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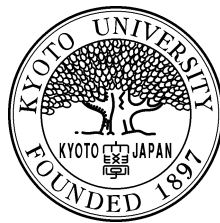
A transcendental approach to Kollár's injectivity theorem II

by

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A TRANSCENDENTAL APPROACH TO KOLLÁR'S INJECTIVITY THEOREM II

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ABSTRACT. We treat a relative version of the main theorem in [F1]: A transcendental approach to Kollár's injectivity theorem. More explicitly, we give a curvature condition that implies Kollár type cohomology injectivity theorems in the relative setting. To carry out this generalization, we use Ohsawa–Takegoshi's twisted version of Nakano's identity.

CONTENTS

1. INTRODUCTION

The following theorem is the main theorem of this paper, which is a relative version of the main theorem in [F1]. It is a generalization of Kollár's injectivity theorem (cf. [K1, Theorem 2.2]). More precisely, it is a generalization of Enoki's injectivity theorem, which is an analytic version of Kollár's theorem (see [En, Theorem 0.2] and [F1, Corollary 1.4]). We note that Kollár's proofs and Esnault–Viehweg's approach are geometric (cf. [K1], [K2, Chapters 9 and 10], and [EV]) and are not related to curvature conditions. Therefore, we do not know the true relationship between Kollár's injectivity theorem and Enoki's one.

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Theorem 1.1 (Main Theorem)**.** *Let $f : X \to Y$ be a proper surjective morphism from a Kähler manifold X to a complex variety Y. Let (E, h_E) (resp. (L, h_L)) be a holomorphic vector (resp. line) bundle on X with a smooth hermitian metric h_E (resp. h_L). Let F be a holomorphic line bundle on X with a singular hermitian metric h_F. Assume the following conditions.*

  (i) *There exists a subvariety Z of X such that h_F is smooth on $X \setminus Z$.*
  (ii) *$\sqrt{-1}\Theta(F) \geq -\widetilde{\gamma}$ in the sense of currents, where $\widetilde{\gamma}$ is a smooth $(1,1)$-form on X.*
  (iii) *$\sqrt{-1}(\Theta(E) + \mathrm{Id}_E \otimes \Theta(F)) \geq_{\mathrm{Nak}} 0$ on $X \setminus Z$.*
  (iv) *$\sqrt{-1}(\Theta(E) + \mathrm{Id}_E \otimes \Theta(F) - \varepsilon_0 \mathrm{Id}_E \otimes \Theta(L)) \geq_{\mathrm{Nak}} 0$ on $X \setminus Z$ for some positive constant ε_0.*

Here, $\geq_{\mathrm{Nak}} 0$ means the Nakano semi-positivity. Let s be a nonzero holomorphic section of L. Then the multiplication homomorphism

$$\times s : R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \to R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F) \otimes L)$$

is injective for every $q \geq 0$, where K_X is the canonical line bundle of X and $\mathcal{J}(h_F)$ is the multiplier ideal sheaf associated to the singular hermitian metric h_F of F. Note that $\times s$ is the sheaf homomorphism induced by the tensor product with s.

We note that Theorem 1.1 will be generalized slightly in Proposition 4.1 below. For the absolute case and the background of Kollár type cohomology injectivity theorems, see the introduction of [F1]. The reader who reads Japanese may find [F2] also useful. The essential part of Theorem 1.1 is contained in Ohsawa's injectivity theorem (see [O]). Our formulation is much more suitable for geometric applications than Ohsawa's (cf. [F1, 4. Applications]). We note that the main ingredient of our proof is Ohsawa–Takegoshi's twisted version of Nakano's identity (cf. Proposition 2.20).

The next corollary directly follows from Theorem 1.1. It contains a generalization of the Grauert–Riemenschneider vanishing theorem.

Corollary 1.2 (Torsion-freeness)**.** *Let $f : X \to Y$ be a proper surjective morphism from a Kähler manifold X to a complex variety Y. Let (E, h_E) (resp. (F, h_F)) be a holomorphic vector (resp. line) bundle on X with a smooth hermitian metric h_E (resp. a singular hermitian metric h_F). Assume the conditions* (i)*,* (ii)*, and* (iii) *in Theorem 1.1. Then, $R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F))$ is torsion-free for every $q \geq 0$. In particular, $R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) = 0$ for $q > \dim X - \dim Y$.*

We will describe the proof of Theorem 1.1 in Section 3, which may help the reader to understand [O].

From now on, we discuss various vanishing theorems as applications of Theorem 1.1.

Corollary 1.3 (Kawamata–Viehweg–Nadel type vanishing theorem). *Let $f : X \to Y$ be a proper surjective morphism from a complex manifold X to a complex variety Y. Let E be a Nakano semi-positive vector bundle on X and let $\mathcal{L}$ be a holomorphic line bundle on X such that $\mathcal{L}^{\otimes m} \simeq \mathcal{M} \otimes \mathcal{O}_X(D)$ where m is a positive integer, $\mathcal{M}$ is an f-nef-big line bundle, and D is an effective Cartier divisor on X. Then*

$$R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$$

for every $q > 0$ where $\mathcal{J} = \mathcal{J}(\frac{1}{m}D)$ is the multiplier ideal sheaf associated to $\frac{1}{m}D$.

In the log minimal model program for projective morphisms between complex varieties, the Kawamata–Viehweg vanishing theorem plays crucial roles. It was first obtained by Nakayama (cf. [N1, Theorem 3.7]).

Corollary 1.4 (Kawamata–Viehweg vanishing theorem for proper morphisms). *Let $f : X \to Y$ be a proper surjective morphism from a complex manifold X to a complex variety Y. Let H be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $\mathrm{Supp}\{H\}$ is a normal crossing divisor on X and that $\mathcal{O}_X(mH)$ is an f-nef-big line bundle for some positive integer m. Then*

$$R^q f_*(K_X \otimes \mathcal{O}_X(\ulcorner H \urcorner)) = 0$$

for every $q > 0$.

We can also prove a Kollár type vanishing theorem from Theorem 1.1. The proof of Corollary 1.5 is a routine work for experts.

Corollary 1.5 (Kollár type vanishing theorem). *Let $f : X \to Y$ be a proper surjective morphism from a Kähler manifold X to a complex variety Y. Let $g : Y \to Z$ be a proper morphism between complex varieties. Let E be a Nakano semi-positive vector bundle on X such that $\mathcal{L}^{\otimes m} \simeq f^* \mathcal{N} \otimes \mathcal{O}_X(D)$ where m is a positive integer, $\mathcal{N}$ is a g-nef-big line bundle, and D is an effective Cartier divisor on X. Then*

$$R^p g_* R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$$

for every $p > 0$ and every $q \geq 0$ where $\mathcal{J} = \mathcal{J}(\frac{1}{m}D)$ in the multiplier ideal sheaf associated to $\frac{1}{m}D$.

All the statements in the introduction may look complicated. So it seems to be worth mentioning that the following well-known vanishing theorems easily follow from the main theorem: Theorem 1.1. The proofs of Corollary 1.6 and Corollary 1.7 explain one of the reasons why we think that injectivity theorems are generalizations of vanishing theorems (cf. [F3], [F4]) and that the formulation of Theorem 1.1 is useful for various applications.

Corollary 1.6 (Nakano vanishing theorem (cf. [D2, (4.9)])). *Let X be a compact complex manifold and let E be a Nakano positive vector bundle on X. Then $H^q(X, K_X \otimes E) = 0$ for every $q > 0$.*

Proof. Since E is Nakano positive, $L = \det E$ is a positive line bundle. Therefore, X is projective and L is ample by Kodaira's embedding theorem. If ε is a small positive number, then

$$\sqrt{-1}(\Theta(E) - \varepsilon \mathrm{Id}_E \otimes \Theta(L)) \geq_{\mathrm{Nak}} 0.$$

Thus, by using Theorem 1.1, $H^q(X, K_X \otimes E)$ can be embedded into $H^q(X, K_X \otimes E \otimes L^{\otimes m})$ for sufficiently large positive integer m. By Serre's vanishing theorem, $H^q(X, K_X \otimes E) = 0$ for every $q > 0$. $\square$

Corollary 1.7 (Kawamata–Viehweg vanishing theorem). *Let X be a smooth projective variety and let L be a nef and big line bundle on X. Then $H^q(X, K_X \otimes L) = 0$ for every $q > 0$.*

Proof. By Kodaira's lemma, we can write $L^{\otimes m} \simeq A \otimes \mathcal{O}_X(D)$ such that m is a positive integer, A is an ample line bundle, and D is an effective Cartier divisor on X with $\mathcal{J}(\frac{1}{m}D) = \mathcal{O}_X$. Let h_D be the singular hermitian metric of $\mathcal{O}_X(D)$ naturally associated to D (cf. [F1, Example 2.3]) and let h_A be a smooth hermitian metric of A whose curvature is positive. We put $h_L = h_A^{\frac{1}{m}} h_D^{\frac{1}{m}}$. Then $\sqrt{-1}\Theta(L) \geq 0$ in the sense of currents, h_L is smooth on $X \setminus D$, and $\sqrt{-1}(\Theta(L) - \varepsilon\Theta(A)) \geq 0$ on $X \setminus D$ for $0 < \varepsilon \ll 1$. Therefore, we have inclusions $H^q(X, K_X \otimes L) \subset H^q(X, K_X \otimes L \otimes A^{\otimes l})$ for every q and every sufficiently large positive integer l by Theorem 1.1. Thus $H^q(X, K_X \otimes L) = 0$ for every $q > 0$ by Serre's vanishing theorem. $\square$

For related topics, see [N1, §3], [N2, II. §5.c, V. §3], and [T]. We think that one of the most important open problems on vanishing theorems for the log minimal model program is to prove the results in [F6, Sections 6 and 8] and [F7, Chapter 2] for projective morphisms between complex varieties. We can not directly use the arguments in this paper because the L^2-method does not work for log canonical pairs. We note that the arguments in [F5], [F6], and [F7] are geometric.

We summarize the contents of this paper. In Section 2, we collect basic definitions and results in the algebraic and analytic geometries. In this section, we discuss Ohsawa–Takegoshi's twisted version of Nakano's identity. It is a key ingredient of the proof of the main theorem: Theorem 1.1. Section 3 is devoted to the proof of the main theorem: Theorem 1.1. In Section 4, we discuss the proofs of the corollaries in Section 1 and some applications. In the final section: Section 5, we discuss various examples of nef, semi-positive, and semi-ample line bundles. It is very important to understand the differences in the notion of semi-ample, semi-positive, and nef line bundles.

2. Preliminaries

In this section, we collect basic definitions and results in the algebraic and analytic geometries.

2.1 (Projective morphisms)**.** For details of *projective morphisms* of complex varieties and ample line bundles, see, for example, [N1, §1] and [N2, II. 1.10. Definition, Remark].

2.2 (Big line bundles)**.** In this paper, we will freely use *Iitaka's D-dimension κ*, the *numerical D-dimension ν*, and so on, for algebraic varieties.

Let us recall the definition of *f-nef-big line bundles* for proper morphisms between complex varieties, which we need in corollaries in Section 1.

Definition 2.3 (cf. [N1, Definition])**.** Let $f : X \to Y$ be a proper surjective morphism from a complex variety X onto a complex variety Y. Let L be a line bundle on X. Then L is called *f-big* if the relative Iitaka D-dimension $\kappa(X/Y, L) = \dim X - \dim Y$. Furthermore, if $L \cdot C \geq 0$ for every irreducible curve C such that $f(C)$ is a point, then L is called *f-nef-big*.

2.4 (Q-divisors). Let $D = \sum_i d_i D_i$ be a Q-divisor on a normal complex variety X where D_i is a prime divisor for every i and $D_i \neq D_j$ for $i \neq j$. Then we define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. the *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where for every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$.

2.5 (Singular hermitian metric). Let L be a holomorphic line bundle on a complex manifold X.

Definition 2.6 (Singular hermitian metric). A *singular hermitian metric* on L is a metric which is given in any trivialization $\theta : L|_\Omega \simeq \Omega \times \mathbb{C}$ by
$$\|\xi\| = |\theta(\xi)| e^{-\varphi(x)}, \quad x \in \Omega, \ \xi \in L_x,$$
where $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is an arbitrary function, called the *weight* of the metric with respect to the trivialization θ. Here, $L^1_{\mathrm{loc}}(\Omega)$ is the space of the locally integrable functions on Ω.

2.7 (Multiplier ideal sheaf). The notion of multiplier ideal sheaves introduced by Nadel is very important. First, we recall the notion of (quasi-)plurisubharmonic functions.

Definition 2.8 (Plurisubharmonic function). Let X be a complex manifold. A function $\varphi : X \to [-\infty, \infty)$ is said to be *plurisubharmonic* (psh, for short) if, on each connected component of X,

1. φ is upper semi-continuous, and
2. φ is locally integrable and $\sqrt{-1}\partial\bar{\partial}\varphi$ is positive semi-definite as a $(1,1)$-current,

or $\varphi \equiv -\infty$. A *smooth strictly plurisubharmonic function* ψ on X is a smooth function on X such that $\sqrt{-1}\partial\bar{\partial}\psi$ is a positive definite smooth $(1,1)$-form.

Definition 2.9. A *quasi-plurisubharmonic* (quasi-psh, for short) function is a function φ which is locally equal to the sum of a psh function and of a smooth function.

Next, we define multiplier ideal sheaves.

Definition 2.10 (Multiplier ideal sheaf). If φ is a quasi-psh function on a complex manifold X, the *multiplier ideal sheaf* $\mathcal{J}(\varphi) \subset \mathcal{O}_X$ is defined by
$$\Gamma(U, \mathcal{J}(\varphi)) = \{f \in \mathcal{O}_X(U); \ |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\}$$
for every open set $U \subset X$. Then it is known that $\mathcal{J}(\varphi)$ is a coherent ideal sheaf of $\mathcal{O}_X$. See, for example, [D2, (5.7) Proposition].

Finally, we note the definition of $\mathcal{J}(h_F)$ in Theorem 1.1.

Remark 2.11. By the assumption (ii) in Theorem 1.1, the weight of the singular hermitian metric h_F is a quasi-psh function on any trivialization. So, we can define multiplier ideal sheaves locally and check that they are independent of trivializations. Thus, we can define the multiplier ideal sheaf globally and denote it by $\mathcal{J}(h_F)$, which is an abuse of notation. It is a coherent ideal sheaf on X.

2.12 (Kähler geometry). We collects the basic notion and results of the hermitian and Kähler geometries (see also [D2]).

Definition 2.13 (Chern connection and its curvature form). Let X be a complex hermitian manifold and let (E, h) be a holomorphic hermitian vector bundle on X. Then there exists the *Chern connection* $D = D_{(E,h)}$, which can be split in a unique way as a sum of a $(1, 0)$ and of a $(0, 1)$-connection, $D = D'_{(E,h)} + D''_{(E,h)}$. By the definition of the Chern connection, $D'' = D''_{(E,h)} = \bar{\partial}$. We obtain the *curvature form* $\Theta(E) = \Theta_{(E,h)} = \Theta_h := D^2_{(E,h)}$. The subscripts might be suppressed if there is no danger of confusion.

Definition 2.14 (Inner product). Let X be an n-dimensional complex manifold with the hermitian metric g. We denote by ω the *fundamental form* of g. Let (E, h) be a hermitian vector bundle on X and let u, v be E-valued (p, q)-forms with measurable coefficients. We set

$$\|u\|^2 = \int_X |u|^2 dV_\omega, \ \langle\langle u, v \rangle\rangle = \int_X \langle u, v \rangle dV_\omega,$$

where $|u|$ is the pointwise norm induced by g and h on $\Lambda^{p,q}T^*_X \otimes E$, and $dV_\omega = \frac{1}{n!}\omega^n$. More explicitly, $\langle u, v \rangle dV_\omega = {}^t u \wedge H \overline{*v}$, where ${}^t u$ is the transposed matrix of u, $*$ is the *Hodge star operator* relative to ω, and H is the (local) matrix representation of h. When we need to emphasize the metrics, we write $|u|_{g,h}$, and so on.

Lemma 2.15 (Adjoint). *Let* $\theta \in C^{s,t}(X)$ *where* $C^{s,t}(X)$ *is the space of smooth* (s, t)*-forms on* X. *Then* $\theta^* = (-1)^{(p+q)(s+t+1)}*\bar{\theta}*$ *on* $C^{p,q}(X, E)$ *where* $C^{p,q}(X, E)$ *is the space of smooth* E*-valued* (p, q)*-forms on* X. *In particular, if* θ *is a 1-form, then* $\theta^* = *\bar{\theta}*$.

Proof. We take $u \in C^{p,q}(X, E)$ and $v \in C^{p-s,q-t}(X, E)$. Then

$$\langle v, \theta^* u \rangle dV_\omega = \langle \theta \wedge v, u \rangle dV_\omega$$

by the definition of θ^*. Let H be the local matrix representation of h. Then we have

$$\begin{aligned}
\langle \theta \wedge v, u \rangle dV_\omega &= {}^t(\theta \wedge v) H \overline{*u} \\
&= (-1)^{(p+q-s-t)(s+t)} ({}^t v) H \theta \overline{*u} \\
&= (-1)^{(p+q-s-t)(s+t)+(2n-p-q+s+t)} ({}^t v) H * * \overline{\bar{\theta} * u} \\
&= (-1)^{(p+q)(s+t+1)} ({}^t v) H \overline{* * \bar{\theta} * u}.
\end{aligned}$$

Therefore, $\theta^* u = (-1)^{(p+q)(s+t+1)} * \bar{\theta} * u$. $\qquad\qquad\square$

Let $L_{(2)}^{p,q}(X,E)(= L_{(2)}^{p,q}(X,(E,h)))$ be the space of square integrable E-valued (p,q)-forms on X. The inner product was defined in Definition 2.14. When we emphasize the metrics, we write $L_{(2)}^{p,q}(X,E)_{g,h}$, where g (resp. h) is the hermitian metric of X (resp. E). As usual one can view D' and D'' as closed and densely defined operators on the Hilbert space $L_{(2)}^{p,q}(X,E)$. The formal adjoints D'^*, D''^* also have closed extensions in the sense of distributions, which do not necessarily coincide with the Hilbert space adjoints in the sense of Von Neumann, since the latter ones may have strictly smaller domains. It is well known, however, that the domains coincide if the hermitian metric of X is complete. See Lemma 2.16 below.

Lemma 2.16 (Density Lemma). *Let X be a complex manifold with the complete hermitian metric g and let (E,h) be a holomorphic hermitian vector bundle on X. Then $C_0^{p,q}(X,E)$ is dense in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(D''^*_{(E,h)})$ with respect to the graph norm $\|v\| + \|\bar{\partial}v\| + \|D''^*_{(E,h)}v\|$, where $\mathrm{Dom}(\bar{\partial})$ (resp. $\mathrm{Dom}(D''^*_{(E,h)})$) is the domain of $\bar{\partial}$ (resp. $D''^*_{(E,h)}$).*

Suppose that (E,h) is a holomorphic hermitian vector bundle and that (e_λ) is a holomorphic frame for E over some open set U. Then the metric h is given by the $r \times r$ hermitian matrix $H = (h_{\lambda\mu})$, where $h_{\lambda\mu} = h(e_\lambda, e_\mu)$ and $r = \mathrm{rank}E$. Then we have $h(u,v) = {}^t u H \bar{v}$ on U for smooth sections u, v of $E|_U$. This implies that $h(u,v) = \sum_{\lambda,\mu} u_\lambda h_{\lambda\mu} \bar{v}_\mu$ for $u = \sum e_i u_i$ and $v = \sum e_j v_j$. Then we obtain that $\sqrt{-1}\Theta_h(E) = \sqrt{-1}\bar{\partial}(\overline{H}^{-1}\partial\overline{H})$ and ${}^t\overline{(\sqrt{-1}{}^t\Theta_h(E)H)} = \sqrt{-1}{}^t\Theta_h(E)H$ on U. Let $C^{p,q}(X,E)$ (resp. $C_0^{p,q}(X,E)$) be the space of smooth E-valued (p,q)-forms (resp. smooth E-valued (p,q)-forms with compact supports) on X. We define $\{u,v\} = {}^t u \wedge H\bar{v}$ for $u \in C^{p,q}(X,E)$ and $v \in C^{r,s}(X,E)$, where ${}^t u$ is the transposed matrix of u. We will use $\{\cdot,\cdot\}$ in Section 3.

Definition 2.17 (Nakano positivity and semi-positivity). Let (E,h) be a holomorphic vector bundle on a complex manifold X with a smooth

hermitian metric h. Let Ξ be a $\mathrm{Hom}(E, E)$-valued $(1, 1)$-form such that $\overline{{}^t({}^t\Xi h)} = {}^t\Xi h$. Then Ξ is said to be *Nakano positive* (resp. *Nakano semi-positive*) if the hermitian form on $T_X \otimes E$ associated to ${}^t\Xi h$ is positive definite (resp. semi-definite). We write $\Xi >_{\mathrm{Nak}} 0$ (resp. $\geq_{\mathrm{Nak}} 0$). We note that $\Xi_1 >_{\mathrm{Nak}} \Xi_2$ (resp. $\Xi_1 \geq_{\mathrm{Nak}} \Xi_2$) means that $\Xi_1 - \Xi_2 >_{\mathrm{Nak}}$ 0 (resp. $\geq_{\mathrm{Nak}} 0$). A holomorphic vector bundle (E, h) is said to be *Nakano positive* (resp. *semi-positive*) if $\sqrt{-1}\Theta(E) >_{\mathrm{Nak}} 0$ (resp. $\geq_{\mathrm{Nak}}$ 0). We usually omit "Nakano" when E is a line bundle. We often simply say that a holomorphic line bundle L is *semi-positive* if there exists a smooth hermitian metric h_L on L such that $\sqrt{-1}\Theta(L) \geq 0$.

The space of harmonic forms will play important roles in the proof of Theorem 1.1. See also the introduction of [F1].

Definition 2.18 (Harmonic forms)**.** Let X be an n-dimensional complete Kähler manifold with a complete Kähler metric g. Let (E, h_E) be a holomorphic hermitian vector bundle on X. We put

$$\mathcal{H}^{p,q}(X, (E, h_E))_g = \{u \in L^{p,q}_{(2)}(X, E) | \bar\partial u = 0 \text{ and } D''^*_{(E,h_E)}u = 0\}.$$

Note that $\mathcal{H}^{p,q}(X, (E, h_E))_g \subset C^{p,q}(X, E)$ by the regularization theorem for elliptic partial differential equations of second order.

2.19 (Ohsawa–Takegoshi twist)**.** The following formula is a *twisted* version of Nakano's identity, which is now well known to the experts.

Proposition 2.20 (Ohsawa–Takegoshi twist)**.** *Let (E, h) be a holomorphic hermitian vector bundle on an n-dimensional Kähler manifold X. Let η be any smooth positive function on X. Then, for every $u \in C_0^{n,q}(X, E)$, the equality*

$$(\spadesuit) \qquad \|\sqrt\eta D''^*_{(E,h)}u\|^2 + \|\sqrt\eta \bar\partial u\|^2 - \|\sqrt\eta D'^*u\|^2$$
$$= \langle\langle \sqrt{-1}(\eta\Theta_h - \mathrm{Id}_E \otimes \partial\bar\partial\eta)\Lambda u, u\rangle\rangle + 2\mathrm{Re}\langle\langle \bar\partial\eta \wedge D''^*_{(E,h)}u, u\rangle\rangle$$

holds true. Here, we denote by Λ the adjoint operator of $\omega \wedge \cdot$. Note that $D'' = D''_{(E,h)} = \bar\partial$ and D'^* are independent of the hermitian metric h.

Sketch of the proof. We quickly review the proof of this proposition for the reader's convenience. If A, B are the endomorphisms of pure degree of the graded module $C^{\bullet,\bullet}(X, E)$, their *graded Lie bracket* is defined by

$$[A, B] = AB - (-1)^{\deg A \deg B}BA.$$

Let

$$\Delta' = D'D'^* + D'^*D'$$

and
$$\Delta'' = D'' D''^* + D''^* D''$$
be the complex Laplace operators. Then it is well known that
$$\Delta'' = \Delta' + [\sqrt{-1}\Theta(E), \Lambda],$$
which is sometimes called Nakano's identity. Let us consider the *twisted* Laplace operators
$$D' \eta D'^* + D'^* \eta D' = \eta \Delta' + (\partial \eta) D'^* - (\partial \eta)^* D',$$
and
$$D'' \eta D''^* + D''^* \eta D'' = \eta \Delta'' + (\bar{\partial} \eta) D''^* - (\bar{\partial} \eta)^* D''.$$
On the other hand, we can easily check that
$$[\sqrt{-1}\partial \bar{\partial} \eta, \Lambda] = [D'', (\bar{\partial} \eta)^*] + [D'^*, \partial \eta]$$
by $(\bar{\partial} \eta)^* = -\sqrt{-1}[\partial \eta, \Lambda]$ and $D'^* = -\sqrt{-1}[\bar{\partial}, \Lambda]$. Combining these equalities, we find
$$D'' \eta D''^* + D''^* \eta D'' - D' \eta D'^* - D'^* \eta D' + [\sqrt{-1}\partial \bar{\partial} \eta, \Lambda]$$
$$= \eta[\sqrt{-1}\Theta(E), \Lambda] + (\bar{\partial} \eta) D''^* + D''(\bar{\partial} \eta)^* + (\partial \eta)^* D' + D'^*(\partial \eta).$$
Apply this identity to a form $u \in C_0^{n,q}(X, E)$ and take the inner product with u. Then we obtain the desired formula. $\qquad\square$

The next proposition is [O, Lemma 2.1]. The proof is a routine work. It easily follows from Lemmas 2.16, 2.22, and 2.23.

Proposition 2.21. *Fix a complete Kähler metric g on X. We put*
$$D^{n,q} = \{u \in L_{(2)}^{n,q}(X, E) \mid \bar{\partial} u \in L_{(2)}^{n,q+1}(X, E) \text{ and } D''^* u \in L_{(2)}^{n,q-1}(X, E)\},$$
that is, $D^{n,q} = \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(D''^*_{(E,h)}) \subset L_{(2)}^{n,q}(X, E)$. Suppose that η is bounded and that there exists a constant $\varepsilon > 0$ such that
$$\sqrt{-1}(\eta \Theta_h - \mathrm{Id}_E \otimes \partial \bar{\partial} \eta - \varepsilon \mathrm{Id}_E \otimes \partial \eta \wedge \bar{\partial} \eta) \geq_{\mathrm{Nak}} 0$$
holds everywhere. Then the equality ($\spadesuit$) in Proposition 2.20 holds for all $u \in D^{n,q}$.

Lemma 2.22. *For every $u \in C^{n,q}(X, E)$ and any positive real number δ, we have*
$$\begin{aligned} |2\mathrm{Re}\langle\!\langle \bar{\partial} \eta \wedge D''^*_{(E,h)} u, u \rangle\!\rangle| &= |2\mathrm{Re}\langle\!\langle D''^*_{(E,h)} u, (\bar{\partial} \eta)^* u \rangle\!\rangle| \\ &\leq \frac{1}{\delta}\|D''^*_{(E,h)} u\|^2 + \delta\|(\bar{\partial} \eta)^* u\|^2, \text{ and} \end{aligned}$$
$$\begin{aligned} \|(\bar{\partial} \eta)^* u\|^2 &= \langle\!\langle (\bar{\partial} \eta)^* u, (\bar{\partial} \eta)^* u \rangle\!\rangle \\ &= \langle\!\langle \sqrt{-1}\partial \eta \wedge \bar{\partial} \eta \Lambda u, u \rangle\!\rangle \end{aligned}$$

since $(\bar{\partial}\eta)^*u = -\sqrt{-1}\partial\eta\Lambda u$ *for* $u \in C^{n,q}(X, E)$. *Note that* $(\bar{\partial}\eta)^*$ *is the adjoint operator of* $\bar{\partial}\eta \wedge \cdot$ *relative to the inner product* $\langle\ ,\ \rangle$.

By combining Proposition 2.20 with Lemma 2.22, we obtain the next lemma.

Lemma 2.23. *We use the same notation as in* Proposition 2.20. *Assume that*

$$\sqrt{-1}(\eta\Theta_h - \mathrm{Id}_E \otimes \partial\bar{\partial}\eta - \varepsilon\mathrm{Id}_E \otimes \partial\eta \wedge \bar{\partial}\eta) \geq_{\mathrm{Nak}} 0$$

holds everywhere for some positive constant ε. *Then, for every* $u \in C_0^{n,q}(X, E)$, *we have*

$$\|\sqrt{\eta}D'^*u\|^2 \leq \|\sqrt{\eta}D''^*_{(E,h)}u\|^2 + \|\sqrt{\eta}\bar{\partial}u\|^2 + \frac{1}{\varepsilon}\|\sqrt{\eta}D''^*_{(E,h)}u\|^2,$$

and

$$\|\sqrt{\eta}D''^*_{(E,h)}u\|^2 + \|\sqrt{\eta}\bar{\partial}u\|^2 - \|\sqrt{\eta}D'^*u\|^2 + \frac{1}{\delta}\|\sqrt{\eta}D''^*_{(E,h)}u\|^2$$
$$\geq (\varepsilon - \delta)\|(\bar{\partial}\eta)^*u\|^2$$

for any positive real number δ.

We close this section by the following remark on [T].

Remark 2.24. By Proposition 2.21, we can prove [T, Theorem 3.4 (ii)] under the slightly weaker assumption that φ is a *bounded* smooth psh function on M. We do not have to assume that $|d\varphi|$ is bounded on M. For the notations, see [T]. In this case, there are positive constants C_1 and C_2 such that $\varphi + C_1 > 0$ on M and $C_2 - (\varphi + C_1)^2 > 0$ on M. We can use Proposition 2.21 (and Lemmas 2.22, 2.23) for $\eta := C_2 - (\varphi + C_1)^2$ and $\varepsilon := \frac{1}{2C_2} > 0$. Then we obtain $(\bar{\partial}\varphi)^*u = 0$ and $\langle\sqrt{-1}\partial\bar{\partial}\varphi\Lambda u, u\rangle_h = 0$.

3. Proof of the main theorem

In this section, we prove Theorem 1.1. So, we freely use the notation in Theorem 1.1. Let $W \Subset Y$ be any Stein open subset. We put $V = f^{-1}(W)$. Then V is a holomorphically convex weakly 1-complete Kähler manifold. To prove Theorem 1.1, it is sufficient to show that

$$\times s : H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F)) \to H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F) \otimes L)$$

is injective for every $q \geq 0$. Note that the above cohomology groups are separated topological vector spaces since V is holomorphically convex (cf. [R, Theoreme 10]).

Remark 3.1. A weakly 1-complete manifold X is called a *weakly pseudoconvex manifold* in [D2]. A weakly 1-complete manifold is a complex manifold equipped with a *smooth* plurisubharmonic exhaustion function. More explicitly, there exists a smooth plurisubharmonic function φ on X such that $X_c = \{x \in X | \varphi(x) < c\}$ is relatively compact in X for every c.

We define bounded smooth functions from the given nonzero holomorphic section s of L.

Definition 3.2. Take a smooth plurisubharmonic exhaustion function φ on V. Without loss of generality, we can assume that $\min_{x \in V} \varphi(x) = 0$. Let s be a holomorphic section of L. Let $|s|$ be the pointwise norm of s with respect to the fiber metric h_L. Let $\lambda : [0, \infty) \to [0, \infty)$ be a smooth convex increasing function such that $5|s|^2 < e^{\lambda(\varphi)}$. Thus, $|s|^2_{\lambda(\varphi)} < \frac{1}{5} < \frac{1}{4}$, where $|s|_{\lambda(\varphi)}$ is the pointwise norm of s with respect to the fiber metric $h_L e^{-\lambda(\varphi)}$. We put $\mu(x) := \lambda(x) + x$. Obviously, μ is also a smooth convex increasing function.

Definition 3.3. We put $\chi(t) = t - \log(-t)$ for $t < 0$. We define

$$\sigma_{\varepsilon,\lambda} = \log(|s|^2_{\lambda(\varphi)} + \varepsilon), \text{ and}$$

$$\eta_{\varepsilon,\lambda} = \frac{1}{\epsilon} - \chi(\sigma_{\varepsilon,\lambda})$$

$$= -\log(|s|^2_{\lambda(\varphi)} + \varepsilon) + \log(-\log(|s|^2_{\lambda(\varphi)} + \varepsilon)) + \frac{1}{\varepsilon}.$$

We can also define $\sigma_{\varepsilon,\mu}$ and $\eta_{\varepsilon,\mu}$ similarly. Note that $\eta_{\varepsilon,\lambda}$ and $\eta_{\varepsilon,\mu}$ are smooth bounded functions on V with $\eta_{\varepsilon,\mu} \geq \eta_{\varepsilon,\lambda} > \frac{1}{\varepsilon}$. The subscripts λ, μ, and ε might be suppressed if there is no danger of confusion.

We note the following obvious remark before we start various calculations.

Remark 3.4. We note that $e < 2\sqrt{2}$. Thus, $\frac{3}{2}\log 2 > 1$. Therefore, $\sigma_{\varepsilon,\mu} \leq \sigma_{\varepsilon,\lambda} < 2\log\frac{1}{2} < -\frac{4}{3}$ if ε is small since $|s|^2_{\mu(\varphi)} + \varepsilon \leq |s|^2_{\lambda(\varphi)} + \varepsilon < \frac{1}{4}$ by $|s|^2_{\lambda(\varphi)} < \frac{1}{5}$. Of course, $\log(-\sigma_{\varepsilon,\mu}) \geq \log(-\sigma_{\varepsilon,\lambda}) > \log\frac{4}{3} > 0$. We have $\chi'(t) = 1 - \frac{1}{t}$ and $\chi''(t) = \frac{1}{t^2}$. Thus, $1 < \chi'(\sigma_{\varepsilon,\mu}) \leq \chi'(\sigma_{\varepsilon,\lambda}) = 1 + \frac{1}{(-\sigma_{\varepsilon,\lambda})} < \frac{7}{4}$.

3.5 (Basic calculations)**.** We calculate various differentials of $\eta_{\varepsilon,\lambda}$. The same arguments work for $\eta_{\varepsilon,\mu}$.

Definition 3.6. Let $u \in C^{p,q}(V, L)$ and $v \in C^{r,s}(V, L)$. We define

$$\{u, v\}_{\lambda(\varphi)} = u \wedge H_L e^{-\lambda(\varphi)}\bar{v},$$

where H_L is the local matrix representation of h_L.

We have

$$\partial \sigma_{\varepsilon,\lambda} = \frac{\{D's, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)} + \varepsilon}.$$

In the above equation, D' is the $(1,0)$ part of the Chern connection of $L' = (L, h_L e^{-\lambda(\varphi)})$, that is, $D' = D'_{(L, h_L e^{-\lambda(\varphi)})}$. Thus, $\Theta(L') = \Theta(L) + \partial \bar{\partial} \lambda(\varphi)$. We obtain the following equation by the direct computation.

$$\sqrt{-1}\partial \bar{\partial} \sigma_{\varepsilon,\lambda} = -\frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)} + \varepsilon}$$
$$+ \frac{\sqrt{-1}\{D's, D's\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)} + \varepsilon} - \frac{\sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)}}{(|s|^2_{\lambda(\varphi)} + \varepsilon)^2},$$

where $L' = (L, h_L e^{-\lambda(\varphi)})$. By the Cauchy-Schwarz inequality, we have

$$\sqrt{-1}\{D's, D's\}_{\lambda(\varphi)}|s|^2_{\lambda(\varphi)} \geq \sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)}.$$

Substituting the Cauchy-Schwarz inequality into the above equation, we obtain

$$\sqrt{-1}\partial \bar{\partial} \sigma_{\varepsilon,\lambda} \geq \frac{\varepsilon}{|s|^2_{\lambda(\varphi)}(|s|^2_{\lambda(\varphi)} + \varepsilon)^2}\sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)}$$
$$- \frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)} + \varepsilon}$$
$$= \frac{\varepsilon}{|s|^2_{\lambda(\varphi)}}\sqrt{-1}\partial \sigma_{\varepsilon,\lambda} \wedge \bar{\partial} \sigma_{\varepsilon,\lambda} - \frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)} + \varepsilon}.$$

Lemma 3.7. *The next equations follow easily from the definition.*

$$\partial \eta_{\varepsilon,\lambda} = -\chi'(\sigma_{\varepsilon,\lambda})\partial \sigma_{\varepsilon,\lambda},$$
$$\bar{\partial} \eta_{\varepsilon,\lambda} = -\chi'(\sigma_{\varepsilon,\lambda})\bar{\partial} \sigma_{\varepsilon,\lambda},$$
$$\partial \bar{\partial} \eta_{\varepsilon,\lambda} = -\chi''(\sigma_{\varepsilon,\lambda})\partial \sigma_{\varepsilon,\lambda} \wedge \bar{\partial} \sigma_{\varepsilon,\lambda} - \chi'(\sigma_{\varepsilon,\lambda})\partial \bar{\partial} \sigma_{\varepsilon,\lambda}.$$

Combining the above (in)equalities, we have

$$
\begin{aligned}
-\sqrt{-1}\partial\bar\partial\eta_{\varepsilon,\lambda} &= \chi'(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\bar\partial\sigma_{\varepsilon,\lambda} + \chi''(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\sigma_{\varepsilon,\lambda}\wedge\bar\partial\sigma_{\varepsilon,\lambda} \\
&\geq \frac{\varepsilon\chi'(\sigma_{\varepsilon,\lambda})}{|s|^2_{\lambda(\varphi)}}\sqrt{-1}\partial\sigma_{\varepsilon,\lambda}\wedge\bar\partial\sigma_{\varepsilon,\lambda} + \chi''(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\sigma_{\varepsilon,\lambda}\wedge\bar\partial\sigma_{\varepsilon,\lambda} \\
&\quad - \chi'(\sigma_{\varepsilon,\lambda})\frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)}+\varepsilon} \\
&= \left(\frac{\varepsilon}{\chi'(\sigma_{\varepsilon,\lambda})|s|^2_{\lambda(\varphi)}} + \frac{\chi''(\sigma_{\varepsilon,\lambda})}{\chi'(\sigma_{\varepsilon,\lambda})^2}\right)\sqrt{-1}\partial\eta_{\varepsilon,\lambda}\wedge\bar\partial\eta_{\varepsilon,\lambda} \\
&\quad - \chi'(\sigma_{\varepsilon,\lambda})\frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)}+\varepsilon}.
\end{aligned}
$$

Lemma 3.8. *We have the following inequality.*

$$
\frac{\chi''(\sigma_{\varepsilon,\lambda})}{\chi'(\sigma_{\varepsilon,\lambda})^2} \geq \eta_{\varepsilon,\lambda}^{-2}
$$

Proof. By the definition, it is easy to see that

$$
\frac{\chi'(\sigma_{\varepsilon,\lambda})}{\chi''(\sigma_{\varepsilon,\lambda})} = \frac{(1-\frac{1}{\sigma_{\varepsilon,\lambda}})^2}{\frac{1}{\sigma_{\varepsilon,\lambda}^2}} = (\sigma_{\varepsilon,\lambda}-1)^2.
$$

On the other hand, $\eta_{\varepsilon,\lambda} = \frac{1}{\varepsilon} - \sigma_{\varepsilon,\lambda} + \log(-\sigma_{\varepsilon,\lambda}) > 1 - \sigma_{\varepsilon,\lambda}$. Thus, we obtain the desired inequality. $\square$

By the above calculations, we obtain a very important inequality.

Proposition 3.9. *Under the curvature conditions*

$$
\sqrt{-1}(\Theta(E) + \mathrm{Id}_E\otimes\Theta(F)) \geq_{\mathrm{Nak}} 0
$$

on $X\setminus Z$ and

$$
\sqrt{-1}(\Theta(E) + \mathrm{Id}_E\otimes\Theta(F) - \varepsilon_0\mathrm{Id}_E\otimes\Theta(L)) \geq_{\mathrm{Nak}} 0
$$

on $X\setminus Z$ for some positive real number ε_0, we have a small positive real number ε_1 such that

$$
\sqrt{-1}(\eta\Theta_{(E\otimes F, h_E h_F e^{-\mu(\varphi)})} - \mathrm{Id}_E\otimes\partial\bar\partial\eta) \geq_{\mathrm{Nak}} \sqrt{-1}(\mathrm{Id}_E\otimes\eta^{-2}\partial\eta\wedge\bar\partial\eta)
$$

holds on $V\setminus Z$ for $0 < \varepsilon < \varepsilon_1$, where $\eta = \eta_{\varepsilon,\lambda}$ or $\eta_{\varepsilon,\mu}$.

Proof. By the definitions of λ and μ, $\partial\bar{\partial}\mu(\varphi) = \partial\bar{\partial}\lambda(\varphi) + \partial\bar{\partial}\varphi$, and $\lambda(\varphi)$ and $\mu(\varphi)$ are plurisubharmonic. Note that

$$
\begin{aligned}
\Theta_{(E\otimes F, h_E h_F e^{-\mu(\varphi)})} &= \Theta(E) + \mathrm{Id}_E \otimes \Theta(F) + \mathrm{Id}_E \otimes \partial\bar{\partial}\mu(\varphi) \\
&= \Theta(E) + \mathrm{Id}_E \otimes \Theta(F) + \mathrm{Id}_E \otimes \partial\bar{\partial}\lambda(\varphi) + \mathrm{Id}_E \otimes \partial\bar{\partial}\varphi, \\
\Theta_{(L, h_L e^{-\mu(\varphi)})} &= \Theta(L) + \partial\bar{\partial}\mu(\varphi), \text{ and} \\
\Theta_{(L, h_L e^{-\lambda(\varphi)})} &= \Theta(L) + \partial\bar{\partial}\lambda(\varphi).
\end{aligned}
$$

We also note that

$$
0 \le \chi'(\sigma_{\varepsilon,\lambda}) \frac{|s|^2_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)} + \varepsilon} < \frac{7}{4}
$$

and

$$
0 \le \chi'(\sigma_{\varepsilon,\mu}) \frac{|s|^2_{\mu(\varphi)}}{|s|^2_{\mu(\varphi)} + \varepsilon} < \frac{7}{4}
$$

by Remark 3.4. If ε_1 is small, then

$$
\eta \ge \max\left\{ \frac{7}{4\varepsilon_0}, \frac{7}{4} \right\}
$$

since $\eta > \frac{1}{\varepsilon} > \frac{1}{\varepsilon_1}$. Therefore,

$$
\sqrt{-1}(\eta\Theta_{(E\otimes F, h_E h_F e^{-\mu(\varphi)})} - \mathrm{Id}_E \otimes \partial\bar{\partial}\eta) \ge_{\mathrm{Nak}} \sqrt{-1}(\mathrm{Id}_E \otimes \eta^{-2}\partial\eta \wedge \bar{\partial}\eta)
$$

holds on $V \setminus Z$ for $0 < \varepsilon < \varepsilon_1$ where $\eta = \eta_{\varepsilon,\lambda}$ or $\eta_{\varepsilon,\mu}$. $\qquad\square$

We note that we need no assumptions on the sign of the curvature $\sqrt{-1}\Theta(L)$ in Proposition 3.9. It is a very important remark.

In the next lemma, we obtain the relationship between the Chern connections of (L, h_L) and $(L, h_L e^{-\lambda(\varphi)})$.

Lemma 3.10. *Let $\gamma : [0, \infty) \to \mathbb{R}$ be any smooth $\mathbb{R}$-valued function. Then we have the following equation by the definition of the Chern connection.*

$$
\begin{aligned}
D'_{(L, h_L e^{-\gamma(\varphi)})} &= (H_L e^{-\gamma(\varphi)})^{-1}\partial(H_L e^{-\gamma(\varphi)} \cdot\ ) \\
&= \partial + \partial\log(H_L e^{-\gamma(\varphi)}) \wedge \cdot \\
&= \partial + \partial\log H_L \wedge \cdot - \gamma'(\varphi)\partial\varphi \wedge \cdot \\
&= D'_{(L, h_L)} - \gamma'(\varphi)\partial\varphi \wedge \cdot\ .
\end{aligned}
$$

We note that $H_L = \overline{H}_L$ since L is a line bundle.

3.11 (Complete Kähler metrics)**.** There exists a complete Kähler metric g on V since V is weakly 1-complete. Let ω be the fundamental form of g. We note the following well-known lemma (cf. [D3, Lemma 5]).

Lemma 3.12. *There exists a quasi-psh function ψ on X such that $\psi = -\infty$ on Z with logarithmic poles along Z and ψ is smooth outside Z.*

Without loss of generality, we can assume that $\psi < -e$ on $V \Subset X$. We put $\widetilde{\psi} = \frac{1}{\log(-\psi)}$. Then $\widetilde{\psi}$ is a quasi-psh function on V and $\widetilde{\psi} < 1$. Thus, we can take a positive constant α such that $\sqrt{-1}\partial\bar{\partial}\widetilde{\psi} + \alpha\omega > 0$ on $V \setminus Z$. Let g' be the Kähler metric on $V \setminus Z$ whose fundamental form is $\omega' = \omega + (\sqrt{-1}\partial\bar{\partial}\widetilde{\psi} + \alpha\omega)$. We note that we can check that

$$\omega' \geq \sqrt{-1}\partial(\log(\log(-\psi))) \wedge \bar{\partial}(\log(\log(-\psi)))$$

if we choose $\alpha \gg 0$. It is because

$$\partial\bar{\partial}\widetilde{\psi} = 2\frac{\frac{-\partial\psi}{-\psi} \wedge \frac{-\bar{\partial}\psi}{-\psi}}{(\log(-\psi))^3} + \frac{\frac{\partial\bar{\partial}\psi}{-\psi}}{(\log(-\psi))^2} + \frac{\frac{-\partial\psi\wedge(-\bar{\partial}\psi)}{(-\psi)^2}}{(\log(-\psi))^2},$$

and

$$\partial(\log(\log(-\psi))) = \frac{\frac{-\partial\psi}{-\psi}}{\log(-\psi)}.$$

Therefore, g' is a *complete* Kähler metric on $V \setminus Z$ by Hopf–Rinow because $\log(\log(-\psi))$ tends to $+\infty$ on Z. For similar arguments, see [F1, Section 3]. We fix these Kähler metrics throughout this proof.

3.13 (Key Results)**.** The following three propositions are the heart of the proof of Theorem 1.1.

Proposition 3.14. *For every $u \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$, $(\bar{\partial}\eta)^* u = 0$ for $\eta = \eta_{\varepsilon,\lambda}$ and $\eta_{\varepsilon,\mu}$. This implies that $\partial\eta \wedge *u = 0$ for $\eta = \eta_{\varepsilon,\lambda}$ and $\eta_{\varepsilon,\mu}$. Thus, we obtain $D'_{(L,h_L e^{-\lambda(\varphi)})}s \wedge *u = 0$ and $D'_{(L,h_L e^{-\mu(\varphi)})}s \wedge *u = 0$.*

Proof. The definition of $\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$ implies that $\bar{\partial}u = 0$ and $D''^*_{(E\otimes F,h_E h_F e^{-\mu(\varphi)})}u = 0$. By Propositions 2.20, 2.21, and 3.9, we have

$$0 \geq -\|\sqrt{\eta}D'^* u\|^2 \geq \langle\!\langle \sqrt{-1}\eta^{-2}\partial\eta \wedge \bar{\partial}\eta\Lambda u, u\rangle\!\rangle \geq 0.$$

Thus, we have $(\bar{\partial}\eta)^* u = 0$ (cf. Lemma 2.22). Therefore, we obtain $\partial\eta \wedge *u = 0$ because $(\bar{\partial}\eta)^* = *\partial\eta \wedge *$ by Lemma 2.15. By the definition of η, we obtain $D'_{(L,h_L e^{-\lambda(\varphi)})}s \wedge *u = 0$ (resp. $D'_{(L,h_L e^{-\mu(\varphi)})}s \wedge *u = 0$) if $\eta = \eta_{\varepsilon,\lambda}$ (resp. $\eta = \eta_{\varepsilon,\mu}$). $\qquad\square$

Proposition 3.15. *If $D'_{(L,h_L e^{-\lambda(\varphi)})}s \wedge *u = 0$ and $D'_{(L,h_L e^{-\mu(\varphi)})}s \wedge *u = 0$, then $D'_{(L,h_L)}s \wedge *u = 0$ and $\partial\varphi \wedge *u = 0$. Therefore, $D'_{(L,h_L e^{-\nu(\varphi)})}s \wedge *u = 0$ for any smooth $\mathbb{R}$-valued function ν defined on $[0,\infty)$.*

Proof. We note that $D'_{(L,h_L e^{-\lambda(\varphi)})} = D'_{(L,h_L)} - \lambda'(\varphi)\partial\varphi \wedge \cdot$ and

$$D'_{(L,h_L e^{-\mu(\varphi)})} = D'_{(L,h_L)} - \lambda'(\varphi)\partial\varphi \wedge \cdot - \partial\varphi \wedge \cdot$$

since $\mu(x) = \lambda(x) + x$. $\qquad\square$

Proposition 3.16. *If $D''^*_{(E\otimes F,h_E h_F e^{-\mu(\varphi)})}u = 0$, then we obtain*

$$D''^*_{(E\otimes F\otimes L,h_E h_F h_L e^{-\mu(\varphi)-\nu(\varphi)})}(su) = 0$$

for any smooth $\mathbb{R}$-valued function ν defined on $[0,\infty)$.

Proof. Let H_E (resp. H_F) be the local matrix representation of h_E (resp. h_F). The condition $D''^*_{(E\otimes F,h_E h_F e^{-\mu(\varphi)})}u = 0$ implies that

$$\bar{\partial}(e^{-\mu(\varphi)}H_E H_F \overline{*u}) = 0.$$

To prove $D''^*_{(E\otimes F\otimes L,h_E h_F h_L e^{-\mu(\varphi)-\nu(\varphi)})}(su) = 0$, it is sufficient to check that $\bar{\partial}(H_E H_F e^{-\mu(\varphi)-\nu(\varphi)}H_L\overline{*su}) = 0$. We note that

$$\bar{\partial}(H_E H_F e^{-\mu(\varphi)-\nu(\varphi)}H_L\overline{*su}) = \bar{\partial}(H_L \bar{s}e^{-\nu(\varphi)}) \wedge e^{-\mu(\varphi)}H_E H_F \overline{*u}$$

by the above condition. The right hand side is zero since $D'_{(L,H_L e^{-\nu(\varphi)})}s\wedge *u = 0$. $\qquad\square$

The next theorem is a key result.

Theorem 3.17 (cf. [O, Proposition 3.1]). *For any smooth $\mathbb{R}$-valued function defined on $[0,\infty)$ such that $\nu \geq C$ for some constant C,*

$$s\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$$

is contained in

$$\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))_{g'}$$

for every q.

Proof. Let $u \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$. Then it is obvious that $su \in L^{n,q}_{(2)}(E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)})$ because $|s|^2_{\lambda(\varphi)} < \frac{1}{5}$ and $0 < e^{-\nu(\varphi)} \leq e^{-C}$. So, the claim is a direct consequence of Proposition 3.16. Note that $\bar{\partial}(su) = 0$ for $u \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$ since s is holomorphic and $\bar{\partial}u = 0$. $\qquad\square$

3.18 (Cohomology groups). Before we start the proof of the main theorem: Theorem 1.1, we represent the cohomology groups on V by the objects on $V \setminus Z$.

Definition 3.19 (Space of locally square integrable forms)**.** We define the space of locally (in V) square integrable $E \otimes F$-valued (n, q)-forms on $V \setminus Z$. It is denoted by $L_{loc,V}^{n,q}(V \setminus Z, E \otimes F)$ or $L_{loc,V}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F))$. The vector space $L_{loc,V}^{n,q}(V \setminus Z, E \otimes F)$ is spanned by (n, q)-forms u on $V \setminus Z$ with measurable coefficients such that

$$\int_U |u|_{g',h_E h_F}^2 dV_{\omega'} < \infty$$

for every $U \Subset V$ (not $U \Subset V \setminus Z$), where $|\cdot|_{g',h_E h_F}$ is the pointwise norm with respect to g' and $h_E h_F$. We note the following obvious remark. Let $h : V \to (0, \infty)$ be a smooth positive function. Then $L_{loc,V}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F)) = L_{loc,V}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F h))$. We can define $L_{loc,V}^{n,q}(V \setminus Z, E \otimes F \otimes L)$ similarly.

The next lemma is essentially the same as [F1, Claim 1], which is more or less known to experts (cf. [T, Proposition 4.6]).

Lemma 3.20. *The following isomorphism holds.*

$$H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F))$$
$$\simeq \quad H_{loc,V}^{n,q}(V \setminus Z, E \otimes F)_{g'}$$
$$:= \quad \frac{\mathrm{Ker}\bar{\partial} \cap L_{loc,V}^{n,q}(V \setminus Z, E \otimes F)}{L_{loc,V}^{n,q}(V \setminus Z, E \otimes F) \cap \bar{\partial} L_{loc,V}^{n,q-1}(V \setminus Z, E \otimes F)}.$$

Sketch of the proof. Let $V = \bigcup_{i \in I} U_i$ be a locally finite Stein cover of V such that each U_i is sufficiently small and $U_i \Subset V$. We denote this cover by $\mathcal{U} = \{U_i\}_{i \in I}$. By Cartan and Leray, we obtain

$$H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F)) \simeq \check{H}^q(\mathcal{U}, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F)),$$

where the right hand side is the Čech cohomology group calculated by $\mathcal{U}$. By using a partition of unity $\{\rho_i\}_{i \in I}$ associated to $\mathcal{U}$, we can construct a homomorphism

$$\rho : \check{H}^q(\mathcal{U}, K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \to H_{loc,V}^{n,q}(V \setminus Z, E \otimes F)_{g'}.$$

See Remark 3.21 below. We can check that ρ is an isomorphism. Note the following facts: (a) The open set $U_{i_0} \cap \cdots \cap U_{i_k}$ is Stein. So, $U_{i_0} \cap \cdots \cap U_{i_k} \setminus Z$ is a complete Kähler manifold (cf. [D1, Théorème 0.2]). Therefore, $E \otimes F$-valued $\bar{\partial}$-equations can be solved with suitable L^2 estimates on $U_{i_0} \cap \cdots \cap U_{i_k} \setminus Z$ by Lemma 3.22 below. (b) Let U be an open subset of V. An $E \otimes F$-valued holomorphic $(n, 0)$-form on $U \setminus Z$ with a finite L^2 norm can be extended to an $E \otimes F$-valued holomorphic $(n, 0)$-form on U (cf. Remark 3.21). $\qquad\square$

Remark 3.21 (cf. [D1, Lemme 3.3])**.** Let u be an $E \otimes F$-valued (n, q)-form on $V \setminus Z$ with measurable coefficients. Then, we have $|u|^2_{g', h_E h_F} dV_{\omega'} \leq |u|^2_{g, h_E h_F} dV_\omega$, where $|u|_{g', h_E h_F}$ (resp. $|u|_{g, h_E h_F}$) is the pointwise norm induced by g' and $h_E h_F$ (resp. g and $h_E h_F$) since $g' > g$ on $V \setminus Z$. If u is an $E \otimes F$-valued $(n, 0)$-form, then $|u|^2_{g', h_E h_F} dV_{\omega'} = |u|^2_{g, h_E h_F} dV_\omega$.

The following lemma is [F1, Lemma 3.2], which is a reformulation of the classical L^2-estimates for $\bar{\partial}$-equations for our purpose.

Lemma 3.22 (L^2-estimates for $\bar{\partial}$-equations on complete Kähler manifolds)**.** *Let U be a sufficiently small Stein open set of V. If $u \in L^{n,q}_{(2)}(U \setminus Z, E \otimes F)_{g', h_E h_F}$ with $\bar{\partial} u = 0$, then there exists $v \in L^{n,q-1}_{(2)}(U \setminus Z, E \otimes F)_{g', h_E h_F}$ such that $\bar{\partial} v = u$. Moreover, there exists a positive constant C independent of u such that*

$$\int_{U \setminus Z} |v|^2_{g', h_E h_F} \leq C \int_{U \setminus Z} |u|^2_{g', h_E h_F}.$$

We note that g' is not a complete Kähler metric on $U \setminus Z$ but $U \setminus Z$ is a complete Kähler manifold (cf. [D1, Théorème 0.2]).

By the same arguments, the isomorphism in Lemma 3.20 holds even when we replace $(E \otimes F, h_E h_F)$ with $(E \otimes F \otimes L, h_E h_F h_L)$.

3.23 (Proof of the main theorem: Theorem 1.1)**.** Let us start the proof of Theorem 1.1 (cf. [O]).

Proof of Theorem 1.1. Let u be any $\bar{\partial}$-closed locally square integrable $E \otimes F$-valued (n, q)-form on $V \setminus Z$ such that $su = \bar{\partial} v$ for some $v \in L^{n,q-1}_{loc, V}(V \setminus Z, E \otimes F \otimes L)$. We choose λ such that $|s|^2_{\lambda(\varphi)} < \frac{1}{5}$ and $u \in L^{n,q}_{(2)}(V \setminus Z, (E \otimes F, h_E h_F e^{-\lambda(\varphi)}))$. Since $\mu(\varphi) = \lambda(\varphi) + \varphi$, $u \in L^{n,q}_{(2)}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))$. By choosing ν suitably, we can assume that $v \in L^{n,q-1}_{(2)}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-\nu(\varphi)}))$. In particular, $v \in L^{n,q-1}_{(2)}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))$. Let Pu be the orthogonal projection of u to $\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$. We note that

$$L^{n,q}_{(2)}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))$$
$$= \overline{\mathrm{Im}\bar{\partial}} \oplus \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'} \oplus \overline{\mathrm{Im}D''^*_{(E \otimes F, h_E h_F e^{-\mu(\varphi))}}},$$

and

$$\mathrm{Ker}\bar{\partial} = \overline{\mathrm{Im}\bar{\partial}} \oplus \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}.$$

Here, $\overline{\mathrm{Im}\bar\partial}$ (resp. $\overline{\mathrm{Im}D''^*_{(E\otimes F,h_Eh_Fe^{-\mu(\varphi)})}}$) denotes the closure of $\bar\partial C_0^{n,q-1}(V\setminus Z,E\otimes F)$ (resp. $D''^*_{(E\otimes F,h_Eh_Fe^{-\mu(\varphi)})}C_0^{n,q+1}(V\setminus Z,E\otimes F)$) in $L_{(2)}^{n,q}(V\setminus Z,(E\otimes F,h_Eh_Fe^{-\mu(\varphi)}))$. Note that the fixed Kähler metric g' is complete. Therefore, $u-Pu$ is in the closure of the image of $\bar\partial$. Thus, so is $s(u-Pu)$ since s is holomorphic. On the other hand,

$$sPu\in\mathcal{H}^{n,q}(V\setminus Z,(E\otimes F\otimes L,h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))_{g'}$$

by Theorem 3.17. So, sPu coincides with the orthogonal projection of su to $\mathcal{H}^{n,q}(V\setminus Z,(E\otimes F\otimes L,h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))_{g'}$, which must be equal to zero since $v\in L_{(2)}^{n,q-1}(V\setminus Z,(E\otimes F\otimes L,h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))$. Therefore, $Pu=0$. Since $H^q(V,K_V\otimes E\otimes F\otimes\mathcal{J}(h_F))$ is a separated topological vector space (cf. [R, Theoreme 10]) and $u\in\overline{\mathrm{Im}\bar\partial}$, there exists $w\in L_{loc,V}^{n,q-1}(V\setminus Z,E\otimes F)$ such that $u=\bar\partial w$ (cf. [T, Proposition 4.6] and [F1, Claim 1]). This means that u represents zero in $H^q(V,K_V\otimes E\otimes F\otimes\mathcal{J}(h_F))$. $\square$

4. COROLLARIES AND APPLICATIONS

In this section, we discuss the proofs of corollaries in Section 1 and some applications of Theorem 1.1.

First, we give a proof of Corollary 1.2, which is obvious if we apply Theorem 1.1 for $L=\mathcal{O}_X\simeq f^*\mathcal{O}_Y$.

Proof of Corollary 1.2. The statement is local. So, we can assume that Y is Stein. Let $s\in H^0(Y,\mathcal{O}_Y)$ be an arbitrary nonzero section. By Theorem 1.1,

$$\times s:R^qf_*(K_X\otimes E\otimes F\otimes\mathcal{J}(h_F))\to R^qf_*(K_X\otimes E\otimes F\otimes\mathcal{J}(h_F))$$

is injective for every $q\geq 0$. Thus, $R^qf_*(K_X\otimes E\otimes F\otimes\mathcal{J}(h_F))$ is torsion-free for every $q\geq 0$. $\square$

The following proposition is a slight generalization of Theorem 1.1.

Proposition 4.1. *In* Theorem 1.1, *we can weaken the assumption that X is Kähler as follows. For any point $P\in Y$, there exist an open neighborhood U of P and a proper bimeromorphic morphism $g:W\to V:=f^{-1}(U)$ from a Kähler manifold W.*

Sketch of the proof. We put $Z'=g^{-1}(Z\cap V)\subset W$. We can apply Corollary 1.2 to $g:W\to V$, Z', (g^*E,g^*h_E), and (g^*F,g^*h_F). Then we obtain $R^qg_*(K_W\otimes g^*E\otimes g^*F\otimes\mathcal{J}(g^*h_F))=0$ for every $q>0$ and it is well known (and easy to check) that

$$g_*(K_W\otimes g^*E\otimes g^*F\otimes\mathcal{J}(g^*h_F))\simeq K_V\otimes E\otimes F\otimes\mathcal{J}(h_F)$$

(cf. [D2, (5.8) Proposition]). Therefore, by Leray's spectral sequence,

$$R^q(f \circ g)_*(K_W \otimes g^*E \otimes g^*F \otimes \mathcal{J}(g^*h_F)) \simeq R^q f_*(K_V \otimes E \otimes F \otimes \mathcal{J}(h_F))$$

for every $q \geq 0$. Apply Theorem 1.1 to $f \circ g : W \to U$, Z', (g^*E, g^*h_E), (g^*L, g^*h_L), and (g^*F, g^*h_F). Then we obtain that

$$\times s : R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \to R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F) \otimes L)$$

is injective for every $q \geq 0$ by the above isomorphisms. $\qquad\square$

The next result is related to the main theorem in [Le].

Corollary 4.2. *Let $f : X \to \Delta$ be a smooth* projective *surjective morphism from a Kähler manifold X to a disk Δ and let E be a Nakano semi-positive vector bundle on X. Assume that there exists $D \in |K_{X_0}^{\otimes l}|$ such that $\mathcal{J}_{X_0}(cD) \simeq \mathcal{O}_{X_0}$ for every $0 \leq c < 1$, where $0 \in Y$ and $X_0 = f^{-1}(0)$. Then there exists an open set $U \subset \Delta$ such that $0 \in U$ and $R^q f_*(K_X^{\otimes m} \otimes E)$ is locally free on U for every $q \geq 0$ and $1 \leq m \leq l$. Equivalently, $\dim_{\mathbb{C}} H^q(X_t, K_{X_t}^{\otimes m} \otimes E)$ is constant for every $q \geq 0$ by the base change theorem, where $t \in U$ and $X_t = f^{-1}(t)$.*

Proof. In this proof, we shrink Δ without mentioning it for simplicity of notation. Let $s_0 \in H^0(X_0, K_{X_0}^{\otimes l})$ such that $D = (s_0 = 0)$. By Siu's extension theorem (see [S, Theorem 0.1]), there exists $s \in H^0(X, K_X^{\otimes l})$ such that $s|_{X_0} = s_0$. We consider the singular hermitian metric $h_{K_X^{\otimes(m-1)}} = (\frac{1}{|s|^2})^{\frac{m-1}{l}}$ of $K_X^{\otimes(m-1)}$. By the Ohsawa–Takegoshi L^2-extension theorem and the assumption $\mathcal{J}_{X_0}(cD) \simeq \mathcal{O}_{X_0}$ for $0 \leq c < 1$, we obtain that $\mathcal{J}_X(h_{K_X^{\otimes m-1}}) \simeq \mathcal{O}_X$ in a neighborhood of X_0. Therefore, we obtain that $R^q f_*(K_X^{\otimes m} \otimes E) = R^q f_*(K_X \otimes K_X^{\otimes(m-1)} \otimes E \otimes \mathcal{J}_X(h_{K_X^{\otimes m-1}}))$ is locally free by Corollary 1.2. $\qquad\square$

Let us start the proof of the Kawamata–Viehweg–Nadel type vanishing theorem: Corollary 1.3.

Proof of Corollary 1.3. Let P be a point of Y. The problem is local. So we repeatedly shrink Y around P without mentioning it explicitly. Since $\mathcal{M}$ is f-big, we have a bimeromorphic map $\Phi : X \dashrightarrow X' \subset Y \times \mathbb{P}^N$ over Y. By applying Hironaka's Chow lemma (cf. [Hi, Corollary 2]), we can construct a bimeromorphic map $\varphi : Z \to X$ from a complex manifold such that $f \circ \varphi : Z \to Y$ is projective. It is easy to see that

$$\varphi_* \left(K_Z \otimes \varphi^*E \otimes \varphi^*\mathcal{L} \otimes \mathcal{J}\left(\frac{1}{m}\varphi^*D\right) \right) \simeq K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}\left(\frac{1}{m}D\right).$$

by the definition of $\mathcal{J}$ (cf. [La2, Theorem 9.2.33]).

Claim. *We have*

$$R^q \varphi_* \left(K_Z \otimes \varphi^* E \otimes \varphi^* \mathcal{L} \otimes \mathcal{J} \left(\frac{1}{m} \varphi^* D \right) \right) = 0$$

for every $q > 0$

Proof of Claim. The problem is local. So we can shrink X and assume that $\mathcal{M}$ is trivial. In particular, $(\varphi^* \mathcal{L})^{\otimes m} \simeq \mathcal{O}_Z(\varphi^* D)$. Thus, by Corollary 1.2, we obtain that $R^q \varphi_*(K_Z \otimes \varphi^* E \otimes \varphi^* \mathcal{L} \otimes \mathcal{J}(\frac{1}{m}\varphi^* D))$ is torsion-free for every q. Thus, $R^q \varphi_*(K_Z \otimes \varphi^* E \otimes \varphi^* \mathcal{L} \otimes \mathcal{J}(\frac{1}{m}\varphi^* D)) = 0$ for every $q > 0$ since φ is bimeromorphic. $\square$

Therefore, by replacing X with Z, we can assume that $f : X \to Y$ is projective. By Kodaira's lemma, we can write $\mathcal{L}^{\otimes a} \simeq \mathcal{A} \otimes \mathcal{O}_X(G)$ where a is a positive integer, $\mathcal{A}$ is an f-ample line bundle on X, and G is an effective Cartier divisor on X. Then $\mathcal{L}^{\otimes(mb+a)} \simeq (\mathcal{M}^{\otimes b} \otimes \mathcal{A}) \otimes \mathcal{O}_X(G + bD)$. Note that $\mathcal{M}^{\otimes b} \otimes \mathcal{A}$ is f-ample and that $\mathcal{J}(\frac{G+bD}{mb+a}) = \mathcal{J}(\frac{1}{m}D)$ if $b \gg 0$. Therefore, we can further assume that $\mathcal{M}$ is f-ample. Then, by Theorem 1.1, we can construct inclusions

$$R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) \subset R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J} \otimes \mathcal{M}^{\otimes k})$$

for every $k > 0$ with $|\mathcal{M}^{\otimes k}| \neq \emptyset$. Thus, by Serre's vanishing theorem (cf. [N2, p. 25 Remark (4)]), we obtain $R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$ for every $q > 0$. $\square$

Proof of Corollary 1.4. Note that $m\{H\}$ is Cartier. We have $m\ulcorner H \urcorner \sim mH + m\{-H\}$. We put $E = \mathcal{O}_X$, $\mathcal{L} = \mathcal{O}_X(\ulcorner H \urcorner)$, $\mathcal{M} = \mathcal{O}_X(mH)$, and $D = m\{-H\}$, and apply Corollary 1.3. Then we obtain $R^q f_*(K_X \otimes \mathcal{O}_X(\ulcorner H \urcorner)) = 0$ for every $q > 0$. We note that $\mathcal{J}(\{-H\}) = \mathcal{O}_X$ since $\text{Supp}\{H\}$ is a normal crossing divisor (cf. [La2, Lemma 9.3.44]). $\square$

The proof of Corollary 1.5 is a routine work. So we only sketch the proof here. For details, see, for example, the proofs of [F5, Theorem 1.1 (ii)], [F6, Theorem 6.3 (ii)], and [F7, Theorem 2.39 (ii)].

Sketch of the proof of Corollary 1.5. We can repeatedly shrink Z without mentioning it. By Hironaka's Chow lemma and Hironaka's flattening theorem (cf. [Hi, Corollary 2, Flattening Theorem]), we can assume that $g : Y \to Z$ is projective (cf. the proof of Corollary 1.3). By Kodaira's lemma, we can assume that $\mathcal{N}$ is g-ample. Let A be a general smooth sufficiently g-ample Cartier divisor on Y. We put $B = f^* A$. We consider the following short exact sequence

$$0 \to K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J} \to K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J}$$
$$\to K_B \otimes E|_B \otimes \mathcal{L}|_B \otimes \mathcal{J}|_B \to 0.$$

Since A is general, $\mathcal{J}|_B = \mathcal{J}(\frac{1}{m}D|_B)$, and

$$0 \to R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) \to R^q f_*(K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J})$$
$$\to R^q f_*(K_B \otimes E|_B \otimes \mathcal{L}|_B \otimes \mathcal{J}|_B) \to 0$$

is exact for every q. By taking the long exact sequence, we obtain

$$R^p g_* R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$$

for every $p \geq 2$ and every $q \geq 0$ because A is sufficiently g-ample and the induction on dimension. Then we obtain the following commutative diagram.

$$
\begin{array}{ccc}
R^1 g_* R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) & \xrightarrow{\ \ \alpha\ \ } & R^{1+q}(g \circ f)_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) \\
\downarrow & & \downarrow{\scriptstyle \beta} \\
R^1 g_* R^q f_*(K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J}) & \longrightarrow & R^{1+q}(g \circ f)_*(K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J})
\end{array}
$$

Note that α is injective by the above vanishing result and that β is injective by Theorem 1.1. Since A is sufficiently g-ample, we have $R^1 g_* R^q f_*(K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J}) = 0$. Thus, $R^1 g_* R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$ for every $q \geq 0$. So we finish the proof. $\qquad\square$

5. EXAMPLES: NEF, SEMI-POSITIVE, AND SEMI-AMPLE LINE BUNDLES

In this section, we collect some examples of nef, semi-positive, and semi-ample line bundles. These examples help us understand our results in [F1] and this paper. We think that it is very important to understand the differences in the notion of semi-ample, semi-positive, and nef line bundles.

First, we recall the following well-known example. It implies that there exists a nef line bundle that has no smooth hermitian metrics with semi-positive curvature.

Example 5.1 (cf. [DPS, Example 1.7]). Let C be an elliptic curve and let $\mathcal{E}$ be the rank two vector bundle on C which is defined by the unique non-splitting extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

We consider the ruled surface $X := \mathbb{P}_C(\mathcal{E})$. On that surface there is a unique section $D := \mathbb{P}_C(\mathcal{O}_C) \subset X$ of $X \to C$ such that $\mathcal{O}_D(D) \simeq \mathcal{O}_D$ and $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is a nef line bundle (cf. [Ha2, V Proposition 2.8]). It is not difficult to see that $H^1(X, K_X \otimes \mathcal{O}_X(2D)) \to H^1(X, K_X \otimes \mathcal{O}_X(3D))$ is a zero map, $H^1(X, K_X \otimes \mathcal{O}_X(2D)) \simeq \mathbb{C}$, and $H^1(X, K_X \otimes \mathcal{O}_X(3D)) \simeq \mathbb{C}$. We note that $K_X \sim \mathcal{O}_X(-2D)$. Therefore,

$\mathcal{O}_X(D)$ has no smooth hermitian metrics with semi-positive curvature by Enoki's injectivity theorem (see [En, Theorem 0.2], [F1, Corollary 1.4]). Note that $\kappa(X, \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)) = 0$ and $\nu(X, \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)) = 1$. We also note that Kollár's injectivity theorem implies nothing since $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is not semi-ample.

The next one is an example of nef and big line bundles that have no smooth hermitian metrics with semi-positive curvature. I learned the following construction from Dano Kim.

Example 5.2. We use the same notation as in Example 5.1. Let $P \in C$ be a closed point. We put $\mathcal{F} := \mathcal{E} \oplus \mathcal{O}_C(P)$ and $Y := \mathbb{P}_C(\mathcal{F})$. Then it is easy to see that $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$ is nef and big (cf. [La2, Example 6.1.23]). Since $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ has no smooth hermitian metrics with semi-positive curvature, neither has $\mathcal{L}$. In this case, $H^i(Y, K_Y \otimes \mathcal{L}^{\otimes k}) = 0$ for $i > 0$ and every $k \geq 1$ by the Kawamata–Viehweg vanishing theorem.

Let us recall some examples of semi-positive line bundles that are not semi-ample.

Example 5.3 (cf. [DEL, p.145]). Let C be a smooth projective curve with the genus $g(C) \geq 1$. Let $L \in \mathrm{Pic}^0(C)$ be non-torsion. We put $\mathcal{E} := \mathcal{O}_C \oplus L$ and $X := \mathbb{P}_C(\mathcal{E})$. Then $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is semi-positive, but not semi-ample. We note that $\kappa(X, \mathcal{L}) = 0$ since $H^0(X, \mathcal{L}^{\otimes k}) = H^0(C, S^k(\mathcal{E})) = H^0(C, \mathcal{O}_C) = \mathbb{C}$ for every $k \geq 0$, where $S^k(\mathcal{E})$ is the k-th symmetric product of $\mathcal{E}$. We can easily check that

$$K_X = \pi^*(K_C \otimes \det \mathcal{E}) \otimes \mathcal{L}^{\otimes(-2)} = \pi^*(K_C \otimes L) \otimes \mathcal{L}^{\otimes(-2)},$$

where $\pi : X \to C$ is the projection. Let m be an integer with $m \geq 2$. Then

$$H^i(X, K_X \otimes \mathcal{L}^{\otimes m}) = H^i(X, \pi^*(K_C \otimes L) \otimes \mathcal{L}^{\otimes(m-2)})$$
$$= \bigoplus_{k=1}^{m-1} H^i(C, K_C \otimes L^{\otimes k}).$$

Thus, $h^0(X, K_X \otimes \mathcal{L}^{\otimes m}) = (m-1)(g-1)$, $h^1(X, K_X \otimes \mathcal{L}^{\otimes m}) = 0$, and $h^2(X, K_X \otimes \mathcal{L}^{\otimes m}) = 0$ for $m \geq 2$. So, we obtain no interesting results from injectivity theorems. Note that $H^2(X, K_X \otimes \mathcal{L}^{\otimes m}) = 0$ for $m \geq 1$ also follows from the Kawamata–Viehweg vanishing theorem since $\nu(X, \mathcal{L}) = 1$ and $\dim X = 2$.

Example 5.4 (Cutkosky). We use the same notation as in Example 5.3. Let $P \in C$ be a closed point. We put $\mathcal{F} := \mathcal{O}_C(P) \oplus L$ and

$Y := \mathbb{P}_C(\mathcal{F})$. Then it is easy to see that $\mathcal{M} := \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$ is big and semi-positive, but not semi-ample. We note that

$$\bigoplus_{m \geq 0} H^0(Y, \mathcal{M}^{\otimes m})$$

is not finitely generated. For details, see, for example, [La1, Example 2.3.3].

The following example shows the difference between Enoki's injectivity theorem and Kollár's one.

Example 5.5. Let C be a smooth projective curve with the genus $g(C) = g \geq 1$. Let $L \in \mathrm{Pic}^0(C)$ be non-torsion. We put $\mathcal{E} := \mathcal{O}_C \oplus L \oplus L^{-1}$, $X := \mathbb{P}_C(\mathcal{E})$, and $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. It is obvious that $\mathcal{E}$ has a smooth hermitian metric whose curvature is Nakano semi-positive. Thus, $\mathcal{L}$ is semi-positive since $\mathcal{L}$ is a quotient line bundle of $\pi^*\mathcal{E}$, where $\pi : X \to C$ is the projection. In particular, $\mathcal{L}$ is nef. On the other hand, it is not difficult to see that $\mathcal{L}$ is not semi-ample. We have

$$K_X = \pi^*(K_C \otimes \det \mathcal{E}) \otimes \mathcal{L}^{\otimes(-3)} = \pi^*K_C \otimes \mathcal{L}^{\otimes(-3)}.$$

We can easily check that

$$S^m(\mathcal{E}) = \bigoplus_{\substack{0 \leq a+b \leq m \\ a,b \geq 0}} L^{a-b}.$$

Note that the rank of $S^m(\mathcal{E})$ is $\frac{1}{2}(m+2)(m+1)$. Let m be an integer with $m \geq 3$. Then it is easy to see that

$$H^i(X, K_X \otimes \mathcal{L}^{\otimes m}) = H^i(X, \pi^*K_C \otimes \mathcal{L}^{\otimes(m-3)}) = H^i(C, K_C \otimes S^{m-3}(\mathcal{E})).$$

for all i. We need the following obvious lemma.

Lemma 5.6. *We have $h^0(C, K_C) = g$ and $h^1(C, K_C) = 1$. Moreover, $h^0(C, K_C \otimes L^{\otimes k}) = g - 1$ and $h^1(C, K_C \otimes L^{\otimes k}) = 0$ for $k \neq 0$.*

Therefore, we obtain $H^3(X, K_X \otimes \mathcal{L}^{\otimes m}) = H^2(X, K_X \otimes \mathcal{L}^{\otimes m}) = 0$, $h^1(X, K_X \otimes \mathcal{L}^{\otimes m}) = \lfloor \frac{m-3}{2} \rfloor + 1 = \lfloor \frac{m-1}{2} \rfloor$, and

$$
\begin{aligned}
&h^0(X, K_X \otimes \mathcal{L}^{\otimes m}) \\
&= g\lfloor \frac{m-1}{2} \rfloor + (g-1)\left(\frac{(m+2)(m+1)}{2} - \lfloor \frac{m-1}{2} \rfloor \right) \\
&= \lfloor \frac{m-1}{2} \rfloor + (g-1)\frac{(m+2)(m+1)}{2}.
\end{aligned}
$$

On the other hand, $h^0(X, \mathcal{L}^{\otimes k}) = h^0(C, S^k(\mathcal{E})) = \lfloor \frac{k}{2} \rfloor + 1$ for $k \geq 0$. Let $s \in |\mathcal{L}^{\otimes k}|$ be a non-zero holomorphic section of $\mathcal{L}^{\otimes k}$ for $k \geq 0$. Then

$$\times s : H^1(X, K_X \otimes \mathcal{L}^{\otimes m}) \rightarrow H^1(X, K_X \otimes \mathcal{L}^{\otimes(m+k)})$$

is injective by Enoki's injectivity theorem (cf. [En, Theorem 0.2], [F1, Corollary 1.4]). Note that $h^1(X, K_X \otimes \mathcal{L}^{\otimes m}) = \lfloor \frac{m-1}{2} \rfloor$ and $h^1(X, K_X \otimes \mathcal{L}^{\otimes(m+k)}) = \lfloor \frac{m+k-1}{2} \rfloor$. We have $\kappa(X, \mathcal{L}) = 1$ by the above calculation. Since $\mathcal{L}^2 \cdot F = 1$, where F is a fiber of $\pi : X \rightarrow C$, $\nu(X, \mathcal{L}) = 2$. Thus, the nef line bundle $\mathcal{L}$ is not abundant. So, we think that there are no algebraic proofs for the above injectivity theorem. Note that $H^3(X, K_X \otimes \mathcal{L}^{\otimes m}) = H^2(X, K_X \otimes \mathcal{L}^{\otimes m}) = 0$ for $m \geq 1$ follows from the Kawamata–Viehweg vanishing theorem since $\nu(X, \mathcal{L}) = 2$ and $\dim X = 3$.

The following two examples are famous ones due to Mumford and Ramanujam.

Example 5.7 (Mumford). Let us recall the construction of Mumford's example (see [Ha1, Example 10.6]). We use the same notation as in [Ha1, Example 10.6]. Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Then there exists a stable vector bundle E of rank two and $\deg E = 0$ such that its symmetric powers $S^m(E)$ are stable for all $m \geq 1$. We consider the ruled surface $X := \mathbb{P}_C(E)$. Let D be the divisor corresponding to $\mathcal{O}_X(1)$. Since E is a unitary flat vector bundle, $\mathcal{L} := \mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}_C(E)}(1)$ is semi-positive by $\pi^* E \rightarrow \mathcal{L} \rightarrow 0$, where $\pi : X \rightarrow C$ is the projection. We know that $H^0(X, \mathcal{L}^{\otimes m}) = H^0(C, S^m(E)) = 0$ since $S^m(E)$ is stable and $c_1(S^m(E)) = 0$ for every $m \geq 1$. Thus, $\kappa(X, \mathcal{L}) = -\infty$. On the other hand, $\mathcal{L} \cdot \mathcal{L} = 0$ and $\mathcal{L} \cdot C' > 0$ for every curve C' on X. Then $\nu(X, \mathcal{L}) = 1$.

Example 5.8 (Ramanujam). Let us recall the construction of Ramanujam's example (see [Ha1, Example 10.8]). We use the same notation as in [Ha1, Example 10.8]. Let X be the ruled surface obtained in Example 5.7. We assume that D is the divisor given in [Ha1, Example 10.6] (see Example 5.7 above). Let H be an effective ample divisor on X. We define $\overline{X} := \mathbb{P}_X(\mathcal{O}_X(D - H) \oplus \mathcal{O}_X)$, and let $\pi : \overline{X} \rightarrow X$ be the projection. Let X_0 be the section of π corresponding to $\mathcal{O}_X(D - H) \oplus \mathcal{O}_X \rightarrow \mathcal{O}_X(D - H) \rightarrow 0$ and $\overline{D} := X_0 + \pi^* H$. We put $\mathcal{M} := \mathcal{O}_{\overline{X}}(\overline{D})$. We write $\mathcal{O}_{\overline{X}}(1) = \mathcal{O}_{\mathbb{P}_X(\mathcal{O}_X(D-H) \oplus \mathcal{O}_X)}(1)$. Then $\mathcal{O}_{\overline{X}}(1) \simeq \mathcal{O}_{\overline{X}}(X_0)$. Therefore, $\mathcal{M} \simeq \mathcal{O}_{\overline{X}}(1) \otimes \pi^* \mathcal{O}_X(H)$ and $\pi^*(\mathcal{O}_X(D) \oplus \mathcal{O}_X(H)) \rightarrow \mathcal{M} \rightarrow 0$. Thus, it is easy to see that $\mathcal{M}$ is semi-positive, nef and big. By the construction, $\mathcal{M} \cdot C' > 0$ for every curve C' on $\overline{X}$. However, $\mathcal{M}$ is not semi-ample since $\mathcal{M}|_{X_0} \simeq \mathcal{O}_{X_0}(D)$

does not have sections on X_0. In particular, $\bigoplus_{m \geq 0} H^0(\overline{X}, \mathcal{O}_{\overline{X}}(m\overline{D}))$ is not finitely generated.

I learned the following construction from the referee.

Example 5.9. Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be as in Example 5.1. Let A be a very ample divisor on X. We take a smooth member $B \in |2A|$ and take a double cover $\widetilde{C} \to C$ by $B \sim 2A$. We consider the base change diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\ \varphi\ } & \widetilde{X} \\
\downarrow & & \downarrow \\
C & \longleftarrow & \widetilde{C}
\end{array}
$$

and $\mathcal{A} := \varphi^* \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1) \simeq \mathcal{O}_{\widetilde{X}}(\varphi^* D)$. The natural map

$$\alpha : H^1(\widetilde{X}, K_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(2\varphi^* D)) \to H^1(\widetilde{X}, K_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(3\varphi^* D))$$

contains

$$\beta : H^1(X, K_X \otimes \mathcal{O}_X(2D)) \to H^1(X, K_X \otimes \mathcal{O}_X(3D))$$

as a direct summand by the construction. Since β is zero (see Example 5.1), α is not injective. By Enoki's injectivity theorem (cf. [En, Theorem 0.2], [F1, Corollary 1.4]), $\mathcal{A}$ is not semi-positive. We note that $\widetilde{C}$ is a smooth projective curve of genus $g \geq 2$. Then there exists a stable vector bundle F of rank two and $\deg F = 0$ such that its symmetric powers $S^m(F)$ are stable for all $m \geq 1$ (cf. Example 5.7). We put $Y = \mathbb{P}_{\widetilde{C}}(F)$ and $\mathcal{B} = \mathcal{O}_{\mathbb{P}_{\widetilde{C}}(F)}(1)$. We take the fiber product

$$
\begin{array}{ccc}
\widetilde{X} & \xleftarrow{\ p_1\ } & V = \widetilde{X} \times_{\widetilde{C}} Y \\
\downarrow & & \downarrow{\scriptstyle p_2} \\
\widetilde{C} & \longleftarrow & Y
\end{array}
$$

and put $\mathcal{M} = p_1^* \mathcal{A} \otimes p_2^* \mathcal{B}$. Note that V is a smooth projective variety. Then it is easy to see that $\mathcal{M} \cdot C' > 0$ for every curve C' on V. On the other hand, $\mathcal{M}$ has no smooth hermitian metrics with semi-positive curvature. It is because $\mathcal{A}$ is not semi-positive.

Remark 5.10. By the same way as in Example 5.9, we can construct a smooth projective 5-fold V and a line bundle $\mathcal{N}$ on V such that $\mathcal{N} \cdot C' > 0$ for every curve C' on V, $\mathcal{N}$ is nef and big, and $\mathcal{N}$ is not semi-positive by using Example 5.2 and Example 5.8. Of course, $\bigoplus_{m \geq 0} H^0(V, \mathcal{N}^{\otimes m})$ is not finitely generated.

References

[D1]   J-P. Demailly, Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holo-
       morphe semi-positif au-dessus d'une variété kählérienne complète, Ann. Sci.
       École Norm. Sup. (4) **15** (1982), no. 3, 457–511.

[D2]   J-P. Demailly, Multiplier ideal sheaves and analytic methods in algebraic
       geometry, School on Vanishing Theorems and Effective Results in Algebraic
       Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, **6**, Abdus Salam Int.
       Cent. Theoret. Phys., Trieste, 2001.

[D3]   J-P. Demailly, Cohomology of q-convex spaces in top degrees, Math. Z. **204**
       (1990), no. 2, 283–295.

[DEL] J-P. Demailly, L. Ein, R. Lazarsfeld, A subadditivity property of multiplier
       ideals. Dedicated to William Fulton on the occasion of his 60th birthday.
       Michigan Math. J. **48** (2000), 137–156.

[DPS] J-P. Demailly, T. Peternell, M. Schneider, Compact complex manifolds with
       numerically effective tangent bundles, J. Algebraic Geom. **3** (1994), no. 2,
       295–345.

[En]   I. Enoki, Kawamata–Viehweg vanishing theorem for compact Kähler man-
       ifolds, *Einstein metrics and Yang–Mills connections* (Sanda, 1990), 59–68,
       Lecture Notes in Pure and Appl. Math., **145**, Dekker, New York, 1993.

[EV]   H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, **20**.
       Birkhäuser Verlag, Basel, 1992.

[F1]   O. Fujino, A transcendental approach to Kollár's injectivity theorem,
       preprint (2007).

[F2]   O. Fujino, On Kollár's injectivity theorem (Japanese), Sūrikaisekikenkyūsho
       Kōkyūroku No. 1550 (2007), 131-140.

[F3]   O. Fujino, Multiplication maps and vanishing theorems for toric varieties,
       Math. Z. **257** (2007), no. 3, 631–641.

[F4]   O. Fujino, Vanishing theorems for toric polyhedra, Higher dimensional alge-
       braic varieties and vector bundles, 81–95, RIMS Kôkyûroku Bessatsu, B9,
       Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

[F5]   O. Fujino, On injectivity, vanishing and torsion-free theorems for algebraic
       varieties, Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), no. 8, 95–100.

[F6]   O. Fujino, Fundamental theorems for the log minimal model program, to
       appear in Publ. Res. Inst. Math. Sci.

[F7]   O. Fujino, Introduction to the log minimal model program for log canonical
       pairs, preprint (2009). arXiv:0907.1506v1

[Ha1] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in
       Math. **156**, Springer-Verlag, Heidelberg (1970).

[Ha2] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. **52**.
       Springer-Verlag, New York-Heidelberg, 1977.

[Hi]   H. Hironaka, Flattening theorem in complex-analytic geometry, Amer. J.
       Math. **97** (1975), 503–547.

[K1]   J. Kollár, Higher direct images of dualizing sheaves, I. Ann. of Math. (2)
       123 (1986), no. 1, 11–42.

[K2]   J. Kollár, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures.
       Princeton University Press, Princeton, NJ, 1995.

[La1] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bun-
       dles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3.

Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], **48**. Springer-Verlag, Berlin, 2004.

[La2] R. Lazarsfeld, *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], **49**. Springer-Verlag, Berlin, 2004.

[Le] M. Levine, Pluri-canonical divisors on Kähler manifolds, Invent. Math. **74** (1983), no. 2, 293–303.

[N1] N. Nakayama, The lower semicontinuity of the plurigenera of complex varieties, *Algebraic geometry*, Sendai, 1985, 551–590, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

[N2] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, **14**. Mathematical Society of Japan, Tokyo, 2004.

[O] T. Ohsawa, On a curvature condition that implies a cohomology injectivity theorem of Kollár–Skoda type, Publ. Res. Inst. Math. Sci. **41** (2005), no. 3, 565–577.

[R] J. P. Ramis, Théorèmes de séparation et de finitude pour l'homologie et la cohomologie des espaces (p,q)-convexes-concaves, Ann. Scuola Norm. Sup. Pisa (3) **27** (1973), 933–997 (1974).

[S] Y.-T. Siu, Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type, *Complex geometry* (Göttingen, 2000), 223–277, Springer, Berlin, 2002.

[T] K. Takegoshi, Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms, Math. Ann. **303** (1995), no. 3, 389–416.

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