Kyoto University

Kyoto-Math 2011-05

Abundance of nonuniform hyperbolicity in bifurcations of surface endomorphisms

by

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March 2011

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ABUNDANCE OF NON-UNIFORM HYPERBOLICITY IN BIFURCATIONS OF SURFACE ENDOMORPHISMS

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Abstract. We study an interplay between homoclinic behavior and singularities in surface endomorphisms. We show that appropriate rescalings near homoclinic orbits intersecting fold singularities yield families of non-invertible Hénon-like maps. Then we construct positive measure sets of parameters corresponding to maps which exhibit nonuniformly hyperbolic behavior. This implies an extension of the celebrated theorem of Benedicks and Carleson, and that of Mora and Viana to surface endomorphisms.

1. Introduction

It is well-known that unfoldings of non-transverse homoclinic orbits of diffeomorphisms unleash incredibly rich arrays of complicated behaviors. A program was proposed by Palis [14], which aims to understand all dynamical complexities beyond uniform hyperbolicity through the study of homoclinic bifurcations. The aim of this paper is to contribute to advancing this program to endomorphisms.

For endomorphisms, interactions of invariant manifolds with singularities (points where the Jacobian of the map is singular) bring new phenomena which are not well-understood (see e.g. [10, 16]). In this paper we reveal an interplay between homoclinic points and fold singularities which leads to the emergence of chaotic attractors.

Landmark theorems on the abundance of nonuniform hyperbolicity, or chaotic attractors, were obtained by Jakobson [11] for one-dimensional maps with critical points, and by Benedicks and Carleson [2] for the Hénon map. Mora and Viana [13], Díaz, Rocha and Viana [8] pushed their argument further and proved the existence of chaotic attractors in very general global bifurcations of diffeomorphisms. See Wang and Young [22] for more advanced properties of the attractor. For other subsequent developments in the creation of the theory of nonuniformly hyperbolic dynamics for Hénon-like maps, see [3, 4, 5, 6, 21, 23].

A characteristic of these developments is an almost exclusive concentration on invertible systems. Unfortunately, substantial overhauls are necessary for extensions to non-invertible systems. Moreover, in many applications of practical interest, (e.g. control theory, economics, electronics, neural networks, etc.) systems are often modelled by non-invertible maps. We aim to narrow this gap.

1.1. **Homoclinic points on singularities.**

Let M be a surface and f_0 a C^∞ endomorphism on M. Let $p_0 \in M$ be a hyperbolic fixed point of f_0. Let λ, ρ denote the eigenvalues of $Df_0(p_0)$. We assume $0 < \lambda < 1 < \rho$ and $\lambda\rho < 1$. Let $W^u(p_0)$ denote the unstable manifold of p_0, which is a smoothly immersed real line possibly with self-intersections. Let $W^s_{\rm loc}(p_0)$ denote the local stable manifold of p_0. The stable set of p_0 is defined by $W^s(p_0) = \bigcup_{n\geq 0} f_0^{-n} W^s_{\rm loc}(p_0)$, which is not necessarily a manifold. Let S_0 denote the set of singularities of f_0.

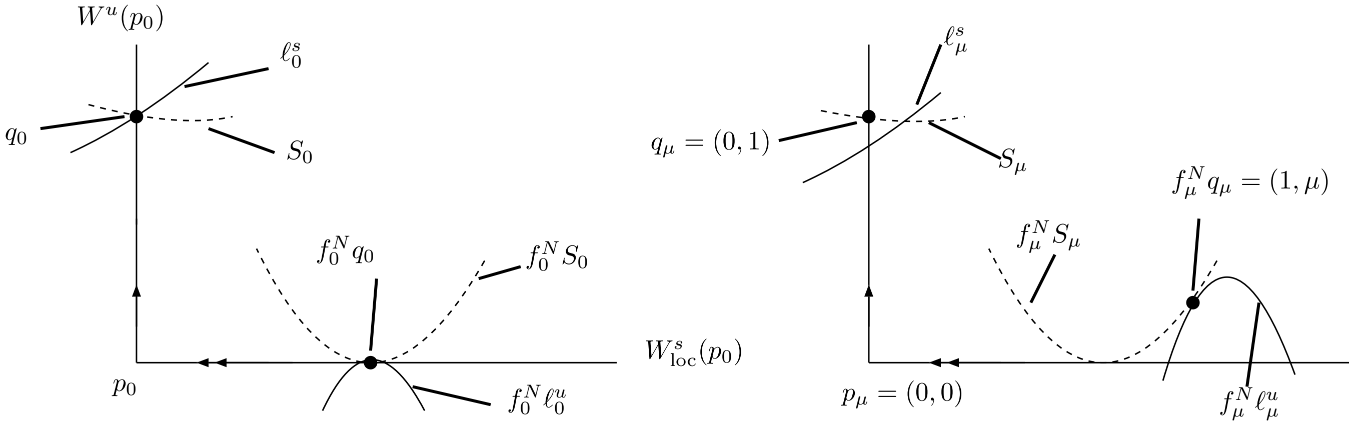

FIGURE 1. $\mu = 0$ (left, $\ell_0^u \subset W^u(p_0)$, $f_0^N \ell_0^s \subset W^s_{\mathrm{loc}}(p_0)$); $\mu \neq 0$ (right)

Definition 1.1. (Simple homoclinic points) A homoclinic point $q \in W^u(p_0) \cap W^s(p_0)$ is *simple* if there exists a sequence $\{q_n\}_{n\in\mathbb{Z}}$ such that $q_0 = q$, $f_0 q_n = q_{n+1}$ for every $n \in \mathbb{Z}$ and $\sharp\{n \in \mathbb{Z} \colon q_n \in S_0\} = 1$.

Let $q \in W^u(p_0) \cap W^s(p_0)$ be a simple homoclinic point. We may assume $q \in S_0$. We write q_0 for q, and assume that q_0 is a *fold singularity*, namely there exist neighborhoods U, V in $\mathbb{R}^2$ of the origin and C^∞ orientation-preserving diffeomorphisms $\phi \colon U \to M$, $\psi \colon V \to M$ such that $\phi(0,0) = q_0$, $\psi(0,0) = f_0(q_0)$, and

$$\psi^{-1} \circ f_0 \circ \phi(x,y) = (x^2, y).$$

Since q_0 is simple, there exist a short compact curve ℓ_0^u in $W^u(p_0)$ which contains q_0 in its interior and has an well-defined sequence of preimages not intersecting S_0. We assume there exists a short smooth compact curve ℓ_0^s in $W^s(p_0)$ which contains q_0 in its interior. Since q_0 is simple, ℓ_0^s is obtained as a preimage of a compact curve in $W^s_{\mathrm{loc}}(p_0)$. We assume the following, the geometric meanings of which are in the parentheses:

(T1) the tangent direction of ℓ_0^u (resp. ℓ_0^s) at q_0 is transverse to that of S_0 at q_0, and to the kernel of $Df_0(q_0)$ ($f_0\ell_0^u$ (resp. $f_0\ell_0^s$) makes a quadratic tangency to f_0S_0 at f_0q_0);
(T2) ℓ_0^u and ℓ_0^s meet transversely to each other at q_0 (the curvature of $f_0\ell_0^u$ at f_0q_0 is different from that of $f_0\ell_0^s$ at f_0q_0).

Let N denote the smallest positive integer such that $f_0^N q_0 \in W^s_{\mathrm{loc}}(p_0)$. In order to get recurrent dynamics involving S_0, we assume the component of $f_0^N S_0$ containing $f_0^N q_0$ is on the same side of $W^s_{\mathrm{loc}}(p_0)$ as that of the branch of $W^u_{\mathrm{loc}}(p_0)$ containing q_0 (See Figure 1).

We consider a generic arc (f_μ) of endomorphisms on M through f_0. Unlike the case of homoclinic tangencies of surface diffeomorphisms, it is not possible two pull the two parabolas $f_0\ell_0^u$, $f_0\ell_0^s$ meeting tangentially at f_0q_0 apart. Nevertheless, it is possible to slide one to the other, as indicated in Figure 2.

For a generic (f_μ), we show that an appropriate re-scaling near the f_0-orbit of q_0 yields a family of Hénon-like endomorphisms of the form

$$(x,y) \mapsto (1 - ax^2, 0) + b \cdot R(a,b,x,y), \tag{1}$$

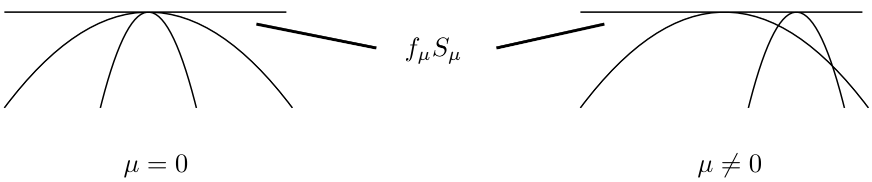

FIGURE 2. $f_0\ell_0^u$ and $f_0\ell_0^s$ are tangent to f_0S_0 (left).

where the components of R are bounded continuous functions. Moreover, there is a vertical line close to the y-axis which is the set of fold singularities. Hence the map is non-invertible, and similar to the "twisted horseshoe map", considered in [9, 11].

Unfortunately, the non-uniform theory mentioned previously fails to apply for this class of families. For instance, the existence of singularities is an intrinsic hurdle for an extension of the solution of the *basin problem*, given by Benedicks and Viana [3], and subsequently Wang and Young [22]. Some regularity conditions of the Jacobian of the maps were assumed in these papers and they no longer hold for maps with singularities. This leads us to the study of *Hénon-like endomorphisms*, a broad class of planar endomorphisms including the above as a prime example.

Mora and Viana [13], adapting the idea of Benedicks and Carleson [2], proved the abundance of non-uniform hyperbolicity in generic unfoldings of dissipative quadratic homoclinic tangencies of surface diffeomorphisms. The next theorem extends their result to surface endomorphisms. What we mean by "chaotic attractors" in the statement is explained in Sect.1.3.

Theorem A. *Let (f_μ) be a C^∞ arc of endomorphisms on surfaces through f_0 as above. Under open and dense assumptions, there exists a positive measure set E accumulating $\mu = 0$ such that for $\mu \in E$ the corresponding f_μ exhibits a "chaotic attractor" near the f_0-orbit of q_0.*

1.2. **Re-scaling near simple homoclinic points.** To prove Theorem A, we introduce a re-scaling near the simple homoclinic point q_0 which converges uniformly to families as in (1). Let p_μ denote the continuation of p_0 for f_μ. Let $r \geq 4$ be an integer. By the linearization theorem [17], under open and dense conditions on λ, ρ there exists a C^r coordinate (x, y) near p_μ such that $f_\mu(x, y) = (\lambda x, \rho y)$. Moreover, these coordinates are taken to be C^r in μ. We extend the domain of linearization so that it contains q_0 and $f_0^N q_0$. In what follows we suppress any linearizing coordinate from notation.

For μ small, let ℓ_μ^s denote the continuation of ℓ_0^s. This makes sense because $W^s_{\rm loc}(p_\mu)$ depends on μ in a continuous way. Analogously, let ℓ_μ^u denote the continuation of ℓ^u. Let S_μ denote the set of singularities of f_μ. Let q_μ denote the point of intersection between ℓ_μ^u and S_μ. Since q_0 is a simple homoclinic point, q_μ is a fold singularity of f_μ^N. For all $\mu \neq 0$ we assume $q_\mu \notin \ell_\mu^s$. This gives rise to a displacement of the two parabolas, indicated in Figure 2.

We adapt our μ-dependent linearizing coordinates in such a way that for all μ, $q_\mu = (0, 1)$ and the x-coordinate of $f_\mu^N q_\mu$ is 1. We re-parametrize μ in such a way that the y-coordinate of $f_\mu^N q_\mu$ is μ. Since q_μ is a fold singularity, there exist μ-dependent local coordinates ϕ_μ near q_μ and φ_μ near $f_\mu^N q_\mu$ such that $\phi_\mu(0,0) = (0,1)$, $\varphi_\mu(0,0) = (1,\mu)$ and $\varphi_\mu^{-1} \circ f_\mu^N \circ \phi_\mu(\tilde{x}, \tilde{y}) = (\tilde{x}^2, \tilde{y})$.

Clearly, ϕ_μ and φ_μ are taken to be C^r in μ. Let

$$D\phi_\mu^{-1}(0,1) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \text{ and } D\varphi_\mu(0,0) = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

For (x,y) near $(0,0)$ we have

$$\phi_\mu^{-1}(x, y+1) = \left(a_1x + a_2y + G_1(x,y), a_3x + a_4y + G_2(x,y)\right),$$

and at $(x,y) = (0,0)$ we have

$$G_1 = G_2 = \partial_x G_1 = \partial_y G_1 = \partial_x G_2 = \partial_y G_2 = 0. \tag{2}$$

Similarly, for $(\hat{x},\hat{y})$ near $(0,0)$ we have

$$\varphi_\mu(\hat{x},\hat{y}) = \left(1,\mu\right) + \left(b_1\hat{x} + b_2\hat{y} + H_1(\hat{x},\hat{y}), b_3\hat{x} + b_4\hat{y} + H_2(\hat{x},\hat{y})\right),$$

and at $(\hat{x},\hat{y}) = (0,0)$,

$$H_1 = H_2 = \partial_{\hat{x}} H_1 = \partial_{\hat{y}} H_1 = \partial_{\hat{x}} H_2 = \partial_{\hat{y}} H_2 = 0. \tag{3}$$

Keep in mind that $a_1, a_2, \cdots, b_3, b_4$ are functions of μ, and G_1, G_2, H_1, H_2 are functions of μ, x, y. For $\mu = 0$, since $\varphi_0^{-1} f_0^{N_0} S_0 = \{(0,\hat{y})\}$, the second component of $D\varphi_0(0,\hat{y})\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ attains a local minimum or maximum at $\hat{y} = 0$. This implies

$$\partial_{\hat{y}\hat{y}} H_2(0,0) = 0 \text{ for } \mu = 0. \tag{4}$$

Lemma 1.1. *We have:* (a) $a_2(0)b_2(0)b_3(0) \neq 0$*;* (b) $b_4(0) = \frac{db_4}{d\mu}(0) = 0$.

Proof. The tangent direction of S_0 at q_0 is spanned by $D\phi_0^{-1}(0,0)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}a_2(0)\\a_4(0)\end{smallmatrix}\right)$. By (T1), it is transverse to the y-direction $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ in the linearizing coordinate. Hence $a_2(0) \neq 0$ holds. The tangent direction of $f_\mu^N S_\mu$ at $f_\mu^N q_\mu$ is spanned by $D\varphi_\mu(0,0)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}b_2(\mu)\\b_4(\mu)\end{smallmatrix}\right)$. For $\mu = 0$, it is tangent to the x-direction and thus $b_4(0) = 0$. Since $b_4(\mu)$ attains a local minimum or maximum at $\mu = 0$, we have $\frac{db_4}{d\mu}(0) = 0$. Since φ_0 is a diffeomorphism, $\det D\varphi_0(0,0) = -b_2(0)b_3(0) \neq 0$ holds. $\square$

We now define our rescaling as follows:

$$\zeta_n(\xi,\eta) = \left(\rho^{-n/2}\xi + 1, \rho^{-2n}\eta + \rho^{-n}\right), \quad \mu_n(\theta) = \rho^{-2n}\theta + \rho^{-n},$$

and

$$\psi_n(\theta,\xi,\eta) = \zeta_n^{-1} \circ f_{\mu_n(\theta)}^{n+N_0} \circ \zeta_n(\xi,\eta).$$

Let $\alpha_\mu = a_2(\mu)^2 b_3(\mu)$. (a) in Lemma 1.1 gives $\alpha_\mu \neq 0$.

Proposition 1.1. *The map $(\theta,\xi,\eta) \to \psi_n(\theta,\xi,\eta)$ converges uniformly on any compact set in $\mathbb{R}^3$ in the C^r topology to $(\theta,\xi,\eta) \to (0, \alpha_0\eta^2 + \theta)$ as $n \to \infty$.*

We finish the proof of Theorem A assuming the conclusion of this proposition (and Theorem B). Set $b = \rho^{-n/2}$ and write $\psi_n(\theta,\xi,\eta) = \left(bT_1(\theta,b,\xi,\eta), \alpha_0\eta^2 + \theta + bT_2(\theta,b,\xi,\eta)\right)$. By the substitution $\xi = x$, $\eta = -ay/\alpha_0$, $\theta = -a/\alpha_0$, this transforms into

$$(a,x,y) \to \left(b\tilde{T}_1(a,b,x,y), 1 - ay^2 + b\tilde{T}_2(a,b,x,y)\right).$$

By the definition of the rescaling, this map has a line of fold singularities close to the x-axis.

Proof of Proposition 1.1. Let $(\tilde{x},\tilde{y}) = \phi_\mu^{-1} \circ f^n_{\mu_n(\theta)} \circ \zeta_n(\xi,\eta)$. Then

$$\tilde{x} = a_1\lambda^n(\rho^{-n/2}\xi + 1) + a_2\rho^{-n}\eta + G_1(\lambda^n(\rho^{-n/2}\xi+1), \rho^{-n}\eta),$$
$$\tilde{y} = a_3\lambda^n(\rho^{-n/2}\xi + 1) + a_4\rho^{-n}\eta + G_2(\lambda^n(\rho^{-n/2}\xi+1), \rho^{-n}\eta),$$

Let $(\hat{x},\hat{y}) = (\tilde{x}^2,\tilde{y})$. A direct computation shows $\psi_n(\theta,\xi,\eta) = (0, \alpha_\mu\eta^2 + \theta) + (I, II)$, where

$$I = \rho^{n/2}(b_1\hat{x} + b_2\hat{y} + H_1(\hat{x},\hat{y})),$$
$$II = \rho^{2n}(b_3(\hat{x} - a_2^2\rho^{-2n}\eta^2) + b_4\hat{y} + H_2(\hat{x},\hat{y})).$$

It suffices to show that I, II converge uniformly to zero (the null function) as $n \to \infty$, in the C^r topology on any compact set.

C^0-convergence. By $G_1(0,0) = G_2(0,0) = 0$ in (2) we have $\tilde{x} = O(\rho^{-n})$ and $\tilde{y} = O(\rho^{-n})$. Moreover, (2) gives $G_1(x,y) = O(\rho^{-2n})$ and $G_2(x,y) = O(\rho^{-2n})$. Hence $\hat{x} = O(\rho^{-2n})$. Using (3) and the estimates for $\hat{x}$, $\hat{y} = \tilde{y}$ we have $H_1(\hat{x},\hat{y}) = O(\rho^{-2n})$. Altogether these yield $I = \rho^{n/2} \cdot o(\rho^{-n/2})$, and thus the uniform C^0-convergence of I as $n \to \infty$.

Regarding II, $\hat{x} = \tilde{x}^2$ and the estimate for G_1 give $\hat{x} - a_2^2\rho^{-2n}\eta^2 = O(\lambda^n\rho^{-n})$, which is $o(\rho^{-2n})$ by $\lambda\rho < 1$. Lemma 1.1 gives $b_4 = O(\rho^{-2n})$. For the last term H_2, (3) gives

$$H_2(\hat{x},\hat{y}) = \frac{1}{2}\partial_{\hat{x}\hat{x}}H_2(0,0)\hat{x}^2 + \partial_{\hat{x}\hat{y}}H_2(0,0)\hat{x}\hat{y} + \frac{1}{2}\partial_{\hat{y}\hat{y}}H_2(0,0)\hat{y}^2 + \text{h.o.t.}, \tag{5}$$

where h.o.t. denotes the higher order terms in x, y, with coefficients depending on μ. Substituting the previous estimates of $\hat{x}$, $\hat{y}$ and then using (4) gives $H_2(\hat{x},\hat{y}) = o(\rho^{-2n})$. Consequently we obtain $II = \rho^{2n} \cdot o(\rho^{-2n})$.

C^1-convergence. (2) gives $G_1(x,y) = A_\mu x^2 + B_\mu xy + C_\mu y^2$ +h.o.t. Substituting $(x,y) = (\lambda^n(\rho^{-n/2}\xi+1), \rho^{-n}\eta)$ into this and then differentiating the result gives $\partial G_1 = O(\rho^{-2n})$, where $\partial = \partial_\xi, \partial_\eta, \partial_\theta$. The same estimates hold for G_2. Hence $\partial\tilde{x} = O(\rho^{-n})$ and $\partial\tilde{y} = O(\rho^{-n})$, and therefore $\partial\hat{x} = o(\rho^{-3n})$. An estimate analogous to that of G_1 gives $\partial H_1 = o(\rho^{-n})$. We obtain $\partial I = \rho^{n/2} \cdot O(\rho^{-n})$.

The estimate of $\partial\hat{x}$ gives $\partial(\hat{x} - a_2^2\rho^{-2n}\eta^2) = o(\rho^{-2n})$. By (b) in Lemma 1.1, we have $\partial_\theta(b_4\hat{y}) = o(\rho^{-2n})$ and $\partial_\xi(b_4\hat{y}) = o(\rho^{-2n}) = \partial_\eta(b_4\hat{y})$. For II, an analogous estimate to that of ∂H_1 (i.e. substituting the formula for $\hat{x}$, $\hat{y}$ into (5) and differentiating the result) we obtain $\partial H_2 = o(\rho^{-2n})$. Hence $\partial II = \rho^{2n} \cdot o(\rho^{-2n})$ holds.

C^s-convergence, $2 \le s \le r$. Let (i,j,k) be such that $2 \le i+j+k \le s$. By the chain rule and $\lambda\rho < 1$, for any function $F = \tilde{x}, \tilde{y}, \hat{x}, G_1, G_2, H_1, H_2$ we have $\partial_\theta^i\partial_\xi^j\partial_\eta^k F = O(\rho^{-2ni - \frac{3}{2}nj - nk})$. This readily yields the desired uniform convergence on I.

For II we also have the uniform convergence, except for the case $(i,j,k) = (0,0,2)$. In this exceptional case, the worst factor in the first term of II is G_1^2. A direct computation gives $\partial_\eta^2(G_1^2) = 2G_1\partial_{\eta\eta}G_1 + 2(\partial_\eta G_1)^2 = o(\rho^{-2n})$. For the second term, Lemma 1.1 gives $\partial_\eta^2(b_4\hat{y}) = b_4\partial_\eta^2\hat{y} = o(\rho^{-2n})$. For the last H_2, it suffices to show that the derivatives of the first three terms in (5) are $o(\rho^{-2n})$. The previous estimates give $\partial_\eta^2\hat{x}^2 = o(\rho^{-2n})$ and $\partial_\eta^2(\hat{x}\hat{y}) = o(\rho^{-2n})$. For the third term, (4) gives $\partial_{\hat{y}\hat{y}}H_2(0,0) \cdot \partial_\eta^2(\hat{y}^2) = o(\rho^{-2n})$. This completes the proof of Proposition 1.1. □

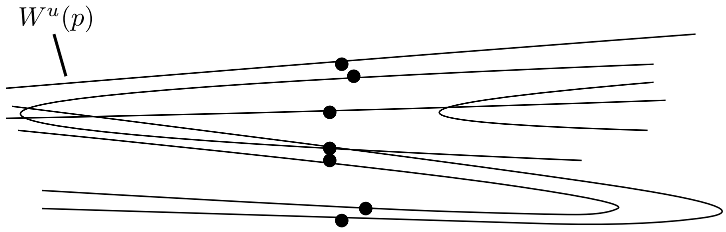

FIGURE 3. critical points on the unstable manifold with self-intersection

1.3. **Nonuniform hyperbolicity for Hénon-like endomorphisms.** We are concerned with a parametrized family $f_a\colon [-2,2]^2 \to \mathbb{R}^2$ of maps of the form

$$f_a\colon (x,y) \mapsto (g_a x, 0) + b \cdot R(a,b,x,y), \tag{6}$$

where (a,b) is close to $(a^*,0)$, and R is bounded, continuous, C^4 in (a,x,y). The g_a is a map on $[-2,2]$ with the following properties:

(A1) $g_a[-1,1] \subset [-1,1]$ for $a \leq a^*$ and $(a,x) \to g_a x$ is C^4;
(A2) g_a has a nonempty critical set $\mathrm{Crit} = \{x_0 \in [-1,1]\colon g'_a x_0 = 0\}$ in $(-1,1)$. We assume Crit does not depend on a and $g''_a x_0 \neq 0$ for each $x_0 \in \mathrm{Crit}$.

For g_{a^*} we assume:

(A3) $\overline{\bigcup_{i\geq 1} g^i_{a^*}(\mathrm{Crit})} \bigcap \mathrm{Crit} = \emptyset$;
(A4) all periodic points of g_{a^*} are hyperbolic repelling;
(A5) by (A3) (A4), $g_{a^*}(\mathrm{Crit})$ belongs to a hyperbolic set K_{a^*}. Let K_a denote the continuation of K_{a^*}, which is a hyperbolic set of g_a. For each $x_0 \in \mathrm{Crit}$, let $x_1(a) = g_a x_0$. Let $r(a)$ denote the point in K_a whose kneading sequence relative to Crit is the same as that of $x_1(a^*)$. We assume for each $x_0 \in \mathrm{Crit}$,

$$p(x_0,a^*) := \frac{dx_1}{da}(a^*) - \frac{dr}{da}(a^*) \neq 0. \tag{7}$$

The next theorem asserts the abundance of non-uniformly hyperbolic parameters, extending the theorem of Benedicks and Carleson [2] to Hénon-like endomorphisms. Let $|\cdot|$ denote the one-dimensional Lebesgue measure.

Theorem B. *For sufficiently small $b > 0$ there exists a set $\Delta = \Delta_b$ of a-values near 0 for which $|\Delta| > 0$ and the following holds for all $f \in \{f_a\colon a \in \Delta\}$: for each fixed saddle p of f with $W^u(p) \subset [-2,2]^2$, there exists a countable set $\mathcal{C} \subset W^u(p)$ near* $\mathrm{Crit} \times \{0\}$ *such that each $\zeta \in \mathcal{C}$ satisfies:*

(a) $\|Df^n(f\zeta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \geq e^{\lambda n}$ *for every $n \geq 0$, where $\lambda > 0$ is a constant which depends only on g_{a^*};*
(b) *ζ admits a tangent direction which is exponentially contracted by both positive and negative iterations.*

Elements of $\mathcal{C}$ are called (dynamically) *critical points* [2, 13]. Each $\zeta \in \mathcal{C}$ is obtained as a limit: there exist a monotone increasing sequence $n_1 < n_2 < \cdots$ of positive integers, and a sequence $\zeta_{n_1}, \zeta_{n_2}, \cdots$ of points on the unstable manifold with $\zeta = \lim_{i\to\infty} \zeta_{n_i}$, and ζ_{n_i} is a *critical point of order n_i* (see Sect.3.2).

For the family (1), it is not hard to show the existence of a positively invariant region which contains the fixed saddle near 1/2 in its interior. Theorem B applies. In addition, along the line of [2] one can show that the closure of the unstable manifold contains a dense orbit. (b) implies that this set is not uniformly hyperbolic, that we call a "chaotic attractor" in the statement of Theorem A. To really deserve the name of attractor, the basin of attraction should have nonempty interior. We do not know if this is the case.

1.4. **Sketch of the contents.** The rest of this paper consists of five sections and one appendix, entirely for the proof of Theorem B. In the first two sections we pursue the study of one fixed map. In the remaining sections we deal with parameter issues. To keep the length of this paper within reasonable bounds, we put the emphasis on those of our arguments which are new or differ non-trivially from previous ones, giving precise references to published computations in [2, 13, 22].

In the context of one-dimensional maps on intervals or the circle, the worst enemy for nonuniform hyperbolicity is the set of critical points. It is now classical [7] that, an exponential growth of derivatives along the orbits of critical points, called Collet-Eckmann condition, implies the existence of a nonuniformly hyperbolic behavior. It is also classical [1, 11] that, by excluding undesirable parameters inductively (looking at the recurrence of the critical points), one can construct a positive measure set of parameters corresponding to nonuniformly hyperbolic behavior.

In the work [2] of pivotal historic importance, Benedicks and Carleson extended their parameter exclusion argument in one dimension [1] to the Hénon family. As the Hénon map is a diffeomorphism, there is no critical point in the usual sense. Nevertheless, they showed that it is possible to define *dynamical critical points* for certain Hénon maps, allowing them to perform a parameter exclusion with some resemblance to the one-dimensional case. At this point, a significant difference from the one-dimensional case is that, the construction of critical points constitutes an integral component of the whole inductive scheme.

For the purpose of presenting a clearer perspective, we elect to recover the one-dimensional scenario to the extent that is possible. We do this in the following steps:

- define (approximations of) critical points explicitly;
- introduce three conditions $(\mathrm{G1})_n$, $(\mathrm{G2})_n$, $(\mathrm{G3})_n$ in terms of derivatives along the orbits of these critical points ("temporal Collet-Eckmann condition for two-dimensional maps");
- show that these conditions to hold for every n ensures the existence of a recognizable source of nonuniform hyperbolicity;
- show that the set of parameters for which this holds has positive Lebesgue measure.

The contents of each section are briefly outlined as follows. In Sect.2 we develop preliminary estimates and constructions, including the (partial) definition of critical points. In Sect.3, under the assumptions of $(\mathrm{G1}\text{-}3)_n$ on certain critical points we develop a procedure for choosing *binding points*, to recover the loss of hyperbolicity due to returns to critical regions. We also show that these assumptions to hold for every n ensures the existence of the set $\mathcal{C}$ as in Theorem B.

In Sect.4 we commence the study of the dependence of critical points on parameter. This preliminary step is important to handle the issue that critical points do not persist when the parameter is varied, because of their dynamical definition. This issue was successfully tackled in [2, 13, 22], by introducing *continuations of critical points*. However, their construction of continuations is deeply rooted in the whole inductive scheme. We introduce continuations (deformations in our terms) in a different way, well-adapted to our critical points. In Sect.5, Sect.6 we construct a parameter set Δ as in Theorem B and show $|\Delta| > 0$. At this point we follow the combinatorics of Tsujii [19, 20] instead of [2, 13, 22], primarily because the extension of this approach is more transparent in our dealing with multiple critical points, and moreover allows us to dispense with a large deviation argument in parameter space altogether.

Proofs of some lemmas originating in [2, 13, 22] necessitate slight adaptations, because of the differences of formulations and the non-invertibility. These proofs are given in appendix, in which we closely follow the ideas or the arguments of those of the published papers.

Acknowledgments. I thank Masato Tsujii for a personal communication which led me to this subject. Most of this work has been done while I was visiting IMPA, Rio de Janeiro, Brazil. I thank Marcelo Viana for his hospitality during this visit.

2. Preliminaries

In this section we develop preliminary estimates and constructions needed for later sections.

2.1. **Hyperbolicity, quadratic behavior and curvature estimate.** For $r > 0$, define

$$I(r) = \bigcup_{x_0 \in \mathrm{Crit}} (x_0 - r, x_0 + r) \times [-\sqrt{b}, \sqrt{b}].$$

The next lemma follows from the properties of the interval map g_{a^*}.

Lemma 2.1. *There exist $c, \lambda_0 > 0$ independent of M, δ such that the following holds for $f = f_a$ with (a, b) close to $(a^*, 0)$: let $z \in [-2, 2] \times [-\sqrt{b}, \sqrt{b}]$ and v be a tangent vector at z with slope $\leq \sqrt{b}$.*

(i) *if $z \in fI(\sqrt{b})$, then $\|Df^j v\| \geq c\|Df^i v\|$ for $0 \leq i < j \leq M$;*

(ii) *if $z, fz, \cdots, f^{n-1}z \notin I(\delta)$, then:*

 (a) $\mathrm{slope}(Df^n v) \leq \sqrt{b}$ *and $\|Df^n v\| \geq c\delta e^{\lambda_0 n}\|v\|$;*

 (b) *if, in addition, $f^n z \in I(\delta)$, then $\|Df^n v\| \geq ce^{\lambda_0 n}\|v\|$.*

2.2. **Constants.** Fix $C_0 > 0$ once and for all so that the norms of all the partial derivatives of $(a, z) \mapsto f_a z$ are bounded by C_0. The letter C is used to denote generic constants which only depends on (f_a).

We are concerned with positive constants $\lambda, \alpha, M, \delta, b$, chosen in this order. Sufficiently small b is chosen last. Some of the purposes of these are the following:

- $\lambda = \frac{99}{100}\lambda_0$ are concerned with rates of growth of derivatives along critical orbits;
- $\alpha \ll 1$ determines the rate of approach to criticalities;
- $M \gg 1$ is the minimal order of critical points, and is chosen so that $2C_0 n e^{-3\alpha n} \leq \log 2$ holds for $n \geq M$;
- $\delta \ll 1$ determines the size of a critical region.

Set $\kappa_0 = C_0^{-10}$, $\theta = \alpha^3$ and $N = \left[\frac{\log 1/\delta}{\theta}\right]$, where the square bracket denotes the integer part. Some of the purposes of these are the following:

- κ_0 is the rate of growth of derivatives needed for various constructions:
- θ bounds the number of critical points needed to be considered at step n of induction for the construction of the parameter set;
- $N \gg M$ is the minimal order of critical points needed to deal with returns to $I(\delta)$.

In the next lemma, proved in Appendix A.1, we assume γ is a *horizontal curve*, namely, a C^2-curve such that the slopes of its tangent directions are $\leq 1/10$ and the curvature is everywhere $\leq 1/10$. For $z \in \gamma$, let $t(z)$ denote any unit vector tangent to γ at z. We assume $\mathrm{slope}(Dft(\zeta)) \geq C/\sqrt{b}$ holds for some $\zeta \in \gamma$. Let e denote any unit vector tangent to $f\gamma$ at $f\zeta$. Split $Dft(z) = A(z)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B(z)e$. Let us agree that for two positive numbers, $a \approx b$ indicates that there exists $C_1 > 0$, $C_2 > 0$ such that $C_1 \leq a/b \leq C_2$.

Lemma 2.2. *For all $z \in \gamma \cap I(\delta)$, $|A(z)| \approx |z - \zeta|$ and $|B(z)| \leq C\sqrt{b}$.*

A version of the next curvature estimate was in [[22], Lemma 2.4] for Hénon-like diffeomorphisms. It is straightforward to check that the same proof works for endomorphisms.

Lemma 2.3. *Let γ be a C^2 curve tangent to a nonzero vector v at z. Let $i \geq 0$ and suppose that $Dfv, \cdots, Df^iv$ are nonzero. Let $\kappa_i(z)$ denote the curvature of $f^i\gamma$ at f^iz. Then*

$$\kappa_i(z) \leq (Cb)^i \frac{\|v\|^3}{\|Df^iv\|^3}\kappa_0(z) + \sum_{j=1}^{i}(Cb)^j \frac{\|Df^{i-j}v\|^3}{\|Df^iv\|^3}.$$

2.3. **Most contracting directions.** Some versions of results in this subsection were obtained in [2, 13, 22]. Although diffeomorphisms are treated in these papers, it is straightforward to check that they hold for endomorphisms. Our presentation closely follows [[22], Section 2.1].

Let M be a 2×2 matrix. Denote by e the unit vector (up to sign) such that $\|Me\| \leq \|Mu\|$ holds for any unit vector u. We call e, when it exists, the *most contracting direction* of M. For a sequence of matrices M_1, $M_2 \cdots$, we use $M^{(i)}$ to denote the matrix product $M_i \cdots M_2M_1$, and e_i to denote the mostly contracting direction of $M^{(i)}$.

Hypothesis for Sect.2.2. The matrices M_i satisfy $|\det M_i| \leq Cb$ and $\|M_i\| \leq C_0$.

Lemma 2.4. ([22] Lemma 2.1) Let $i \geq 2$, and suppose that $\|M^{(i)}\| \geq \kappa^i$ and $\|M^{(i-1)}\| \geq \kappa^{i-1}$ for some $\kappa \geq b^{1/10}$. Then e_i and e_{i-1} are well-defined, and satisfy

$$\|e_i \times e_{i-1}\| \leq \left(\frac{Cb}{\kappa^2}\right)^{i-1}.$$

Corollary 2.1. ([22] Corollary 2.1) *If $\|M^{(i)}\| \geq \kappa^i$ for $1 \leq i \leq n$, then:*
(a) $\|e_n - e_1\| \leq \frac{Cb}{\kappa^2}$*;*
(b) $\|M^{(i)}e_n\| \leq \left(\frac{Cb}{\kappa^2}\right)^i$ *holds for $1 \leq i \leq n$.*

Next we consider for each i a parametrized family of matrices $M_i(s_1, s_2, s_3)$ such that $\|\partial^j \det M_i(s_1, s_2, s_3)\| \leq C_0^i b$, and $|\partial^j M_i(s_1, s_2, s_3)| \leq C_0^i$ for each $0 \leq j \leq 3$. Here, ∂^j represents any one of the partial derivatives of order j with respect to s_1, s_2, or s_3.

Corollary 2.2. ([22] Corollary 2.2) Suppose that $\|M^{(i)}(s_1, s_2, s_3)\| \geq \kappa^i$ for $1 \leq i \leq n$. Then for $j = 1, 2, 3$ and $2 \leq i \leq n$,

$$|\partial^j(e_i \times e_{i-1})| \leq \left(\frac{Cb}{\kappa^{2+j}}\right)^{i-1}, \tag{8}$$

$$\|\partial^j(M^{(i)}e_i)\| \leq \left(\frac{Cb}{\kappa^{2+j}}\right)^{i}. \tag{9}$$

Let $e_1(z)$ denote the most contracting direction of $Df(z)$ when it makes sense. From the form of our map (6), $e_1(z)$ is defined for all $z \notin I(\sqrt{b})$. In view of [[13] pp. 21], we have

$$\text{slope}(e_1) \geq C/\sqrt{b} \quad \text{and} \quad \|\partial e_1\| \leq C\sqrt{b}. \tag{10}$$

We say z is *κ-expanding up to time n*, or simply *expanding*, if there exists a tangent vector v at z and $\kappa \geq b^{1/10}$ such that for every $1 \leq i \leq n$,

$$\|Df^i v\| \geq \kappa^i \|v\|.$$

With a slight abuse of language, we also say v is κ-expanding up to time n. For $n \geq 1$, let $e_n(z)$ denote the most contracting direction of $Df^n(z)$ when it makes sense. From Corollaries 2.1, 2.2 and (10) we get

Corollary 2.3. *If z is κ-expanding up to time n, then* $\text{slope}(e_n) \geq C/\sqrt{b}$ *and* $\|\partial e_n\| \leq \frac{Cb}{\kappa^3}$.

2.4. **Long stable leaves.** A C^2-curve Γ of the form

$$\Gamma = \{(x(y), y)\colon |y| \leq \sqrt{b}, |x'(y)| \leq C\sqrt{b}, |x''(y)| \leq C\sqrt{b}\}.$$

is called a *vertical curve.* By a *vertical strip of radius $r > 0$ around* Γ we mean the region $\{(x, y)\colon |x - x(y)| \leq r, |y| \leq \sqrt{b}\}$. A C^2-distance between two vertical curves is measured by regarding them as C^2-functions on $[-\sqrt{b}, \sqrt{b}]$.

Lemma 2.5. *Let $\kappa \geq C_0^{-10}$. If z is κ-expanding up to time n, then for $1 \leq i \leq n$, the maximal integral curve $\Gamma_i(z)$ of e_i through z contains a vertical curve. In addition, for $1 < i \leq n$, $d_{C^2}(\Gamma_i(z), \Gamma_{i-1}(z)) \leq \left(\frac{Cb}{\kappa^4}\right)^{i-1}$.*

Proof. For the construction of $\Gamma_i(z)$, see [[13] Section 6]. The bound on the C^2-distance follows from this construction and Lemma 2.4, Corollary 2.2. $\square$

By a *long stable leaf of order i through z* we mean the curve $\Gamma_i(z)$ in the statement.

2.5. **Bounded Distortion.** In the next lemma, we assume v is a unit tangent vector at z which is κ-expanding up to time $n \geq M$. Let

$$D_n(v) = e^{-3\alpha n} \min_{i \in [0, n-1]} \min_{j \in [i, n]} \frac{\|Df^j v\|^2}{\|Df^i v\|^3}. \tag{11}$$

Let γ be a C^2 curve tangent to v such that $\text{length}(\gamma) \leq D_n(v)$, and the curvature is everywhere ≤ 1.

Lemma 2.6. (Bounded distortion on properly sized curves) *For all $\xi_1, \xi_2 \in \gamma$ we have*

$$\frac{\|Df^n t(\xi_1)\|}{\|Df^n t(\xi_2)\|} \leq 2 \quad \textit{and} \quad \left| \frac{\|Df^n t(\xi_1)\|}{\|Df^n t(\xi_2)\|} - 1 \right| \leq \frac{|\xi_1 - \xi_2|}{D_n(v)},$$

where $t(\xi_\sigma)$ denotes any unit vector tangent to γ at ξ_σ, $\sigma = 1, 2$.

Proof. For $i \geq 0$, let $v_i = Df^i v$ and $\gamma_i = f^i \gamma$. Let κ_i denote the maximum of the curvature of γ_i. The first inequality would hold if for $0 \leq i < n$,

$$(1 + \kappa_i) \cdot \text{length}(\gamma_i) \leq 2C_0 e^{-3\alpha n} \frac{\|v_{i+1}\|}{\|v_i\|}. \tag{12}$$

Indeed, for all $\xi \in \gamma$ we have

$$\sum_{i=0}^{n-1} \left| \log \frac{\|v_{i+1}\|}{\|v_i\|} - \log \frac{\|Df^{i+1}t(\xi)\|}{\|Df^i t(\xi)\|} \right| \leq 2C_0 \sum_{i=0}^{n-1} \frac{(1+\kappa_i)\text{length}(\gamma_i)}{\frac{\|v_{i+1}\|}{\|v_i\|}},$$

and therefore

$$\log \frac{\|Df^n t(\xi_1)\|}{\|Df^n t(\xi_2)\|} \leq 4C_0^2 n e^{-3\alpha n} \leq \log 2.$$

We prove (12) by induction on i. Let

$$d_n(i) = \min_{j \in [i,n]} \frac{\|v_j\|^2}{\|v_i\|^3}.$$

We have $\text{length}(\gamma_0) \leq D_n(v) d_n(0)^{-1} d_n(0) \leq D_n(v) d_n(0)^{-1} \|v_1\|^2 \leq C_0 e^{-3\alpha n} \|v_1\|$. This and the assumption $\kappa_0 \leq 1$ give (12) for $i = 0$.

Assume (12) holds for $0 \leq i < k$. The choice of M In Sect.2.2 ensures $\frac{\|Df^k t(\xi)\|}{\|Df^k t(\eta)\|} \leq 2$ for all $\xi, \eta \in \gamma$, and therefore

$$\text{length}(\gamma_k) \leq 2 \cdot \|v_k\| \text{length}(\gamma) \leq 2 \cdot D_n(v_0) \|v_k\|.$$

Lemma 2.3 gives $(1+\kappa_k) \cdot \text{length}(\gamma_k) \leq D_n(v_0)\left(I + II + III\right)$, where

$$I = 2\|v_k\|,\ II = \frac{2^2(Cb)^k}{\|v_k\|^2},\ III = 2^8 \sum_{i=1}^{k} (Cb)^i \frac{\|v_{k-i}\|^3}{\|v_k\|^2}.$$

By definition,

$$1 = d_n(k)^{-1} d_n(k) \leq d_n(k)^{-1} \frac{\|v_{k+1}\|^2}{\|v_k\|^3}, \tag{13}$$

and thus

$$I \leq C_0 d_n(k)^{-1} \frac{\|v_{k+1}\|}{\|v_k\|}.$$

Multiplying (13) with the definition of II,

$$II \leq 4(Cb)^k d_n(k)^{-1} \frac{\|v_{k+1}\|^2}{\|v_k\|^5} \leq 4C_0 (Cb)^k b^{\frac{-3k}{4}} d_n(k)^{-1} \frac{\|v_{k+1}\|}{\|v_k\|} \leq b^{\frac{1}{5}} d_n(k)^{-1} \frac{\|v_{k+1}\|}{\|v_k\|},$$

where we have used the assumption on v_0 and $\|Df\| \leq C_0$ for the second inequality.

The most problematic term III is treated as follows. First,

$$d_n(k-i)^{-1} d_n(k-i) \frac{\|v_{k-i}\|^3}{\|v_k\|^2} = d_n(k-i)^{-1} \min_{k-i \leq j \leq n} \frac{\|v_j\|^2}{\|v_k\|^2} \leq d_n(k-i)^{-1} \frac{\|v_{k+1}\|}{\|v_k\|},$$

where the last inequality follows from $\min_{k-i \leq j \leq n} \|v_j\|^2 \leq \|v_k\| \|v_{k+1}\|$. Consequently,

$$III \leq \frac{\|v_{k+1}\|}{\|v_k\|} \sum_{i=1}^{k} (Cb)^i \cdot d_n(k-i)^{-1}.$$

Plugging the three inequalities into the previous one and then using the definition of $D_n(v)$ yields (12) for $i = k$.

For the second inequality, let γ_i' denote the curve in γ_i which connects $f^i\xi_1$ and $f^i\xi_2$. The same reasoning as above, replacing γ_i by γ_i', and $D_n(v)$ by $|\xi_1-\xi_2|$, shows for $0\leq i<n$,

$$(1+\kappa_i)\mathrm{length}(\gamma_i')\leq \frac{2C_0|\xi_1-\xi_2|}{\min_{0\leq j\leq n-1}d_n(j)}. \tag{14}$$

This yields

$$\log\frac{\|Df^nt(\xi_1)\|}{\|Df^nt(\xi_2)\|}\leq\sum_{i=0}^{n-1}(1+\kappa_i)\mathrm{length}(\gamma_i')\leq\frac{2C_0ne^{-3\alpha n}|\xi_1-\xi_2|}{e^{-3\alpha n}\min_{0\leq i\leq n-1}d_n(i)}\leq\frac{|\xi_1-\xi_2|}{D_n(v_0)},$$

and the second inequality holds. □

For $z\in[-2,2]^2$, let us write $v(z)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in T_z\mathbb{R}^2$.

Lemma 2.7. (Bounded distortion in vertical strips) *Let $\kappa\geq C_0^{-10}$, and let z be κ-expanding up to time $n\geq M$. For all ξ_1, ξ_2 in the vertical strip of radius $D_n(v(z))$ around $\Gamma_n(z)$ and for $1\leq i\leq n$,*

$$\frac{\|Df^iv(\xi_1)\|}{\|Df^iv(\xi_2)\|}\leq 3.$$

Proof. Let η_σ denote the point on $\Gamma_n(z)$ with the same y-coordinate as that of ξ_σ $(\sigma=1,2)$. By a result of [[13], Section 6], we have for $1\leq i\leq n$, $\|Df^iv(\eta_1)\|/\|Df^iv(\eta_2)\|\leq 1+\epsilon$, $\epsilon\ll 1$. It follows that $|\xi_\sigma-\eta_\sigma|\leq D_n(v(\xi_\sigma))$. Hence, the desired inequality follows from Lemma 2.6. □

2.6. **Recovering expansion.** Let γ be a horizontal curve in $I(\delta)$ and $n\geq M$. We say $z\in\gamma$ is a *critical point of order n* on γ if:

(i) $f^{i+1}\zeta\in[-2,2]^2$ for $1\leq i<n$ and $\|Df^i(fz)\|\geq c/10$ for $1\leq i\leq n$;
(ii) $e_n(fz)$ is tangent to $Dft(z)$, where $t(z)$ is any unit vector tangent to γ at z.

We now introduce three conditions on derivatives along orbits of critical points, which will be taken as assumptions of induction for the construction of the parameter set Δ. Let ζ be a critical point of order n and assume that $f^{i+1}\zeta\in[-2,2]^2$ for $1\leq i\leq 20n$. For $i\geq 1$, let $w_i(\zeta)=Df^{i-1}(f\zeta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. We say ζ *has good critical behavior up to time $k\geq M$* if the following holds:

(G1) $\|w_i(\zeta)\|\geq e^{\lambda(i-1)}$ for $M\leq i\leq k$;
(G2) $\|w_j(\zeta)\|\geq e^{-2\alpha i}\|w_i(\zeta)\|$ for $M\leq i<j\leq k$;
(G3) there exists a monotone increasing integer-valued function on $\chi\colon[M,20n]\cap\mathbb{N}$ such that for each $j\in[M,20n]$ there exists $\chi(j)\in[(1-\sqrt{\alpha})j,j]$ such that $\|w_{\chi(j)}(\zeta)\|\geq c\delta\|w_i(\zeta)\|$ holds for $1\leq i<\chi(j)$.

Besides the mere exponential growth in (G1), a certain information on oscillations of derivatives are necessary. (G2) is a variant of *basic assumption* in [1, 2]. One implication of (G3) is that the set $\{i\in[1,20n]\colon \mathrm{slope}(w_i)\leq\sqrt{b}\}$ is quite dense in $[1,20n]$.

Hypothesis for the rest of Sect.2.6: ζ is a critical point of order n on a horizontal curve γ, with good critical behavior up to time $20n$.

Under this hypothesis, we establish key analytic estimates. For each $M \leq k \leq 20n$, write D_k for $D_k(w_1(\zeta))$. Write $\Gamma_n(f\zeta) = \{(x_n(y), y)\colon |y| \leq \sqrt{b}\}$. Let

$$V_k = \left\{(x,y)\colon |x - x_n(y)| \leq \frac{1}{2}D_k, |y| \leq \sqrt{b}\right\}.$$

Take a monotone increasing function χ satisfying the condition in (G3). Let v denote any nonzero vector tangent to γ at z. If $fz \in V_k \setminus V_{k+1}$, then we say v is in *admissible position relative to* ζ. Define a *bound period* $p = p(\zeta, z)$ by

$$p = \chi(k),$$

and a *fold period* $q = q(\zeta, z)$ by

$$q = \min\left\{1 \leq i < p\colon |\zeta - z|^{\beta} \cdot \|w_{j+1}(\zeta)\| \geq 1 \text{ for every } i \leq j < p\right\},$$

where

$$\beta = \frac{2\log C_0}{\log 1/b} \ll 1. \tag{15}$$

It is easy to check that q makes sense, by (G1-3) and the assumption on z. If $fz \in V_{20n-1}$, then we say v is in *critical position relative to* ζ.

Proposition 2.1. *Let γ, ζ, z, v be as above.*

(i) *If v is in admissible position relative to ζ and $z \in V_k \setminus V_{k+1}$, then:*
  (a) $\log|\zeta - z|^{-\frac{3}{\log C_0}} \leq p \leq \log|\zeta - z|^{-\frac{3}{\lambda}}$.
  (b) $\log|\zeta - z|^{-\frac{\beta}{\log C_0}} \leq q \leq \log|\zeta - z|^{-\frac{2\beta}{\lambda}}$.
  (c) $\|Df^i v\| \approx |\zeta - z| \cdot \|w_i(\zeta)\|$ *for* $q < i \leq k$;
  (d) $|\zeta - z|\|v\| \leq \|Df^q v\| \leq |\zeta - z|^{1-\beta}\|v\|$;
  (e) $\|Df^p v\| \geq |\zeta - z|^{-1+\frac{\alpha}{\log C_0}}\|v\| \geq e^{\frac{\lambda p}{3}}\|v\|$;
  (f) $\|Df^i v\| < \|v\|$ *for* $1 \leq i \leq q$;
  (g) $\|Df^p v\| \geq (c\delta/10)\|Df^i v\|$ *for* $0 \leq i < p$;
  (h) $|f^i\zeta - f^i z| \leq e^{-2\alpha p}$ *for* $1 \leq i \leq p$:

(ii) *If v is in critical position relative to ζ, then* $\|Df^n v\| \leq e^{-8\lambda n}\|v\|$.

Proof of Proposition 2.1. A central idea follows the well-known line [2, 13, 22]. We split Dfv into the direction of e_k and that of $\binom{1}{0}$, iterate them separately, and put them together after the fold period is expired. We divide the proof of (i) into five steps.

Step 1(Estimate of the horizontal distance between fz and $\Gamma(f\zeta)$). For a point r near $f\zeta$, write

$$r = f\zeta + \xi(r)w_1(\zeta)^T + \eta(r)e_n(f\zeta)^T,$$

where T denotes the transpose. Integrations of the inequalities in Lemma 2.2 along γ from ζ to z give

$$|\xi(fz)| \approx |z - \zeta|^2, \quad |\eta(fz)| \leq C\sqrt{b}|z - \zeta|.$$

Write $fz = (x_0, y_0)$. Let y_1 denote the y-coordinate of $f\zeta$. Since $f\gamma$ is tangent to the vertical curve $\Gamma_n(f\zeta)$ at $f\zeta$,

$$\frac{d\xi(x_n(y), y)}{dy}(y_1) = 0, \quad \left|\frac{d^2\xi(x_n(y), y)}{dy^2}\right| \leq C\sqrt{b}.$$

Then

$$|\xi(x_n(y_0), y_0)| \leq C\sqrt{b}|y_0 - y_1|^2 \leq C\sqrt{b}|\eta(fz)|^2 \leq Cb^{\frac{3}{2}}|\xi(fz)|.$$

Since $|x_0 - x_n(y_0)| = |\xi(fz) - \xi(x_n(y_0), y_0)|$, we get

$$|x_0 - x_n(y_0)| \approx |z - \zeta|^2. \tag{16}$$

Step 2(Proofs of (a), (b)*).* (G1) gives

$$D_k \leq \|w_{k-1}(\zeta)\|^{-1} \leq e^{-\lambda(k-2)}.$$

(G2) and the definition (11) give

$$D_{k+1} \geq e^{-3\alpha(k+1)} C_0^{-k} \min(c^2, e^{-4\alpha k}) \geq C_0^{-2k}.$$

By the assumption on z and (16),

$$C_0^{-2k} \leq D_{k+1} \leq C|\zeta - z|^2 \leq CD_k \leq Ce^{-\lambda(k-2)}.$$

Taking logs we obtain

$$\frac{1}{\log C_0} \log|\zeta - z|^{-1} \leq k \leq \frac{2}{\lambda} \log|\zeta - z|^{-1}. \tag{17}$$

The definition of p and (G3) give $(1 - \sqrt{\alpha})k \leq p \leq k$, and thus (a) holds.

(G1) and the definition of q give

$$e^{\lambda(q-1)} \leq \|w_q(\zeta)\| < |\zeta - z|^{-\beta}.$$

Taking logs and then rearranging the result yields the upper estimate in (b). The lower estimate follows from

$$1 \leq |\zeta - z|^\beta \|w_{q+1}(\zeta)\| \leq |\zeta - z|^\beta C_0^q.$$

Step 3(Existence of contractive fields). Write $\Gamma_k(f\zeta) = \{(x_k(y), y)\colon |y| \leq \sqrt{b}\}$. Using the assumption on z, Lemma 2.7 and $k \leq 20n$ we have

$$|x_0 - x_k(y_0)| \leq |x_0 - x_n(y_0)| + |x_n(y_0) - x_k(y_0)| \leq \frac{1}{2}D_k + (Cb)^{\frac{k}{20}} \leq D_k.$$

Hence, the contractive fields $e_1, \cdots, e_k$ are well-defined in a neighborhood containing fz, $f\zeta$ and all the estimates in Sect.2.3 are in place.

Step 4(Correctness of splitting). Split $Dfv = A\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + Be_k(fz)$. Write $e_n(z) = \left(\begin{smallmatrix}\cos\theta_n(z)\\ \sin\theta_n(z)\end{smallmatrix}\right)$ and $\rho \cdot Dfv = \left(\begin{smallmatrix}\cos\psi\\ \sin\psi\end{smallmatrix}\right)$, $\theta_n, \psi \in [0, \pi)$, $\rho > 0$ being the normalizing constant. Lemma 2.2 gives

$$|\theta_n(f\zeta) - \psi| \approx \rho^{-1}|\zeta - z|\|v\| \gg |\zeta - z|.$$

Thus $|\theta_n(f\zeta) - \theta_n(fz)| \leq C|f(\zeta) - f(z)| \ll |\theta_n(f\zeta) - \psi|$, which implies

$$|\theta_n(fz) - \psi| \approx |\theta_n(f\zeta) - \psi|.$$

We also have $|\theta_n(fz) - \theta_k(fz)| \leq (Cb)^n \ll |\zeta - z|$, where the first inequality follows from Lemma 2.4 and the last one from the assumptionon z. Hence $|\theta_k(fz) - \psi| \approx |\theta_n(fz) - \psi|$. Consequently we obtain

$$|A| \approx \rho|\theta_k(fz) - \psi| \approx \rho|\theta_n(fz) - \psi| \approx \rho|\theta_n(f\zeta) - \psi| \approx |\zeta - z|\|v\|. \tag{18}$$

Step 5(Proofs of (c-h)). Let $q < i \leq k$. We have

$$|A| \cdot \|Df^{i-1}(fz) \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \| \geq C|\zeta - z|\|v\| \cdot \|w_i(\zeta)\| \geq C|\zeta - z|^{1-\beta}\|v\|,$$

where we have used Lemma 2.6 and (18) for the first inequality; the definition of q for the second. We also have

$$|B| \cdot \|Df^{i-1}e_k(fz)\| \leq (Cb)^i\|v\| \leq (Cb)^q\|v\| \leq |\zeta - z|^{\frac{3}{2}}\|v\|,$$

where we have used the lower estimate of q and (15) for the last inequality. These two estimates yield (c). (c) for $i = q$ and the definition of q gives the upper estimate in (d). The lower estimate follows from $\|w_q(\zeta)\| \geq 1$.

Proof of (e). We have

$$\|Df^k v\| \geq C\|w_k(\zeta)\| \cdot |\zeta - z|\|v\| \geq |\zeta - z|^{-1}e^{-10\alpha k}\|v\|,$$

where we have used (18) for the first inequality; $\|w_k(\zeta)\||\zeta - z|^2 \geq e^{-9\alpha k}$ for the second inequality which follows from (G2) and the assumption on ζ. Hence we obtain

$$\|Df^p v\| \geq C_0^{-\sqrt{\alpha}k}\|Df^k v\| \geq |\zeta - z|^{-1+\frac{\alpha}{\log C_0}}\|v\|.$$

For the last inequality we have used the second inequality in (17). This yields the first inequality. Substituting $|\zeta - z|^{-1} \geq e^{\lambda k/2}$ into this yields the second one.

Proof of (f). Let $1 \leq i \leq q$. The definition of q and (G2) give

$$|A| \cdot \|Df^{i-1}(fz)\| \leq |\zeta - z| \cdot \|v\| \cdot \|w_q(\zeta)\| \frac{\|w_i(\zeta)\|}{\|w_q(\zeta)\|} \leq |\zeta - z|^{1-2\beta}\|v\| \ll \|v\|.$$

The other component of $Df^i v$ is exponentially contracted, and hence (f) holds.

Proof of (g). The ratio of the two quantities in (18) can be made arbitrarily close to a uniform constant, by choosing sufficiently small δ and b. With this and the bounded distortion in Proposition 2.4, for $q < i < p$,

$$\frac{\|Df^p v\|}{\|Df^i v\|} \geq \frac{1}{10} \cdot \frac{\|w_p(\zeta)\|}{\|w_i(\zeta)\|} \geq \frac{c\delta}{10},$$

where the last inequality follows from (G3). For $1 \leq i \leq q$, using (e,f),

$$\frac{\|Df^p v\|}{\|Df^i v\|} \geq e^{\frac{\lambda p}{3}} \frac{\|v\|}{\|Df^i v\|} \geq e^{\frac{\lambda p}{3}}.$$

Proof of (h). Let $z' = (x_n(y_0), y_0)$. (16) gives $|fz - fz'| \leq D_k(\zeta)$, and thus $|f^i z - f^i z'| \leq C\|w_i(\zeta)\||fz - fz'| \leq Ce^{-3\alpha p}$. Here, we have used Lemma 2.6 for the first inequality and the definition of $D_k(\zeta)$ for the second. We also have $|f^i\zeta - f^i z'| \leq (Cb)^{i-1}|f\zeta - fz'|$. Hence $|f^i\zeta - f^i z| \leq |f^i\zeta - f^i z'| + |f^i z' - f^i z| \leq Ce^{-3\alpha p} \leq e^{-2\alpha p}$.

Finally we prove (ii). Split $Dfv = A \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + Be_n(f\zeta)$. Lemma 2.2 gives $|A| \approx |\zeta - z|\|v\|$. The assumption on z and $\|Df^{n-1}(fz)\| \leq C\|w_n(\zeta)\|$ give $\|A \cdot Df^{n-1}(fz) \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \| \leq e^{-9\lambda n}$. We also have

$$\begin{aligned}\|B \cdot Df^{n-1}(fz)e_n(f\zeta)\| &\leq \|Df^{n-1}(fz)e_n(fz)\| + \|Df^{n-1}(fz)(e_n(f\zeta) - e_n(fz))\| \\ &\leq (Cb)^n + C\|w_n(\zeta)\||\zeta - z| \leq e^{-9\lambda n}.\end{aligned}$$

This completes the proof of Proposition 2.1. $\square$

3. Choice of binding points

To recover the loss of hyperbolicity due to returns to $I(\delta)$, we carry on the same strategy as in [2, 13, 22]: *look for a suitable critical point and use it as a guide.* Such a critical point, if exists, is called a *binding point.* The aim of this section is to establish the choice of binding points.

3.1. Binding points for returns near the boundary of $I(\delta)$.

Let $I^{(j)}(\delta)$, $j = 1, 2, \cdots, \sharp\mathrm{Crit}$ denote the components of $I(\delta)$. Using Corollary 2.2 (and borrowing some arguments in Sect.4), it is possible to construct for each $I^{(j)}(\delta)$ a smooth map $a \to c_j(a)$ defined in a neighborhood of a^* such that:

- $c_j = c_j(a)$ is a critical point of order N of f_a with good critical behavior;
- $\left|\frac{dc_j}{da}\right| \leq C.$

These critical points will be chosen as binding points for returns near the boundary of $I(\delta)$. For returns deep inside $I(\delta)$, we construct other critical points and choose them as binding points.

3.2. Construction of new critical points.

A C^2 curve is called a $C^2(b)$*-curve* if the slopes of all its tangent vectors are $\leq \sqrt{b}$ and the curvature is everywhere $\leq \sqrt{b}$. The next two lemmas, the proofs of which are given in appendix, are used to construct new critical points around the existing ones. For corresponding versions, see: [2] p.113, Lemma 6.1; [13] Sect.7A, 7B; [22] Lemma 2.10, 2.11.

Lemma 3.1. *Let γ be a $C^2(b)$-curve in $I(\delta)$ parameterized by arc length and such that $\gamma(0)$ is a critical point of order n. Suppose that:*

(i) *$\gamma(s)$ is defined for $s \in [-b^{\frac{n}{4}}, b^{\frac{n}{4}}]$;*
(ii) *there exists $m \in [n/3, 20n]$ such that $\|Df^i(f\gamma(0))\| \geq c$ for $1 \leq i \leq m$.*

There exists $s_0 \in [-b^{\frac{n}{4}}, b^{\frac{n}{4}}]$ such that $\gamma(s_0)$ is a critical point of order m on γ.

Next we consider two $C^2(b)$-curves γ_1, γ_2 in $I(\delta)$ parametrized by arc length, in a way that the x-coordinate of $\gamma_1(0)$ coincide with that of $\gamma_2(0)$. Let $t_\sigma(s)$ denote any unit vector tangent to γ_σ at $\gamma_\sigma(s)$, $\sigma = 1, 2$.

Lemma 3.2. *Let γ_1, γ_2 be as above and suppose that:*

(i) *$\gamma_1(s)$, $\gamma_2(s)$ are defined for $s \in [-\varepsilon^{\frac{n}{2}}, \varepsilon^{\frac{n}{2}}]$, $\varepsilon \leq C_0^{-5}$;*
(ii) *$\gamma_1(0)$ is a critical point of order n on γ_1 and $\|Df^i(f\gamma_1(0))\| \geq c$ for $1 \leq i \leq n$;*
(iii) *$|\gamma_1(0) - \gamma_2(0)| \leq \varepsilon^n$ and $\mathrm{angle}(t_1(0), t_2(0)) \leq \varepsilon^n$.*

There exists $s_0 \in [-\varepsilon^{\frac{n}{2}}, \varepsilon^{\frac{n}{2}}]$ such that $\gamma_2(s_0)$ is a critical point of order n on γ_2.

Remark 3.1. The corresponding versions to Lemma 3.2 in [2, 13, 22] assume that γ_1 and γ_2 are pairwise disjoint. The smallness of the angle as in (iii) automatically follows from this. We allow γ_1 to intersect γ_2, and therefore need to take the smallness of the angle as an independent assumption.

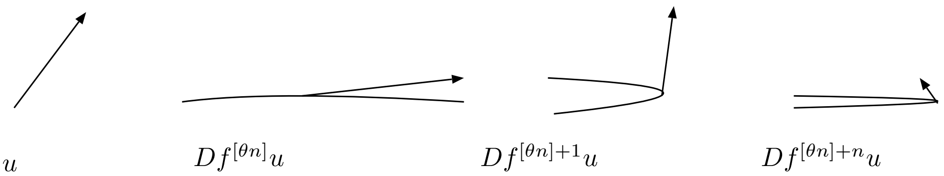

FIGURE 4. The evolution of u under iteration. the horizontal segment tangent to $Df^{[\theta n]}u$ indicates γ, and the curves indicate images of γ.

3.3. **Hyperbolic times and nice critical points.** Let[1]

$$r_0 = c \cdot \min\left\{\frac{ce^{\lambda_0}}{10}, 1\right\}. \tag{19}$$

Definition 3.1. (Hyperbolic times)*Let v be a tangent vector at z and let $m \geq 1$. We say v is r-regular up to time m if for $0 \leq i < m$,*

$$\|Df^m v\| \geq r\delta \|Df^i v\|.$$

We say $\mu \in [0, m]$ is an m-hyperbolic time of v if $Df^{\mu+i}v$ is $\kappa_0^{\frac{1}{2}}$-expanding up to time $m - \mu$.

The next lemma, the proof of which is given in appendix, ensures the existence of hyperbolic times. See [2] Lemma 6.6, [13] Lemma 9.1, [22] Claim 5.1 for related issues.

Lemma 3.3. *Let $m \geq \log(1/\delta)$ and suppose that a tangent vector v at z is $r_0/10$-regular up to time m. There exist $s \geq 2$ and a sequence $\mu_1 < \mu_2 < \cdots < \mu_s$ of m-hyperbolic times of v such that:*

(a) *$Df^{\mu_j}v$ is $\kappa_0^{\frac{1}{4}}$-expanding up to time $m - \mu_j$;*
(b) *$1/16 \leq (m - \mu_{j+1})/(m - \mu_j) \leq 1/4$ for $1 \leq j \leq s - 1$;*
(c) *$0 \leq \mu_1 < m/2$ and $m - \log(1/\delta) \leq \mu_s \leq m - \log(1/\delta)/2$.*

Definition 3.2. (Nice critical points) Let γ be a horizontal curve in $I(\delta)$. A critical point ζ of order $n \geq N$ on γ is *nice* if (cf. FIGURE 4):

(C1) $\|Df^i(f\zeta)\| \geq c$ for $1 \leq i \leq n$;
(C2) there exist $\xi \in f^{-[\theta n]}\zeta$ and a unit vector u at ξ such that:
- u is $\kappa_0^{\frac{1}{3}}$-expanding and $r_0/10$-regular, both up to time $[\theta n]$;
- $Df^{[\theta n]}u$ is tangent to γ.

3.4. **Binding procedure.** For the rest of this section we assume m, n are integers with $m \geq \log(1/\delta)$, $n \geq N$, and:

(H1) each nice critical point ζ of order $\leq n$ has a good critical behavior;
(H2) a tangent vector v at z is r_0-regular up to time m, and $f^m z \in I^{(j)}(\delta) \subset I(\delta)$.

We indicate how to choose a binding point for $Df^m v$. First of all, if $Df^m v$ is in admissible position relative to c_j, then we choose c_j as a binding point. If $Df^m v$ is in critical position

[1] In the case $g_a = 1 - ax^2$, one can take $c = 1$, $r_0 = 1/10$.

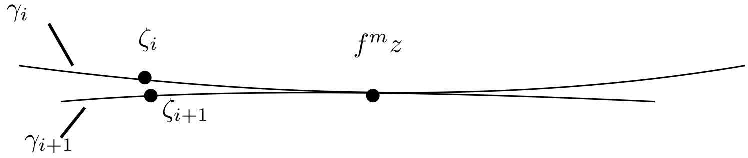

FIGURE 5. critical points on $C^2(b)$-curves

relative to c_j, then in view of Lemma 3.3, fix once and for all a sequence $\mu_1 < \mu_2 < \cdots < \mu_s$ of m-hyperbolic times of v satisfying

$$(20) \qquad m-\mu_1 \leq \theta n, \quad \frac{1}{2}\log(1/\delta) \leq m-\mu_s \leq \log(1/\delta), \quad \frac{1}{16} \leq \frac{m-\mu_{i+1}}{m-\mu_i} \text{ for } 1 \leq i < s.$$

Correspondingly, fix once and for all a sequence $n \geq n_1 > \cdots > n_s$ of integers such that

$$(21) \qquad m-\mu_i = [\theta n_i] \text{ for } 1 \leq i \leq s.$$

Let l_i denote the straight segment of length $2\kappa_0^{3\theta n_i}$ centered at $f^{\mu_i}z$ and tangent to $Df^{\mu_i}v$. Let $\gamma_i = f^{m-\mu_i}l_i$. By Lemma 2.6, the distortion of $f^j|l_i$ $(1 \leq j \leq m-\mu_i)$ is uniformly bounded and consequently γ_i is a $C^2(b)$-curve extending to both sides around $f^m z$ to length $\geq \kappa_0^{4\theta n_i}$. In particular, γ_s extends to both sides around $f^m z$ to length $\geq \kappa_0^{4\log(1/\delta)}$. By Lemma 3.2, there exists a nice critical point of order n_s on γ_s, which we denote by ζ_s. If $Df^m v$ is in admissible position relative to ζ_s, then we choose ζ_s as a binding point. Otherwise we appeal to the next

Lemma 3.4. (Existence of nice critical points of higher order) *Let $i \in [2,s]$ and suppose that there exists a nice critical point ζ_i of order n_i on γ_i relative to which $Df^m v$ is in critical position. Then there exists a nice critical point ζ_{i-1} of order n_{i-1} on γ_{i-1} relative to which $Df^m v$ is in admissible or critical position.*

Recursively using Lemma 3.4, we end up with two cases as below. The choice of binding points splits into these two cases:

Case 1; there exist $j \in [1,s]$, and for each $i \in [j,s]$ a nice critical point ζ_i of order n_i on γ_i such that $Df^m v$ is in critical position relative to $\zeta_s, \cdots, \zeta_{j+1}$, and in admissible position relative to ζ_j. In this case, choose ζ_j as a binding point.

Case 2; there exists a nice critical point of order n_1 on γ_1 relative to which $Df^m v$ is in critical position. In this case, choose ζ_1 as a binding point.

As a corollary we obtain

Corollary 3.1. *Let ζ_0 denote the binding point for $Df^m v$ and let k_0 denote the order of ζ. If $Df^m v$ is in admissible position relative to ζ_0 and $\zeta_0 \notin \{c_1, \cdots, c_{\sharp\mathrm{Crit}}\}$, then*

$$-\log|\zeta_0 - z| \approx k_0.$$

3.5. **Recovering hyperbolicity.** Having established the choices of binding points, we now apply Proposition 2.1. If $Df^m v$ is in admissible position relative to the binding point, then all the estimates in Proposition 2.1(i) are in place: the loss of hyperbolicity and regularity suffered from the return to $I(\delta)$ are recovered at the end of the bound period.

In addition, in this case one can repeat the binding procedure in the following manner. Write $m = m_1$. Let ζ denote any binding point for $Df^m v$ and let $p_1 = p(\zeta, f^m z)$ denote the bound period. (e,g) Proposition 2.1 implies that v is $c/10$-regular up to time $m_1 + p_1$.

Let $m_2 \geq m_1 + p_1$ denote the smallest such that $f^{m_2}z \in I(\delta)$. By Lemma 2.1, v is r_0-regular up to time m_2. Subsequently the binding procedure is performed once again, replacing $m, f^m z, Df^m v$ by $m_2, f^{m_2}v, Df^{m_2}v$ correspondingly.

In this way, one may define integers

$$m_1 < m_1 + p_1 \leq m_2 < m_2 + p_2 \leq m_3 < \cdots$$

inductively as follows: for $k \geq 1$, p_k is the bound period of $f^{m_k}z$; n_{k+1} is the smallest $j \geq m_k + p_k$ such that $f^j z \in I(\delta)$. (Note that an orbit may return to $I(\delta)$ during its bound periods, i.e. (m_k) are not the only return times to $I(\delta)$.) This decomposes the orbit of z into segments corresponding to time intervals $(m_k, m_k + p_k)$ and $[m_k + p_k, m_{k+1}]$, during which we describe the orbit of z as being"bound" and "free" states respectively; m_k are called *free return times of z*.

Proof of Lemma 3.4. Let $\gamma = f^{\mu_i - \mu_{i-1}} l_{i-1}$. Parametrize γ by arc length and assume $\gamma(0) = f^{\mu_i}z$. Then $\gamma(s)$ is well-defined for $s \in [-\kappa_0^{60\theta n_i}, \kappa_0^{60\theta n_i}]$, because

$$(1/2)\kappa^{3\theta n_{i-1}} \kappa_0^{\frac{1}{4}(\mu_i - \mu_{i-1})} \geq (1/2)\kappa_0^{(3+\frac{1}{4})(m-\mu_{i-1})} \geq \kappa_0^{60\theta n_i}.$$

The last inequality follows from (20).

We use "$\cdot$" to denote the differentiation on s. Let $\varphi(s) = \text{angle}(e_{m-\mu_{i-1}}(\gamma(s)), \dot\gamma(s))$. For all $s \in [-\kappa_0^{500\theta n_i}, \kappa_0^{500\theta n_i}]$, we show

$$\varphi(s) \geq \kappa_0^{\frac{1}{3}\theta n_i}. \tag{22}$$

We finish the proof of Lemma 3.4 assuming this estimate. Let $\xi = f^{-(m-\mu_i)}\zeta_i \cap l_i$. We have

$$|\xi - f^{\mu_i}z| \leq 2\kappa_0^{-\frac{1}{4}(m-\mu_i)} |\zeta_i - f^m z| \ll \int_0^{\kappa_0^{500\theta n_i}} \varphi(s) ds.$$

For the second inequality we have used (22), $\theta \ll 1$ and the assumption that $Df^m v$ is in critical position relative to ζ_i. This implies that the long stable leaf of order $m - \mu_i$ through ξ intersects γ. Let $\gamma(s_0)$ denote any point of the intersection. Then

$$|\zeta_i - f^{m-\mu_i}\gamma(s_0)| \leq (Cb)^{m-\mu_i}. \tag{23}$$

By the bounded distortion and Sublemma 3.1 below,

$$\text{angle}(Df^{m-\mu_i}t(\xi), Df^{m-\mu_i}\dot\gamma(s_0)) \leq (Cb)^{m-\mu_i}. \tag{24}$$

Here, $t(\xi)$ is any unit vector tangent to l_i at ξ. By Lemma 3.2, there exists a critical point of order n_i on $\gamma_{i-1} = f^{m-\mu_{i-1}} l_{i-1}$, denoted by ζ'_{i-1}, such that $|\gamma(s_0) - \zeta'_{i-1}| \leq (Cb)^{\frac{1}{2}(m-\mu_i)}$. By the bounded distortion, the exponential growth in (G1) for the orbit of ζ_i is passed onto that of ζ'_{i-1} up to time $20n_i$, which is $> n_{i-1}$ by (20). Lemma 3.1 yields a critical point of order n_{i-1} on γ_{i-1}, which we denote by ζ_{i-1}. We claim that ζ_{i-1} is a nice critical point of order n_{i-1} on γ_{i-1}. Indeed, (C1) holds as a consequence of the above exponential growth, and (C2) follows from the bounded distortion and Lemma 3.3.

It is left to prove (22). We have $\varphi(s) \geq \varphi(0) - I - II$, where

$$I = \text{angle}(e_{m-\mu_{i-1}}(\gamma(s)), e_{m-\mu_{i-1}}(\gamma(0))), \quad II = \text{angle}(\dot\gamma(0), \dot\gamma(s)).$$

We estimate the right-hand-side term by term. Note that $\dot\gamma(0)$ is collinear to $Df^{\mu_i}v$. Since μ_i is an m-hyperbolic time, $\varphi(0)$ is bounded from below as follows. Splitting $\dot\gamma(0)$ into the direction of $e_{m-\mu_i}(\gamma(0))$ and the direction orthogonal to it,

$$\kappa_0^{\frac{1}{4}(m-\mu_i)} \leq \frac{\|Df^m v\|}{\|Df^{\mu_i}v\|} \leq \sqrt{(Cb)^{m-\mu_i} + C_0^2 \sin^2\varphi(0)},$$

which implies $\varphi(0) \geq \kappa_0^{\frac{2}{7}\theta n_i}$.

To conclude, it suffices to show $\max(I, II) \ll \varphi(0)$. This holds for I from Lemma 2.4 and $|s| \leq \kappa_0^{5000\theta n_i}$. Regarding II, as l_i is a straight segment, Lemma 2.3 gives

$$|\ddot\gamma(s)| \leq \sum_{j=\mu_{i-1}}^{\mu_i-1} (Cb)^{\mu_i-j} \frac{\|Df^j v\|^3}{\|Df^{\mu_i}v\|^3}.$$

We have

$$\|Df^j v\| \leq C_0^{j-\mu_{i-1}} \|Df^{\mu_{i-1}}v\| \leq C_0^{\mu_i-\mu_{i-1}} \|Df^{\mu_{i-1}}v\|$$

and

$$\|Df^{\mu_i}v\| \geq \kappa_0^{\frac{1}{4}(\mu_i-\mu_{i-1})} \|Df^{\mu_{i-1}}v\|.$$

Replacing these in the fraction and the using Lemma 3.3, $|\ddot\gamma(s)| \leq C_0^{3(\mu_i-\mu_{i-1})} \leq C_0^{3(m-\mu_{i-1})} \leq \kappa_0^{-4800\theta n_i}$. Therefore $III \leq |\ddot\gamma(s)||s| \ll \varphi(0)$ holds.

Sublemma 3.1. *If $|f^i\xi - f^i\eta| \leq (\frac{Cb}{\kappa})^i$ for $0 \leq i < n$, then for any nonzero tangent vectors v, w at ξ, η,*

$$\text{angle}(Df^n v, Df^n w) \leq \left(\frac{Cb}{\kappa}\right)^{n-1} \sum_{i=0}^{n-1} \frac{\|Df^i v\|}{\|Df^n v\|} \frac{\|Df^i w\|}{\|Df^n w\|}.$$

Proof. From the proof of [[22] Claim 5.3]. □

3.6. **Source of nonuniform hyperbolicity: the set $\mathcal{C}$.** In this subsection we assume that every critical point has good critical behavior, and show that this assumption implies the occurrence of nonuniformly hyperbolic behavior. The issue on the abundance of parameters for which this assumption holds is adduced to later sections.

Proposition 3.1. *The following statement holds for all $f = f_a$ with (a, b) sufficiently close to $(a^*, 0)$; if all nice critical points of f of order $\geq N$ have good critical behavior, then for each fixed saddle p of f with $W^u(p) \subset [-2, 2]^2$, there exists a countable set $\mathcal{C} \subset W^u(p)$ near* Crit $\times \{0\}$ *such that each $\zeta \in \mathcal{C}$ satisfies:*

(a) $\|Df^n(f\zeta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \geq e^{\lambda n}$ *for every $n \geq 0$;*
(b) *ζ admits a tangent direction which is exponentially contracted by both positive and negative iterations.*

Proof. Fix a fundamental domain $\mathcal{F}$ in $W^u_{\rm loc}(p)$, and let $z \in \mathcal{F}$. Let $t(z)$ denote a nonzero unit vector tangent to $W^u_{\rm loc}(p)$ at z. Define a sequence $n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq n_3 < \cdots$ inductively as follows: n_1 is the smallest such that $f^{n_1}z \in I(\delta)$ and p_1 is the bound period of $f^{n_1}z$; $n_k \geq n_{k-1} + p_{k-1}$ is the smallest such that $f^{n_k}z \in I(\delta)$, and p_k is the bound period of $f^{n_k}z$. From the fact that p is a fixed saddle, it follows that this sequence is defined indefinitely, or else there exists an integer m such that $Df^m t(z)$ is in critical position relative to critical points of arbitrarily high order. If the latter case occurs, we let $f^m z \in \mathcal{C}$.

The set $\mathcal{C}$ thus defined is a countable set, because the defining map $\mathcal{F} \to \mathcal{C}$ is surjective and any point of the inverse image of the map is isolated in $\mathcal{F}$. (a)(b) follow from the fact that each element of $\mathcal{C}$ is accumulated by nice critical points for which (G1) holds. $\square$

4. Parameter dependence of critical points

In the remaining three sections we construct a set Δ of positive Lebesgue measure for which every critical point has good critical behavior. The construction of Δ is done by induction. At each step we exclude all parameters from further consideration for which some nice critical points may not have good critical behavior and necessary analytic estimates fail for proceeding to the next step. At this point we face an intrinsic difficulty, not present in one-dimension: *critical points do not persist when the parameter is varied, because they are dynamically defined.* In this section, we resolve this difficulty by introducing a parametrized family of critical points.

4.1. **Deformations of quasi critical points.** We relax the definition of nice critical points as follows.

Definition 4.1. (Quasi critical points) Let γ be a $C^2(b)$-curve in $I(\delta)$. Let $n \geq N$, and let ζ be a critical point of order n on γ. We say ζ is a *quasi critical point of order n on γ* if there exist $\xi \in f^{-[\theta n]}\zeta$ and a unit vector u at ξ such that:

(i) $Df^{[\theta n]}u$ is tangent to γ;
(ii) u is $\kappa_0^{\frac{1}{2}}$-expanding up to time $[\theta n]$.

Hypothesis for the rest of Sect.4.1: ζ is a quasi critical point of order $n \geq N$ on a $C^2(b)$-curve γ such that:

$(\mathrm{Q1})_n$ $\|Df^i(f\zeta)\| \geq c$ for $1 \leq i \leq n$;
$(\mathrm{Q2})_n$ there exist $\xi \in f^{-[\theta n]}\zeta$ and a unit vector u at ξ such that:
- $Df^{[\theta n]}u$ is tangent to γ;
- u is $\kappa_0^{\frac{1}{3}}$-expanding and $r_0/160$-regular, both up to time $[\theta n]$.

Let $H = [-2, 2] \times \{\sqrt{b}\}$. Let r denote the point of intersection between H and the long stable leaf of order $[\theta n]$ through ξ. Let $l \subset H$ denote the horizontal of length $2\kappa_0^{3\theta n}$ centered at r. Let

$$I_n(\hat{a}) = [\hat{a} - \kappa_0^n, \hat{a} + \kappa_0^n].$$

We now introduce a C^3 map $a \in I_n(\hat{a}) \mapsto \zeta(a)$ called a *deformation of* ζ. Here, $\zeta(a)$ is a quasi critical point of order n of f_a given by the next

Proposition 4.1. *The following holds for all $a \in I_n(\hat{a})$:*

(a) *$f_a^{[\theta n]}l$ is a $C^2(b)$-curve extending both sides around $f_a^{[\theta n]}r$ to length $\geq \kappa_0^{5\theta n}$;*
(b) *there exists a quasi critical point $\zeta(a)$ of order n on $f_a^{[\theta n]}l$. In addition, $|\zeta - \zeta(\hat{a})| \leq (Cb)^{\frac{\theta n}{4}}$;*
(c) *$a \in I_n(\hat{a}) \mapsto \zeta(a)$ is C^3, and there exists $C > 1$ such that for $j = 1, 2, 3$, $\left\| \frac{d^j}{da}\zeta(a) \right\| \leq C^{\theta n}$.*

Before entering a proof of this proposition we prove the next lemma on iterates of $f_{\hat{a}}$.

Lemma 4.1. *$f_{\hat a}^{[\theta n]}l$ is a $C^2(b)$-curve extending both sides around $f_{\hat a}^{[\theta n]}r$ to length $\geq \kappa_0^{4\theta n}$. Moreover, there exists a quasi critical point $\zeta(\hat a)$ of order n on $f_{\hat a}^{[\theta n]}l$. Furthermore, $|\zeta-\zeta(\hat a)|\leq (Cb)^{\frac{\theta n}{4}}$.*

Proof. Write f for $f_{\hat a}$. For a point $p\in[-2,2]^2$, let us write $v(p)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. The next comparison of derivatives is used: $\|Df^iv(\xi)\|\approx\|Df^iv(z)\|$ for $z\in l$ and $1\leq i\leq[\theta n]$. This follows from the bounded distortion in a neighborhood containing ξ and l in consequence of the assumption on ξ and Lemma 2.7. Lemma 4.1 follows from this and $\|Df^{[\theta n]}v(r)\|\geq\frac{1}{2}\|Df^{[\theta n]}v(\xi)\|\geq\frac{1}{2}\|Df^{[\theta n]}u\|\geq\frac{1}{2}\kappa_0^{\frac{\theta n}{3}}$.

By Lemma 2.3, the curvature of $f^{[\theta n]}l$ is bounded from above by

$$C\sum_{i=0}^{[\theta n]-1}(Cb)^{[\theta n]-i}\frac{\|Df^iv(\xi)\|^3}{\|Df^{[\theta n]}v(\xi)\|^3}.$$

We evaluate the fraction as follows. For $0\leq i\leq\theta n/2$,

$$\frac{\|Df^iv(\xi)\|}{\|Df^{[\theta n]}v(\xi)\|}\leq C_0^i\kappa_0^{-\theta n/2}\leq\kappa_0^{-2([\theta n]-i)}.$$

For $\theta n/2<i<[\theta n]$, split $u=A\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+Be_i(\xi)$. An analogous reasoning to the proof of (22) shows $\|Df^iu\|\approx|A|\cdot\|Df^iv(\xi)\|$. By the bounded distortion and (Q2)$_n$,

$$\frac{\|Df^iv(\xi)\|}{\|Df^{[\theta n]}v(\xi)\|}\leq C\cdot\frac{\|Df^iu\|}{\|Df^{[\theta n]}u\|}\leq\frac{C}{\delta}.$$

Replacing all these in the summand, we obtain the curvature is everywhere $\leq\sqrt{b}$. The second inequality with $i=[\theta n]-1$ implies that the slopes of the tangent directions of $f^{[\theta n]}l$ are $\leq\sqrt{b}$.

Subemma 3.1 gives angle$(Df^{[\theta n]}u,Df^{[\theta n]}v(r))\leq(Cb)^{\frac{\theta n}{2}}$, and also $|f^{[\theta n]}\xi-f^{[\theta n]}r|\leq(Cb)^{\frac{\theta n}{2}}$. By Lemma 3.2, there exists a critical point of order n on $f^{[\theta n]}l$. The bounded distortion and (Q2)$_n$ together imply that this critical point is a quasi critical point of order n. The last assertion follows from Lemma 3.2. ☐

Proof of Proposition 4.1. Let $z\in l$, $a\in I_n(\hat a)$ and $1\leq i\leq[\theta n]$. Then

$$\|Df_{\hat a}^iv(r)-Df_a^iv(r)\|\leq\kappa_0^{\frac{9n}{10}}.$$

(Q2)$_n$ and the bounded distortion give

$$\|Df_{\hat a}^iv(r)\|\geq C\|Df_{\hat a}^iv(\xi)\|\geq C\kappa_0^{\frac{i}{2}}.$$

Hence, $\|Df_{\hat a}^iv(r)\|\approx\|Df_a^iv(r)\|$ holds. The bounded distortion in Lemma 2.6 gives $\|Df_a^iv(r)\|\approx\|Df_a^iv(z)\|$, and consequently $\|Df_{\hat a}^iv(r)\|\approx\|Df_a^iv(z)\|$. From this and the proof of Lemma 4.1 we obtain (a).

We divide the rest of the proof of Proposition 4.1 into three steps. In the first two steps we prove (b). In the last step we prove (c).

Step 1(Construction of a critical point of $f_{\hat a}$ on $f_a^{[\theta n]}l$). Parametrize l by arc length s. For $a\in I_n(\hat a)$, let $x(a)\in l$ denote the point such that the x-coordinate of $f_a^{[\theta n]}x(a)$ coincides with that of $\zeta(\hat a)$. Then

$$|f_a^{[\theta n]}x(a)-f_{\hat a}^{[\theta n]}x(\hat a)|\leq 2|f_a^{[\theta n]}x(\hat a)-f_{\hat a}^{[\theta n]}x(\hat a)|\leq\kappa_0^{\frac{9n}{10}},\tag{25}$$

$$|f_a^{[\theta n]}x(a) - f_a^{[\theta n]}x(\hat{a})| \leq |f_a^{[\theta n]}x(a) - f_{\hat{a}}^{[\theta n]}x(\hat{a})| + |f_{\hat{a}}^{[\theta n]}x(\hat{a}) - f_a^{[\theta n]}x(\hat{a})| \leq 2\kappa_0^{\frac{9n}{10}}.$$

By the $C^2(b)$-property,

$$\mathrm{angle}(Df_a^{[\theta n]}v(x(a)), Df_a^{[\theta n]}v(x(\hat{a}))) \leq 2\sqrt{b}\kappa_0^{\frac{9n}{10}}.$$

The proof of Lemma 4.1 implies

$$\mathrm{angle}(Df_a^{[\theta n]}v(x(\hat{a})), Df_{\hat{a}}^{[\theta n]}v(x(\hat{a}))) \leq \kappa_0^{\frac{9n}{10}}.$$

Hence

$$\mathrm{angle}(Df_a^{[\theta n]}v(x(a)), Df_{\hat{a}}^{[\theta n]}v(x(\hat{a}))) \leq 2\kappa_0^{\frac{9n}{10}}. \tag{26}$$

(25) (26) permit us to use Lemma 3.2 to construct a critical point of $f_{\hat{a}}$ of order n on $f_a^{[\theta n]}l$, which we denote by z, located within $\kappa_0^{\frac{4n}{5}}$ of $\zeta(\hat{a})$. By the bounded distortion, $\|Df_{\hat{a}}^i(f_{\hat{a}}z)\| \geq c/3$ holds for $1 \leq i \leq n$.

Step 2(Construction of a quasi critical point of f_a on $f_a^{[\theta n]}l$). Let γ denote the $C^2(b)$-curve in $f_a^{[\theta n]}l$ which extends both sides around z to length $\kappa_0^{\frac{n}{2}}$. Since $|f_{\hat{a}}z - f_a z| \leq C_0\kappa_0^n$, the bounded distortion gives, for $1 \leq i \leq n$,

$$\|Df_{\hat{a}}^i v(f_a z)\| \geq (1/2)\|Df_{\hat{a}}^i v(f_{\hat{a}}z)\| \geq c/6.$$

As $a \in I_n(\hat{a})$,

$$\|Df_a^i v(f_a z)\| \geq \|Df_{\hat{a}}^i v(f_a z)\| - \|Df_a^i(f_a z) - Df_{\hat{a}}^i(f_a z)\| \geq c/6 - \kappa_0^{\frac{9n}{10}}.$$

Namely, $f_a z$ is expanding up to time n under the iteration of Df_a. By Proposition 2.4 and $\mathrm{diam}(f_a\gamma) \leq C\kappa_0^{\frac{n}{2}}$, the most contracting direction of Df_a^i, denoted by $e_{a,i}$, is well-defined in a neighborhood of $f_a\gamma$.

Parametrize γ by arc length and assume $\gamma(0) = z$. Let $t(s)$ denote any unit vector tangent to γ at $\gamma(s)$. Split

$$Df_{\hat{a}}t(s) = A(s)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B(s)e_{\hat{a},n}(f_{\hat{a}}z),$$
$$Df_a t(s) = A'(s)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B'(s)e_{a,n}(f_a\gamma(s)).$$

These two equalities and $\|Df_a(\gamma(s)) - Df_{\hat{a}}(\gamma(s))\| \leq C|\hat{a} - a|$ altogether imply

$$A'(s) = A(s) + B(s)\cos\theta_{\hat{a}}(f_{\hat{a}}z) - B'(s)\cos\theta_a(f_a\gamma(s)) + R,$$

$$0 = B(s)\sin\theta_{\hat{a}}(f_{\hat{a}}z) - B'(s)\sin\theta_a(f_a\gamma(s)) + R,$$

where $e_{a,n}(\cdot) = \begin{pmatrix}\cos\theta_a(\cdot)\\ \sin\theta_a(\cdot)\end{pmatrix}$ and $|R| \leq C|\hat{a} - a| \leq C\kappa_0^n$. Letting $\psi(s) = |\theta_{\hat{a}}(f_{\hat{a}}z) - \theta_a(f_a\gamma(s))|$,

$$|B(s) - B'(s)| \leq C\psi(s) + C|R| \quad \text{and} \quad |A(s) - A'(s)| \leq C\psi(s) + C|R|.$$

From the results in Sect.2.3,

$$\psi(s) \leq |\theta_{\hat{a}}(f_{\hat{a}}z) - \theta_{\hat{a}}(f_a\gamma(s))| + |\theta_{\hat{a}}(f_a\gamma(s)) - \theta_a(f_a\gamma(s))| \leq C\sqrt{b}\left(|s| + |\hat{a} - a|\right).$$

Lemma 2.2 gives $|A(\pm\kappa_0^{\frac{n}{2}})| \approx \kappa_0^{\frac{n}{2}}$, $A(\kappa_0^{\frac{n}{2}})A(-\kappa_0^{\frac{n}{2}}) < 0$ and $|B(s)| \leq C\sqrt{b}$, and hence

$$|A(\pm\kappa_0^{\frac{n}{2}}) - A'(\pm\kappa_0^{\frac{n}{2}})| \leq C\sqrt{b}\kappa_0^{\frac{n}{2}} + C\kappa_0^n < |A(\pm\kappa_0^{\frac{n}{2}})|.$$

This implies $A'(\kappa_0^{\frac{n}{2}})A'(-\kappa_0^{\frac{n}{2}}) < 0$, and therefore $A'(s) = 0$ has a solution. Lemma 2.2 implies that this solution is unique, and by definition, it corresponds to a critical point of f_a of order n on γ, denoted by $\zeta(a)$. By construction, $\zeta(a)$ is a quasi critical point of f_a of order n.

Step 3(Derivative estimates). Parametrize l by arc length s. Let $\varrho(a)$ denote the unique parameter such that

$$\zeta(a) = f_a^{[\theta n]} l(\varrho(a)). \tag{27}$$

Consider the unit vector $F(s, a) = \rho \cdot Df_a^{[\theta n]+1} v(l(s))$, where $\rho > 0$ is the normalizing constant. Let $G(s, a)$ denote the most contracting direction of Df_a^n at $f_a^{[\theta n]+1} l(s)$, so that

$$F(\varrho(a), a) - G(\varrho(a), a) = 0.$$

Let $v_0 = Df_a^{[\theta n]} v(l(\varrho(a)))$ and $v_1 = Df_a v_0$. Let κ_0 denote the curvature of $f^{[\theta n]} l$ at $\zeta(a)$. We claim

$$\kappa_0 \geq C\|v_0\|^2 / \|v_1\|^2, \tag{28}$$

and

$$\|\partial_a F\| \geq \kappa_0^{-5\theta n}, \quad \|\partial_s F\| = \kappa_0 \|v_1\| \geq \|v_0\|^2 \|v_1\|^{-1} \gg \|v_0\|,$$

$$\|\partial_a G\| \leq C\sqrt{b}, \quad \|\partial_s G\| \leq C\sqrt{b}\|v_0\|,$$

where all the partial derivatives are taken at $(\varrho(a), a)$. The factor $\sqrt{b}$ in the upper estimate of $\|\partial_s G\|$ comes from (10) and Corollary 2.2. Hence $\|\partial_s (F - G)\| \geq C\kappa_0^{\frac{\theta n}{2}}$ holds. The implicit function theorem yields

$$\left| \frac{d}{da} \varrho \right| \leq \kappa_0^{-9\theta n}.$$

Differentiating (27) with a and using this we obtain the desired bound of $\left|\frac{d}{da}\zeta\right|$. Higher order derivatives are bounded in the same way.

It is left to prove (28). Write $\gamma_0 = f_a^{[\theta n]} l$. Parametrize γ_0 by arc length s so that $\gamma(0) = \zeta(a)$, and let $\gamma_1(s) = f\gamma_0(s)$. Let "$\cdot$" denote the differentiation with respect to s. We have $\ddot{\gamma}_1(0) = Df(\gamma_0(0))\ddot{\gamma}_0(0) + X\dot{\gamma}_0(0)$, where

$$Df(\gamma_0(0)) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } X = \begin{pmatrix} \langle \nabla A, \dot{\gamma}_0(0) \rangle & \langle \nabla B, \dot{\gamma}_0(0) \rangle \\ \langle \nabla C, \dot{\gamma}_0(0) \rangle & \langle \nabla D, \dot{\gamma}_0(0) \rangle \end{pmatrix}.$$

From the form of our map (6) and the fact that γ_0 is $C^2(b)$, we have $\|\ddot{\gamma}_0(0)\| \leq C\sqrt{b}$ and $|\langle \nabla A, \dot{\gamma}_0(0) \rangle| \geq C > 0$. In addition, all the other entries of X are $\leq Cb$ in norm. Hence $\|\ddot{\gamma}_1(0)\| \geq C > 0$ and $\mathrm{slope}(\ddot{\gamma}_1(0)) \leq C\sqrt{b}$ hold. Since $\mathrm{slope}(\dot{\gamma}_1(0)) \geq C/\sqrt{b}$, the curvature is

$$\frac{\|\dot{\gamma}_1(0) \times \ddot{\gamma}_1(0)\|}{\|\dot{\gamma}_1(0)\|^3} \geq \frac{C}{\|\dot{\gamma}_1(0)\|^2}.$$

This proves (28). $\square$

4.2. **Uniform derivative estimates.** From this point on, we use "$\cdot$" to denote the a-derivatives. Since the construction of deformations of quasi critical points of order n involve n iterations, and now n is arbitrary, the next uniform bounds on derivatives of deformations are highly nontrivial.

Proposition 4.2. *Let ζ be a quasi critical point of order $n \geq N$ of $f_{\hat{a}}$ on a $C^2(b)$-curve γ. Assume:*

(i) $\|Df^i(f\zeta)\| \geq 2c$ *for* $1 \leq i \leq n$;

(ii) *there exist* $\xi \in f^{-[\theta n]}\zeta$ *and a unit vector* u *at* ξ *such that:*

- $Df^{[\theta n]}u$ *is tangent to* γ;
- u *is* $\kappa_0^{\frac{i}{3}}$*-expanding and* $r_0/40$*-regular, both up to time* $[\theta n]$.

For the deformation $a \in I_n(\hat{a}) \mapsto \zeta(a)$ *and* $j = 1, 2$ *we have*

$$\|\dot{\zeta}(a)\| \leq \kappa_0^{-10\log(1/\delta)}.$$

Proof. We divide the proof into three steps. First, in a slightly different way from Proposition 4.1 we construct a smooth map $a \in I_n(\hat{a}) \to z(a)$ such that $z(a)$ is a quasi critical point of order n of f_a. Next, we repeat similar constructions for lower orders. Finally we put these together and complete the proof.

Step 1(Construction of a parametrized quasi critical point of order n). Let γ denote the straight segment of length $\kappa_0^{5\theta n}$ centered at ξ and tangent to u.

Lemma 4.2. *For all* $a \in I_n(\hat{a})$ *we have:*

(a) $|\log \|Df^i_{\hat{a}}(\xi)u\| - \log \|Df^i_a(\xi)u\|| \leq 1$ *for* $1 \leq i \leq [\theta n]$;

(b) $f_a^{[\theta n]}\gamma$ *is a* $C^2(b)$*-curve extending both sides around* $f_a^{[\theta n]}\xi$ *to length* $\geq \kappa_0^{6\theta n}$;

(c) *there exists a quasi critical approximation* $z(a)$ *of order* n *on* $f_a^{[\theta n]}\gamma$;

(d) *for all* $\eta \in \gamma$,

$$|f_a^i\eta - f_{\hat{a}}^i\xi| \leq \kappa_0^{4\theta n} \quad 0 \leq i \leq [\theta n]. \tag{29}$$

Proof. (a-c) follow from slight modifications of the arguments in Sect.4.1, 4.1. The Hausdorff distance between $f_{\hat{a}}^i\gamma$ and $f_a^i\gamma$ is $\leq \kappa_0^{4\theta n}$, and (d) follows. $\square$

Step 2(Construction of parametrized quasi critical points of lower order). In view of the assumption $n \geq N$ and Lemma 3.3, fix once and for all a maximal sequence $0 = \mu_1 < \mu_2 < \cdots < \mu_s$ of $[\theta n]$-hyperbolic times of the tangent vector u at ξ under the iteration of $f_{\hat{a}}$. Correspondingly, fix once and for all a sequence $n =: n_1 > n_2 > \cdots > n_s$ of integers such that $[\theta n_i] = [\theta n] - \mu_i$ holds for $1 \leq i \leq s$. Let $\xi(a) \in \gamma$ be such that $f_a^{[\theta n]}\xi(a) = z(a)$. Let $\gamma_i(a)$ denote the straight segment of length $2\kappa_0^{3\theta n_i}$ centered at $f_a^{\mu_i}\xi(a)$ and tangent to $f_a^{\mu_i}\gamma$.

Lemma 4.3. *For every* $1 \leq i \leq s$ *and all* $a \in I_n(\hat{a})$ *we have:*

(a) $f_a^{[\theta n_i]}\gamma_i(a)$ *is a* $C^2(b)$*-curve extending both sides around* $z(a)$ *to length* $\geq \kappa_0^{6\theta n_i}$;

(b) *there exists a quasi critical point* $z_i(a)$ *of order* n_i *on* $f_a^{[\theta n_i]}\gamma_i(a)$ *such that*

$$|z_i(a) - z(a)| \leq \sum_{k=1}^{i} b^{\frac{\theta n_k}{5}}. \tag{30}$$

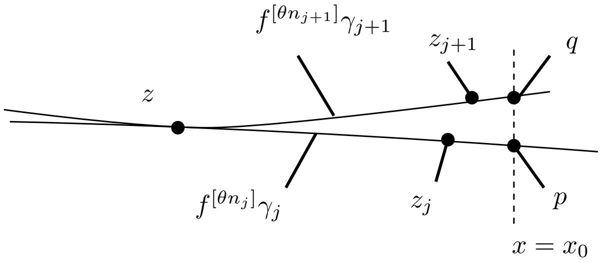

FIGURE 6

Proof. (a) follows from a slight modification of the proof of Lemma **??**. We prove (b) by induction on i. The argument is parameter-independent. So, let us suppress a from notation and write $z(a) = z$, $z_i(a) = z_i$ and so on.

(b) for $i = 1$ follows from the fact $z_1 = z$. Assume (b) for $i = j \geq 1$. The lower estimate of length in (a) for $i = j$ permits us to use Lemma 3.1 to construct a critical point of order n_{j+1} on $f^{[\theta n_j]}\gamma_j$, denoted by p, such that

$$|z_j - p| \leq (Cb)^{\frac{n_{j+1}}{4}}. \tag{31}$$

We regard the two $C^2(b)$-curves $f^{[\theta n_j]}\gamma_j$, $f^{[\theta n_{j+1}]}\gamma_{j+1}$ as graphs of functions $y_j(x)$, $y_{j+1}(x)$ correspondingly. Let x_0 be such that $p = (x_0, y_j(x_0))$. The assumption of the induction gives $\text{length}(f^{[\theta n_j]}\gamma_j) \gg |z - p|$. Hence, $y_{j+1}(x_0)$ makes sense. Let $q = (x_0, y_{j+1}(x_0))$. Let $s \in \gamma$ be such that $f^{[\theta n_j]}s = p$.

The bounded distortion on γ gives $|f^{\mu_{j+1}}\xi - s| \leq 2\kappa_0^{-\frac{1}{3}([\theta n_{j+1}]}|z - p| \leq (Cb)^{\frac{1}{10}\theta n_{j+1}}$. From this and the lower estimate of the length of γ_{j+1}, it follows that the long stable leaf of order $[\theta n_{j+1}]$ through s intersects γ_{j+1}. Then $|y_j(x_0) - y_{j+1}(x_0)| \leq (Cb)^{\theta n_{j+1}}$, and Sublemma 3.1 gives $|y_j'(x_0) - y_{j+1}'(x_0)| \leq (Cb)^{\frac{\theta n_{j+1}}{2}}$. By Lemma 3.2, there exists a quasi critical point of order n_j on γ_{j+1}, denoted by z_{j+1}, such that $|z_{j+1} - q| \leq (Cb)^{\frac{\theta n_{j+1}}{4}}$. Consequently we obtain

$$\begin{aligned} |z_{j+1} - z| &\leq |z_{j+1} - q| + |q - p| + |p - z_j| + |z_j - z| \\ &\leq (Cb)^{\frac{\theta n_{j+1}}{4}} + (Cb)^{\frac{\theta n_{j+1}}{2}} + (Cb)^{\frac{n_{j+1}}{4}} + \sum_{k=1}^{j} b^{\frac{\theta n_k}{5}} \leq \sum_{k=1}^{j+1} b^{\frac{\theta n_k}{5}}. \end{aligned}$$

This proves (b) for $i = j + 1$. □

Step 3(Overall estimates). Put $a = \hat{a}$ in Lemma 4.3. Then we obtain a sequence $\zeta_1, \cdots, \zeta_s$ of quasi critical points of $f_{\hat{a}}$, of order $n_1 > \cdots > n_s$ correspondingly. By the initial assumption on ζ, ξ, u, $(\mathrm{Q1})_{n_i}$, $(\mathrm{Q2})_{n_i}$ holds for ζ_i, for each $i \in [1, s]$. Hence, the deformation $a \in I_{n_i}(\hat{a}) \mapsto \zeta_i(a)$ of ζ_i is well-defined by virtue of Proposition 4.1. As $n_1 = n$, $\zeta_1 = \zeta$, and $\zeta_1(a) = \zeta(a)$ holds for all $a \in I_n(\hat{a})$.

Lemma 4.4. *For each $i \in [1, s]$ and for all $a \in I_{n_i}(\hat{a})$, $|\zeta_i(a) - z_i(a)| \leq (Cb)^{\frac{\theta n_i}{4}}$.*

We finish the proof of Proposition 4.2 assuming the conclusion of this lemma. We appeal to the next

Lemma 4.5. (Hadamard) *Let $g \in C^2[0, L]$ be such that $|g| \leq M_0$ and $|g''| < M_2$. If $4M_0 < L^2$ then $|g'| \leq \sqrt{M_0}(1 + M_2)$.*

Write $\zeta_i(a) = \zeta_i$, $\dot{\zeta}_i(a) = \dot{\zeta}_i$, $\ddot{\zeta}_i(a) = \ddot{\zeta}_i$ and $z_i(a) = z_i$. Proposition 4.1 gives $|\ddot{\zeta}_{i+1} - \ddot{\zeta}_i| \leq 2\kappa_0^{-3\theta n_i}$. Lemma 4.3 and Lemma 4.4 give

$$|\zeta_{i+1} - \zeta_i| \leq |\zeta_{i+1} - z_{i+1}| + |\zeta_i - z_i| + |z_{i+1} - z_i| \leq (Cb)^{\frac{\theta n_i}{5}}.$$

This permits us to use Lemma 4.5 to get $\|\dot{\zeta}_{i+1} - \dot{\zeta}_i\| \leq (Cb)^{\frac{\theta n_i}{6}}$. Summing this over all $1 \leq i < s$ and $\|\dot{\zeta}_s\| \leq \kappa_0^{-3\theta n_s} \leq \kappa_0^{3\log\delta}$ from Proposition 4.1,

$$\|\dot{\zeta}\| \leq \|\dot{\zeta}_s\| + \sum_{i=1}^{s-1} \|\dot{\zeta}_{i+1} - \dot{\zeta}_i\| \leq \kappa_0^{4\log\delta}.$$

For the second order derivative estimate, use Lemma 4.5 with respect to $\dot{\zeta}_{i+1} - \dot{\zeta}_i$ together with the third order derivative estimate in Proposition 4.1.

It is left to prove Lemma 4.4. To this and we need some notation. Let $\xi_i(a) \in \gamma_i(a)$ be such that $f_a^{[\theta n_i]}\xi_i(a) = z_i(a)$. Let $a, a' \in I_n(\hat{a})$. Let $x_i(a, a')$ denote the point of intersection between H and the long stable leaf of f_a of order $[\theta n_i]$ through $\xi_i(a')$. Let $\delta_i(a)$ denote the horizontal of length $2\kappa_0^{3\theta n_i}$ centered at $x_i(a, a)$. Analogously to the proof of Lemma 4.1, it is possible to show that $f_a^{[\theta n_i]}\delta_i(a)$ is a $C^2(b)$-curve, and there exists a critical point $\bar{z}_i(a)$ of order n_i on $\delta_i(a)$ such that $|\bar{z}_i(a) - z_i(a)| \leq (Cb)^{\frac{\theta n_i}{4}}$.

To conclude, it suffices to show $\bar{z}_i(a) = \zeta_i(a)$. Let $I = |x_i(\hat{a}, \hat{a}) - x_i(a, \hat{a})|$ and $II = |x_i(a, \hat{a}) - x_i(a, a)|$. Corollary 2.2 gives $I \leq C|\hat{a} - a| \leq 2\kappa_0^n$. Meanwhile we have $II \leq C|\xi_i(\hat{a}) - \xi_i(a)| \leq C\kappa_0^{4\theta n}$. Here, the first inequality follows from the Lipschitz continuity of $e_{[\theta n_i]}$, and the second from (d) in Lemma 4.2. We obtain

$$|x_i(\hat{a}, \hat{a}) - x_i(a, a)| \leq C\kappa_0^{4\theta n_i}. \tag{32}$$

By the construction of the deformation, there exists a horizontal $l_i \subset H$ of length $2\kappa_0^{3\theta n_i}$ centered at $x_i(\hat{a}, \hat{a})$ such that $f_a^{[\theta n_i]} l_i$ is a $C^2(b)$-curve on which $\zeta_i(a)$ lies. By (32), l_i intersects δ_i. Therefore $f_a^{[\theta n_i]}(l_i \cup \delta_i)$ is a $C^2(b)$-curve as well, on which lie two critical points $\bar{z}_i(a)$ and $\zeta_i(a)$. As they are of order n_i, they coincide with each other. □

5. Parameter exclusion I: preliminaries

In this last two sections we define the parameter set Δ in Theorem B and show that it has positive Lebesgue measure. In this section we do some preliminary works.

The definition of Δ is inductive: at step n, we define a parameter set Δ_n by excluding from Δ_{n-1} all those undesirable parameters for which some nice critical points do not behave in a good manner, in a possible violation of (G1-3). We set $\Delta = \bigcap_{n\geq 0} \Delta_n$. In Sect.5.1 we give a formal definition of Δ_n.

To conclude $|\Delta| > 0$, the main step is to show that $|\Delta_{n-1} \setminus \Delta_n|$ decreases exponentially in n. Our strategy is briefly outlined as follows. We first decompose $\Delta_{n-1} \setminus \Delta_n$ into a finite number of subsets, based on certain combinatorics describing *itineraries* of critical points. We then estimate the measure of each subset separately, and unify them at the end. In Sect.5.2, 5.3 we introduce an integral part of this combinatorics.

5.1. **Definition of parameter sets.** We give a formal inductive definition of Δ_n. Choose small $\varepsilon > 0$, so that if b is small, then for any $f \in \{f_a \colon a \in [a^* - 2\varepsilon, a^* - \varepsilon]\}$ and any critical point of ζ of f we have:

(a) $\|w_i(\zeta)\| \geq e^{\lambda(i-1)}$ for $M \leq i \leq 20N$;
(b) $\|w_j(\zeta)\| \geq e^{-2\alpha i}\|w_i(\zeta)\|$ for $M \leq i < j \leq 20N$.

This choice is feasible by the fact that any critical point is contained in $I(\sqrt{b})$. Set $\Delta_n = [a^* - 2\varepsilon, a^* - \varepsilon]$ for $0 \leq n \leq N$.

Let $n > N$, $a \in \Delta_{n-1}$ and suppose that every nice critical point of f_a of order $< n$ has a good critical behavior. Let $20(n-1) \leq m < 20n$. We say a nice critical point ζ of f_a of order $\geq n$ satisfies $(G)_m$ if:

(i) the orbit $f\zeta, f^2\zeta, \cdots, f^m\zeta$ into is decomposed into bound and free segments in the sense of Sect.3.4;
(ii) let $n_1 < n_2 < \cdots < n_s \leq m$ denote all the free return times of ζ, with $z_1, \cdots, z_s$ the corresponding binding points. They are of order $< n$ and

$$\sum_{1 \leq i \leq s} \log|f^{n_i}\zeta - z_i| \geq -\alpha m. \tag{33}$$

For $n > N$, define Δ_n to be the set of all $a \in \Delta_{n-1}$ for which every nice critical point of order $\geq n$ satisfies $(G)_{20n-1}$. In other words,

$$\Delta_{n-1} \setminus \Delta_n = \left\{ \begin{array}{r} a \in \Delta_{n-1} \colon\ (G)_m \text{ fails for some } m \in [20(n-1), 20n) \\ \text{and some nice critical point of order} \geq n \text{ of } f_a \end{array} \right\}.$$

The next proposition indicates that, for parameters in Δ_n, nice critical points of order n can be used as binding points, and thus allows us to proceed to the definition of Δ_{n+1}.

Proposition 5.1. *Let $n > N$, $a \in \Delta_{n-1}$ and let ζ be a critical point of order $\geq n$ of f_a. If $(G)_{20n-1}$ holds for ζ, then:*

(a) $\|w_i(\zeta)\| \geq e^{\lambda(i-1)}$ *for* $M \leq i \leq 20n$,
(b) $\|w_j(\zeta)\| \geq e^{-2\alpha i}\|w_i(\zeta)\|$ *for* $M \leq i < j \leq 20n$;
(c) *if ζ is of order n, then it has good critical behavior.*

Let $f \in \{f_a \colon a \in \Delta\}$. By the definition of Δ and Proposition 5.1, every critical point of f has good critical behavior. Then Proposition 3.1 ensures the existence of the set $\mathcal{C}$ as in Theorem B. Hence, to complete the proof of Theorem B it is left to show $|\Delta| > 0$.

Proof of Proposition 5.1. We divide the proof into three steps. In the first two steps we show (G1-2) for every $20(n-1) < k \leq 20n$. Lastly we show (G3).

Step 1(Proof of (G1)). We begin with the elementary case where there is no return to $I(\delta)$ before time k. In this case, Lemma 2.1 and $c\delta e^{\alpha(k-1)} \geq 1$ from the definition of N give $\|w_k(\zeta)\| \geq c\delta e^{(\lambda_0-\alpha)(k-1)}e^{\alpha(k-1)} \geq e^{\lambda(k-1)}$, and in addition, $\|w_k(\zeta)\| \geq c\delta e^{\lambda_0(k-i)}\|w_i(\zeta)\| \geq c\delta e^{\alpha(k-i)}\|w_i(\zeta)\| \geq e^{-\alpha i}\|w_i(\zeta)\|$ for $i < k$.

Proceeding to the general case, let $n_1 < \cdots < n_s$ denote all the free return times of ζ before k, with $p_1, \cdots, p_s$, $q_1, \cdots, q_s$ the corresponding bound and fold periods. Proposition 2.1 and condition (G) give

$$\sum_{i=1}^{s} p_i \leq \frac{3}{\lambda}\alpha(k-1). \tag{34}$$

The chain rule gives

$$\|w_{n_s+p_s}(\zeta)\| = \|w_{n_1}(\zeta)\| \prod_{l=1}^{s-1} \frac{\|w_{n_{l+1}}(\zeta)\|}{\|w_{n_l+p_l}(\zeta)\|} \prod_{l=1}^{s} \frac{\|w_{n_l+p_l}(\zeta)\|}{\|w_{n_l}(\zeta)\|},$$

where

$$\|w_{n_1}(\zeta)\| \geq \delta^{-1}e^{\lambda n_1}, \quad \frac{\|w_{n_{l+1}}(\zeta)\|}{\|w_{n_l+p_l}(\zeta)\|} \geq ce^{\lambda_0(n_{l+1}-n_l-q_l)}, \quad \frac{\|w_{n_l+p_l}(\zeta)\|}{\|w_{n_l}(\zeta)\|} \geq c^{-1}.$$

The first inequality holds for (a, b) sufficiently close to $(a^*, 0)$; the second follows from Lemma 2.1; the last from Proposition 2.1. Putting all these together,

$$\|w_{n_s+p_s}(\zeta)\| \geq (c\delta)^{-1}e^{\lambda_0\left(n_s+p_s-\sum_{i=1}^{s} p_i\right)}. \tag{35}$$

If $f^k\zeta$ is bound, namely $n_s + p_s > k$, then using $C_0^{-p_s} \geq C_0^{\frac{3\alpha(k-1)}{\lambda}}$,

$$\|w_k(\zeta)\| \geq C_0^{-p_s}\|w_{n_s+p_s}(\zeta)\| \geq e^{\lambda_0\left(-\left(\frac{\log C_0}{\lambda_0}+1\right)\frac{3\alpha}{\lambda}(k-1)+k\right)} \geq e^{\lambda(k-1)},$$

where we have used (34) for the third inequality. If $f^k\zeta$ is free, namely $n_s + p_s \leq k$, then Proposition 2.1 gives $\|w_k(\zeta)\| \geq c\delta e^{\lambda(k-n_s-p_s)}\|w_{n_s+p_s}(\zeta)\|$. Combining this with (35) we obtain $\|w_k(\zeta)\| \geq e^{\lambda_0(k-\sum p_i)} \geq e^{\lambda_0(k-\alpha k)} \geq e^{\lambda(k-1)}$, and hence (G1) holds.

Step2(Proof of (G2)). We deal with five cases separately.

Case I: both $f^i\zeta$ and $f^j\zeta$ are free. Suppose that no return takes place in $[i, k]$. This case can be covered by the argument in the beginning of the proof. Otherwise, we split the orbit into free and bound segments. Using Lemma 2.1 for each free segment and Lemma 2.1 for each bound segment we have

$$\|w_j(\zeta)\| \geq c\delta e^{\frac{\lambda}{3}(j-i)}\|w_i(\zeta)\| \geq e^{-\alpha i}\|w_i(\zeta)\|. \tag{36}$$

The last inequality is because $c\delta e^{\frac{\lambda j}{3}} \geq 1$ because j is large as there is a return time.

Case II: $f^i\zeta$ is free and $j \in (n_l+q_l, n_l+p_l)$ for some $l \in [1, s]$. Let z denote the binding point for $f^{n_l}\zeta$. Then

$$\begin{aligned}\|w_j(\zeta)\| &\geq C|f^{n_l}\zeta - z|e^{\lambda(j-n_l)}\|w_{\hat{n}}(\zeta)\| \geq Ce^{\frac{\lambda}{3}(j-i)-\alpha n_l}\|w_i(\zeta)\| \\ &\geq Ce^{\frac{\alpha}{2}j-\frac{3}{2}\alpha i}e^{(\frac{\lambda}{3}-\frac{3}{2}\alpha)(j-i)}\|w_i(\zeta)\| \geq e^{-\frac{3}{2}\alpha i}\|w_i(\zeta)\|,\end{aligned}$$

where the first inequality is because j is out of fold period; for the second inequality we have used (36) from time i to n_l (δ is dropped by Lemma 2.1) and $|f^{n_l}\zeta - z| \geq e^{-\alpha n_l}$ from (G); $n_l < j$ for the third; the last inequality is because j is large.

Case III: $f^i\zeta$ is free and $j \in [n_l + 1, n_l + q_l]$ for some $l \in [1, s]$. The upper estimate of fold periods in Proposition 2.1 and condition (G) give $\|w_j(\zeta)\| \geq C_0^{-\frac{2\alpha\tilde{\alpha}}{\lambda}j}\|w_{n_l+q_l}(\zeta)\|$. For the segment from time i to $n_l + q_l$, Case II applies and therefore

$$\frac{\|w_{n_l+q_l}(\zeta)\|}{\|w_i(\zeta)\|}\frac{\|w_j(\zeta)\|}{\|w_{n_l+q_l}(\zeta)\|} \geq Ce^{\frac{\alpha}{2}j-\frac{3}{2}\alpha i}e^{(\frac{\lambda}{3}-\frac{3}{2}\alpha)(n_l+q_l-i)}C_0^{-\frac{2\alpha\tilde{\alpha}}{\lambda}j} \geq e^{-\frac{3}{2}\alpha i}.$$

Case V: $i \in (n_l, n_l + q_l)$ for some $l \in [1, s]$. From the proof of Proposition 2.1, $\|w_i(\zeta)\| < \|w_{n_l}(\zeta)\|$ holds. This and the estimates in Cases II, III yield the desired one.

Case IV: $i \in [n_l+q_l, n_l+p_l)$ for some $l \in [1,s]$. If $j \in [n_l+q_l, n_l+p_l)$, then the bounded distortion gives $\frac{\|w_j(\zeta)\|}{\|w_i(\zeta)\|} \geq \frac{c\delta}{10} \geq e^{-\alpha i}$. Otherwise, $n_l+p_l \leq (1+\frac{3\alpha}{\lambda})n_l$ from (G) and from Cases I, II, III,

$$\frac{\|w_{n_l+p_l}(\zeta)\|}{\|w_i(\zeta)\|}\frac{\|w_j(\zeta)\|}{\|w_{n_l+p_l}(\zeta)\|} \geq \frac{c\delta}{10}e^{-\frac{3}{2}\alpha(n_l+p_l)} \geq e^{-2\alpha i}.$$

Step 3(Proof of (G3)). Let $j \in [M, 20n]$. Define a finite sequence

$$j =: h_0 > h_1 > \cdots > h_{t(j)}$$

of free return times of ζ inductively as follows. Let $\hat{h}_{k+1}$ denote the largest free return time before h_k, when it makes sense. Let p_{k+1} denote the corresponding bound period. If

$$h_k - \hat{h}_{k+1} - p_{k+1} \leq \frac{1}{\lambda_0}\log(10/(c\delta)), \tag{37}$$

then let $h_{k+1} = \hat{h}_{k+1}$. In all other cases, h_{k+1} is undefined, namely $k = t(j)$. Define $\chi(j) = h_{t(j)}$. Obviously, $\chi(j) \leq j$ holds. It is left to show for $1 \leq i \leq \chi(j)$,

$$\|w_{\chi(j)}(\zeta)\| \geq c\delta\|w_i(\zeta)\|, \tag{38}$$

and

$$(1-\sqrt{\alpha})j \leq \chi(j). \tag{39}$$

If there exists no return time before $\chi(j)$, then (38) follows from Lemma 2.1. Otherwise, we first observe $\|w_{\chi(j)}(\zeta)\| \geq c\delta\|w_i(\zeta)\|$ for $\hat{h}_{t(j)+1}+p_{t(j)+1} \leq i \leq \chi(j)$, from Lemma 2.1. For all other i, $\|w_{\hat{h}_{t(j)+1}+p_{t(j)+1}}(\zeta)\| \geq (c\delta/10)\|w_i(\zeta)\|$ holds. Using all these and the reverse inequality of (37) for $k = t(j)$,

$$\|w_{\chi(j)}(\zeta)\| \geq c\delta e^{\lambda_0(\chi(j)-\hat{h}_{t(j)+1}-p_{t(j)+1})}\|w_{\hat{h}_{t(j)+1}+p_{t(j)+1}}(\zeta)\| \geq 10\|w_{\hat{h}_{t(j)+1}+p_{t(j)+1}}\|.$$

Hence (38) holds for $1 \leq i < \hat{h}_{t(j)+1}+p_{t(j)+1}$.

We show (39). If $t(j)=0$, there is nothing to prove. If $t(j)=1$, then the inequality follows from a condition (G). Suppose $t(j)>1$, and that $h_{t(j)} < (1-\sqrt{\alpha})j$. We derive a contradiction. Let $k_0 \in [0, t(j)]$ denote the smallest such that $h_{k_0} < (1-\sqrt{\alpha})j$. Condition (G) and (37) together implies $k_0 > 1$. Let $B = \{i \in [(1-\sqrt{\alpha})j, j]\colon f^i\zeta \text{ is bound}\}$ and $F = \{i \in [(1-\sqrt{\alpha})j, j]\colon f^i\zeta \text{ is free}\}$. By definition and the assumption, $[h_k, h_k+p_k] \subset [(1-\sqrt{\alpha})j, j]$ holds for every $i \in [1, k_0-1]$. The lower estimate of bound periods in Proposition 2.1 gives $\sharp B \geq \frac{1}{\log C_0}(k_0-1)\log(1/\delta)$. Summing (37) over all $i = 0, 1, \cdots, k_0-2$ gives $\sharp F \leq C\frac{(k_0-1)}{\lambda_0}\log 1/\delta$. Hence $\sqrt{\alpha}j \leq C\sharp B$ holds, where this C depends only on C_0, λ_0, c. On the other hand, condition (G) and (a) Proposition 2.1 give $\sharp B \leq \frac{3\alpha j}{\lambda}$. These two estimates are incompatible, if α is chosen sufficiently small, depending only on C_0, λ_0, c. $\square$

5.2. **Expansion at deep returns.** Let $n > N$ and $f \in \{f_a \colon a \in \Delta_{n-1} \setminus \Delta_n\}$. Let ζ be a nice critical point of f order $\geq n$, having $\nu < 20n$ as its free return time. If ν is not the first return time to $I(\delta)$, then let $n_1 < \cdots < n_t < \nu$ denote all the free return times of ζ before ν, with $z_1, \cdots, z_t$ and $p_1, \cdots, p_t$ the corresponding binding points and the bound periods. For each $i \in [1, \nu) \setminus \bigcup_{1\leq s\leq t}[n_s, n_s+p_s-1]$, let

$$\sigma_i(\zeta) = \frac{\|w_{i+1}(\zeta)\|}{\|w_i(\zeta)\|^2},$$

and let

$$\sigma_{n_s}(\zeta)=\frac{|f^{n_s}\zeta-z_s|^{\frac{10}{9}}}{\|w_{n_s}(\zeta)\|}.$$

Let $K_0=\inf_{c\in\mathrm{Crit},n>0}d(g^n_{a^*}c,\mathrm{Crit})$, where d denotes the minimal distance apart. Define

$$\Theta_\nu(\zeta)=\frac{K_0}{10}\left[\sum_{i=1}^{\nu-1}\sigma_i(\zeta)^{-1}\right]^{-1}. \tag{40}$$

It is understood that the sum runs over all i such that $f^i\zeta$ is free. The g_{a^*} is the interval map in Sect.1.3 with the critical set Crit and d denotes the minimal distance apart. By (A3), the infimum is nonzero.

Lemma 5.1. *For the above f,ζ,ν, $\|w_\nu(\zeta)\|\Theta_\nu(\zeta)\geq e^{-2\alpha(\beta-1)\nu}$.*

Proof. We estimate $\|w_\nu(\zeta)\|^{-1}\sigma_i(\zeta)^{-1}$ for each $1\leq i<\nu$ such that $f^i\zeta$ is free.

Step1 (estimates for free returns): Let $n_{t+1}=\nu$. For $1\leq s\leq t$ we have

$$\begin{aligned}\|w_{n_{s+1}}(\zeta)\|^{-1}\sigma_{n_s}(\zeta)^{-1}&=\frac{\|w_{n_s+p_s}(\zeta)\|}{\|w_{n_{s+1}}(\zeta)\|}\frac{\|w_{n_s}(\zeta)\|}{\|w_{n_s+p_s}(\zeta)\|}|f^{n_s}\zeta-z_s|^{-\frac{10}{9}}\\&\leq\frac{\|w_{n_s}(\zeta)\|}{\|w_{n_s+p_s}(\zeta)\|}|f^{n_s}\zeta-z_s|^{-\frac{10}{9}}\leq|f^{n_s}\zeta-z_s|^{-\frac{1}{10}}.\end{aligned}$$

For the last inequality we have used (d,e) Proposition 2.1. As $\|w_\nu(\zeta)\|\geq\|w_{n_{s+1}}(\zeta)\|$ we obtain

$$\|w_\nu(\zeta)\|^{-1}\sigma_{n_s}(\zeta)^{-1}\leq|f^{n_s}\zeta-z_s|^{-\frac{1}{10}}\leq e^{\alpha(\beta-1)n_s},$$

where the last inequality follows from (G). Summing this over all s gives

$$\sum_{s=1}^{t}\|w_\nu(\zeta)\|^{-1}\sigma_{n_s}(\zeta)^{-1}\leq 2e^{\alpha(\beta-1)\nu}. \tag{41}$$

Step2 (estimates for free segments): Let $F:=[0,n_1)\bigcup\cup_{1\leq s\leq t}[n_s+p_s,n_{s+1})$. Let

$$C_1=\max\left\{2,4\lambda^{-1}\log C_0\right\},\quad C_2=1-1/C_1\in(0,1),\quad C_3=\min\{1/2,C_2\}. \tag{42}$$

Put $s_0=-\frac{\log\delta}{\lambda C_1}$. For each $i\in F$, Lemma 2.1 gives

$$\|w_\nu(\zeta)\|\sigma_i(\zeta)=\frac{\|w_\nu(\zeta)\|\|w_{i+1}(\zeta)\|}{\|w_i(\zeta)\|^2}\geq c\delta e^{\lambda(\nu-i)}=c\delta^{C_2}e^{\lambda(\nu-i-s_0)}.$$

Split $F=F_1\cup F_2$, where $F_1=\{i\in F\colon i\leq\nu-s_0\}$ and $F_2=\{i\in F\colon i>\nu-s_0\}$. Summing the reciprocals of the above inequality over all $i\in F_1$,

$$\sum_{i\in F_1}\|w_\nu(\zeta)\|^{-1}\cdot\sigma_i(\zeta)^{-1}\leq\frac{C}{\delta^{C_2}}.$$

We claim $f^i\zeta\notin I(^3\sqrt{\delta})$ for each $i\in F_2$. Indeed, if this is not the case, then $\frac{\|w_\nu(\zeta)\|}{\|w_i(\zeta)\|}\leq{}^3\sqrt{\delta}\cdot CC_0^{\nu-i}$ holds. On the other hand, Lemma 2.1 and Proposition 2.1 give $\frac{\|w_\nu(\zeta)\|}{\|w_i(\zeta)\|}\geq c$. These two inequalities yield $\nu - i \geq -\frac{\log\delta}{4\log C_0}$, a contradiction to the assumption $i \in F_2$. Hence the claim holds and

$$\sum_{i\in F_2} \|w_\nu(\zeta)\|^{-1}\cdot\sigma_i^{-1} \leq \frac{C\sharp F_2}{\sqrt[3]{\delta}} \leq \frac{Cs_0}{\sqrt[3]{\delta}} \leq \frac{1}{\sqrt{\delta}}.$$

These two estimates yield

$$\sum_{i\in F} \|w_\nu(\zeta)\|^{-1}\cdot\sigma_i(\zeta)^{-1} \leq \frac{C}{\delta^{C_3}}. \tag{43}$$

Step3 (Overall estimate): (41) (43) give

$$\sum_{i=1}^{\nu-1} \|w_\nu(\zeta)\|^{-1}\cdot\sigma_i(\zeta)^{-1} \leq 2e^{\alpha(\beta-1)\nu} + \frac{C}{\delta^{C_3}} \leq 3e^{\alpha(\beta-1)\nu},$$

where the last inequality is because of the fact that ν is a return time of ζ. Taking reciprocals we obtain the desired inequality. □

The expansion estimate in Lemma 5.1 does not reflect the depth of the return at time ν. Hence, it is useless for our purpose if the depth of the return is shallow, compared with $\alpha\nu$. However, the exclusion rule in (33) does allow this case to occur. A solution to this problem is to introduce a particular type of returns for which another expansion estimate is available, and do exclusions only at these returns.

Definition 5.1. (Deep return times) Let $f \in \{f_a\colon a \in \Delta_{n-1}\setminus\Delta_n\}$. Let ζ be a nice critical point of f order $\geq n$, having $\nu < 20n$ as a free return time, with z the binding point. If ν is not the first return time to $I(\delta)$, then let $n_1 < \cdots < n_t < \nu$ denote all the free return times of ζ before ν, with $z_1, \cdots, z_t$ the corresponding binding points. Write $n_{t+1} = \nu$ and $z_{t+1} = z$. We say ν is a *deep return time*, if it is the first return time to $I(\delta)$, or else for $1 \leq s \leq t$,

$$\sum_{j=s+1}^{t+1} 2\log|f^{n_j}\zeta - z_j| \leq \log|f^{n_s}\zeta - z_s|.$$

Let

$$C_4 = \max\{1/5, C_3\} \in (0,1). \tag{44}$$

Lemma 5.2. *For the above f, ζ, ν, z, if ν is a deep return time of ζ, then*

$$\|w_\nu(\zeta)\|\Theta_\nu(\zeta) \geq |f^\nu\zeta - z|^{C_4}.$$

Proof. If ν is the first return time, then the desired estimate is a consequence of (43). Assume that ν is not the first return time. As ν is an deep return,

$$|f^{n_s}\zeta - z_s|^{-1} \leq \prod_{j=s+1}^{t+1} |f^{n_j}\zeta - z_j|^{-2}.$$

The proof of Lemma 5.1 gives $\|w_{n_{s+1}}(\zeta)\|^{-1}\sigma_{n_s}^{-1}(\zeta) \leq |f^{n_s}\zeta - z_s|^{-\frac{1}{10}}$, and so

$$\|w_{n_{s+1}}(\zeta)\|^{-1}\sigma_{n_s}^{-1}(\zeta) \leq \prod_{j=s+1}^{t+1} |f^{n_j}\zeta - z_j|^{-\frac{1}{5}}. \tag{45}$$

For $1 \leq s < t$,

$$\frac{\|w_{n_{s+1}}(\zeta)\|}{\|w_\nu(\zeta)\|} = \prod_{j=s+1}^{t} \frac{\|w_{n_j+p_j}(\zeta)\|}{\|w_{n_{j+1}}(\zeta)\|} \frac{\|w_{n_j}(\zeta)\|}{\|w_{n_j+p_j}(\zeta)\|} \leq \prod_{j=s+1}^{t} \frac{\|w_{n_j}(\zeta)\|}{\|w_{n_j+p_j}(\zeta)\|}.$$

Multiplying these,

$$\|w_\nu(\zeta)\|^{-1}\sigma_{n_s}^{-1}(\zeta) \leq |f^\nu\zeta - z|^{-\frac{1}{5}} \prod_{j=s+1}^{t} \frac{\|w_{n_j}(\zeta)\|}{\|w_{n_j+p_j}(\zeta)\|}|f^{n_j}\zeta - z_j|^{-\frac{1}{5}}. \tag{46}$$

For each term in the product, (e) Proposition 2.1 gives

$$\frac{\|w_{n_j}(\zeta)\|}{\|w_{n_j+p_j}(\zeta)\|}|f^{n_j}\zeta - z_j|^{-\frac{1}{5}} \leq |f^{n_j}\zeta - z_j|^{\frac{1}{2}} \leq \sqrt{\delta}.$$

Hence

$$\|w_\nu(\zeta)\|^{-1}\sigma_{n_s}(\zeta)^{-1} \leq |f^\nu\zeta - z|^{-\frac{1}{5}}\delta^{\frac{t-s}{2}}.$$

Summing this over all $1 \leq s < t$ and (45) for $s = t$ gives

$$\|w_\nu(\zeta)\|^{-1} \cdot \sum_{s=1}^{t} \sigma_{n_s}^{-1}(\zeta) \leq |f^\nu\zeta - z|^{-\frac{1}{5}} \sum_{s=1}^{t} \delta^{\frac{t-s}{2}} \leq |f^\nu\zeta - z|^{-\frac{1}{5}}.$$

The estimate for free segments in (43) and the above inequality yield

$$\|w_\nu(\zeta)\|^{-1} \cdot \sum_{i=1}^{\nu-1} \sigma_i^{-1}(\zeta) \leq \frac{C}{\delta^{C_3}} + |f^\nu\zeta - z|^{-\frac{1}{5}} \leq |f^\nu\zeta - z|^{-C_4}.$$

Taking the reciprocals of both sides yields the desired inequality. □

5.3. **Grid coordinates.** For each $\mu \geq \theta N$, fix a subdivision of $\mathbb{R} \times \{\sqrt{b}\}$ into right-open horizontals of equal length κ_0^μ. We label all of them intersecting H with $l = 1, 2, 3, \cdots$, from the left to the right. By a *μ-grid coordinate* of a point x on H we mean the integer l which is a label of the horizontal containing x.

In general, let ζ be a nice critical point of order n on a horizontal curve γ. By definition, there exists $\xi \in f^{-[\theta n]}\zeta$ and a tangent vector u at ξ for which (C2) in Definition 3.2 holds. Let μ be any $[\theta n]$-hyperbolic time of u. We call μ a *hyperbolic time of ζ*. The long stable leaf through $f^\mu\xi$ of order $[\theta n] - \mu$ intersects H exactly at one point. Let $A(\zeta, \mu)$ denote the $([\theta n] - \mu)$-grid coordinate of the point of the intersection.

6. Parameter exclusion II: positive measure

In this last section we show $|\Delta| > 0$. In Sect.6.1 we decompose $\Delta_{n-1} \setminus \Delta_n$ into a finite number of subsets, based on the combinatorics introduced in the previous sections. Assuming a key measure estimate (Proposition 6.1) on each of these subsets, we conclude $|\Delta| > 0$. All the remaining subsections is devoted to a proof of the key measure estimate.

6.1. **Decomposition of parameter sets excluded at step n.** We decompose $\Delta_{n-1} \setminus \Delta_n$ as follows. Fix the following combinatorics:

(D1) positive integers $m \in [20(n-1), 20n)$, s, t, R;
(D2) sequences $(\mu_1, \cdots, \mu_s), (x_1, \cdots, x_s)$ of s positive integers;
(D3) sequences $(\nu_1, \cdots, \nu_t)$, $(l_1, \cdots, l_t)$, $(n_1, \cdots, n_t)$, $(r_1, \cdots, r_t)$, $(y_1, \cdots, y_t)$ of t positive integers.

Let $E_n(*) = E_n(m, s, t, R, \cdots)$ denote the set of all $a \in \Delta_{n-1} \setminus \Delta_n$ for which there exists a nice critical point ζ of $f_a = f$ of order $\geq n$ such that the following holds:

(Z1) $(G)_{m-1}$ holds, and $(G)_m$ fails;
(Z2) $\{\mu_1 < \cdots < \mu_s\} \subset [0, [\theta n]]$ is a sequence of hyperbolic times of ζ satisfying

$$\frac{1}{2} \leq \frac{[\theta n] - \mu_s}{\log(1/\delta)} \leq 1, \quad [\theta n] - \mu_1 \geq \frac{1}{2}\theta n, \quad \frac{1}{16} \leq \frac{[\theta n] - \mu_{i+1}}{[\theta n] - \mu_i} \leq \frac{1}{4} \text{ for } 1 \leq i < s. \tag{47}$$

Lemma 3.3 ensures the existence of such a sequence;
(Z3) $x_i = A(\zeta, \mu_i)$ for every $1 \leq i \leq s$;
(Z4) $\nu_1 < \cdots < \nu_t = m$ are all the free return times in the first m iterates of ζ, with $z_1, \cdots, z_t$ the corresponding binding points;
(Z) for each $k \in [1, t]$, $l_k \in [1, \sharp\mathrm{Crit}]$ is such that $f^{\nu_k}\zeta \in I^{(l_k)}(\delta)$;
(Z) $n_k < n$, and

$$n_k = \begin{cases} \text{the order of } z_k \text{ if } z_k \neq c_{l_k} \\ 0 \text{ if } \ z_k = c_{l_k}. \end{cases}$$

(Z5) If $\nu_k < m$, then $|f^{\nu_k}\zeta - z_k| \in [e^{-r_k}, e^{-r_k+1})$. If $\nu_k = m$ (which means $k = t$ and $\nu_t = m$), then r_t is defined as follows. If $|f^m\zeta - z_t| > e^{-\alpha m}$, then r_t is such that $|f^m\zeta - z_t| \in [e^{-r_t}, e^{-r_t+1})$ holds. Otherwise, $r_t = \alpha m$;
(Z6) If $n_k \neq 0$, then $y_k = A(z_k, 0)$. Otherwise, $y_k = 0$.

If $a \in E_n(*)$, then any nice critical point of f_a of order $\geq n$ for which (Z1-6) hold is called *responsible for a*, or a *responsible critical point of f_a*. The parameter set $E_n(*)$ is called an *n-class*. By definition, any parameter in $\Delta_{n-1} \setminus \Delta_n$ belongs to some n-class.

Before proceeding let us record constraints on the above integers. Corollary 3.1 gives

$$n_k \approx r_k \quad \text{if} \quad \nu_k < m. \tag{48}$$

By the definition of r_t in (Z5),

$$n_t \leq \alpha^{-1} r_t \quad \text{if} \quad \nu_k = \nu_t = m. \tag{49}$$

(G) and the definition of r_k in (Z5) give

$$r_k \leq \alpha\nu_k \text{ for } 1 \leq k \leq t. \tag{50}$$

Proposition 6.1. $|E_n(*)| < e^{-\frac{1}{3}R}|\Delta_0|$, *where* $R = r_1 + r_2 \cdots + r_t$.

We finish the proof of Theorem B assuming the conclusion of Proposition 6.1. We begin by counting the number of all feasible n-classes. The number of all feasible $(\mu_1, \cdots, \mu_s)$ is bounded by the number of ways of choosing s objects from $[\theta n]$ objects, which is $\binom{[\theta n]}{s}$. For one such way, there are at most $\prod_{i=1}^{s} \kappa_0^{-(m-\mu_i)}$ number of ways to choose $(x_1, \cdots, x_s)$. (47)

gives $m-\mu_i\leq 4^{i-s}(m-\mu_s)$, and therefore

$$\sum_{i=1}^{s}(m-\mu_j)\leq\sum_{i=1}^{s}4^{j-s}(m-\mu_s)\leq 2(m-\mu_s)\leq 2\theta n.$$

Hence, it is possible to choose $C>1$ such that the number of all feasible sequences in (D2) is

$$\leq\binom{[\theta n]}{s}\prod_{i=1}^{s}\kappa_0^{-(m-\mu_i)}\leq C^{\theta n}.$$

The number of all feasible $(\nu_1,\cdots,\nu_t)$ is $\leq\binom{n}{t}$. The number of all feasible $(r_1,\cdots,r_t)$ is equal to the total number of combinations of dividing R objects into t groups, which is $\binom{R+t}{t}$. (48) (49) give $n_1+\cdots+n_t\leq C\alpha^{-1}R$. Hence, the number of all feasible $(n_1,\cdots,n_t)$ and that of $(y_1,\cdots,y_t)$ are correspondingly $\leq\binom{\frac{R}{20\lambda}+t}{t}$ and $\leq\kappa_0^{-\theta\sum_{k=1}^{t}n_k}\leq e^{C\theta\alpha^{-1}R}$. Using $\max\{t/R,t/n\}\leq C/\log(1/\delta)$ and Stirling's formula for factorials, we have that the number of all feasible sequences in (D3) is

$$\leq\binom{n}{t}\binom{R+t}{t}\binom{\frac{R}{20\lambda}+t}{t}e^{C\theta\alpha^{-1}R}\leq e^{\tau(\delta)n+C\theta\alpha^{-1}R},$$

where $\tau(\delta)\to 0$ as $\delta\to 0$.

The next lemma asserts that the the sum of deep return depths has a positive definite proportion.

Lemma 6.1. *$R\geq\alpha m/2$.*

Proof. Let $a\in E_n(*)$ and ζ be a responsible critical point of f_a. Write f for f_a. If the orbit of ζ does not return to $I(\delta)$ before time m, then necessarily $f^m\zeta\in I(\delta)$ holds, and $r_1=r_t\geq\alpha m$, because of (Z1). Hence the desired inequality holds in this case.

Suppose that there exist return times of ζ in $(0,m)$. For a non deep return time $\eta\in(0,m)$, let η' denote the smallest integer in $[0,\eta-1]$ such that

$$\sum_{\substack{\eta'+1\leq i\leq\eta\\ \text{free return}}}2\log|f^i\zeta-\tilde{\zeta}_i|>\log|f^{\eta'}\zeta-\tilde{\zeta}_{\eta'}|,\tag{51}$$

where $\tilde{\zeta}_i$ denotes the binding point for $f^i\zeta$. By definition, there exists no deep return time in $[\eta'+1,\eta]$. Define a strictly decreasing sequence $\eta_1>\eta_2>\cdots>\eta_u$ of integers in $(0,m]$ as follows: η_1 is the largest non deep return time in $(0,m]$. Given η_l, let η_{l+1} denote the largest non deep return time which is $<\eta_l'+1$. By definition, the intervals $[\eta_l'+1,\eta_l]$ for $(l=1,\cdots,u)$ are mutually disjoint and cover all the non deep return times in $(0,m]$. In view of (51) we have

$$\sum_{0<i\leq m\text{: non deep}}2\log|f^i\zeta-\tilde{\zeta}_i|>\sum_{l=1}^{u}\log|f^{\eta_l'}\zeta-\tilde{\zeta}_{\eta_l'}|\geq\sum_{0<i\leq m}\log|f^i\zeta-\tilde{\zeta}_i|.$$

Hence we obtain

$$\sum_{k=1}^{t}r_k\geq\sum_{k=1}^{t}-\log|f^{\nu_k}\zeta-\zeta_k|\geq-\frac{1}{2}\sum_{0<i\leq m}\log|f^i\zeta-\tilde{\zeta}_i|\geq\frac{\alpha m}{2}.$$

The last inequality follows from (33). $\square$

Using $R \geq \alpha m/2 \geq 10\alpha(n-1)$ and $\max(\tau(\delta),\theta) \ll \alpha$, we have

$$e^{\tau(\delta)n+\frac{\theta R}{\alpha}}e^{-C_5R} \leq e^{\tau(\delta)\frac{R}{9\alpha}+\frac{\theta R}{\alpha}}e^{-C_5R} \leq e^{-\frac{1}{2}C_5R}.$$

Hence

$$\begin{aligned}|\Delta_{n-1}\setminus\Delta_n| &\leq |\Delta_0|\sum_{m,s,t,R}\sum_{R\geq\alpha m/2}\sum_{r_1+\cdots+r_t=R}|E_n(*)| \leq |\Delta_0|e^{\tau(\delta)n+\frac{\theta R}{\alpha}}\sum_{R\geq\alpha m/2}e^{-\frac{C_5R}{2}}\\ &\leq |\Delta_0|e^{-\frac{C_5\alpha m}{5}} \leq |\Delta_0|e^{-4C_5\alpha(n-1)}.\end{aligned}$$

Since $\Delta_0=\Delta_N$ we obtain

$$|\Delta| = |\Delta_N| - \sum_{n>N}|\Delta_{n-1}\setminus\Delta_n| \geq |\Delta_N|\left(1-\sum_{n>N}e^{-4C_5\alpha(n-1)}\right) > 0.$$

6.2. **Structure of the rest of this section.** The rest of this section is entirely devoted to the proof of Proposition 6.1. The main step is to analyze the parameter dependence of positions of responsible critical points at each return time $\nu_1,\cdots\nu_t$ in the definition of $E_n(*)$. In the next three subsections we treat this main step. Building on this we give combinatorial considerations. In Sect. we complete the proof of Proposition 6.1.

Hypothesis for Sect.6.3, 6.4, 6.5: $\hat{a}\in E_n(*)$, and ζ is a responsible critical point of $f_{\hat{a}}$ of order $\geq n$.

6.3. **Critical curves.** We need to consider all responsible nice critical points of order $\geq n$, while bad parameters are excluded at each deep return time $\nu_1,\cdots,\nu_t$, which are $\leq 20n$. This necessitates working with deformations commensurate with each ν_k. We argue as follows.

Fix once and for all sequence $m_1>\cdots>m_s$ of integers such that for each $i\in[1,s]$,

$$[\theta m_i] = [\theta n]-\mu_i.$$

A slight modification of the proof of Lemma 4.3 shows the existence of a sequence $\zeta^{(1)},\cdots,\zeta^{(s)}$ of quasi critical points of order $m_1,\cdots,m_s$ such that for $1\leq i\leq s$,

$$|\zeta-\zeta^{(i)}| \leq (Cb)^{\frac{\theta m_i}{10}}. \tag{52}$$

For each ν_k, let

$$\eta_k = \min\left\{1\leq i\leq s\colon e^{-\lambda\nu_k/2}\leq \kappa_0^{m_i}\right\}. \tag{53}$$

Definition 6.1. (Adapted deformations) The deformation $a\in I_{m_{\eta_k}}(\hat{a})\mapsto\zeta^{(\eta_k)}(a)$ of $\zeta^{(\eta_k)}$ is called a *ν_k-adapted deformation of ζ.*

We prove a couple of lemmas surrounding the ν_k-adapted deformation of ζ. The next lemma indicates that the $f_{\hat{a}}$-orbits of ζ and $\zeta^{(\eta_k)}$ are indistinguishable up to time ν_k.

Lemma 6.2. $|f_{\hat{a}}^i\zeta - f_{\hat{a}}^i\zeta^{(\eta_k)}| \leq (Cb)^{\frac{\theta\nu_k}{200}}$ *for* $0\leq i\leq\nu_k$.

Proof. Suppose $\eta_k>1$. The definition gives $e^{-\lambda\nu_k/2}>\kappa_0^{m_{\eta_k-1}}$, and thus $m_{\eta_k-1}\geq\lambda\nu_k/(2\log(1/\kappa_0))$. (47) gives $\frac{1}{16}\leq\frac{m_{\eta_k}}{m_{\eta_k-1}}$, and hence $m_{\eta_k}\geq\lambda\nu_k/(32\log(1/\kappa_0))$. This yields

$$|f_{\hat{a}}^i\zeta - f_{\hat{a}}^i\zeta^{(\eta_k)}| \leq C_0^i|\zeta-\zeta^{(\eta_k)}| \leq C_0^{\nu_k}(Cb)^{\frac{\theta m_{\eta_k}}{10}} \leq (Cb)^{\frac{\theta\nu_k}{200}}.$$

Suppose $\eta_k=1$. Then $m_{\eta_k}=m_1\geq n$. Since $20n\geq\nu_k$, we get $m_{\eta_k}\geq\nu_k/20$, and the same inequality holds. □

Let

$$J_{\nu_k}(\hat{a},\zeta)=[\hat{a}-\Theta_{\nu_k}(\zeta),\hat{a}+\Theta_{\nu_k}(\zeta)]. \tag{54}$$

Lemma 6.3. *$J_{\nu_k}(\hat{a},\zeta)\subset I_{m_{\eta_k}}(\hat{a})$. If moreover $n_k\neq 0$, then $I_{m_{\eta_k}}(\hat{a})\subset I_{n_k}(\hat{a})$. In particular, the deformation of the binding point for $f_{\hat{a}}^{\nu_k}\zeta$ is well-defined on $J_{\nu_k}(\hat{a},\zeta)$.*

Proof. (G) implies that there is some $i\in[(2/3)\nu_k,\nu_k]$ such that $f^i\zeta$ is free. Hence $\Theta_{\nu_k}(\zeta)\leq e^{-\lambda\nu_k/2}$ holds. On the other hand, (53) gives $e^{-\lambda\nu_k/2}\leq\kappa_0^{m_{\eta_k}}$. Hence the first inclusion holds.

For the second inclusion, it suffices to show $m_{\eta_k}\geq n_k$. This holds for the case $\nu_k=\nu_t=m$, from $n_k<n$ and $m_{\eta_k}=m_1\geq n$. Suppose $\nu_k<m$. (G) gives $n_k\leq C\alpha\nu_k\leq C\alpha n$. If $\eta_k=1$, then $m_{\eta_k}=m_1\geq n$, and hence $m_{\eta_k}\geq n_k$. If $\eta_k>1$, then the inequality in the proof of Lemma 6.2 gives the same inequality. □

In what follows, we consider the evolution of parametrized curves:

$$a\in J_{\nu_k}(\hat{a},\zeta)\mapsto\zeta_i(a,k)=:f_a^i(\zeta^{(\eta_k)}(a)),\quad i=0,1,2,\cdots,\nu_k,$$

and show that this evolution is similar to that of a curve under the iteration of the fixed map $f_{\hat{a}}$. A central idea follows the well-known line [2, 13, 22] and consists of two parts; to establish an equivalence between space and a-derivatives (Sect.6.4) and then; to transfer phase-space analyses to parameter space (Sect.6.5).

6.4. **Equivalence between space and a-derivatives.** Recall that (g_a) is the unperturbed family of maps on $[-1,1]$. For each $x_0\in$ Crit and $i\geq 0$, let $x_i(a):=g_a^ix_0$. Let

$$\mathcal{Q}_k(x_0,a):=\frac{\frac{dx_k}{da}(a)}{(g_a^{k-1})'x_1(a)}.$$

According to [18], we have

$$\mathcal{Q}_k(x_0,a^*)\to p(x_0,a^*)\neq 0\text{ as }k\to\infty, \tag{55}$$

where $p(x_0,a^*)$ is the one in (7). Pick a positive integer k_0 such that $|\mathcal{Q}_k(x_0)|\geq p(x_0,a^*)/2>0$ holds for all $k\geq k_0$ and each $x_0\in$ Crit. For $i\geq 1$, write $w_i(\zeta)=Df_{\hat{a}}^{i-1}(f_{\hat{a}}\zeta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

Lemma 6.4. *There exist $C_1>0$, $C_2>0$ such that for $a\in I_{m_{\tilde{k}}}(\hat{a})\to\zeta_0(a,k)$ we have*

$$C_1\|w_i(\zeta)\|\leq\|\dot{\zeta}_i(\hat{a},k)\|\leq C_2\|w_i(\zeta)\|\quad\textit{for } k_0\leq i\leq\nu_k.$$

In addition, the second inequality remains to hold for all $1\leq i<k_0$.

Let $\theta_i=\text{angle}(w_i(\zeta),\dot{\zeta}_i(\hat{a},k))$.

Lemma 6.5. *For every $i\geq k_0$ such that $f_{\hat{a}}^i\zeta$ is free, $\theta_i\leq\frac{C}{\delta\|w_i(\zeta)\|}$.*

Proofs of these lemmas are given in Appendices A.5, A.6.

6.5. **Evolution of critical curves.** For $i\in[k_0+1,\nu_k]$, define

$$\rho_i=\sum_{k_0<j<i\colon\text{free}}e^{-\frac{\lambda j}{3}}+\sigma_j^{-1}(\zeta)\Theta_{\nu_k}(\zeta)+\sum_{k_0<j<i\colon\text{free return}}\text{length}(\gamma_j)^{\frac{1}{10}}.$$

For $i\geq 1$, write $w_i(a):=Df_a^{i-1}(\zeta_1(a,k))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

Lemma 6.6. *The following holds for $k_0<i\leq\nu_k$ such that $f_{\hat{a}}^i\zeta$ is free:*

(a) $\left|\log\|\dot\zeta_i(\hat a,k)\|-\log\|\dot\zeta_i(a,k)\|\right|\leq\rho_i\leq 1$ *for* $a\in J_{\nu_k}(\hat a,\zeta)$*;*
(b) $\|\ddot\zeta_j(a,k)\|\leq(C\delta)^{-3(i-j)}\|\dot\zeta_i(a,k)\|^3$ *for* $a\in J_{\nu_k}(\hat a,\zeta)$ *and* $k_0\leq j\leq i$*;*
(c) *the curvature of* $\gamma_i:=\{\zeta_i(a,k)\colon a\in J_{\nu_k}(\hat a,\zeta)\}$ *is everywhere* $\leq\frac{1}{100}$.

We postpone a lengthy proof of this lemma to Sect.6.9 and instead derive two corollaries. For $r\in(0,1)$ and a compact interval J centered at $\hat a$, denote by $r\cdot J$ the interval of length $r|J|$ centered at $\hat a$. Fix $C_5\in(0,1)$ such that

$$C_4+C_5\in(0,1). \tag{56}$$

Corollary 6.1. *For all* $a\in J_{\nu_k}(\hat a,\zeta)\setminus e^{-C_5r_k}\cdot J_{\nu_k}(\hat a,\zeta)$*,*

$$|\zeta_{\nu_k}(\hat a,k)-\zeta_{\nu_k}(a,k)|\geq e^{-(C_4+C_5)r_k}.$$

Proof. From Lemma 6.6(d), γ_{ν_k} is a horizontal curve. Lemma 6.6(a) gives

$$|\zeta_{\nu_k}(\hat a,k)-\zeta_{\nu_k}(a,k)|\geq C\left\|\dot\zeta_{\nu_k}(\hat a,k)\right\||\hat a-a|\geq C\|w_{\nu_k}(\zeta)\||\hat a-a|,$$

where the second inequality follows from Lemma 6.4. From the assumption on a, the right hand side is $\geq C\|w_{\nu_k}(\zeta)\||J_{\nu_k}(\hat a,\zeta)|\cdot e^{-C_5r_k}$. If $\nu_k<m$, then Lemma 5.2 gives $\|w_{\nu_k}(\zeta)\||J_{\nu_k}(\hat a,\zeta)|\geq e^{-C_4r_k}$. If $\nu_k=m$, which means $k=t$ and $\nu_t=m$, then $r_t\leq\alpha m$, $\beta=10/9$, $C_4\geq 1/5$ and Lemma 5.1 give

$$\|w_{\nu_k}(\zeta)\||J_{\nu_k}(\hat a,\zeta)|\geq e^{-2\alpha(\beta-1)\nu_k}=e^{-2\alpha(\beta-1)m}\geq e^{-C_4r_t}.$$

Consequently, in either of the two cases we obtain the desired inequality. □

Definition 6.2. (Critical parameter) From the second inclusion in Lemma 6.3, the deformation $a\in I_{n_k}(\hat a)\mapsto z_k(a)$ of the binding point for $f_{\hat a}^{\nu_k}\zeta$ is well-defined on $J_{\nu_k}(\hat a,\zeta)$. Proposition 4.2 and Corollary 6.1 together imply the existence of a unique parameter $c_0\in e^{-C_5r_k}\cdot J_{\nu_k}(\hat a,\zeta)$ such that the x-coordinate of $\zeta_{\nu_k}(c_0,k)$ coincides with that of $z_k(c_0)$. We call c_0 a *critical parameter* in $J_{\nu_k}(\hat a,\zeta)$.

6.6. **Combinatorial lemmas.** We shall reduce the measure estimate of $E_n(*)$ to elementary combinatorial considerations. To this end we need three key lemmas, based primarily on the expansion estimate in Corollary 6.1 and the notion of critical parameters.

Lemma 6.7. *Let* a_1*,* $a_2\in E_n(*)$ *and let* ζ_1*,* ζ_2 *be responsible critical points correspondingly. If* $k<t$ *and* $a_1\in e^{-r_k/10}\cdot J_{\nu_k}(a_2,\zeta_2)$*, then* $J_{\nu_{k+1}}(a_1,\zeta_1)\subset 2e^{-r_k/10}\cdot J_{\nu_k}(a_2,\zeta_2)$*.*

Proof. The next sublemma allows us to "relate" critical points responsible for different parameters through their deformations.

Sublemma 6.1. *Let* $a_1,a_2\in E_n(*)$ *and* ζ^1,ζ^2 *be responsible critical points correspondingly. Let* z_k^σ *denote the binding point for* $f_{a_\sigma}^{\nu_k}\zeta^\sigma$ *and let* $z_k^\sigma(\cdot)$ *denote its deformation* ($\sigma=1,2$)*. If* $J_{\nu_k}(a_1,\zeta_1)\cap J_{\nu_k}(a_2,\zeta_2)\neq\emptyset$*, then for all* $c\in J_{\nu_k}(a_1,\zeta_1)\cap J_{\nu_k}(a_2,\zeta_2)$*,* $\zeta_0^1(c,k)=\zeta_0^2(c,k)$ *and* $z_k^1(c)=z_k^2(c)$*.*

Proof. By the construction of deformations in Sect.4, there exists a horizontal $l^1\subset H$ of length $2\kappa_0^{3\theta m_{\eta_k}}$ such that $f_c^{[\theta m_{\eta_k}]}l^1$ is a $C^2(b)$-curve and $\zeta_0^1(c,k)$ lies on it. Correspondingly, there exists a horizontal $l^2\subset H$ of length $2\kappa_0^{3\theta m_{\eta_k}}$ such that $f_c^{[\theta m_{\eta_k}]}l^2$ is $C^2(b)$ and $\zeta_0^2(c,k)$ lies on it. By (Z3), the midpoints of l^1,l^2 have the same $[\theta m_{\eta_k}]$-grid coordinate. Hence, l^1 intersects l^2 and

$f_c^{[\theta\lambda_{\eta_k}]}(l^1 \cup l^2)$ is $C^2(b)$. From the elementary fact that one $C^2(b)$-curve does not admit more than two critical points of the same order, $\zeta_0^1(c,k) = \zeta_0^2(c,k)$ follows. An analogous argument with (Z6) in the place of (Z3) gives $z_k^1(c) = z_k^2(c)$. □

Returning to the proof of Lemma 6.7, let c_0 denote the critical parameter in $J_{\nu_k}(a_2, \zeta^2)$. We claim $c_0 \notin J_{\nu_{k+1}}(a_1, \zeta^1)$. This claim and the assumption on a together imply that one of the components of $J_{\nu_{k+1}}(a_1, \zeta_1) \setminus \{a_1\}$ is contained in $e^{-r_k/10} \cdot J_{\nu_k}(a_2, \zeta^2)$. This yields the inclusion.

It is left to prove the claim. We argue by contradiction assuming $c_0 \in J_{\nu_{k+1}}(a_1, \zeta^1)$. The last inequality in (61) implies that $\zeta^1_{\nu_k}(c_0, k+1)$ is in admissible position relative to $z_k^1(c_0)$. Hence, $\zeta^1_{\nu_k}(c_0, k)$ is in admissible position relative to $z_k^1(c_0)$ as well. The assumption $c_0 \in J_{\nu_k}(a_1, \zeta^1) \cap J_{\nu_k}(a_2, \zeta^2)$ and Sublemma 6.1 give $z_k^1(c_0) = z_k^2(c_0)$ and $\zeta^1_{\nu_k}(c_0, k) = \zeta^2_{\nu_k}(c_0, k)$. Hence, $\zeta^2_{\nu_k}(c_0, k)$ is in admissible position relative to $z_k^2(c_0)$. This means that c_0 is not a critical parameter in $J_{\nu_k}(a_2, \zeta^2)$, a contradiction. □

Lemma 6.8. *Let a_1, $a_2 \in E_n(*)$ and let ζ^1, ζ^2 be responsible critical points correspondingly. If $a_2 \notin J_{\nu_k}(a_1, \zeta^1)$, then $J_{\nu_k}(a_1, \zeta^1) \cap J_{\nu_k}(a_2, \zeta^2) = \emptyset$.*

Proof. We derive a contradiction assuming the intersection is nonempty. Using Sublemma 6.1 and Lemma 6.6, it is possible to show $|J_{\nu_k}(a_1, \zeta^1)| \approx |J_{\nu_k}(a_2, \zeta^2)|$. Let c_σ denote the critical parameter in $J_{\nu_k}(a_\sigma, \zeta^\sigma)$ $(\sigma = 1, 2)$. Since $a_2 \notin J_{\nu_k}(a_1, \zeta^1)$, $c_1 \neq c_2$ holds.

Let z_k^σ denote the binding point for $f_{a_\sigma}^{\nu_k}\zeta^\sigma$ and let $z_k^\sigma(\cdot)$ denote its deformation $(\sigma = 1, 2)$. Sublemma 6.1 gives $z_k^1(c_2) = z_k^2(c_2)$. Hence

$$|z_k^1(c_1) - z_k^2(c_2)| = |z_k^1(c_1) - z_k^1(c_2)| \le C^{-\log\delta}|c_1 - c_2|,$$

where we have used Proposition 4.2 for the last inequality. On the other hand, Lemma 6.6 and (G1) give

$$|\zeta^1_{\nu_k}(c_1, k) - \zeta^2_{\nu_k}(c_2, k)| = |\zeta^1_{\nu_k}(c_1, k) - \zeta^2_{\nu_k}(c_2, k)| \ge Ce^{\lambda\nu_k}|c_1 - c_2|.$$

Since $c_1 \neq c_2$, $|\zeta^1_{\nu_k}(c_1, k) - \zeta^2_{\nu_k}(c_2, k)| \gg |z_k^1(c_1) - z_k^2(c_2)|$ holds. This yields a contradiction to the fact that c_1 and c_2 are critical parameters. □

Lemma 6.9. *Let $a_1 \in E_n(*)$ and let ζ^1 denote any responsible critical point for a_1. Then $J_{\nu_k}(a_1, \zeta^1) \setminus e^{-C_5 r_k} \cdot J_{\nu_k}(a_1, \zeta^1)$ does not intersect $E_n(*)$.*

Proof. Let $a_2 \in J_{\nu_k}(a_1, \zeta^1) \setminus e^{-C_5 r_k} \cdot J_{\nu_k}(a_1, \zeta^1)$. We argue by contradiction assuming $a_2 \in E_n(*)$. Let ζ^2 denote any critical point responsible for a_2. Let z_k^σ denote the binding point for $f_{a_\sigma}^{\nu_k}\zeta^\sigma$ and let $z_k^\sigma(\cdot)$ denote its deformation $(\sigma = 1, 2)$. As $a_2 \in J_{\nu_k}(a_1, \zeta^1) \cap J_{\nu_k}(a_2, \zeta^2)$, Sublemma 6.1 gives

$$\zeta_0^1(a_2, k) = \zeta_0^2(a_2, k), \quad z_k^1(a_2) = z_k^2(a_2). \tag{57}$$

By the construction of deformations in Sect.4,

$$|z_k^\sigma - z_k^\sigma(a_\sigma)| \le (Cb)^{\frac{\theta n_k}{4}} \le e^{-r_k}. \tag{58}$$

If $\nu_k < m$, then the last inequality follows from (48). If $\nu_k = \nu_t = m$, it follows from the definition of r_t.

Claim 6.1. *For $\sigma = 1, 2$, $|\zeta^\sigma_{\nu_k}(a_\sigma, k) - f_{a_\sigma}^{\nu_k}\zeta^\sigma| \le e^{-r_k}$.*

Proof. Lemma 6.2 and (50) give $|\zeta^\sigma_{\nu_k}(a_\sigma, k) - f_{a_\sigma}^{\nu_k}\zeta^\sigma| \le (Cb)^{\frac{\theta\nu_k}{200}} \le e^{-\nu_k} \le e^{-r_k}$. □

Using (58) and Claim 6.1, we have

$$\begin{aligned}|f_{a_2}^{\nu_k}\zeta^2 - z_k^2| &\geq |\zeta_{\nu_k}^2(a_2,k) - z_k^2(a_2)| - |\,\zeta_{\nu_k}^2(a_2,k) - f_{a_2}^{\nu_k}\zeta^2| - |\,z_k^2(a_2) - z_k^2| \\ &\geq |\zeta_{\nu_k}^2(a_2,k) - z_k^2(a_2)| - 2e^{-r_k}.\end{aligned}$$

For the first term of the last line,

$$\begin{aligned}|\zeta_{\nu_k}^2(a_2,k) - z_k^2(a_2)| &\geq |\zeta_{\nu_k}^2(a_2,k) - \zeta_{\nu_k}^1(a_1,k)| - |\zeta_{\nu_k}^1(a_1,k) - z_k^1(a_1)| - |\,z_k^1(a_1) - z_k^2(a_2)| \\ &= |\zeta_{\nu_k}^1(a_2,k) - \zeta_{\nu_k}^1(a_1,k)| - |\,\zeta_{\nu_k}^1(a_1,k) - z_k^1(a_1)| - |\,z_k^1(a_1) - z_k^1(a_2)|,\end{aligned}$$

where the equality follows from (57). We estimate the three terms in the last line one by one. For the first term, Corollary 6.1 gives

$$|\zeta_{\nu_k}^1(a_2,k) - \zeta_{\nu_k}^1(a_1,k)| \geq Ce^{-(C_4+C_5)r_k}.$$

For the second term, (58) Claim 6.1 give

$$|\zeta_{\nu_k}^1(a_1,k) - z_k^1(a_1)| \leq |\zeta_{\nu_k}^1(a_1,k) - f_{a_1}^{\nu_k}\zeta^1| + |f_{a_1}^{\nu_k}\zeta^1 - z_k^1| + |z_k^1 - z_k^1(a_1)| \leq 3e^{-r_k}.$$

For the third term, Proposition 4.2 gives

$$|z_k^1(a_1) - z_k^2(a_2)| \leq C^{-\log\delta}|a_1 - a_2| \leq e^{-\frac{\lambda\nu_k}{2}} \leq e^{-\frac{\lambda}{2\alpha}r_k}.$$

For the last inequality we have used (50). Consequently we obtain $|f_{a_2}^{\nu_k}\zeta^2 - z_k^2| \geq Ce^{-(C_4+C_5)r_k}$. It follows that ζ^2 is not a responsible critical point for a_2, a contradiction. □

6.7. **Proof of Proposition 6.1.** By induction, for each $k \in [1,t]$ we choose a finite sequence $J_{k,1}, J_{k,2}, \cdots$, of parameter intervals with the following properties:

(i) each $J_{k,i}$ has the form $J_{k,i} = J_{\nu_k}(a_{k,i}, z_{k,i})$, where $a_{k,i} \in E_n(*)$ and $z_{k,i}$ is a critical point responsible for $a_{k,i}$;
(ii) $J_{k,1}, J_{k,2}, \cdots$ are pairwise disjoint and $E_n(*) \subset \bigcup_i e^{-C_5 r_k} \cdot J_{k,i}$;
(iii) if $t > 1$, then for each $k \in [2,t]$ and $(a_{k,i}, z_{k,i})$ there exists $(a_{k-1,j}, z_{k-1,j})$ such that $J_{k,i} \subset 2e^{-C_5 r_{k-1}} \cdot J_{k-1,j}$;
(iv) $\sum_i |J_{1,i}| \leq 10|\Delta_0|$.

A simple computation gives

$$|E_n(*)| \leq 2^t e^{-C_5 R} \sum_i |J_{1,i}| \leq e^{-\frac{C_5 R}{2}}|\Delta_0|.$$

To choose the intervals as required, start with $k = 1$. We claim that it is possible to choose $a_{1,1}, a_{1,2}, \cdots$, in $E_n(*)$ and responsible critical points $z_{1,1}, z_{1,2}, \cdots$ correspondingly, for which the intervals $J_{1,1}, J_{1,2}, \cdots$ satisfy (ii). Indeed, choose some $a_{1,1} \in E_n(*)$ and define $J_{1,1}$ choosing some responsible critical point for $a_{1,1}$. If $J_{1,1}$ covers $E_n(*)$, then the claim holds. Otherwise, choose some $a_{1,2} \in E_n(*) - J_{1,1}$, and define $J_{2,1}$ choosing some responsible critical point for $a_{2,1}$. By Lemma 6.8, $J_{1,1}, J_{1,2}$ are pairwise disjoint. Repeat this. As the length of these intervals are uniformly bounded from below, there must come a point at which our claim is fulfilled.

Given $J_{k-1,1}, J_{k-1,2}, \cdots$ for which (ii) (iii) hold, $J_{k,1}, J_{k,2}, \cdots$ are defined as follows. For each $J_{k-1,i}$, in the same way as the previous paragraph it is possible to choose a finite number of parameters $a_{k,1}, a_{k,2}, \cdots$ in $E(*) \cap e^{-C_5 r_{k-1}} \cdot J_{k-1,i}$ such that the corresponding intervals $J_{k,1}, J_{k,2}, \cdots$ are pairwise disjoint and satisfy $E_n(*) \cap e^{-C_5 r_{k-1}} \cdot J_{k-1,i} \subset \bigcup_j J_{k,j}$. Lemma 6.7 gives $\bigcup_j J_{k,j} \subset 2e^{-C_5 r_{k-1}} \cdot J_{k-1,i}$. Repeat the same construction for every $J_{k-1,i}$. (ii) (iii) for

$J_{k,1}, J_{k,2}, \cdots$ follow from the construction. (iv) follows from the pairwise disjointness of the intervals and the next

Lemma 6.10. *For every i, $|J_{1,i}| \leq 4|\Delta_0|$.*

Proof. Recall that $J_{1,i} = J_{\nu_1}(a_{1,i}, \zeta_{1,i})$, where $a_{1,i} \in \Delta_{n-1} \setminus \Delta_n$ and $\zeta_{1,i}$ is a critical point responsible for $a_{1,i}$. If $\nu_1 - 1 \geq -\log \varepsilon/\lambda$, then $|J_{1,i}| \leq \|w_{\nu_1}(\zeta_{1,i})\|^{-1} \leq e^{-\lambda(\nu_1-1)} \leq \varepsilon$. As $\Delta_0 = [a^* - 2\varepsilon, a^* - \varepsilon]$, the desired inequality follows.

Suppose that $\nu_1 - 1 < -\log \varepsilon/\lambda$. As $a_{1,i} \in \Delta_0$, it suffices to show $a^* \notin J_{1,i}$. We derive a contradiction assuming $a^* \in J_{1,i}$. By condition (A3) on the interval map g_{a^*}, it is possible to choose sufficiently small b depending only on ε so that all quasi critical points of f_{a^*} are apart from $I(\delta)$ in a distance by at least $\frac{1}{2}K_0$ during their first $[-\log \varepsilon/\lambda]$ iterates. Consider the ν_1-adapted deformation $a \in J_{1,i} \mapsto z(a)$ of $\zeta_{1,i}$, and write $z_{\nu_1}(a) = f_a^{\nu_1} z(a)$. Since ν_1 is a return time of $\zeta_{1,i}$, $z_{\nu_1}(a_{1,i}) \in I(2\delta)$ holds. Hence $|z_{\nu_1}(a^*) - z_{\nu_1}(a_{1,i})| \geq \frac{1}{3}K_0$. On the other hand, Lemma 6.6 and (40) together imply $|z_{\nu_1}(a^*) - z_{\nu_1}(a_{1,i})| \leq \frac{1}{5}K_0$. We reach a contradiction. $\square$

6.8. **Hölder distortion.** For the proof of Lemma 6.6 we need the next distortion estimate. We assume ζ is a critical point on a horizontal curve γ. Let ω be a curve in γ containing a point having p with its bound period, and length$(\omega) \leq d(\zeta, \omega)^{1+\epsilon}$. Here, d denotes the minimal distance apart and $\delta \ll \epsilon$. For our purpose, $\epsilon = 1/3$ suffices. For $z \in \omega$, let $t(z)$ denote any unit vector tangent to ω at z.

Sublemma 6.2. *For all $\xi, \eta \in \omega$,*

$$\left| \frac{\|Df^p t(\xi)\|}{\|Df^p t(\eta)\|} - 1 \right| \leq C|f^p\xi - f^p\eta|^{\frac{\epsilon}{1+\epsilon}}.$$

Proof. From the assumption, the contractive fields e_i, $1 \leq i < p$ are well-defined in a neighborhood of $f\omega$. Let z denote both ξ and η. Split $Dft(z) = A(z)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B(z)e_{p-1}(fz)$. Then $\|Df^p t(\xi) - Df^p t(\eta)\| \leq I_1 + I_2 + I_3 + I_4$, where

$$\begin{aligned}
I_1 &= |A(\xi) - A(\eta)|\|Df^{p-1}(f\xi)\|, \\
I_2 &= |B(\xi) - B(\eta)|\|Df^{p-1}(f\xi)\|, \\
I_3 &= |B(\eta)|\|Df^{p-1}(f\xi)e_{p-1}(f\xi) - Df^{p-1}(f\eta)e_{p-1}(f\eta)\|, \\
I_4 &= |A(\eta)|\|Df^{p-1}(f\xi)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - Df^{p-1}(f\eta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|.
\end{aligned}$$

We divide the rest of the proof into three steps. First we estimate I_1, I_2, I_3. Next we estimate I_4. In the last step we glue all these estimate together and complete the proof.

Step 1(Estimates of I_1, I_2, I_3). The proof of Lemma 2.2 implies $|A(\xi) - A(\eta)| \leq C|\xi - \eta|$. Hence

$$I_1 \leq C|\xi - \eta|\|w_p(\zeta)\| \leq Cd(\zeta, \omega)|\xi - \eta|^{\frac{\epsilon}{1+\epsilon}}\|w_p(\zeta)\|.$$

The last inequality follows from the assumption on ω. The same reasoning gives

$$I_2 \leq Cd(\zeta, \omega)|\xi - \eta|^{\frac{\epsilon}{1+\epsilon}}\|w_p(\zeta)\|.$$

The second estimate in Corollary 2.2 gives

$$I_3 \leq (Cb)^{p-1}|\xi - \eta| \leq |\xi - \eta|\|w_p(\zeta)\|.$$

Step 2(Estimate of I_4). Take a point r such that the long stable leaf of order $p-1$ through $f\eta$ intersects the horizontal line through $f\xi$ at fr. For a point y and $i\geq 1$, let $w_i(y) = Df^{i-1}(fy)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Let

$$\theta_i = \text{angle}(w_i(\xi), w_i(\eta),\ \theta_i' = \text{angle}(w_i(\eta), w_i(r)),\ \theta_p'' = \text{angle}(w_p(\xi), w_p(r)).$$

Integrations of the two inequalities as in Lemma 2.2 along the path in ω connecting ξ and η give $|f\xi - fr| \leq Cd(\zeta,\omega)|\xi-\eta|$ and $|f\eta - fr|\leq C\sqrt{b}|\xi-\eta|$. The second estimate in Lemma 2.6 give

$$\left|\frac{\|w_p(r)\|}{\|w_p(\xi)\|}-1\right| \leq C\frac{|f\xi-fr|}{d^2(\zeta,\omega)} \leq C|\xi-\eta|^{\frac{\epsilon}{1+\epsilon}}.$$

(G2) on ζ and the bounded distortion give $\|w_{i+1}(\eta)\|\geq Ce^{-\alpha i}\|w_i(\eta)\|$ and $\|w_{i+1}(r)\|\geq Ce^{-\alpha i}\|w_i(r)\|$ for $1\leq i<p$. Hence

$$\left|\log\frac{\|w_p(\eta)\|}{\|w_p(r)\|}\right| \leq \sum_{i=1}^{p-1}\left|\log\frac{\|w_{i+1}(\eta)\|}{\|w_i(\eta)\|} - \log\frac{\|w_{i+1}(r)\|}{\|w_i(r)\|}\right| \leq C\sum_{i=1}^{p-1} e^{\alpha i}(|f^i\eta - f^ir| + \theta_i').$$

Using $|f^i\eta - f^ir|\leq (Cb)^{i-1}|f\eta - fr|$ and $\theta_i'\leq (Cb)^{i-1}|f\eta-fr|$ which follows from the proof of Sublemma 3.1, we get

$$\left|\log\frac{\|w_p(\eta)\|}{\|w_p(r)\|}\right| \leq C|f\eta - fr|\sum_{i=1}^{p-1}e^{\alpha i}(Cb)^{i-1}\leq C|\xi-\eta|.$$

These two estimates yield

$$\left|\frac{\|w_p(\eta)\|}{\|w_p(\xi)\|}-1\right|\leq C|\xi-\eta|^{\frac{\epsilon}{1+\epsilon}}.$$

Let l denote the horizontal connecting $f\xi$ and fr. Then $f^{p-1}l$ is $C^2(b)$ and

$$\theta_p''\leq\sqrt{b}|f^p\xi - f^pr|\leq C\sqrt{b}|f^p\xi - f^p\eta|.$$

The second is because of the definition of r and the fact that $f^p\omega$ is $C^2(b)$. Together with the upper estimate of θ_p' and $|f^p\xi - f^p\eta|\geq|\xi-\eta|$, we obtain

$$\theta_p\leq\theta_p'+\theta_p''\leq C\sqrt{b}|f^p\xi - f^p\eta|.$$

Using $|A(\eta)|\leq Cd(\zeta,\omega)$ and $\|w_p(z)\|\approx\|w_p(\zeta)\|$,

$$\begin{aligned} I_4 &\leq Cd(\zeta,\omega)\|w_p(\xi)-w_p(\eta)\|\leq Cd(\zeta,\omega)\|w_p(\zeta)\|\left(\theta_p + \left|\frac{\|w_p(\xi)\|}{\|w_p(\eta)\|}-1\right|\right)\\ &\leq Cd(\zeta,\omega)\|w_p(\zeta)\||f^p\xi - f^p\eta|^{\frac{\epsilon}{1+\epsilon}}.\end{aligned}$$

Step 3(Overall estimate). Gluing all the estimates together, we obtain

$$\|Df^pt(\xi) - Df^pt(\eta)\|\leq C\|w_p(\zeta)\|d(\zeta,\omega)|f^p\xi - f^p\eta|^{\frac{\epsilon}{1+\epsilon}}.$$

Combining this with $\|Df^pt(z)\|\geq C\|w_p(\zeta)\|d(\zeta,\omega)$ yields the desired inequality. $\square$

6.9. **Proof of Lemma 6.6.** We proceed by induction on i. To ease notation, let us write $\zeta(a,k)=z(a)$, and for $i\geq 0$, $f_a^iz(a)=z_i(a)$. When no ambiguity arises, we drop a from notation and write $f_a=f$, $z_i(a)=z_i$.

Step1($i=k_0+1$). (a) for $i=k_0+1$ follows from Lemma 6.4.

Proof of (b). It suffices to show the next

Sublemma 6.3. *For $i=k_0,k_0+1$ and all $a\in I_n(\hat{a})$, $\|\ddot{z}_i\|\leq 10^{-3}\|\dot{z}_i\|^3$.*

Proof. We have $\dot{z}_i=Df(z_{i-1})\dot{z}_{i-1}+\psi(z_{i-1})$, where $\psi(z)=\frac{\partial(f_{\tilde{a}}z)}{\partial\tilde{a}}(a)$. Using this inductively,

$$\dot{z}_i=Df^{i-1}(z_1)\dot{z}_1+\sum_{s=1}^{i-1}Df^{i-s-1}(z_{s+1})\psi(z_s). \tag{59}$$

For each $0\leq s<i-1$, using $\prod_{j=s+1}^{i-1}\|Df(z_j)\|\approx\|Df^{i-s-1}(z_{s+1})\|\approx\|w_i(\zeta)\|/\|w_{s+1}(\zeta)\|$ because of $i\in\{k_0,k_0+1\}$,

$$\left\|\frac{d}{da}Df^{i-s-1}(z_{s+1})\right\|\leq\|Df^{i-s-1}(z_{s+1})\|\sum_{j=s+1}^{i-1}\frac{C+C\|\dot{z}_j\|}{\|Df(z_j)\|}\leq C\frac{\|w_i(\zeta)\|}{\|w_{s+1}(\zeta)\|}\sum_{j=s+1}^{i-1}\|w_j(\zeta)\|.$$

From $\|\dot{z}_i\|\geq C\|w_i(\zeta)\|$ in Lemma 6.4, we have

$$\frac{1}{\|\dot{z}_i\|^2}\left\|\frac{d}{da}Df^{i-s-1}(z_{s+1})\right\|\leq\frac{C}{\|w_{s+1}(\zeta)\|}\sum_{j=s+1}^{i-1}\frac{\|w_j(\zeta)\|}{\|w_i(\zeta)\|}\leq C.$$

Using this for $s=0$ and the uniform boundedness of $\|\dot{z}_1\|$, $\|\ddot{z}_1\|$ from Proposition 4.2,

$$\frac{1}{\|\dot{z}_i\|^2}\left(\left\|\frac{d}{da}Df^{i-1}(z_1)\right\|\|\dot{z}_1\|+\|Df^{i-1}(z_1)\|\|\ddot{z}_1\|\right)\leq C\kappa_0^{-10\log(1/\delta)}.$$

On the other hand, for each $1\leq s\leq i-1$ we have

$$\left\|\frac{d}{da}\psi(z_s)\right\|\leq C\|\dot{z}_s\|\leq C\|w_s(\zeta)\|.$$

Hence

$$\frac{1}{\|\dot{z}_i\|^2}\left\|Df^{i-s-1}(z_{s+1})\cdot\frac{d}{da}\psi(z_s)\right\|\leq\frac{C\|w_s(\zeta)\|}{\|w_i(\zeta)\|\|w_{s+1}(\zeta)\|}\leq C.$$

Differentiating (59) and substituting these estimates yields

$$\frac{\|\ddot{z}_i\|}{\|\dot{z}_i\|^3}\leq\frac{C}{\|\dot{z}_i\|}\left(\kappa_0^{-10\log(1/\delta)}+i\right)\leq 10^{-3}.$$

The last inequality holds for sufficiently large k_0. □

Proof of (c) *for $i=k_0+1$.* Let $j\geq k_0$ and let A_j denote the curvature of γ_j at z_j. Let

$$A'_{j+1}=\frac{\|Df(z_j)\dot{z}_j\times\ddot{z}_{j+1}\|}{\|\ddot{z}_{j+1}\|^3},\quad A''_{j+1}=\frac{\|\psi(z_j)\times\ddot{z}_{j+1}\|}{\|\ddot{z}_{j+1}\|^3},$$

Note that $A_{j+1}\leq A'_{j+1}+A''_{j+1}$.

Sublemma 6.4. *For every $j \geq k_0$,*

$$A'_{j+1} \leq Cb \frac{\|\dot{z}_j\|^3}{\|\dot{z}_{j+1}\|^3} \left(A'_j + A''_j + 1\right).$$

Proof. Write $F(a,z) = f_a z$. Differentiating $z_{j+1} = F(a, z_j)$ twice and then substituting the result into the definition of A'_{j+1}, we have $A'_{j+1} \leq I + II + III$, where

$$\begin{aligned} I &= \|\dot{z}_{j+1}\|^{-3} \left\|Df(z_j)\dot{z}_j \times (\partial_{za} F \dot{z}_j + \partial_{aa} F)\right\|, \\ II &= \|\dot{z}_{j+1}\|^{-3} \left\|Df(z_j)\dot{z}_j \times \left(D^2 f(\dot{z}_j) + \partial_{az} F\right) \dot{z}_j\right\|, \\ III &= \|\dot{z}_{j+1}\|^{-3} \left\|Df(z_j)\dot{z}_j \times Df(z_j)\ddot{z}_j\right\|. \end{aligned}$$

All the partial derivatives are taken at (a, z_j). The $D^2 f(\dot{z}_j)$ in II is defined as follows. Let $Df(z_j) = \left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right)$, $\nabla = \partial_x + \partial_y$, and

$$D^2 f(\dot{z}_j) = \begin{pmatrix} \langle \nabla f_{11}, \dot{z}_j \rangle & \langle \nabla f_{12}, \dot{z}_j \rangle \\ \langle \nabla f_{21}, \dot{z}_j \rangle & \langle \nabla f_{22}, \dot{z}_j \rangle \end{pmatrix},$$

where $\langle \cdot, \cdot \rangle$ denotes the scholar product.

The second components of the vectors involved in the product in I has a factor b. Hence

$$I \leq Cb \frac{\|\dot{z}_j\|^2 + \|\dot{z}_j\|}{\|\dot{z}_{j+1}\|^3} \leq b \frac{\|\dot{z}_j\|^3}{\|\dot{z}_{j+1}\|^3}.$$

For the last inequality we have used $\|\dot{z}_j\| \gg 1$ which follows from Lemma 6.4. In the same way,

$$II \leq b \frac{\|\dot{z}_j\|^3}{\|\dot{z}_{j+1}\|^3}.$$

For the last term,

$$III \leq Cb \frac{\|\dot{z}_j\|^3}{\|\dot{z}_{j+1}\|^3} \frac{\|\dot{z}_j \times \ddot{z}_j\|}{\|\dot{z}_j\|^3} \leq Cb \frac{\|\dot{z}_j\|^3}{\|\dot{z}_{j+1}\|^3} (A'_j + A''_j).$$

Putting these three inequalities together we obtain the desired one. □

Lemma 6.3 for $i = k_0$ gives $A'_{k_0} \leq C$, $A''_{k_0} \leq C$. Hence Sublemma 6.4 gives $A'_{k_0+1} \leq Cb$. Together with $A''_{k_0+1} \leq 1/1000$ which follows from Lemma 6.3 we obtain $A_{k_0+1} \leq 1/100$.

Step2($j \to j+p$). Suppose that (a), (b), (c) hold for some $j \in [k_0 + 1, \nu_k)$ such that $f^j_{\hat{a}}\zeta$ is free. If $f^j_{\hat{a}}\zeta \in I(\delta)$, then let p denote the bound period. Otherwise, let $p = 1$. In either of the two cases, $f^{j+p}_{\hat{a}}\zeta$ is free and $j + p \leq \nu_k$.

Proof of (a) *for $i = j + p$.*

Sublemma 6.5. *For all $a \in J_{\nu_k}(\hat{a}, \zeta)$,*

$$\left\| Df^p(z_j) \frac{\dot{z}_j}{\|\dot{z}_j\|} - \frac{w_{j+p}(\zeta)}{\|w_j(\zeta)\|} \right\| \leq \frac{1}{4} \left(\rho_{j+p} - \rho_j\right) \frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|}.$$

Proof. The left hand side is $\leq I + II + III$, where

$$\begin{aligned} I &= \left\| Df_{\hat{a}}^p(z_j(\hat{a})) \frac{\dot{z}_j(\hat{a})}{\|\dot{z}_j(\hat{a})\|} - \frac{w_{j+p}(\zeta)}{\|w_j(\zeta)\|} \right\|, \\ II &= \|Df_{\hat{a}}^p(z_j(a)) - Df_a^p(z_j(a))\|, \\ III &= \left\| Df_{\hat{a}}^p(z_j(a)) \frac{\dot{z}_j(a)}{\|\dot{z}_j(a)\|} - Df_{\hat{a}}^p(z_j(\hat{a})) \frac{\dot{z}_j(\hat{a})}{\|\dot{z}_j(\hat{a})\|} \right\|. \end{aligned}$$

Recall that $\theta_j(\hat{a})$ is the angle made by $w_j(\zeta)$ and $\dot{z}_j(\hat{a})$. Then

$$I \leq C\theta_j(\hat{a})\|Df_{\hat{a}}^p(z_j(\hat{a}))\| + \|Df_{\hat{a}}^p(z_j(\hat{a})) - Df_{\hat{a}}^p(f_{\hat{a}}^j\zeta)\| \leq e^{-\frac{\lambda j}{4}}. \tag{60}$$

To bound the first term of the right-hand-side, we have used (d) for $i = j$ and $p \leq C\alpha j \ll j$. By Lemma 6.2, the second term is bounded by $(Cb)^{\frac{\theta\nu_k}{200}}$.

To estimate I we deal with two cases separately.

Case (i): $p = 1$. Since the curvature of γ_j is $\leq 1/100$ from the inductive assumption (c),

$$III \leq \frac{1}{20}|z_j(\hat{a}) - z_j(a)| + II.$$

$$II + III \leq \frac{1}{20}|z_j(\hat{a}) - z_j(a)| + 2II \leq \frac{1}{10}|z_j(\hat{a}) - z_j(a)|.$$

By the definition of $\Theta_{\nu_k}(\zeta)$,

$$II + III \leq \frac{1}{10}\text{length}(\gamma_j) \leq \frac{1}{5}\|\dot{z}_j(\hat{a})\|\Theta_{\nu_k}(\zeta) \leq \|w_j(\zeta)\|\Theta_{\nu_k}(\zeta) = \frac{\|w_{j+1}(\zeta)\|}{\|w_j(\zeta)\|}\sigma_j^{-1}(\zeta)\Theta_{\nu_k}(\zeta).$$

Combining this with (60) we get the desired inequality.

Case (ii): $p > 1$. Let z denote the binding point for $f_{\hat{a}}^j\zeta$. As $p \ll j$ we have

$$II \leq C^p|\hat{a} - a| \leq |z_j(\hat{a}) - z_j(a)| \leq C \cdot \text{length}(\gamma_j) \leq |z - f_{\hat{a}}^j\zeta|^{\frac{10}{9}}\sigma_j^{-1}(\zeta)\Theta_{\nu_k}(\zeta). \tag{61}$$

$\text{length}(\gamma_j) \leq |z - f_{\hat{a}}^j\zeta|^{\frac{10}{9}}$. It is possible to choose a horizontal curve $\tilde{\gamma}$ containing γ_j, on which z lies. This allows us to use Sublemma 6.2 with $\epsilon = 1/3$ to get

$$III \leq \frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|}\text{length}(\gamma_j)^{\frac{1}{9}}.$$

This and (60) yield the desired estimate. □

Sublemma 6.5 and $\rho_{j+p} - \rho_j \leq 1$ gives

$$\left\| Df^p(z_j)\frac{\dot{z}_j}{\|\dot{z}_j\|} \right\| \geq \frac{3}{4}\frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|}. \tag{62}$$

Hence

$$\left| \log \left\| Df^p(z_j)\frac{\dot{z}_j}{\|\dot{z}_j\|} \right\| - \log\frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|} \right| \leq \frac{1}{3}(\rho_{j+p} - \rho_j). \tag{63}$$

Dividing the both sides of $\|\dot{z}_{j+p} - Df^p(z_j)\dot{z}_j\| \leq C^p$ by $\|\dot{z}_j\| \approx \|w_j(\zeta)\|$ and then using $p \ll j$,

$$\left| \frac{\|\dot{z}_{j+p}\|}{\|\dot{z}_j\|} - \left\| Df^p(z_j)\frac{\dot{z}_j}{\|\dot{z}_j\|} \right\| \right| \leq \|w_j(\zeta)\|^{-1/2}. \tag{64}$$

This and (62) together imply

$$\frac{\|\dot{z}_{j+p}\|}{\|\dot{z}_j\|} \geq \frac{3}{4}\frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|} - \|w_j(\zeta)\|^{-\frac{1}{2}} \geq \frac{1}{2}\frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|}. \tag{65}$$

This implies

$$\left|\log\left\|Df^p(z_j)\frac{\dot{z}_j}{\|\dot{z}_j\|}\right\| - \log\frac{\|\dot{z}_{j+p}\|}{\|\dot{z}_j\|}\right| \leq 2\|w_j(\zeta)\|^{-\frac{1}{2}}\frac{\|w_j(\zeta)\|}{\|w_{j+p}(\zeta)\|} \leq \frac{1}{6}\|w_j(\zeta)\|^{-\frac{1}{2}} \leq \frac{1}{6}(\rho_{j+p}-\rho_j).$$

Hence, for all $a \in J_{\nu_k}(\hat{a},\zeta)$,

$$\left|\log\frac{\|\dot{z}_{j+p}(a)\|}{\|\dot{z}_j(a)\|} - \log\frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|}\right| \leq \frac{1}{2}\left(\rho_{j+p}-\rho_j\right).$$

This yields

$$\left|\log\frac{\|\dot{z}_{j+p}(a)\|}{\|\dot{z}_{j+p}(\hat{a})\|} - \log\frac{\|\dot{z}_j(a)\|}{\|\dot{z}_j(\hat{a})\|}\right| \leq \rho_{j+p}-\rho_j.$$

This and the assumption $\left|\log\frac{\|\dot{z}_j(a)\|}{\|\dot{z}_j(\hat{a})\|}\right| \leq \rho_j$ yield $\left|\log\frac{\|\dot{z}_{j+p}(a)\|}{\|\dot{z}_{j+p}(\hat{a})\|}\right| \leq \rho_{j+p}$. This proves the first half of (a) for $i=j+p$.

For every free return time $i<j$, (64) implies $\|\dot{z}_j\| \geq e^{\frac{\lambda}{4}(j-i)}\|\dot{z}_i\|$, and thus $\mathrm{length}(\gamma_j) \geq e^{\frac{\lambda}{4}(j-i)}\mathrm{length}(\gamma_i)$. This yields

$$\sum_{\substack{i\leq j\\ \text{free return}}} \mathrm{length}(\gamma_i)^{\frac{1}{10}} \leq \mathrm{length}(\gamma_j)^{\frac{1}{10}}\sum_{i\leq j} e^{-\frac{\lambda}{40}(j-i)},$$

which implies $\rho_{j+p}\leq 1$. This proves the second half of (a) for $i=j+p$.

Proof of (b) *for $i=j+p$.* For every $k_0\leq i\leq j$ we have

$$\|\ddot{z}_i\| \leq (C\delta)^{-3(j-i)}\|\dot{z}_j\|^3 \leq (C\delta)^{-3(j-i+1)}\|\dot{z}_{j+p}\|^3 \leq (C\delta)^{-3(j+p-i)}\|\dot{z}_{j+p}\|^3,$$

where we have used: (b) for the previous step for the first inequality; $\|\dot{z}_j\| \leq (C\delta)^{-1}\|\dot{z}_{j+p}\|$ for the second inequality. Hence, it suffices to show for $j+1\leq i\leq j+p$,

$$\|\ddot{z}_i\| \leq \|\dot{z}_{j+p}\|^3. \tag{66}$$

Write $G(a,z)=f_a^{i-j}z$. Let $\partial_z G = \left(\begin{smallmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{smallmatrix}\right)$, $\nabla = \partial_x+\partial_y$, and define

$$\partial_{zz}G(\cdot) = \begin{pmatrix} \langle\nabla g_{11},\cdot\rangle & \langle\nabla g_{12},\cdot\rangle \\ \langle\nabla g_{21},\cdot\rangle & \langle\nabla g_{22},\cdot\rangle\end{pmatrix},$$

where $\langle\cdot,\cdot\rangle$ denotes the scholar product. Differentiating $z_i = G(a,z_j)$ gives

$$\ddot{z}_i = \partial_{za}G\dot{z}_j + \partial_{aa}G + \left(\partial_{zz}G(\dot{z}_j) + \partial_a\left(\partial_z G\right)\right)\dot{z}_j + \partial_z G\ddot{z}_j,$$

where all the partial derivatives are taken at (a,z_j). We have:

$$\|\partial_{za}G\| \leq C^p, \|\partial_{aa}G\| \leq C^p, \|\partial_a\left(\partial_z G\right)\| \leq C^p, \|\partial_{zz}G(\dot{z}_j)\| \leq C^p\|\dot{z}_j\|;$$

$$\|\partial_z G\| \leq C|z-f_{\hat{a}}^j\zeta|^{-1}\frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|} \quad \text{if } p>1;$$

$$\|\partial_z G\| \leq C \quad \text{if } p=1.$$

We first treat the case $p>1$. Using the above estimates and $\|\ddot{z}_j\|\leq\|\dot{z}_j\|^3\leq C\|w_j(\zeta)\|^3$ from the assumption of the induction,

$$\begin{aligned}\|\ddot{z}_i\|&\leq C^p\|\dot{z}_j\|^2+C|z-f_{\hat{a}}^j\zeta|^{-1}\frac{\|w_{j+p}(\zeta)\|}{\|w_j(\zeta)\|}\|\ddot{z}_j\|\\&\leq C^p\|w_j(\zeta)\|^2+C|z-f_{\hat{a}}^j\zeta|^{-1}\|w_{j+p}(\zeta)\|\|w_j(\zeta)\|^2.\end{aligned}$$

On the first term of the right-hand-side, $p\ll j$ gives

$$C^p\|w_j(\zeta)\|^2\leq\frac{1}{10}\|w_{j+p}(\zeta)\|^3.$$

On the second term, (e) Proposition 2.1 gives

$$\begin{aligned}|z-f_{\hat{a}}^j\zeta|^{-1}\|w_{j+p}(\zeta)\|\|w_j(\zeta)\|^2&=|z-f_{\hat{a}}^j\zeta|^{-1}\|w_{j+p}(\zeta)\|^3\frac{\|w_j(\zeta)\|^2}{\|w_{j+p}(\zeta)\|^2}\\&\leq|z-f_{\hat{a}}^j\zeta|^{\frac{1}{3}}\|w_{j+p}(\zeta)\|^3\leq\delta^{\frac{1}{3}}\|w_{j+p}(\zeta)\|^3.\end{aligned}$$

Plugging these into the right-hand-side yields (66). In the case $p=1$, use the alternative estimate of $\|\partial_zG\|$.

Proof of (c) *for $i=j+p$.* Using Sublemma 6.4 inductively,

$$A'_{j+p}\leq(Cb)^{j+p-k_0}\frac{\|\dot{z}_{k_0}\|^3}{\|\dot{z}_{j+p}\|^3}\cdot A'_{k_0}+\sum_{i=1}^{j+p-k_0}(Cb)^i\frac{\|\dot{z}_{j+p-i}\|^3}{\|\dot{z}_{j+p}\|^3}\left(A''_{j+p-i}+C\right).$$

Lemma 6.4 gives

$$\frac{\|\dot{z}_{k_0}\|}{\|\dot{z}_{j+p}\|}\leq C\frac{\|w_{k_0}(\zeta)\|}{\|w_{j+p}(\zeta)\|}\leq C\delta^{-1}.$$

(b) gives

$$\frac{\|\dot{z}_{j+p-i}\|^3}{\|\dot{z}_{j+p}\|^3}\cdot A''_{j+p-i}\leq(C\delta)^{-i}.$$

Plugging these into the above inequality gives $A'_{j+p}\leq Cb$. Combining this with $A''_{j+p}\leq1$ which follows from (b), we obtain $A_{j+p}\leq1/100$. This recovers the assumption of induction and completes the proof of Lemma 6.6. □

Appendix: computational proofs

A.1. Proof of Lemma 2.2. We regard γ as a graph of a function γ_0 and write $z=(x,\gamma_0(x))$. Let $e=\left(\begin{smallmatrix}e_1\\e_2\end{smallmatrix}\right)$ and $S=\left(\begin{smallmatrix}1&e_1\\0&e_2\end{smallmatrix}\right)^{-1}$. Let $R(x)$ denote the rotation matrix by the angle made by $t(z)$ and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, which we denote by $\theta(x)$. Then $A(z)$, $B(z)$ are equal to the $(1,1)$, $(2,1)$ entries of the matrix $S\cdot Df(z)\cdot R(x)^{-1}$ correspondingly. Write $S=\left(\begin{smallmatrix}1+\epsilon_1&\epsilon_2\\\epsilon_3&1+\epsilon_4\end{smallmatrix}\right)$ and $Df(z)=\left(\begin{smallmatrix}g'_a(x)+\alpha_1&\alpha_2\\\alpha_3&\alpha_4\end{smallmatrix}\right)$. A direct computation gives

$$A(z)=(1+\epsilon_1)(g'_a(x)\cos\theta+\alpha_1\sin\theta)+\epsilon_2(\alpha_3\cos\theta+\alpha_4\sin\theta),$$

$$B(z)=\epsilon_3(g'_a(x)\cos\theta+\alpha_1\sin\theta)+(1+\epsilon_4)(\alpha_3\cos\theta+\alpha_4\sin\theta).$$

To evaluate $A'=dA/dx$, we use $|\theta|\leq1/10$, $|\theta'|\leq1/5$, $|\epsilon_i|\leq C\sqrt{b}$, $|\alpha_i|\leq Cb\quad(i=1,2,3,4)$, and the non-degeneracy of Crit. Then $|A'|\approx1$ holds. Since $A(\zeta)=0$, the mean value theorem gives the desired estimate of $|A|$. The estimate of $|B|$ is straightforward from the formula. □

A.2. Proof of Lemma 3.1. By (i) (ii), the vector fields e_i ($i=1,2,\cdots,\max\{m,n\}$) are well-defined in a neighborhood of $f\gamma$. Let $t(s)$ denote any unit vector tangent to γ at $\gamma(s)$. Let $\hat\gamma(s)=f\gamma(s)$. Split $Dft(s)$ in two different ways:

$$A(s)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+B(s)e_n(\hat\gamma(0))=Dft(s)=A'(s)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+B'(s)e_m(\hat\gamma(s)).$$

Let $\psi(s)=\mathrm{angle}(e_n(\hat\gamma(0)),e_m(\hat\gamma(s)))$. Comparing the two components of the vectors on both sides,

$$|A(s)-A'(s)|\leq 2|B(s)|\psi(s)\leq C\sqrt{b}\psi(s), \tag{67}$$

where the second inequality follows from Lemma 2.2. From the results in Sect.2.3,

$$\psi(s)\leq \mathrm{angle}(e_n(\hat\gamma(0)),e_n(\hat\gamma(s)))+\mathrm{angle}(e_n(\hat\gamma(s)),e_m(\hat\gamma(s)))\leq C\sqrt{b}|s|+(Cb)^{\frac{n}{3}}.$$

This gives $\psi(\pm b^{\frac{n}{4}})\leq Cb^{\frac{n}{4}}$, and therefore $|A(\pm b^{\frac{n}{4}})-A'(\pm b^{\frac{n}{4}})|\leq Cb^{\frac12+\frac{n}{4}}$. Lemma 2.2 gives $|A(\pm b^{\frac{n}{4}})|\approx b^{\frac{n}{4}}$ and $A(-b^{\frac{n}{4}})A(b^{\frac{n}{4}})<0$. Then $A'(-b^{\frac{n}{4}})A'(b^{\frac{n}{4}})<0$ follows. Hence there exists $s_0\in[-b^{\frac{n}{4}},b^{\frac{n}{4}}]$ such that $A'(s_0)=0$. In other words, $\gamma(s_0)$ is a critical approximation of order m on γ. □

A.3. Proof of Lemma 3.2. Let $\hat\gamma_\sigma(s)=f\gamma_\sigma(s)$, $\sigma=1,2$. Split

$$Dft_2(s)=A(s)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+B(s)e_n(\hat\gamma_1(0)).$$

Since γ_2 is $C^2(b)$, it is possible to choose a horizontal curve which is tangent to $t_1(0)$, $t_2(\varepsilon^{\frac{n}{2}})$, $t_2(-\varepsilon^{\frac{n}{2}})$. Lemma 2.2 applied to this curve implies $A(\varepsilon^{\frac{n}{2}})A(-\varepsilon^{\frac{n}{2}})<0$. Hence, $A(s_0)=0$ holds for some s_0. Since γ_2 is a horizontal curve, the uniqueness of such s_0 follows from Lemma 2.2.

By (i) (ii), the contractive fields $e_1,\cdots,e_n$ are well-defined in a neighborhood of $f(\gamma_2)$. Split

$$Dft_2(s)=A'(s)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+B'(s)e_n(\hat\gamma_2(s)).$$

Let $\psi(s)=\mathrm{angle}(e_n(\hat\gamma_1(0)),e_n(\hat\gamma_2(s)))$. Comparing the components of the above two equalities,

$$|A(s)-A'(s)|\leq 2|B(s)|\psi(s)\leq C\sqrt{b}\psi(s), \tag{68}$$

where the last inequality follows from Lemma 2.2. By the results in Sect.2.3,

$$\begin{aligned}\psi(s)&\leq \mathrm{angle}(e_n(\hat\gamma_1(0)),e_n(\hat\gamma_1(s)))+\mathrm{angle}(e_n(\hat\gamma_1(s)),e_n(\hat\gamma_2(s)))\\&\leq C\sqrt{b}|s|+C\sqrt{b}(|s|+\varepsilon^n).\end{aligned}$$

To estimate the second term of the right-hand-side of the first inequality we have used

$$|\hat\gamma_1(s)-\hat\gamma_2(s)|\leq|\hat\gamma_1(s)-\hat\gamma_1(0)|+|\hat\gamma_1(0)-\hat\gamma_2(0)|+|\hat\gamma_2(0)-\hat\gamma_2(s)|\leq C|s|+C\varepsilon^n,$$

which follows from (iii). Then $\psi(\pm\varepsilon^{\frac{n}{2}})\leq C\sqrt{b}\varepsilon^{\frac{n}{2}}$, and hence $|A(\pm\varepsilon^{\frac{n}{2}})-A'(\pm\varepsilon^{\frac{n}{2}})|\ll\varepsilon^{n/2}$ follows. Lemma 2.2 gives $|A(\pm\varepsilon^{\frac{n}{2}})|\approx\varepsilon^{\frac{n}{2}}$, and therefore $A'(-\varepsilon^{\frac{n}{2}})A'(\varepsilon^{\frac{n}{2}})<0$ follows. Hence $A'(s_0)=0$ holds for some $s_0\in[-\varepsilon^{\frac{n}{2}},\varepsilon^{\frac{n}{2}}]$. □

A.4. Proof of Lemma 3.3. Let $\hat{\mathcal{H}}=\{\hat\mu_1<\hat\mu_2<\cdots<\hat\mu_{\hat s}\}$ denote any sequence of integers in $[0,m]$ with the following properties:

(i) $\hat\mu_1<m/2$ and $\hat\mu_s\geq m-\log(1/\delta)$;
(ii) $\|Df^{\hat\mu_i+j}v\|\geq\kappa_0^{\frac{j}{4}}\|Df^{\hat\mu_i}v\|$ for $1\leq j\leq m-\hat\mu_i$;
(iii) $4(m-\hat\mu_i)\geq m-\hat\mu_{i-1}$.

We finish the proof of the lemma assuming the existence of such a sequence. Define a subsequence $\mathcal{H}$ of $\hat{\mathcal{H}}$ inductively as follows. Let $\hat{\mu}_{\hat{s}} \in \mathcal{H}$. If $\hat{\mu}_j \in \mathcal{H}$, let $\psi(j) < j$ denote the largest such that $4(m-\hat{\mu}_j) \geq m - \hat{\mu}_{\psi(j)}$. Let $\hat{\mu}_{j-1}, \cdots, \hat{\mu}_{\psi(j)} \notin \mathcal{H}$. Unless $\psi(j) = 1$, let $\hat{\mu}_{\psi(j)-1} \in \mathcal{H}$.

Write $\mathcal{H} = \{\mu_1 < \mu_2 < \cdots < \mu_s\}$. By definition, $\mu_s \geq m - \log(1/\delta)$, $4(m-\mu_1) \geq m/2$ and $4(m-\mu_{i+1}) < m - \mu_i$. To finish, we prove the lower estimate in (b). Let $\mu_{i+1} = \hat{\mu}_j$. Then $\mu_i = \hat{\mu}_{\psi(j)-1}$, and

$$m - \mu_{i+1} = m - \hat{\mu}_j \geq (1/4)(m - \hat{\mu}_{\psi(j)}) \geq (1/16)(m - \hat{\mu}_{\psi(j)-1}) \geq (1/16)(m-\mu_i).$$

To prove the existence of such a sequence, we borrow an argument in the proof of [[22], Claim 5.1]].

Sublemma 6.6. *For each $i \in [\log(1/\delta), m]$ there exists $i' \in [m-i, m-[i/2]]$ such that $\|Df^{i'+j}v\| \geq \kappa_0^{\frac{j}{4}} \|Df^{i'}v\|$ holds for $1 \leq j \leq m - i'$.*

Proof. Let $\mathcal{G}$ denote the graph of the function $k \in [0,m] \to \log\|Df^k v\|$. Let L denote the infinite line through the point $(m, \log\|Df^m v\|)$ with slope $\log C_0$. All points of $\mathcal{G}$ lies above L. Let P denote the point of intersection between L and the vertical line $\{x = m - [i/2]\}$. Let L be pivoted at P and rotate it clockwise until it hits $\mathcal{G}$. Let i' be such that $(i', \log\|Df^{i'}v\|)$ belongs to the set of points of the first hit. We clearly have $i' \in [m-i, m-[i/2]]$. The slope of the rotated L in its final position is bigger than

$$\log(1/C_0) + (m-i')^{-1}\log\frac{\|Df^m v\|}{\|Df^{i'}v\|} \geq \frac{1}{4}\log\kappa_0,$$

where we have used $m - i' \geq i \geq \log(1/\delta)$ and $\|Df^m v\| \geq (r_0\delta/10)\cdot\|Df^{i'}v\|$ for the first inequality. Since $\mathcal{G}$ lies above L in its final position, the desired inequality holds. ☐

Consider the maximal monotone decreasing sequence in $\{i'\}_{\log(1/\delta)\leq i \leq m}$. By Sublemma 6.6 and $m \geq 3\log(1/\delta)$, it contains multiple integers and satisfies (i) (ii). It also satisfies (iii), by the next

Sublemma 6.7. *If $i' < j'$ and $k' \notin (i', j')$ for every $k \in [\log(1/\delta), m]$, then $4(m-j') \geq m - i'$.*

Proof. We have $i' \leq m - [i/2] \leq [i/2]'$. Hence $j' \leq [i/2]' \leq m - i/4$, and thus $4(m-j') \geq i$. We also have $i \geq m - i'$. ☐

A.5. Proof of Lemma 6.4. We adapt the proof of [[22] Proposition 6.1] to our setting. We have $\dot{z}_i = Df(z_{i-1})\dot{z}_{i-1} + \psi(z_{i-1})$, where $\psi(z) = \frac{\partial(f_a z)}{\partial a}(\hat{a})$. Using this inductively,

$$\dot{z}_i = Df^{i-1}(z_1)\dot{z}_1 + \sum_{s=1}^{i-1} Df^{i-s-1}(z_{s+1})\psi(z_s).$$

Sublemma 6.8. *For each $i \in [k_0, \nu_k]$ we have*

$$\|Df^{i-s}(f^s\zeta)\| \leq e^{-\lambda s/2}\|w_i(\zeta)\| \quad \text{for } 0 \leq s \leq i.$$

By the sublemma and the uniform boundedness of $\dot{z}_1$ from Proposition 4.2,

$$\frac{\|\dot{z}_i\|}{\|w_i(\zeta)\|} \leq C\sum_{s=0}^{\infty} e^{-\lambda s/2} \leq C.$$

Hence the second inequality holds.

To prove the first inequality, split $\dot{z}_i = I + II$, where

$$I = Df^{i-1}(z_1)\dot{z}_1 + \sum_{s=1}^{k_0} Df^{i-s-1}(z_{s+1})\psi(z_s),$$

$$II = \sum_{s=k_0+1}^{i-1} Df^{i-s-1}(z_{s+1})\psi(z_s).$$

Write

$$I = Df^{i-k_0}(z_{k_0})V,$$

where

$$V = Df^{k_0-1}(z_1)\dot{z}_1 + \sum_{s=1}^{k_0-1} Df^{k_0-s-1}(z_{s+1})\psi(z_s).$$

Sublemma 6.9. *There exists $C > 0$ such that $\|V\| \geq C\|w_{k_0}(\zeta)\|$.*

Proof. Let $x_0 \in \mathrm{Crit}$ be such that $(x_0, 0)$ and ζ belong to the same component of $I(\delta)$. Let $x_i = g^i_{a^*}x_0$. As $(a,b) \to (a^*, 0)$ we have $z_1 \to (x_1, 0)$, $\|w_{k_0}(\zeta)\| \to \pm(g^{k_0-1}_{a^*})'x_1$, $\dot{z}_1 \to \left(\frac{dx_1}{da}(a^*), 0\right)$. The last convergence is because of $\dot{z}_1 = Df(z_0)\dot{z}_0 + \psi(z_0)$ and the uniform boundedness of $\dot{z}_0$. Hence

$$\frac{1}{\|w_{k_0}(\zeta)\|} Df^{k_0-1}(z_1)\dot{z}_1 \to \left(\pm\frac{dx_1}{da}(a^*), 0\right).$$

We also have $\psi(z_s) \to \frac{\partial g}{\partial a}(a^*, x_s)$, where $g(a,x) = g_a x$. Hence

$$\frac{1}{\|w_{k_0}(\zeta)\|}\sum_{s=1}^{k_0-1} Df^{k_0-s-1}(z_{s+1})\psi(z_s) \to \left(\pm\sum_{s=1}^{k_0-1}\frac{\frac{\partial g}{\partial a}(a^*, x_s)}{(g^s_{a^*})'x_1}, 0\right),$$

and therefore

$$\frac{1}{\|w_{k_0}(\zeta)\|}V \to \left(\pm\sum_{s=0}^{k_0-1}\frac{\frac{\partial g}{\partial a}(a^*, x_s)}{(g^s_{a^*})'x_1}, 0\right) = (\pm\mathcal{Q}_{k_0}(x_0), 0).$$

To get the equality, differentiate $x_{k_0}(a) = g(a, x_{k_0-1}(a))$, divide the result by $(g^{k_0-1}_a)'x_1 = g'_a x_{k_0-1}\cdots g'_a x_1$ and use the result inductively. By (55), the claim holds. ☐

Sublemma 6.8 gives

$$\frac{\|II\|}{\|w_i(\zeta)\|} \leq C\sum_{s=k_0+1}^{i} e^{-\lambda s/2}.$$

Taking k_0 sufficiently large and then taking (a,b) close to $(a^*, 0)$, we obtain

$$\|I\| \geq C\|Df^{i-k_0}(z_{k_0})\|\|V\| \geq C\|Df^{i-k_0}(z_{k_0})\| \cdot \|w_{k_0}(\zeta)\| \geq C\|w_i(\zeta)\| \gg \|II\|.$$

This proves the first inequality.

Proof of Sublemma 6.8. Let q_t denote the bound period of a free return $t \leq i$, and let $I_t = [t - q_t, t + q_t]$.

Claim 6.2. *For each $s \notin \cup_t I_t$, $\|w_{s+j}(\zeta)\| \geq \delta\min\left(c, C_0^{-j}\right)\|w_s(\zeta)\|$ for $1 \leq j \leq i - s$.*

Proof. If $s+j$ is free, then, as s is free, (36) and Lemma 2.1 give $\|w_{s+j}(\zeta)\| \geq c\delta\|w_s(\zeta)\|$. If $s+j \in (r, r+q_r)$ for some free return r, then $r-s \leq j$. Since $s \notin I_r$, $s < r-q_r$ holds, and hence $q_r \leq j$. It follows that $\|w_{r+q_r}(\zeta)\| \geq \|w_r(\zeta)\|$ and $\|w_r(\zeta)\| \geq c\delta\|w_s(\zeta)\|$, and therefore $\|w_{s+j}(\zeta)\| \geq C_0^{-(r+q_r-s-j)}\|w_{r+q_r}(\zeta)\| \geq C_0^{-q_r}\|w_{r+q_r}(\zeta)\| \geq \delta C_0^{-j}\|w_s(\zeta)\|$. □

Returning to the proof of Sublemma 6.8, we argue with subdivision into cases.

Case I: $s \notin \cup_t I_t$. By the claim, $e_j(z_s)$ is well-defined for $1 \leq j \leq i-s$. Since s is free, $\mathrm{slope}(w_s(\zeta)) \leq \sqrt{b}$ holds. Hence we obtain

$$\|Df^{i-s}(f^s\zeta)\| \leq C\frac{\|w_i(\zeta)\|}{\|w_s(\zeta)\|} \leq Ce^{-\lambda s}\|w_i(\zeta)\|.$$

Case II: $s \in \cup_t I_t$. Let r_0 denote the last free return such that $s \in I_{r_0}$. Condition (G) gives $q_{r_0} \leq 3\alpha r_0/\lambda$, and hence $(1-3\alpha/\lambda)r_0 \leq r_0 - q_{r_0} \leq s$. We get $q_{r_0} \leq C\alpha s$.

Case II-a: $i \in I_{r_0}$. Since $i-s \leq q_{r_0}$, $\|Df^{i-s}(f^s\zeta)\| \leq C_0^{q_{r_0}} \leq e^{-\lambda s/2}\|w_i(\zeta)\|$.

Case II-b: $i \notin I_{r_0}$ and $i-s \leq 3\alpha i/\lambda$. We have $\|Df^{i-s}(f^s\zeta)\| \leq C_0^{10\alpha i} \leq C_0^{10\alpha s} \leq e^{-\lambda s/2}\|w_i(\zeta)\|$.

Case II-c: $i \notin I_{r_0}$ and $i-s > 3\alpha i/\lambda$. Define a strictly increasing sequence $s_0 < s_1 < \cdots$ of integers inductively as follows: Start with $s_0 := s$. Given s_k, let r_k denote the last free return such that $s_k \in I_{r_k}$. Put $s_{k+1} = r_k + q_{r_k}$. If $s_k \notin \cup I_t$, then s_{k+1} is undefined. By definition, $s_{k+1} - s_k \leq 2q_{r_k}$ holds.

Suppose that $s_\ell \geq i$ holds for some ℓ. Then $2\sum_{k=0}^{\ell-1} q_{r_k} \geq s_\ell - s_0 \geq i - s_0 > 3\alpha i/\lambda$. On the other hand, (G) gives $\sum_{k=0}^{\ell-1} q_{r_k} \leq 3\alpha i/\lambda$. We reach a contradiction. Hence, for the largest integer in the sequence, denoted by s_ℓ, $s_\ell \notin \cup I_t$ and $s_\ell < i$ hold. Then the estimate in Case I gives $\|Df^{i-s_\ell}(f^{s_\ell}\zeta)\| \leq Ce^{-\lambda s_\ell}\|w_i(\zeta)\|$, and

$$\begin{aligned}\|Df^{i-s}(f^s\zeta)\| &\leq \|Df^{i-s_\ell}(f^{s_\ell}\zeta)\| \prod_{k=0}^{\ell-1} \|Df^{s_{k+1}-s_k}(f^{s_k}\zeta)\| \\ &\leq e^{-\lambda s_\ell}\|w_i(\zeta)\| C_0^{2\sum_{k=0}^{\ell-1} q_k} \leq e^{-\frac{\lambda s}{2}}\|w_i(\zeta)\|.\end{aligned}$$

This completes the proof of Sublemma 6.8.

A.6. Proof of Lemma 6.5. Since $\|\dot{z}_j \times w_j(\zeta)\| = \|\dot{z}_j\|\|w_j(\zeta)\|\sin\theta_j$,

$$\begin{aligned}\sin\theta_j &\leq \frac{1}{\|\dot{z}_j\|}\left(\sum_{s=1}^{j} \frac{1}{\|w_j(\zeta)\|}\|w_j(\zeta) \times Df^{j-s}(z_s)\psi(z_{s-1})\| + \frac{\|w_j(\zeta) \times Df^j(z_0)\dot{z}_0\|}{\|w_j(\zeta)\|}\right) \\ &\leq \frac{1}{\|\dot{z}_j\|}\left(\sum_{s=1}^{j} \frac{\|w_s(\zeta)\|}{\|w_j(\zeta)\|}\left\|\frac{w_s(\zeta)}{\|w_s(\zeta)\|} \times \psi(z_{s-1})\right\|(Cb)^{j-s} + \frac{\|\dot{z}_0\|}{\|w_j(\zeta)\|}(Cb)^j\right) \\ &\leq \frac{C}{\delta\|\dot{z}_j\|}\sum_{s=0}^{\infty}(Cb)^s + \frac{\|\dot{z}_0\|}{\|\dot{z}_j\|\|w_j(\zeta)\|}(Cb)^j \leq \frac{C}{\delta\|\dot{z}_j\|},\end{aligned}$$

where the third inequality follows from $\|w_j(\zeta)\| \geq C\delta\|w_s(\zeta)\|$. For the last inequality we have used the boundedness of $\|\dot{z}_0\|$ in Proposition 4.2 and that k_0 is a large integer.

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