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A UNIQUENESS THEOREM FOR GLUING SPECIAL LAGRANGIAN SUBMANIFOLDS

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1. Introduction

Let M_1, M_2 be special Lagrangian submanifolds of a Calabi–Yau manifold N, and suppose M_1 intersects M_2 transversally at a point P. One can construct another special Lagrangian submanifold M by gluing a Lawlor neck [6] into $M_1 \cup M_2$ at P; see Butscher [2], D. Lee [7], Y. Lee [8], and Joyce [5]. By construction, M is close to the Lawlor neck near P, and to $M_1 \cup M_2$ away from P. Here is a problem: uniqueness of a special Lagrangian submanifold which is close to the Lawlor neck near P and to $M_1 \cup M_2$ away from P. The main result of this paper is a uniqueness theorem in the case where M_1, M_2 are flat special Lagrangian tori of real dimension 3, and N is a flat complex torus of complex dimension 3; see Theorem 6.1 for the precise statement. The author plans to prove a uniqueness theorem for more general M_1, M_2 in the sequel of this paper.

In section 2 we make statement of the key step to the proof of the main result. Section 3 and Section 4 provide what we shall need in the proof of Theorem 2.3. In Section 5 we prove Theorem 2.3 with the help of results in Section 3 and Section 4. In Section 6 we state the main result, and prove it by making a direct use of Theorem 2.3.

Theorem 2.3 is similar to Simon's theorem [11, Theorem 5, p563], which was originally applied to the unique tangent cone problem for minimal submanifolds. The proof of Theorem 2.3 is almost similar to Simon's. There is however a significant difference between Lemma 3.5 and Simon's lemma [11, Lemma 3, p561]. The condition (3.5) in Lemma 3.5 is closely related to Hofer's analysis [4, pp534–539] of pseudo-holomorphic curves in symplectizations of contact manifolds.

2. Statement of Theorem 2.3

An m-form ϕ on a Riemannian manifold N is said to be of comass ≤ 1 if

$$\phi(v_1, \dots, v_m) \leq 1$$

for every orthonormal vector fields $v_1, \dots, v_m$ on N. For each m-form ϕ of comass ≤ 1 on a Riemannian manifold N, a ϕ-submanifold M of N is defined to be an m-dimensional oriented submanifold with volume form $\phi|_M$. Harvey and Lawson [3] prove that ϕ-submanifolds are minimal submanifolds of the Riemannian manifold if ϕ is a closed form of comass ≤ 1. A closed form ϕ of comass ≤ 1 is called a calibration on the Riemannian manifold. A calibration is said to be parallel if it is a parallel differential form on the Riemannian manifold.

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Let ϕ be a parallel calibration of degree $m > 1$ on the Euclidean space $\mathbb{R}^n$. Set

$$\psi = (\partial_r \lrcorner \phi)\,|_{\mathbb{S}^{n-1}}, \tag{2.1}$$

where ∂_r is the vector field in the direction of the radial coordinate $r = |\bullet|$ on $\mathbb{R}^n \setminus \{0\}$, where $\lrcorner$ is the interior product of vector fields with differential forms, and where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

Proposition 2.1. *ψ is an $(m-1)$-form of comass ≤ 1 on $\mathbb{S}^{n-1}$; in particular*

$$\int_X \psi \le \mathrm{Vol}(X)$$

for every compact $(m-1)$-dimensional oriented submanifold X of $\mathbb{S}^{n-1}$.

Proof. For every orthonormal vector fields $v_1, \ldots, v_{m-1}$ on $\mathbb{S}^{n-1}$,

$$\psi(v_1, \ldots, v_{m-1}) = \phi(\partial_r, v_1, \ldots, v_{m-1}) \le 1$$

since $\partial_r, v_1, \ldots, v_{m-1}$ are orthonormal. □

Let b_0, b_1 be real numbers with $b_0 < b_1$. Let $A(b_0, b_1; \mathbb{S}^{n-1})$ be the pre-image of $(b_0, b_1) \times \mathbb{S}^{n-1}$ under the polar coordinates $\mathbb{R}^n \setminus \{0\} \to (0, \infty) \times \mathbb{S}^{n-1} : a \mapsto (|a|, a/|a|)$. Let X be a compact submanifold of $\mathbb{S}^{n-1}$, and $A(b_0, b_1; X)$ be the pre-image of $(b_0, b_1) \times X$ under the polar coordinates.

Remark 2.1. X is a ψ-submanifold if and only if $A(b_0, b_1; X)$ is a ϕ-submanifold of $A(b_0, b_1; \mathbb{S}^{n-1})$.

Proposition 2.2. *ψ-submanifolds are minimal submanifolds of $\mathbb{S}^{n-1}$.*

Proof. This follows from Remark 2.1 and the fact that X is a minimal submanifold of $\mathbb{S}^{n-1}$ if and only if $A(b_0, b_1; X)$ is a minimal submanifold of $A(b_0, b_1; \mathbb{S}^{n-1})$. □

Let ν be a normal vector field on $A(b_0, b_1; X)$ in $A(b_0, b_1; \mathbb{S}^{n-1})$. Set

$$\begin{aligned} \|\nu\|_{C^0_{\mathrm{cyl}}} &= \sup_{A(b_0,b_1;X)} |\nu|/r, \\ \|\nu\|_{C^1_{\mathrm{cyl}}} &= \sup_{A(b_0,b_1;X)} \big(|\nu|/r + |D\nu|\big), \end{aligned}$$

where r is the radial coordinate, and $D\nu$ is the covariant derivative of ν. These are induced by the cylindrical metric $dr^2/r^2 + ds^2$, where ds^2 is the metric on $\mathbb{S}^{n-1}$ induced by the Euclidean metric on $\mathbb{R}^n$. Set

$$G(\nu) = \Big\{ \frac{|a|}{\sqrt{|a|^2 + |\nu(a)|^2}} \big(a + \nu(a)\big) \;\Big|\; a \in A(b_0, b_1; X) \Big\}.$$

The following theorem will be the key step to the proof of the main result; see the proof of Proposition 6.2 in Section 6.

Theorem 2.3. *Let m, n be integers with $1 < m < n$, let ϕ be a parallel calibration of degree m on $\mathbb{R}^n$, let ψ be the $(m-1)$-form (2.1) on $\mathbb{S}^{n-1}$, let X be a compact ψ-submanifold of $\mathbb{S}^{n-1}$, and let β be a positive real number < 1. Then, there exist real numbers $\theta = \theta(m, n, X) \in (0, 1/2)$, $C = C(m, n, X, \phi) > 0$, $\epsilon = \epsilon(m, n, X, \phi, \beta) > 0$ such that the following holds:*

If b_0, b_1 are real numbers with $b_0 < b_1\beta$, if M is a closed ϕ-submanifold of $A(b_0\beta, b_1; \mathbb{S}^{n-1})$, and if for each $i = 0, 1$ there exists a normal vector field ν_i on $A(b_i\beta, b_i; X)$ in $A(b_i\beta, b_i; \mathbb{S}^{n-1})$ such that

$$M \cap A(b_i\beta, b_i; \mathbb{S}^{n-1}) = G(\nu_i),$$
$$\|\nu_i\|_{C^1_{\mathrm{cyl}}} \le \epsilon \text{ for each } i = 0, 1,$$

then there exists a normal vector field ν on $A(b_0\beta, b_1; X)$ in $A(b_0\beta, b_1; \mathbb{S}^{n-1})$ such that

$$M = G(\nu),$$
$$\nu|_{A(b_i\beta, b_i; X)} = \nu_i \text{ for each } i = 0, 1,$$
$$\|\nu\|_{C^1_{\mathrm{cyl}}} \le C\epsilon^\theta.$$

Theorem 2.3 will be proved in Section 5 with the help of results in Section 3 and Section 4.

3. First Lemma for Theorem 2.3: Lemma 3.5

Let m, n be integers with $1 < m < n$. Let $r : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ be the map $a \mapsto |a|$, and $s : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ be the map $a \mapsto a/|a|$. Let ϕ be a parallel calibration of degree m on $\mathbb{R}^n$, and ψ be the $(m-1)$-form (2.1) on $\mathbb{S}^{n-1}$.

Proposition 3.1.

$$m\phi|_{\mathbb{R}^n \setminus \{0\}} = d(r^m s^* \psi). \tag{3.1}$$

Proof. Since ϕ is an m-form with constant components in the Euclidean coordinates on $\mathbb{R}^n$, the Lie derivative of ϕ along $r\partial_r$ is equal to $m\phi$. Therefore, by Cartan's formula,

$$m\phi = d(r\partial_r \lrcorner \phi).$$

Therefore,

$$m\phi|_{\mathbb{R}^n \setminus \{0\}} = d(r^m \widetilde{\psi}), \tag{3.2}$$

where $\widetilde{\psi} = r^{1-m}\partial_r \lrcorner (\phi|_{\mathbb{R}^n \setminus \{0\}})$. Then, $\partial_r \lrcorner \widetilde{\psi} = 0$, and $\widetilde{\psi}$ is invariant under the flow generated by $r\partial_r$. Therefore, $\widetilde{\psi} = s^*(\widetilde{\psi}|_{\mathbb{S}^{n-1}})$. On the other hand, $\widetilde{\psi}|_{\mathbb{S}^{n-1}} = \psi$ by (2.1). Thus, $\widetilde{\psi} = s^*\psi$. Therefore, (3.2) gives (3.1). □

Remark 3.1. By the calculation above, $s^*\psi = r^{1-m}\partial_r \lrcorner (\phi|_{\mathbb{R}^n \setminus \{0\}})$.

Proposition 3.2. *Let M be a ϕ-submanifold of $\mathbb{R}^n$. Then, $(\nu \lrcorner \phi)|_M = 0$ if ν is normal to M.*

Proof. Suppose $v_1, \dots, v_{m-1} \in T_P M$ are orthonormal. It suffices to prove:

$$\phi(\nu, v_1, \dots, v_{m-1}) = 0. \tag{3.3}$$

There exists $v \in T_P M$ such that $\phi(v, v_1, \dots, v_{m-1}) = 1$. The function $t \mapsto \phi((\sin t)\nu + (\cos t)v, v_1, \dots, v_{m-1})$ attains maximum 1 at $t = 0$. Differentiating it at $t = 0$, therefore, gives (3.3). □

Proposition 3.3 (Harvey and Lawson). *Let M be a ϕ-submanifold of $\mathbb{R}^n$. Then,*

$$\langle \overrightarrow{TM}, d\psi \rangle = |\partial_r \wedge \overrightarrow{TM}|^2 m r^{-m},$$

where $\langle \bullet, \bullet \rangle$ is the pairing of m-vector fields and m-forms, and $\overrightarrow{TM}$ is the unit simple m-vector field dual to the volume form of M.

Proof. See Harvey and Lawson [3, (5.13), Lemma 5.11, p65]. □

Corollary 3.4. *Let M be a closed ϕ-submanifold of $A(b_0, b_1; \mathbb{S}^{n-1})$. If $b_0 < c_0 < c_1 < b_1$, if c_0, c_1 are regular values of $r|_M : M \to I$, and if $r^{-1}(c_0)$, $r^{-1}(c_1)$ are non-empty, then*

$$\int_{M \cap r^{-1}(c_0)} \psi \leq \int_{M \cap r^{-1}(c_1)} \psi. \tag{3.4}$$

Proof. By Proposition 3.3,

$$\int_{M \cap r^{-1}([c_0, c_1])} d\psi \geq 0.$$

This, by Stokes' theorem, gives (3.4). □

The following lemma will be important in the proof of Theorem 2.3; see Section 5.

Lemma 3.5. *Let m, n be integers with $1 < m < n$, let ϕ be a parallel calibration of degree m on $\mathbb{R}^n$, let ψ be the $(m-1)$-form (2.1) on $\mathbb{S}^{n-1}$, let X be a compact ψ-submanifold of $\mathbb{S}^{n-1}$, let β be a positive real numbers < 1, and let ϵ_* be a positive real number. Then, there exists a positive real number ϵ_{**} such that the following holds:*

If M is a closed ϕ-submanifold of $A(\beta^2, 1; \mathbb{S}^{n-1})$, if

$$\int_M d\psi \leq \epsilon_{**}, \tag{3.5}$$

and if there exists a normal vector field ν on $A(\beta, 1; X)$ in $A(\beta, 1; \mathbb{S}^{n-1})$ such that

$$M \cap A(\beta, 1; \mathbb{S}^{n-1}) = G(\nu),$$
$$\|\nu\|_{C^1_{\mathrm{cyl}}} \leq \epsilon_{**},$$

then there exists a normal vector field ν' on $A(\beta^{3/2}, 1; X)$ in $A(\beta^{3/2}, 1; \mathbb{S}^{n-1})$ such that

$$M = G(\nu'),$$
$$\|\nu'\|_{C^{1,1/2}_{\mathrm{cyl}}(\beta^{3/2}, \beta)} \leq \epsilon_*,$$

where $C^{1,1/2}_{\mathrm{cyl}}(\beta^{3/2}, \beta)$ is the Hölder space on $A(\beta^{3/2}, \beta; X)$ with respect to the metric $dr^2/r^2 + ds^2$ on $A(\beta^{3/2}, \beta; X)$, where r is the radial coordinate, and ds^2 is the metric on $\mathbb{S}^{n-1}$ induced by the Euclidean metric on $\mathbb{R}^n$.

Proof. Suppose there does not exist such ϵ_{**}. Then, there exists a sequence $(M_i)_{i=2,3,4,\ldots}$ of closed ϕ-submanifolds of $A(\beta^2, 1; \mathbb{S}^{n-1})$ with the following properties:

(P1) $\int_{M_i} d\psi \leq 1/i$;

(P2) for each $i = 2, 3, 4, \ldots$, there exists a normal vector field ν_i on $A(\beta, 1; X)$ in $A(\beta, 1; \mathbb{S}^{n-1})$ such that

$$\begin{aligned} &M_i \cap A(\beta, 1; \mathbb{S}^{n-1}) = G(\nu_i), \\ &\|\nu_i\|_{C^1_{\mathrm{cyl}}} \le 1/i; \end{aligned}$$

(P3) for each $i = 1, 2, \ldots$, there does not exist any normal vector field ν_i' on $A(\beta^{3/2}, \beta; X)$ in $A(\beta^{3/2}, \beta; \mathbb{S}^{n-1})$ such that

$$\begin{aligned} &M_i = G(\nu_i'), \\ &\|\nu_i'\|_{C^{1,1/2}_{\mathrm{cyl}}} \le \epsilon_*. \end{aligned}$$

By Proposition 3.1 and (P1),

$$\begin{aligned} \mathrm{Vol}(M_i) &= \int_{M_i} \phi \\ &= \int_{M_i} m^{-1} d(r^m s^* \psi) \\ &= \int_{M_i} r^{m-1} dr \wedge s^*\psi + m^{-1} r^m d\psi \\ &\le \left(\int_{M_i} dr \wedge s^*\psi \right) + (mi)^{-1}, \end{aligned}$$

whereas by Proposition 3.4 and (P2),

$$\begin{aligned} \int_{M_i} dr \wedge s^*\psi &= \int_{\beta^2}^1 db \int_{M_i \cap r^{-1}(b)} s^*\psi \\ &\le \int_{\beta^2}^1 db \limsup_{b \to \rho} \int_{M_i \cap r^{-1}(b)} s^*\psi \\ &\le \left(1 - \beta^2\right) \limsup_{b \to \rho} \int_{M_i \cap r^{-1}(b)} s^*\psi \\ &\le C \end{aligned}$$

for some C independent of $i = 2, 3, 4, \ldots$. Therefore,

$$\sup_{i=2,3,4,\ldots} \mathrm{Vol}(M_i) < \infty. \tag{3.6}$$

Therefore, by Allard's compactness [1, Theorem 5.6], there exists a subsequence $(M_{i_j})_{j=2,3,4,\ldots}$ such that $(M_{i_j})_{j=2,3,4,\ldots}$ converges as varifolds to some rectifiable varifold M_∞ in $A(\beta^2, 1; \mathbb{S}^{n-1})$. Let $\|M_\infty\|$ denote the Radon measure on $A(\beta^2, 1; \mathbb{S}^{n-1})$ induced by M_∞. For each $\|M_\infty\|$-measurable subset E of $A(\beta^2, 1; \mathbb{S}^{n-1})$, let $\|M_\infty\| \llcorner E$ denote the restriction of $\|M_\infty\|$ to E, i.e.,

$$\|M_\infty\| \llcorner E(E') = \|M_\infty\|(E \cap E').$$

Proposition 3.6. *$a^m \|M_\infty\| \llcorner (a^{-1}E) = \|M_\infty\| \llcorner E$ for every $\|M_\infty\|$-measurable subset E of $A(\beta^2, 1; \mathbb{S}^{n-1})$ and every positive real number a with $aE \subset A(\beta^2, 1; \mathbb{S}^{n-1})$.*

Proof. By an approximation argument, it suffices to prove the following:

$$\frac{d}{da} a^m \int_{(r,s) \in A(\beta^2, 1; \mathbb{S}^{n-1})} f(ar) g(s) d\|M_\infty\| = 0 \tag{3.7}$$

for every $f \in C^\infty_{\mathrm{c}}(\beta^2, 1)$ with $a(\mathrm{supp} f) \subset (\beta^2, 1)$ and every $g \in C^\infty(\mathbb{S}^{n-1})$. By (P1),

$$
\begin{aligned}
&\text{the left-hand side of (3.7)}\\
&= \frac{d}{da} \lim_{j\to\infty} a^m \int_{M_{i_j}} f(ar) g \,\mathrm{dVol}(M_{i_j})\\
&= \lim_{j\to\infty} \int_{M_{i_j}} \frac{d}{da}((ar)^m f(ar)) dr/r \wedge g s^*\psi\\
&= \lim_{j\to\infty} \int_{M_{i_j}} a^{-1} \frac{d}{dr}((ar)^m f(ar)) dr \wedge g s^*\psi\\
&= \lim_{j\to\infty} -\int_{M_{i_j}} a^{-1}(ar)^m f(ar) dg \wedge s^*\psi.
\end{aligned}
\tag{3.8}
$$

By Remark 3.1, Proposition 3.2, Proposition 3.3 and (3.6),

$$
\begin{aligned}
&\lim_{j\to\infty} -\int_{M_{i_j}} a^{-1}(ar)^m f(ar) dg \wedge s^*\psi\\
&= \lim_{j\to\infty} \int_{M_{i_j}} a^{-1}(ar)^m f(ar) r^{1-m} \partial_r \lrcorner (dg \wedge \phi),\\
&= \lim_{j\to\infty} \int_{M_{i_j}} a^{-1}(ar)^m f(ar) r^{1-m} \langle \mathrm{pr}_{TM_{i_j}^\perp} \partial_r, dg\rangle \phi\\
&\le \lim_{j\to\infty} a^{m-1} \sup|f| \sqrt{\int_{M_{i_j}} m r^{-m} |\mathrm{pr}_{TM_{i_j}^\perp} \partial_r|^2 \phi} \sqrt{\int_{M_{i_j}} |dg|^2 \phi}\\
&\le \lim_{j\to\infty} \text{const.} \sqrt{\int_{M_{i_j}} d\psi}\\
&= 0,
\end{aligned}
\tag{3.9}
$$

where $\mathrm{pr}_{TM_{i_j}^\perp}$ is the projection of $\mathbb{R}^n$ onto the normal bundle of M_{i_j} in $\mathbb{R}^n$. By (3.8) and (3.9), the left-hand side of (3.7) is zero. This completes the proof of Proposition 3.6. □

By (P2), on the other hand, $\|M_\infty\| \llcorner A(\beta, 1; \mathbb{S}^{n-1})$ is equal to $\|A(\beta, 1; X)\|$ as Radon measures on $A(\beta^2, 1; \mathbb{S}^{n-1})$, where

$$\|A(\beta, 1; X)\|(E) = \mathrm{Vol}(A(\beta, 1, ; X) \cap E).$$

Therefore, by Proposition 3.6, $M_\infty = A(\beta^2, 1; X)$ as varifolds in $A(\beta^2, 1; \mathbb{S}^{n-1})$. Therefore, $(M_{i_j})_{j=2,3,4,\ldots}$ converges to a submanifold $A(\beta^2, 1; X)$ as varifolds in $A(\beta^2, 1; \mathbb{S}^{n-1})$. Therefore, by Allard's regularity [1, Theorem 8.19], $(M_{i_j})_{j=2,3,4,\ldots}$ converges to a submanifold $A(\beta^2, 1; X)$ in the local $C^{1,1/2}$-sense. This contradicts (P3), which completes the proof of Lemma 3.5. □

4. Second Lemma for Theorem 2.3: Lemma 4.3

Let X be a compact smooth Riemannian manifold, and V be a smooth real vector bundle over X with a fibre metric and a metric connection. Let C^∞_x denote the space of all smooth sections of V over X. For every $v, v' \in C^\infty_x$, set

$$(v, v')_{L^2_x} = \int_X \big(v(x), v'(x)\big) dx,$$
$$\|v\|_{L^2_x} = \sqrt{\int_X \big(v(x), v(x)\big) dx},$$

where $\big(v(x), v'(x)\big)$ is the inner product on the fibre $V|_x$ over $x \in X$. Let $F : C^\infty_x \to \mathbb{R}$ be a functional of the following form:

$$Fv = \int_X f(x, v, D_x v) dx \tag{4.1}$$

for every $v \in C^\infty_x$ with covariant derivative $D_x v$, where $f = f(x, v, p)$ is a $\mathbb{R}$-valued smooth function of $x \in X$, $v \in V|_x$, and $p \in T^*_x X \otimes V|_x$. Suppose f satisfies the following conditions:

(C1) $(v, p) \mapsto f(x, v, p)$ is a real-analytic function on the vector space $V|_x \oplus (T^*_x X \otimes V|_x)$ for every $x \in X$;

(C2) F satisfies the Legendre-Hadamard condition at $0 \in C^\infty_x$, i.e.,

$$\frac{d^2}{dh^2} f(x, 0, h^2 \xi \otimes v)\Big|_{h=0} > c|\xi|^2|v|^2$$

for every $x \in X, \xi \in T^*_x X, v \in V|_x$ for some $c > 0$.

Let $-\operatorname{grad} F : C^\infty_x \to C^\infty_x$ be the Euler–Lagrange operator of F, i.e.,

$$\big(\operatorname{grad} F(v), v'\big)_{L^2_x} = \frac{d}{dh} F(v + hv')\Big|_{h=0}$$

for every $v, v' \in C^\infty_x$. Suppose

$$\operatorname{grad} F(0) = 0,$$

where $0 \in C^\infty_x$.

Let t_0, t_∞ be real numbers with $t_0 < t_\infty$. Consider the product $(t_0, t_\infty) \times X$, and the pull-back $\mathrm{pr}_2^* V$ over $(t_0, t_\infty) \times X$, where pr_2^* is the projection of $(t_0, t_\infty) \times X$ onto X. Let $C^\infty_{t,x}(t_0, t_\infty)$ denote the space of all smooth sections of $\mathrm{pr}_2^* V$ over $(t_0, t_\infty) \times X$. For every $u = u(t, x) \in C^\infty_{t,x}(t_0, t_\infty)$, set $u(t) = u(t, \bullet) \in C^\infty_x$ for each $t \in (t_0, t_\infty)$. For every $u = u(t, x) \in C^\infty_{t,x}(t_0, t_\infty)$ and for every non-negative integers k, let $\|u\|_{C^{k,\mu}_{t,x}(t_0,t_\infty)}$ be the Hölder norm with respect to the product metric $dt^2 + dx^2$ on $(t_0, t_\infty) \times X$. For every $u = u(t, x) \in C^\infty_{t,x}(t_0, t_\infty)$, set

$$\|u\|_{L^2_{t,x}(t_0,t_\infty)} = \sqrt{\int_{t_0}^{t_\infty} \|u(t)\|^2_{L^2_x} dt}.$$

Theorem 4.1 (Simon)**.** *There exist real numbers $\delta_0 > 0$, $\theta \in (0, 1/2)$ depending only on X, V, F such that if $t_3, t_4 \in (t_0, t_\infty)$ with $t_3 < t_4$, if $u \in C^\infty_{t,x}(t_0, t_\infty)$, if $\delta > 0$, and if for each $t \in [t_3, t_4]$,*

$$\|u(t)\|_{C^{2,1/2}_x} \le \delta_0,$$
$$F(0) - F\big(u(t)\big) \le \delta,$$
$$\|\partial_t u(t) + \operatorname{grad} F\big(u(t)\big)\|_{L^2_x} \le (3/4)\|\partial_t u(t)\|_{L^2_x},$$

then

$$\int_{t_3}^{t_4} \|\partial_t u(t)\|_{L^2_x} dt \le (4/\theta)\big(\big|F\big(u(t_3)\big) - F(0)\big|^{\theta} + \delta^{\theta}\big).$$

Proof. The Simon condition implies the Łojasiewicz inequality in the sense of Simon [11, Theorem 3], which leads to Theorem 4.1; see Simon [11, Lemma 1, p542]. □

Simon [11] studies smooth functions $u = u(t,x) \in C^{\infty}_{t,x}(t_0, t_\infty)$ satisfying the partial differential equation

$$\partial_t^2 u - \partial_t u - \operatorname{grad} F(u) + R(u, \partial_t u, \partial_t^2 u) = 0, \tag{4.2}$$

where $R : C^{\infty}_x \times C^{\infty}_x \times C^{\infty}_x \to C^{\infty}_x$ is a remainder of the following form:

$$\begin{aligned} R(v, v^{(1)}, v^{(2)}) =& \big(A(x, v, D_x v, v^{(1)}) D_x^2 v\big) v^{(1)} \\ &+ \sum_{(k,l)=(0,1),(1,1),(0,2)} B_{kl}(x, v, D_x v, v^{(1)}) D_x^l v^{(k)} \end{aligned} \tag{4.3}$$

for every $v, v^{(1)}, v^{(2)} \in C^{\infty}_x$, where $A = A(x,v,p,q)$, $B_{kl} = B_{kl}(x,v,p,q)$ are smooth functions of $x \in X$, $v \in V|_x$, $p \in T^*_x X \otimes V|_x$, $q \in V|_x$ with $A(x,v,p,q) \in \bigotimes^2 T_x X \otimes V|^*_x$, $B_{kl}(x,v,p,q) \in \bigotimes^l T_x X$, $B_{kl}(x,0,0,0) = 0$ for every $x \in X$, for every $(k,l) = (1,0),(1,1),(2,0)$.

Remark 4.1. Let R be of the form (4.3) with A, B as above. Then, for every $C_2 > 0$ there exists $\delta_5 = \delta_5(X, V, F, R, C_2) > 0$ such that if $\|u\|_{C^{1,1/2}_{t,x}} \le \delta_5$, then

$$\begin{aligned} &|R(u(t), \partial_t u(t), \partial_t^2 u(t))| \\ &\le \frac{1}{16}|\partial_t u(t)| + \frac{1}{16}|\partial_t u(t)| + \frac{1}{16C_2}|D_x \partial_t u(t)| + \frac{1}{8}|\partial_t^2 u(t)|. \end{aligned}$$

Remark 4.2. Suppose X is a minimal submanifold of $\mathbb{S}^{n-1}$, and V is the normal bundle of X in $\mathbb{S}^{n-1}$. For each normal vector field v on X in $\mathbb{S}^{n-1}$, set

$$G(v) = \Big\{ \frac{|x|}{\sqrt{|x|^2 + |v(x)|^2}} \big(x + v(x)\big) \;\Big|\; x \in X \Big\},$$

which is a compact submanifold of $\mathbb{S}^{n-1}$ if $\sup_{x\in X} |v(x)| \le \epsilon_0$ for some $\epsilon_0 > 0$ depending only on $X \subset \mathbb{S}^{n-1}$. Let C^{∞}_x be the space of all normal vector fields v on X in $\mathbb{S}^{n-1}$ with $\sup_{x\in X} |v(x)| \le \epsilon_0$. Let $F : C^{\infty}_x \to \mathbb{R}$ be the following functional:

$$F(v) = \operatorname{Vol}(G(v)). \tag{4.4}$$

F is of the form (4.1) with integrand f satisfying (C1), (C2), and $\operatorname{grad} F(0) = 0$. Set $b_0 = e^{-t_\infty/m}$, $b_1 = e^{-t_0/m}$; $b_0 < b_1$. Let $A(b_0, b_1; \mathbb{S}^{n-1})$, $A(b_0, b_1; X)$ be as in Section 2. Let ν be a normal vector field on $A(b_0, b_1; X)$ in $A(b_0, b_1; \mathbb{S}^{n-1})$. Set $u(t,x) = e^{t/m}\nu(e^{-t/m}x)$; $u \in C^{\infty}_{t,x}(t_0, t_\infty)$. Let $G(\nu)$ be as in Section 2. If $G(\nu)$ is a minimal submanifold, then u satisfies Simon's equation (4.2), where F is the functional (4.4), and R is some remainder of the form (4.3) depending only on m, n, X.

Let $H : C^{\infty}_x \to C^{\infty}_x$ be the linearized operator of $\operatorname{grad} F$ at $0 \in C^{\infty}_x$. Simon's equation (4.2) is of the following form:

$$\partial_t^2 u - \partial_t u - Hu = \sum_{0 \le k+l \le 2} E_{kl}(x, u, D_x u, \partial_t u) D_x^l \partial_t^k u,$$

where $E_{kl} = E_{kl}(x, u, p, q)$ are smooth functions of $x \in X$, $v \in V|_x$, $p \in T_x^*X \otimes V|_x$, $q \in V|_x$ with $E_{kl}(x, v, p, q) \in \bigotimes^l T_x X$, $E_{kl}(x, 0, 0, 0) = 0$ for every $x \in X$, for every k, l with $0 \leq k + l \leq 2$.

Remark 4.3. Let t_1, t_6 be real numbers with $t_1 < t_6$. There exists $\delta_2 = \delta_2(X, V, F, R) > 0$ such that if $u \in C^\infty_{t,x}$ with $\|u\|_{C^{1,1/2}_{t,x}(t_1,t_6)} \leq \delta_2$, then

$$\max_{0 \leq k+l \leq 2} \|E_{kl}\|_{C^{0,1/2}_{t,x}(t_1,t_6)} \leq \delta_1,$$

where $\delta_1 = \delta_1(X, V, F)$ is as in Remark 4.4 below.

Remark 4.4. Let $T > 0$. The Legendre-Hadamard condition (C2) implies that $\partial_t^2 - \partial_t - H$ is uniformly elliptic on $C^\infty_{t,x}(t_0, t_\infty)$. Therefore, there exists $\delta_1 = \delta_1(X, V, F) > 0$ be such that if $w \in C^\infty_{t,x}(-T/3, T/3)$, if $a_{kl}(t, x) \in C^\infty_{t,x}(-T/3, T/3)$, and if

$$\partial_t^2 w - \partial_t w - Hw = \sum_{0 \leq k+l \leq 2} a_{kl}(t, x) D^l_x \partial^k_t w,$$

$$\max_{0 \leq k+l \leq 2} \|a_{kl}\|_{C^{0,1/2}_{t,x}(-T/3,T/3)} \leq \delta_1,$$

then,

$$\|w\|_{C^{2,1/2}_{t,x}(-T/5,T/5)} \leq C_1 \|w\|_{L^2_{t,x}(-T/4,T/4)} \tag{4.5}$$

for some $C_1 = C_1(X, V, F, T) > 0$. This is a Schauder estimate; see Morrey [10, Chapter 6].

Theorem 4.2 (Simon)**.** *There exist $h > 0$, $T > 0$, $\delta_3 > 0$ depending only on X, V, F such that for any $j \in \{3, 4, 5, \dots\}$ the following holds:*

If $w \in C^\infty_{t,x}(0, jT)$, $a_{kl}(t, x) \in C^\infty_{t,x}(0, jT)$, and

$$\partial_t^2 w - \partial_t w - Hw = \sum_{0 \leq k+l \leq 2} a_{kl}(t, x) D^l_x \partial^k_t w,$$

$$\max_{0 \leq k+l \leq 2} \sup_{(t,x) \in (0,jT) \times X} |a_{kl}(t, x)| \leq \delta_3,$$

$$\|w\|_{L^2_{t,x}(0,jT)} < \infty,$$

then, there exist integers i_1, i_2 with $0 \leq i_1 \leq i_2 \leq j$ such that: if $1 < i_1$, then

$$\|w\|_{L^2_{t,x}(iT,(i+1)T)} \leq e^{-hT} \|w\|_{L^2_{t,x}((i-1)T,iT)}$$

for each $i \in \{1, \dots, i_1 - 1\}$; if $i_1 < i_2$, then

$$\|w(t)\|_{L^2_x} \leq (3/2) \|w(t')\|_{L^2_x}$$

for $t, t' \in (i_1 T, i_2 T)$ with $|t' - t| \leq T$, and

$$\|\partial_t w(t)\|_{L^2_x} \leq (1/2) \|w(t)\|_{L^2_x}$$

for each $t \in (i_1 T, i_2 T)$; if $i_2 < j - 1$, then

$$\|w\|_{L^2_{t,x}((i-1)T,iT)} \leq e^{-hT} \|w\|_{L^2_{t,x}(iT,(i+1)T)}$$

for each $i \in \{i_2 + 1, \dots, j - 1\}$.

Proof. See Simon [12, Theorem 3.4]. □

The following lemma will be important in the proof of Theorem 2.3; see Section 5.

Lemma 4.3. *Let X be a compact smooth Riemannian manifold, V be a smooth real vector bundle over X with a fibre metric and a metric connection, $F : C_x^\infty \to \mathbb{R}$ be a functional with* $\operatorname{grad} F(0) = 0$ *and of the form* (4.1) *with integrand f satisfying* (C1), (C2). Then, there exist real numbers $C > 0$ and $\theta \in (0, 1/2)$ such that for every remainder R of the form (4.3), there exists $\delta_* > 0$ such that the following holds:

If $t_0 < t_*$, if $0 < \delta < 1$, if $u \in C_{t,x}^\infty(t_0, t_*)$ satisfies Simon's equation (4.2), and if

$$\|u\|_{C_{t,x}^{1,1/2}(t_0,t_*)} \leq \delta_*, \tag{4.6}$$

$$\limsup_{t \to t_0} \|u(t)\|_{L_x^2} \leq \delta, \tag{4.7}$$

$$\sup_{t \in (t_0,t_*)} \Big(F(0) - F\big(u(t)\big) \Big) \leq \delta, \tag{4.8}$$

$$\|\partial_t u\|_{L_{t,x}^2(t_0,t_*)} \leq \sqrt{\delta}, \tag{4.9}$$

then

$$\sup_{t \in (t_0,t_*)} \|u(t)\|_{L_x^2} < C_* \delta^\theta.$$

Proof. By (4.7), it suffices to prove the following:

$$\int_{t_0}^{t_*} \|\partial_t u(t)\|_{L_x^2} dt < C_* \delta^\theta. \tag{4.10}$$

Let $T = T(X, V, F) > 0$ be as in Theorem 4.2. If $t_* - t_0 < 5T$, then

$$\begin{aligned} \int_{t_0}^{t_*} \|\partial_t u(t)\|_{L_x^2} dt &\leq \sqrt{t_* - t_0} \|\partial_t u\|_{L_{t,x}^2(t_0,t_*)} \\ &\leq \sqrt{5T} \delta, \end{aligned} \tag{4.11}$$

which gives the conclusion (4.10).

Suppose $t_* - t_0 \geq 5T$. Let $t_1, t_6 \in (t_0, t_*)$ be such that $T/2 \leq t_1 - t_0 \leq T$, $T/2 \leq t_* - t_6 \leq T$ and $t_6 - t_1 = jT$ for some $j \in \{3, 4, 5, \dots\}$. Then,

$$\int_{t_0}^{t_1} \|\partial_t u(t)\|_{L_{t,x}^2} \leq \sqrt{T} \delta. \tag{4.12}$$

$$\int_{t_6}^{t_*} \|\partial_t u(t)\|_{L_{t,x}^2} \leq \sqrt{T} \delta. \tag{4.13}$$

in the same way as (4.11).

Suppose $\delta_* < \delta_2$, where $\delta_2 = \delta_2(X, V, F, R) > 0$ as in Remark 4.3. By (4.6), Remark 4.3 and Remark 4.4,

$$\|u\|_{C_{t,x}^{2,1/2}(t_1,t_6)} \leq C_1 \delta_*,$$

where $C_1 = C_1(X, V, F; T) > 0$ be as in Remark 4.4, so that $C_1 = C_1(X, V, F)$ since $T = T(X, V, F) > 0$. It suffices therefore to prove (4.10) under the assumption that

$$\|u\|_{C_{t,x}^{2,1/2}(t_1,t_6)} \leq \delta_{**}$$

for some $\delta_{**} = \delta_{**}(X, V, F, R) > 0$.

Set $u' = \partial_t u$. Then,

$$\partial_t^2 u' - \partial_t u' - Hu' = \sum_{0 \leq k+l \leq 2} \tilde{a}_{kl}(t, x) D_x^l \partial_t^k u' \tag{4.14}$$

for some smooth functions $\tilde{a}_{kl}(t,x)$ of $t \in (t_0, t_*)$, $x \in X$ with $\tilde{a}_{kl}(t,x) \in \bigotimes^l T_x X$ for every k, l with $0 \leq k+l \leq 2$. By an argument similar to Remark 4.3, there exists $\delta_4 = \delta_4(X,V,F,R) > 0$ such that if $\|u\|_{C^{2,1/2}_{t,x}(t_1,t_6)} \leq \delta_4$, then

$$\max_{0\leq k+l\leq 2} \|E_{kl}\|_{C^{0,1/2}_{t,x}(t_1,t_6)} \leq \delta_1, \tag{4.15}$$

$$\max_{0\leq k+l\leq 2} \|\tilde{a}_{kl}\|_{C^{0,1/2}_{t,x}(t_1,t_6)} \leq \min\{\delta_1, \delta_3\}, \tag{4.16}$$

where $\delta_1 = \delta_1(X,V,F)$ as in Remark 4.4, and $\delta_3 = \delta_3(X,V,F)$ as in Theorem 4.2.

Suppose $\|u\|_{C^{2,1/2}_{t,x}(t_1,t_6)} \leq \delta_4$, where $\delta_4 = \delta_4(X,V,F,R) > 0$ as above. By (4.14) and (4.16), u' satisfies the assumption in Theorem 4.2. Therefore, there exists integers i_1, i_2 with $0 \leq i_1 \leq i_2 \leq j$ such that: if $1 < i_1$, then

$$\|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \leq e^{-hT} \|\partial_t u\|_{L^2_{t,x}(t_1+(i-1)T, t_1+iT)} \tag{4.17}$$

for each $i \in \{1, \ldots, i_1 - 1\}$; if $i_1 < i_2$, then

$$\|\partial_t u(t)\|_{L^2_x} \leq (3/2)\|\partial_t u(t')\|_{L^2_x} \tag{4.18}$$

for $t, t' \in (t_1 + i_1 T, t_1 + i_2 T)$ with $|t' - t| \leq T$, and

$$\|\partial_t^2 u(t)\|_{L^2_x} \leq (1/2)\|\partial_t u(t)\|_{L^2_x} \tag{4.19}$$

for each $t \in (t_1 + i_1 T, t_1 + i_2 T)$; if $i_2 < j - 1$, then

$$\|\partial_t u\|_{L^2_{t,x}(t_1+(i-1)T, t_1+iT)} \leq e^{-hT} \|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \tag{4.20}$$

for each $i \in \{i_2 + 1, \ldots, j-1\}$; here $h = h(X,V,F) > 0$ as in Theorem 4.2. By the method in (4.11), and by (4.17),

$$\begin{aligned}
\int_{t_1}^{t_1+i_1T} \|\partial_t u(t)\|_{L^2_x} dt &\leq \sum_{i=1}^{i_1-1} \int_{t_1+iT}^{t_1+(i+1)T} \|\partial_t u(t)\|_{L^2_x} dt \\
&\leq \sum_{i=0}^{i_1-1} \sqrt{T} \|\partial_t u\|_{L^2_{t,x}(t_1+iT, t_1+(i+1)T)} \\
&\leq \sum_{i=0}^{i_1-1} \sqrt{T} e^{-ihT} \|\partial_t u\|_{L^2_{t,x}(t_1, t_1+T)} \\
&\leq \sqrt{T}(1-e^{-hT})^{-1} \|\partial_t u\|_{L^2_{t,x}(t_1, t_1+T)} \\
&\leq \sqrt{T}(1-e^{-hT})^{-1} \delta.
\end{aligned} \tag{4.21}$$

Set $t_5 = t_1 + i_2 T$. As (4.17) gives (4.21), so (4.20) gives the following:

$$\int_{t_5}^{t_6} \|\partial_t u(t)\|_{L^2_x} dt \leq \sqrt{T}(1-e^{-hT})^{-1} \delta. \tag{4.22}$$

If $i_1 = i_2$, then (4.12), (4.21) and (4.22) imply the conclusion (4.10).

Suppose $i_1 < i_2$. Set $t_2 = t_1 + i_1 T$, $t_3 = t_2 + T/3$, $t_4 = t_5 - T/3$. Then,

$$\int_{t_2}^{t_3} \|\partial_t u(t)\|_{L^2_x} dt \le \sqrt{T/3}\delta, \tag{4.23}$$

$$\int_{t_2}^{t_3+T/4} \|\partial_t u(t)\|_{L^2_x} dt \le \sqrt{7T/12}\delta, \tag{4.24}$$

$$\int_{t_4}^{t_5} \|\partial_t u(t)\|_{L^2_x} dt \le \sqrt{T/3}\delta. \tag{4.25}$$

in the same way as (4.11).

For each $t \in [t_3, t_4]$, on the other hand, by (4.15), (4.16) and Remark 4.4,

$$\|\partial_t u\|_{C^{2,1/2}_{t,x}(t-T/5,t+T/5)} \le C_1 \|\partial_t u\|_{L^2_{t,x}(t-T/4,t+T/4)}, \tag{4.26}$$

$$\|u\|_{C^{2,1/2}_{t,x}(t-T/5,t+T/5)} \le C_1 \|u\|_{L^2_{t,x}(t-T/4,t+T/4)}, \tag{4.27}$$

where $C_1 = C_1(X, V, F; T) > 0$ as in Remark 4.4, so that $C_1 = C_1(X, V, F)$ since $T = T(X, V, F)$. By (4.26) and (4.18), for each $t \in [t_3, t_4]$,

$$\begin{aligned} \|D_x \partial_t u(t)\|_{L^2_x} &\le \sqrt{\mathrm{Vol}(X)} \sup_{x \in X} |D_x \partial_t u(t,x)| \\ &\le \sqrt{\mathrm{Vol}(X)} \|\partial_t u\|_{C^{2,1/2}_{t,x}(t-T/5,t+T/5)} \\ &\le \sqrt{\mathrm{Vol}(X)} C_1 \|\partial_t u\|_{L^2_{t,x}(t-T/4,t+T/4)} \\ &\le \sqrt{\mathrm{Vol}(X)} C_1 (3/2) \sqrt{T/2} \|\partial_t u(t)\|_{L^2_x} \\ &\le C_2 \|\partial_t u(t)\|_{L^2_x}, \end{aligned} \tag{4.28}$$

where $C_2 = \sqrt{\mathrm{Vol}(X)} C_1 (3/2)\sqrt{T/2}$, so that $C_2 = C_2(X, V, F) > 0$. Therefore, by Remark 4.1, (4.19) and Simon's equation (4.2), for each $t \in [t_3, t_4]$,

$$\begin{aligned} \|\partial_t u(t) + \mathrm{grad}\, F\big(u(t)\big)\|_{L^2_x} &\le \|\partial_t^2 u(t) + R\left(u(t), \partial_t u(t), \partial_t^2 u(t)\right)\|_{L^2_x} \\ &\le \left(\frac{1}{2} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16 C_2} C_2 + \frac{1}{8}\frac{1}{2}\right) \|\partial_t u(t)\|_{L^2_x} \\ &\le (3/4) \|\partial_t u(t)\|_{L^2_x}. \end{aligned} \tag{4.29}$$

if $\|u\|_{C^{2,1/2}_{t,x}(t_1,t_6)} \le \delta_5$, where $\delta_5 = \delta_5(X, V, F, R, C_2) > 0$ as in Remark 4.1, so that $\delta_5 = \delta_5(X, V, F, R)$.

Suppose $\|u\|_{C^{2,1/2}_{t,x}(t_1,t_6)} \le \min\{\delta_5, \delta_0\}$, where $\delta_5 = \delta_5(X, V, F, R)$ as above, and $\delta_0 = \delta_0(X, V, F) > 0$ as in Theorem 4.1. By Theorem 4.1, (4.29) and (4.8),

$$\int_{t_3}^{t_4} \|\partial_t u(t)\|_{L^2_x} dt \le (4/\theta) \left(\left| F\big(u(t_3)\big) - F(0) \right|^\theta + \delta^\theta \right),$$

where $\theta = \theta(X, V, F) > 0$ as in Theorem 4.1. Since $\mathrm{grad}\, F(0) = 0$, there exists $C_0 = C_0(X, V, F) > 0$ such that if $v \in C^\infty_x$, $\|v\|_{C^{2,1/2}_x} < \delta_0$, then

$$|F(v) - F(0)| \le C_0 \|v\|^2_{C^1_x}.$$

Therefore, by (4.27),

$$\begin{aligned}\big|F\big(u(t_3)\big)-F(0)\big| &\le C_0\|u(t_3)\|^2_{C^1_x}\\ &\le C_0C_1\|u\|^2_{L^2_{t,x}(t_3-T/4,t_3+T/4)}\\ &\le C_0C_1(T/2)\sup_{t\in(t_3-T/4,t_3+T/4)}\|u(t)\|^2_{L^2_x},\end{aligned}$$

whereas by (4.7), (4.12), (4.21) and (4.24),

$$\begin{aligned}\sup_{t\in(t_3-T/4,t_3+T/4)}\|u(t)\|_{L^2_x} &\le \limsup_{t\to t_0}\|u(t)\|_{L^2_x}+\int_{t_0}^{t_3+T/4}\|\partial_t u(t)\|_{L^2_x}\,dt\\ &\le \Big(1+\sqrt{T}+\sqrt{T}(1-e^{-hT})^{-1}+\sqrt{7T/12}\Big)\,\delta,\end{aligned}$$

Thus,

$$\int_{t_3}^{t_4}\|\partial_t u(t)\|_{L^2_x}\,dt\le C_*\delta^\theta \tag{4.30}$$

for some constant C_* depending only on X, V, F. Thus, (4.12), (4.21), (4.23), (4.30), (4.25), (4.22) and (4.13) imply the conclusion (4.10).

The proof of Lemma 4.3 is thus complete. □

5. Proof of Theorem 2.3

We now prove Theorem 2.3.

Let ϕ be a parallel calibration of degree $m>1$ on the Euclidean space $\mathbb{R}^n$. Let ψ be the $(m-1)$-form $(\partial_r\lrcorner\phi)|_{\mathbb{S}^{n-1}}$ on the unit sphere $\mathbb{S}^{n-1}$. Let X be a compact ψ-submanifold of $\mathbb{S}^{n-1}$.

By Proposition 2.2, X is a minimal submanifold of $\mathbb{S}^{n-1}$. Let F, R be as in Remark 4.2. Let $\theta\in(0,1/2)$, $C_*>0$, $\delta_*>0$ be as in Lemma 4.3, so that $\theta=\theta(m,n,X)$, $C_*(m,n,X)>0$, $\delta_*(m,n,X)>0$.

Let β be a positive real number <1. Suppose $\epsilon_*>0$ is so small that if $\|\nu\|_{C^0_{\mathrm{cyl}}}\le\epsilon_*$, then:

$$\begin{aligned}&G(\nu)\text{ is a closed submanifold of }A(b_0,b_1;\mathbb{S}^{n-1}),\\ &\text{and if }G(\nu)=G(\nu')\text{ for some }\nu',\text{ then }\nu=\nu'\end{aligned}\tag{5.1}$$

in the notation of Section 2. Let $\epsilon_{**}>0$ be as in Lemma 3.5, so that $\epsilon_{**}=\epsilon_{**}(m,n,X,\phi,\beta,\epsilon_*)=$.

Suppose now that $\epsilon>0$, that $b_0>0$, $b_1>0$ with $b_0<b_1\beta$, that M is a closed ϕ-submanifold of $A(b_0\beta,b_1;\mathbb{S}^{n-1})$, and that for each $i=0,1$ there exists a normal vector field ν_i on $A(b_i\beta,b_i;X)$ in $A(b_i\beta,b_i;\mathbb{S}^{n-1})$ such that

$$\begin{aligned}&M\cap A(b_i\beta,b_i;\mathbb{S}^{n-1})=G(\nu_i),\\ &\|\nu_i\|_{C^1_{\mathrm{cyl}}}\le\epsilon\text{ for each }i=0,1.\end{aligned}\tag{5.2}$$

For each $i=0,1$, by $\|\nu_i\|_{C^1_{\mathrm{cyl}}}\le\epsilon$,

$$\sup_{b\in(b_i\beta,b_i)}\left|\int_X\psi-\int_{M\cap r^{-1}(b)}s^*\psi\right|\le c_0\epsilon.\tag{5.3}$$

for some $c_0 = c_0(m, n, X, \phi) > 1$. Therefore, by Stokes's theorem,

$$\int_M s^* d\psi \leq 2c_0\epsilon. \tag{5.4}$$

Therefore, if $\epsilon <\epsilon_{**}$, then by (5.2) with $i = 1$, and by Lemma 3.5, there exists a normal vector field ν on $A(b_1\beta^{3/2}, b_1; X)$ in $A(b_1\beta^{3/2}, b_1; \mathbb{S}^{n-1})$ such that

$$\begin{aligned} &M \cap A(b_1\beta^{3/2}, b_1; \mathbb{S}^{n-1}) = G(\nu), \\ &\|\nu\|_{C^{1,1/2}_{\mathrm{cyl}}(b_1\beta^{3/2}, b_1)} \leq \epsilon_*. \end{aligned} \tag{5.5}$$

Suppose $\epsilon <\epsilon_{**}$. Let S_* be the set of all $b_* \in [b_0, b_1\beta)$ such that there exists a normal vector field ν on $A(b_*, b_1; X)$ in $A(b_*, b_1; \mathbb{S}^{n-1})$ such that

$$\begin{aligned} &M \cap A(b_*, b_1; \mathbb{S}^{n-1}) = G(\nu), \\ &\|\nu\|_{C^{1,1/2}_{\mathrm{cyl}}(b_*, b_1)} \leq \epsilon_*. \end{aligned} \tag{5.6}$$

S_* is non-empty since $b_1\beta^{3/2} \in S_*$ by (5.5).

Proposition 5.1. *Suppose $b_* \in S_* \cap [b_0, b_1)$, and let ν be as in* (5.6)*. Set $u(t, x) = e^{t/m}\nu(e^{-t/m}x)$, $t_0 = -\log b_1/m$ and $t_* = -\log b_*/m$. Then,*

$$\|u\|_{L^2_{t,x}(t_0,t_*)} \leq C_*(4c_0\epsilon)^\theta$$

if $\epsilon < (4c_0)^{-1}$ and $\epsilon_* < \epsilon_1$ for some $\epsilon_1 = \epsilon_1(m, n, X) > 0$.

Proof. By Remark 4.2, u satisfies Simon's equation (4.2) with respect to F and R. By Proposition 2.1, Corollary 3.4 and (5.3),

$$\begin{aligned} &\sup_{b\in(b_*,b_1)} \mathrm{Vol}(X) - \mathrm{Vol}\left(s\left(M \cap r^{-1}(b)\right)\right) \\ &\leq \sup_{b\in(b_*,b_1)} \int_X \psi - \int_{M\cap r^{-1}(b)} s^*\psi \\ &\leq \int_X \psi - \int_{M\cap r^{-1}(b_0)} s^*\psi \\ &\leq c_0\epsilon. \end{aligned} \tag{5.7}$$

By calculation, if $\|\nu\|_{C^1_{\mathrm{cyl}}} \leq \epsilon_1$ for some $\epsilon_1 = \epsilon_1(m, n, X) > 0$, then

$$|r\partial_r\nu|^2 \leq 2|\partial_r \wedge \overrightarrow{TM}|^2, \tag{5.8}$$

where $M = G(\nu)$, and $\overrightarrow{TM}$ is as in Proposition 3.3. Suppose $\epsilon_* < \epsilon_1$. Then, by Proposition 3.3, (5.4) and (5.8),

$$\begin{aligned} \int_M |r\partial_r\nu|^2/r^m \,\mathrm{dVol}(M) &\leq 2\int_M |\partial_r \wedge \overrightarrow{TM}|^2/r^m \,\mathrm{dVol}(M) \\ &= 2\int_M s^*d\psi \\ &\leq 4c_0\epsilon. \end{aligned} \tag{5.9}$$

Now, (5.6), (5.2) with $i = 1$, (5.7), (5.9) imply (4.6), (4.7), (4.8), (4.9) in Lemma 4.3, respectively. Therefore, by Lemma 4.3,

$$\sup_{t\in(t_0,t_*)} \|u(t)\|_{L^2_x} < C_*(4c_0\epsilon)^\theta$$

if $\epsilon < (4c_0)^{-1}$. □

Suppose $\epsilon < (4c_0)^{-1}$ and $\epsilon_* < \epsilon_1$. Suppose $b_* \in S_*$. Set $\tau = \log\beta/m$. Then, by interpolation and Proposition 5.1,

$$\begin{aligned}
\|\nu\|_{C^1_{\mathrm{cyl}}(b_*,b_*/\beta)} &= \|u\|_{C^1_{t,x}(t_*-\tau,t_*)} \\
&\leq \epsilon_{**}(2\epsilon_*)^{-1}\|u\|_{C^{1,1/2}_{t,x}(t_*-\tau,t_*)} + c_1 \sup_{t\in(t_*-\tau,t_*)} \|u(t)\|_{L^2_x} \\
&\leq \epsilon_{**}/2 + c_1 C_*(4c_0\epsilon)^\theta \\
&\leq \epsilon_{**}
\end{aligned} \tag{5.10}$$

if $\epsilon <\epsilon_{**}(2c_1C_*)^{-1}(4c_0\epsilon)^{-\theta}$, where $c_1{=}c_1(m,n,X,\tau,\epsilon_{**}(2\epsilon_*)^{-1}) > 0$.

Suppose $\epsilon <\epsilon_{**}(2c_1C_*)^{-1}(4c_0\epsilon)^{-\theta}$. By (5.10), (5.4) and Lemma 3.5, $b_*\beta^{1/2} \in S_*$. Therefore, b_* is an interior point in S_*. Thus, S_* is an open subset of $[b_0, b_1\beta)$.

By (5.1), S_* is a closed subset of $[b_0, b_1\beta)$. Thus, S_* is a non-empty open closed subset of $[b_0, b_1\beta)$. Therefore, $S_* = [b_0, b_1\beta)$. In particular, $b_0 \in S_*$. Set $t_\infty = -\log b_0/m$. Then, by (5.2) with $i = 0$ and Proposition 5.1,

$$\sup_{t\in(t_0,t_\infty+\tau)} \|u(t)\|_{L^2_x} \leq C_*(4c_0\epsilon)^\theta.$$

Therefore, by the Schauder estimate in Remark 4.4,

$$\|u\|_{C^1_{t,x}(t_0,t_\infty+\tau)} \leq C\epsilon^\theta$$

for some $C = C(C_*, c_0) = C(m, n, X, \phi) > 0$. This completes the proof of Theorem 2.3.

6. The Main Result

Let $(x^1, y^1, x^2, y^2, x^3, y^3)$ be the coordinates on $\mathbb{R}^6$, and ω_0 the symplectic form $dx^1 \wedge dy^1 + dx^2 \wedge dy^2 + dx^3 \wedge dy^3$ on $\mathbb{R}^6$. Let J_0 be the complex structure on $\mathbb{R}^6$ which maps $\partial/\partial x^\alpha$ to $\partial/\partial y^\alpha$ for every $\alpha = 1, 2, 3$, and Ω_0 the complex volume form $dz^1 \wedge dz^2 \wedge dz^3$ on $\mathbb{R}^6$, where $z^\alpha = x^\alpha + iy^\alpha$ for every $\alpha = 1, 2, 3$. Harvey and Lawson [3] prove that $\mathrm{Re}\,\Omega_0$ is a calibration with respect to the metric $g_0 = \sum_{\alpha=1}^3 dz^\alpha \overline{dz^\alpha}$ on $\mathbb{R}^6$. $\mathrm{Re}\,\Omega_0$-submanifolds of $(\mathbb{R}^6, g_0)$ are called special Lagrangian submanifolds of $(\mathbb{R}^6, \omega_0, J_0, \Omega_0)$; this is well-defined since $g_0 = \omega_0(\bullet, J_0\bullet)$.

Let $\mathbb{R}^3_1$, $\mathbb{R}^3_2$ be special Lagrangian planes in $(\mathbb{R}^6, \omega_0, J_0, \Omega_0)$ such that $\mathbb{R}^3_1 \oplus \mathbb{R}^3_2 = \mathbb{R}^6$. Lawlor [6] gives an explicit construction of L, f with the following properties:

(L1) L is a closed special Lagrangian submanifold of $(\mathbb{R}^6, \omega_0, J_0, \Omega_0)$;
(L2) f is a diffeomorphism of $\mathbb{R} \times \mathbb{S}^2$ into $\mathbb{R}^6$;
(L3) L is the image of $f : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}^6$;
(L4) there exists $R > 0$ such that

$$|f_R - i_R|/r + |d(f_R - i_R)| = O(r^{-3}) \text{ with respect to the metric } g_0 \text{ on } \mathbb{R}^6,$$

where r is the projection of $\mathbb{R} \times \mathbb{S}^2$ onto $\mathbb{R}$, f_R is the restriction of f to $(\mathbb{R} \setminus [-R, R]) \times \mathbb{S}^2$, and i_R is the inclusion of $(\mathbb{R} \setminus [-R, R]) \times \mathbb{S}^2$ into $\mathbb{R}^6$ whose image is $\mathbb{R}^3_1 \cup \mathbb{R}^3_2 \setminus B(R)$.

Here, $B(R)$ is the ball of radius R centred at 0 in $(\mathbb{R}^6, g_0)$.

Let $\mathbb{T}^6$ be the torus $\mathbb{R}^6/\mathbb{Z}^6$. By abuse of notation, ω_0, J_0, g_0, Ω_0 denote the symplectic form, the complex structure, the metric, the complex volume form, respectively on $\mathbb{T}^6$ as well as on $\mathbb{R}^6$. Likewise, $B(R)$ denotes the ball of radius R centred at 0 in $(\mathbb{T}^6, g_0)$ as well as in $(\mathbb{R}^6, g_0)$. Set $\mathbb{T}^3_1 = \mathbb{R}^3_1/(\mathbb{R}^3_1 \cap \mathbb{Z}^6)$, $\mathbb{T}^3_2 = \mathbb{R}^3_2/(\mathbb{R}^3_2 \cap \mathbb{Z}^6)$. $\mathbb{T}^3_1$, $\mathbb{T}^3_2$ are special Lagrangian submanifolds of $(\mathbb{T}^6, \omega_0, J_0, \Omega_0)$.

In view of the proof of the theorem of D. Lee [7, Theorem 3] or Joyce [5, Theorem 9.10], for every $\delta > 0$ there exist $(J_\delta, \Omega_\delta)$, M_δ, ϵ_δ, b_δ and b'_δ with the following properties:

- (P1) $J_\delta = f_\delta^* J_0$, $\Omega_\delta = f_\delta^* \Omega_0$ for some $f_\delta \in GL(3, \mathbb{R})$ such that $f_\delta^* \omega_0 = \omega_0$;
- (P2) M_δ is a compact special Lagrangian submanifold of $(\mathbb{T}^3, \omega_0, J_\delta, \Omega_\delta)$, i.e., a $\operatorname{Re}\Omega_\delta$-submanifold of $(\mathbb{T}^6, g_\delta)$, where $g_\delta = f_\delta^* g_0$;
- (P3) $\epsilon_\delta > 0$, $0 < b_\delta < b'_\delta < 1$, $\lim_{\delta\to 0} \epsilon_\delta = 0$, $\lim_{\delta\to 0} b_\delta/\delta = \infty$;
- (P4) $M_\delta \cap B(b_\delta)$ is the graph of some normal vector field on $\delta L \cap B(b_\delta)$ in $B(b_\delta)$ with C^1-norm less than $b_\delta \epsilon_\delta$, where $\delta L \cap B(b_\delta)$ is embedded in $\mathbb{T}^6$;
- (P5) $M_\delta \setminus \overline{B(b'_\delta)}$ is the graph of some normal vector field on $(\mathbb{T}^3_1 \cup \mathbb{T}^3_2) \setminus \overline{B(b'_\delta)}$ in $(\mathbb{T}^6 \setminus \overline{B(b'_\delta)}, g_\delta)$ with C^1-norm less than $b_\delta \epsilon_\delta$;
- (P6) there exists a normal vector field ν on $(\mathbb{T}^3_1 \cup \mathbb{T}^3_2) \cap A(b, b')$ in $(A(b, b'), g_\delta)$ such that $M_\delta \cap A(b, b') = G(\nu)$ with $\|\nu\|_{C^1_{\mathrm{cyl}}} \le \epsilon_\delta$ for some b, b' with $0 < b < b_\delta < b'_\delta < b' < 1$ in the notation of Section 2, where $A(b, b') = B(b') \setminus \overline{B(b)}$;
- (P7) M_δ is diffeomorphic to the connected sum $\mathbb{T}^3_1 \# \mathbb{T}^3_2$.

The main result of this paper is the following:

Theorem 6.1. *Let $(J_\delta, \Omega_\delta)$, M_δ, ϵ_δ, b_δ and b'_δ be as above. Let M'_δ be such that:*

- (P2') *M'_δ is a compact special Lagrangian submanifold of $(\mathbb{T}^6, \omega_0, J_\delta, \Omega_\delta)$;*
- (P4') *$M'_\delta \cap B(b_\delta)$ is the graph of some normal vector field on $\delta L \cap B(b_\delta)$ in $(B(b_\delta), g_\delta)$ with C^1-norm less than $b_\delta \epsilon_\delta$;*
- (P5') *$M'_\delta \setminus \overline{B(b'_\delta)}$ is the graph of some normal vector field on $(\mathbb{T}^3_1 \cup \mathbb{T}^3_2) \setminus \overline{B(b'_\delta)}$ in $(\mathbb{T}^6 \setminus \overline{B(b'_\delta)}, g_\delta)$ with C^1-norm less than $b_\delta \epsilon_\delta$.*

Then, $M'_\delta = M_\delta + t$ for some $t \in \mathbb{T}^6$ whenever δ is sufficiently small.

The proof of Theorem 6.1 is divided into the following two propositions:

Proposition 6.2. *M'_δ is sufficiently close to M_δ in the C^1-topology induced by the metric g_δ on $\mathbb{T}^6$ whenever δ is sufficiently small.*

Proposition 6.3. *If a special Lagrangian submanifold M''_δ of $(\mathbb{T}^6, \omega_0, J_\delta, \Omega_\delta)$ is sufficiently close to M_δ in the C^1-topology induced by the metric g_δ on $\mathbb{T}^6$, then $M''_\delta = M_\delta + t$ for some $t \in \mathbb{T}^6$ whenever δ is sufficiently small.*

Proof of Proposition 6.2. By (L4), (P3), (P4) and (P5), there exist b_0, b_1, β, ν_0 and ν_1 such that:

- $b_0 > 0$, $b_1 > 0$, $0 < \beta < 1$, $b_0 < b_\delta < b'_\delta < b_1\beta$;
- for each $i = 0, 1$, ν_i is a normal vector field on $(\mathbb{T}^3_1 \cup \mathbb{T}^3_2) \cap A(b_i\beta, b_i)$ in $A(b_i\beta, b_i)$;
- $M_\delta \cap A(b_i\beta, b_i) = G(\nu_i)$ with $\|\nu_i\|_{C^1_{\mathrm{cyl}}} \le \epsilon/2$, for each $i = 0, 1$ in the notation of Section 2 whenever δ is sufficiently small;

here $A(b, b') = B(b') \setminus \overline{B(b)}$ for each b, b' with $0 < b < b' < 1$, and $\epsilon > 0$ is as in Theorem 2.3. Therefore, by (P4') and (P5'), there exist normal vector fields ν'_i on $((\mathbb{T}^3_1 \cup \mathbb{T}^3_2) \cap A(b_i\beta, b_i), g_\delta)$ in $A(b_i\beta, b_i)$ for each $i = 0, 1$ such that

$$M'_\delta \cap A(\beta b_i, b_i) = G(\nu'_i) \text{ with } \|\nu'_i\|_{C^1_{\mathrm{cyl}}} \le \epsilon \text{ for each } i = 0, 1$$

whenever δ is sufficiently small. By (P1) and (P2'), therefore, $M'_\delta \cap A(b_0\beta, b_1)$ satisfies the assumption of Theorem 2.3. By Theorem 2.3, therefore, there exists a normal vector field ν' on $(\mathbb{T}^3_1 \cup \mathbb{T}^3_2) \cap A(b_0\beta, b_1)$ in $(A(b_0\beta, b_1), g_\delta)$ such that

$M'_\delta \cap A(b_0\beta, b_1) = G(\nu')$ with $\|\nu'\|_{C^1_{\rm cyl}} \le \epsilon'_\delta$ for some ϵ'_δ converging to 0 as $\delta \to 0$. This, together with (P4), (P5), (P6), (P4') and (P5'), proves Proposition 6.2. □

Proof of Proposition 6.3. Let $\mathcal{D}(M_\delta)$ be the space of all special Lagrangian submanifolds of $(\mathbb{T}^6, \omega_0, J_\delta, \Omega_\delta)$ which are sufficiently close to M_δ in the C^1-topology induced by g_δ on $\mathbb{T}^6$. By the deformation theory of Mclean [9, Section 3], $\mathcal{D}(M_\delta)$ is a manifold of dimension $b^1(M_\delta)$, where $b^1(M_\delta)$ is the first Betti number of M_δ. By (P7), $b^1(M_\delta) = 6$. Thus,

$$\dim \mathcal{D}(M_\delta) = 6. \tag{6.1}$$

For each $t \in \mathbb{T}^6$, by (P1), $M_\delta + t$ is a special Lagrangian submanifold of $(\mathbb{T}^6, \omega_0, J_\delta, \Omega_\delta)$. Therefore, $f : t \mapsto M_\delta + t$ maps a neighbourhood of $0 \in \mathbb{T}^6$ into $\mathcal{D}(M_\delta)$. By (P5), if δ is sufficiently small, then $df_0 : T_0\mathbb{T}^6 \to T_{f(0)}\mathcal{D}(M_\delta)$ is one-to-one. Therefore, by (6.1) and Inverse Function Theorem, f is a diffeomorphism of a neighbourhood of $0 \in \mathbb{T}^6$ onto a neighbourhood of $f(0) = M_\delta$ in $\mathcal{D}(M_\delta)$. This completes the proof of Proposition 6.3. □

Proof of Theorem 6.1. Proposition 6.2 and Proposition 6.3 imply Theorem 6.1. □

Remark 6.1. The key step to the proof of Theorem 6.1 is the proof of Proposition 6.2, where we have made a direct use of Theorem 2.3, which assumes that the ambient space is a flat Euclidean space. This is why we considered the flat torus $(\mathbb{T}^6, g_\delta)$. It is very likely, however, that modification of Theorem 2.3 leads to an extension of Theorem 6.1 to more general Calabi–Yau manifolds. The author plans to work it out in the near future.

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