A remarkable σ-finite measure unifying supremum penalisations for a stable Lévy process

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November 22, 2010

Abstract

The σ-finite measure $\mathcal{P}_{\mathrm{sup}}$ which unifies supremum penalisations for a stable Lévy process is introduced. Silverstein's coinvariant and coharmonic functions for Lévy processes and Chaumont's h-transform processes with respect to these functions are utilized for the construction of $\mathcal{P}_{\mathrm{sup}}$.

Key words
Lévy processes, Stable Lévy processes, Reflected processes,
Penalisation, Path decomposition,
Conditioning to stay negative/positive, Conditioning to hit 0 continuously.

1 Introduction

Roynette-Vallois-Yor ([18] and [19], see also [20] and [21]) have considered the limit laws of Wiener measure weighted by various processes (Γ_t), and they call these studies *Brownian penalisations*. Especially we call the case where the weight process is given by a function of its supremum, i.e., **(S)** $\Gamma_t = f(S_t)$, *supremum penalisation*. Concerning the Brownian supremum penalisations, the authors [19] have obtained the following result: Let $X = ((X_t), (\mathscr{F}_t), \mathbb{W})$ be the canonical representation of a 1-dimensional standard Brownian motion with $\mathbb{W}(X_0 = 0) = 1$ and let $\mathscr{F}_\infty = \sigma(\bigvee_t \mathscr{F}_t)$. Put $S_t = \sup_{s \le t} X_s$. If f is a non-negative Borel function which satisfies

$$\int_0^\infty f(x)\mathrm{d}x = 1, \tag{1.1}$$

then there exists a unique probability law $\mathbb{W}^{(f)}$ on $\mathscr{F}_\infty$ such that

$$\frac{\mathbb{W}[f(S_t)F_s]}{\mathbb{W}[f(S_t)]} \longrightarrow \mathbb{W}^{(f)}[F_s] \quad \text{as} \quad t \to \infty, \tag{1.2}$$

for any fixed $s > 0$ and for any bounded $\mathscr{F}_s$-measurable functional F_s. Moreover the limit measure $\mathbb{W}^{(f)}$ is characterized by

$$\mathbb{W}^{(f)}|_{\mathscr{F}_s} = M_s^{(f)} \cdot \mathbb{W}|_{\mathscr{F}_s}, \tag{1.3}$$

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where $(M_s^{(f)}, s \geq 0)$ is a $((\mathscr{F}_s), \mathbb{W})$-martingale which has the form

$$M_s^{(f)} = f(S_s)(S_s - X_s) + \int_{S_s}^{\infty} f(x)\mathrm{d}x. \tag{1.4}$$

We remark that these martingales $(M_s^{(f)})$ which are known as the Azéma-Yor martingales were applied to solve the Skorokhod embedding problem; see [1] and [2], and [15] and references therein. In [19] the authors have also obtained the description of the probability measure $\mathbb{W}^{(f)}$ as follows.

Theorem 1.1 (Roynette-Vallois-Yor [19]). *The following holds:*

(i) $\mathbb{W}^{(f)}(S_\infty \in \mathrm{d}x) = f(x)\mathrm{d}x$.

(ii) Let $g = \sup\{t \geq 0 : X_t = S_\infty\}$. *Then* $\mathbb{W}^{(f)}(g < \infty) = 1$ *and, under* $\mathbb{W}^{(f)}$, *we have*

(a) $(X_u, u \leq g)$ *and* $(X_g - X_{g+u}, u \geq 0)$ *are independent;*

(b) conditional on $S_\infty = x$, *the pre-supremum process* $(X_u, u \leq g)$ *is distributed as a Brownian motion starting from* 0 *and stopped at its first hitting time of* x;

(c) the post-supremum process $(X_g - X_{g+u}, u \geq 0)$ *is distributed as a* 3*-dimensional Bessel process starting from* 0.

Theorem 1.1 implies that, under the limit measure $\mathbb{W}^{(f)}$, the time g when the process attains its overall supremum is finite, so that the supremum penalisation procedure can be interpreted as looking for probabilities on canonical space, which are close to $\mathbb{W}$, and such that $S_\infty < \infty$ a.s.

Roynette-Vallois-Yor considered Brownian penalisations for many other kinds of weighted processes. For instance, **(L)** $\Gamma_t = f(L_t)$ where L_t denotes the local time of X at the origin, and **(K)** $\Gamma_t = \exp(-\int L(t,x)V(\mathrm{d}x))$ where $L(t,x)$ denotes the local time of X at x; we call the former case *local time penalisation* and the latter case *Kac killing penalisation.* Meanwhile Najnudel-Roynette-Yor [14] have introduced a certain σ-finite measure $\mathcal{W}$ defined as follows:

$$\mathcal{W} = \int_0^\infty \frac{\mathrm{d}u}{\sqrt{2\pi u}}(\Pi^{(u)} \bullet P^{3B}), \tag{1.5}$$

where $\Pi^{(u)}$ denotes the law of Brownian bridge from 0 to 0 of length u and $P^{3B} = (P^{3B,+} + P^{3B,-})/2$ denotes the law of symmetrized 3-dimensional Bessel process; $P^{3B,+}$ is the law of 3-dimensional Bessel process starting from 0, $BES(3)$, whereas $P^{3B,-}$ is the law of $(-BES(3))$. The authors in [14] have shown that the Brownian penalisations including **(S)(L)(K)** can be understood in a unified manner, thanks to this measure $\mathcal{W}$. Especially in the supremum penalisation case, they have shown the following absolute continuity relationship between $\mathcal{W}$ and $\mathbb{W}^{(f)}$:

$$f(S_\infty) \cdot \mathcal{W}^- = \mathbb{W}^{(f)} \quad \text{on} \quad \mathscr{F}_\infty, \tag{1.6}$$

where

$$\mathcal{W}^- = \mathbf{1}_{\{S_\infty < \infty\}} \cdot \mathcal{W} = \int_0^\infty \frac{\mathrm{d}u}{\sqrt{2\pi u}} \left(\Pi^{(u)} \bullet \frac{P^{3B,-}}{2} \right). \tag{1.7}$$

As a generalisation of these studies, Yano-Yano-Yor [26] have considered the two kinds of penalisations **(L)** and **(K)** in the case of symmetric α-stable Lévy process with index $\alpha \in (1, 2]$. Let us denote by $((X_t), \mathbb{P})$ such a stable Lévy process with $\mathbb{P}(X_0 = 0) = 1$. The authors have introduced a σ-finite measure $\mathcal{P}$ defined as follows, which is the analogue of $\mathcal{W}$:

$$\mathcal{P} = \int_0^\infty \frac{\Gamma(1/\alpha)}{\alpha\pi} \frac{\mathrm{d}u}{u^{1/\alpha}} (\mathbb{Q}^{(u)} \bullet \mathbb{P}^\times), \tag{1.8}$$

where $\mathbb{Q}^{(u)}$ denotes the law of the stable bridge from 0 to 0 of length u and $\mathbb{P}^\times$ denotes the h-transform process with respect to the harmonic function $|x|^{\alpha-1}$ of the process killed at the first hitting time of 0. We should remark that the process under the measure $\mathbb{P}^\times$ is called *conditioned to avoid* 0, because of the following property obtained by K. Yano [24]: If a functional Z is of the form $Z = f(X_{t_1}, \cdots, X_{t_n})$ for some $0 < t_1 < \cdots < t_n$ and some continuous function $f : \mathbb{R}^n \to \mathbb{R}$ which vanishes at ∞, then one has

$$\mathbb{P}^\times[Z] = \lim_{t\to\infty} \lim_{\varepsilon\to 0+} \mathbb{P}\left[Z \circ \theta_\varepsilon \;\middle|\; \forall u \le t, X_u \circ \theta_\varepsilon \neq 0 \right], \tag{1.9}$$

where $\theta.$ is the shift operator: $X_u \circ \theta. = X_{\cdot+u}$. Moreover the following long-time behavior of path under $\mathbb{P}^\times$ is also obtained by K. Yano [25]:

$$\mathbb{P}^\times \left(\limsup_{t\to\infty} X_t = \limsup_{t\to\infty} (-X_t) = \lim_{t\to\infty} |X_t| = \infty \right) = 1. \tag{1.10}$$

Thus we can see immediately that, under $\mathcal{P}$, $S_\infty = \infty$ a.e. That is, $\mathcal{P}$ cannot unify the supremum penalisations **(S)** in the stable case.

Yano-Yano-Yor [27] have studied the supremum penalisation for a (α, ρ)-stable Lévy process with index $\alpha \in (0, 2]$ and positivity parameter $\rho \in (0, 1)$. The authors have introduced a generalised Azéma-Yor martingale $(M_s^{(f)})$ which is defined as

$$M_s^{(f)} = f(S_s)(S_s - X_s)^{\alpha\rho} + \alpha\rho \int_{S_s}^\infty f(x)(x - X_s)^{\alpha\rho-1} \mathrm{d}x, \tag{1.11}$$

for any non-negative Borel function f satisfying

$$0 < \int_0^\infty f(x) x^{\alpha\rho-1} \mathrm{d}x < \infty, \tag{1.12}$$

and also introduced the probability measure $\mathbb{P}^{(f)}$ given as

$$\mathbb{P}^{(f)}|_{\mathscr{F}_s} = \frac{M_s^{(f)}}{M_0^{(f)}} \cdot \mathbb{P}|_{\mathscr{F}_s}. \tag{1.13}$$

The authors obtained the following result:

Theorem 1.2 (Yano-Yano-Yor [27]). *Let f be a non-negative function which satisfies either of the following two conditions:*

(i) $f(x) = \mathbf{1}_{\{x \le a\}}$ *for some* $a > 0$*;*

(ii) f is absolutely continuous with respect to the Lebesgue measure and satisfies

$$\lim_{x\to\infty} f(x) = 0 \quad \textit{and} \quad 0 < \int_0^\infty |f'(x)| x^{\alpha\rho} \mathrm{d}x < \infty. \tag{1.14}$$

Then it holds that, for any $s > 0$ and any bounded $\mathscr{F}_s$-measurable functional F_s,

$$\frac{\mathbb{P}[f(S_t)F_s]}{\mathbb{P}[f(S_t)]} \longrightarrow \mathbb{P}^{(f)}[F_s] \quad \textit{as} \quad t \to \infty. \tag{1.15}$$

We remark that the condition (ii) in Theorem 1.2 is stronger than the condition (1.12) because we have

$$\begin{aligned}\int_0^\infty f'(x) x^{\alpha\rho} \mathrm{d}x &= \alpha\rho \int_0^\infty f'(x)\mathrm{d}x \int_0^x y^{\alpha\rho-1}\mathrm{d}y \\ &= \alpha\rho \int_0^\infty y^{\alpha\rho-1}\mathrm{d}y \int_y^\infty f'(x)\mathrm{d}x = -\alpha\rho \int_0^\infty f(y) y^{\alpha\rho-1}\mathrm{d}y.\end{aligned}$$

One may conjecture that the assumption of Theorem 1.2 can be weakened to the condition (1.12) that is sufficient to define the generalised Azéma-Yor martingale and the measure $\mathbb{P}^{(f)}$; however, this is still an open problem.

In the present paper we introduce a certain σ-finite measure $\mathcal{P}_{\mathrm{sup}}$ by using Chaumont's h-transform processes for Lévy processes (cf. Theorem 5.1 below):

$$\mathcal{P}_{\mathrm{sup}} = \int_0^\infty \mathrm{d}x \psi(x) (\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}),$$

where ψ is the function stated below in (2.10), $\mathbb{P}_{0\nearrow x}$ denotes the law of the process starting from 0 and conditioned to hit x continuously, and $\mathbb{P}_{x\downarrow x}$ denotes the law of the process starting from x and conditioned to stay below level x. $\mathcal{P}_{\mathrm{sup}}$ is another analogue of $\mathcal{W}$ and $\mathcal{P}$, and it is a generalisation of $\mathcal{W}^-$ given in (1.7). We remark that, in the Brownian case, $\mathcal{P}_{\mathrm{sup}}^{\mathrm{BM}}$ is given by the following:

$$\mathcal{P}_{\mathrm{sup}}^{\mathrm{BM}} = \int_0^\infty \mathrm{d}x \left(\mathbb{W}_{0\nearrow x} \bullet P_x^{3B,-}\right) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}u \frac{x}{\sqrt{\pi u^3}} \mathrm{e}^{-\frac{x^2}{2u}} \left(\mathbb{W}_{0\nearrow x}^{(u)} \bullet P_x^{3B,-}\right), \tag{1.16}$$

where $\mathbb{W}_{0\nearrow x}$ denotes the law of Brownian motion killed at the first hitting time at x and $\mathbb{W}_{0\nearrow x}^{(u)}(\cdot) = \mathbb{W}_{0\nearrow x}(\cdot | T_{\{x\}} = u)$, and $P_x^{3B,-}$ denotes the law of the translation by x of $(-BES(3))$. The latter equality is obtained from the well-known fact (see, e.g., [11]) that

$$\mathbb{W}(T_{\{x\}} \in \mathrm{d}u) = \mathrm{d}u \frac{x}{\sqrt{2\pi u^3}} \mathrm{e}^{-\frac{x^2}{2u}}. \tag{1.17}$$

We note that the measure $\mathcal{P}_{\mathrm{sup}}^{\mathrm{BM}}$ equals $\mathcal{W}^-$ by the agreement formula obtained by Pitman-Yor [16].

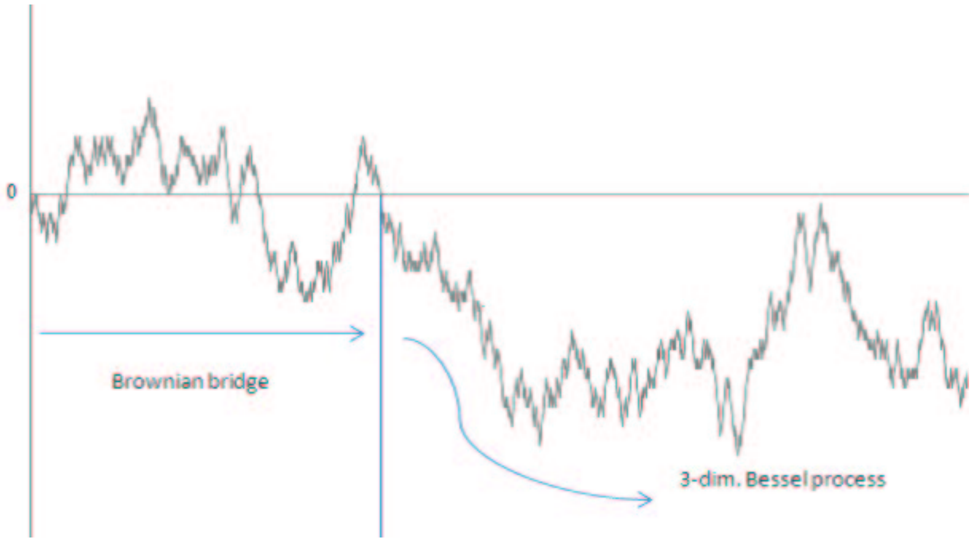

Figure 1. Sample path of $\Pi^{(u)} \bullet P^{3B,-}$

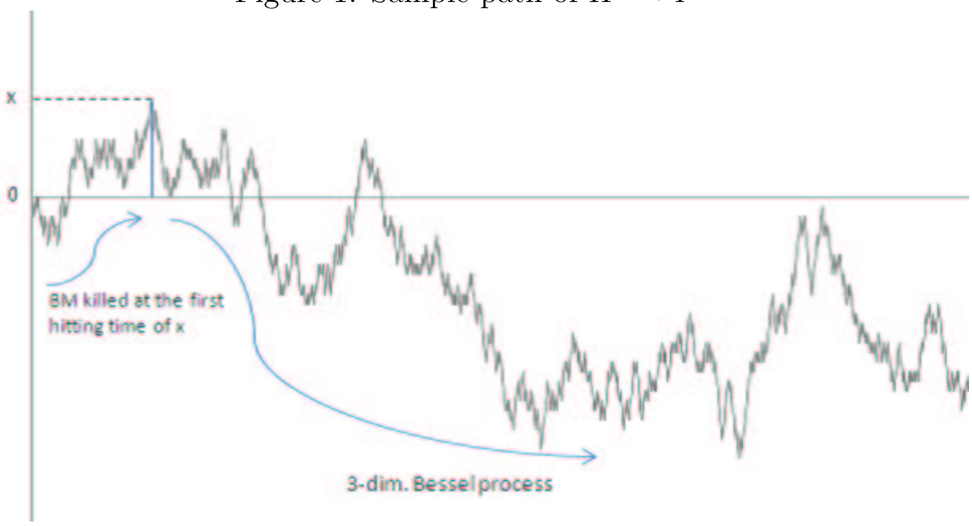

Figure 2. Sample path of $\mathbb{W}^{(u)}_{0\nearrow x} \bullet P^{3B,-}_x$

We then show that the measure $\mathcal{P}_{\sup}$ unifies the supremum penalisations. More precisely, we shall define a probability measure $\mathbb{P}^{(f)}$ as the transformation of the law $\mathbb{P}$ of a Lévy process by the generalised Azéma-Yor martingale defined as (6.2) below. This measure $\mathbb{P}^{(f)}$ is the generalisation of (1.13) for a general Lévy process. We then prove the absolute continuity relationship between $\mathcal{P}_{\sup}$ and $\mathbb{P}^{(f)}$ in the Lévy case, which is the analogue of (1.6) (cf. Theorem 7.3 below):

$$\frac{f(S_\infty) \cdot \mathcal{P}_{\sup}}{\mathcal{P}_{\sup}[f(S_\infty)]} = \mathbb{P}^{(f)} \quad \text{on} \quad \mathscr{F}_\infty.$$

We obtain a detailed description of $\mathbb{P}^{(f)}$ as a consequence of this result (cf. Theorem 7.4 below). To prove the absolute continuity relationship between $\mathcal{P}_{\sup}$ and $\mathbb{P}^{(f)}$, we shall introduce a path decomposition of the law $\mathbb{P}$ of a Lévy process up to a fixed time t with respect to the position and the time where the process attains its supremum before time t.

The organization of the present paper is as follows. In Section 2 and 3, we recall some preliminary facts about Lévy processes and (α, ρ)-stable Lévy processes, respectively. If a reader needs to see details, he/she may refer to, e.g., [3], [10], [12] and [22]. In Section 4, we review Chaumont's two kinds of h-transform processes for a Lévy process. In Section 5, we establish a path decomposition of the law of a Lévy process at the position and the time where the Lévy process attains its supremum up to a fixed time t. In Section 6, we introduce the generalised Azéma-Yor martingale in the general Lévy case, which is the generalisation of (1.4) and (1.11). A certain probability measure which should appear as the limit measure of the supremum penalisation is also introduced in this section. In Section 7, we introduce the σ-finite measure $\mathcal{P}_{\sup}$ which unifies the supremum penalisations and give some properties of the measure $\mathcal{P}_{\sup}$. In Section 8, we compare $\mathcal{P}_{\sup}$ with $\mathcal{P}$ and give some remarks on these measures.

2 Preliminaries about Lévy processes

Let $\mathscr{D}([0,\infty))$ be the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R} \cup \{\delta\}$ with lifetime $\zeta(\omega) = \inf\{s : \omega(s) = \delta\}$ where δ is a cemetery point. Let (X_t) denote the coordinate process, $X_t(\omega) = \omega_t$, and let $(\mathscr{F}_t)$ denote its natural filtration with $\mathscr{F}_\infty = \vee_{t\ge 0}\mathscr{F}_t$. Let $\mathbb{P}$ be the law of a Lévy process $X = (X_t, t \ge 0)$ with $\mathbb{P}(X_0 = 0) = 1$ such that

$$\mathbb{P}\left[\exp\{i\lambda X_t\}\right] = \mathrm{e}^{-t\Psi(\lambda)}, \qquad t \ge 0,\ \lambda \in \mathbb{R}, \tag{2.1}$$

where

$$\Psi(\lambda) = i\gamma\lambda + \frac{\sigma^2\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(1 - \mathrm{e}^{i\lambda x} + i\lambda x 1_{\{|x|<1\}}\right) \nu(\mathrm{d}x) \tag{2.2}$$

for some constants γ, σ, and Lévy measure ν on $\mathbb{R} \setminus \{0\}$ which satisfies

$$\int_{\mathbb{R}\setminus\{0\}} (x^2 \wedge 1)\nu(\mathrm{d}x) < \infty. \tag{2.3}$$

We denote by $\mathbb{P}_x$ the law of $X + x$ under $\mathbb{P}$ for every $x \in \mathbb{R}$. Throughout this paper we assume the following absolute continuity condition **(A1)**:

(A1) For each $\alpha > 0$, there exists an integrable function u_α such that

$$\mathbb{P}_x\left[\int_0^\infty \mathrm{e}^{-\alpha t} f(X_t)\mathrm{d}t\right] = \int_{-\infty}^\infty u_\alpha(y) f(x+y)\mathrm{d}y, \tag{2.4}$$

for every non-negative Borel function f.

Let S_t and I_t be respectively the supremum and the infimum processes up to time t, that is, for all $t < \zeta(\omega)$,

$$S_t = \sup\{X_s : 0 \le s \le t\} \qquad \text{and} \qquad I_t = \inf\{X_s : 0 \le s \le t\}. \tag{2.5}$$

Let T_A denote the first entrance time of a Borel set $A \subset \mathbb{R}$ of X, i.e.,

$$T_A = \inf\{s > 0 : X_s \in A\}. \tag{2.6}$$

Define

$$R = S - X. \tag{2.7}$$

The process $R = (R_t, t \geq 0)$ is called *the reflected process of X at the supremum.* We recall that R is a strong Markov process (Bingham [5], see also [4]). We consider the following condition **(A2)**:

(A2) 0 is regular for $(0, \infty)$ with respect to X under $\mathbb{P}$, i.e., $\mathbb{P}(T_{(0,\infty)} = 0) = 1$.

Then 0 is regular for itself with respect to R, and hence we can define a local time $L = (L_t, t \geq 0)$ at level 0 of R. We denote by τ the right-continuous inverse of L and let $H = X(\tau) = I(\tau)$. We recall that the pair (τ, H) is a bivariate subordinator, called the (upwards) ladder process, in particular, τ and H are separately also subordinators, called the (upwards) ladder time and the (upwards) ladder height process, respectively. Denote by X^* the dual process of X, i.e., $X^* = -X$. Consider

(A2*) 0 is regular for $(-\infty, 0)$ with respect to X under $\mathbb{P}$.

Then we can define a local time L^* at level 0 of $R^* = S^* - X^* = X - I$, and also get the (downwards) ladder time τ^* and the (downwards) ladder height time H^* of R^*.

We denote by E the set of càdlàg paths $e : [0, \infty) \to \mathbb{R} \cup \{\delta\}$ such that

$$e(t) \begin{cases} \in \mathbb{R} \setminus \{0\}, & 0 < t < \zeta_e; \\ = \delta, & t \geq \zeta_e, \end{cases}$$

where

$$\zeta_e = \inf\{t > 0 : e(t) = \delta\}. \tag{2.8}$$

We call E the set of excursions and an element $e \in E$ an excursion path. For $e \in E$, we call ζ_e the lifetime of the excursion e. Set $D = \{l : \tau_l - \tau_{l-} > 0\}$. For each $l \in D$, we set

$$e_l(t) = \begin{cases} R_{t+\tau_{l-}}, & 0 \leq t < \tau_l - \tau_{l-}; \\ \delta, & t \geq \tau_l - \tau_{l-}. \end{cases}$$

By Itô's theorem, the point process $(e_l, l \in D)$ which takes values on E is a Poisson point process, and its characteristic measure $\boldsymbol{n}$ is called *the Itô measure of excursions.* Similarly, we can introduce excursions e^* with respect to R^* and denote by $\boldsymbol{n}^*$ its Ito measure.

We recall the following important formula, see also p. 7 in [4], and Proposition (1.10) in Chapter XII in [17]. Denote by $\mathscr{P}(\mathscr{F}_t)$ the predictable σ-field relative to $(\mathscr{F}_t)$ (cf. p. 47 in [17]), and let $\mathscr{E} = \sigma\{e(t)\}$.

Theorem 2.1 (Compensation formula). *Let $F = F(t, \omega, e)$ be a positive process defined on $[0, \infty) \times \mathscr{D} \times E$, measurable with respect to $\mathscr{P}(\mathscr{F}_t) \otimes \mathscr{E}$ and vanishing at δ. Then one has*

$$\mathbb{P}\left[\sum_{l \in D} F(\tau_{l-}, X, e_l)\right] = \mathbb{P} \otimes \widehat{\boldsymbol{n}}\left[\int_0^\infty \mathrm{d}L_t F(t, X, \widehat{X})\right], \tag{2.9}$$

where the symbol $\widehat{\ }$ *means independence.*

Under **(A1)** and **(A2)**, there exists a unique coexcessive function ψ for the killed process, i.e., $\mathbb{P}_{-x}[\psi(X_t^*)\mathbf{1}_{\{t<T_{(0,\infty)}\}}] \le \psi(x)$ for $x \ge 0$, which satisfies

$$\int_0^\infty \psi(y)f(y)\mathrm{d}y = \mathbb{P}\left[\int_0^\infty f(S_{\tau_s})\mathrm{d}s\right] = \mathbb{P}\left[\int_0^\infty f(S_t)\mathrm{d}L_t\right], \tag{2.10}$$

for any non-negative Borel function f on $[0,\infty)$. We remark that ψ is continuous and satisfies that $0 < \psi(x) < \infty$ for $x \in (0,\infty)$. Thanks to Silverstein [23], the function ψ is *coharmonic* on $(0,\infty)$, that is,

$$\mathbb{P}_{-x}\left[\psi(X_{T_M}^*)\mathbf{1}_{\{T_M<T_{(0,\infty)}\}}\right] = \psi(x), \qquad x > 0, \tag{2.11}$$

where M denotes a subinterval of $(-\infty,0)$ whose complement $(-\infty,0)\setminus M$ is open and has compact closure. We assume further that

(A3) $\mathbb{P}_x(T_{(-\infty,0)} < \infty) = 1$ for $x > 0$ $\quad(\overset{\text{iff}}{\Longleftrightarrow}\ I_\infty = -\infty\ \ \mathbb{P}\text{-a.s.})$.

Then the function h given by

$$h(x) = \int_0^x \psi(y)\mathrm{d}y = \mathbb{P}\left[\int_0^\infty \mathbf{1}_{\{S_t\le x\}}\mathrm{d}L_t\right] \tag{2.12}$$

is *coinvariant* by Silverstein [23], that is,

$$\mathbb{P}_{-x}\left[h(X_t^*)\mathbf{1}_{\{t<T_{(0,\infty)}\}}\right] = h(x), \qquad x > 0; \tag{2.13}$$

$$\boldsymbol{n}\left[h(X_t)\mathbf{1}_{\{t<\zeta\}}\right] = 1. \tag{2.14}$$

We remark that the function h is finite, continuous, increasing and subadditive on $[0,\infty)$, and that $h(0) = 0$ by **(A2)**. We remark that every positive coinvariant function is also coharmonic.

Similarly, under **(A1)** and **(A2*)**, there exists a version of the potential density of the subordinator $(I_{\tau_s^*})_{s\ge 0}$. That is, there exists a unique coexcessive function ψ^* for the killed process, i.e., $\mathbb{P}_x[\psi^*(X_t)\mathbf{1}_{\{t<T_{(-\infty,0)}\}}] \le \psi^*(x)$ for $x \ge 0$, which satisfies

$$\int_0^\infty \psi^*(y)f(y)\mathrm{d}y = \mathbb{P}\left[\int_0^\infty f(I_{\tau_s^*})\mathrm{d}s\right] = \mathbb{P}\left[\int_0^\infty f(I_t)\mathrm{d}L_t^*\right], \tag{2.15}$$

for any non-negative Borel function f on $(0,\infty)$. Also thanks to Silverstein [23], the function ψ^* is coharmonic on $(0,\infty)$, that is,

$$\mathbb{P}_x\left[\psi^*(X_{T_{M'}})\mathbf{1}_{\{T_{M'}<T_{(-\infty,0)}\}}\right] = \psi^*(x), \qquad x > 0, \tag{2.16}$$

where M' denotes a subinterval of $(0,\infty)$ whose complement $(0,\infty)\setminus M'$ is open and has the compact closure. If we assume further that

(A3*) $\mathbb{P}_{-x}(T_{(0,\infty)} < \infty) = 1$ for $x > 0$ $\quad(\overset{\text{iff}}{\Longleftrightarrow}\ S_\infty = \infty\ \ \mathbb{P}\text{-a.s.})$.

Then the function h^* given by

$$h^*(x) = \int_0^x \psi^*(y)\mathrm{d}y = \mathbb{P}\left[\int_0^\infty \mathbf{1}_{\{I_t \le x\}}\mathrm{d}L_t^*\right] \tag{2.17}$$

is coinvariant, that is,

$$\mathbb{P}_x\left[h^*(X_t)\mathbf{1}_{\{t<T_{(-\infty,0)}\}}\right] = h^*(x), \qquad x > 0; \tag{2.18}$$

$$\boldsymbol{n}^*\left[h^*(X_t)\mathbf{1}_{\{t<\zeta\}}\right] = 1. \tag{2.19}$$

3 Preliminaries about (α,ρ)-stable Lévy processes

Consider a probability measure $\mathbb{P}$ on $\mathscr{D}([0,\infty))$ with respect to which X is a strictly stable Lévy process of index $\alpha \in (0,2]$ with $\mathbb{P}(X_0 = 0) = 1$. That is,

$$\mathbb{P}[\mathrm{e}^{i\lambda X_t}] = \mathrm{e}^{-t\Psi(\lambda)}, \qquad t \ge 0,\ \lambda \in \mathbb{R},$$

where

$$\Psi(\lambda) = \begin{cases} c|\lambda|^\alpha \left(1 - i\beta \mathrm{sgn}(\lambda)\tan\dfrac{\pi\alpha}{2}\right), & \alpha \in (0,1)\cup(1,2), \\ c|\lambda| + di\lambda, & \alpha = 1, \\ c\lambda^2, & \alpha = 2, \end{cases}$$

for some constants $c > 0$, $d \in (-\infty,\infty)$ and $\beta \in [-1,1]$. The Lévy measure ν is given by

$$\nu(\mathrm{d}x) = \begin{cases} (c_+\mathbf{1}_{\{x>0\}} + c_-\mathbf{1}_{\{x<0\}})|x|^{-\alpha-1}\mathrm{d}x, & \alpha \in (0,1)\cup(1,2), \\ \widetilde{c}|x|^{-2}\mathrm{d}x, & \alpha = 1, \\ 0, & \alpha = 2, \end{cases}$$

where $\beta = (c_+ - c_-)/(c_+ + c_-)$, and for some constant $\widetilde{c} > 0$. When $c_{+[-]} = 0$, the process is spectrally negative[positive] (or, has no positive[negative] jumps). We remark that the condition **(A1)** is also valid in the stable Lévy case because of the scaling property of X.

Put $\rho = \mathbb{P}(X_t \ge 0)$. By the scaling property of X, ρ does not depend on $t > 0$. We call ρ *the positivity parameter*. It is well-known that the value of ρ for $\alpha \ne 1, 2$ can be represented in terms of the parameter β as

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\left(\beta \tan\frac{\pi\alpha}{2}\right). \tag{3.1}$$

See Section 2.6 in [28], and p. 218 in [3]. The range of the value of ρ is classified as follows:

$$\rho \begin{cases} \in [0,1] \qquad \text{if} \quad \alpha \in (0,1) \\ \quad \text{(when } \rho = 0 \text{ or } 1, \text{the process is a subordinator or a negative subordinator)}, \\ \in (0,1) \qquad \text{if} \quad \alpha = 1, \\ \in [1-1/\alpha, 1/\alpha] \qquad \text{if} \quad \alpha \in (1,2) \\ \quad \text{(when } \rho = 1-1/\alpha \text{ or } 1/\alpha, \text{ the process is spectrally positive or spectrally negative)}, \\ = 1/2 \qquad \text{if} \quad \alpha = 2. \end{cases}$$

Assume that

(B) $\rho \in (0,1)$ $\quad$ ($\overset{\text{iff}}{\Longleftrightarrow}$ $|X|$ is not a subordinator).

Then $\alpha\rho \in (0,1)$. We note that the condition **(B)** for the stable Lévy case implies the conditions **(A2)** and **(A2*)**, that is, 0 is regular for both $(0,\infty)$ and $(-\infty,0)$ with respect to X. Therefore we can define the local times L, L^*, etc. for the reflected and dual reflected processes in this case. Moreover the condition **(B)** also implies the conditions **(A3)** and **(A3*)**: More precisely, when $\alpha \in (1,2]$, **(A3)** and **(A3*)** hold since X is strictly stable; when $\alpha \in (0,1]$, they hold because of the condition **(B)**.

Assuming **(B)**, the function h is given by

$$h(x) = Cx^{\alpha\rho}, \qquad x > 0 \tag{3.2}$$

for some constant $C > 0$. This is obtained from the fact that the ladder time process τ is a subordinator of index ρ and the ladder height process H is a stable process of index $\alpha\rho$ (see Lemma VIII 1 in [4]). Furthermore, in this case, we have

$$\psi(x) = C\alpha\rho x^{\alpha\rho-1}, \qquad x > 0. \tag{3.3}$$

Similarly, we have

$$h^*(x) = Dx^{\alpha(1-\rho)} \qquad \text{and} \qquad \psi^*(x) = D\alpha(1-\rho)x^{\alpha(1-\rho)-1}, \qquad x > 0 \tag{3.4}$$

for some constant $D > 0$. These constants C and D may depend upon the choice of the local time L and L^*, respectively. In what follows we choose the versions of the local times L and L^* so that $C = D = 1$ for the sake of simplicity.

Example 3.1 (Brownian case). When $\alpha = 2$ and $\rho = 1/2$, X is a 1-dimensional Brownian motion up to a multiplicative constant. In this case we have

$$h(x) = x \qquad \text{and} \qquad \psi(x) = 1, \qquad x > 0. \tag{3.5}$$

4 Chaumont's two kinds of conditionings for a Lévy process

In this section we shall review two kinds of conditionings for a Lévy process introduced by Chaumont [7, 6], which are obtained by Doob's h-transform.

Let $X = ((X_t), \mathbb{P})$ be a Lévy process with the conditions **(A1)**, **(A2)** and **(A3)**. The functions ψ and h are stated as (2.10) and (2.17), respectively.

1° The process conditioned to stay negative.

For non-negative $\mathscr{F}_t$-measurable functional F_t, define $(\mathbb{P}_{-x\downarrow 0}, x \geq 0)$ as

$$\mathbb{P}_{-x\downarrow 0}[F_t(X)] = \frac{1}{h(x)}\mathbb{P}_{-x}\left[h(X_t^*)\mathbf{1}_{\{t<T_{(0,\infty)}\}}F_t(X)\right], \qquad x > 0, \tag{4.1}$$

$$\mathbb{P}_{0\downarrow 0}[F_t(X)] = \boldsymbol{n}\left[h(X_t)\mathbf{1}_{\{t<\zeta_e\}}F_t(X^*)\right]. \tag{4.2}$$

The family $(\mathbb{P}_{-x\downarrow 0}|_{\mathscr{F}_t}, t \geq 0)$ is proved to be consistent by the coinvariance of the function h and hence $\mathbb{P}_{-x\downarrow 0}$ is well-defined as a probability measure on $\mathscr{F}_\infty$. It is proved by Chaumont-Doney [9] that $\mathbb{P}_{-x\downarrow 0}$ converges in the Skorokhod sense to $\mathbb{P}_{0\downarrow 0}$ as $x \to 0$. The process $(X, \mathbb{P}_{-x\downarrow 0})$ is called *the process starting from* $(-x)$ *and conditioned to stay negative* since it has the following property:

Theorem 4.1 ([6, Theorem 1]). *Let $\boldsymbol{e}$ be an independent exponential random variable with index* 1*. Then, for any $x > 0, t \geq 0$ and any $\mathscr{F}_t$-measurable functional F_t, it holds that*

$$\lim_{\varepsilon \to 0} \mathbb{P}_{-x}\left[\mathbf{1}_{\{t<\boldsymbol{e}/\varepsilon\}} F_t \;\middle|\; X_s < 0,\ 0 \leq s \leq \boldsymbol{e}/\varepsilon\right] = \mathbb{P}_{-x\downarrow 0}[F_t]. \tag{4.3}$$

Theorem 4.1 implies the following: For $x \geq 0$,

$$\mathbb{P}_{-x\downarrow 0}(X_0 = -x;\ \zeta = \infty;\ X_t < 0 \text{ for all } t > 0;\ \lim_{t\to\infty} X_t = -\infty) = 1. \tag{4.4}$$

Here ζ denotes the lifetime.

For $b \leq a$, denote by $\mathbb{P}_{b\downarrow a}$ the law of $X + a$ under $\mathbb{P}_{b-a\downarrow 0}$, that is, $(X, \mathbb{P}_{b\downarrow a})$ is the process starting from b and conditioned to stay below level a.

2° The process conditioned to hit 0 continuously.

Define $(\mathbb{P}_{-x\nearrow 0}, x > 0)$ as

$$\mathbb{P}_{-x\nearrow 0}\left[\mathbf{1}_{\{t<\zeta\}} F_t(X)\right] := \frac{1}{\psi(x)} \mathbb{P}_{-x}\left[\psi(X_t^*)\mathbf{1}_{\{t<T_{(0,\infty)}\}} F_t(X)\right], \tag{4.5}$$

for non-negative $\mathscr{F}_t$-measurable functional F_t. The process $(X, \mathbb{P}_{-x\nearrow 0})$ is called *the process starting from* $(-x)$ *and conditioned to hit* 0 *continuously*, or also called *the process conditioned to die at* 0, and has the following property:

Theorem 4.2 ([6, Proposition 2]). *For $x > 0$, it holds that*

$$\mathbb{P}_{-x\nearrow 0}(X_0 = -x;\ \zeta < \infty;\ X_t < 0 \text{ for all } t < \zeta;\ X_{\zeta-} = 0) = 1, \tag{4.6}$$

where ζ denotes the lifetime.

The following result is also shown by Chaumont [6]:

Theorem 4.3 ([6, Proposition 3]). *For any $x > 0, k > 0, t \geq 0$ and any $\mathscr{F}_t$-measurable functional F_t,*

$$\lim_{\varepsilon \to 0} \mathbb{P}_{-x}\left[\mathbf{1}_{\{t<T_{(-k,\infty)}\}} F_t \;\middle|\; S_{T_{(0,\infty)}-} \geq -\varepsilon\right] = \mathbb{P}_{-x\nearrow 0}\left[\mathbf{1}_{\{t<T_{(-k,0)}\}} F_t\right]. \tag{4.7}$$

Denote by $\mathbb{P}_{0\nearrow x}$ the law of $X + x$ under $\mathbb{P}_{-x\nearrow 0}$, that is, $(X, \mathbb{P}_{0\nearrow x})$ is the process starting from 0 and conditioned to hit x continuously. For later use, we rewrite (4.5) by translation to obtain

$$\mathbb{P}_{0\nearrow x}\left[\mathbf{1}_{\{t<\zeta\}} F_t(X)\right] = \frac{1}{\psi(x)} \mathbb{P}\left[\psi(x - X_t)\mathbf{1}_{\{t<T_{(x,\infty)}\}} F_t(X)\right], \tag{4.8}$$

since we have

$$\begin{aligned}
\mathbb{P}_{0\nearrow x}\left[\mathbf{1}_{\{t<\zeta\}} F_t(X)\right] &= \mathbb{P}_{-x\nearrow 0}\left[\mathbf{1}_{\{t<\zeta\}} F_t(X + x)\right] \\
&= \frac{1}{\psi(x)} \mathbb{P}_{-x}\left[\psi(X_t^*)\mathbf{1}_{\{t<T_{(0,\infty)}\}} F_t(X + x)\right] \\
&= \frac{1}{\psi(x)} \mathbb{P}\left[\psi(x + X_t^*)\mathbf{1}_{\{t<T_{(x,\infty)}\}} F_t(X)\right].
\end{aligned}$$

5 Path decomposition at the position and the time where the Lévy process attains its supremum up to time t

Our aim in this section is to prove Theorem 5.1, which consists of a path decomposition with respect to the position and the time where the Lévy process attains its supremum up to time $t > 0$.

Let us denote by $X^{(u)}$ the coordinate process considered up to time u, i.e.,

$$X_t^{(u)} = \begin{cases} X_t, & t < u; \\ \delta, & t \geq u, \end{cases}$$

and denote by $\mathbb{P}_x^{(u)}$ the law of $X^{(u)}$ under $\mathbb{P}_x$. We denote the concatenation between two independent processes $X^{(u)}$ and $\widehat{X}^{(v)}$ by $X^{(u)} \bullet \widehat{X}^{(v)}$, i.e.,

$$(X^{(u)} \bullet \widehat{X}^{(v)})_t = \begin{cases} X_t^{(u)}, & 0 \leq t < u; \\ \widehat{X}_{t-u}^{(v)}, & u \leq t < u+v; \\ \delta, & t \geq u+v. \end{cases}$$

We define the measure $\mathbb{P}_x^{(u)} \bullet \mathbb{P}_y^{(v)}$ as the law of the concatenation $X^{(u)} \bullet \widehat{X}^{(v)}$ between two independent processes $X^{(u)}$ and $\widehat{X}^{(v)}$ where $(X^{(u)}, \widehat{X}^{(v)})$ is considered under the product measure $\mathbb{P}_x^{(u)} \otimes \widehat{\mathbb{P}}_y^{(v)}$.

For $t > 0$, we denote the last time when the process attains its supremum before t by

$$g_t = \sup\{s \leq t : X_s = S_s\}. \tag{5.1}$$

Theorem 5.1. *Let $X = ((X_t), \mathbb{P})$ be a Lévy process with $\mathbb{P}(X_0 = 0) = 1$ and assume* **(A1)***, as well as both* **(A2)** *and* **(A2*)***. Let $F_t(X^{(t)}) = F(t, X_{t\wedge\cdot})$. Then it holds that*

$$\mathbb{P}\left[F_t(X^{(t)})\right] = \int \rho_t(\mathrm{d}x\mathrm{d}u) \left(\mathbb{P}_{0\nearrow x}^{(u)} \bullet \mathbb{M}_x^{(t-u)}\right) \left[F_t(X^{(t)})\right], \tag{5.2}$$

where the integral is taken over $[0,\infty) \times [0,t)$ and

$$\rho_t(\mathrm{d}x\mathrm{d}u) = \mathrm{d}x\psi(x)\mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u)\boldsymbol{n}(\zeta_e > t-u); \tag{5.3}$$

$$\mathbb{P}_{0\nearrow x}^{(u)}(\cdot) = \mathbb{P}_{0\nearrow x}(\cdot|\zeta = u) \quad (\zeta \text{ denotes the lifetime}); \tag{5.4}$$

$$\mathbb{M}_x^{(s)}[F(X)] = \frac{\boldsymbol{n}\left[F(x - X^{(s)}); \zeta_e > s\right]}{\boldsymbol{n}(\zeta_e > s)} \quad (\zeta_e \text{ denotes the lifetime}). \tag{5.5}$$

In other words, the following statements hold:

(i) $\rho_t(\mathrm{d}x\mathrm{d}u)$ gives the joint distribution of S_t and g_t, i.e.,

$$\rho_t(\mathrm{d}x\mathrm{d}u) = \mathbb{P}(S_t \in \mathrm{d}x, g_t \in \mathrm{d}u); \tag{5.6}$$

(ii) given $g_t = u$, the pre-supremum process $(X_s, s \leq u)$ and the post-supremum process $(X_u - X_{u+s}, 0 \leq s \leq t-u)$ are independent under $\mathbb{P}$;

(iii) *given $S_t = x$ and $g_t = u$, $(X_s, s \le u)$ under $\mathbb{P}$ is distributed as $\mathbb{P}^{(u)}_{0\nearrow x}$; the process conditioned to hit x continuously, with duration u;*

(iv) *given $S_t = x$ and $g_t = u$, $(x - X_{u+s}, 0 \le s \le t-u)$ under $\mathbb{P}$ is distributed as the meander $\mathbb{M}^{(t-u)} := \mathbb{M}^{(t-u)}_0$.*

Remark 5.2. The fact (ii) in Theorem 5.1 is well-known and can be found in Lemma VI 6 in [4].

Remark 5.3. We can also see that $X_{g_t} = X_{g_t-}$, that is, the process does not jump at g_t; the last hitting time of its supremum up to time t. This fact is guaranteed by the conditions **(A2)** and **(A2*)**, see also [4], p. 160.

Remark 5.4. Theorem 5.1 is obtained independently by Chaumont [8] in his recent work for some purpose different from ours.

Before the proof of Theorem 5.1, we recall the following lemma from Chaumont [7]:

Lemma 5.5 ([7, Lemma 3]). *Let $X = ((X_t), \mathbb{P})$ be a Lévy process with $\mathbb{P}(X_0 = 0) = 1$ satisfying conditions* **(A1)**, **(A2)** *and* **(A2*)**. *Denote by L the local time at 0 of the reflected process $R = S - X$. Let H be a predictable functional. Then it holds that*

$$\mathbb{P}\left[\int_0^\infty H_t(X)\mathrm{d}L_t\right] = \int_0^\infty \mathbb{P}_{-x\nearrow 0}[H_\zeta(X+x)]\psi(x)\mathrm{d}x. \tag{5.7}$$

The proof of Lemma 5.5 for the stable Lévy process is given in [7]. Lemma 5.5 for the general Lévy process is proved in the same way, so we omit the proof.

Proof of Theorem 5.1. We have

$$\int_0^\infty \mathrm{d}t F_t(X^{(t)}) = \sum_{l\in D}\int_0^{\zeta(e_l)} F_{\tau_{l-}+r}(X^{(\tau_{l-})} \bullet (X_{\tau_{l-}} - e_l))\mathrm{d}r. \tag{5.8}$$

Hence we have

$$\begin{aligned}\mathbb{P}\left[\int_0^\infty \mathrm{d}t F_t(X^{(t)})\right] &= \mathbb{P}\left[\sum_{l\in D}\int_0^{\zeta(e_l)} F_{\tau_{l-}+r}\left(X^{(\tau_{l-})} \bullet (X_{\tau_{l-}} - e_l)\right)\mathrm{d}r\right] \\ &= \mathbb{P}\otimes\widehat{\boldsymbol{n}}\left[\int_0^\infty \mathrm{d}L_s \int_0^{\widehat{\zeta}_e} F_{s+r}(X^{(s)} \bullet (X_s - \widehat{X}^{(r)}))\mathrm{d}r\right], \end{aligned} \tag{5.9}$$

by the compensation formula (Theorem 2.1). By Lemma 5.5, we have

$$\begin{aligned}(5.9) &= \int_0^\infty \mathrm{d}x\psi(x)\,(\mathbb{P}_{-x\nearrow 0}\otimes\widehat{\boldsymbol{n}})\left[\int_0^\infty F_{\zeta+r}\left((X^{(\zeta)}+x)\bullet(X_\zeta + x - \widehat{X}^{(r)})\right)\mathbf{1}_{\{r<\widehat{\zeta}_e\}}\mathrm{d}r\right] \\ &= \int_0^\infty \mathrm{d}x\psi(x)\,(\mathbb{P}_{-x\nearrow 0}\otimes\widehat{\boldsymbol{n}})\left[\int_0^\infty F_{\zeta+r}\left((X^{(\zeta)}+x)\bullet(x - \widehat{X}^{(r)})\right)\mathbf{1}_{\{r<\widehat{\zeta}_e\}}\mathrm{d}r\right]. \end{aligned} \tag{5.10}$$

Here we use the fact that $X_\zeta = 0$. By translation by x of $\mathbb{P}_{-x\nearrow 0}$ and then changing of variable $\zeta + r = u$, we have

$$
\begin{aligned}
(5.10) &= \int_0^\infty \mathrm{d}x\psi(x)\,(\mathbb{P}_{0\nearrow x}\otimes \widehat{\boldsymbol{n}})\left[\int_0^\infty F_{\zeta+r}\left(X^{(\zeta)}\bullet(x-\widehat{X}^{(r)})\right)\mathbf{1}_{\{r<\widehat{\zeta}_e\}}\mathrm{d}r\right] \\
&= \int_0^\infty \mathrm{d}x\psi(x)\,(\mathbb{P}_{0\nearrow x}\otimes \widehat{\boldsymbol{n}})\left[\int_0^\infty F_{u}\left(X^{(\zeta)}\bullet(x-\widehat{X}^{(u-\zeta)})\right)\mathbf{1}_{\{u-\zeta<\widehat{\zeta}_e\}}\mathbf{1}_{\{u>\zeta\}}\mathrm{d}u\right].
\end{aligned} \tag{5.11}
$$

This identity holds with F_t replaced by $\mathrm{e}^{-qt}F_t$ for any $q>0$, and hence, by uniqueness of the Laplace transform, we obtain

$$
\mathbb{P}\left[F_t(X^{(t)})\right] = \int_0^\infty \mathrm{d}x\psi(x)\,(\mathbb{P}_{0\nearrow x}\otimes \widehat{\boldsymbol{n}})\left[F_t\left(X^{(\zeta)}\bullet(x-\widehat{X}^{(t-\zeta)})\right)\mathbf{1}_{\{t-\zeta<\widehat{\zeta}_e\}}\mathbf{1}_{\{t>\zeta\}}\right] \tag{5.12}
$$

$$
= \int_0^\infty \mathrm{d}x\psi(x)\int_0^t \mathbb{P}_{0\nearrow x}(\zeta\in\mathrm{d}u)\left(\mathbb{P}^{(u)}_{0\nearrow x}\otimes\widehat{\boldsymbol{n}}\right)\left[F_t\left(X^{(u)}\bullet(x-\widehat{X}^{(t-u)})\right)\mathbf{1}_{\{t-u<\widehat{\zeta}_e\}}\right] \tag{5.13}
$$

$$
= \int_0^\infty \mathrm{d}x\psi(x)\int_0^t \mathbb{P}_{0\nearrow x}(\zeta\in\mathrm{d}u)\boldsymbol{n}(\zeta_e>t-u)\left(\mathbb{P}^{(u)}_{0\nearrow x}\otimes\widehat{\mathbb{M}}^{(t-u)}_x\right)\left[F_t\left(X^{(u)}\bullet\widehat{X}^{(t-u)}\right)\right], \tag{5.14}
$$

which completes the proof. □

Remark 5.6. (i) In the (α,ρ)-stable Lévy case with $\alpha\in(0,2]$ and $\rho\in(0,1)$, it is well-known that (see Lemma 3.2 in [13])

$$
\boldsymbol{n}(\zeta_e>t) = \frac{K\cdot t^{-\rho}}{\Gamma(1-\rho)}, \tag{5.15}
$$

where $K>0$ is some constant, and hence we obtain from (3.3) and (5.3) that

$$
\mathbb{P}\left(S_t\in\mathrm{d}x, g_t\in\mathrm{d}u\right) = K\cdot\mathrm{d}x x^{\alpha\rho-1}\mathbb{P}_{0\nearrow x}(\zeta\in\mathrm{d}u)\frac{(t-u)^{-\rho}}{\Gamma(1-\rho)}. \tag{5.16}
$$

Furthermore, together with the following well-known fact (see, e.g., [4]) that

$$
\mathbb{P}(g_t\in\mathrm{d}u) = \frac{1}{\Gamma(1-\rho)\Gamma(\rho)}u^{\rho-1}(t-u)^{-\rho}\mathrm{d}u, \tag{5.17}
$$

then we obtain

$$
\mathbb{P}(S_t\in\mathrm{d}x|g_t=u)\mathrm{d}u = K\cdot\mathrm{d}x x^{\alpha\rho-1}\mathbb{P}_{0\nearrow x}(\zeta\in\mathrm{d}u)\Gamma(\rho)u^{1-\rho}. \tag{5.18}
$$

(ii) In the Brownian case, i.e., $\alpha=2$ and $\rho=1/2$, we note that $X_t \overset{\text{law}}{=} W_{2t}$ for a 1-dimensional standard Brownian motion (W_t), and we have the following:

$$
\mathbb{P}(S_t\in\mathrm{d}x, g_t\in\mathrm{d}u) = \mathrm{d}x\mathrm{d}u\frac{x}{2\pi\sqrt{u^3(t-u)}}\mathrm{e}^{-\frac{x^2}{4u}}; \tag{5.19}
$$

$$
\mathbb{P}_{0\nearrow x}(\zeta\in\mathrm{d}u) = \mathbb{P}(T_{\{x\}}\in\mathrm{d}u) = \mathrm{d}u\frac{x}{2\sqrt{\pi u^3}}\mathrm{e}^{-\frac{x^2}{4u}}, \tag{5.20}
$$

because of the following well-known facts (see, e.g., p. 102 and p. 80 in [11], respectively):

$$\mathbb{P}(\widetilde{S}_t \in \mathrm{d}x, \widetilde{g}_t \in \mathrm{d}u) = \mathrm{d}x\mathrm{d}u \frac{x}{\pi\sqrt{u^3(t-u)}} \mathrm{e}^{-\frac{x^2}{2u}}; \tag{5.21}$$

$$\mathbb{P}(\widetilde{T}_{\{x\}} \in \mathrm{d}u) = \mathrm{d}u \frac{x}{\sqrt{2\pi u^3}} \mathrm{e}^{-\frac{x^2}{2u}}, \tag{5.22}$$

where $\widetilde{S}_t = \sup_{s\le t} W_s$, $\widetilde{g}_t = \sup\{s \le t : W_s = \widetilde{S}_t\}$, and $\widetilde{T}_A = \inf\{s > 0 : W_s \in A\}$ for a Borel set $A \subset \mathbb{R}$. Thus we can easily check that the equality (5.16) is valid.

Remark 5.7. Assume moreover **(A3)**. Then, thanks to Bertoin's result; Corollary 3.2 in [3], it holds that

$$\lim_{t\to\infty} \mathbb{M}^{(t)}[F(X)] = \mathbb{P}_{0\downarrow 0}[F(X)], \tag{5.23}$$

where

$$\mathbb{M}^{(t)}[F(X)] = \mathbb{M}_0^{(t)}[F(X)] = \frac{\boldsymbol{n}\left[F(-X^{(t)}); \zeta_e > t\right]}{\boldsymbol{n}(\zeta_e > t)}. \tag{5.24}$$

6 Generalised Azéma-Yor martingales and definition of a probability measure $\mathbb{P}^{(f)}$

Let us introduce a generalisation of (1.4) and (1.11). Let $X = ((X_t), \mathbb{P})$ be a Lévy process with notation given in Section 2 and assume **(A1)**, **(A2)** and **(A3)**. Let ψ and h be the functions given by (2.10) and (2.17), respectively. Let f be a non-negative Borel function on $[0, \infty)$ satisfying

$$(0 <) \int_0^\infty f(x)\psi(x)\mathrm{d}x < \infty. \tag{6.1}$$

We introduce the process $(M_t^{(f)}, t \ge 0)$ by

$$M_t^{(f)} = f(S_t)h(S_t - X_t) + \int_{S_t}^\infty f(x)\psi(x - X_t)\mathrm{d}x. \tag{6.2}$$

Theorem 6.1. *$(M_t^{(f)}, t \ge 0)$ is a $((\mathscr{F}_t), \mathbb{P})$-martingale.*

The proof of Theorem 6.1 is done in the same way as in [27] in the stable Lévy case; the coinvariance of the function h plays a key role. Thus we omit it.

We introduce the probability measure $\mathbb{P}^{(f)}$ on $\mathscr{F}_\infty$ as follows:

$$\mathbb{P}^{(f)}|_{\mathscr{F}_t} = \frac{M_t^{(f)}}{M_0^{(f)}} \cdot \mathbb{P}|_{\mathscr{F}_t}. \tag{6.3}$$

Since $(M_t^{(f)})$ is a martingale, the consistency holds, and hence $\mathbb{P}^{(f)}$ is well-defined.

7 The σ-finite measure which unifies the supremum penalisations

Let us consider a Lévy process $X = ((X_t), \mathbb{P})$ with $\mathbb{P}(X_0 = 0) = 1$. In this section we assume:

(A1) absolute continuity condition for the resolvent;

(A2) & **(A2***)** 0 is regular for both $(0, \infty)$ and $(-\infty, 0)$ with respect to X;

(A3) & **(A3***)** $I_\infty = -\infty$ and $S_\infty = \infty$ $\mathbb{P}$-a.s.,

where I_∞ and S_∞ are the overall infimum and supremum of X_t, respectively, i.e., $I_\infty = \inf\{X_t : t \geq 0\}$ and $S_\infty = \sup\{X_t : t \geq 0\}$. Remark again that the condition **(B)** in the (α, ρ)-stable Lévy case implies all the above conditions.

We introduce $\mathcal{P}_{\sup}$ as follows.

Definition 7.1. Define

$$\mathcal{P}_{\sup} = \int_0^\infty \mathrm{d}x \psi(x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}), \tag{7.1}$$

where $\mathbb{P}_{0\nearrow x}$ denotes the law of $X + x$ under $\mathbb{P}_{-x\nearrow 0}$, i.e., $\mathbb{P}_{0\nearrow x}$ denotes the law of the process starting from 0 and conditioned to hit x continuously, and $\mathbb{P}_{x\downarrow x}$ denotes the law of $X + x$ under $\mathbb{P}_{0\downarrow 0}$, i.e., $\mathbb{P}_{x\downarrow x}$ denotes the law of the process starting from x and conditioned to stay below level x.

Denote

$$g = \sup\{t \geq 0 : X_t = S_\infty\}. \tag{7.2}$$

Theorem 7.2. *The following statements hold:*

(i) $\mathcal{P}_{\sup}(S_\infty \in \mathrm{d}x, g \in \mathrm{d}u) = \mathrm{d}x\psi(x)\mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u)$,
in particular, $\mathcal{P}_{\sup}(S_\infty \in \mathrm{d}x) = \mathrm{d}x\psi(x)$*;*

(ii) $\mathcal{P}_{\sup}$ *is a* σ*-finite measure on* $\mathscr{F}_\infty$*;*

(iii) $\mathcal{P}_{\sup}$ *is singular to* $\mathbb{P}$ *on* $\mathscr{F}_\infty$*.*

(iv) For each $t > 0$ *and* $A \in \mathscr{F}_t$*, it holds that*

$$\mathcal{P}_{\sup}(A) = \begin{cases} 0, & \textit{if} \quad \mathbb{P}(A) = 0; \\ \infty, & \textit{if} \quad \mathbb{P}(A) > 0. \end{cases} \tag{7.3}$$

Consequently, $\mathcal{P}_{\sup}$ *is not* σ*-finite on* $\mathscr{F}_t$ *for* $t < \infty$*.*

Proof. (i) We have

$$\mathcal{P}_{\sup} = \int_0^\infty \mathrm{d}x \psi(x) \int_0^\infty \mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u) \left(\mathbb{P}^{(u)}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x} \right), \tag{7.4}$$

and hence

$$\begin{aligned} \mathcal{P}_{\sup}[F(S_\infty)G(g)] &= \int_0^\infty \mathrm{d}x\psi(x) \int_0^\infty \mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u) \left(\mathbb{P}^{(u)}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x} \right) [F(S_\infty)G(g)] \\ &= \int_0^\infty \mathrm{d}x\psi(x)F(x) \int_0^\infty \mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u)G(u), \end{aligned}$$

for any test functions F and G. Thus we obtain the desired result.

(ii) For each $x > 0$, $\mathcal{P}_{\sup}(S_\infty < x) = \int_0^x \psi(y)\mathrm{d}y$ is finite, which shows the desired conclusion.

(iii) We have $\mathcal{P}_{\sup}(S_\infty = \infty) = 0$. On the other hand, we have $\mathbb{P}(S_\infty < \infty) = 0$ by our assumption **(A3*)**. This implies that $\mathcal{P}_{\sup}$ is singular to $\mathbb{P}$ on $\mathscr{F}_\infty$.

(iv) Suppose that $\mathbb{P}(A) = 0$ for $A \in \mathscr{F}_t$. We have

$$\begin{aligned} \mathcal{P}_{\sup}(A) &= \int_0^\infty \mathrm{d}x\psi(x) \left(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}\right)(A) \\ &= \int_0^\infty \mathrm{d}x\psi(x) \left(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}\right)[\mathbf{1}_A; t < \zeta] + \int_0^\infty \mathrm{d}x\psi(x) \left(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}\right)[\mathbf{1}_A; t \geq \zeta] \\ &=: I_1 + I_2. \end{aligned}$$

On one hand, we have

$$\begin{aligned} I_1 &= \int_0^\infty \mathrm{d}x\psi(x)\mathbb{P}_{0\nearrow x}[\mathbf{1}_A; t < \zeta] \\ &= \int_0^\infty \mathrm{d}x\mathbb{P}\left[\psi(x - X_t)\mathbf{1}_{\{t < T_{(x,\infty)}\}}\mathbf{1}_A\right] \quad \text{(by (4.8))} \\ &= 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} I_2 &= \int_0^\infty \mathrm{d}x\psi(x) \left(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}\right)[\mathbf{1}_A(X); t \geq \zeta] \\ &= \int_0^\infty \mathrm{d}x\psi(x) \left(\mathbb{P}_{0\nearrow x} \otimes \widehat{\mathbb{P}}_{0\downarrow 0}\right) \left[\mathbf{1}_A \left(X^{(\zeta)} \bullet \left(x + \widehat{X}^{(t-\zeta)}\right)\right) \mathbf{1}_{\{t \geq \zeta\}}\right] \\ &= \int_0^\infty \mathrm{d}x\psi(x) \left(\mathbb{P}_{0\nearrow x} \otimes \widehat{\boldsymbol{n}}\right) \left[h(\widehat{X}_{t-\zeta})\mathbf{1}_{\{t-\zeta < \widehat{\zeta}_e\}}\mathbf{1}_A \left(X^{(\zeta)} \bullet \left(x - \widehat{X}^{(t-\zeta)}\right)\right) \mathbf{1}_{\{t \geq \zeta\}}\right], \end{aligned} \tag{7.5}$$

by the definition of $\mathbb{P}_{0\downarrow 0}$. Then

$$\begin{aligned} (7.5) &= \int_0^\infty \mathrm{d}x\psi(x) \left(\mathbb{P}_{0\nearrow x} \otimes \widehat{\boldsymbol{n}}\right) \left[h\left(x - (x - \widehat{X}_{t-\zeta})\right) \mathbf{1}_A(X)\mathbf{1}_{\{0 \leq t-\zeta < \widehat{\zeta}_e\}}\right] \\ &= \int_0^\infty \mathrm{d}x\psi(x) \left(\mathbb{P}_{0\nearrow x} \otimes \widehat{\boldsymbol{n}}\right) \left[h\left(x - \left(X^{(\zeta)} \bullet (x - \widehat{X}^{(t-\zeta)})\right)_t\right) \mathbf{1}_A(X)\mathbf{1}_{\{0 \leq t-\zeta < \widehat{\zeta}_e\}}\right] \\ &= \mathbb{P}\left[h(S_t - X_t)\mathbf{1}_A\right] \quad \text{(by Theorem 5.1)} \\ &= 0. \end{aligned}$$

Thus we obtain $\mathcal{P}_{\sup}(A) = 0$.

Conversely, suppose that $\mathbb{P}(A) > 0$ for $A \in \mathscr{F}_t$. Then we see that

$$\begin{aligned}
\mathcal{P}_{\sup}(A) &\geq \int_0^\infty \mathrm{d}x\psi(x)\mathbb{P}_{0\nearrow x}\left[\mathbf{1}_A; t < \zeta\right] \\
&= \int_0^\infty \mathrm{d}x\mathbb{P}\left[\psi(x - X_t)\mathbf{1}_{\{t<T_{(x,\infty)}\}}\mathbf{1}_A\right] \\
&\geq \int_1^\infty \mathrm{d}x\mathbb{P}\left[\psi(x - X_t)\mathbf{1}_{\{t<T_{(1,\infty)}\}}\mathbf{1}_A\right] \\
&= \mathbb{P}\left[\{h(\infty) - h(1 - X_t)\}\mathbf{1}_{\{t<T_{(1,\infty)}\}}\mathbf{1}_A\right].
\end{aligned}$$

Since we have

$$h(\infty) = \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \mathbb{P}\left[\int_0^\infty \mathbf{1}_{\{S_t \leq x\}}\mathrm{d}L_t\right] = \mathbb{P}\left[\int_0^\infty \mathrm{d}L_t\right] = \mathbb{P}[L_\infty] = \infty,$$

thus $\mathcal{P}_{\sup}(A) = \infty$. Therefore the proof is completed. □

We shall give some relationships between the measures $\mathcal{P}_{\sup}$, $\mathbb{P}$ and $\mathbb{P}^{(f)}$.

Theorem 7.3. *It holds that*

$$\mathcal{P}_{\sup}\left[f(S_\infty)F_t(X)\right] = \mathbb{P}\left[M_t^{(f)}F_t(X)\right]. \tag{7.6}$$

Consequently, one has

$$\frac{\mathcal{P}_{\sup}\left[f(S_\infty)F_t(X)\right]}{\mathcal{P}_{\sup}\left[f(S_\infty)\right]} = \mathbb{P}\left[\frac{M_t^{(f)}}{M_0^{(f)}}F_t(X)\right] = \mathbb{P}^{(f)}[F_t(X)], \tag{7.7}$$

and

$$\frac{f(S_\infty)\cdot\mathcal{P}_{\sup}}{\mathcal{P}_{\sup}[f(S_\infty)]} = \mathbb{P}^{(f)} \quad \text{on} \quad \mathscr{F}_\infty. \tag{7.8}$$

Proof. Recall the computation in the proof of Theorem 7.2 (iv). We have

$$\begin{aligned}
\mathcal{P}_{\sup}\left[f(S_\infty)F_t(X)\right] &= \int_0^\infty \mathrm{d}x\psi(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\left[f(S_\infty)F_t(X)\right] \\
&= \int_0^\infty \mathrm{d}x\psi(x)f(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\left[F_t(X)\right],
\end{aligned} \tag{7.9}$$

since $S_\infty = x$ under the measure $\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x}$. Then

$$\begin{aligned}
(7.9) = &\int_0^\infty \mathrm{d}x\psi(x)f(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\left[F_t(X); t < \zeta\right] \\
&+ \int_0^\infty \mathrm{d}x\psi(x)f(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\left[F_t(X); t \geq \zeta\right] \\
=:& \, I_1 + I_2.
\end{aligned}$$

On one hand, we have

$$\begin{aligned} I_1 &= \int_0^\infty \mathrm{d}x \psi(x) f(x) \mathbb{P}_{0\nearrow x}\left[F_t(X); t<\zeta\right] = \int_0^\infty \mathrm{d}x f(x) \mathbb{P}\left[\psi(x-X_t)\mathbf{1}_{\{t<T_{(x,\infty)}\}} F_t(X)\right] \\ &= \mathbb{P}\left[F_t(X)\int_0^\infty \mathrm{d}x f(x)\psi(x-X_t)\mathbf{1}_{\{S_t\le x\}}\right]. \end{aligned} \tag{7.10}$$

On the other hand, we obtain from the same computation in the proof of (iv) in the previous theorem that

$$\begin{aligned} I_2 &= \int_0^\infty \mathrm{d}x \psi(x) f(x) (\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x})\left[F_t(X); t\ge\zeta\right] \\ &= \int_0^\infty \mathrm{d}x \psi(x) f(x) (\mathbb{P}_{0\nearrow x} \otimes \widehat{\boldsymbol{n}})\left[h(x-X_t)F_t(X)\mathbf{1}_{\{0\le t-\zeta<\widehat{\zeta}_e\}}\right] \\ &= \int_0^\infty \mathrm{d}x \psi(x) (\mathbb{P}_{0\nearrow x} \otimes \widehat{\boldsymbol{n}})\left[f(S_t)h(S_t-X_t)\mathbf{1}_{\{t-\zeta<\widehat{\zeta}_e\}} F_t(X)\mathbf{1}_{\{t\ge\zeta\}}\right]. \end{aligned} \tag{7.11}$$

By Theorem 5.1, we get

$$(7.11) = \mathbb{P}\left[f(S_t)h\left(S_t-X_t\right)F_t\left(X\right)\right]. \tag{7.12}$$

Combining (7.10) and (7.12), we obtain

$$\begin{aligned} \mathcal{P}_{\sup}\left[f(S_\infty)F_t\right] &= \mathbb{P}\left[F_t(X)\int_{S_t}^\infty \mathrm{d}x f(x)\psi(x-X_t)\right] + \mathbb{P}\left[F_t(X)f(S_t)h\left(S_t-X_t\right)\right] \\ &= \mathbb{P}\left[F_t(X)\left\{\int_{S_t}^\infty \mathrm{d}x f(x)\psi(x-X_t) + f(S_t)h\left(S_t-X_t\right)\right\}\right], \end{aligned} \tag{7.13}$$

that is,

$$\mathcal{P}_{\sup}\left[f(S_\infty)F_t\right] = \mathbb{P}\left[M_t^{(f)}F_t\right]. \tag{7.14}$$

Especially, when $t=0$, we have

$$\mathcal{P}_{\sup}\left[f(S_\infty)\right] = \int_0^\infty \mathrm{d}x f(x)\psi(x). \tag{7.15}$$

Therefore we obtain

$$\frac{\mathcal{P}_{\sup}\left[f(S_\infty)F_t(X)\right]}{\mathcal{P}_{\sup}\left[f(S_\infty)\right]} = \mathbb{P}\left[\frac{M_t^{(f)}}{M_0^{(f)}}F_t(X)\right] = \mathbb{P}^{(f)}[F_t(X)]. \tag{7.16}$$

This completes the proof. □

The measure $\mathcal{P}_{\sup}$ does not depend upon f. Recall that $\mathbb{P}^{(f)}$ is the limit measure of supremum penalisation. The measure $\mathcal{P}_{\sup}$ implies the following fact that gives the detailed description of $\mathbb{P}^{(f)}$.

Theorem 7.4. *One has*

$$\mathbb{P}^{(f)} = \int_0^\infty \mathbb{P}^{(f)}(S_\infty \in \mathrm{d}x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}). \tag{7.17}$$

That is, it holds that, under $\mathbb{P}^{(f)}$,

(i) $\mathbb{P}^{(f)}(S_\infty \in \mathrm{d}x) = \frac{1}{M_0^{(f)}}\psi(x)f(x)\mathrm{d}x$ *where* $M_0^{(f)} = \int_0^\infty \psi(x)f(x)\mathrm{d}x$;

(ii) *given* $g = u$, $(X_s, s \le u)$ *and* $(X_u - X_{u+s}, s \ge 0)$ *are independent;*

(iii) *given* $S_\infty = x$ *and* $g = u$, $(X_s, s \le u)$ *is distributed as the process conditioned to hit* x *continuously with duration* u*;*

(iv) *given* $S_\infty = x$ *and* $g = u$, $(x - X_{u+s}, s \ge 0)$ *is distributed as the process conditioned to stay negative.*

Under our assumption in this section, the following result for the martingale $(M_t^{(f)})$ can be proved.

Theorem 7.5. *Let* $X = ((X_t), \mathbb{P})$ *be a Lévy process with* **(A1)**, **(A2)**, **(A2*)**, **(A3)** *and* **(A3*)**, *and let* $M_t^{(f)}$ *be the process given in* (6.2). *Then* $M_t^{(f)}$ *converges to* 0 $\mathbb{P}$*-a.s. as* $t \to \infty$.

Proof. We show that $M_t^{(f)} \to 0$ a.s. through the measure $\mathcal{P}_{\sup}$. Since $(M_t^{(f)})$ is a non-negative $\mathbb{P}$-martingale as proved before, there exists a $\mathscr{F}_\infty$-measurable functional $M_\infty^{(f)}$ such that $M_t^{(f)} \to M_\infty^{(f)}$ $\mathbb{P}$-a.s. by the martingale convergence theorem. For $a > 0$,

$$\begin{aligned}
\mathbb{P}\left[M_\infty^{(f)}\right] &= \mathbb{P}\left[M_\infty^{(f)}\mathbf{1}_{\{S_\infty \ge a\}}\right] \quad \text{(by the fact that } \mathbb{P}(S_\infty = \infty) = 1\text{)} \\
&\le \liminf_{t\to\infty} \mathbb{P}\left[M_t^{(f)}\mathbf{1}_{\{S_t \ge a\}}\right] \quad \text{(by Fatou's lemma)} \\
&= \liminf_{t\to\infty} \mathcal{P}_{\sup}\left[f(S_\infty)\mathbf{1}_{\{S_t \ge a\}}\right] \quad \text{(by (7.7))} \\
&= \mathcal{P}_{\sup}\left[f(S_\infty)\mathbf{1}_{\{S_\infty \ge a\}}\right]. \quad \text{(by the dominated convergence theorem)}
\end{aligned}$$

Letting $a \to \infty$, then $\mathcal{P}_{\sup}\left[f(S_\infty)\mathbf{1}_{\{S_\infty \ge a\}}\right] \to 0$. Thus $\mathbb{P}[M_\infty^{(f)}] = 0$, and therefore we obtain $\mathbb{P}(M_\infty^{(f)} = 0) = 1$. ☐

Finally, we mention the following relationship between $\mathcal{P}_{\sup}$ and the law $\mathbb{P}$.

Proposition 7.6. *It holds that*

$$\mathcal{P}_{\sup}[\mathbf{1}_{\{g \le t\}}F_t(X)] = \mathbb{P}[h(S_t - X_t)F_t(X)], \tag{7.18}$$

for every $\mathscr{F}_t$*-measurable functional* F_t.

Proof. For $\lambda > 0$, let $f_\lambda(x) = \mathrm{e}^{-\lambda x}$. We note that

$$M_t^{(f_\lambda)} = \mathrm{e}^{-\lambda S_t} h(S_t - X_t) + \int_{S_t}^{\infty} \mathrm{e}^{-\lambda x} \psi(x - X_t) \mathrm{d}x, \tag{7.19}$$

where $M_t^{(f)}$ is defined as (6.2). By Theorem 7.3, we have

$$\begin{aligned}
\mathbb{P}\left[M_t^{(f_\lambda)} \mathrm{e}^{\lambda S_t} F_t(X)\right] &= \mathcal{P}_{\sup}\left[f_\lambda(S_\infty) \mathrm{e}^{\lambda S_t} F_t(X)\right] \\
&= \mathcal{P}_{\sup}\left[\mathrm{e}^{-\lambda(S_\infty - S_t)} F_t(X)\right] \\
&= \mathcal{P}_{\sup}\left[\mathrm{e}^{-\lambda(S_\infty - S_t)} \mathbf{1}_{\{g \le t\}} F_t(X)\right] + \mathcal{P}_{\sup}\left[\mathrm{e}^{-\lambda(S_\infty - S_t)} \mathbf{1}_{\{g > t\}} F_t(X)\right] \\
&= \mathcal{P}_{\sup}\left[\mathbf{1}_{\{g \le t\}} F_t(X)\right] + \mathcal{P}_{\sup}\left[\mathrm{e}^{-\lambda(S_\infty - S_t)} \mathbf{1}_{\{g > t\}} F_t(X)\right].
\end{aligned}$$

Letting $\lambda \to \infty$, we obtain the desired conclusion. □

8 Some remarks on $\mathcal{P}$ and $\mathcal{P}_{\sup}$

Recall the σ-finite measure $\mathcal{P}$ which is given in [26] (see also [24]):

$$\mathcal{P} = \int_0^\infty \mathbb{P}[\mathrm{d}L_u^X](\mathbb{Q}^{(u)} \bullet \mathbb{P}^\times), \tag{8.1}$$

where L_t^X denotes the local time at 0 of X itself, $\mathbb{Q}^{(u)}$ denotes the law of the stable bridge from 0 to 0 with length u and $\mathbb{P}^\times$ denotes the h-transform process with respect to the harmonic function $|x|^{\alpha-1}$ of the process killed at the first hitting time of 0. On comparison, it becomes clear that the two σ-finite measures $\mathcal{P}_{\sup}$ and $\mathcal{P}$ are quite different: $\mathcal{P}_{\sup}$ is based on the excursion theory for the reflected process of a Lévy process, whereas $\mathcal{P}$ comes from the excursion theory for a Lévy process itself. We stress that this difference cannot appear in the Brownian case because of the fact that $(S_t, S_t - X_t)_{t \ge 0} \overset{\text{law}}{=} (L_t^X, |X_t|)_{t \ge 0}$ which is known as Lévy's theorem.

Finally, we mention the relationship between $\mathcal{P}_{\sup}$ and $\mathcal{P}$ as follows:

(i) $\mathcal{P} \perp \mathcal{P}_{\sup}$ on $\mathscr{F}_\infty$;

(ii) if $A \in \mathscr{F}_t$, then

$$\mathcal{P}(A) > 0 \quad \Longleftrightarrow \quad \mathcal{P}_{\sup}(A) > 0, \tag{8.2}$$

and both are infinite.

Acknowledgement. The author would like to thank Professor Marc Yor and Professor Loïc Chaumont for giving her many valuable comments.

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