New series in the Johnson cokernels of the mapping class groups of surfaces

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Dedicated to the memory of Midori Kato

Abstract

Let $\Sigma_{g,1}$ be a compact oriented surface of genus g with one boundary component, and $\mathcal{M}_{g,1}$ its mapping class group. Morita showed that the image of the k-th Johnson homomorphism $\tau_k^{\mathcal{M}}$ of $\mathcal{M}_{g,1}$ is contained in the kernel $\mathfrak{h}_{g,1}(k)$ of an Sp-equivariant surjective homomorphism $H \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1) \to \mathcal{L}_{2g}(k+2)$, where $H := H_1(\Sigma_{g,1}, \mathbf{Z})$ and $\mathcal{L}_{2g}(k)$ is the degree k-part of the free Lie algebra $\mathcal{L}_{2g}$ generated by H.

In this paper, we study the Sp-module structure of the cokernel $\mathfrak{h}_{g,1}^{\mathbf{Q}}(k)/\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ of the rational Johnson homomorphism $\tau_{k,\mathbf{Q}}^{\mathcal{M}} := \tau_k^{\mathcal{M}} \otimes \mathrm{id}_{\mathbf{Q}}$ where $\mathfrak{h}_{g,1}^{\mathbf{Q}}(k) := \mathfrak{h}_{g,1}(k) \otimes_{\mathbf{Z}} \mathbf{Q}$. In particular, we show that the irreducible Sp-module corresponding to a partition $[1^k]$ appears in the k-th Johnson cokernel for any $k \equiv 1 \pmod 4$ and $k \geq 5$ with multiplicity one. We also give a new proof of the fact due to Morita that the irreducible Sp-module corresponding to a partition $[k]$ appears in the Johnson cokernel with multiplicity one for odd $k \geq 3$.

The strategy of the paper is to give explicit descriptions of maximal vectors with highest weight $[1^k]$ and $[k]$ in the Johnson cokernel. Our construction is inspired by the Brauer-Schur-Weyl duality between $\mathrm{Sp}(2g, \mathbf{Q})$ and the Brauer algebras, and our previous work for the Johnson cokernel of the automorphism group of a free group.

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1 Introduction

Dennis Johnson established a new remarkable method to investigate the group structure of the mapping class group of a surface and the Torelli group in a series of his pioneer works [Joh1], [Joh2], [Joh3] and [Joh4] in 1980's. Especially, he gave a finite set of generators of the Torelli group, and constructed a homomorphism τ to determine the abelianization of the Torelli group. Now, his homomorphism τ is called the first Johnson homomorphism, and it is generalized to the Johnson homomorphisms of higher degrees. Over the last two decades, the study of the Johnson homomorphisms of the mapping class group has achieved a good progress by many authors including Morita [Mo2], Hain [Ha] and so on.

To put it plainly, the Johnson homomorphism are used to describe "one by one approximations" of the Torelli group as follows. To explain it, let us fix some notations. For a compact oriented surface $\Sigma_{g,1}$ of genus g with one boundary component, let $\mathcal{M}_{g,1}$ be its mapping class group. Namely, $\mathcal{M}_{g,1}$ is a group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,1}$ which fix the boundary component pointwise. The fundamental group $\pi_1(\Sigma_{g,1},*)$ of $\Sigma_{g,1}$ is isomorphic to a free group F_{2g} of rank $2g$. In this paper we fix an isomorphism $\pi_1(\Sigma_{g,1},*) \cong F_{2g}$. Let $\Gamma_{2g}(k)$ be the lower central series of F_{2g} beginning with $\Gamma_{2g}(1) = F_{2g}$, and set $\mathcal{L}_{2g}(k) := \Gamma_{2g}(k)/\Gamma_{2g}(k+1)$. For each $k \geq 1$ let $\mathcal{M}_{g,1}(k)$ be a normal subgroup of $\mathcal{M}_{g,1}$ consisting of elements which act $F_{2g}/\Gamma_{2g}(k+1)$ trivially. Then we have a descending filtration

$$\mathcal{M}_{g,1}(1) \supset \mathcal{M}_{g,1}(2) \supset \cdots \supset \mathcal{M}_{g,1}(k) \supset \cdots$$

of $\mathcal{M}_{g,1}$ such that the first term $\mathcal{M}_{g,1}(1)$ is just the Torelli group $\mathcal{I}_{g,1}$. This filtration is called the Johnson filtration of $\mathcal{M}_{g,1}$. Set $\mathrm{gr}^k(\mathcal{M}_{g,1}) := \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)$ for each $k \geq 1$. Then each of $\mathrm{gr}^k(\mathcal{M}_{g,1})$ is an $\mathrm{Sp}(2g,\mathbf{Z})$-equivariant free abelian group of finite rank, and they are considered as one by one approximations of the Torelli group. Although to clarify the $\mathrm{Sp}(2g,\mathbf{Z})$-module structure of each of $\mathrm{gr}^k(\mathcal{M}_{g,1})$ plays an important role on various studies of the Torelli group, even to determine its rank is quite a difficult problem in general.

In order to study each graded quotients $\mathrm{gr}^k(\mathcal{M}_{g,1})$, the Johnson homomorphisms

$$\tau_k^{\mathcal{M}} : \mathrm{gr}^k(\mathcal{M}_{g,1}) \hookrightarrow H^* \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1)$$

of $\mathcal{M}_{g,1}$ are valuable tools where $H := H_1(\Sigma_{g,1},\mathbf{Z})$ and $H^* := \mathrm{Hom}_{\mathbf{Z}}(H,\mathbf{Z})$. Here we remark that H^* is canonically isomorphic to H by the Poincaré duality. In general, the k-th Johnson homomorphism is denoted by τ_k simply. In this paper, however, to distinguish the Johnson homomorphism of the mapping class group from that of the automorphism group of a free group, we attach a subscript $\mathcal{M}$ to that of the mapping class group. (See Subsection 3.3 for details.) Since each of $\tau_k^{\mathcal{M}}$ is an $\mathrm{Sp}(2g,\mathbf{Z})$-equivariant injective homomorphism, to determine the image $\mathrm{Im}(\tau_k^{\mathcal{M}})$ of $\tau_k^{\mathcal{M}}$ is one of the most basic problems. In particular, from a representation theoretic view, it is important to clarify the irreducible decomposition of $\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ as an $\mathrm{Sp}(2g,\mathbf{Q})$-module where $\tau_{k,\mathbf{Q}}^{\mathcal{M}} := \tau_k^{\mathcal{M}} \otimes \mathrm{id}_{\mathbf{Q}}$. In the following, the subscript $\mathbf{Q}$ always means tensoring with $\mathbf{Q}$ over $\mathbf{Z}$. Now, we have $\mathrm{Im}(\tau_1^{\mathcal{M}}) \cong \Lambda^3 H$ due to Johnson [Joh1]. Furthermore the $\mathrm{Sp}(2g,\mathbf{Q})$-module structure of $\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ are completely determined for $1 \leq k \leq 4$. (See a table in Subsection 3.3.)

On the other hand, Morita [Mo2] began to study the Johnson images systematically, and gave many remarkable results. Here we recall some of them. First, Morita [Mo2] showed that $\mathrm{Im}(\tau_k^{\mathcal{M}})$ is contained in the kernel $\mathfrak{h}_{g,1}(k)$ of $H \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1) \to \mathcal{L}_{2g}(k+2)$ for any $k \geq 2$. (See Subsection 3.3.) Second, he also showed that $\mathrm{Im}(\tau_k^{\mathcal{M}})$ does not coincide with $\mathfrak{h}_{g,1}(k)$ in general. Namely, the Johnson homomorphism $\tau_k^{\mathcal{M}} : \mathrm{gr}^k(\mathcal{M}_{g,1}) \hookrightarrow \mathfrak{h}_{g,1}(k)$ is not surjective in general. More precisely, he constructed an $\mathrm{Sp}(2g,\mathbf{Q})$-equivariant surjective homomorphisms

$$\mathrm{Tr}_k : \mathfrak{h}_{g,1}^{\mathbf{Q}}(k) \to S^k H_{\mathbf{Q}}$$

such that $\mathrm{Tr}_k \circ \tau_{k,\mathbf{Q}}^{\mathcal{M}} \equiv 0$ for any odd $k \geq 3$ using the Magnus representation of $\mathcal{M}_{g,1}$. Here $S^k H_{\mathbf{Q}}$ is the symmetric tensor product of $H_{\mathbf{Q}}$ of degree k, and is isomorphic to the irreducible $\mathrm{Sp}(2g,\mathbf{Q})$-module with highest weight $[k]$. Hence $S^k H_{\mathbf{Q}}$ appears in the irreducible decomposition of the cokernel $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}}) := \mathfrak{h}_{g,1}^{\mathbf{Q}}(k)/\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ for odd $k \geq 3$. We should remark that throughout the paper $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ denotes $\mathfrak{h}_{g,1}^{\mathbf{Q}}(k)/\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$, not $H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)/\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$. Now, the map Tr_k is called the Morita trace, and $S^k H_{\mathbf{Q}}$ the Morita obstruction. Here the term "obstruction" means an obstruction for the surjectivity of the Johnson homomorphism $\tau_{k,\mathbf{Q}}^{\mathcal{M}}$. We also remark that Hiroaki Nakamura, partially Asada and Nakamura [AN], showed that the multiplicity of $S^k H_{\mathbf{Q}}$ in $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ is exactly one in his unpublished work.

From results for the irreducible decomposition of $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ for low degrees, it seems that the number of the irreducible components in $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ grows rapidly as degree increases. At the present stage, however, there are few results for obstructions other than the Morita obstruction for a general degree k. Thus, to establish a new method to detect a non-trivial irreducible component in $\mathrm{Coker}(\tau_k^{\mathcal{M}})$ other than the Morita obstruction is an important problem in the study of the Johnson homomorphisms.

The main purpose of the paper is to detect new series of obstructions in the Johnson cokernels. To state our theorem, we will use the following notations. First, we remark that for each $k \geq 1$ the symmetric group $\mathfrak{S}_{k+2}$ of degree $k+2$ naturally acts on the space $H_{\mathbf{Q}}^{\otimes k+2}$ from the right as a permutation of the components. For each $1 \leq i \leq k+1$, denote by $s_i \in \mathfrak{S}_{k+2}$ the adjacent transposition between i and $i+1$, and by σ_{k+2} the cyclic permutation $s_{k+1}s_k \cdots s_2 s_1$. Let P be a subgroup of $\mathfrak{S}_{k+2}$ which fixes 1. The group P is isomorphic to $\mathfrak{S}_{k+1}$. The Dynkin-Specht-Wever element θ_P for P in the group algebra $\mathbf{Q}\mathfrak{S}_{k+2}$ is defined to be

$$\theta_P := (1 - s_2)(1 - s_3 s_2) \cdots (1 - s_{k+1} s_k \cdots s_2).$$

Our main theorem is

Theorem 1. *(= Theorem 7.7.) Suppose $k \equiv 1 \pmod 4$, $k \geq 5$ and $g \geq k+2$. An element*

$$\varphi_{[1^k]} \ := \ (\omega \otimes (e_1 \wedge \cdots \wedge e_k)) \cdot \theta_P \cdot (1 + \sigma_{k+2} + \cdots + \sigma_{k+2}^{k+1})$$

is an Sp*-maximal vector of weight $[1^k]$ in $\mathfrak{h}_{g,1}^{\mathbf{Q}}(k)$. Moreover this gives a unique* Sp*-irreducible component with highest weight $[1^k]$ in* $\mathrm{Coker}\,\tau_{k,\mathbf{Q}}^{\mathcal{M}}$.

In addition to this, we also give a new proof of the fact that the Morita obstruction uniquely appears in $\mathrm{Coker}(\tau_k^{\mathcal{M}})$ for odd $k \geq 3$, due to Morita [Mo2] and Nakamura. (See Theorem 7.6.)

In order to prove these, we use two key facts. The first one is a remarkable work with respect to $\mathrm{gr}^k(\mathcal{M}_{g,1})$ due to Hain [Ha]. In general, the graded sum $\mathrm{gr}(\mathcal{M}_{g,1}) := \oplus_{k\geq 1}\mathrm{gr}^k(\mathcal{M}_{g,1})$ has a Lie algebra structure induced from the commutator bracket of $\mathcal{I}_{g,1}$. In [Ha], Hain showed that the Lie algebra $\mathrm{gr}_{\mathbf{Q}}(\mathcal{M}_{g,1})$ is generated by the degree one part $\mathrm{gr}^1_{\mathbf{Q}}(\mathcal{M}_{g,1})$ as a Lie algebra. This shows the following. Let $\mathcal{M}'_{g,1}(k)$ be the lower central series of $\mathcal{I}_{g,1}$ and set $\mathrm{gr}^k(\mathcal{M}'_{g,1}) := \mathcal{M}'_{g,1}(k)/\mathcal{M}'_{g,1}(k+1)$. Then we can define the Johnson homomorphism like homomorphism

$$\tau_k'^{\mathcal{M}} : \mathrm{gr}^k(\mathcal{M}'_{g,1}) \to \mathfrak{h}_{g,1}(k).$$

(See Subsection 3.3.) Then Hain's result above induces $\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}}) = \mathrm{Im}(\tau_{k,\mathbf{Q}}'^{\mathcal{M}})$ for any $k \geq 1$.

The second is our previous result for the cokernel of the Johnson homomorphism of the automorphism group of a free group. By a classical work of Dehn and Nielsen, it is known that a natural homomorphism $\mathcal{M}_{g,1} \to \mathrm{Aut}\,F_{2g}$ induced from the action of $\mathcal{M}_{g,1}$ of the fundamental group $\pi_1(\Sigma_{g,1}, *) \cong F_{2g}$ is injective. Namely, we can consider $\mathcal{M}_{g,1}$ as a subgroup of $\mathrm{Aut}\,F_{2g}$. From this view point, we can apply results for the Johnson homomorphisms of $\mathrm{Aut}\,F_{2g}$ to the study of that of $\mathcal{M}_{g,1}$. For any $n \geq 2$, in general, a subgroup IA_n consisting of automorphisms of a free group F_n which acts on $H_1(F_n, \mathbf{Z})$ trivially is called the IA-automorphism group of F_n. Let $\mathcal{A}'_n(k)$ be the lower central series of IA_n, and set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$ for any $k \geq 1$. Then we can define the Johnson homomorphism $\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ for each $k \geq 1$. Then, in our paper [Sa], we showed that for $k \geq 2$ and $n \geq k+2$,

$$\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) \cong \mathcal{C}_n^{\mathbf{Q}}(k)$$

where $\mathcal{C}_n(k) := H^{\otimes k}/\langle a_1 \otimes \cdots \otimes a_k - a_2 \otimes \cdots \otimes a_k \otimes a_1 \,|\, a_i \in H\rangle$. (See Subsection 3.3 for details.)

In our previous paper [ES], we gave the irreducible decomposition of $\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) \cong \mathcal{C}_n^{\mathbf{Q}}(k)$ as a $\mathrm{GL}(n, \mathbf{Q})$-module. Especially, we showed that $S^k H_{\mathbf{Q}}$, which is also called the

Morita obstruction, appears in $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ with multiplicity one for any $k \geq 2$, and that $\Lambda^k H_{\mathbf{Q}}$ appears with multiplicity one for odd $k \geq 3$. Combining Hain's result above and the fact $\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) \cong \mathcal{C}_n^{\mathbf{Q}}(k)$ for $n \geq k+2$, we can establish a new method to detect non-trivial Sp-irreducible components in $\mathrm{Coker}(\tau_k^{\mathcal{M}})$. (For more details, see Section 7.1.) The present paper produces the first successful results for the use of such method.

2 Notations

Throughout the paper, we use the following notations. Let G be a group and N a normal subgroup of G.

- The binomial coefficient $\binom{n}{r}$ is denoted by ${}_nC_r$.
- For any real number x, we set $\lfloor x \rfloor := \max\{n \in \mathbf{Z} \,|\, n \leq x\}$.
- For any integer p, set

$$\delta_{p \equiv a \,(\mathrm{mod}\, m)} := \begin{cases} 1 & \text{if} \quad p \equiv a \,(\mathrm{mod}\, m), \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

- The automorphism group $\mathrm{Aut}\, F_n$ of F_n acts on F_n from the right unless otherwise noted. For any $\sigma \in \mathrm{Aut}\, F_n$ and $x \in F_n$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- The automorphism group $\mathrm{Aut}\, F_n$ acts on F_n from the right unless otherwise noted. For any $\sigma \in \mathrm{Aut}\, F_n$ and $x \in F_n$, the action of σ on x is denoted by x^σ.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.
- For elements $g_1, \ldots, g_k \in G$, a left-normed commutator

$$[[\cdots[[g_1, g_2], g_3], \cdots], g_k]$$

  of weight k is denoted by $[g_{i_1}, g_{i_2}, \cdots, g_{i_k}]$.
- For any $\mathbf{Z}$-module M and a commutative ring R, we denote $M \otimes_{\mathbf{Z}} R$ by the symbol obtained by attaching a subscript R to M, like M_R or M^R. Similarly, for any $\mathbf{Z}$-linear map $f : A \to B$, the induced R-linear map $A_R \to B_R$ is denoted by f_R or f^R.
- For a semisimple G-module M and an irreducible G-module N, we denote by $[N : M]$ the multiplicity of N in the irreducible decomposition of M.

3 Johnson homomorphisms of the mapping class groups and the automorphism group of free groups

3.1 Mapping class groups of surfaces

Here we recall some properties of the mapping class groups of surfaces. For any integer $g \geq 1$, let $\Sigma_{g,1}$ be the compact oriented surface of genus g with one boundary component. We denote by $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma_{g,1}$. Namely, $\mathcal{M}_{g,1}$ is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which fix the boundary pointwise.

The mapping class group $\mathcal{M}_{g,1}$ has an important normal subgroup called the Torelli group. Let $\mu_{\mathcal{M}} : \mathcal{M}_{g,1} \to \mathrm{Aut}(H_1(\Sigma_{g,1}, \mathbf{Z}))$ be the classical representation of $\mathcal{M}_{g,1}$ induced from the action of $\mathcal{M}_{g,1}$ on the integral first homology group $H_1(\Sigma_{g,1}, \mathbf{Z})$ of $\Sigma_{g,1}$. The kernel of $\mu_{\mathcal{M}}$ is called the Torelli group, denoted by $\mathcal{I}_{g,1}$. Namely, $\mathcal{I}_{g,1}$ consists of mapping classes of $\Sigma_{g,1}$ which act on $H_1(\Sigma_{g,1}, \mathbf{Z})$ trivially.

Let us observe the image of $\mu_{\mathcal{M}}$. Take a base point $*$ of $\Sigma_{g,1}$ on the boundary. Then the fundamental group $\pi_1(\Sigma_{g,1}, *)$ of $\Sigma_{g,1}$ is a free group of rank $2g$. We fix a basis $x_1, \ldots, x_{2g}$ of $\pi_1(\Sigma_{g,1}, *)$ as shown Figure 1.

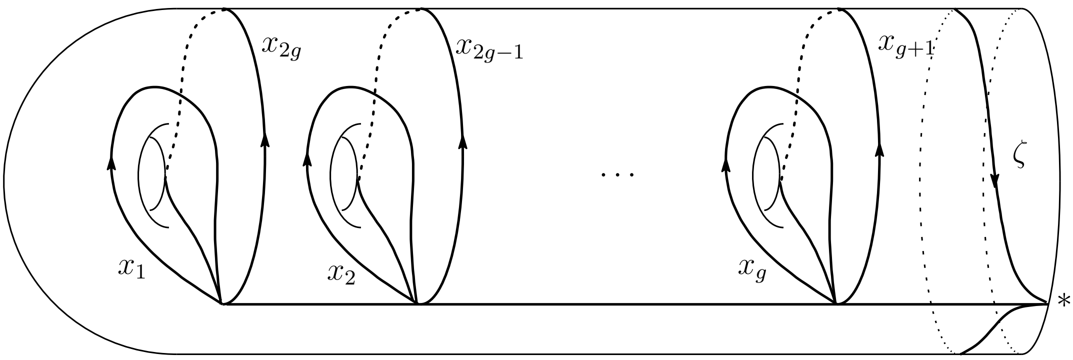


Figure 1: generators $x_1, \ldots, x_{2g}$ of $\pi_1(\Sigma_{g,1}, *)$ and a simple closed curve ζ

Then the homology classes $e_1, \ldots, e_{2g}$ of $x_1, \ldots, x_{2g}$ form a symplectic basis of the homology group $H_1(\Sigma_{g,1}, \mathbf{Z})$. Using this symplectic basis, we can identify $\mathrm{Aut}(H_1(\Sigma_{g,1}, \mathbf{Z}))$ as the general linear group $\mathrm{GL}(2g, \mathbf{Z})$. Under this identification, the image of μ_M is considered as the symplectic group

$$\mathrm{Sp}(2g, \mathbf{Z}) := \{X \in \mathrm{GL}(2g, \mathbf{Z}) \mid {}^tXJX = J\} \text{ for } J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

where I_g is the identity matrix of degree g.

Next, we consider an embedding of the mapping class group $\mathcal{M}_{g,1}$ into the automorphism group of a free group of rank $2g$. For $n \geq 2$ let F_n be a free group of rank n

with basis $x_1, \ldots, x_n$. We denote by $\mathrm{Aut}\, F_n$ the automorphism group of F_n. Let H be the abelianization $H_1(F_n, \mathbf{Z})$ of F_n and $\mu : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ a natural homomorphism induced from the abelianization map $F_n \to H$. Throughout the paper, we identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing a basis $e_1, \ldots, e_n$ of H induced from the basis $x_1, \ldots, x_n$ of F_n. By a classical work of Nielsen [Ni], a finite presentation of $\mathrm{Aut}\, F_n$ is obtained. Observing the images of the generators of Nielsen's presentation, we see that μ is surjective. The kernel IA_n of ρ is called the IA-automorphism group of F_n. The IA-automorphism group IA_n is a free group analogue of the Torelli group $\mathcal{I}_{g,1}$.

Now, throughout the paper, we identify $\pi_1(\Sigma_{g,1}, *)$ with F_{2g}, and $H_1(\Sigma_{g,1}, \mathbf{Z})$ with H for $n = 2g$ using the basis above. Then the action of $\mathcal{M}_{g,1}$ on $\pi_1(\Sigma_{g,1}, *) = F_{2g}$ induces a natural homomorphism

$$\varphi : \mathcal{M}_{g,1} \to \mathrm{Aut}\, F_{2g}.$$

By a classical work due to Dehn and Nielsen, it is known that φ is injective. More precisely, we have

Theorem 3.1 (Dehn and Nielsen)**.** *For any $g \geq 1$, we have*

$$\varphi(\mathcal{M}_{g,1}) = \{\sigma \in \mathrm{Aut}\, F_{2g} \mid \zeta^\sigma = \zeta\}$$

where $\zeta = [x_1, x_{2g}][x_2, x_{2g-1}] \cdots [x_g, x_{g+1}] \in F_{2g}$, namely ζ is a homotopy class of a simple closed curve on $\Sigma_{g,1}$ parallel to the boundary.

For $n = 2g$, we have $\mu_{\mathcal{M}} = \mu \circ \varphi : \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbf{Z})$, and a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{IA}_{2g} \arrow[r] & \mathrm{Aut}\, F_{2g} \arrow[r, "\mu"] & \mathrm{GL}(2g, \mathbf{Z}) \arrow[r] & 1 \\
1 \arrow[r] & \mathcal{I}_{g,1} \arrow[r] \arrow[u, hook, "\varphi|_{\mathcal{I}_{g,1}}"'] & \mathcal{M}_{g,1} \arrow[r, "\mu_{\mathcal{M}}"] \arrow[u, hook, "\varphi"'] & \mathrm{Sp}(2g, \mathbf{Z}) \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

3.2 Free Lie algebras

In this subsection, we recall the free Lie algebra generated by H, and its derivation algebra. (See [Se] and [Re] for basic material concerning the free Lie algebra for instance.)

Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by the rule

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2.$$

We denote by $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ the k-th graded quotient of the lower central series of F_n, and by $\mathcal{L}_n := \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ the associated graded sum. The degree 1 part $\mathcal{L}_n(1)$ of $\mathcal{L}_n$ is just H. Classically, Magnus showed that each of $\mathcal{L}_n(k)$ is a free abelian, and Witt [W] gave its rank as follows.

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{L}_n(k)) = \frac{1}{k} \sum_{d|k} \mathrm{M\ddot{o}b}(d) n^{\frac{k}{d}} \tag{1}$$

where Möb is the Möbius function. For any k, $l \geq 1$, let us consider a bilinear alternating map

$$[\ ,\ ]_{\mathrm{Lie}} : \mathcal{L}_n(k) \times \mathcal{L}_n(l) \to \mathcal{L}_n(k+l)$$

defined by $[\,[\alpha],[\beta]\,]_{\mathrm{Lie}} := [\,[\alpha,\beta]\,]$ for any $[\alpha] \in \mathcal{L}_n(k)$ and $[\beta] \in \mathcal{L}_n(l)$, where $[\alpha,\beta]$ is a commutator in F_n, and $[\,[\alpha,\beta]\,]$ is a coset class of $[\alpha,\beta]$ in $\mathcal{L}_n(k+l)$. Then $[\ ,\ ]_{\mathrm{Lie}}$ induces a graded Lie algebra structure of the graded sum $\mathcal{L}_n$. By a classical work of Magnus, the Lie algebra $\mathcal{L}_n$ is isomorphic to the free Lie algebra generated by H.

The Lie algebra $\mathcal{L}_n$ is considered as a Lie subalgebra of the tensor algebra generated by H as follows. Let

$$T(H) := \mathbf{Z} \oplus H \oplus H^{\otimes 2} \oplus \cdots$$

be the tensor algebra of H over $\mathbf{Z}$. Then $T(H)$ is the universal enveloping algebra of the free Lie algebra $\mathcal{L}_n$, and the natural map $\iota : \mathcal{L}_n \to T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for X, $Y \in \mathcal{L}_n$ is an injective graded Lie algebra homomorphism. We denote by ι_k the homomorphism of degree k part of ι, and consider $\mathcal{L}_n(k)$ as a submodule $H^{\otimes k}$ through ι_k.

Here, we recall the derivation algebra of the free Lie algebra. Let $\mathrm{Der}(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. Namely,

$$\mathrm{Der}(\mathcal{L}_n) := \{f : \mathcal{L}_n \xrightarrow{\mathbf{Z}-\text{linear}} \mathcal{L}_n \mid f([a,b]) = [f(a),b] + [a,f(b)],\ \ a,b \in \mathcal{L}_n\}.$$

For $k \geq 0$, the degree k part of $\mathrm{Der}(\mathcal{L}_n)$ is defined to be

$$\mathrm{Der}(\mathcal{L}_n)(k) := \{f \in \mathrm{Der}(\mathcal{L}_n) \mid f(a) \in \mathcal{L}_n(k+1),\ \ a \in H\}.$$

Then, we have

$$\mathrm{Der}(\mathcal{L}_n) = \bigoplus_{k \geq 0} \mathrm{Der}(\mathcal{L}_n)(k),$$

and can consider $\mathrm{Der}(\mathcal{L}_n)(k)$ as

$$\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

for each $k \geq 1$ by the universality of the free Lie algebra. Let $\mathrm{Der}^+(\mathcal{L}_n)$ be a graded Lie subalgebra of $\mathrm{Der}(\mathcal{L}_n)(k)$ with positive degree. (See Section 8 of Chapter II in [Bou].)

3.3 (Higher) Johnson homomorphisms

First we recall the Johnson filtration and the Johnson homomorphisms of the automorphism group of a free group. Then we consider those of the mapping class group.

For each $k \geq 1$, let $N_{n,k} := F_n/\Gamma_n(k+1)$ of F_n be the free nilpotent group of class k and rank n, and $\mathrm{Aut}\, N_{n,k}$ its automorphism group. Since the subgroup $\Gamma_n(k+1)$ is characteristic in F_n, the group $\mathrm{Aut}\, F_n$ naturally acts on $N_{n,k}$ from the right. This action induces a homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, N_{n,k}$. Let $\mathcal{A}_n(k)$ be the kernel of this homomorphism. Then the groups $\mathcal{A}_n(k)$ define a descending filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

This filtration is called the Johnson filtration of $\mathrm{Aut}\, F_n$. Set $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$. Andreadakis [An] originally studied the Johnson filtration, and obtained basic and important properties of it as follows:

Theorem 3.2 (Andreadakis, [An])**.**

(i) *For any* $k,\ l \geq 1$, $\sigma \in \mathcal{A}_n(k)$ *and* $x \in \Gamma_n(l)$, $x^{-1}x^{\sigma} \in \Gamma_n(k+l)$.

(ii) *For any* $k,\ l \geq 1$, $[\mathcal{A}_n(k), \mathcal{A}_n(l)] \subset \mathcal{A}_n(k+l)$. *In other words, the Johnson filtration is a descending central filtration of* IA_n.

(iii) *For any* $k \geq 1$, $\mathrm{gr}^k(\mathcal{A}_n)$ *is a free abelian group of finite rank.*

In order to study the structure of $\mathrm{gr}^k(\mathcal{A}_n)$, the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$ is defined as follows.

Definition 3.3. For each $k \geq 1$, define a homomorphism $\tilde{\tau}_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \ \mapsto \ \big(x \bmod \Gamma_n(2) \mapsto x^{-1}x^{\sigma} \bmod \Gamma_n(k+2)\big), \quad x \in F_n.$$

Then the kernel of $\tilde{\tau}_k$ is just $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

This homomorphism is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$.

Here we consider actions of $\mathrm{GL}(n, \mathbf{Z}) = \mathrm{Aut}\, F_n/\mathrm{IA}_n$. First, since each term of the lower central series of F_n is a characteristic subgroup, $\mathrm{Aut}\, F_n$ naturally acts on it, and hence each of the graded quotient $\mathcal{L}_n(k)$. By (i) of Theorem 3.2, we see that the action of IA_n on $\mathcal{L}_n(k)$ is trivial. Thus the action of $\mathrm{GL}(n, \mathbf{Z}) = \mathrm{Aut}\, F_n/\mathrm{IA}_n$ on $\mathcal{L}_n(k)$ is well-defined. On the other hand, since each term of the Johnson filtration is a normal subgroup of $\mathrm{Aut}\, F_n$, the group $\mathrm{Aut}\, F_n$ naturally acts on $\mathcal{A}_n(k)$ by conjugation, and hence each of the graded quotient $\mathrm{gr}^k(\mathcal{A}_n)$. By (ii) of Theorem 3.2, we see that the action of IA_n on $\mathrm{gr}^k(\mathcal{A}_n)$ is trivial. Namely, we may consider $\mathrm{gr}^k(\mathcal{A}_n)$ as a $\mathrm{GL}(n, \mathbf{Z}) = \mathrm{Aut}\, F_n/\mathrm{IA}_n$-module. With respect to the actions above, we see that The Johnson homomorphism τ_k is $\mathrm{GL}(n, \mathbf{Z})$-equivariant for each $k \geq 1$.

Furthermore, we remark that the sum of the Johnson homomorphisms forms a Lie algebra homomorphism as follows. Let $\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ be the graded sum of $\mathrm{gr}^k(\mathcal{A}_n)$. The graded sum $\mathrm{gr}(\mathcal{A}_n)$ has a graded Lie algebra structure induced from the commutator bracket on IA_n by an argument similar to that of the free Lie algebra $\mathcal{L}_n$. Then the sum of the Johnson homomorphisms

$$\tau := \bigoplus_{k \geq 1} \tau_k : \mathrm{gr}(\mathcal{A}_n) \to \mathrm{Der}^+(\mathcal{L}_n)$$

is a graded Lie algebra homomorphism. (See also Theorem 4.8 in [Mo2].)

In the following, we consider three central subfiltration of the Johnson filtration of $\mathrm{Aut}\, F_n$, and "restrictions" of the Johnson homomorphism τ_k.

The first one is the lower central series of IA_n. Let $\mathcal{A}'_n(k)$ be the lower central series of IA_n with $\mathcal{A}'_n(1) = \mathrm{IA}_n$. Since the Johnson filtration is central, $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for each $k \geq 1$. Set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. Then $\mathrm{GL}(n, \mathbf{Z})$ naturally acts on each of $\mathrm{gr}^k(\mathcal{A}'_n)$, and the restriction of $\tilde{\tau}_k$ to $\mathcal{A}'_n(k)$ induces a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

We also call τ'_k the Johnson homomorphism of $\mathrm{Aut}\, F_n$. We remark that if we denote by $i_k : \mathrm{gr}^k(\mathcal{A}'_n) \to \mathrm{gr}^k(\mathcal{A}_n)$ the homomorphism induced from the inclusion $\mathcal{A}'_n(k) \hookrightarrow \mathcal{A}_n(k)$, then $\tau'_k = \tau_k \circ i_k$ for each $k \geq 1$. Similarly to the sum τ of τ_ks, the sum $\tau' := \oplus_{k \geq 1} \tau'_k : \mathrm{gr}(\mathcal{A}'_n) \to \mathrm{Der}^+(\mathcal{L}_n)$ is a graded Lie algebra homomorphism.

Let $\mathcal{C}_n(k)$ be a quotient module of $H^{\otimes k}$ by the action of cyclic group Cyc_k of order k on the components:

$$\mathcal{C}_n(k) = H^{\otimes k} \big/ \langle a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_2 \otimes a_3 \otimes \cdots \otimes a_k \otimes a_1 \,|\, a_i \in H \rangle.$$

In [Sa], we determined the cokernel of the rational Johnson homomorphisms τ'_k in stable range. Namely, we have

Theorem 3.4 (Satoh, [Sa])**.** *For any $k \geq 2$ and $n \geq k+2$,*

$$\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) \cong \mathcal{C}_n^{\mathbf{Q}}(k).$$

We also remark that in our previous paper [ES], we studied the GL-irreducible decomposition of $\mathcal{C}_n^{\mathbf{Q}}(k)$. For more details, see Proposition 7.2 and Proposition 7.3.

Next, we consider the Johnson filtration of the mapping class group. By Dehn and Nielsen's classical work, we can consider $\mathcal{M}_{g,1}$ as a subgroup of $\mathrm{Aut}\, F_{2g}$ as above. Under this embedding, set $\mathcal{M}_{g,1}(k) := \mathcal{M}_{g,1} \cap \mathcal{A}_{2g}(k)$ for each $k \geq 1$. Then we have a descending filtration

$$\mathcal{I}_{g,1} = \mathcal{M}_{g,1}(1) \supset \mathcal{M}_{g,1}(2) \supset \cdots$$

of the Torelli group $\mathcal{I}_{g,1}$. This filtration is called the Johnson filtration of $\mathcal{M}_{g,1}$. Set $\mathrm{gr}^k(\mathcal{M}_{g,1}) := \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)$. For each $k \geq 1$, the mapping class group $\mathcal{M}_{g,1}$ acts on $\mathrm{gr}^k(\mathcal{M}_{g,1})$ by conjugation. This action induces that of $\mathrm{Sp}(2g,\mathbf{Z}) = \mathcal{M}_{g,1}/\mathcal{I}_{g,1}$ on it.

By an argument similar to that of $\mathrm{Aut}\,F_n$, the Johnson homomorphisms of $\mathcal{M}_{g,1}$ are defined as follows. For $n = 2g$ and $k \geq 1$, consider a restriction of $\tilde{\tau}_k : \mathcal{A}_{2g}(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_{2g}(k+1))$ to $\mathcal{M}_{g,1}(k)$. Then its kernel is just $\mathcal{M}_{g,1}(k+1)$. Hence we obtain an injective homomorphism

$$\tau_k^{\mathcal{M}} : \mathrm{gr}^k(\mathcal{M}_{g,1}) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_{2g}(k+1)) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1).$$

The homomorphism $\tau_k^{\mathcal{M}}$ is $\mathrm{Sp}(2g,\mathbf{Z})$-equivariant, and is called the k-th Johnson homomorphism of $\mathcal{M}_{g,1}$. If we consider a $\mathrm{GL}(2g,\mathbf{Z})$-module H as a $\mathrm{Sp}(2g,\mathbf{Z})$-module, then $H^* \cong H$ by the Poincaré duality. Hence, in the following, we canonically identify the target $H^* \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1)$ of $\tau_k^{\mathcal{M}}$ with $H \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1)$.

Historically, the Johnson filtration of $\mathrm{Aut}\,F_n$ was originally studied by Andreadakis [An] in 1960's as mentioned above. On the other hand, the Johnson filtration and the Johnson homomorphisms of $\mathcal{M}_{g,1}$ were begun to study by D. Johnson [Joh1] in 1980's who determined the abelianization of the Torelli subgroup of the mapping class group of a surface in [Joh4]. In particular, he showed that $\mathrm{Im}(\tau_1^{\mathcal{M}}) \cong \Lambda^3 H$ as an $\mathrm{Sp}(2g,\mathbf{Z})$-module, and it gives the free part of $H_1(\mathcal{I}_{g,1},\mathbf{Z})$.

Now, let us recall the fact that the image of $\tau_k^{\mathcal{M}}$ is contained in a certain $\mathrm{Sp}(2g,\mathbf{Z})$-submodule of $H \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1)$, due to Morita [Mo2]. In general, for any $n \geq 1$, let $H \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \mathcal{L}_n(k+2)$ be a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism defined by

$$a \otimes X \mapsto [a, X], \quad \text{for} \quad a \in H, \;\; X \in \mathcal{L}_n(k+1).$$

For $n = 2g$, we denote by $\mathfrak{h}_{g,1}(k)$ the kernel of this homomorphism:

$$\mathfrak{h}_{g,1}(k) := \mathrm{Ker}(H \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1) \to \mathcal{L}_{2g}(k+2)).$$

Then Morita [Mo2] showed that the image $\mathrm{Im}(\tau_k^{\mathcal{M}})$ is contained in $\mathfrak{h}_{g,1}(k)$. Therefore, to determine how different is $\mathrm{Im}(\tau_k^{\mathcal{M}})$ from $\mathfrak{h}_{g,1}(k)$ is one of the most basic problems. Throughout the paper, the cokernel $\mathrm{Coker}(\tau_k^{\mathcal{M}})$ of $\tau_k^{\mathcal{M}}$ always means the quotient $\mathrm{Sp}(2g,\mathbf{Z})$-module $\mathfrak{h}_{g,1}(k)/\mathrm{Im}(\tau_k^{\mathcal{M}})$. So far, the Sp-module structure of $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ is determined for $1 \leq k \leq 4$ as follows.

k	$\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$	$\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$	
1	$[1^3] \oplus [1]$	0	Johnson [Joh1]
2	$[2^2] \oplus [1^2] \oplus [0]$	0	Morita [Mo1], Hain [Ha]
3	$[3,1^2] \oplus [2,1]$	$[3]$	Asada-Nakamura [AN], Hain [Ha]
4	$[4,2] \oplus [3,1^3] \oplus [2^3] \oplus 2[3,1] \oplus [2,1^2] \oplus 2[2]$	$[2,1^2] \oplus [2]$	Morita [Mo3]

Morita [Mo2] showed that the symmetric tensor product $S^k H_{\mathbf{Q}}$ appears in the Sp-irreducible decomposition of $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ for each $k \geq 2$ using the Morita trace map. In general, however, to determine the cokernel of $\tau_k^{\mathcal{M}}$ is a difficult problem.

Here, we recall a remarkable result of Hain. As an $\mathrm{Sp}(2g, \mathbf{Z})$-module, we consider $\mathfrak{h}_{g,1}(k)$ as a submodule of the degree k part $\mathrm{Der}(\mathcal{L}_n)(k)$ of the derivation algebra of $\mathcal{L}_n$. On the other hand, the graded sum

$$\mathfrak{h}_{g,1} := \bigoplus_{k \geq 1} \mathfrak{h}_{g,1}(k)$$

naturally has a Lie subalgebra structure of $\mathrm{Der}^+(\mathcal{L}_n)$. Therefore we obtain a graded Lie algebra homomorphism

$$\tau^{\mathcal{M}} := \bigoplus_{k \geq 1} \tau_k^{\mathcal{M}} : \mathrm{gr}(\mathcal{M}_{g,1}) \to \mathfrak{h}_{g,1}.$$

Then we have

Theorem 3.5 (Hain [Ha]). *The Lie subalgebra* $\mathrm{Im}(\tau_{\mathbf{Q}}^{\mathcal{M}})$ *is generated by the degree one part* $\mathrm{Im}(\tau_{1,\mathbf{Q}}^{\mathcal{M}}) = \Lambda^3 H_{\mathbf{Q}}$ *as a Lie algebra.*

Finally, we consider the lower central series of the Torelli group, and reformulate Hain's result above. Let $\mathcal{M}'_{g,1}(k)$ be the lower central series of $\mathcal{I}_{g,1}$, and set $\mathrm{gr}^k(\mathcal{M}'_{g,1}) := \mathcal{M}'_{g,1}(k)/\mathcal{M}'_{g,1}(k+1)$ for $k \geq 1$. Let $\tau_k'^{\mathcal{M}} : \mathrm{gr}^k(\mathcal{M}'_{g,1}) \to H \otimes_{\mathbf{Z}} \mathcal{L}_{2g}(k+1)$ be an Sp-equivariant homomorphism induced from the restriction of $\tilde{\tau}_k$ to $\mathcal{M}'_{g,1}(k)$. Then we have

Proposition 3.6 (Hain, [Ha]). *We have* $\mathrm{Im}(\tau_{k,\mathbf{Q}}^{\mathcal{M}}) = \mathrm{Im}(\tau_{k,\mathbf{Q}}'^{\mathcal{M}})$ *for each* $k \geq 1$.

For $n = 2g$, we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Im}\,\tau'_{k,\mathbf{Q}} \arrow[rr, hook] & & H^*_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{2g}(k+1) \arrow[r, two heads] & H_{\mathbf{Q}}^{\otimes k} \arrow[r, two heads] & \mathcal{C}^{\mathbf{Q}}_{2g}(k) \\
\mathrm{Im}\,\tau^{\mathcal{M}}_{k,\mathbf{Q}} \arrow[r, equal] & \mathrm{Im}\,\tau'^{\mathcal{M}}_{k,\mathbf{Q}} \arrow[u, hook] \arrow[r, hook] & \mathfrak{h}^{\mathbf{Q}}_{g,1}(k) \arrow[r, hook] & H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{2g}(k+1) \arrow[u, "\wr"'] \arrow[rr, two heads] & & \mathcal{L}^{\mathbf{Q}}_{2g}(k+1)
\end{tikzcd}$$

4 Highest weight theory for $\mathrm{Sp}(2g, \mathbf{Q})$

4.1 Irreducible highest weight modules for $\mathrm{Sp}(2g, \mathbf{Q})$

Let us consider the general linear group $\mathrm{GL}(n, \mathbf{Q})$ and the symplectic group

$$\mathrm{Sp}(2g, \mathbf{Q}) := \{X \in \mathrm{GL}(2g, \mathbf{Q}) \mid {}^t X J X = J\} \text{ for } J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

where I_g is the identity matrix of degree g. We fix a maximal torus

$$T_n = \{\mathrm{diag}(x_1, \ldots, x_n) \mid x_j \neq 0,\ 1 \leq j \leq n\}$$

of $\mathrm{GL}(n, \mathbf{Q})$. The intersection $\mathrm{Sp}(2g, \mathbf{Q}) \cap T_{2g} = \{\mathrm{diag}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1})\}$ gives a maximal torus of $\mathrm{Sp}(2g, \mathbf{Q})$. We also fix this maximal torus and write T_{2g}^{Sp}.

We define one-dimensional representations ε_i of T_n by $\varepsilon_i(\mathrm{diag}(x_1, \ldots, x_n)) = x_i$. Then

$$\begin{aligned} P_{\mathrm{GL}(n,\mathbf{Q})} &:= \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda_i \in \mathbb{Z},\ 1 \leq i \leq n\} \cong \mathbb{Z}^n, \\ P^+_{\mathrm{GL}(n,\mathbf{Q})} &:= \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in P_{\mathrm{GL}(n,\mathbf{Q})} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\} \end{aligned}$$

give the weight lattice and the set of dominant integral weights of $\mathrm{GL}(n, \mathbf{Q})$ respectively. If $n = 2g$, we can restrict ε_i to T_{2g}^{Sp} for $1 \leq i \leq g$. Then

$$\begin{aligned} P_{\mathrm{Sp}(2g,\mathbf{Q})} &:= \{\lambda_1\varepsilon_1 + \cdots + \lambda_g\varepsilon_g \mid \lambda_i \in \mathbb{Z},\ 1 \leq i \leq g\} \cong \mathbb{Z}^g, \\ P^+_{\mathrm{Sp}(2g,\mathbf{Q})} &:= \{\lambda_1\varepsilon_1 + \cdots + \lambda_g\varepsilon_g \in P_{\mathrm{Sp}(2g,\mathbf{Q})} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0\} \end{aligned}$$

give the weight lattice and the set of dominant integral weights of $\mathrm{Sp}(2g, \mathbf{Q})$ respectively. In particular, there exists a bijection between $P^+_{\mathrm{Sp}(2g,\mathbf{Q})}$ and the set of partitions such that $\ell(\lambda) \leq g$.

Let G be a classical group $\mathrm{GL}(n, \mathbf{Q})$ or $\mathrm{Sp}(2g, \mathbf{Q})$, T its fixed maximal torus, P its weight lattice and P^+ the set of dominant integral weight with respect to T. For a rational representation V of G, there exists an irreducible decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$ as a T-module where $V_\lambda := \{v \in V \mid tv = \lambda(t)v$ for any $t \in T\}$. We call this decomposition a weight decomposition of V with respect to T. If $V_\lambda \neq \{0\}$, then we call λ a weight of V. For a weight λ, a non-zero vector $v \in V_\lambda$ is call a weight vector of weight λ.

Let U be the subgroup of G consists of all upper unitriangular matrices in G. For a rational representation V of G, we define $V^U := \{v \in V \mid uv = v$ for all $u \in U\}$. We call a non-zero vector $v \in V^U$ a maximal vector of V. This subspace V^U is T-stable. Thus, as a T-module, V^U has a irreducible decomposition $V^U = \bigoplus_{\lambda \in P} V^U_\lambda$ where $V^U_\lambda := V^U \cap V_\lambda$.

Theorem 4.1 (Cartan-Weyl's highest weight theory)**.**

(i) Any rational representation of V is completely reducible.

(ii) Suppose V is an irreducible rational representation of G. Then V^U is one-dimensional, and the weight λ of $V^U = V^U_\lambda$ belongs to P^+. We call this λ the highest weight of V, and any non-zero vector $v \in V^U_\lambda$ is called a highest weight vector of V.

(iii) For any $\lambda \in P^+$, there exists a unique (up to isomorphism) irreducible rational representation L^λ of G with highest weight λ. Moreover, for two $\lambda, \mu \in P^+$, $L^\lambda \cong L^\mu$ if and only if $\lambda = \mu$.

(iv) *The set of isomorphism classes of irreducible rational representations of G is parametrized by the set P^+ of dominant integral weights.*

(v) *Let V be a rational representation of G and χ_V a character of V as a T-module. Then for two rational representation V and W, they are isomorphic as G-modules if and only if $\chi_V = \chi_W$.*

Remark 4.2. We can parametrize the set of isomorphism classes of irreducible rational representations of $\mathrm{GL}(n, \mathbf{Q})$ by $P^+_{\mathrm{GL}(n,\mathbf{Q})}$. On the other hand, we define the determinant representation by $\det^e : \mathrm{GL}(n, \mathbf{Q}) \ni X \to \det X^e \in \mathbf{Q}^\times$. The highest weight of this representation is given by $(e, e, \cdots, e) \in P^+_{\mathrm{GL}(n,\mathbf{Q})}$. If $\lambda \in P^+$ satisfies $\lambda_n < 0$, then $L^\lambda \cong \det^{-\lambda_n} \otimes L^{(\lambda_1-\lambda_n, \lambda_2-\lambda_n, \ldots, 0)}$. Moreover the set of isomorphism classes of polynomial irreducible representations is parametrized by the set of partitions λ such that $\ell(\lambda) \leq n$. We denote the polynomial representations corresponding to a partition λ by L^λ_{GL}, $L^{(\lambda)}$ or simply (λ).

Remark 4.3. We can parametrize the set of isomorphism classes of irreducible rational representations of $\mathrm{Sp}(2g, \mathbf{Q})$ by $P^+_{\mathrm{Sp}(2g,\mathbf{Q})} \cong \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0 \mid \lambda_i \in \mathbb{Z}, 1 \leq i \leq n\}$, namely the set of partitions λ such that $\ell(\lambda) \leq g$. In this paper, we denote the irreducible representation corresponding to λ by L^λ_{Sp}, $L^{[\lambda]}$ or simply $[\lambda]$.

Note that the natural representation $H_{\mathbf{Q}} = \mathbf{Q}^{2g}$ of $\mathrm{Sp}(2g, \mathbf{Q})$ is irreducible with highest weight $(1, 0, \ldots, 0)$ and $H^*_{\mathbf{Q}} \cong H_{\mathbf{Q}}$ by the Poincaré duality. More precisely, we set $i' := 2g - i + 1$ for each integer $1 \leq i \leq 2g$. Then for the standard basis $\{e_i\}_{i=1}^{2g}$ of $H_{\mathbf{Q}}$, we see

$$\langle e_i, e_j \rangle = 0 = \langle e_{i'}, e_{j'} \rangle, \quad \langle e_i, e_{j'} \rangle = \delta_{ij} = -\langle e_{j'}, e_i \rangle, \quad (1 \leq i \leq g). \tag{2}$$

There is an isomorphism $H_{\mathbf{Q}} \to H^*_{\mathbf{Q}}$ as $\mathrm{Sp}(2g, \mathbf{Q})$-modules given by

$$H_{\mathbf{Q}} \ni v \mapsto \langle \bullet, v \rangle \in H^*_{\mathbf{Q}}. \tag{3}$$

In general, all irreducible rational representation $[\lambda]$ is isomorphic to its dual.

Let us recall Pieri's formula, the simplest version of the decomposition of tensor product representations. For two partition λ and μ satisfying $\lambda \supset \mu$, the skew shape $\lambda \backslash \mu$ is a vertical strip if there is at most one box in each row.

Theorem 4.4 (Pieri's formula)**.** *Let μ be a partition such that $\ell(\mu) \leq n$. Then*

$$L^{(1^k)}_{\mathrm{GL}} \otimes L^\mu_{\mathrm{GL}} \cong \bigoplus_\lambda L^\lambda_{\mathrm{GL}},$$

where λ runs over the set of partitions obtained by adding a vertical k-strip to μ such that $\ell(\lambda) \leq n$.

4.2 Branching rules from $\mathrm{GL}(2g, \mathbf{Q})$ to $\mathrm{Sp}(2g, \mathbf{Q})$

We regard $\mathrm{Sp}(2g, \mathbf{Q})$ as a subgroup of $\mathrm{GL}(2g, \mathbf{Q})$. We consider the restriction of an irreducible polynomial representation $L^{\lambda}_{\mathrm{GL}}$ to $\mathrm{Sp}(2g, \mathbf{Q})$. We can give its irreducible decomposition using the Littlewood-Richardson coefficients $\mathrm{LR}^{\nu}_{\lambda\mu}$ as follows.

Theorem 4.5 ([FH, 25.39],[KT, Proposition 2.5.1])**.** *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0)$ be a partition such that $\ell(\lambda) \leq g$. Then we have*

$$\mathrm{Res}^{\mathrm{GL}(2g,\mathbf{Q})}_{\mathrm{Sp}(2g,\mathbf{Q})}(L^{\lambda}_{\mathrm{GL}}) \cong \bigoplus_{\bar{\lambda}} N_{\lambda\bar{\lambda}} L^{\bar{\lambda}}_{Sp}$$

where $\bar{\lambda}$ runs over all partitions such that $\ell(\bar{\lambda}) \leq g$. Here

$$N_{\lambda\bar{\lambda}} = \sum_{\eta} \mathrm{LR}^{\lambda}_{\eta\bar{\lambda}}$$

where η runs over all partitions $\eta = (\eta_1 = \eta_2 \geq \eta_3 = \eta_4 \geq \cdots)$ with each part occurring an even number of times, namely η' even. Here η' is a conjugate partition of η.

Remark 4.6. We give a combinatorial description of the Littlewood-Richardson coefficients. (e.g. [FH], [Mac].) For two Young diagrams λ and μ satisfying $\lambda \subset \mu$, we denote by $\lambda\backslash\mu$ a skew Young diagram, which is the difference of λ and μ. For a skew Young diagram $\lambda\backslash\mu$ of size m, a semistandard tableau of shape $\lambda\backslash\mu$ is an array T of positive integers $1, 2, \ldots, m$ of shape $\lambda\backslash\mu$ that is weakly increasing in every row and strictly increasing in every column.

(i) For two partitions $\lambda \supset \mu$, a *semi-standard tableau* on $\lambda\backslash\mu$ is a numbering on $\lambda\backslash\mu \to \mathbb{Z}_{\geq 1}$ such that the numbers inserted in $\lambda\backslash\mu$ must increase strictly down each column and weakly from left to right along each row. For a semistandard tableau on $\lambda\backslash\mu$, we denote the number of i appearing in this semistandard tableau by m_i. We call $(m_1, m_2, \ldots)$ a *weight* of the semistandard tableau.

(ii) For a semistandard tableau T on $\lambda\backslash\mu$, we define a sequence $w(T)$ of integers by reading the numbers inserted in $\lambda\backslash\mu$ from right to left in successive rows, starting with top row.

(iii) For a sequence $w = (a_1 a_2 \cdots)$, we denote the number of i appearing in a subsequence $(a_1 a_2 \cdots a_r)$ by $m_i(a_1 a_2 \cdots a_r)$. A sequence w is a *lattice permutation* if $m_1(a_1 a_2 \cdots a_r) \geq m_2(a_1 a_2 \cdots a_r) \geq \cdots$ for any $r \geq 1$.

The Littlewood-Richardson coefficients $\mathrm{LR}^{\lambda}_{\mu\nu}$ is the number of semi-standard tableaux T on $\lambda\backslash\mu$ with weight ν such that $w(T)$ is a lattice permutation.

4.3 Review on the classical Schur-Weyl duality

For the natural representation $H_{\mathbf{Q}} \cong L^{(1)}$ of $\mathrm{GL}(n, \mathbf{Q})$, we consider the k-th tensor product representation $\rho_k : \mathrm{GL}(n, \mathbf{Q}) \to \mathrm{GL}(H_{\mathbf{Q}}^{\otimes k})$ of $H_{\mathbf{Q}}$. For each $k \geq 1$, the symmetric group $\mathfrak{S}_k$ of degree k naturally acts on the space $H_{\mathbf{Q}}^{\otimes k}$ from the right as a permutation of the components. Since these two actions are commutative, we can decompose $H_{\mathbf{Q}}^{\otimes k}$ as a $(\mathrm{GL}(n, \mathbf{Q}) \times \mathfrak{S}_k)$-module. Let us recall this irreducible decomposition, called the Schur-Weyl duality for $\mathrm{GL}(n, \mathbf{Q})$ and $\mathfrak{S}_k$.

Theorem 4.7 (Schur-Weyl's duality for $\mathrm{GL}(n, \mathbf{Q})$ and $\mathfrak{S}_k$)**.**

(i) Let λ be a partition of k such that $\ell(\lambda) \leq n$. There exists a non-zero maximal vector v_λ with weight λ satisfying the following three conditions:

(a) The $\mathfrak{S}_k$-invariant subspace $S^\lambda := \sum_{\sigma \in \mathfrak{S}_k} \mathbf{Q} v_\lambda \cdot \sigma$ gives an irreducible representation of $\mathfrak{S}_k$.

(b) The subspace $(H_{\mathbf{Q}}^{\otimes k})_\lambda^U$ of weight λ coincides with the subspace S^λ, where U is the fixed unipotent subgroup of $\mathrm{GL}(n, \mathbf{Q})$ consisting of upper unitriangular matrices.

(c) The $\mathrm{GL}(n, \mathbf{Q})$-module generated by v_λ is isomorphic to the irreducible representation $L_{\mathrm{GL}}^{(\lambda)}$ of $\mathrm{GL}(n, \mathbf{Q})$ with highest weight λ.

(ii) We have the irreducible decomposition:

$$H_{\mathbf{Q}}^{\otimes k} \cong \bigoplus_{\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0) \vdash k} L^\lambda \boxtimes S^\lambda.$$

as $(\mathrm{GL}(n, \mathbf{Q}) \times \mathfrak{S}_k)$-modules.

(iii) Suppose $n \geq k$. Then $\{S^\lambda \mid \lambda \vdash k\}$ gives a complete representatives of irreducible representations of $\mathfrak{S}_k$.

Remark 4.8.

(i) The irreducible representation S^λ of $\mathfrak{S}_k$ is isomorphic to the following $\mathfrak{S}_k$-module.

For a partition λ of k, we define two special Young subgroups $C_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots$ and $R_\lambda := \mathfrak{S}_{\lambda'_1} \times \mathfrak{S}_{\lambda'_2} \times \cdots$ of $\mathfrak{S}_k$. Here a partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the conjugate partition of λ. In the group algebras of these two groups, we find idempotents

$$a_\lambda = \frac{1}{|R_\lambda|} \sum_{\sigma \in R_\lambda} \sigma \in \mathbf{Q} R_\lambda, \text{ and } b_\lambda = \frac{1}{|C_\lambda|} \sum_{\sigma \in C_\lambda} \mathrm{sgn}(\sigma) \sigma \in \mathbf{Q} C_\lambda.$$

Then $c_\lambda = |R_\lambda||C_\lambda| a_\lambda b_\lambda$ gives an idempotent in $\mathbf{Q}\mathfrak{S}_k$, called the Young symmetrizer for λ. The right ideal $c_\lambda \cdot \mathbf{Q}\mathfrak{S}_k$ in $\mathbf{Q}\mathfrak{S}_k$ gives an irreducible $\mathfrak{S}_k$-module which is isomorphic to S^λ above.

(ii) We construct v_λ appearing in the theorem above by the following way.

First, we define $v_1 \wedge v_2 \wedge \cdots \wedge v_r$ to be an anti-symmetrizer

$$\sum_{\sigma \in \mathfrak{S}_r} \mathrm{sgn}(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_r) \cdot \sigma \in H_{\mathbf{Q}}^{\otimes r}.$$

For the natural base $\{e_i\}_{i=1}^n$ of $H_{\mathbf{Q}}$, we define

$$v_\lambda := (e_1 \wedge \cdots \wedge e_{\lambda'_1}) \otimes (e_1 \wedge \cdots \wedge e_{\lambda'_2}) \otimes \cdots \in H_{\mathbf{Q}}^{\otimes k}. \tag{4}$$

Note that v_λ is a maximal vector of weight λ and

$$v_\lambda = (e_1 \otimes \cdots \otimes e_{\lambda'_1} \otimes e_1 \otimes \cdots \otimes e_{\lambda'_2} \otimes \cdots) \cdot c_\lambda.$$

This v_λ gives our desirable vector in the theorem above.

4.4 Brauer-Schur-Weyl's duality

The first two subsection is based on [HY] and [Hu]. The last one is based on [Ra].

4.4.1 Brauer algebras

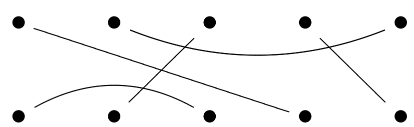
Let us define the Brauer algebra $B_k(-2g)$ with a parameter $-2g$ and size k.

Definition 4.9. The Brauer algebra $B_k(-2g)$ over $\mathbf{Q}$ is a unital associative $\mathbf{Q}$-algebra with the following generators and defining relations:

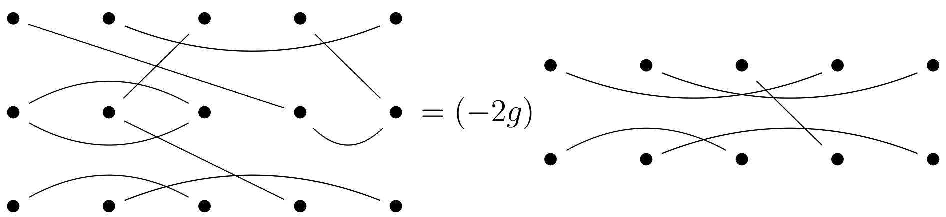
$$\begin{aligned}
\textit{generators} &: \quad s_1, \ldots, s_{k-1}, \gamma_1, \ldots, \gamma_{n-1}, \\
\textit{relations} &: \quad s_i^2 = 1, \quad \gamma_i^2 = (-2g)\gamma_i, \quad \gamma_i s_i = \gamma_i = s_i \gamma_i, \quad (1 \le i \le k-1), \\
&\quad\ s_i s_j = s_j s_i, \quad s_i \gamma_j = \gamma_j s_i, \quad \gamma_i \gamma_j = \gamma_j \gamma_i, \quad (1 \le i < j-1 \le k-2), \\
&\quad\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad \gamma_i \gamma_{i+1} \gamma_i = \gamma_i, \quad \gamma_{i+1} \gamma_i \gamma_{i+1} = \gamma_{i+1}, \quad (1 \le i \le k-2), \\
&\quad\ s_i \gamma_{i+1} \gamma_i = s_{i+1} \gamma_i, \quad \gamma_{i+1} \gamma_i s_{i+1} = \gamma_{i+1} s_i, \quad (1 \le i \le k-2).
\end{aligned}$$

Remark 4.10. The Brauer algebra $B_k(-2g)$ is obtained by the following diagrammatic way.

First of all, the Brauer k diagram is a diagram with specific $2k$ vertices arranged in two rows of k each, the top rows and the bottom rows, and exactly k edges such that every vertex is joined to another vertex (distinct from itself) by exactly one edge.

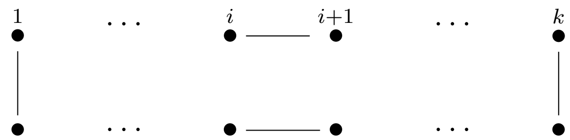


We define a multiplication of two diagrams as follows. We compose two diagrams D_1 and D_2 by identifying the bottom row of D_1 with the top row of D_2 such that the i-th vertex in the bottom row of D_1 is coincided with the i-th vertex in the top row of D_2. The result is a graph, with a certain number, $n(D_1, D_2)$, of interior loops. After removing the interior loops and the identified vertices, retaining the edges and remaining vertices, we obtain a new Brauer k-diagram $D_1 \circ D_2$. Then we define a multiplication $D_1 \cdot D_2$ by $(-2g)^{n(D_1,D_2)} D_1 \circ D_2$.

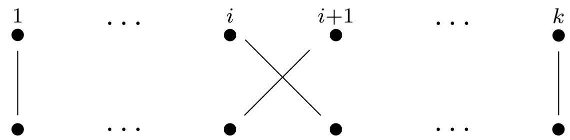


The Brauer algebra $B_k(-2g)$ is defined as $\mathbb{Q}$-linear space with a basis being the set of the Brauer k-diagrams and the multiplication of two elements given by the linear extension of a product above.

The generators s_i and γ_i correspond to the following diagrams.

$\gamma_i \;=$ 

$(1 \leq i \leq k-1)$

$s_i \;=$ 

$(1 \leq i \leq k-1)$

4.4.2 Decomposition of tensor spaces (Brauer-Schur-Weyl's duality)

Let us recall the inner product on $H_{\mathbf{Q}}$ defined by (2). Set $i' := 2g - i + 1$ for each integer $1 \leq i \leq 2g$. For the standard basis $\{e_i\}_{i=1}^{2g}$ of $H_{\mathbf{Q}}$, we see

$$\langle e_i, e_j \rangle = 0 = \langle e_{i'}, e_{j'} \rangle, \quad \langle e_i, e_{j'} \rangle = \delta_{ij} = -\langle e_{j'}, e_i \rangle, \quad (1 \leq i \leq g).$$

For each integer $1 \leq i \leq 2g$, we define

$$e_i^* = \begin{cases} e_{i'}, & (1 \leq i \leq g), \\ -e_{i'}, & (g+1 \leq i \leq 2g). \end{cases} \tag{5}$$

Then both of $\{e_i\}_{i=1}^{2g}$ and $\{e_i^*\}_{i=1}^{2g}$ are basis for $H_{\mathbf{Q}}$ such that one is dual to the other in the sense that $\langle e_i, e_j^* \rangle = \delta_{ij}$ for any i, j.

The following lemma is obvious, but important to generalize the Schur-Weyl duality for $\mathrm{Sp}(2g, \mathbf{Q})$.

Lemma 4.11. *An element*

$$\omega := \sum_{i=1}^{2g} e_i \otimes e_i^* \in H_{\mathbf{Q}}^{\otimes 2}$$

is invariant under the action of $\mathrm{Sp}(2g, \mathbf{Q})$ *on* $H_{\mathbf{Q}}^{\otimes 2}$*.*

We define a right action of $B_k(-2g)$ on $H_{\mathbf{Q}}^{\otimes k}$ as follows.

Proposition 4.12. *There is a right action of* $B_k(-2g)$ *on* $H_{\mathbf{Q}}^{\otimes k}$ *which is defined on generators by*

$$\begin{aligned}
(v_{i_1} \otimes \cdots \otimes v_{i_k}) \cdot \gamma_j &:= -v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes \left(\sum_{r=1}^{2g} e_k \otimes e_k^* \right) \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_k}, \\
(v_{i_1} \otimes \cdots \otimes v_{i_k}) \cdot s_j &:= -v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_k},
\end{aligned}$$

for any $v_{i_1}, \ldots, v_{i_k} \in H_{\mathbf{Q}}$*. Moreover, this action commutes with that of* $\mathrm{Sp}(2g, \mathbf{Q})$*.*

Here we state the Brauer-Schur-Weyl duality.

Theorem 4.13 (Brauer-Schur-Weyl's duality for $\mathrm{Sp}(2g, \mathbf{Q})$ and $B_k(-2g)$)**.**

(i) Let λ *be a partition of* $k - 2j$ *for* $0 \le j \le \lfloor \frac{k}{2} \rfloor$ *such that* $\ell(\lambda) \le g$*. Then there exists a maximal vector* $v_\lambda \in H_{\mathbf{Q}}^{\otimes k}$ *with highest weight* λ *satisfying the following three conditions:*

(a) A $B_k(-2g)$*-submodule*

$$D^\lambda := \sum_{\sigma \in B_k(-2g)} \mathbf{Q} v_\lambda \cdot \sigma$$

of $H_{\mathbf{Q}}^{\otimes k}$ *gives an irreducible representation of* $B_k(-2g)$*.*

(b) The subspace $(H_{\mathbf{Q}}^{\otimes k})_\lambda^U$ *of* $H_{\mathbf{Q}}^{\otimes k}$ *coincides with* D^λ*. Here* U *is the fixed unipotent subgroup for* $\mathrm{Sp}(2g, \mathbf{Q})$*.*

(c) The $\mathrm{Sp}(2g, \mathbf{Q})$*-module generated by* v_λ *is isomorphic to the irreducible representation* $L_{Sp}^{[\lambda]}$ *of* $Sp(2g, \mathbf{Q})$ *with highest weight* λ*.*

(ii) We have the irreducible decomposition

$$H_{\mathbf{Q}}^{\otimes k} \cong \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{\lambda \vdash k-2j, \ell(\lambda) \le g} L_{Sp}^{[\lambda]} \boxtimes D^\lambda.$$

as an $(\mathrm{Sp}(2g, \mathbf{Q}) \times B_k(-2g))$*-module.*

(iii) Suppose $g \geq k$. Then $\{D^\lambda \mid \lambda \vdash k-2j\ (0 \leq j \leq \lfloor \frac{k}{2} \rfloor)\}$ gives a complete representatives of irreducible representations of $B_k(-2g)$.

In our purpose of this paper, to observe an explicit construction of v_λ and a description of D^λ is important.

Theorem 4.14 ([Hu, Definition 3.9, Lemma 3.10, Lemma 4.8])**.**

(i) For a partition λ of $k-2j$ for $0 \leq j \leq \lfloor \frac{k}{2} \rfloor$ such that $\ell(\lambda) \leq g$, a maximal vector v_λ is given by

$$v_\lambda := \omega^{\otimes j} \otimes (e_1 \wedge \cdots \wedge e_{\lambda'_1}) \otimes (e_1 \wedge \cdots \wedge e_{\lambda'_2}) \otimes \cdots .$$

(ii) We regard a subalgebra generated by s_i $(1 \leq i \leq k-1)$ in $B_k(-2g)$ as a group algebra $\mathbf{Q}\mathfrak{S}_k$. Then the right module $v_\lambda \cdot B_k(-2g)$ coincides with $v_\lambda \cdot \mathbf{Q}\mathfrak{S}_k$ as a $\mathbf{Q}$-vector space.

4.4.3 Character values and decompositions of D^λ as an $\mathfrak{S}_k$-module

We give a branching low of the irreducible $B_k(-2g)$-modules D^λ as $\mathfrak{S}_k$-modules. But confusingly, the algebra $\mathbf{Q}\mathfrak{S}_k$ has an involution $\iota : \sigma \mapsto \mathrm{sgn}(\sigma)\sigma$, and the action of a subalgebra generated by s_i's in $B_k(-2g)$ on $H_{\mathbf{Q}}^{\otimes k}$ is twisted by this involution. Therefore a $\mathbf{Q}\mathfrak{S}_k$-module D is isomorphic to $\mathbf{sgn} \otimes D$ as an $\iota(\mathbf{Q}\mathfrak{S}_k)$-module. Here $\mathbf{sgn}$ is the signature representation of $\mathfrak{S}_k$. Note that an irreducible $\mathfrak{S}_k$-module $S^{\nu'}$ is isomorphic to $\mathbf{sgn} \otimes S^\nu$.

In our purpose, we consider the ordinary (untwisted) action of $\mathfrak{S}_k$ on $H_{\mathbf{Q}}^{\otimes k}$ in the following theorem (ii).

Theorem 4.15 ([Ra, Theorem 5.1])**.**

(i) For a partition λ of $k-2j$ for $0 \leq j \leq \lfloor \frac{k}{2} \rfloor$ such that $\ell(\lambda) \leq g$, let $\chi^\lambda_{B_k(-2g)}$ be the irreducible character of D^λ. Then we have

$$\chi^\lambda_{B_k(-2g)}(\sigma) = \sum_{\nu \vdash k, \nu \supset \lambda'} \left(\sum_{\beta: even} \mathrm{LR}^\nu_{\lambda'\beta} \right) \chi^\nu_{\mathfrak{S}_k}(\sigma).$$

for any $\sigma \in \mathfrak{S}_k \subset$ (a subalgebra generated by $\{s_i\}_{i=1}^{k-1}$). Here $\chi^\nu_{\mathfrak{S}_k}$ is an irreducible character of $\mathfrak{S}_k$ associated to a partition ν of k. The number LR *is the Littlewood-Richardson coefficient. The even partition $\beta = (\beta_1, \beta_2, \ldots)$ is a partition such that any parts β_i are even.*

(ii) We have the irreducible decomposition of D^λ is given by

$$\bigoplus_{\nu \vdash k, \nu \supset \lambda'} (S^{\nu'})^{\oplus \sum_{\beta: even} \mathrm{LR}^\nu_{\lambda'\beta}}$$

with respect to the ordinary $\mathfrak{S}_k$-action on $H_{\mathbf{Q}}^{\otimes k}$.

Remark 4.16. For a partition $\lambda \vdash k-2j$, we have the following dimension formula:

$$\dim D^{\lambda} = {}_kC_{2j}(2j-1)!! \cdot \dim S^{\lambda}.$$

This gives the multiplicity of $L^{\lambda}_{\mathrm{Sp}}$ in $H_{\mathbf{Q}}^{\otimes k}$.

5 Dynkin-Specht-Weyman's idempotent and the free Lie algebras

Let us consider the right action of $\mathfrak{S}_{k+2}$ on $H_{\mathbf{Q}}^{\otimes(k+2)}$. Set $\sigma_i := s_{i-1}s_{i-2}\cdots s_1$ for each $2 \leq i \leq k+2$, and

$$\theta_{k+2} := (1-\sigma_2)\cdots(1-\sigma_{k+2}) \in \mathbf{Q}\mathfrak{S}_{k+2}.$$

This element characterizes the degree $(k+2)$-nd part $\mathcal{L}^{\mathbf{Q}}_{2g}(k+2)$ of the free Lie algebra $\mathcal{L}^{\mathbf{Q}}_{2g}$ generated by $H_{\mathbf{Q}} = \mathbf{Q}^{2g}$ as follows. (e.g., [Ga, Theorem 2.1], [Re, Theorem 8.16], [Mo3, Lemma 4.5].)

Theorem 5.1 (Dynkin-Specht-Wever)**.**

(i) $\theta_{k+2}^2 = (k+2)\theta_{k+2}$*. We call an element* $\frac{1}{k+2}\theta_{k+2}$ *the Dynkin-Specht-Wever idempotent.*

(ii) For $v_1 \otimes v_2 \otimes \cdots \otimes v_{k+2} \in H_{\mathbf{Q}}^{\otimes k+2}$*, a left-normed element* $[v_1, v_2, \ldots, v_{k+2}] \in \mathcal{L}^{\mathbf{Q}}_{2g}(k+1)$ *coincides with* $(v_1 \otimes v_2 \otimes \cdots \otimes v_{k+2}) \cdot \theta_{k+2}$*. Hence the right action of* θ_{k+2} *on* $H_{\mathbf{Q}}^{\otimes k+2}$ *induces a projection* $H_{\mathbf{Q}}^{\otimes k+2} \to \mathcal{L}^{\mathbf{Q}}_{2g}(k+1)$*, and* $H_{\mathbf{Q}}^{\otimes k+2} \cdot \theta_{k+2}$ *is isomorphic to* $\mathcal{L}^{\mathbf{Q}}_{2g}(k+2)$*.*

(iii) For $v \in H_{\mathbf{Q}}^{\otimes(k+2)}$*, the following two conditions are equivalent;*

(a) $v \in \mathcal{L}^{\mathbf{Q}}_{2g}(k+2)$*,*

(b) $v \cdot \theta_{k+2} = (k+2)v$*.*

Recall that we need to consider the $\mathrm{Sp}(2g, \mathbf{Q})$-module

$$\mathfrak{h}^{\mathbf{Q}}_{g,1}(k) = \mathrm{Ker}(H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{2g}(k+1) \to \mathcal{L}^{\mathbf{Q}}_{2g}(k+2)).$$

To characterize $\mathfrak{h}^{\mathbf{Q}}_{g,1}(k)$ in $H_{\mathbf{Q}}^{\otimes k+2}$, let us consider a subgroup P of $\mathfrak{S}_{k+2}$ which fixes 1. Namely, P is isomorphic to $\mathfrak{S}_{k+1}$. Set

$$\theta_P := (1-s_2)(1-s_3s_2)\cdots(1-s_{k+1}s_k\cdots s_2).$$

We can regard this element in $\mathbf{Q}P$ as the Dynkin-Specht-Wever idempotent for P. Using this element, we obtain a characterization of $\mathfrak{h}^{\mathbf{Q}}_{g,1}(k)$ as the following theorem.

Proposition 5.2 ([Mo3, Proposition 4.6]). *For $v \in H_{\mathbf{Q}}^{\otimes(k+2)}$, the following two conditions are equivalent;*

(i) $v \in \mathfrak{h}_{g,1}^{\mathbf{Q}}(k)$,

(ii) $v \cdot \theta_P = (k+1)v$ *and* $v \cdot \sigma_{k+2} = v$.

Corollary 5.3. *We have*

$$\theta_P \cdot (1 + \sigma_{k+2} + \sigma_{k+2}^2 + \cdots + \sigma_{k+2}^{k+1}) \cdot \theta_P = (k+1)\theta_P \cdot (1 + \sigma_{k+2} + \sigma_{k+2}^2 + \cdots + \sigma_{k+2}^{k+1})$$

on $H_{\mathbf{Q}}^{\otimes k+2}$. Thus we obtain

$$v \cdot \theta_P(1 + \sigma_{k+2} + \sigma_{k+2}^2 + \cdots + \sigma_{k+2}^{k+1}) \in \mathfrak{h}_{g,1}^{\mathbf{Q}}(k)$$

for any $v \in H_{\mathbf{Q}}^{\otimes k+2}$.

Proof. Let us recall the following expansions of a left-normed element in the free Lie algebra:

$$[x_1, x_2, \ldots, x_m] = \sum (-1)^r x_{i_1} \otimes \cdots \otimes x_{i_r} \otimes x_1 \otimes x_{j_1} \otimes \cdots \otimes x_{j_{m-r-1}} \tag{6}$$

where the sum runs over all integers r and tuples $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_{m-r-1})$ of integers satisfying the conditions

$$0 \leq r \leq m-1, \quad m \geq i_1 > \cdots > i_r \geq 2, \quad 2 \leq j_1 < \cdots < j_{m-r-1} \leq m.$$

(See e.g., [Re, Lemma 1.1].) The expansion above is equivalent to

$$\sum (-1)^{r-1} x_{i_1} \otimes \cdots \otimes x_{i_r} \otimes x_2 \otimes x_{j_1} \otimes \cdots \otimes x_{j_{m-r-1}} \tag{7}$$

where the sum runs over all integers r and tuples $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_{m-r-1})$ of integers satisfying the conditions

$$0 \leq r \leq m-1, \quad m \geq i_1 > \cdots > i_r \geq 1, \quad 1 \leq j_1 < \cdots < j_{m-r-1} \leq m$$

and $i_1, \ldots, i_r, j_1, \ldots, j_{m-r-1} \neq 2$.

Note that $(v_1 \otimes \cdots \otimes v_{k+2}) \cdot \theta_P = v_1 \otimes [v_2, \ldots, v_{k+2}]$ for any $v_1, \ldots, v_{k+2} \in H_{\mathbf{Q}}$. To prove our statement, we shall prove

$$\begin{aligned} &(v_1 \otimes \cdots \otimes v_{k+2}) \cdot \theta_P \cdot (1 + \sigma + \cdots + \sigma^{k+1}) \\ = \; & v_1 \otimes [v_2, \ldots, v_{k+2}] - \sum_{j=2}^{k+2} v_j \otimes [[v_2, v_3, \ldots, v_{j-1}], [v_{j+1}, [v_{j+2}, \cdots, [v_{k+2}, v_1] \cdots]]]. \end{aligned} \tag{8}$$

In the formula above, the righthand side is contained in $H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)$. Therefore if (8) is true, by Theorem 5.1, we obtain our claim.

To prove the formula (8), we set

$$x_1 = [v_1, \dots, v_{j-1}],\ x_2 = v_j,\ x_3 = v_{j+1},\ \dots,\ x_{k+4-s} = v_{k+2}.$$

Then applying the formula (7), we expand $(v_1 \otimes \cdots \otimes v_{k+2}) \cdot \theta_P$ like as

$$v_1 \otimes \sum (-1)^{r-1} x_{i_1} \otimes \cdots \otimes x_{i_r} \otimes x_2 \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+3-s-r}}$$

satisfying the similar condition for (7). Hence, in $(v_1 \otimes \cdots \otimes v_{k+2}) \cdot \theta_P \cdot (1 + \sigma + \cdots + \sigma^{k+1})$, the terms which first part is equal to v_j are given by

$$v_j \otimes \sum (-1)^{r-1} x_{j_1} \otimes \cdots \otimes x_{j_{k+3-s-r}} \otimes v_1 \otimes x_{i_1} \otimes \cdots \otimes x_{i_r} \tag{9}$$

satisfying the conditions

$$0 \leq r \leq k+3-s, \quad 1 \leq j_1 < \cdots < j_{k+3-s-r} \leq k+2, \quad k+2 \geq i_1 > \cdots > i_r \geq 1$$

and i_1, …, i_r, j_1, …, $j_{k+3-s-r} \neq 2$.

On the other hand, note that the following expansion of a right-normed element in a free Lie algebra:

$$[x_1, [x_2, \cdots, [x_{m-1}, x_m] \cdots]] = \sum (-1)^r x_{j_1} \otimes \cdots \otimes x_{j_{m-r-1}} \otimes x_m \otimes x_{i_1} \otimes \cdots \otimes x_{i_r},$$

where the sum runs over all integers r, tuples $(i_1, \dots, i_r)$ and $(j_1, \dots, j_{m-r-1})$ of integers satisfying the conditions

$$0 \leq r \leq m-1, \quad m \geq i_1 > \cdots > i_r \geq 1, \quad 1 \leq j_1 < \cdots < j_{m-r-1} \leq m.$$

Applying this formula to (9), we obtain

$$-v_j \otimes [x_1, [x_2, \cdots, [x_{k+4-s}, v_1]]]$$

for $x_1 = [v_1, \dots, v_{j-1}]$, $x_2 = v_j$, $x_3 = v_{j+1}$, …, $x_{k+4-s} = v_{k+2}$. Thus we have the formula (8). $\square$

6 Multiplicities in $\mathrm{Res}_{\mathrm{Cyc}_k}^{\mathfrak{S}_k} S^\lambda$ via Kraśkiewicz-Weyman's combinatorial description

Let Cyc_k be a cyclic group of order k. Take a generator σ_k of Cyc_k and a primitive k-th root $\zeta_k \in \mathbf{C}$ of unity. In this section, we consider representations of the cyclic group Cyc_k

over an intermediate field $\mathbf{Q}(\zeta_k) \subset \mathbf{K} \subset \mathbf{C}$.

To begin with, we define one-dimensional representations (or characters) $\chi_k^j : \mathrm{Cyc}_k \to \mathbf{K}^\times$ by $\chi_k^j(\sigma_k) = \zeta_k^j$ for $0 \leq j \leq k-1$. Especially, we denote the trivial representation χ_k^0 by $\mathbf{triv}_k$. The set of isomorphism classes of irreducible representations of Cyc_k is given by $\{\chi_k^j,\ 0 \leq j \leq k-1\}$. Consider Cyc_k as a subgroup of $\mathfrak{S}_k$ by an embedding $\sigma_k^i \mapsto (1\,2\,\cdots\,k)^i$ for $0 \leq i \leq k-1$. Let us recall Kraśkiewicz-Weyman's combinatorial description for the branching rules of irreducible $\mathfrak{S}_k$-modules S^λ to the cyclic subgroup Cyc_k. To do this, first we define a major index of a standard tableau.

Definition 6.1. For a standard tableau T, we define the descent set of T to be the set of entries i in T such that $i+1$ is located in a lower row than that which i is located. We denote by $D(T)$ the descent set of T. The major index of T is defined by

$$\mathrm{maj}(T) := \sum_{i \in D(T)} i.$$

If $D(T) = \phi$, we set $\mathrm{maj}(T) = 0$.

Theorem 6.2 ([KW], [Re, Theorem 8.8, 8.9], [Ga, Theorem 8.4])**.** *The multiplicity of χ_k^j in $\mathrm{Res}_{\mathrm{Cyc}_k}^{\mathfrak{S}_k} S^\lambda$ is equal to the number of standard tableaux with shape λ satisfying $\mathrm{maj}(T) \equiv j$ modulo k.*

Example 6.3. For $k \geq 2$, we have the following table on the multiplicities of $\mathbf{triv}_k = \chi_j^0$

and χ_j^1.

λ	T	major index	mult. of $\mathbf{triv}_m$	mult. of χ_m^1
(m)	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 2 & \cdots & m \\ \hline \end{array}$	0	1	0
$(m-1,1)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 2 & \cdots & m \\ \hline p \\ \cline{1-1} \end{array}$ $(2 \leq p \leq m)$	$p-1$	0	1
(1^m)	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline m \\ \hline \end{array}$	$\frac{m(m-1)}{2} \equiv \begin{cases} 0, & m : \text{odd} \\ -\frac{m}{2}, & m : \text{even} \end{cases}$	$\begin{cases} 1, & m : \text{odd} \\ 0, & m : \text{even} \end{cases}$	$\begin{cases} 1, & m = 2 \\ 0, & m \neq 2 \end{cases}$
$(2,1^{m-2})$	$\begin{array}{\|c\|c\|}\hline 1 & p \\ \hline 2 \\ \cline{1-1} \vdots \\ \cline{1-1} m \\ \cline{1-1} \end{array}$ $(2 \leq p \leq m)$	$\frac{m(m-1)}{2} - (p-1) \equiv \begin{cases} 1-p, & m : \text{odd} \\ 1-p-\frac{m}{2}, & m : \text{even} \end{cases}$	$\begin{cases} 1, & m : \text{even} \\ 0, & m : \text{odd} \end{cases}$	$\begin{cases} 1, & m \neq 2 \\ 0, & m = 2 \end{cases}$

Example 6.4. For $m \geq 3$ and a partition $\lambda = (m-2, 1^2)$, we have

(i) $[\mathbf{triv}_m : \mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m} S^\lambda] = \begin{cases} (m-2)/2 & \text{if } m : \text{even}, \\ (m-1)/2 & \text{if } m : \text{odd}. \end{cases}$

(ii) $[\chi^1_m : \mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m} S^\lambda] = \begin{cases} (m-3)/2 & \text{if } m : \text{odd}, \\ (m-2)/2 & \text{if } m : \text{even}. \end{cases}$

In fact, for a partition

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & m \\ \hline p \\ \cline{1-1} q \\ \cline{1-1} \end{array} ,$$

its major index is given by $\mathrm{maj}(T) = p + q - 2$ for $2 \leq p < q \leq m$. Then $\mathrm{maj}(T) \equiv 0 \pmod{m}$ if and only if $p + q = m + 2$. Hence we have the number of standard tableaux of shape λ is equal to $\frac{m}{2} - 1$ for odd m and $\frac{m-1}{2}$ for even m. On the other hand, $\mathrm{maj}(T) \equiv 1 \pmod{m}$ if and only if $p + q = m + 3$. Hence the number of standard tableaux of shape λ is equal to $\frac{m-3}{2}$ for odd m and $\frac{m-2}{2}$ for even m.

Example 6.5. For $m \geq 4$ and a partition $\lambda = (2^2, 1^{m-4})$, we have

$$[\chi_m^1 : \mathrm{Res}_{\mathrm{Cyc}_m}^{\mathfrak{S}_m} S^\lambda] = \begin{cases} \frac{m-3}{2} & \text{if } m \text{ is odd,} \\ \frac{m-4}{2} & \text{if } m \equiv 0 \pmod 4, \\ \frac{m-2}{2} & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

To prove this, we consider the following two kind of standard tableaux of shape λ:

$$T_{p,q} = \begin{array}{|c|c|} \hline 1 & p \\ \hline 2 & q \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} m & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad (2 \leq p < p+1 < q \leq m), \quad T_p = \begin{array}{|c|c|} \hline 1 & p \\ \hline 2 & p+1 \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} m & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad (3 \leq p \leq m-1).$$

Their major indices are given by

$$\mathrm{maj}(T_{p,q}) = \frac{m(m-1)}{2} + 2 - p - q \quad \text{and} \quad \mathrm{maj}(T_p) = \frac{m(m-1)}{2} + 1 - p.$$

If m is odd, $\frac{m(m-1)}{2} \equiv 0 \pmod m$. Thus $\mathrm{maj}(T_{p,q}) \equiv 1 \pmod m$ if and only if $p+q = m+1$. The number of such (p,q)s is $\frac{m-3}{2}$. There is no T_p such that $\mathrm{maj}(T_p) \equiv 1 \pmod m$. If m is even, $\frac{m(m-1)}{2} \equiv \frac{m}{2} \pmod m$. Since $m \neq 2$, $\mathrm{maj}(T_p) \equiv 1 \pmod m$ if and only if $p = \frac{m}{2}$ for $m > 4$. If $m = 4$, there is no such T_p.

On the other hand, $\mathrm{maj}(T_{p,q}) \equiv 1 \pmod m$ if and only if $p+q = m+1+\frac{m}{2}$ for $m = 4, 6, 8$ and $p+q = m+1+\frac{m}{2}$, or $1+\frac{m}{2}$ for $m \geq 10$. If $m = 4$, 6 or 8, the number of such (p,q)s is 0, 1 or 1 respectively. Suppose $m \geq 10$. If $m = 4M$, $\mathrm{maj}(T_{p,q}) \equiv 1 \pmod m$ if and only if $p+q = 6M+1$ or $2M+1$. The number of such (p,q)s is $(M-1)+(M-2) = 2M-3 = \frac{m}{2}-3$. If $m = 4M+2$, $\mathrm{maj}(T_{p,q}) \equiv 1 \pmod m$ if and only if $p+q = 6M+4$ or $2M+2$. The number of such (p,q)s is $M+(M-1) = 2M-1 = \frac{m}{2}-2$. Therefore we obtain the claim.

7 Sp-irreducible components of the Johnson cokernels

7.1 Our strategy for detecting Sp-irreducible components

In the rest of this paper, we assume $g \geq k+2$. To explain our strategy for detecting Sp-irreducible components in the Johnson cokernel of the mapping class group, let us recall the following diagram as mentioned above:

$$\begin{tikzcd}
 & \mathrm{Im}\,\tau'_{k,\mathbf{Q}} \arrow[rr, hook] & & H^*_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{2g}(k+1) \arrow[r, two heads] & H^{\otimes k}_{\mathbf{Q}} \arrow[r, two heads] & \mathcal{C}^{\mathbf{Q}}_{2g}(k) \\
\mathrm{Im}\,\tau^{\mathcal{M}}_{k,\mathbf{Q}} \arrow[r, equal] & \mathrm{Im}\,\tau'^{\mathcal{M}}_{k,\mathbf{Q}} \arrow[r, hook] \arrow[u, hook] & \mathfrak{h}^{\mathbf{Q}}_{g,1}(k) \arrow[r, hook] & H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{2g}(k+1) \arrow[rr, two heads] \arrow[u, "\wr"'] & & \mathcal{L}^{\mathbf{Q}}_{2g}(k+1)
\end{tikzcd}$$

Here we may regard it as a diagram of $\mathrm{Sp}(2g, \mathbf{Q})$-modules and $\mathrm{Sp}(2g, \mathbf{Q})$-equivariant homomorphisms. By Theorem 3.4, we see $\mathrm{Coker}(\mathrm{Im}\, \tau'_{k,\mathbf{Q}} \hookrightarrow H^*_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{2g}(k+1))$ coincides with $\mathcal{C}^{\mathbf{Q}}_{2g}(k)$ for $2g \geq k+2$. Observing a natural isomorphism $H^* \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{2g}(k+1) \cong H \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{2g}(k+1)$ induced from the Poincaré duality, we obtain $\mathrm{Sp}(2g, \mathbf{Q})$-equivariant homomorphism $c_k : \mathfrak{h}^{\mathbf{Q}}_{g,1}(k) \to \mathcal{C}^{\mathbf{Q}}_{2g}(k)$. Note that $\mathrm{Im}\, \tau'^{\mathcal{M}}_{k,\mathbf{Q}} \subset \mathrm{Im}\, \tau'_{k,\mathbf{Q}}$. Then we have the following criterion for detecting Sp-irreducible components in the Johnson cokernel $\mathrm{Coker}(\mathrm{Im}\, \tau'^{\mathcal{M}}_{k,\mathbf{Q}} \to \mathfrak{h}^{\mathbf{Q}}_{g,1}(k))$.

Proposition 7.1. *Let V be an irreducible $\mathrm{Sp}(2g, \mathbf{Q})$-submodule of $\mathfrak{h}^{\mathbf{Q}}_{g,1}(k)$. If $c_k(V)$ is a non-trivial (then automatically irreducible) component of $\mathcal{C}^{\mathbf{Q}}_{2g}(k)$, then V is an irreducible $\mathrm{Sp}(2g, \mathbf{Q})$-module in $\mathrm{Coker}(\mathrm{Im}\, \tau'^{\mathcal{M}}_{k,\mathbf{Q}})$. In particular, if there is a maximal vector v of weight λ in $\mathfrak{h}^{\mathbf{Q}}_{g,1}(k)$ such that $c_k(v) \neq 0$ (then $c_k(v)$ is a maximal in $\mathcal{C}^{\mathbf{Q}}_{2g}(k)$), then v gives an $\mathrm{Sp}(2g, \mathbf{Q})$-irreducible component in $\mathrm{Coker}(\mathrm{Im}\, \tau'^{\mathcal{M}}_{k,\mathbf{Q}})$ which is isomorphic to the irreducible $\mathrm{Sp}(2g, \mathbf{Q})$-module $L^{[\lambda]}_{\mathrm{Sp}}$.*

To find such a maximal vector, we use Theorem 4.14 and Corollary 5.3. Namely, for a maximal vector v_λ as in Theorem 4.14, we consider $\phi_\lambda := v_\lambda \cdot \theta_P \cdot (1 + \sigma_{k+2} + \cdots + \sigma^{k+2}_{k+1})$. If $\phi_\lambda \neq 0$, this is a maximal vector of weight λ such that $\phi_\lambda \in \mathfrak{h}^{\mathbf{Q}}_{g,1}(k)$ by Corollary 5.3. Then we investigate whether $c_k(\phi_\lambda) \in \mathcal{C}^{\mathbf{Q}}_{2g}(k)$ is 0 or not.

7.2 Some multiplicity formulae

In this subsection, we give some explicit multiplicity formulae for $[k]$ and $[1^k]$ in $\mathfrak{h}^{\mathbf{Q}}_{g,1}(k)$ and $\mathcal{C}^{\mathbf{Q}}_{2g}(k)$. First, let us recall the multiplicity formulae in our previous paper [ES].

Proposition 7.2. *Suppose $n \geq k+2$.*

(i) For a partition λ of k,

$$[L^{\lambda}_{\mathrm{GL}} : \mathcal{C}^{\mathbf{Q}}_{n}(k)] = [\mathbf{triv}_k : \mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^{\lambda}].$$

(ii) For a partition λ of $k+2$,

$$[L^{\lambda}_{\mathrm{GL}} : \mathcal{L}^{\mathbf{Q}}_{n}(k+2)] = [\chi^1_k : \mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^{\lambda}].$$

(iii) For a partition λ of $k+2$,

$$[L^{\lambda}_{\mathrm{GL}} : H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}^{\mathbf{Q}}_{n}(k+1)] = \sum_{\mu} [L^{\lambda}_{\mathrm{GL}} : \mathcal{L}^{\mathbf{Q}}_{n}(k+1)]$$

where μ runs over all partitions obtained by removing a single node.

Proposition 7.3.

(i) *The multiplicities of the* $\mathrm{Sp}(2g,\mathbf{Q})$*-irreducible representation* $[k]$ *in* $\mathfrak{h}_{g,1}^{\mathbf{Q}}(k)$ *and* $\mathcal{C}_{2g}^{\mathbf{Q}}(k)$ *are given by*

$$[L_{\mathrm{Sp}}^{[k]} : \mathfrak{h}_{g,1}^{\mathbf{Q}}(k)] = \begin{cases} 1 & \text{if} \quad k : \text{odd}, \\ 0 & \text{if} \quad k : \text{even}, \end{cases} \quad [L_{\mathrm{Sp}}^{[k]} : \mathcal{C}_{2g}^{\mathbf{Q}}(k)] = 1.$$

(ii) *The multiplicities of the* $\mathrm{Sp}(2g,\mathbf{Q})$*-irreducible representation* $[1^k]$ *in* $\mathfrak{h}_{g,1}^{\mathbf{Q}}(k)$ *and* $\mathcal{C}_{2g}^{\mathbf{Q}}(k)$ *are given by*

$$[L_{\mathrm{Sp}}^{[1^k]} : \mathfrak{h}_{g,1}^{\mathbf{Q}}(k)] = \begin{cases} 1 & \text{if} \ \ k \equiv 1,2 \pmod 4, \\ 0 & \text{if} \ \ \text{otherwise}, \end{cases} \quad [L_{\mathrm{Sp}}^{[1^k]} : \mathcal{C}_{2g}^{\mathbf{Q}}(k)] = \begin{cases} 1 & \text{if} \ \ k : \text{odd}, \\ 0 & \text{if} \ \ k : \text{even}. \end{cases}$$

Proof. We will use irreducible decompositions of the restriction $\mathrm{Res}_{\mathrm{Sp}}^{\mathrm{GL}}$ (See Theorem 4.5.) and Pier's rule (See Theorem 4.4.).

(i) If $\mathrm{Res}_{\mathrm{Sp}(2g,\mathbf{Q})}^{\mathrm{GL}(2g,\mathbf{Q})} L_{\mathrm{GL}}^{(\lambda)}$ has an Sp-irreducible component $L_{\mathrm{Sp}}^{[k]}$, then a partition λ is either $\lambda = (k+1,1)$ or $(k,1^2)$. We have

$$\begin{aligned} [L_{\mathrm{GL}}^{(k+1,1)} : H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] &= [L_{\mathrm{GL}}^{(k+1)} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] + [L_{\mathrm{GL}}^{(k,1)} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] = 1, \\ [L_{\mathrm{GL}}^{(k+1,1)} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+2)] &= 1, \end{aligned}$$

$$\begin{aligned} [L_{\mathrm{GL}}^{(k,1^2)} : H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] &= [L_{\mathrm{GL}}^{(k-1,1^2)} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] + [L_{\mathrm{GL}}^{(k,1)} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)], \\ &= \begin{cases} \frac{k-2}{2}+1 & \text{if} \ \ k : \text{even}, \\ \frac{k-1}{2}+1 & \text{if} \ \ k : \text{odd}, \end{cases} \\ [L_{\mathrm{GL}}^{(k,1^2)} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+2)] &= \begin{cases} \frac{k}{2} & \text{if} \ \ k : \text{even}, \\ \frac{k-1}{2} & \text{if} \ \ k : \text{odd}, \end{cases} \\ [L_{\mathrm{Sp}}^{[k]} : \mathcal{C}_{2g}^{\mathbf{Q}}(k)] &= [L_{\mathrm{GL}}^{(k)} : \mathcal{C}_{2g}^{\mathbf{Q}}(k)] = 1. \end{aligned}$$

Thus we obtain the claim.

(ii) If $\mathrm{Res}_{\mathrm{Sp}(2g,\mathbf{Q})}^{\mathrm{GL}(2g,\mathbf{Q})} L_{\mathrm{GL}}^{(\lambda)}$ has an Sp-irreducible component $L_{\mathrm{Sp}}^{[1^k]}$, then a partition λ is either $\lambda = (2^2,1^{k-2})$, $(2,1^k)$ or (1^{k+2}). We have

$$\begin{aligned} [L_{\mathrm{GL}}^{(1^{k+2})} : H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] &= [L_{\mathrm{GL}}^{(1^{k+1})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] = 0, \\ [L_{\mathrm{GL}}^{(1^{k+2})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+2)] &= 0, \\ [L_{\mathrm{GL}}^{(2,1^k)} : H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] &= [L_{\mathrm{GL}}^{(1^{k+1})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] + [L_{\mathrm{GL}}^{(2,1^{k-1})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] = 1, \\ [L_{\mathrm{GL}}^{(2,1^k)} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+2)] &= 1, \\ [L_{\mathrm{Sp}}^{[1^k]} : \mathcal{C}_{2g}^{\mathbf{Q}}(k)] &= [L_{\mathrm{GL}}^{(1^k)} : \mathcal{C}_{2g}^{\mathbf{Q}}(k)] = \begin{cases} 1 & \text{if} \quad k : \text{odd}, \\ 0 & \text{if} \quad k : \text{even}. \end{cases} \end{aligned}$$

Suppose $k \equiv 1, 3 \pmod 4$. Then

$$\begin{aligned}
[L_{\mathrm{GL}}^{(2^2,1^{k-2})} : H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] &= [L_{\mathrm{GL}}^{(2^2,1^{k-3})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] + [L_{\mathrm{GL}}^{(2,1^{k-1})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)], \\
&= \begin{cases} \frac{k-1}{2} & \text{if} \quad k \equiv 3 \pmod 4, \\ \frac{k+1}{2} & \text{if} \quad k \equiv 1 \pmod 4, \end{cases} \\
[L_{\mathrm{GL}}^{(2^2,1^{k-2})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+2)] &= \frac{k-1}{2}.
\end{aligned}$$

Suppose $k \equiv 0, 2 \pmod 4$. Then

$$\begin{aligned}
[L_{\mathrm{GL}}^{(2^2,1^{k-2})} : H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] &= [L_{\mathrm{GL}}^{(2^2,1^{k-3})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)] + [L_{\mathrm{GL}}^{(2,1^{k-1})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+1)], \\
&= \frac{k}{2}, \\
[L_{\mathrm{GL}}^{(2^2,1^{k-2})} : \mathcal{L}_{2g}^{\mathbf{Q}}(k+2)] &= \begin{cases} \frac{k-2}{2} & \text{if} \quad k \equiv 2 \pmod 4, \\ \frac{k}{2} & \text{if} \quad k \equiv 0 \pmod 4. \end{cases}
\end{aligned}$$

Hence we obtain the claim.

□

Remark 7.4. By the argument above, the Sp-irreducible component $[1^k]_{\mathrm{Sp}}$ appears in the restriction of the GL-irreducible component $(2^2, 1^{k-2})_{\mathrm{GL}}$.

Remark 7.5. Our calculation above gives a combinatorial description of the GL (and Sp) irreducible decomposition of $\mathfrak{h}_{g,1}^{\mathbf{Q}}$ obtained by Kontsevich in [Kon1] and [Kon2].

7.3 Descriptions of maximal vectors

To give an explicit description of maximal vectors, we use an (i, j)-expansion operator $D_{ij} : H_{\mathbf{Q}}^{\otimes k} \to H_{\mathbf{Q}}^{\otimes (k+2)}$ defined by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_k) \cdot D_{ij} := \sum_{r=1}^{2g} v_1 \otimes \cdots \otimes v_{i-1} \otimes e_r \otimes v_i \otimes \cdots \otimes v_{j-2} \otimes e_r^* \otimes v_{j-1} \otimes \cdots \otimes v_k$$

for $1 \leq i < j \leq k+2$. Using this, we obtain several maximal vectors satisfying the condition of Proposition 7.1. First we consider a maximal vector which defines the Morita obstruction $[k]$ in $\mathrm{Coker}(\mathrm{Im}\, \tau_{k,\mathbf{Q}}^{\mathcal{M}})$.

Theorem 7.6 (Morita)**.** *Let k be an odd integer such that $k \geq 3$. Suppose $g \geq k+2$. An element*

$$\begin{aligned}
\varphi_{[k]} &:= (\omega \otimes e_1^{\otimes k}) \cdot \theta_P \cdot (1 + \sigma_{k+2} + \cdots + \sigma_{k+2}^{k+1}) \\
&= 2 \left(\sum_{i=1}^{k+1} \sum_{r=1}^{k-i+2} (-1)^{r-1} {}_k C_{r-1} (e_1^{\otimes k}) \cdot D_{i,i+r} \right).
\end{aligned}$$

is a maximal vector with highest weight $[k]$ *in* $\mathfrak{h}_{g,1}^{\mathbf{Q}}(k)$*. Moreover this gives a unique irreducible component of* $[k]$ *in* $\operatorname{Coker}\tau_{k,\mathbf{Q}}^{\mathcal{M}}$*.*

Second we consider a maximal vector which defines the $\mathrm{Sp}(2g,\mathbf{Q})$-module with highest weight $[1^k]$ in $\operatorname{Coker}(\operatorname{Im}\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ for $k \equiv 1 \pmod 4$ and $k \geq 5$.

Theorem 7.7. *Suppose* $k \equiv 1 \pmod 4$*,* $k \geq 5$ *and* $g \geq k+2$*. An element*

$$\begin{aligned}
\varphi_{[1^k]} &:= (\omega \otimes (e_1 \wedge \cdots \wedge e_k)) \cdot \theta_P \cdot (1 + \sigma_{k+2} + \cdots + \sigma_{k+2}^{k+1}) \\
&= 2\left(\sum_{i=1}^{k+1}\sum_{r=1}^{k-i+2}(-1)^{\delta_{r\equiv 2,3 \,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{r-1}{2}\rfloor}(e_1 \wedge \cdots \wedge e_k)\cdot D_{i,i+r}\right)
\end{aligned}$$

is a maximal vector with highest weight $[1^k]$ *in* $\mathfrak{h}_{g,1}^{\mathbf{Q}}(k)$*. Moreover this gives a unique irreducible component of* $[1^k]$ *in* $\operatorname{Coker}\tau_{k,\mathbf{Q}}^{\mathcal{M}}$*.*

7.4 Proofs of main theorems

We will give proofs of Theorem 7.6 and Theorem 7.7. But, since our proof for Theorem 7.6 is easier than that of Theorem 7.7, we omit the details for Theorem 7.6.

7.4.1 Proof of Theorem 7.7

Step.1 For $r \equiv 2 \pmod 4$, we prove

$$\begin{aligned}
&(e_1 \wedge \cdots \wedge e_k)D_{12}(1-s_2)(1-s_3s_2)\cdots(1-s_r\cdots s_3s_2) \\
=\ &\sum_{j=1}^{r}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{r-2}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}(e_1 \wedge \cdots \wedge e_k)D_{1,1+j}
\end{aligned}$$

by the induction on r.

Indeed, if $p=2$, the both side of the formula above coincide with $(e_1\wedge\cdots\wedge e_k)(D_{12}-D_{13})$. Suppose $p>2$ and $p+4 \leq k+1$. For simplicity we denote $(e_1 \wedge \cdots \wedge e_k)D_{ij}$ by $D_{i,j}^{\mathrm{sgn}}$. We have

$$\begin{aligned}
& D^{\mathrm{sgn}}_{1,1+j}(1-s_{p+1}\cdots s_2)(1-s_{p+2}\cdots s_2)(1-s_{p+3}\cdots s_2)(1-s_{p+4}\cdots s_2) \\
= \;& (D^{\mathrm{sgn}}_{1,1+j}-(-1)^{p+1}D^{\mathrm{sgn}}_{1,2+j})(1-s_{p+2}\cdots s_2)(1-s_{p+3}\cdots s_2)(1-s_{p+4}\cdots s_2) \\
\overset{p:even}{=} \;& (D^{\mathrm{sgn}}_{1,1+j}+D^{\mathrm{sgn}}_{1,2+j})(1-s_{p+2}\cdots s_2)(1-s_{p+3}\cdots s_2)(1-s_{p+4}\cdots s_2) \\
= \;& (D^{\mathrm{sgn}}_{1,1+j}+D^{\mathrm{sgn}}_{1,2+j}-(-1)^{p+2}D^{\mathrm{sgn}}_{1,2+j}-(-1)^{p+2}D^{\mathrm{sgn}}_{1,3+j})(1-s_{p+3}\cdots s_2)(1-s_{p+4}\cdots s_2) \\
\overset{p:even}{=} \;& (D^{\mathrm{sgn}}_{1,1+j}-D^{\mathrm{sgn}}_{1,3+j})(1-s_{p+3}\cdots s_2)(1-s_{p+4}\cdots s_2) \\
= \;& (D^{\mathrm{sgn}}_{1,1+j}-D^{\mathrm{sgn}}_{1,3+j}-(-1)^{p+3}D^{\mathrm{sgn}}_{1,2+j}+(-1)^{p+3}D^{\mathrm{sgn}}_{1,4+j})(1-s_{p+4}\cdots s_2) \\
\overset{p:even}{=} \;& (D^{\mathrm{sgn}}_{1,1+j}+D^{\mathrm{sgn}}_{1,2+j}-D^{\mathrm{sgn}}_{1,3+j}-D^{\mathrm{sgn}}_{1,4+j})(1-s_{p+4}\cdots s_2) \\
= \;& D^{\mathrm{sgn}}_{1,1+j}+D^{\mathrm{sgn}}_{1,2+j}-D^{\mathrm{sgn}}_{1,3+j}-D^{\mathrm{sgn}}_{1,4+j}-(-1)^{p+4}(D^{\mathrm{sgn}}_{1,2+j}+D^{\mathrm{sgn}}_{1,3+j}-D^{\mathrm{sgn}}_{1,4+j}-D^{\mathrm{sgn}}_{1,5+j}) \\
\overset{p:even}{=} \;& D^{\mathrm{sgn}}_{1,1+j}-2D^{\mathrm{sgn}}_{1,3+j}+D^{\mathrm{sgn}}_{1,5+j}.
\end{aligned}$$

Therefore, the action of $(1-s_{p+1}\cdots s_2)(1-s_{p+2}\cdots s_2)(1-s_{p+3}\cdots s_2)(1-s_{p+4}\cdots s_2)$ on $\sum_{j=1}^{r}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{r-2}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}D^{\mathrm{sgn}}_{1,1+j}$ is obtained by the following way:

$$\begin{aligned}
& \sum_{j=1}^{r}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{r-2}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}(D^{\mathrm{sgn}}_{1,1+j}-2D^{\mathrm{sgn}}_{1,3+j}+D^{\mathrm{sgn}}_{1,5+j}) \\
= \;& \sum_{j=5}^{p}\left\{(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{p-2}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}-2(-1)^{\delta_{j\equiv 0,1\,(\mathrm{mod}\,4)}}\,{}_{\frac{p-2}{2}}C_{\lfloor\frac{j-3}{2}\rfloor}+(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{p-2}{2}}C_{\lfloor\frac{j-5}{2}\rfloor}\right\}D^{\mathrm{sgn}}_{1,1+j} \\
& +D^{\mathrm{sgn}}_{12}-D^{\mathrm{sgn}}_{13}-\frac{p-2}{2}D^{\mathrm{sgn}}_{14}+\frac{p-2}{2}D^{\mathrm{sgn}}_{15}-2(D^{\mathrm{sgn}}_{14}-D^{\mathrm{sgn}}_{15}+D^{\mathrm{sgn}}_{1,p+2}-D^{\mathrm{sgn}}_{1,p+3}) \\
& -\frac{p-2}{2}D^{\mathrm{sgn}}_{1,p+2}+\frac{p-2}{2}D^{\mathrm{sgn}}_{1,p+3}+D^{\mathrm{sgn}}_{1,p+4}-D^{\mathrm{sgn}}_{1,p+5} \\
= \;& \sum_{j=1}^{p+4}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{p+2}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}D^{\mathrm{sgn}}_{1,1+j}.
\end{aligned}$$

Step.2 We have

$$\begin{aligned}
& (e_1\wedge\cdots\wedge e_k)D_{ij}s_{k+1}\cdots s_2 s_1 \\
& \quad = \begin{cases} (e_1\wedge\cdots\wedge e_k)(-1)^{k-1}D_{i+1,j+1} \overset{k:even}{=} (e_1\wedge\cdots\wedge e_k)D_{i+1,j+1} & \text{if} \quad j\neq k+2, \\ -(e_1\wedge\cdots\wedge e_k)D_{1,i+1} & \text{if} \quad j=k+2 \end{cases}
\end{aligned}$$

for $k\equiv 1\ (\mathrm{mod}\,4)$. Hence we obtain an explicit formula

$$\begin{aligned}
& (\omega\otimes(e_1\wedge\cdots\wedge e_k))\cdot\theta_P\cdot(1+\sigma_{k+2}+\cdots+\sigma_{k+2}^{k+1}) \\
& \quad = 2\sum_{i=1}^{k+1}\sum_{j=1}^{k-i+2}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}(e_1\wedge\cdots\wedge e_k)\cdot D_{i,i+j}.
\end{aligned}$$

In fact,

$$
\begin{aligned}
&\sum_{j=1}^{k+1}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}D^{\mathrm{sgn}}_{1,1+j}(1+\sigma_{k+2}+\cdots+\sigma_{k+2}^{k+1})\\
=\;&\sum_{j=1}^{k+1}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}\left(\sum_{i=1}^{k+2-j}D^{\mathrm{sgn}}_{i,i+j}-\sum_{i=1}^{j}D^{\mathrm{sgn}}_{i,i+k+2-j}\right)\\
=\;&\sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}D^{\mathrm{sgn}}_{i,i+j}-\sum_{i=1}^{k+1}\sum_{j=i}^{k+1}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}D^{\mathrm{sgn}}_{i,i+k+2-j}\\
=\;&\sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}D^{\mathrm{sgn}}_{i,i+j}-\sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i}(-1)^{\delta_{k+2-j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{k+1-j}{2}\rfloor}D^{\mathrm{sgn}}_{i,i+j}\\
=\;&\sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}D^{\mathrm{sgn}}_{i,i+j}+\sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{k+1-j}{2}\rfloor}D^{\mathrm{sgn}}_{i,i+j}\\
=\;&2\sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i}(-1)^{\delta_{j\equiv 2,3\,(\mathrm{mod}\,4)}}\,{}_{\frac{k-1}{2}}C_{\lfloor\frac{j-1}{2}\rfloor}D^{\mathrm{sgn}}_{i,i+j}.
\end{aligned}
$$

Step.3 Let us consider a surjective Sp-homomorphism

$$\mathrm{cont}_k : H_{\mathbf{Q}}^{\otimes(k+2)}\xrightarrow{\sim}H_{\mathbf{Q}}^*\otimes H_{\mathbf{Q}}^{\otimes(k+1)}\twoheadrightarrow H_{\mathbf{Q}}^{\otimes k}$$

by composing an Sp-isomorphism $H_{\mathbf{Q}}^{\otimes(k+2)}\to H_{\mathbf{Q}}^*\otimes H_{\mathbf{Q}}^{\otimes(k+1)}$ induced from $H_{\mathbf{Q}}\xrightarrow{\sim}H_{\mathbf{Q}}^*$ given by (3) and a contraction homomorphism. Then we obtain

$$\mathrm{cont}_k((e_1\wedge\cdots\wedge e_k)D_{ij})=\begin{cases}(-2g)(e_1\wedge\cdots\wedge e_k) & \text{if}\quad i=1,\ j=2,\\ (-1)^{j-2}(e_1\wedge\cdots\wedge e_k) & \text{if}\quad i=1,\ j\geq 3,\\ (-1)^{j-3}(e_1\wedge\cdots\wedge e_k) & \text{if}\quad i=2,\ j\geq 3,\\ 0 & \text{if}\quad \text{otherwise.}\end{cases}$$

To prove these formulae, let us recall that

$$\langle e_i,e_j\rangle=0=\langle e_{i'},e_{j'}\rangle,\quad \langle e_i,e_{j'}\rangle=\delta_{ij}=-\langle e_{j'},e_i\rangle,\quad (1\leq i\leq g).$$

and

$$e_i^*=\begin{cases}e_{i'}, & (1\leq i\leq g),\\ -e_{i'}, & (g+1\leq i\leq 2g).\end{cases}$$

where $i' := 2g - i + 1$ for each integer $1 \leq i \leq 2g$.

Then we have

$$\begin{aligned}
\mathrm{cont}_k(D_{12}^{\mathrm{sgn}}) &= \mathrm{cont}_k\left(\sum_{r=1}^{2g} e_r \otimes e_r^* \otimes (e_1 \wedge \cdots \wedge e_k)\right) \\
&= \sum_{r=1}^{2g} \langle e_r^*, e_r\rangle e_1 \wedge \cdots \wedge e_k = (-2g) e_1 \wedge \cdots \wedge e_k.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathrm{cont}_k(D_{1j}^{\mathrm{sgn}}) &= \mathrm{cont}_k\left(\sum_{r=1}^{2g}\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma) e_r \otimes e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes \overset{\overset{j}{\vee}}{e_r^*} \otimes \cdots \otimes e_{\sigma(k)}\right) \\
&= \sum_{r=1}^{2g}\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma)\langle e_{\sigma(1)}, e_r\rangle \otimes e_{\sigma(2)} \otimes \cdots \otimes \overset{\overset{j-2}{\vee}}{e_r^*} \otimes \cdots \otimes e_{\sigma(k)} \\
&= \sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma) e_{\sigma(2)} \otimes \cdots \otimes \overset{\overset{j-2}{\vee}}{e_{\sigma(1)'}^*} \otimes \cdots \otimes e_{\sigma(k)} \\
&= -\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma) e_{\sigma(2)} \otimes \cdots \otimes \overset{\overset{j-2}{\vee}}{e_{\sigma(1)}} \otimes \cdots \otimes e_{\sigma(k)} \\
&= -(e_1 \wedge \cdots \wedge e_k) \cdot s_1 s_2 \cdots s_{j-3} \\
&= (-1)^{j-2} e_1 \wedge \cdots \wedge e_k,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\mathrm{cont}_k(D_{2j}^{\mathrm{sgn}}) &= \mathrm{cont}_k\left(\sum_{r=1}^{2g}\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma) e_{\sigma(1)} \otimes e_r \otimes e_{\sigma(2)} \otimes \cdots \otimes \overset{\overset{j}{\vee}}{e_r^*} \otimes \cdots \otimes e_{\sigma(k)}\right) \\
&= \sum_{r=1}^{2g}\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma)\langle e_r, e_{\sigma(1)}\rangle \otimes e_{\sigma(2)} \otimes \cdots \otimes \overset{\overset{j-2}{\vee}}{e_r^*} \otimes \cdots \otimes e_{\sigma(k)} \\
&= \sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma) e_{\sigma(2)} \otimes \cdots \otimes \overset{\overset{j-2}{\vee}}{e_{\sigma(1)'}^*} \otimes \cdots \otimes e_{\sigma(k)} \\
&= \sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma) e_{\sigma(2)} \otimes \cdots \otimes \overset{\overset{j-2}{\vee}}{e_{\sigma(1)}} \otimes \cdots \otimes e_{\sigma(k)} \\
&= (e_1 \wedge \cdots \wedge e_k) \cdot s_1 s_2 \cdots s_{j-3} \\
&= (-1)^{j-3} e_1 \wedge \cdots \wedge e_k.
\end{aligned}$$

For $i \geq 3$, because of $g > k$, it is clear that $\mathrm{cont}_k((e_1 \wedge \cdots \wedge e_k)D_{ij}) = 0$.

Step.4 We obtain $c_k(\varphi_{[1^k]}) \neq 0$.

Indeed, for the natural surjection $\mathrm{pr} : H_{\mathbf{Q}}^{\otimes k} \to \mathcal{C}_{2g}^{\mathbf{Q}}(k)$, we have

$$\begin{aligned}
& c(\varphi_{[1^k]}) \\
= \;& 2\left(\begin{array}{l} \displaystyle\sum_{j=1}^{k+1} (-1)^{\delta_{j\equiv 2,3 \pmod 4}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} c_k(e_1 \wedge \cdots \wedge e_k D_{1,1+j}) \\ \displaystyle+\sum_{j=1}^{k} (-1)^{\delta_{j\equiv 2,3 \pmod 4}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} c_k(e_1 \wedge \cdots \wedge e_k D_{2,2+j}) \end{array}\right) \\
= \;& 2\left(\begin{array}{l} \displaystyle -2g + \sum_{j=2}^{k+1} (-1)^{\delta_{j\equiv 2,3 \pmod 4}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} (-1)^{j-1} \\ \displaystyle+\sum_{j=1}^{k} (-1)^{\delta_{j\equiv 2,3 \pmod 4}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} (-1)^{j-1} \end{array}\right) \mathrm{pr}(e_1 \wedge \cdots \wedge e_k) \\
= \;& 2\left(-2g + 2 + 2\sum_{j=2}^{k} (-1)^{j-1+\delta_{j\equiv 2,3 \pmod 4}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor}\right) \mathrm{pr}(e_1 \wedge \cdots \wedge e_k) \\
= \;& 2\left(-2g - 2 + 2\sum_{j=1}^{k+1} (-1)^{j-1+\delta_{j\equiv 2,3 \pmod 4}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor}\right) \mathrm{pr}(e_1 \wedge \cdots \wedge e_k).
\end{aligned}$$

Here, we claim that

$$\sum_{j=1}^{k+1} (-1)^{j-1+\delta_{j\equiv 2,3 \pmod 4}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} = 0.$$

In fact, by setting $k = 4K+1$, we have

$$
\begin{aligned}
& \sum_{j=1}^{k+1} (-1)^{j-1+\delta_{j\equiv 2,3 \,(\mathrm{mod}\, 4)}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} \\
= & \sum_{j=1}^{k+1} (-1)^{\delta_{j\equiv 0,3 \,(\mathrm{mod}\, 4)}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} \\
= & \sum_{\substack{1\le j\le k+1 \\ j:\mathrm{odd}}} (-1)^{\delta_{j\equiv 3 \,(\mathrm{mod}\, 4)}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} + \sum_{\substack{1\le j\le k+1 \\ j:\mathrm{even}}} (-1)^{\delta_{j\equiv 0 \,(\mathrm{mod}\, 4)}} {}_{\frac{k-1}{2}}C_{\lfloor \frac{j-1}{2} \rfloor} \\
= & \sum_{p=0}^{2K} (-1)^{\delta_{p\equiv 1 \,(\mathrm{mod}\, 2)}} {}_{2K}C_p + \sum_{q=1}^{2K+1} (-1)^{\delta_{q\equiv 0 \,(\mathrm{mod}\, 2)}} {}_{2K}C_{q-1} \\
= & \ 2\sum_{p=0}^{2K} (-1)^{\delta_{p\equiv 1 \,(\mathrm{mod}\, 2)}} {}_{2K}C_p = 2(1-1)^{2K} = 0.
\end{aligned}
$$

Hence, we conclude $c_k(\varphi_{[1^k]}) = -4(g+1)\,\mathrm{pr}(e_1 \wedge \cdots \wedge e_k)$.

Since $[L^{[1^k]} : H_{\mathbf{Q}}^{\otimes k}] = [L^{[1^k]} : \mathcal{C}_{2g}^{\mathbf{Q}}(k)] = 1$ and $e_1 \wedge \cdots \wedge e_k$ is a maximal vector with highest weight (1^k) of $H_{\mathbf{Q}}^{\otimes k}$, we have $\mathrm{pr}(e_1 \wedge \cdots \wedge e_k) \neq 0$.

Step.5 By Proposition 7.1 and Proposition 7.3, the maximal vector $\varphi_{[1^k]}$ gives a unique irreducible component of $[1^k]$ in $\mathrm{Coker}\, \tau_{k,\mathbf{Q}}^{\mathcal{M}}$.

This completes the proof of Theorem 7.7. □

7.4.2 Outline of proof of Theorem 7.6

To begin with, we can show

$$(e_1^{\otimes k} D_{12})(1-s_2)(1-s_3s_2)\cdots(1-s_r\cdots s_3s_2) = \sum_{j=1}^{r} (-1)^{j-1}{}_{r-1}C_{j-1}(e_1^{\otimes k})D_{1,1+j}$$

by using the induction on r. Secondly, we have

$$(e_1^{\otimes k} D_{ij}) s_{k+1}s_k\cdots s_2s_1 = \begin{cases} e_1^{\otimes k} D_{i+1,j+1}, & \text{if } j \neq k+2, \\ -e_1^{\otimes k} D_{1,i+1}, & \text{if } j = k+2. \end{cases}$$

Hence we get an explicit formula

$$(\omega \otimes e_1^{\otimes k}) \cdot \theta_P \cdot (1 + \sigma_{k+2} + \cdots + \sigma_{k+2}^{k+1}) = \sum_{i=1}^{k+1} \sum_{r=1}^{k-i+2} (-1)^{r-1}{}_kC_{r-1}(e_1^{\otimes k}) \cdot D_{i,i+r}.$$

Thirdly, we have

$$\mathrm{cont}_k(e_1^{\otimes k} D_{ij}) = \begin{cases} (-2g)(e_1^{\otimes k}) & \text{if} \quad i=1,\ j=2, \\ -(e_1^{\otimes k}) & \text{if} \quad i=1,\ j\geq 3, \\ (e_1^{\otimes k}) & \text{if} \quad i=2,\ j\geq 3, \\ 0 & \text{if} \quad \text{otherwise}, \end{cases}$$

and $\mathrm{pr}(e_1^{\otimes k}) \neq 0$. Thus we obtain

$$\begin{aligned} c_k(\varphi_{[k]}) &= \sum_{j=1}^{k+1}(-1)^{j-1}{}_kC_{j-1}\ c_k(e_1^{\otimes k}D_{1j}) + \sum_{j=1}^{k}(-1)^{j-1}{}_kC_{j-1}\ c_k(e_1^{\otimes k}D_{2j}) \\ &= \left(-2g - \sum_{j=2}^{k+1}(-1)^{j-1}{}_kC_{j-1} + \sum_{j=1}^{k}(-1)^{j-1}{}_kC_{j-1}\right)\mathrm{pr}(e_1^{\otimes k}) \\ &= \left(-2g + (-1)^{k+1} + \sum_{j=2}^{k}\left\{(-1)^j{}_kC_j + (-1)^{j-1}{}_kC_{j-1}\right\} + 1\right)\mathrm{pr}(e_1^{\otimes k}) \\ &= (2-2g)\,\mathrm{pr}(e_1^{\otimes k}) \neq 0. \end{aligned}$$

Therefore, by Proposition 7.1 and Proposition 7.3, the maximal vector $\varphi_{[k]}$ gives a unique irreducible component of $[k]$ in $\mathrm{Coker}\,\tau_{k,\mathbf{Q}}^{\mathcal{M}}$.

This completes the proof of Theorem 7.6. □

7.5 A conjecture for the Johnson cokernels

Finally, we conclude by suggesting a conjecture for the Johnson cokernels of the mapping class group.

By observing the table of $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}})$ for $1 \leq k \leq 4$ in Subsection 3.3, we see that $\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}}) \cong \mathrm{Im}(c_k)$ for $1 \leq k \leq 4$ as an $\mathrm{Sp}(2g,\mathbf{Q})$-module, where $c_k : \mathfrak{h}_{g,1}^{\mathbf{Q}}(k) \to \mathcal{C}_n^{\mathbf{Q}}(k)$ is an $\mathrm{Sp}(2g,\mathbf{Q})$-equivariant homomorphism defined in Subsection 7.1. These facts let us conjecture that

Conjecture 7.8. *For any $k \geq 1$, as an $\mathrm{Sp}(2g,\mathbf{Q})$-module,*

$$\mathrm{Coker}(\tau_{k,\mathbf{Q}}^{\mathcal{M}}) \cong \mathrm{Im}(c_k).$$

Namely, we conjecture that all of the Sp-irreducible components of the Johnson cokernels of the mapping class group can be detected by the map c_k.

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