrow U_0$ ($0 \leq t \leq 1$) by

$$f_t(\mathbf{z}) = \begin{cases} (\mathbf{x} + t \cdot \tau(\mathbf{z})\mathbf{y}) + \sqrt{-1}\mathbf{y} & \text{if } \mathbf{z} \in U_0 \setminus U_1 \\ \mathbf{x} + \sqrt{-1}\mathbf{y} & \text{if } \mathbf{z} \in U_1. \end{cases}$$

Then by Lemma 7.2, $f_t(\mathbf{z}) \in U$. Hence $\rho_0 = f_1 : U_0 \rightarrow U_1$ is a deformation retraction. (Figure 13.)

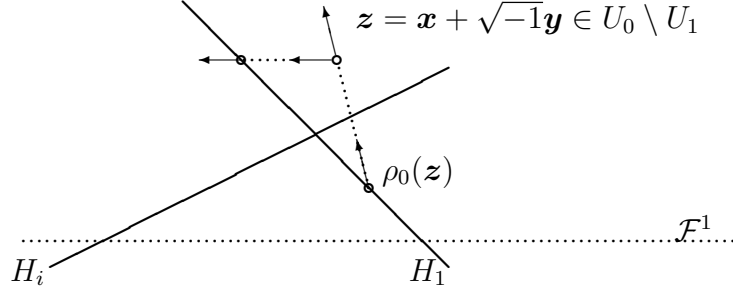


Figure 13: $\rho_0 : U_0 \rightarrow U_1$.

7.3.3 The case that H_k and H_{k+1} are parallel

Here, we assume that H_k and H_{k+1} are parallel ($1 \leq k \leq n-1$).

Definition 7.7. (1) Define the closed subset $D_k \subset \mathbb{R}^2$ by

$$D_k = \{\mathbf{x} \in \mathbb{R}^2 \mid \alpha_1(\mathbf{x}) \leq 0, \alpha_2(\mathbf{x}) \leq 0, \dots, \alpha_k(\mathbf{x}) \leq 0, \text{ and } \alpha_{k+1}(\mathbf{x}) \geq 0\}.$$

- (2) Denote the upper roof of D_k by R_k . More precisely, R_k is the closure of $\partial(D_k) \setminus (H_k \cup H_{k+1})$.
- (3) Suppose $\alpha_k(\mathbf{x}) \leq 0$ and $\alpha_{k+1}(\mathbf{x}) \geq 0$. Then denote the line passing through \mathbf{x} which is parallel to H_k by $L_{\mathbf{x}}$.

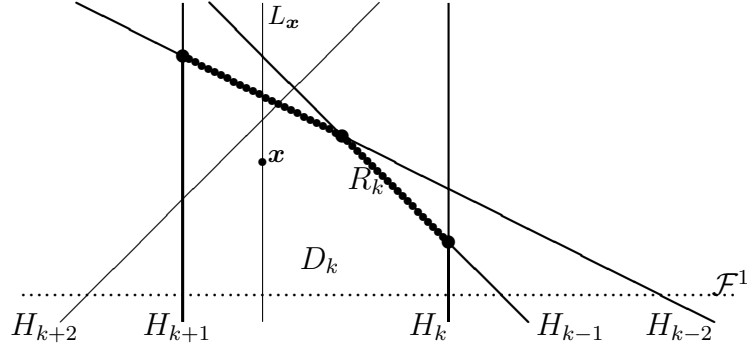


Figure 14: D_k and its roof R_k .

Remark 7.8. By definitions, if $\mathbf{x} + \sqrt{-1}\mathbf{y} \in U_k \setminus U_{k+1}$, then $\mathbf{x} \in D_k$.

The set $\{\mathbf{x} \in \mathbb{R}^2 \mid \alpha_k(\mathbf{x}) \leq 0 \leq \alpha_{k+1}(\mathbf{x})\}$ is a strip with boundaries H_k and H_{k+1} . We can define a deformation retract of this strip to D_k by

$$\text{pr}_k(\mathbf{x}) = \begin{cases} R_k \cap L_{\mathbf{x}} & \text{if } \mathbf{x} \notin D_k, \\ \mathbf{x} & \text{if } \mathbf{x} \in D_k. \end{cases}$$

Suppose $\mathbf{z} = \mathbf{x} + \sqrt{-1}\mathbf{y} \in U_k \setminus U_{k+1}$. Since $\mathbf{z} \notin S_k$, \mathbf{y} is neither zero nor parallel to H_{k+1} . Hence there exists a unique real number $\tau(\mathbf{x}, \mathbf{y}) \in \mathbb{R}$ such that $\mathbf{x} + \tau(\mathbf{x}, \mathbf{y}) \cdot \mathbf{y} \in H_{k+1}$.

Define the family of continuous map F_t ($0 \leq t \leq 1$) by

$$F_t(\mathbf{x} + \sqrt{-1}\mathbf{y}) = \text{pr}_k(\mathbf{x} + t \cdot \tau(\mathbf{x}, \mathbf{y}) \cdot \mathbf{y}) + \sqrt{-1}\mathbf{y} \quad (12)$$

for $\mathbf{x} + \sqrt{-1}\mathbf{y} \in U_k$ with $\mathbf{x} \in D_k$. (Figure 15.)

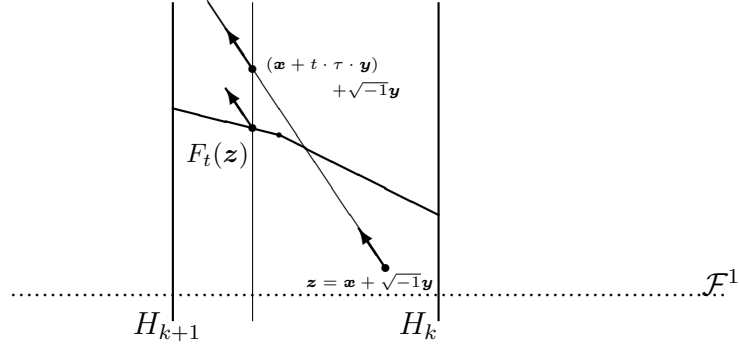


Figure 15: Retraction F_t .

Proposition 7.9. Let us extend the above F_t by

$$F_t(\mathbf{x} + \sqrt{-1}\mathbf{y}) = \begin{cases} F_t(\mathbf{z}) \text{ (as above)} & \text{if } \mathbf{z} \in U_k \setminus U_{k+1}, \\ \mathbf{z} & \text{if } \mathbf{z} \in U_{k+1}, \end{cases}$$

Then $F_t(\mathbf{z}) \in U_k$ for any $\mathbf{z} \in U_k$ and hence F_t determines a deformation retract $U_k \rightarrow U_{k+1}$.

Proof. Let $\mathbf{z} = \mathbf{x} + \sqrt{-1}\mathbf{y} \in U_k$ and $F_t(\mathbf{z}) = \mathbf{z}' = \mathbf{x}' + \sqrt{-1}\mathbf{y}$. Suppose that $\mathbf{z}' \notin U_k$. If $\mathbf{x} + t \cdot \tau \cdot \mathbf{y} \in D_k$, then $F_t(\mathbf{z}) = (\mathbf{x} + t \cdot \tau \cdot \mathbf{y}) + \sqrt{-1}\mathbf{y}$. By Lemma 7.2 $F_t(\mathbf{z}) \in M(\mathcal{A})$, hence contained in U_k . Thus we may assume that $\mathbf{x} + t \cdot \tau \cdot \mathbf{y} \notin D_k$ and $\mathbf{x}' \in R_k$. Furthermore, we may assume that $\mathbf{y} \in T_{\mathbf{x}'}\mathbb{R}^2$ is contained in a line $H_j \subset T_{\mathbf{x}'}$ with $\mathbf{x}' \in H_j$ for some $1 \leq j < k$. Then $\mathbf{x} + t \cdot \tau \cdot \mathbf{y}$ must be contained in the domain $\{\alpha_j > 0\}$. However, this contradicts $\mathbf{x} \in \{\alpha_j \leq 0\}$ and the fact that \mathbf{y} is parallel to H_j . Hence $F_t(\mathbf{z}) \in U_k$. \square

7.3.4 LQ-curves

The remaining case is the construction of deformation retract $U_k \rightarrow U_{k+1}$ when H_k and H_{k+1} are not parallel. The idea is similar to the previous case, however, it requires more technicality.

An Linear-Quadric curve on \mathbb{R}^2 is a curve which is linear when $x_1 \leq 1$ and quadric when $x_1 \geq 1$. Precise definition is as follows.

Definition 7.10. An *LQ-curve* (Linear-Quadric-curve) C on the real plane \mathbb{R}^2 is either a vertical line $C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = t\}$ or the graph $\{(x, f(x)) \mid x \in \mathbb{R}\}$ of a C^1 -function $f(x)$ such that

$$f(x_1) = \begin{cases} ax_1 + b & \text{for } x_1 \leq 1, \\ cx_1^2 + dx_1 & \text{for } x_1 \geq 1, \end{cases}$$

where $t, a, b, c, d \in \mathbb{R}$.

Remark 7.11. (1) Since $f(x_1)$ is C^1 at $x_1 = 1$, $f(x_1)$ should have the following expression.

$$f(x_1) = \begin{cases} ax_1 + b & \text{for } x_1 \leq 1, \\ -bx_1^2 + (a + 2b)x_1 & \text{for } x_1 \geq 1. \end{cases} \quad (13)$$

(2) $f(x_1)$ and the derivative $f'(x_1)$ for some $x_1 \in \mathbb{R}$ determines the unique LQ-curve.

Let $\mathbf{x} \in \mathbb{R}^2$ be a point in the positive quadrant and $\mathbf{y} \in T_{\mathbf{x}}\mathbb{R}^2 \setminus \{\mathbf{0}\}$ a nonzero tangent vector. Then there exists a unique C^1 -map $X_{\mathbf{x}, \mathbf{y}} : \mathbb{R} \rightarrow \mathbb{R}^2$ such that

- $X(0) = \mathbf{x}, \dot{X}(0) = \mathbf{y}$,
- $\{X(t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2$ is an LQ-curve.
- $|\dot{X}(t)| = |\mathbf{y}|$.

Roughly speaking, $X(t)$ is a motion along an LQ-curve with constant velocity. $X_{\mathbf{x}, \mathbf{y}}(t)$ is continuous with respect to \mathbf{x}, \mathbf{y} and t .

In the remainder of this section, we assume $\mathbf{x} \in (\mathbb{R}_{\geq 0})^2$ and $\mathbf{x} \neq \mathbf{0}$. Then $0 \leq \arg \mathbf{x} \leq \frac{\pi}{2}$. We also assume that $\mathbf{y} \notin \mathbb{R} \cdot \mathbf{x}$. When $\mathbf{x} \neq \mathbf{0}$, we call \mathbf{y} positive (resp. negative) if $\arg \mathbf{x} < \arg \mathbf{y} < \arg \mathbf{x} + \pi$ (resp. $\arg \mathbf{x} - \pi < \arg \mathbf{y} < \arg \mathbf{x}$). It is easily seen if \mathbf{y} is positive (resp. negative), then $\arg X_{\mathbf{x}, \mathbf{y}}(t)$ is increasing (resp. decreasing) in t .

Lemma 7.12. *Let \mathbf{x} and \mathbf{y} as above. Then the LQ-curve $X_{\mathbf{x}, \mathbf{y}}(t)$ intersects the positive x_1 -axis $\{(x_1, 0) \mid x_1 > 0\}$ exactly once.*

Proof. If \mathbf{y} is vertical, the assertion holds obviously. Assume that \mathbf{y} is not vertical. We use the expression (13). From the assumption that $\mathbf{y} \notin \mathbb{R} \cdot \mathbf{x}$, $b \neq 0$. Suppose $b > 0$. The quadric equation $-bx^2 + (a + 2b)x = 0$ has the solution $x = \frac{a+2b}{b}$ (and $x = 0$). If $\frac{a+2b}{b} > 1$, then we have $a + b = f(1) > 0$. Since $f(0) = b > 0$, $f(t) \neq 0$ for $0 \leq t \leq 1$. Hence $x = \frac{a+2b}{b}$ is the unique solution. $\frac{a+2b}{b} = 1$ is equivalent to say $a + b = 0$. Hence $x_1 = 1$ is the unique

solution of $f(x_1) = 0$. $\frac{a+2b}{b} < 1$ implies that $a+b < 0$. Since $f(0) > 0 > f(1)$, there exists the unique solution $f(t) = 0$ with $0 \leq t \leq 1$. The case $b < 0$ is similar. \square

Definition 7.13. Let \mathbf{x} and \mathbf{y} be as above. Denote by $\tau = \tau(\mathbf{x}, \mathbf{y})$ the unique real number such that $X_{\mathbf{x}, \mathbf{y}}(\tau) \in \{(x_1, 0) \mid x_1 > 0\}$. (Figure 16.)

Remark 7.14. $\tau(\mathbf{x}, \mathbf{y})$ is continuous on $\{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in (\mathbb{R}_{\geq 0})^2 \setminus \{\mathbf{0}\}, \mathbf{y} \notin \mathbb{R} \cdot \mathbf{x}\}$.

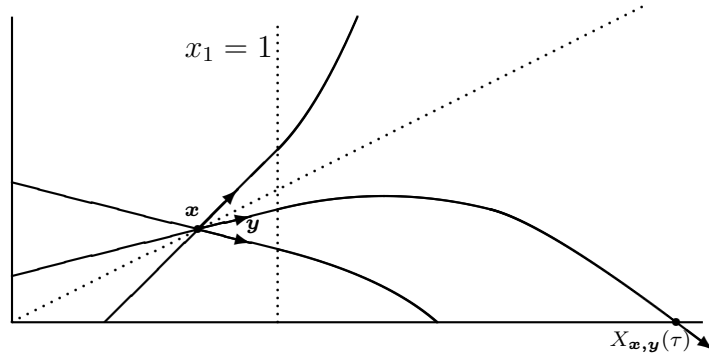


Figure 16: LQ-curves $X_{\mathbf{x}, \mathbf{y}}(t)$.

7.3.5 The case that H_k and H_{k+1} are not parallel

Next we assume that H_k and H_{k+1} are not parallel, and constructing a deformation retraction $\rho_k : U_k \rightarrow U_{k+1}$. The idea is similar to the parallel case (§7.3.3). However we need LQ-curves to construct the retraction.

Here we choose coordinates x_1, x_2 such that $\alpha_k = -x_1$, $\alpha_{k+1} = x_2$ and $\mathcal{F}^1 = \{x_1 + x_2 = 1\}$. Recall Definition 7.7 that $D_k \subset \mathbb{R}^2$ is defined by

$$D_k = \{\mathbf{x} \in \mathbb{R}^2 \mid \alpha_1(\mathbf{x}) \leq 0, \alpha_2(\mathbf{x}) \leq 0, \dots, \alpha_k(\mathbf{x}) \leq 0, \text{ and } \alpha_{k+1}(\mathbf{x}) \geq 0\},$$

and the roof R_k is defined as the closure of $\partial(D_k) \setminus (H_k \cup H_{k+1})$.

Definition 7.15. Suppose $\mathbf{x} \neq \mathbf{0}$. Then denote the line passing through \mathbf{x} and the intersection $\{\mathbf{0}\} = H_k \cap H_{k+1}$ by $L_{\mathbf{x}}$.

We can define a deformation retract of $(\mathbb{R}_{\geq 0})^2 \setminus \{\mathbf{0}\}$ to D_k by

$$\text{pr}_k(\mathbf{x}) = \begin{cases} R_k \cap L_{\mathbf{x}} & \text{if } \mathbf{x} \notin D_k, \\ \mathbf{x} & \text{if } \mathbf{x} \in D_k. \end{cases}$$

Suppose $\mathbf{z} = \mathbf{x} + \sqrt{-1}\mathbf{y} \in U_k \setminus U_{k+1}$. Since $\mathbf{z} \notin S_k$, $\mathbf{y} \notin \mathbb{R} \cdot \mathbf{x}$ (Proposition 7.3). Hence there exists a unique real number $\tau = \tau(\mathbf{x}, \mathbf{y}) \in \mathbb{R}$ such that $X_{\mathbf{x}, \mathbf{y}}(\tau) \in H_{k+1}$.

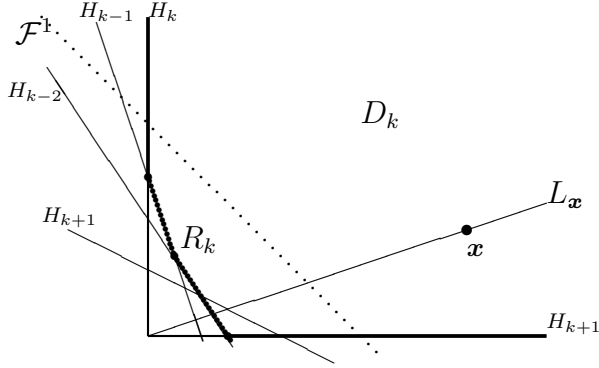


Figure 17: D_k , R_k and L_x .

Define the family of continuous map F_t ($0 \leq t \leq 1$) by

$$F_t(\mathbf{x} + \sqrt{-1}\mathbf{y}) = \text{pr}_k(X_{\mathbf{x},\mathbf{y}}(t \cdot \tau(\mathbf{x}, \mathbf{y}))) + \sqrt{-1}\dot{X}_{\mathbf{x},\mathbf{y}}(t \cdot \tau(\mathbf{x}, \mathbf{y})), \quad (14)$$

for $\mathbf{x} + \sqrt{-1}\mathbf{y} \in U_k$ with $\mathbf{x} \in D_k$. (Figure 18.)

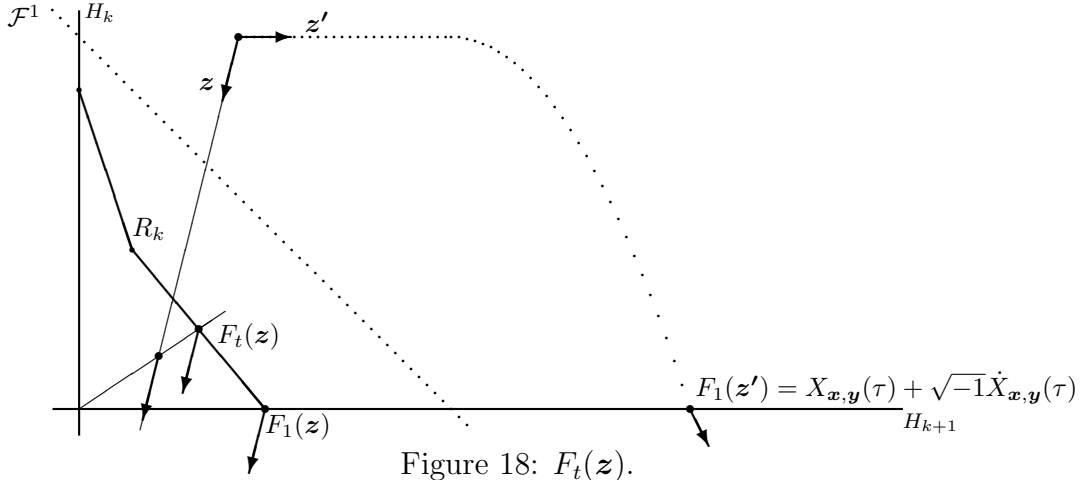


Figure 18: $F_t(\mathbf{z})$.

The next proposition completes the proof of the main result, which is proved in a similar way to the proof of Proposition 7.9.

Proposition 7.16. Let us extend the above F_t by

$$F_t(\mathbf{x} + \sqrt{-1}\mathbf{y}) = \begin{cases} F_t(\mathbf{z}) \text{ (as above)} & \text{if } \mathbf{z} \in U_k \setminus U_{k+1}, \\ \mathbf{z} & \text{if } \mathbf{z} \in U_{k+1}, \end{cases}$$

Then $F_t(\mathbf{z}) \in U_k$ for any $\mathbf{z} \in U_k$ and hence F_t determines a deformation retract $U_k \rightarrow U_{k+1}$.

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References

- [A] W. Arvola, The fundamental group of the complement of an arrangement of complex hyperplanes. *Topology* **31** (1992), 757–765.
- [CS] D. C. Cohen, A. Suciu, The braid monodromy of plane algebraic curves and hyperplane arrangements. *Comment. Math. Helvetici* **72** (1997), no. 2, 285–315.
- [D] E. Delucchi, Shelling-type orderings of regular CW-complexes and acyclic matchings of the Salvetti complex. *Int. Math. Res. Not.* 2008, no. 6, 39pp.
- [DP] A. Dimca, S. Papadima, Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements. *Ann. of Math. (2)* **158**(2003), no. 2, 473–507.
- [EGT1] M. Eliyahu, D. Garber, M. Teicher, Conjugation-free geometric presentations of fundamental groups of arrangements. *Manuscripta Math.* **133** (1-2), 247–271 (2010).
- [EGT2] M. Eliyahu, D. Garber, M. Teicher, A conjugation-free geometric presentation of fundamental groups of arrangements II: Expansion and some properties. [arXiv:1009.1349](https://arxiv.org/abs/1009.1349)
- [F] M. Falk, Homotopy types of line arrangements. *Invent. Math.* **111** (1993), no. 1, 139–150.
- [GS] G. Gaiffi, M. Salvetti, The Morse complex of a line arrangement. *J. of Algebra* **321** (2009), no. 1, 316–337.
- [GP] V. Guillemin, A. Pollack, *Differential topology*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974. xvi+222 pp
- [IY] K. Ito, M. Yoshinaga, Semi-algebraic partition and basis of Borel-Moore homology of hyperplane arrangements. To appear in *Proc. A. M. S.*
- [Li] A. Libgober, On the homotopy type of the complement to plane algebraic curves. *J. Reine Angew. Math.* **367** (1986), 103–114.

- [MT] B. Moishezon, M. Teicher, Braid group technique in complex geometry I: line arrangements in $\mathbb{C}\mathbb{P}^2$. *Contemp. Math.* **78** (1988), 425–555.
- [OS] P. Orlik, L. Solomon, Combinatorics and topology of complements of hyperplanes. *Invent. Math.* **56** (1980), 167–189.
- [OT1] P. Orlik, H. Terao, Arrangements of Hyperplanes. *Grundlehren Math. Wiss.* **300**, Springer-Verlag, New York, 1992.
- [PS] S. Papadima, A. Suciu, Higher homotopy groups of complements of complex hyperplane arrangements. *Adv. Math.* **165** (2002), no. 1, 71–100.
- [R1] R. Randell, The fundamental group of the complement of a union of complex hyperplanes. *Invent. Math.* **69** (1982), no. 1, 103–108. Correction, *Invent. Math.* **80** (1985), 467–468.
- [R2] R. Randell, Homotopy and group cohomology of arrangements, *Topology Appl.* **78** (1997), 201–213
- [R3] R. Randell, Morse theory, Milnor fibers and minimality of hyperplane arrangements. *Proc. Amer. Math. Soc.* **130** (2002), no. 9, 2737–2743.
- [S1] M. Salvetti, Arrangements of lines and monodromy of plane curves. *Compositio Math.* **68** (1988), 103–122.
- [S2] M. Salvetti, Topology of the complement of real hyperplanes in \mathbb{C}^N . *Invent. Math.* **88** (1987), no. 3, 603–618.
- [SS] M. Salvetti, S. Settepanella, Combinatorial Morse theory and minimality of hyperplane arrangements. *Geom. Topol.* **11** (2007), 1733–1766.
- [Y1] M. Yoshinaga, Hyperplane arrangements and Lefschetz’s hyperplane section theorem. *Kodai Math. J.*, **30** (2007) no. 2, 157–194.
- [Y2] M. Yoshinaga, The chamber basis of the Orlik-Solomon algebra and Aomoto complex. *Arkiv för Matematik*, vol. 47 (2009), 393–407.
- [Y3] M. Yoshinaga, Minimality of hyperplane arrangements and basis of local system cohomology. To appear in the proceedings of the 5-th Franco-Japanese Symposium on Singularities, IRMA Lectures in Mathematics and Theoretical Physics.
- [Z] T. Zaslavsky, Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes. *Memoirs Amer. Math. Soc.* **154** 1975.

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