Kyoto University

Kyoto-Math 2011-07

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by

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April 2011

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April 20, 2011

1 Introduction

Before considering gauge groups, let us recall classical results on decompositions of Lie groups. By the theorem of Hopf, for a connected, finite Hopf space X, there is a rational homotopy equivalence

$$X \simeq_{(0)} S^{2n_1+1} \times \cdots \times S^{2n_l+1},$$

where $n_1 \leq \cdots \leq n_l$. We call the increasing sequence $\{n_1, \ldots, n_l\}$ the *type* of X. We give a list of the types of compact, simple Lie groups.

Table 1: Types of simple Lie groups

$\mathrm{SU}(n)$	$1, 2, \ldots, n-1$
$\mathrm{Sp}(n)$	$1, 3, \ldots, 2n-1$
$\mathrm{Spin}(2n)$	$1, 3, \ldots, 2n-3, n-1$
$\mathrm{Spin}(2n+1)$	$1, 3, \ldots, 2n-1$
G_2	$1, 5$
F_4	$1, 5, 7, 11$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$

Let G be a connected Lie group of type $\{n_1, \ldots, n_l\}$. Generalizing the above theorem of Hopf, Serre [S] proved that if G is simply connected and an odd prime p satisfies $p \geq n_l$, then there is a p-local homotopy equivalence:

$$G \simeq_{(p)} S^{2n_1+1} \times \cdots \times S^{2n_l+1}$$

In this case, G is called *p-regular*. Later, this result is generalized to simply connected, finite Hopf spaces. In the special case that G admits an automorphism α satisfying $\alpha^q = 1$ for some positive integer q, Harris [Ha] showed that G decomposes into the product of the fixed point subgroup H of α and G/H if we invert q. More generally, Mimura, Nishida and Toda [MNT] gave a mod p decomposition of G by constructing direct factors explicitly when G has no p-torsion in homology. Later, Wilkerson [Wi] gave a unified simple proof for all of the above decompositions. The mod p decompositions of Lie groups are stated as follows. For $i = 1, \ldots, p-1$, let $T_i(G)$ be the set of all entries n in the type of G such that $n \equiv i \mod (p-1)$.

Theorem 1.1 (Mimura, Nishida and Toda [MNT], Wilkerson [Wi]). *Let G be a compact, connected, semi-simple Lie group such that $H_*(G; \mathbf{Z})$ is p-torsion free. Then there exist spaces B_k for $k = 1, \ldots, p-1$ and a p-local homotopy equivalence*

$$G \simeq_{(p)} \prod_{k=1}^{p-1} B_k,$$

where B_k is of type $T_k(G)$.

For convenience, we will often assume the spaces B_k and the sets $T_i(G)$ are indexed by $\mathbf{Z}/(p-1)$. We refer to [MNT] for more detailed description of B_k. For the mod p decompositions of non-simply connected Lie groups, see [KK1].

Now we consider gauge groups. Let G be a topological group, and let P be a principal G-bundle over K. The gauge group of P, denoted by $\mathcal{G}(P)$, is the group of automorphisms of P. The homotopy types of gauge groups have been extensively studied, especially when G is a low rank Lie group. One naively thinks that $\mathcal{G}(P)$ inherits some structures of G. For example, the higher homotopy commutativity of G yields a splitting of $\mathcal{G}(P)$ as in [KK4]. Then one may expect that the mod p decomposition of G induces that of $\mathcal{G}(P)$. Recently, Theriault [Th] obtained the gauge group analog of Serre's mod p decomposition when $K = S^4$. He also got the gauge group analog of Mimura, Nishida and Toda's decomposition when G is the low rank classical group and $K = S^4$. By a different approach, the authors [KK3] also got the mod p decomposition of $\mathcal{G}(EG|_{BH})$ when G has the mod p decomposition of Harris [Ha], where H is the fixed point subgroup as above and $EG|_{BH}$ means the universal G-bundle pulled back to BH.

The aim of this article is to prove the following theorems which unifies all of the above mod p decompositions of gauge groups $\mathcal{G}(P)$ when the base space of P is a certain sphere including S^4. We also give an example of P such that the adjoint bundle of P is indecomposable but not $\mathcal{G}(P)$, which has been a folklore for a long time.

Theorem 1.2. *Let G be a compact, connected, semi-simple Lie group such that $H_*(G; \mathbf{Z})$ is p-torsion free. Let n be an entry of the type of G, and let $\alpha \in \pi_{2n+1}(G)$. Then for a principal G-bundle P over S^{2n+2} classified by α, we have a p-local homotopy equivalence*

$$\mathcal{G}(P) \simeq_{(p)} \prod_{k=1}^{p-1} X_k,$$

such that there is a homotopy fiber sequence

$$\Omega_0^{2n+2} B_{k+n+1} \to X_k \to B_k,$$

where B_k is as in Theorem 1.1.

Theorem 1.3. *Let G, α, B_k and X_k be as in Theorem 1.2. If the p-localization of the adjoint of the Samelson product $\langle \alpha, 1_G \rangle$, say $\delta : G \to \Omega_0^{2n+1} G$, is a Hopf map, then X_k is the homotopy fiber of the composite*

$$B_k \xrightarrow{\text{incl}} G_{(p)} \xrightarrow{\delta} \Omega_0^{2n+1} G_{(p)} \xrightarrow{\text{proj}} \Omega_0^{2n+1} B_{k+n+1}.$$

Remark 1.1. By an analogous manner using the result of [KK1], we can get similar decompositions of $\mathcal{G}(P)$ when G is a non-simply connected semi-simple Lie group whose universal cover has no p-torsion in homology. We can also get the decompositions of $\mathcal{G}(P)$ for $(G, p) = (E_6, 3), (E_8, 5)$

2 Decomposing gauge groups

Let us recall basic facts on homotopy colimits. If each X_i is cofibrant in a diagram of spaces X_i, then so is also $\operatorname{hocolim} X_i$ [Hi]. In particular, if each X_i is a CW-complex, then $\operatorname{hocolim} X_i$ has the homotopy type of a CW-complex. We will use this fact implicitly. The following classical lemma is the key of our decomposition.

Lemma 2.1 (Farjoun [D-F])**.** *Suppose that each B_n is path-connected in a tower of homotopy fiber sequences:*

$$\begin{array}{ccccccc}
F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots
\end{array}$$

Then

$$\operatorname{hocolim} F_n \to \operatorname{hocolim} E_n \to \operatorname{hocolim} B_n$$

is a homotopy fiber sequence.

Note that filtered homotopy colimits commute with homotopy pullbacks. Then, in particular, we get:

Lemma 2.2. *For a tower of pointed maps $X_0 \to X_1 \to X_2 \to \cdots$, the natural map*

$$\operatorname{hocolim} \Omega X_n \to \Omega(\operatorname{hocolim} X_n)$$

is a weak equivalence.

Hereafter, we will localize all spaces and maps at the odd prime p.

Let us recall and modify Wilkerson's proof [Wi] of Theorem 1.1. Let G be a compact, connected, semi-simple Lie group such that $H_*(G; \mathbf{Z})$ is p-torsion free. Since $H_*(G; \mathbf{Z})$ is p-torsion free, we have

$$H^*(G; \mathbf{Z}_{(p)}) = \Lambda(x_{2n_1+1}, \ldots, x_{2n_l+1}),\ |x_i| = i,$$

where $\{n_1, \ldots, n_l\}$ is the type of G and each x_i is universally transgressive. For an integer q with $(p, q) = 1$, there is the unstable Adams operation $\phi^q : BG \to BG$ satisfying $(\phi^q)^*(y) = q^n y$ for any $y \in H^{2n}(BG)$. Then since x_i are universally transgressive, we have

$$(\Omega\phi^q)^*(x_{2n_i+1}) = q^{n_i+1} x_{2n_i+1}.$$

Let u be an integer whose mod p reduction is the primitive $(p-1)^{\text{st}}$ root of unity in $\mathbf{Z}/p$. We define a map $f(n, k) : G \to G$ by

$$f(n, k) = u^k \circ \Omega\phi^u - u^{k+n+1},$$

where $r : \Omega X \to \Omega X$ stands for the r^{th} power map and the sum is taken by the multiplication of G. Then we have

$$f(n, k)^*(x_{2n_i+1}) = (u^{k+n_i+1} - u^{k+n+1}) x_{2n_i+1}. \tag{2.1}$$

For $i = 1, \ldots, p-1$, let S_i be an arbitrary set of positive integers such that $T_j(G) \subset S_i$ for $j \neq i$ and that each $n \in S_i$ satisfies $n \not\equiv i \mod (p-1)$. We define a map $F(i, k) : G \to G$ to be the composite of all $f(n, k)$ with $n \in S_i$. For a self-map $g : X \to X$, we denote the homotopy colimit of the diagram $X \xrightarrow{g} X \xrightarrow{g} X \to \cdots$ by $\operatorname{hocolim} g$. Put $B(i, k) = \operatorname{hocolim} F(i, k)$. Clearly, we have $B(i, k) \simeq B(i-k, 0)$, and then the choice of S_i does not effect on the homotopy type of $B(i, k)$.

Recall that the Hurewicz homomorphism induces an isomorphism:

$$\operatorname{Hom}(\pi_*(G) \otimes \mathbf{Q}, \mathbf{Q}) \xrightarrow{\cong} H^*(G; \mathbf{Q}) \tag{2.2}$$

Then it follows from (2.1) that the induced map $F(i, k)_* : \pi_{2n+1}(G)/\text{torsion} \to \pi_{2n+1}(G)/\text{torsion}$ is an isomorphism for $n \not\equiv i \mod (p-1)$ and the zero-map for $n \equiv i \mod (p-1)$. Thus $\pi_*(B(i, k))$ is a finitely generated $\mathbf{Z}_{(p)}$-module for each $*$, equivalently, $B(i, k)$ is of finite type at the prime p.

By (2.1), one easily sees that the tower

$$\cdots \to H^*(G;\mathbf{Z}/p) \xrightarrow{F(i,k)^*} H^*(G;\mathbf{Z}/p) \xrightarrow{F(i,k)^*} H^*(G;\mathbf{Z}/p)$$

is Mittag-Leffler, and then $H^*(B(i,k);\mathbf{Z}/p)$ is isomorphic to the limit of this tower. Thus we obtain that the natural map $G \to \prod_{i=1}^{p-1} B(i,k)$ is a mod p cohomology equivalence. Since $B(i,k)$ is of finite type at the prime p, the above map is a homotopy equivalence by the J.H.C. Whitehead theorem. In the above calculation of the mod p homology, one gets that the type of $B(i,0)$ is $T_i(G)$. Therefore we have established Theorem 1.1.

Let n be an entry of the type of G. Then by looking at the decompositions of exceptional Lie groups [MNT], we see that $\pi_{2n+1}(G)$ is p-torsion free for any G. Then the natural map $\pi_{2n+1}(G) \to \pi_{2n+1}(G)\otimes\mathbf{Q}$ is injective, and thus by (2.2), we get:

Proposition 2.1. *Let G and n be as above. Then we have $(\Omega\phi^q)_* = q^{n+1} : \pi_{2n+1}(G) \to \pi_{2n+1}(G)$.*

Proof of Theorem 1.2. Let P be a principal G-bundle over K classified by $\alpha : K \to BG$. Then by [AB], there is a natural homotopy equivalence

$$B\mathcal{G}(P) \simeq \mathrm{map}(K, BG;\alpha), \tag{2.3}$$

where $\mathrm{map}(X,Y;f)$ stands for the component of $\mathrm{map}(X,Y)$ containing f. See [KK4] for details.

Let $\underline{r} : S^m \to S^m$ be the degree r map. By Proposition 2.1, ϕ^q_* maps $\mathrm{map}(S^{2n+2}, BG;\alpha)$ into $\mathrm{map}(S^{2n+2}, BG; q^n\alpha)$. Then we can define a map $g(i) : \Omega\mathrm{map}(S^{2n+2}, BG;\alpha) \to \Omega\mathrm{map}(S^{2n+2}, BG;\alpha)$ by

$$g(i) = (\underline{u^{-n-1}})^* \circ (\phi^u)_* - u^{-n-1+i}.$$

Put $S_i = \bigcup_{j\neq i}(T_j(G) \cup \{k+n+1 \mid k \in T_{j-n-1}(G)\})$. We define a map $G(i) : \Omega\mathrm{map}(S^{2n+2}, BG;\alpha) \to \Omega\mathrm{map}(S^{2n+2}, BG;\alpha)$ to be the composite of all $g(k)$ for $k \in S_i$. Then we get a commutative diagram:

$$\begin{tikzcd}
\Omega_0^{2n+2}G \arrow[r, "\Omega^{2n+2}F(i,-n-1)"] \arrow[d] & \Omega_0^{2n+2}G \arrow[d] \\
\Omega\mathrm{map}(S^{2n+2}, BG;\alpha) \arrow[r, "G(i)"] \arrow[d] & \Omega\mathrm{map}(S^{2n+2}, BG;\alpha) \arrow[d] \\
G \arrow[r, "F(i,0)"] & G
\end{tikzcd}$$

Put $X_i = \operatorname{hocolim} G(i)$. Then it follows from the above modification of Wilkerson's proof together with Lemma 2.1 and Lemma 2.2 that there are homotopy fibrations $\Omega_0^{2n+2}B_{k+n+1} \to$

$X_k \to B_k$ satisfying a homotopy commutative diagram:

$$\begin{tikzcd}
\prod_{k\in\mathbf{Z}/(p-1)} \Omega_0^{2n+2} B_{k+n+1} \arrow[r, "\simeq"] \arrow[d] & \Omega_0^{2n+2} G \arrow[d] \\
\prod_{k\in\mathbf{Z}/(p-1)} X_k \arrow[r] \arrow[d] & \Omega \mathrm{map}(S^{2n+2}, BG; \alpha) \arrow[d] \\
\prod_{k\in\mathbf{Z}/(p-1)} B_k \arrow[r, "\simeq"] & G
\end{tikzcd}$$

Then by (2.3), the proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. By Proposition 2.1, we have

$$(\Omega\phi^q)_*(\langle \alpha, 1_G\rangle) = \langle (\Omega\phi^q)_*(\alpha), \Omega\phi^q\rangle = \langle q^{n+1}\alpha, \Omega\phi^q\rangle = \langle \alpha, 1_G\rangle \circ (\underline{q^{n+1}} \wedge \Omega\phi^q).$$

Then if δ is a Hopf map, we have $\delta \circ G(i) = F(0,i) \circ \delta$. Since $\Omega\mathrm{map}(S^{2n+2}, BG; \alpha)$ is the homotopy fiber of δ by [Wh], the proof is completed. □

3 Investigating the factor spaces X_k

By investigating the factor spaces X_k in Theorem 1.2, we verify that our decompositions include those of Theriault [Th].

Let G be a topological group. For $\alpha \in \pi_k(G)$, we denote the adjoint $G \to \Omega_0^k G$ of the Samelson product $\langle \alpha, 1_G\rangle$ by δ_α. In [Th], a sufficient condition for δ_α being a Hopf map was given for $k = 3$. It is straightforward to generalize this result for any k as:

Theorem 3.1 (Theriault [Th])**.** *Let (G, p) be as in the following table. Then the map δ_α is a Hopf map.*

G	p
$\mathrm{SU}(n)$	$(p-1)(p-2)+1 \geq n$
$\mathrm{Sp}(n)$	$(p-1)(p-2) \geq 2n$
$\mathrm{Spin}(2n+1)$	$(p-1)(p-2) \geq 2n$
$\mathrm{Spin}(2n)$	$(p-1)(p-2) \geq 2(n-1)$
G_2, F_4, E_6	$p \geq 5$
E_7, E_8	$p \geq 7$

Let G be a compact, connected, simple Lie group of type $\{n_1, \ldots, n_l\}$ such that $H_*(G; \mathbf{Z})$ has no p-torsion. In particular, we have $\pi_3(G) \cong \mathbf{Z}$. We fix a generator ϵ of $\pi_3(G)$. Let P_m be the principal G-bundle over S^4 classified by $m\epsilon$. We abbreviate the map $\delta_{m\epsilon}$ by δ_m. In what follows, we will deal only with gauge groups of principal bundles P_m for our purpose, but analogous results for gauge groups of bundles over spheres satisfying the condition in Theorem 1.2 can be obtained quite similarly.

3.1 p-regular case

Let G be a compact, connected Lie group of type $\{n_1, \dots, n_l\}$. Suppose $p \geq n_l$. Then by Serre's result [S], B_k has the homotopy type of S^{2k+1} or a point. Since we localize at the odd prime p, it follows that if a map $B_k \to \Omega_0^3 B_{k+2}$ is essential, we have $k = p-2, p-1$. Thus by Theorem 1.2, Theorem 1.3 and Theorem 3.1, we obtain:

Proposition 3.1 (Theriault [Th])**.** *Let G be a simple Lie group of type $\{n_1, \dots, n_l\}$, and let P_m be the principal G-bundle over S^4 classified by $m\epsilon$. For $p > n_l + 2$, we have*

$$\mathcal{G}(P_m) \simeq \prod_{i=1}^{l} (S^{2n_i-1} \times \Omega_0^4 S^{2n_i-1}).$$

Remark 3.1. Our convention of the types of Lie groups differs by 1 from that of Theriault [Th]. Then the result also differs by 1.

Let us further investigate $\mathcal{G}(P_m)$ in the p-regular case, that is, we consider the case when $n_l = p-2, p-1$. Recall from [To] that the homotopy groups of odd spheres are given as:

$$\pi_{2n+1+k}(S^{2n+1}) \cong \begin{cases} \mathbf{Z} & k = 0 \\ \mathbf{Z}/p & k = 2i(p-1)-1,\ i = 1, \dots, p-1 \\ \mathbf{Z}/p & k = 2i(p-1)-2,\ i = n, \dots, p-1 \\ 0 & \text{other } 1 \leq k \leq 2p(p-1)-3 \end{cases} \tag{3.1}$$

Let λ_{2n_i+1} denote the inclusion $B_{n_i} = S^{2n_i+1} \to G$. Then one can deduce from (3.1) that the composite $B_k \to G \xrightarrow{\delta_m} \Omega_0^3 G \to \Omega_0^3 B_{k+2}$ is trivial if and only if so is the Samelson product $\langle m\epsilon, \lambda_{2k+1} \rangle = m\langle \epsilon, \lambda_{2k+1} \rangle$.

Let us first deal with exceptional groups. For $n_l = p-2, p-1$, it follows from (3.1) and Table 1 that the only possibility for the Samelson product $\langle \epsilon, \lambda_{2k-1} \rangle$ being non-trivial is the case $n_l = p-2$. On the other hand, Hamanaka and the second author [HK] showed that the Samelson product $\langle \epsilon, \lambda_{2p-3} \rangle$ is non-trivial for $n_l = p-2$ and then it is of order p by 3.1. Thus we obtain:

Proposition 3.2. *Let G be a compact, exceptional simple Lie group of type $\{n_1, \dots, n_l\}$, and let P_m be as above.*

1. *If $n_l = p-1$, then*

$$\mathcal{G}(P_m) \simeq \prod_{i=1}^{l} (S^{2n_i-1} \times \Omega_0^4 S^{2n_i-1}).$$

2. *If $n_l = p-2$, then*
$$\mathcal{G}(P_m) \simeq X \times \prod_{i \neq l} S^{2n_i+1} \times \prod_{i \neq 1} \Omega_0^4 S^{2n_i+1}$$
in which X is the homotopy fiber of the map $S^{2p-3} \to \Omega_0^3 S^3$ of order $\frac{p}{(m,p)}$.

Let us next consider $\mathrm{SU}(n)$ for $n = p-2, p-1$. For $n = p-2$, the only possible non-trivial map $B_k \to \Omega_0^3 B_{k+2}$ occurs when $k = p-2$. As above, this map corresponds to the Samelson product $m\langle \epsilon, \lambda_{2p-3} \rangle$. By a classical result of Bott [B], we see that the Samelson product $\langle \epsilon, \lambda_{2p-3} \rangle$ is non-trivial. Thus we see that the map $S^{2p-3} \to \Omega_0^3 S^3$ is of order $\frac{p}{(m,p)}$ by 3.1.

For $n = p-1$, possible non-trivial maps $B_k \to \Omega_0^3 B_{k+2}$ occurs when $k = p-2, p-1$. We can see as above that these maps are of order $\frac{p}{(m,p)}$. Summarizing, we obtain:

Proposition 3.3. *Let P_m be a principal $\mathrm{SU}(n)$-bundle over S^4 classified by $m \in \mathbf{Z} \cong \pi_3(\mathrm{SU}(n))$.*

1. *If $n = p-2$,*
$$\mathcal{G}(P_m) \simeq X \times \prod_{i=1}^{p-3} S^{2i+1} \times \prod_{i=2}^{p-2} \Omega_0^4 S^{2i+1},$$
where X is the homotopy fiber of the map $S^{2p-3} \to \Omega_0^3 S^3$ of order $\frac{p}{(m,p)}$.

2. *If $n = p-1$,*
$$\mathcal{G}(P_m) \simeq X_1 \times X_2 \times \prod_{i=1}^{p-3} S^{2i+1} \times \prod_{i=3}^{p-1} \Omega_0^4 S^{2i+1},$$
where for $i = 2, 3$, X_i is the homotopy fiber of the map $S^{2p+2i-5} \to \Omega_0^3 S^{2i+1}$ of order $\frac{p}{(m,p)}$.

The decomposition of $\mathcal{G}(P_m)$ for $G = \mathrm{Sp}(n)$ can be deduced quite similarly. Using this together with the result of [KK1], we can also deduce the decomposition of $\mathcal{G}(P_m)$ for $G = \mathrm{SO}(n)$.

3.2 The case of $\mathrm{SU}(n)$ for $n \leq (p-2)(p-1)+1$

Let P_m be the principal $\mathrm{SU}(n)$-bundle over S^4 classified by $m\epsilon$, where ϵ is a fixed generator of $\pi_3(\mathrm{SU}(n))$. We investigate the factor spaces X_k in Theorem 1.2 when $n \leq (p-2)(p-1)+1$. To this end, we calculate the homotopy groups of B_k, where B_k is as in Theorem 1.1 for $G = \mathrm{SU}(n)$. Then B_k has the homotopy type of the space $B_k^{l_k}$ for $l_k = \lfloor \frac{n-k}{p-1} \rfloor$ in [MNT] which fits into a homotopy fiber sequence:
$$\Omega S^{2k+1+2l(p-1)} \to B_k^l \to B_k^{l+1}$$
Then by (3.1) together with the above homotopy fiber sequence, we get:

Lemma 3.1. *For $k = 1, \dots, p-1$ and $m \leq (p-l+1)(p-1)-2$, $\pi_{2k+2(l-1)(p-1)+2m}(B^l_k) \neq 0$ if and only if $m \equiv 0 \mod (p-1)$.*

Corresponding to the inclusion $\Sigma \mathbf{C}P^{n-1} \to \mathrm{SU}(n)$, B_k has a subcomplex $A_k = S^{2k+1} \cup e^{2k+1+2(p-1)} \cup \cdots \cup e^{2k+1+2l_k(p-1)}$. That is, there is a homotopy commutative diagram:

$$\begin{tikzcd} \Sigma \mathbf{C}P^{n-1} \arrow[r] \arrow[d, "\simeq"'] & \mathrm{SU}(n) \arrow[d, "\simeq"] \\ \bigvee_{k=1}^{p-1} A_k \arrow[r] & \prod_{k=1}^{p-1} B_k \end{tikzcd}$$

Since $n \leq (p-2)(p-1)+1$, we have a retraction $r : \Sigma B_k \to \Sigma A_k$. We denote by $\lambda_k : \Sigma A_k \to B\mathrm{SU}(n)$ and $\bar{\lambda}_k : \Sigma B_k \to B\mathrm{SU}(n)$ the adjoint of the inclusions $A_k \to \mathrm{SU}(n)$ and $B_k \to \mathrm{SU}(n)$, respectively. As in [KN], one sees that $\bar{\lambda}_k$ factors as $\Sigma B_k \xrightarrow{\alpha} \Sigma B_k \xrightarrow{r} \Sigma A_k \xrightarrow{\lambda_k} B\mathrm{SU}(n)$ for some self-homotopy equivalence α.

Let $\hat{\epsilon} \in \pi_4(B\mathrm{SU}(n))$ be the adjoint of ϵ. Suppose the Whitehead product $[\hat{\epsilon}, \lambda_k]$ is trivial. Then there exists an extension $S^4 \times \Sigma A_k \to B\mathrm{SU}(n)$ of $\hat{\epsilon} \vee \lambda_k$ up to homotopy. Since $\bar{\lambda}_k \simeq \lambda_k \circ r \circ \alpha$, we also have an extension $S^4 \times \Sigma B_k \to B\mathrm{SU}(n)$ of $\hat{\epsilon} \vee \bar{\lambda}_k$ up to homotopy, which is equivalent to that the Whitehead product $[\hat{\epsilon}, \bar{\lambda}_k]$ is trivial. On the other hand, the triviality of $[\hat{\epsilon}, \bar{\lambda}_k]$ obviously implies that of $[\hat{\epsilon}, \lambda_k]$. Thus since Whitehead products are the adjoint of Samelson products, we obtain:

Lemma 3.2. *Let ι_k and $\bar{\iota_k}$ denote the inclusions $A_k \to \mathrm{SU}(n)$ and $B_k \to \mathrm{SU}(n)$, respectively. The order of the Samelson product $\langle \epsilon, \iota_k \rangle$ is equal to that of $\langle \epsilon, \bar{\iota}_k \rangle$.*

Now we calculate the Samelson product $\langle \epsilon, \iota_k \rangle$. Choose u, v such that $\dim A_u = 2n-3$ and $\dim A_v = 2n-1$, respectively. Since $\pi_{2k}(\mathrm{SU}(n)) = 0$ for $k < n$, the Samelson product $\langle \epsilon, \iota_k \rangle$ is trivial unless $k = u, v$. Then it follows that

$$X_k \simeq B_k \times \Omega^4_0 B_{k+2}$$

for $k \neq u, v$. By Lemma 3.1, we have $\pi_{2n}(B_{u+2}) \neq 0$ and $\pi_{2n+2}(B_{v+2}) \neq 0$. Moreover, by Lemma 3.1, if $\pi_{2n}(B_\mu) \neq 0$ and $\pi_{2n+2}(B_\nu) \neq 0$, we have $\mu \equiv u+2,\ \nu \equiv v+2 \mod (p-1)$. This is also seen from the fact that $\pi_{2n}(\mathrm{SU}(n))$ and $\pi_{2n+2}(\mathrm{SU}(n))$ are cyclic. Let δ^u_m and δ^v_m be the composite of maps

$$B_u \to \mathrm{SU}(n) \xrightarrow{\delta_m} \Omega^3_0 \mathrm{SU}(n) \to \Omega^3_0 B_\mu \text{ and } B_v \to \mathrm{SU}(n) \xrightarrow{\delta_m} \Omega^3_0 \mathrm{SU}(n) \to \Omega^3_0 B_\nu,$$

respectively. Then we have obtained:

Theorem 3.2 (Theriault [Th]). *Choose $u, v, \mu, \nu \in \mathbf{Z}/(p-1)$ such that $\dim A_u = 2n-3$, $\dim A_v = 2n-1$, $\pi_{2n}(B_\mu) \neq 0$ and $\pi_{2n+2}(B_\nu) \neq 0$. Let P_m be the principal $\mathrm{SU}(n)$-bundle over S^4 classified by $m\epsilon$. If $n \leq (p-2)(p-1)+1$, we have*

$$\mathcal{G}(P_m) \simeq X \times Y \times \prod_{i \neq u,v} B_i \times \prod_{i \neq \mu,\nu} \Omega_0^4 B_i,$$

where X and Y are the homotopy fiber of the composite

$$B_u \to \mathrm{SU}(n) \xrightarrow{\delta_m} \Omega_0^3 \mathrm{SU}(n) \to \Omega_0^3 B_\mu \text{ and } B_v \to \mathrm{SU}(n) \xrightarrow{\delta_m} \Omega_0^3 \mathrm{SU}(n) \to \Omega_0^3 B_\nu.$$

respectively.

Remark 3.2. Theriault [Th] described the map $B_u \to \Omega_0^3 B_\mu$ and $B_v \to \Omega_0^3 B_\nu$ in a different from, but it is due to his translation of the map δ_m.

Let us calculate the orders of the maps δ_m^u and δ_m^v in the above theorem. By definition, δ_m^s ($s = u, v$) is the adjoint of the composite of the Samelson product $\langle m\epsilon, \bar{\iota}_k \rangle$ and the projection $\pi : SU(n) \to B_t$ for $t = \mu, \nu$ according as $s = u, v$. By Lemma 3.1, we see that the projection π induces an injection $\pi_* : [\Sigma^3 A_s, \mathrm{SU}(n)] \to [\Sigma^3 A_s, B_t]$, and then we obtain that the order of δ_m^s is equal to that of the Samelson product $\langle m\epsilon, \iota_k \rangle$. Note that the inclusion $\mathrm{SU}(n) \to \mathrm{U}(n)$ and the pinch map $q : S^3 \times A_s \to \Sigma^3 A_s$ induce injections $[\Sigma^3 A_s, \mathrm{SU}(n)] \to [\Sigma^3 A_s, \mathrm{U}(n)]$ and $q^* : [\Sigma^3 A_s, \mathrm{U}(n)] \to [S^3 \times A_s, \mathrm{U}(n)]$. Note also that the Samelson product $j \circ \langle \epsilon, \iota_k \rangle = \langle j \circ \epsilon, j \circ \iota_k \rangle$ maps to the commutator $[\pi_1 \circ j \circ (m\epsilon), \pi_2 \circ j \circ \iota_k]$ in the group $[S^3 \times A_s, \mathrm{U}(n)]$ by the pinch map q, where π_i stands for the i^{th} projection for $i = 1, 2$. Then we calculate the order of the commutator $[\pi_1 \circ j \circ (m\epsilon), \pi_2 \circ j \circ \iota_k]$.

Define a map $\Theta : \widetilde{K}(X) \to \bigoplus_{i=0}^{p-2} H^{2n+2i}(X; \mathbf{Z}_{(p)})$ by

$$\Theta(\xi) = \bigoplus_{i=0}^{p-2} (n+i)! \mathrm{ch}_{n+i}(\xi)$$

for $\xi \in \widetilde{K}(X)$, where ch_m denotes the $2m$-dimensional part of the Chern character. Recall that the cohomology of $\mathrm{U}(n)$ is given by

$$H^*(\mathrm{U}(n); \mathbf{Z}_{(p)}) = \Lambda(x_1, x_3, \ldots, x_{2n-1}),\ |x_i| = i,$$

where each x_i is universally transgressive. The following is proved in [H].

Theorem 3.3 (Hamanaka [H]). *Let X be a CW-complexes with $\dim X \leq 2n + 2p - 4$. For maps $\alpha, \beta : X \to \mathrm{U}(n)$, we put*

$$\gamma_k = \sum_{\substack{i+j-1=k \\ 1 \leq i,j \leq n}} \alpha^*(x_{2i-1}) \beta^*(x_{2j-1}).$$

Then the order of the commutator $[\alpha, \beta]$ in the group $[X, \mathrm{U}(n)]$ is equal to the order of the element $(\gamma_n, \ldots, \gamma_{n+p-2})$ in the cokernel of the map $\Theta : \widetilde{K}(X) \to \bigoplus_{i=0}^{p-2} H^{2n+2i}(X; \mathbf{Z}_{(p)})$.

Proposition 3.4. *The orders of the maps δ_m^u and δ_m^v are $\frac{p}{(p,m)}$.*

Proof. Let l be the largest integer such that $H^{2s-1+2l(p-1)}(A_s;\mathbf{Z}_{(p)})\neq 0$, that is, the rank of B_s. Put $u_{2i-1}=(j\circ\iota_s)^*(x_{2i-1})$. Then by definition, $H^*(A_s;\mathbf{Z}_{(p)})$ is a free $\mathbf{Z}_{(p)}$-module generated by $u_{2s-1},u_{2s-1+(p-1)},\dots,u_{2s-1+2l(p-1)}$. As is calculated in [KN], $\widetilde{K}^{-1}(A_s)$ is a free $\mathbf{Z}_{(p)}$-module generated by $\xi_0,\dots,\xi_l$ such that

$$\mathrm{ch}(\xi_k)=\sum_{i=0}^{l}\frac{(s-1+k(p-1))^{s-1+i(p-1)}}{s-1+i(p-1)!}\Sigma u_{2s-1+2i(p-1)}.$$

Notice that $\widetilde{K}(S^3\times A_s)\cong\widetilde{K}^{-1}(S^3)\otimes\widetilde{K}^{-1}(A_s)$. Then we get that $\mathrm{Im}\{\Theta:\widetilde{K}(S^3\times A_s)\to\bigoplus_{i=0}^{p-2}H^{2n+2i}(S^3\times A_s;\mathbf{Z}_{(p)})$ is generated by $p(w\times u_{2n-3})$ and $p(w\times u_{2n-1})$ according to $s=u,v$, where w denotes a generator of $H^3(S^3;\mathbf{Z}_{(p)})$.

On the other hand, we have

$$\sum_{\substack{i+j-1=k\\1\le i,j\le n}}(\pi_1\circ j\circ(m\epsilon))^*(x_{2i-1})(\pi_2\circ j\circ\iota_s)^*(x_{2j-1})=m(w\times u_{2k-1}).$$

Thus the proof is completed by Theorem 3.3. □

The mod p decomposition of P_m for $G=\mathrm{Sp}(n)$ with $2n\le(p-2)(p-1)$ can be quite similarly obtained, and the case $G=\mathrm{SO}(n)$ is also similar. Thus by all of the above observation in this section, we conclude that our mod p decompositions include those of Theriault. One can easily see that our mod p decompositions also include those in [KK3] when the base spaces are spheres satisfying the condition in Theorem 1.2.

4 Example

Recall that a fiberwise space over K is a map $X\to K$ and the fiberwise product of fiberwise spaces $X\to K$ and $Y\to K$ is the pullback of the triad $X\to K\leftarrow Y$. We will assume all fiberwise spaces will be fiberwise localized at the prime p. We say that a fiberwise space $X\to K$ is trivial if it is a weak equivalence.

Let P be a principal G-bundle over K. The adjoint bundle $\mathrm{ad}P$ of P is defined by

$$P\times G/(p,g)\sim(ph^{-1},hgh^{-1})\text{ for }p\in P,\ g,h\in G.$$

Then we know that the gauge group $\mathcal{G}(P)$ is naturally isomorphic to the space of sections of $\mathrm{ad}P$. Then if $\mathrm{ad}P$ is the fiberwise product of two fiberwise spaces over K, $\mathcal{G}(P)$ decomposes into two spaces. This decomposition method is used in [KK3]. One might assume that such a decomposition of the adjoint bundle is, in general, stronger than a decomposition of the gauge group, but this has been a folklore. The aim of this section is to present examples for this.

Let $EG \to BG$ be the universal G-bundle. It is well known that there is a fiberwise homotopy equivalence between $\mathcal{L}BG$ and $\mathrm{ad}EG$ over BG, where $\mathcal{L}X$ denotes the free loop space of X. Then we will identify these two fiberwise spaces over BG.

We calculate the cohomology of $\mathcal{L}B\mathrm{Sp}(n)$. In [KK2], the authors constructed the following map.

Theorem 4.1 (Kishimoto and Kono [KK1]). *Let R be a commutative ring. There is a linear map $\hat{\sigma} : H^*(X;R) \to H^{*-1}(\mathcal{L}X;R)$ satisfying:*

1. *For the inclusion $i : \Omega X \to \mathcal{L}X$, $i^* \circ \hat{\sigma}$ coincides with the cohomology suspension.*
2. *$\hat{\sigma}(xy) = \hat{\sigma}(x)y + (-1)^{|x|}x\hat{\sigma}(y)$ for $x, y \in H^*(X;R)$.*
3. *$\hat{\sigma}$ commutes with the stable cohomology operations.*

Recall that the mod p cohomology of $B\mathrm{Sp}(\frac{p-1}{2})$ is given by

$$H^*(B\mathrm{Sp}(\tfrac{p-1}{2}); \mathbf{Z}/p) = \mathbf{Z}/p[q_1, \ldots, q_{\frac{p-1}{2}}],\ |q_i| = 4i.$$

By an easy calculation, we have

$$\mathcal{P}^1 q_1 = (-1)^{\frac{p+1}{2}} \left(\frac{p+1}{2}\right)! q_1 q_{\frac{p-1}{2}} + \cdots .$$

Then by Theorem 4.1 and the Leray-Hirsch theorem, we get

$$H^*(\mathcal{L}B\mathrm{Sp}(\tfrac{p-1}{2}); \mathbf{Z}/p) = \mathbf{Z}/p[q_1, \ldots, q_{\frac{p-1}{2}}] \otimes \Lambda(x_1, \ldots, x_{\frac{p-1}{2}})$$

and

$$\mathcal{P}^1 x_1 = (-1)^{\frac{p+1}{2}} \left(\frac{p+1}{2}\right)! q_1 x_{\frac{p-1}{2}} + \cdots \tag{4.1}$$

where $x_i = \hat{\sigma}(x_i)$.

Note that since $\mathrm{Sp}(\frac{p-1}{2})$ is p-regular, a decomposition $\mathrm{Sp}(\frac{p-1}{2}) \simeq X_1 \times \cdots \times X_k$ implies that each X_i has the homotopy type of a product of spheres.

Let P be the principal $\mathrm{Sp}(\frac{p-1}{2})$-bundle over S^4 classified by $1 \in \mathbf{Z} \cong \pi_3(\mathrm{Sp}(\frac{p-1}{2}))$. Then the adjoint bundle $\mathrm{ad}P$ is fiberwise homotopy equivalent over S^4 to the pullback of $LB\mathrm{Sp}(\frac{p-1}{2})$ by the inclusion of the bottom cell $S^4 \to B\mathrm{Sp}(\frac{p-1}{2})$. Then we get non-trivial cohomology classes u and $y_1, \ldots, y_{\frac{p-1}{2}}$ of $\mathrm{ad}P$ by pulling back q_1 and $x_1, \ldots, x_{\frac{p-1}{2}}$. Thus by (4.1) and the above observation on decompositions of $\mathrm{Sp}(\frac{p-1}{2})$, we see that $\mathrm{ad}P$ is not fiberwise homotopy equivalent to the fiberwise product of $\frac{p-1}{2}$ non-trivial fiberwise spaces over S^4. On the other hand, it follows from Theorem 1.2 that $\mathcal{G}(P)$ is the product of $\frac{p-1}{2}$ non-contractible spaces. Thus we have established:

Theorem 4.2. *Let P be a principal $\mathrm{Sp}(\frac{p-1}{2})$-bundle over S^4 classified by $1 \in \mathbf{Z} \cong \pi_3(\mathrm{Sp}(\frac{p-1}{2}))$. Then $\mathrm{ad}P$ is not fiberwise homotopy equivalent to the fiberwise product of $\frac{p-1}{2}$ non-trivial fiberwise spaces over S^4 while $\mathcal{G}(P)$ is the product of $\frac{p-1}{2}$ non-contractible spaces.*

Corollary 4.1. *Let P be the principal $\mathrm{Sp}(2)$-bundle over S^4 classified by $1 \in \mathbf{Z} \cong \pi_3(\mathrm{Sp}(2))$, and let $p = 5$. Then $\mathrm{ad}P$ is indecomposable as a fiberwise space over S^4 while $\mathcal{G}(P)$ is decomposable.*

References

[AB] M.F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523-615.

[B] R. Bott, *A note on the Samelson products in the classical groups*, Comment. Math. Helv. **34** (1960), 249-256.

[D-F] E. Dror-Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics **1622**, Springer-Verlag, Berlin, 1996.

[H] H. Hamanaka, *On Samelson products in p-localized unitary groups*, Topology Appl. **154** (2007), 573-583.

[HK] H. Hamanaka and A. Kono, *A note on Samelson products and mod p cohomology of classifying spaces of the exceptional Lie groups*, Topology Appl. **157** (2010), no. 2, 393-400.

[Ha] B. Harris, *On the homotopy groups of the classical groups*, Ann. of Math. (2) **74** (1961), 407-413.

[Hi] P.S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs **99**, American Mathematical Society, Providence, RI, 2003.

[KK1] D. Kishimoto and A. Kono, *Mod p decompositions of non-simply connected Lie groups*, J. Math. Kyoto Univ. **48** (2008), no. 1, 1-5.

[KK2] D. Kishimoto and A. Kono, *On the cohomology of free and twisted loop spaces*, J. Pure Appl. Algebra **214** (2010), no. 5, 646-653.

[KK3] D. Kishimoto and A. Kono, *Note on mod p decompositions of gauge groups*, Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), no. 1, 15-17.

[KK4] D. Kishimoto and A. Kono, *Splitting of gauge groups*, Trans. Amer. Math. Soc. **362** (2010), no. 12, 6715-6731.

[KN] D. Kishimoto and T. Nagao, *Commutativity in special unitary groups at odd primes*, Topology Appl. **157** (2010), no. 12, 1949-1954.

[L] G.E. Lang Jr., *The evaluation map and* EHP *sequences*, Pacific J. Math. **44** (1973), 201-210.

[MNT] M. Mimura, G. Nishida and H. Toda, Mod *p decomposition of compact Lie groups*, Publ. Res. Inst. Math. Sci. **13** (1977/78), no. 3, 627-680.

[S] J.-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2) **58** (1953), 258-294.

[Th] S.D. Theriault, *Odd primary homotopy decompositions of gauge groups*, Algebr. Geom. Topol. **10** (2010), no. 1, 535-564.

[To] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Ann. of Math. Studies **49**, Princeton Univ. Press, Princeton N.J., 1962.

[Wh] G.W. Whitehead, *On products in homotopy groups*, Ann. of Math (2) **47** (1946), 460-475.

[Wi] C. Wilkerson, *Self-maps of classifying spaces*, Localization in group theory and homotopy theory, and related topics (Sympos., Battelle Seattle Res. Center, Seattle, Wash., 1974), pp. 150-157. Lecture Notes in Math., **418**, Springer, Berlin, 1974.