ON THE AUGMENTATION QUOTIENTS OF THE IA-AUTOMPORPHISM GROUP OF A FREE GROUP AND A FREE METABELIAN GROUP

Takao Satoh[1]

Department of Mathematics, Graduate School of Science, Kyoto University,
Kitashirakawaoiwake-cho, Sakyo-ku, Kyoto city 606-8502, Japan

Abstract. In this paper, we study the augmentation quotients of the IA-automorphism group of a free group and a free metabelian group. First, for any group G, we construct a lift of the k-th Johnson homomorphism of the automorphism group of G to the k-th augmentation quotient of the IA-automorphism group of G. Then we study the images of them for the case where G is a free group and a free metabelian group. As a corollary, we detect some $\mathbf{Z}$-free part in each of the augmentation quotients, which can not be detected by the abelianization of the IA-automorphism group.

1. Introduction

Let F_n be a free group of rank $n \geq 2$, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. Let denote $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ the natural homomorphism induced from the abelianization $F_n \to H$. The kernel of ρ is called the IA-automorphism group of F_n, denoted by IA_n. The subgroup IA_n reflects much of the richness and complexity of the structure of $\mathrm{Aut}\, F_n$, and plays important roles on various studies of $\mathrm{Aut}\, F_n$. Although the study of the IA-automorphism group has a long history since its finitely many generators were obtained by Magnus [14] in 1935, the combinatorial group structure of IA_n is still quite complicated. For instance, any presentation for IA_n is not known in general.

We have studied IA_n mainly using the Johnson filtration of $\mathrm{Aut}\, F_n$ so far. The Johnson filtration is one of a descending central series

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

consisting of normal subgroups of $\mathrm{Aut}\, F_n$, which first term is IA_n. (For detail, see Subsection 2.3.) Each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ naturally has a $\mathrm{GL}(n, \mathbf{Z})$-module structure, and from it we can extract some valuable information for IA_n. For example, $\mathrm{gr}^1(\mathcal{A}_n)$ is just the abelianization of IA_n due to Cohen-Pakianathan [6, 7], Farb [9] and Kawazumi [13]. Pettet [19] determined the image of the cup product $\cup_{\mathbf{Q}} : \Lambda^2 H^1(\mathrm{IA}_n, \mathbf{Q}) \to H^2(\mathrm{IA}_n, \mathbf{Q})$ by using the $\mathrm{GL}(n, \mathbf{Q})$-module structure of $\mathrm{gr}^2(\mathcal{A}_n) \otimes_{\mathbf{Z}} \mathbf{Q}$. At the present stage, however, the structures of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ are far from well-known.

On the other hand, compared with the Johnson filtration, the lower central series $\Gamma_{\mathrm{IA}_n}(k)$ of IA_n and its graded quotients $\mathcal{L}_{\mathrm{IA}_n}(k) := \Gamma_{\mathrm{IA}_n}(k)/\Gamma_{\mathrm{IA}_n}(k+1)$ are somewhat

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[1]e-address: takao@math.kyoto-u.ac.jp

easier to handle since we can obtain finitely many generators of $\mathcal{L}_{\mathrm{IA}_n}(k)$ using the Magnus generators of IA_n. Since the Johnson filtration is central, $\Gamma_{\mathrm{IA}_n}(k) \subset \mathcal{A}_n(k)$ for any $k \geq 1$. It is conjectured that $\Gamma_{\mathrm{IA}_n}(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis who showed $\Gamma_{\mathrm{IA}_2}(k) = \mathcal{A}_2(k)$ for each $k \geq 1$. Now, it is known that $\Gamma_{\mathrm{IA}_n}(2) = \mathcal{A}_n(2)$ due to Bachmuth [2], and that $\Gamma_{\mathrm{IA}_n}(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [19].

In this paper, we consider the augmentation quotients of IA_n. Let $\mathbf{Z}[G]$ be the integral group ring of a group G, and $\Delta(G)$ the augmentation ideal of $\mathbf{Z}[G]$. We denote by $Q^k(G) := \Delta^k(G)/\Delta^{k+1}(G)$ the k-th augmentation quotient of G. The augmentation quotients $Q^k(\mathrm{IA}_n)$ of IA_n are seemed to be closely related to the lower central series $\Gamma_{\mathrm{IA}_n}(k)$ as follows. If the Andreadakis's conjecture is true, then each of the graded quotients $\mathcal{L}_{\mathrm{IA}_n}(k)$ is free abelian. Hence using a work of Sandling and Tahara [21], (For details, see Subsection 4.1.), we obtain a conjecture for the $\mathbf{Z}$-module structure of $Q^k(\mathrm{IA}_n)$:

Conjecture 1. *For any $k \geq 1$,*

$$Q^k(\mathrm{IA}_n) \cong \sum \bigotimes_{i=1}^{k} S^{a_i}(\mathcal{L}_{\mathrm{IA}_n}(i))$$

as a $\mathbf{Z}$-module. Here $\sum$ runs over all non-negative integers $a_1, \ldots, a_k$ such that $\sum_{i=1}^{k} i a_i = k$, and $S^a(M)$ means the symmetric tensor product of a $\mathbf{Z}$-module M such that $S^0(M) = \mathbf{Z}$.

We see that this is true for $k = 1$ and 2 from a general argument in group ring theory. (For $k = 2$, see (1) below.) For $k \geq 3$, however, it is still open problem. In general, one of the most standard methods to study the augmentation quotients $Q^k(\mathrm{IA}_n)$ is to consider a natural surjective homomorphism $\pi_k : Q^k(\mathrm{IA}_n) \to Q^k(\mathrm{IA}_n^{\mathrm{ab}})$ induced from the abelianization $\mathrm{IA}_n \to \mathrm{IA}_n^{\mathrm{ab}}$ of IA_n. Furthermore, since $\mathrm{IA}_n^{\mathrm{ab}}$ is free abelian, we have a natural isomorphism $Q^k(\mathrm{IA}_n^{\mathrm{ab}}) \cong S^k(\mathcal{L}_{\mathrm{IA}_n}(1))$. Hence, in the conjecture above, we can detect $S^k(\mathcal{L}_{\mathrm{IA}_n}(1))$ in $Q^k(\mathrm{IA}_n)$ by the abelianization of IA_n.

Then we have a natural question to ask: Determine the structure of the kernel of π_k. More precisely, clarify the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{Ker}(\pi_k)$. In order to attack this problem, in this paper we construct and study a certain homomorphism defined on $Q^k(\mathrm{IA}_n)$ which restriction to $\mathrm{Ker}(\pi_k)$ is non-trivial. For a group G, let $\alpha_k = \alpha_{k,G} : \mathcal{L}_G(k) \to Q^k(G)$ be a homomorphism defined by $\sigma \mapsto \sigma - 1$. One of the main purposes of the paper is to construct a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\mu_k : Q^k(\mathrm{IA}_n) \to \mathrm{Hom}_{\mathbf{Z}}(H, \alpha_{k+1}(\mathcal{L}_n(k+1)))$$

where $\mathcal{L}_n(k)$ is the k-th graded quotient of the lower central series of F_n. Furthermore, for the k-th Johnson homomorphism

$$\tau_k' : \mathcal{L}_{\mathrm{IA}_n}(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$$

defined by $\sigma \mapsto (x \mapsto x^{-1}x^{\sigma})$, (See Subsection 2.3 for details.), we show that $\mu_k \circ \alpha_k = \alpha_{k+1}^* \circ \tau_k'$ where α_{k+1}^* is a natural homomorphism induced from α_{k+1}. Since α_{k,F_n} is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphism for each $k \geq 1$, if we identify $\mathcal{L}_n(k)$ with its image $\alpha_k(\mathcal{L}_n(k))$, we obtain $\mu_k \circ \alpha_k = \tau_k'$. Hence, the homomorphism μ_k can be considered as a lift of the Johnson homomorphism τ_k'. In the following, we naturally identify $\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ for $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$.

Historically, the study of the Johnson homomorphisms was originally begun in 1980 by D. Johnson [11] who determined the abelianization of the Torelli subgroup of the mapping class group of a surface in [12]. Now, there is a broad range of remarkable results for the Johnson homomorphisms of the mapping class group. (For example, see [10] and [15], [16], [17].) These works also inspired the study of the Johnson homomorphisms of $\mathrm{Aut}\,F_n$. Using it, we can investigate the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ and $\mathcal{L}_{\mathrm{IA}_n}(k)$. Recently, it has achieved good progress through the works of many authors, for example, [6], [7], [9], [13], [15], [16], [17] and [19]. In particular, in our previous work [24], we determined the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbf{Q}} := \tau'_k \otimes \mathrm{id}_{\mathbf{Q}}$ for $k \geq 2$ and $n \geq k+2$.

The main theorem of the paper is

Theorem 1. *(See Theorem 4.4.) For $k \geq 3$ and $n \geq k+2$, a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism*

$$\mu_k \oplus \pi_k : Q^k(\mathrm{IA}_n) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1))) \bigoplus Q^k(\mathrm{IA}_n^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

Next, we consider the framework above for a free metabelian group. Let $F_n^M := F_n/[[F_n,F_n],[F_n,F_n]]$ be a free metabelian group of rank n. By the same argument as the free group case, we can consider the IA-automorphism group IA_n^M and the Johnson homomorphism

$$\tau'_k : \mathcal{L}_{\mathrm{IA}_n^M}(k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^M(k+1)$$

of $\mathrm{Aut}\,F_n^M$ where $\mathcal{L}_{\mathrm{IA}_n^M}(k)$ is the k-th graded quotient of the lower central series of IA_n^M, and $\mathcal{L}_n^M(k)$ is that of F_n^M. In our previous work [23], we studied the Johnson homomorphism of $\mathrm{Aut}\,F_n^M$, and determined its cokernel. In particular, we showed that there appears only the Morita obstruction S^kH in $\mathrm{Coker}(\tau'_k)$ for any $k \geq 2$ and $n \geq 4$. We remark that in [23], we determined the cokernel of the Johnson homomorphism τ_k which is defined on the graded quotient of the Johnson filtration of $\mathrm{Aut}\,F_n^M$. Observing our proof, we verify that $\mathrm{Coker}(\tau'_k) = \mathrm{Coker}(\tau_k)$.

Now, similarly to the free group case, we can also construct a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism

$$\mu_k : Q^k(\mathrm{IA}_n^M) \to \mathrm{Hom}_{\mathbf{Z}}(H, \alpha_{k+1}(\mathcal{L}_n^M(k+1)))$$

such that $\mu_k \circ \alpha_k = \alpha^*_{k+1} \circ \tau'_k$. The second purpose of the paper is to show

Theorem 2. *(See Theorem 5.3.) For $k \geq 2$ and $n \geq 4$, a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism*

$$\mu_k \oplus \pi_k : Q^k(\mathrm{IA}_n^M) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))) \bigoplus S^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

In this paper, for arbitrary group G, we construct a lift of the Johnson homomorphism of the automorphism group of G to the augmentation quotients of G. In order to do this, in Section 2, after fixing notation and conventions, we recall the associated graded Lie algebra of a group G, the Johnson homomorphism of the automorphism group of G, and the associated graded ring of the integral group ring $\mathbf{Z}[G]$ of G. In Section 3, we construct an $\operatorname{Aut} G/\mathrm{IA}(G)$-equivariant homomorphism μ_k which is considered as a lift of the Johnson homomorphism. In Sections 4 and 5, we consider the case where G is a free group and a free metabelian group respectively.

Contents

2. Preliminaries

2.1. Notation and conventions.

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group $\operatorname{Aut} G$ of G acts on G from the right. For any $\sigma \in \operatorname{Aut} G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. Associated graded Lie algebra of a group.

For a group G, we define the lower central series of G by the rule

$$\Gamma_G(1) := G, \quad \Gamma_G(k) := [\Gamma_G(k-1), G], \quad k \geq 2.$$

We denote by $\mathcal{L}_G(k) := \Gamma_G(k)/\Gamma_G(k+1)$ the graded quotient of the lower central series of G, and by $\mathcal{L}_G := \bigoplus_{k\geq 1} \mathcal{L}_G(k)$ the associated graded sum. The graded sum $\mathcal{L}_G$

naturally has a graded Lie algebra structure induced from the commutator bracket on G, and called the associated graded Lie algebra of G.

For any $g_1, \ldots, g_t \in G$, a commutator of weight k type of

$$[[\cdots[[g_{i_1}, g_{i_2}], g_{i_3}], \cdots], g_{i_k}], \quad i_j \in \{1, \ldots, t\}$$

with all of its brackets to the left of all the elements occurring is called a simple k-fold commutator among the components $g_1, \ldots, g_t$, and we denote it by

$$[g_{i_1}, g_{i_2}, \cdots, g_{i_k}]$$

for simplicity. In general, if G is generated by $g_1, \ldots, g_t$, then the graded quotient $\mathcal{L}_G(k)$ is generated by the simple k-fold commutators

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}], \quad 1 \leq i_j \leq t$$

as a $\mathbf{Z}$-module.

Let $\rho_G : \operatorname{Aut} G \to \operatorname{Aut} G^{\mathrm{ab}}$ be the natural homomorphism induced from the abelianization of G. The kernel $\mathrm{IA}(G)$ of ρ_G is called the IA-automorphism group of G. Then the automorphism group $\operatorname{Aut} G$ naturally acts on $\mathcal{L}_G(k)$ for each $k \geq 1$, and $\mathrm{IA}(G)$ acts on it trivially. Hence the action of $\operatorname{Aut} G/\mathrm{IA}(G)$ on $\mathcal{L}_G(k)$ is well-defined.

2.3. **Johnson homomorphisms.**

For $k \geq 1$, the action of $\operatorname{Aut} G$ on each nilpotent quotient $G/\Gamma_G(k+1)$ induces a homomorphism

$$\operatorname{Aut} G \to \operatorname{Aut}(G/\Gamma_G(k+1)).$$

For $k = 1$, this homomorphism is just ρ_G. We denote the kernel of the homomorphism above by $\mathcal{A}_G(k)$. Then the groups $\mathcal{A}_G(k)$ define a descending central filtration

$$\mathrm{IA}_G = \mathcal{A}_G(1) \supset \mathcal{A}_G(2) \supset \mathcal{A}_G(3) \supset \cdots .$$

(See [1] for details.) We call it the Johnson filtration of $\operatorname{Aut} G$. For each $k \geq 1$, the group $\operatorname{Aut} G$ acts on $\mathcal{A}_G(k)$ by conjugation, and it naturally induces an action of $\operatorname{Aut} G/\mathrm{IA}(G)$ on $\mathrm{gr}^k(\mathcal{A}_G)$. The graded sum $\mathrm{gr}(\mathcal{A}_G) := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_G)$ has a graded Lie algebra structure induced from the commutator bracket on $\mathrm{IA}(G)$.

To study the $\operatorname{Aut} G/\mathrm{IA}(G)$-module structure of each graded quotient $\mathrm{gr}^k(\mathcal{A}_G)$, we define the Johnson homomorphisms of $\operatorname{Aut} G$ in the following way. For each $k \geq 1$, we consider a homomorphism $\mathcal{A}_G(k) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1))$ defined by

$$\sigma \mapsto (g \mapsto g^{-1} g^{\sigma}), \quad x \in G.$$

Then the kernel of this homomorphism is just $\mathcal{A}_G(k+1)$. Hence it induces an injective homomorphism

$$\tau_k = \tau_{G,k} : \mathrm{gr}^k(\mathcal{A}_G) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1)).$$

The homomorphism τ_k is called the k-th Johnson homomorphism of $\operatorname{Aut} G$. It is easily seen that each τ_k is an $\operatorname{Aut} G/\mathrm{IA}(G)$-equivariant homomorphism. Since each Johnson homomorphism τ_k is injective, it is natural question to determine the image, or equivalently, the cokernel of τ_k in the study of the $\operatorname{Aut} G/\mathrm{IA}(G)$-module $\mathrm{gr}^k(\mathcal{A}_G)$.

Here, we consider another descending filtration of $\mathrm{IA}(G)$. Let $\Gamma_{\mathrm{IA}(G)}(k)$ be the k-th subgroup of the lower central series of $\mathrm{IA}(G)$. Then for each $k \geq 1$, $\Gamma_{\mathrm{IA}(G)}(k)$ is a subgroup of $\mathcal{A}_G(k)$ since the Johnson filtration is a central filtration of $\mathrm{IA}(G)$. In general, it is a natural question to ask whether $\Gamma_{\mathrm{IA}(G)}(k)$ coincides with $\mathcal{A}_G(k)$ or not. For the case where G is a free group F_n of rank n, it is conjectured that $\Gamma_{\mathrm{IA}(F_n)}(k)$ coincides with $\mathcal{A}_{F_n}(k)$ by Andreadakis.

Consider $\mathcal{L}_{\mathrm{IA}(G)}(k) := \Gamma_{\mathrm{IA}(G)}(k)/\Gamma_{\mathrm{IA}(G)}(k+1)$ for each $k \geq 1$. Similarly to $\mathrm{gr}(\mathcal{A}_G)$, the graded sum $\mathcal{L}_{\mathrm{IA}(G)} := \bigoplus_{k\geq 1} \mathcal{L}_{\mathrm{IA}(G)}(k)$ has a graded Lie algebra structure induced from the commutator bracket on $\mathrm{IA}(G)$. The restriction of the homomorphism $\mathcal{A}_G(k) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1))$ to $\Gamma_{\mathrm{IA}(G)}(k)$ also induces an $\mathrm{Aut}\, G/\mathrm{IA}(G)$-equivariant homomorphism

$$\tau'_k = \tau'_{G,k} : \mathcal{L}_{\mathrm{IA}(G)}(k) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1)).$$

In this paper, we also call τ'_k the k-th Johnson homomorphism of $\mathrm{Aut}\, G$.

2.4. **Associated graded ring of a group ring.**

For a group G, let $\mathbf{Z}[G]$ be a group ring of G over $\mathbf{Z}$, and $\Delta(G)$ the augmentation ideal of $\mathbf{Z}[G]$. Namely, $\Delta(G)$ is the kernel of the augmentation map $\varepsilon : \mathbf{Z}[G] \to \mathbf{Z}$ defined by

$$\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g, \quad a_g \in \mathbf{Z}.$$

We denote by $\Delta^k(G) := (\Delta(G))^k$ the k-times product of the augmentation ideal $\Delta(G)$ in $\mathbf{Z}[G]$. For each $k \geq 1$, set

$$Q^k(G) := \Delta^k(G)/\Delta^{k+1}(G),$$
$$\mathrm{gr}(\mathbf{Z}[G]) := \bigoplus_{k\geq 1} Q^k(G).$$

The quotients $Q^k(G)$ are called the augmentation quotients of G. The graded sum $\mathrm{gr}(\mathbf{Z}[G])$ naturally has an associative graded ring structure induced from the product in $\mathbf{Z}[G]$. The ring $\mathrm{gr}(\mathbf{Z}[G])$ is called the associated graded ring of the group ring $\mathbf{Z}[G]$.

In general, one of the most standard methods to study $Q^k(G)$ is to consider a natural surjective homomorphism $\pi_k = \pi_{k,G} : Q^k(G) \to Q^k(G^{\mathrm{ab}})$ induced from the abelianization $G \to G^{\mathrm{ab}}$. Furthermore, if G^{ab} is free abelian, we have an natural isomorphism $Q^k(G^{\mathrm{ab}}) \cong S^k(G^{\mathrm{ab}}) = S^k(\mathcal{L}_G(1))$. (See Corollary 8.2 in [18].) In Subsection 4.2, we study the kernel of π_k for $G = F_n$. We remark that for a group G and $k \geq 1$, $\mathrm{Ker}(\pi_k)$ is generated by elements

$$(g_1 - 1)\cdots(g_k - 1) - (g_{\sigma(1)} - 1)\cdots(g_{\sigma(k)} - 1)$$

as a $\mathbf{Z}$-module for any $g_1, \ldots, g_k \in G$, $1 \leq i_j \leq n$ and $\sigma \in \mathfrak{S}_k$. Here $\mathfrak{S}_k$ denotes the symmetric group of degree k.

Here we consider a relation between $\mathrm{gr}(\mathbf{Z}[G])$ and $\mathcal{L}_G$. For any $g \in \Gamma_G(k)$, it is well known that an element $g - 1 \in \mathbf{Z}[G]$ belongs to $\Delta^k(G)$. Then a map $\Gamma_G(k) \to \Delta^k(G)$ defined by $g \mapsto g - 1$ induces a $\mathbf{Z}$-linear map

$$\alpha_k = \alpha_{k,G} : \mathcal{L}_G(k) \to Q^k(G)$$

and a Lie algebra homomorphism

$$\alpha_G := \bigoplus_{k\geq 1} \alpha_k : \mathcal{L}_G \to \mathrm{gr}(\mathbf{Z}[G])$$

where we consider $\mathrm{gr}(\mathbf{Z}[G])$ as a Lie algebra with a Lie bracket $[x,y] := xy - yx$ for any $x, y \in \mathbf{Z}[G]$. We remark that for any group G, $\alpha_{1,G} : G^{\mathrm{ab}} \to Q^1(G)$ is an isomorphism. Hence, so is π_1. For $k \geq 2$, however, π_k is not injective in general. For $k = 2$, if G is a finitely generated, then we have a split exact sequence of $\mathbf{Z}$-modules:

$$0 \to \mathcal{L}_G(2) \xrightarrow{\alpha_{2,G}} Q^2(G) \xrightarrow{\pi_{2,G}} Q^2(G^{\mathrm{ab}}) \to 0. \tag{1}$$

(For a proof, see Corollary 8.13 of Chapter VIII in [18].) We denote by

$$\alpha_{k+1}^* = \alpha_{k+1,G}^* : \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$$

the natural homomorphism induced from α_{k+1}.

3. A lift of the Johnson homomorphisms to the augmentation quotients

In this section, for a group G, we construct an $\mathrm{Aut}\, G/\mathrm{IA}(G)$-equivariant homomorphism $\mu_k : Q^k(G) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$ such that

$$\mu_k \circ \alpha_{k,\mathrm{IA}(G)} = \alpha_{k+1,G}^* \circ \tau_k'. \tag{2}$$

3.1. **Construction of μ_k.**

For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, set $s_\sigma(x) := x^{-1}x^\sigma \in G$. First, we recall an important and useful lemma due to Andreadakis [1]:

Lemma 3.1. *For any k, $l \geq 1$, $\sigma \in \mathcal{A}_G(k)$ and $x \in \Gamma_G(l)$, we have $s_\sigma(x) \in \Gamma_G(k+l)$.*

For the proof of Lemma 3.1, see in [1]. From this lemma, we see that $s_\sigma(x) - 1 \in \Delta^{k+l}(G)$ for any $\sigma \in \mathcal{A}_G(k)$ and $x \in \Gamma_G(l)$. We often use these facts without any quotation. In order to define a lift of the Johnson homomorphism, we prepare some lemmas.

Lemma 3.2. *For any σ, $\tau \in \mathrm{IA}(G)$ and x, $y \in G$, we have*

(1) $s_{\sigma\tau}(x) = s_\tau(x) \cdot s_\sigma(x)^\tau = s_\tau(x)s_\sigma(x)s_\tau(s_\sigma(x))$.
(2) $s_\sigma(xy) = y^{-1}s_\sigma(x)y \cdot s_\sigma(y) = [y^{-1}, s_\sigma(x)]s_\sigma(x)s_\sigma(y)$.

Proof. The equations follow from

$$\begin{aligned} s_{\sigma\tau}(x) &= x^{-1}x^{\sigma\tau} = x^{-1}x^\tau \cdot (x^{-1}x^\sigma)^\tau = x^{-1}x^\tau \cdot x^{-1}x^\sigma \cdot (x^{-1}x^\sigma)^{-1} \cdot (x^{-1}x^\sigma)^\tau, \\ s_\sigma(xy) &= y^{-1}x^{-1}x^\sigma y^\sigma = y^{-1}x^{-1}x^\sigma y \cdot y^{-1}y^\sigma. \end{aligned}$$

□

Lemma 3.3. *For any $x \in \Gamma_G(k)$ and $\sigma \in \mathrm{IA}(G)$, we have*

$$x^\sigma - x \equiv s_\sigma(x) - 1 \pmod{\Delta^{k+2}(G)}.$$

Proof. This is clear from

$$\begin{aligned} x^\sigma - x &= (x^\sigma - 1) - (x - 1) \\ &= (x(x^{-1}x^\sigma) - 1) - (x - 1) \\ &= (x - 1)(s_\sigma(x) - 1) + (s_\sigma(x) - 1) \end{aligned}$$

and $s_\sigma(x) - 1 \in \Delta^{k+1}(G)$. $\square$

Lemma 3.4. *For any $a \in \Delta^k(G)$ and $\sigma \in \mathrm{IA}(G)$, we have $a^\sigma - a \in \Delta^{k+1}(G)$.*

Proof. Any element of $\Delta^k(G)$ can be written as a $\mathbf{Z}$-linear combination of elements types of

$$(x_1 - 1) \cdots (x_k - 1) \ \text{ or } \ (x_1 - 1) \cdots (x_{k+1} - 1)$$

for $x_i \in G$. Hence it suffices to show the lemma for $a = (x_1 - 1) \cdots (x_k - 1)$. Then we have

$$\begin{aligned} a^\sigma - a &= (x_1(x_1^{-1}x_1^\sigma) - 1) \cdots (x_k(x_k^{-1}x_k^\sigma) - 1) - (x_1 - 1) \cdots (x_k - 1), \\ &= \{(x_1 - 1)(x_1^{-1}x_1^\sigma - 1) + (x_1 - 1) + (x_1^{-1}x_1^\sigma - 1)\} \\ &\quad \cdots \{(x_k - 1)(x_k^{-1}x_k^\sigma - 1) + (x_k - 1) + (x_k^{-1}x_k^\sigma - 1)\} \\ &\quad - (x_1 - 1) \cdots (x_k - 1), \\ &\equiv (x_1 - 1) \cdots (x_k - 1) - (x_1 - 1) \cdots (x_k - 1) = 0 \pmod{\Delta^{k+1}(G)}. \end{aligned}$$

$\square$

For any $x \in G$, consider a $\mathbf{Z}$-linear homomorphism $\varphi_x : \mathbf{Z}[\mathrm{IA}(G)] \to \Delta(G)$ defined by $\sigma \mapsto s_\sigma(x) - 1$ for any $\sigma \in \mathrm{IA}(G)$.

Lemma 3.5. *For any $k, l \geq 1$, $x \in \Gamma_G(l)$, and $\sigma_1, \ldots, \sigma_k \in \mathrm{IA}(G)$, we have*

$$\varphi_x((\sigma_1 - 1) \cdots (\sigma_k - 1)) \equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots (s_{\sigma_1}(x)) \cdots)) - 1 \pmod{\Delta^{k+l+1}(G)}.$$

Proof. We prove this lemma by the induction on $k \geq 1$. For $k = 1$, it is obvious by the definition. Assume that $k \geq 2$. Write

$$(\sigma_1 - 1) \cdots (\sigma_{k-1} - 1) = \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \sigma \in \mathbf{Z}[\mathrm{IA}(G)]$$

for $a_\sigma \in \mathbf{Z}$. Then we have

$$\begin{aligned}
&\varphi_x((\sigma_1 - 1)\cdots(\sigma_{k-1} - 1)(\sigma_k - 1)),\\
&\quad = \varphi_x((\sigma_1 - 1)\cdots(\sigma_{k-1} - 1)\sigma_k - (\sigma_1 - 1)\cdots(\sigma_{k-1} - 1)),\\
&\quad = \varphi_x\Big(\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma\, \sigma\sigma_k - \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \sigma\Big),\\
&\quad = \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \{(s_{\sigma\sigma_k}(x) - 1) - (s_\sigma(x) - 1)\},\\
&\quad = \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \{(s_{\sigma_k}(x) s_\sigma(x)^{\sigma_k} - 1) - (s_\sigma(x) - 1)\},\\
&\quad = \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \{(s_{\sigma_k}(x) - 1)(s_\sigma(x)^{\sigma_k} - 1) + (s_{\sigma_k}(x) - 1)\\
&\qquad\qquad + (s_\sigma(x)^{\sigma_k} - 1) - (s_\sigma(x) - 1)\}.
\end{aligned}$$

Here we see

$$\begin{aligned}
\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma (s_{\sigma_k}(x) - 1)(s_\sigma(x)^{\sigma_k} - 1) &= (s_{\sigma_k}(x) - 1)\Big(\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma (s_\sigma(x) - 1)\Big)^{\sigma_k}\\
&\equiv 0 \pmod{\Delta^{k+l+1}(G)}
\end{aligned}$$

since $s_{\sigma_k}(x) - 1 \in \Delta^2(G)$ and $\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma (s_\sigma(x) - 1) \in \Delta^{k+l-1}(G)$ by the inductive hypothesis, and see

$$\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma (s_{\sigma_k}(x) - 1) = (s_{\sigma_k}(x) - 1) \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma = 0.$$

On the other hand, by the inductive hypothesis, we have

$$\begin{aligned}
&\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \{(s_\sigma(x)^{\sigma_k} - 1) - (s_\sigma(x) - 1)\},\\
&\quad = \Big(\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma (s_\sigma(x) - 1)\Big)^{\sigma_k} - \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma (s_\sigma(x) - 1),\\
&\quad = (s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots) - 1)^{\sigma_k} - (s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots) - 1)\\
&\qquad + a^{\sigma_k} - a
\end{aligned}$$

for some $a \in \Delta^{k+l}(G)$. Then, by Lemmas 3.3 and 3.4, we see

$$\equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots)) - 1 \pmod{\Delta^{k+l+1}(G)}.$$

This completes the proof of Lemma 3.5. □

For each $k \geq 1$, since $\Delta^k(\mathrm{IA}(G))$ is generated by elements types of

$$(\sigma_1 - 1)\cdots(\sigma_k - 1) \text{ or } (\sigma_1 - 1)\cdots(\sigma_{k+1} - 1)$$

for $\sigma_i \in \mathrm{IA}(G)$ as a $\mathbf{Z}$-module, by Lemma 3.5 we obtain

Corollary 3.6. *For any $k, l \geq 1$ and $x \in \Gamma_G(l)$, we have $\varphi_x(\Delta^k(\mathrm{IA}(G))) \subset \Delta^{k+l}(\mathrm{IA}(G))$.*

Remark 3.7. *For any $x \in \Gamma_G(l)$ a homomorphism $\mathbf{Z}[\mathrm{IA}(G)] \to Q^{k+l}(\mathrm{IA}(G))$ defined by $a \mapsto \varphi_x(a)$ is a polynomial map of degree $\leq k$.*

Lemma 3.8. *For any $k,\ l \geq 1$ and $x, y \in \Gamma_G(l)$, we have*

$$\begin{aligned} &s_{\sigma_k}(\cdots(s_{\sigma_1}(xy))\cdots) \\ &\qquad \equiv s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) \pmod{\Gamma_G(k+2l+1)} \end{aligned}$$

for any $\sigma_1, \ldots, \sigma_k \in \mathrm{IA}(G)$.

Proof. We prove this lemma by the induction on $k \geq 1$. If $k=1$, it is trivial from the part (2) of Lemma 3.2. Assume $k \geq 2$. By the inductive hypothesis, we see

$$s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(xy))) = c\, s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))\cdot s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))$$

for some $c \in \Gamma_G(k+2l)$. Then, using the part (2) of Lemma 3.2 we have

$$\begin{aligned} &s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(xy)))) \\ &\quad = s_{\sigma_k}(c\, s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))\cdot s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))), \\ &\quad = [\{s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))\}^{-1}, s_{\sigma_k}(c)] \\ &\qquad\quad \cdot s_{\sigma_k}(c)\cdot s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))\cdot s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))), \\ &\quad \equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))\cdot s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))), \\ &\quad = [s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y))))^{-1}, s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))))] \\ &\qquad\quad \cdot s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))))\cdot s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))), \\ &\quad \equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))))\cdot s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))). \end{aligned}$$

modulo $\Gamma_G(k+2l+1)$. $\square$

Lemma 3.9. *For any $k,\ l \geq 1$, $x,\ y \in \Gamma_G(l)$, and $a \in \Delta^k(\mathrm{IA}(G))$, we have*

$$\varphi_{xy}(a) \equiv \varphi_x(a) + \varphi_y(a) \pmod{\Delta^{k+l+1}(G)}.$$

Proof. First, we consider the case where $a = (\sigma_1 - 1)\cdots(\sigma_k - 1)$ for some $\sigma_i \in \mathrm{IA}(G)$. From Lemmas 3.5 and 3.8, we see

$$\begin{aligned} \varphi_{xy}(a) &\equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(xy))\cdots)) - 1, \\ &= cs_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) - 1 \end{aligned}$$

for some $c \in \Gamma_G(k+2l+1)$. Hence we have

$$\begin{aligned} &= (c-1)(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) - 1), \\ &\qquad\qquad + (c-1) + (s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) - 1), \\ &\equiv s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) - 1, \\ &= (s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots) - 1)(s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) - 1) \\ &\qquad\qquad + (s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots) - 1) + (s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) - 1), \\ &\equiv (s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots) - 1) + (s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) - 1), \\ &= \varphi_x(a) + \varphi_y(a) \end{aligned}$$

modulo $\Delta^{k+l+1}(G)$.

For a general case, $a \in \Delta^k(\mathrm{IA}(G))$ is written as a $\mathbf{Z}$-linear combination of elements types of

$$(\sigma_1 - 1)\cdots(\sigma_k - 1) \text{ or } (\sigma_1 - 1)\cdots(\sigma_{k+1} - 1).$$

Therefore, using the argument above, we obtain the Lemma for any $a \in \Delta^k(\mathrm{IA}(G))$. $\square$

Lemma 3.10. *For any $a \in \Delta^k(\mathrm{IA}(G))$, a map $\mu_k(a) : G^{\mathrm{ab}} \to Q^{k+1}(G)$ defined by $x \mapsto \varphi_x(a)$ is a homomorphism.*

Proof. To begin with, we check that $\mu_k(a)$ is well-defined. Consider elements $x, y \in G$ such that $y = xc$ for some $c \in \Gamma_G(2)$. Then by Lemma 3.9,

$$\varphi_y(a) = \varphi_{xc}(a) \equiv \varphi_x(a) + \varphi_c(a) \qquad (\mathrm{mod}\ \Delta^{k+2}(G)).$$

On the other hand, by Corollary 3.6, we see $\varphi_c(a) \in \Delta^{k+2}(G)$. Hence $\varphi_y(a) = \varphi_x(a) \in Q^{k+1}(G)$.

To show $\mu_k(a)$ is a homomorphism, take any x and $y \in G$. Then by Lemma 3.9,

$$\mu_k(a)(xy) = \varphi_{xy}(a) \equiv \varphi_x(a) + \varphi_y(a) = \mu_k(a)(x) + \mu_k(a)(y)$$

modulo $\Delta^{k+2}(G)$. This completes the proof of Lemma 3.10. $\square$

Now, we are ready to define a lift of the Johnson homomorphism τ'_k. For any $k \geq 1$, define a map

$$\mu_k : \Delta^k(\mathrm{IA}(G)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$$

by

$$a \mapsto (x \mapsto \varphi_x(a)).$$

The map μ_k is a homomorphism. Furthermore $\Delta^{k+1}(\mathrm{IA}(G))$ is contained in $\mathrm{Ker}(\mu_k)$. Hence μ_k induces a homomorphism

$$Q^k(\mathrm{IA}(G)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G)).$$

We also denote by μ_k this induced homomorphism, and call it the k-th Johnson homomorphism of $\mathbf{Z}[\mathrm{IA}(G)]$. We see that the compatibility (2) follows by the definition of τ'_k and μ_k.

3.2. **Actions of** $\mathrm{Aut}\, G$.

Next we consider actions of $\mathrm{Aut}\, G$. Since $\mathrm{IA}(G)$ is a normal subgroup of $\mathrm{Aut}\, G$, the group $\mathrm{Aut}\, G$ acts on $\mathbf{Z}[\mathrm{IA}(G)]$ from the right by

$$\Big(\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \sigma\Big) \cdot \tau := \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma (\tau^{-1}\sigma\tau)$$

for any $\tau \in \mathrm{Aut}\, G$. For each $k \geq 1$, since $\Delta^k(\mathrm{IA}(G))$ is preserved by the action of $\mathrm{Aut}\, G$, the group $\mathrm{Aut}\, G$ also acts on each of the graded quotient $Q^k(\mathrm{IA}(G))$. Then $\mathrm{IA}(G)$ acts on $Q^k(\mathrm{IA}(G))$ trivially. In fact, for any $\tau \in \mathrm{IA}(G)$, we have

$$\begin{aligned}
(\sigma_1 - 1)\cdots(\sigma_k - 1)\cdot\tau &= (\tau^{-1}\sigma_1\tau - 1)\cdots(\tau^{-1}\sigma_k\tau - 1),\\
&= ([\tau^{-1}, \sigma_1]\sigma_1 - 1)\cdots([\tau^{-1}, \sigma_k]\sigma_k\tau - 1),\\
&= \{([\tau^{-1}, \sigma_1] - 1)(\sigma_1 - 1) + ([\tau^{-1}, \sigma_1] - 1) + (\sigma_1 - 1)\}\\
&\qquad \cdots\{([\tau^{-1}, \sigma_k] - 1)(\sigma_k - 1) + ([\tau^{-1}, \sigma_k] - 1) + (\sigma_k - 1)\},\\
&\equiv (\sigma_1 - 1)\cdots(\sigma_k - 1)
\end{aligned}$$

module $\Delta^{k+1}(\mathrm{IA}(G))$ since $[\tau^{-1}, \sigma_i] \in \Gamma_{\mathrm{IA}(G)}(2)$ and $[\tau^{-1}, \sigma_i] - 1 \in \Delta^2(\mathrm{IA}(G))$. Since $Q^k(\mathrm{IA}(G))$ is generated by elements $(\sigma_1 - 1)\cdots(\sigma_k - 1)$ for $\sigma_i \in \mathrm{IA}(G)$ as a $\mathbf{Z}$-module, we verify that the action of $\mathrm{IA}(G)$ on $Q^k(\mathrm{IA}(G))$ is trivial. Hence the quotient group $\mathrm{Aut}\, G/\mathrm{IA}(G)$ naturally acts on each of $Q^k(\mathrm{IA}(G))$ from the right.

Now, $\mathrm{Aut}\, G$ naturally acts on $\mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$. Then it is easily seen that the action of $\mathrm{IA}(G)$ on $\mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$ is trivial. Hence the quotient group $\mathrm{Aut}\, G/\mathrm{IA}(G)$ also acts on it. To show that μ_k is $\mathrm{Aut}\, G/\mathrm{IA}(G)$-equivariant, we prepare

Lemma 3.11. *For any $k \geq 1$, and $\sigma, \sigma_1, \ldots, \sigma_k \in \mathrm{Aut}\, G$, we have*

$$(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma} = s_{\sigma^{-1}\sigma_k\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}))\cdots).$$

We prove this lemma by the induction on $k \geq 1$. For $k = 1$, it is clear by

$$s_{\sigma_1}(x)^{\sigma} = (x^{-1}x^{\sigma_1})^{\sigma} = (x^{\sigma})^{-1}x^{\sigma_1\sigma} = (x^{\sigma})^{-1}(x^{\sigma})^{\sigma^{-1}\sigma_1\sigma} = s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}).$$

Assume $k \geq 2$. Using the inductive hypothesis, we obtain

$$\begin{aligned}
&(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma} \\
&\quad = \big((s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))^{-1}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma_k}\big)^{\sigma}, \\
&\quad = \{(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma}\}^{-1}\{(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma}\}^{\sigma^{-1}\sigma_k\sigma}, \\
&\quad = \{s_{\sigma^{-1}\sigma_{k-1}\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}))\cdots)\}^{-1}\{s_{\sigma^{-1}\sigma_{k-1}\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}))\cdots)\}^{\sigma^{-1}\sigma_k\sigma}, \\
&\quad = s_{\sigma^{-1}\sigma_k\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}))\cdots).
\end{aligned}$$

This completes the proof of Lemma 3.11. □

Proposition 3.12. *For any $k \geq 1$, the Johnson homomorphism μ_k is an* $\mathrm{Aut}\, G/\mathrm{IA}(G)$*-equivariant homomorphism.*

Proof. It suffices to show $\mu_k(a^{\sigma}) = (\mu_k(a))^{\sigma}$ for $\sigma \in \mathrm{IA}(G)$ and $a = (\sigma_1 - 1)\cdots(\sigma_k - 1) \in Q^k(\mathrm{IA}(G))$. Then, for any $x \in G^{\mathrm{ab}}$ we have

$$\begin{aligned}
\mu_k(a^{\sigma})(x) &= \mu_k((\sigma^{-1}\sigma_1\sigma - 1)\cdots(\sigma^{-1}\sigma_k\sigma - 1))(x), \\
&= s_{\sigma^{-1}\sigma_k\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x))\cdots) - 1.
\end{aligned}$$

On the other hand, by Lemma 3.11,

$$\begin{aligned}
(\mu_k(a))^{\sigma}(x) &= (\mu_k(a)(x^{\sigma^{-1}}))^{\sigma} = (s_{\sigma_k}(\cdots(s_{\sigma_1}(x^{\sigma^{-1}}))\cdots) - 1)^{\sigma}, \\
&= s_{\sigma^{-1}\sigma_k\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x))\cdots) - 1.
\end{aligned}$$

for any $x \in G^{\mathrm{ab}}$. This completes the proof of Proposition 3.12. □

3.3. Some properties of μ_k.

Here we observe some properties of μ_k. First, we consider the image of μ_k. In general, μ_k is not surjective.

Lemma 3.13. *For each $k \geq 1$, the image of μ_k is contained in that of $\alpha^*_{k+1,G}$.*

Proof. Since $Q^k(\mathrm{IA}(G))$ is generated by $(\sigma_1 - 1)\cdots(\sigma_k - 1)$ for $\sigma_i \in \mathrm{IA}(G)$ as a $\mathbf{Z}$-module, it suffices to show $\mu_k(a) \in \mathrm{Im}(\alpha^*_{k+1,G})$ for $a = (\sigma_1 - 1)\cdots(\sigma_k - 1)$. On the other hand, using Lemma 3.1 recursively, we see that $s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))$ belongs to $\Gamma_G(k+1)$ for any $x \in G$. Hence

$$s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots)) - 1 \in \alpha_{k+1,G}(\mathcal{L}_G(k+1)).$$

This completes the proof of Lemma 3.13. $\square$

By this lemma, in the following, we write the k-th Johnson homomorphism as

$$\mu_k : Q^k(\mathrm{IA}(G)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \alpha_{k+1,G}(\mathcal{L}_G(k+1))).$$

Next, we consider a calculation of $\mu_{k+1}(a(\tau-1))$ for a given $a \in Q^k(\mathrm{IA}(G))$ and $\tau \in \mathrm{IA}(G)$. Let

$$a = \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}(\sigma_1 - 1)\cdots(\sigma_k - 1)$$

for $m_{\sigma_1,\ldots,\sigma_k} \in \mathbf{Z}$. Then for any $x \in G$, we have

$$\begin{aligned}
\mu_{k+1}(a(\tau-1))(x) &= \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\mu_{k+1}((\sigma_1-1)\cdots(\sigma_k-1)(\tau-1))(x), \\
&\equiv \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\{s_\tau(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)) - 1\}
\end{aligned}$$

modulo $\Delta^{k+3}(G)$. If we set $X := s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots) \in \Gamma_G(k+1)$, then

$$\begin{aligned}
&= \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\{X^{-1}X^\tau - 1\}, \\
&= \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\{(X^{-1}-1)(X^\tau-1) + (X^{-1}-1) + (X^\tau-1)\}, \\
&\equiv \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\{(X^\tau-1) - (X-1)\}, \\
&= \left\{\sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}(X-1)\right\}^\tau - \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}(X-1), \\
&\equiv \{\mu_k(a)(x)\}^\tau - \mu_k(a)(x)
\end{aligned}$$

modulo $\Delta^{k+3}(G)$. Hence we have

$$\mu_{k+1}(a(\tau-1))(x) = \{\mu_k(a)(x)\}^\tau - \mu_k(a)(x) \in Q^{k+2}(\mathrm{IA}(G)).$$

This formula is sometimes convenient for a calculation of the image of μ_k.

4. Free group case

In this section, we mainly consider the case where $G = F_n$. For simplicity, we often omit the capital F from the subscript F_n if there is no confusion. For example, we write $\mathcal{L}_n$, $\mathcal{L}_n(k)$, IA_n, ... for $\mathcal{L}_{F_n}$, $\mathcal{L}_{F_n}(k)$, $\mathrm{IA}(F_n)$, ... respectively. Here, we study the structure of graded quotients $Q^k(\mathrm{IA}_n)$ as a $\mathrm{GL}(n,\mathbf{Z})$-module.

4.1. **Preliminary results for $G = F_n$.**

In this subsection, we recall some well-known properties of the IA-automorphism group IA_n, the graded Lie algebra $\mathcal{L}_n$ and the graded ring $\mathrm{gr}(\mathbf{Z}[F_n])$. Let $H := F_n^{\mathrm{ab}}$ be the abelianization of F_n. The natural homomorphism $\rho = \rho_{F_n} : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ induced from the abelianization of $F_n \to H$ is surjective. Throughout the paper, we identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing a basis of H induced from the basis $x_1, \ldots, x_n$ of F_n. Namely, we have $\mathrm{GL}(n, \mathbf{Z}) \cong \mathrm{Aut}\, F_n/\mathrm{IA}_n$.

Magnus [14] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} {x_j}^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $1 \leq i,\, j \leq n$, and

$$K_{ijl} : x_t \mapsto \begin{cases} x_i [x_j, x_l], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $1 \leq i,\, j,\, l \leq n$ and $j < l$. Recently, Cohen-Pakianathan [6, 7], Farb [9] and Kawazumi [13] independently showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{3}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module. In particular, from their result, we see that $\mathrm{IA}_n^{\mathrm{ab}}$ is a free abelian group of rank $2n^2(n-1)$ with basis the coset classes of the Magnus generators K_{ij} and K_{ijl}.

It is classically known due to Magnus that the graded Lie algebra $\mathcal{L}_n$ is isomorphic to the free Lie algebra generated by H over $\mathbf{Z}$. (See [20], for example, for basic material concerning the free Lie algebra.) Each of the degree k part $\mathcal{L}_n(k)$ of $\mathcal{L}_n$ is a free abelian group, which rank is given by Witt's formula

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{L}_n(k)) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}} \tag{4}$$

where μ is the Möbius function.

Next, we consider an embedding of the free Lie algebra $\mathcal{L}_n$ into the graded sum $\mathrm{gr}(\mathbf{Z}[F_n])$. In general, it is known that the graded Lie algebra homomorphism $\alpha_{F_n} : \mathcal{L}_n \to \mathrm{gr}(\mathbf{Z}[F_n])$ induced from $x \mapsto x - 1$ for any $x \in F_n$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphism, and that $\mathrm{gr}(\mathbf{Z}[F_n])$ is naturally isomorphic to the universal enveloping algebra $\mathcal{U}(\mathcal{L}_n)$ of $\mathcal{L}_n$. (See Theorem 6.2 of Chapter VIII in [18].) For simplicity, in the following, we identify $\mathcal{L}_n(k)$ with its image $\alpha_k(\mathcal{L}_n(k))$ in $Q^k(F_n)$.

Here we observe a conjecture for the $\mathbf{Z}$-module structure of $Q^k(\mathrm{IA}_n)$. For a group G such that each of the graded quotients $\mathcal{L}_G(k)$ is a free abelian group for $k \geq 1$, Sandling and Tahara [21] showed that as a $\mathbf{Z}$-module,

$$Q^k(G) \cong \sum \bigotimes_{i=1}^{k} S^{a_i}(\mathcal{L}_G(i))$$

for each $k \geq 1$. Here $\sum$ runs over all non-negative integers $a_1, \ldots, a_k$ such that

$$\sum_{i=1}^{k} i a_i = k,$$

and $S^a(\mathcal{L}_G(i))$ means the symmetric tensor product of $\mathcal{L}_G(i)$ of degree a such that $S^0(\mathcal{L}_G(i)) = \mathbf{Z}$.

On the other hand, it is conjectured by Andreadakis that the lower central series $\Gamma_{\mathrm{IA}_n}(k)$ coincides with the Johnson filtration $\mathcal{A}_n(k)$. He [1] showed that this is true for $n = 2$. Since each of the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ of the Johnson filtration $\mathcal{A}_n(k)$ is free abelian, the Andreadakis's conjecture let us conjecture

Conjecture 4.1. *For any $k \geq 1$,*

$$Q^k(\mathrm{IA}_n) \cong \sum \bigotimes_{i=1}^{k} S^{a_i}(\mathcal{L}_{\mathrm{IA}_n}(i))$$

as a $\mathbf{Z}$-module. Here $\sum$ runs over all non-negative integers $a_1, \ldots, a_k$ such that $\sum_{i=1}^{k} i a_i = k$.

To study $Q^k(\mathrm{IA}_n)$, to begin with, we consider the surjective homomorphism $\pi_k : Q^k(\mathrm{IA}_n) \to Q^k(\mathrm{IA}_n^{\mathrm{ab}})$ induced from the abelianization of IA_n for $k \geq 1$. We remark that each of π_k is an $\mathrm{GL}(n, \mathbf{Z})$-equivariant surjective homomorphism, and that $Q^k(\mathrm{IA}_n^{\mathrm{ab}}) \cong S^k(\mathrm{IA}_n^{\mathrm{ab}})$ since $\mathrm{IA}_n^{\mathrm{ab}}$ is free abelian as mentioned before. For $k = 1$, $\pi_k : Q^1(\mathrm{IA}_n) \to Q^1(\mathrm{IA}_n^{\mathrm{ab}})$ is an isomorphism, and $Q^1(\mathrm{IA}_n) \cong \mathrm{IA}_n^{\mathrm{ab}} = H^* \otimes_{\mathbf{Z}} \Lambda^2 H$. In general, however, π_k is not injective for $k \geq 2$, and seems to have a large kernel from the conjecture above. In this paper, to investigate the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{Ker}(\pi_k)$, we use the Johnson homomorphism μ_k.

4.2. **The image of $\mu_k|_{\mathrm{Ker}(\pi_k)}$.**

Here we study the image of the Johnson homomorphism

$$\mu_k : Q^k(\mathrm{IA}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \subset H^* \otimes_{\mathbf{Z}} Q^{k+1}(F_n)$$

restricted to the kernel of π_k for a sufficiently large n. We remark that $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) = H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1))$ is generated by elements

$$x_i^* \otimes ([x_{i_1}, \ldots, x_{i_{k+1}}] - 1), \quad 1 \leq i, i_j \leq n$$

as a $\mathbf{Z}$-module. First we consider the case where $k \geq 3$.

Proposition 4.2. *For any $k \geq 3$ and $n \geq k+2$, the homomorphism $\mu_k|_{\mathrm{Ker}(\pi_k)} : \mathrm{Ker}(\pi_k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ is surjective.*

Proof. For any $x_i^* \otimes ([x_{i_1}, \ldots, x_{i_{k+1}}] - 1)$, since $n \geq k+2$, there exists some $1 \leq j \leq n$ such that $j \neq i_1, \ldots, i_{k+1}$.

Case 1. The case where $i_{k+1} \neq i$. Set

$$a := \begin{cases} (K_{iji_{k+1}} - 1)(K_{ji_k} - 1) \cdots (K_{ji_3} - 1)(K_{ji_1 i_2} - 1), & \text{if } j \neq i, \\ (K_{ji_{k+1}} - 1)(K_{ji_k} - 1) \cdots (K_{ji_3} - 1)(K_{ji_1 i_2} - 1), & \text{if } j = i. \end{cases}$$

Then we have $\mu_k(a) = x_i^* \otimes ([x_{i_1}, \ldots, x_{i_{k+1}}] - 1)$. On the other hand, if we set

$$b := \begin{cases} (K_{ji_1i_2} - 1)(K_{ji_3} - 1) \cdots (K_{ji_k} - 1)(K_{iji_{k+1}} - 1), & \text{if } j \neq i, \\ (K_{ji_1i_2} - 1)(K_{ji_3} - 1) \cdots (K_{ji_k} - 1)(K_{ji_{k+1}} - 1), & \text{if } j = i, \end{cases}$$

then $\mu_k(b) = 0$. Hence we obtain $\mu_k(a-b) = x_i^* \otimes ([x_{i_1}, \ldots, x_{i_{k+1}}] - 1)$ for $a - b \in \mathrm{Ker}(\pi_k)$.

Case 2. The case where $i_{k+1} = i$. Set

$$a' := (K_{ij}^{-1} - 1)(K_{ji_k} - 1) \cdots (K_{ji_3} - 1)(K_{ji_1i_2} - 1).$$

Then $\mu_k(a') = x_i^* \otimes ([x_{i_1}, \ldots, x_{i_{k+1}}] - 1)$. On the other hand, if we set

$$b' := (K_{ji_1i_2} - 1)(K_{ji_3} - 1) \cdots (K_{ji_k} - 1)(K_{ij}^{-1} - 1),$$

$\mu_k(b') = 0$. Hence we obtain $\mu_k(a' - b') = x_i^* \otimes ([x_{i_1}, \ldots, x_{i_{k+1}}] - 1)$ for $a' - b' \in \mathrm{Ker}(\pi_k)$. This completes the proof of Proposition 4.2. $\square$

We remark that it seems to difficult to show above for $2 \leq n \leq k+2$ since we can not take $1 \leq j \leq n$ such that $j \neq i_1, \ldots, i_{k+1}$ in general.

As a corollary to Proposition 4.2, we see the surjectivity of μ_k of $\mathbf{Z}[\mathrm{IA}(G)]$ for the case where G is a certain quotient group of F_n. Let C be a characteristic subgroup of F_n such that $C \subset \Gamma_n(2)$, and set $G := F_n/C$. Then we have a natural isomorphism $G^{\mathrm{ab}} \cong H$. The natural projection $\phi : F_n \to G$ induces homomorphisms $Q^k(F_n) \to Q^k(G)$, also denoted by ϕ. Since C is characteristic, $\phi : F_n \to G$ induces a homomorphism $\bar{\phi} : \mathrm{Aut}\, F_n \to \mathrm{Aut}(G)$. Clearly, $\bar{\phi}(\mathrm{IA}_n) \subset \mathrm{IA}(G)$. Furthermore, $\bar{\phi}$ naturally induces homomorphisms $Q^k(\mathrm{IA}_n) \to Q^k(\mathrm{IA}(G))$ which is also denoted by $\bar{\phi}$.

Corollary 4.3. *With the notation above, for any $k \geq 3$ and $n \geq k+2$, the homomorphism $\mu_k : \mathrm{Ker}(\pi_{k,\mathrm{IA}(G)}) \to H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_G(k+1))$ is surjective.*

Proof. It is clear from a commutative diagram

$$\begin{array}{ccc} \mathrm{Ker}(\pi_{k,\mathrm{IA}_n}) & \xrightarrow{\mu_k} & H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1)) \\ {\scriptstyle \bar{\phi}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ \mathrm{Ker}(\pi_{k,\mathrm{IA}(G)}) & \xrightarrow{\mu_k} & H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_G(k+1)) \end{array}$$

where the first row and $\mathrm{id} \otimes \phi$ are surjective. $\square$

For example, if G is a free metabelian group $G = F_n/[\Gamma_n(2), \Gamma_n(2)]$, then the Johnson homomorphism $\mu_k : \mathrm{Ker}(\pi_{k,\mathrm{IA}(G)}) \to H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_G(k+1))$ is surjective for any $k \geq 3$ and $n \geq k+2$. In Section 5, we show that we can improve the condition $k \geq 3$ and $n \geq k+2$ above for $G = F_n/[\Gamma_n(2), \Gamma_n(2)]$.

By Proposition 4.2 and Corollary 4.3, we have

Theorem 4.4. *Let C and G be as above. For $k \geq 3$ and $n \geq k+2$, an* $\mathrm{Aut}(G)/\mathrm{IA}(G)$*-equivariant homomorphism*

$$\mu_k \oplus \pi_k : Q^k(\mathrm{IA}(G)) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1,G}(\mathcal{L}_G(k+1))) \bigoplus Q^k(\mathrm{IA}(G)^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

In particular, for $C = \{1\}$, and hence $G = F_n$, we have a $\mathrm{GL}(n, \mathbf{Z})$-equivariant surjective homomorphism

$$\mu_k \oplus \pi_k : Q^k(\mathrm{IA}_n) \to (H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)) \bigoplus S^k(\mathrm{IA}_n^{\mathrm{ab}})$$

for $k \geq 3$ and $n \geq k+2$.

Finally, we consider the case where $k = 2$. Observing a split exact sequence (1), we see that $\mathrm{Ker}(\pi_2) = \alpha_{2,\mathrm{IA}(G)}(\mathcal{L}_{\mathrm{IA}(G)}(2))$. Hence, from the compatibility (2), we see that $\mathrm{Im}(\mu_2|_{\mathrm{Ker}(\pi_2)}) = \alpha^*_{3,F_n}(\mathrm{Im}(\tau'_2))$. In [22], we showed that for any $n \geq 2$, $\mathrm{Im}(\tau'_2)$, which is equal to $\mathrm{Im}(\tau_2)$, satisfies an exact sequence

$$0 \to \mathrm{Im}(\tau'_2) \xrightarrow{\tau'_2} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to S^2 H \to 0$$

of $\mathrm{GL}(n, \mathbf{Z})$-modules. Hence we see that

Proposition 4.5. *For $n \geq 2$, $\mathrm{Im}(\mu_2|_{\mathrm{Ker}(\pi_2)})$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant proper submodule of $H^* \otimes_{\mathbf{Z}} \alpha_3(\mathcal{L}_n(3))$, which rank is given by*

$$\frac{1}{6} n(n+1)(2n^2 - 2n - 3).$$

Here we remark that μ_2 is surjective.

Lemma 4.6. *For any $n \geq 2$, $\mu_2 : Q^2(\mathrm{IA}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3)$ is surjective.*

Proof. Take an element $x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}] - 1)$. We may assume $i_1 \neq i_2$. If $i_j \neq i$ for $1 \leq j \leq 3$, we see that

$$\mu_2((K_{ii_3} - 1)(K_{ii_1i_2} - 1)) = x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}] - 1).$$

If $i_3 = i$ and $i_1, i_2 \neq i$, then

$$\mu_2((K_{ii_1}^{-1} - 1)(K_{i_1i_2} - 1)) = x_i^* \otimes ([x_{i_1}, x_{i_2}, x_i] - 1).$$

If $i_1 = i$ and $i_2, i_3 \neq i$, then

$$\mu_2((K_{ii_3} - 1)(K_{ii_2} - 1)) = x_i^* \otimes ([x_i, x_{i_2}, x_{i_3}] - 1).$$

If $i_2 = i$ and $i_1, i_3 \neq i$, then

$$\mu_2((K_{ii_3} - 1)(K_{ii_1}^{-1} - 1)) = x_i^* \otimes ([x_{i_1}, x_i, x_{i_3}] - 1).$$

If $i_1 = i_3 = i$, then

$$\mu_2((K_{ii_2}^{-1} - 1)(K_{i_2i}^{-1} - 1)) = x_i^* \otimes ([x_i, x_{i_2}, x_i] - 1).$$

If $i_2 = i_3 = i$, then

$$\mu_2((K_{ii_1}^{-1} - 1)(K_{i_1i}^{-1} - 1)) = x_i^* \otimes ([x_{i_1}, x_i, x_i] - 1).$$

Hence the generators of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3)$ are contained in the image of μ_2. □

5. Free metabelian case

In this section, we mainly consider the case where $G = F_n^M := F_n/[\Gamma_n(2), \Gamma_n(2)]$. For simplicity, we often omit the capital F from the subscript F_n^M if there is no confusion. For example, we write $\mathcal{L}_n^M$, $\mathcal{L}_n^M(k)$, $\mathrm{IA}_n^M, \ldots$ for $\mathcal{L}_{F_n^M}$, $\mathcal{L}_{F_n^M}(k)$, $\mathrm{IA}(F_n^M), \ldots$ respectively. Here, we study the structure of graded quotients $Q^k(\mathrm{IA}_n^M)$ as a $\mathrm{GL}(n, \mathbf{Z})$-module.

5.1. Preliminary results for $G = F_n^M$.

In this subsection, we recall some properties of the IA-automorphism group IA_n^M and the graded Lie algebras $\mathcal{L}_n^M$.

To begin with, we have $(F_n^M)^{\mathrm{ab}} = H$, and hence $\mathrm{Aut}\,(F_n^M)^{\mathrm{ab}} = \mathrm{Aut}(H) = \mathrm{GL}(n, \mathbf{Z})$. Since the surjective map $\rho_{F_n} : \mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbf{Z})$ factors through $\mathrm{Aut}\, F_n^M$, a map $\rho_{F_n^M} : \mathrm{Aut}\, F_n^M \to \mathrm{GL}(n, \mathbf{Z})$ is also surjective. Hence we can identify $\mathrm{Aut}\, F_n^M / \mathrm{IA}(F_n^M)$ with $\mathrm{GL}(n, \mathbf{Z})$.

Let $\nu_n : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, F_n^M$ be the natural homomorphism induced from the action of $\mathrm{Aut}\, F_n$ on F_n^M. Restricting ν_n to IA_n, we obtain a homomorphism $\nu_n|_{\mathrm{IA}_n} : \mathrm{IA}_n \to \mathrm{IA}_n^M$. Bachmuth and Mochizuki [4] showed that $\nu_n|_{\mathrm{IA}_n}$ is surjective for $n \geq 4$. They also showed that in [3] $\nu_3|_{\mathrm{IA}_3}$ is not surjective and IA_3^M is not finitely generated. Hence IA_n^M is finitely generated for $n \geq 4$ by the (coset classes of) Magnus generators K_{ij} and K_{ijl}. We remark that since $\mathrm{Ker}(\nu_n|_{\mathrm{IA}_n})$ is contained in $\mathcal{A}_n(3)$, we have isomorphisms

$$(\mathrm{IA}_n^M)^{\mathrm{ab}} \cong \mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module.

The associated Lie algebra $\mathcal{L}_n^M = \oplus_{k \geq 1} \mathcal{L}_n^M(k)$ is called the free metabelian Lie algebra generated by H or the Chen Lie algebra. It is also classically known due to Chen [5] that each $\mathcal{L}_n^M(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{L}_n^M(k)) := (k-1)\binom{n+k-2}{k}.$$

We remark that $\mathcal{L}_n(k) = \mathcal{L}_n^M(k)$ for $1 \leq k \leq 3$.

By the same argument as that in Subsection 4.1, for each $k \geq 2$, we can detect $S^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$ in $Q^k(\mathrm{IA}_n^M)$ by the $\mathrm{GL}(n, \mathbf{Z})$-equivariant surjective homomorphism $\pi_k^M : Q^k(\mathrm{IA}_n^M) \to Q^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$ induced from the abelianization of IA_n^M. In order to investigate the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{Ker}(\pi_k^M)$, we use the Johnson homomorphism μ_k.

5.2. The image of $\mu_k|_{\mathrm{Ker}(\pi_k^M)}$.

Here we study the image of the Johnson homomorphism

$$\mu_k : Q^k(\mathrm{IA}_n^M) \to H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$$

restricted to the kernel of π_k^M for $n \geq 4$. First, in order to get a reasonable generators of $\mathcal{L}_n^M(k+1)$, we consider some lemmas. Let $\mathfrak{S}_l$ be the symmetric group of degree l. Then we have

Lemma 5.1. *Let $l \geq 2$ and $n \geq 2$. For any element $[x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_l}] \in \mathcal{L}_n^M(l+2)$ and any $\lambda \in \mathfrak{S}_l$,*

$$[x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_l}] = [x_{i_1}, x_{i_2}, x_{j_{\lambda(1)}} \ldots, x_{j_{\lambda(l)}}].$$

Proof. Since $\mathfrak{S}_l$ is generated by transpositions $(m\ m+1)$ for $1 \le m \le l-1$, it suffices to prove the lemma for each $\lambda = (m\ m+1)$. Now we have

$$\begin{aligned}
&[[[x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_{m-1}}], x_{j_m}], x_{j_{m+1}}]] \\
&\qquad = -[[x_{j_m}, x_{j_{m+1}}], [x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_{m-1}}]] \\
&\qquad\qquad - [[x_{j_{m+1}}, [x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_{m-1}}]], x_{j_m}], \\
&\qquad = [[[x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_{m-1}}], x_{j_{m+1}}], x_{j_m}]
\end{aligned}$$

in $\mathcal{L}_n^M(m+3)$ by the Jacobi's identity. Hence,

$$\begin{aligned}
[x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_l}] &= [x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_{m-1}}, x_{j_{m+1}}, x_{j_m}, \ldots, x_{j_l}], \\
&= [x_{i_1}, x_{i_2}, x_{j_{\lambda(1)}} \ldots, x_{j_{\lambda(l)}}].
\end{aligned}$$

in $\mathcal{L}_n^M(l+2)$. $\square$

Similarly to $H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1))$, the $\mathbf{Z}$-module $H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$ is generated by elements

$$x_i^* \otimes ([x_{i_1}, \ldots, x_{i_{k+1}}] - 1), \quad 1 \le i, i_j \le n.$$

On the other hand, using Lemma 5.1, elements $[x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \in \mathcal{L}_n^M(k+1)$ is rewritten as

$$[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_{l-1}}, x_i, x_i, \ldots, x_i]$$

in $\mathcal{L}_n^M(k+1)$ for some l, $3 \le l \le k+2$ such that $i_3, i_4, \ldots, i_{l-1} \ne i$. Hence $H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$ is generated by elements

$$x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_{l-1}}, x_i, x_i, \ldots, x_i] - 1)$$

for some l, $3 \le l \le k+2$ such that $i_3, \ldots, i_{l-1} \ne i$. Furthermore, without loss of generality, we may assume $i_2 \ne i$ in the generators above.

Proposition 5.2. *For any $k \ge 2$ and $n \ge 4$, the homomorphism $\mu_k|_{\mathrm{Ker}(\pi_k^M)} : \mathrm{Ker}(\pi_k^M) \to H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$ is surjective.*

Proof. Take a generator $x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_{l-1}}, x_i, x_i, \ldots, x_i] - 1)$ of $H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$ for some l, $3 \le l \le k+2$ such that $i_2, \ldots, i_{l-1} \ne i$ as mentioned above. Since $n \ge 4$, there exists some $1 \le j \le n$ such that $j \ne i, i_1, i_2$. First, consider an element

$$a := (K_{ij}^{-1} - 1)(K_{ji} - 1) \cdots (K_{ji} - 1) \in \Delta^{k-l+2}(\mathrm{IA}_n^M)$$

where $(K_{ji} - 1)$ appears $k-l+1$ times in the product. Then we see

$$\mu_{k-l+3}(a) = x_i^* \otimes ([x_j, x_i, \ldots, x_i] - 1)$$

where x_i appears $k-l+2$ times among the component.

Next, set

$$b := \begin{cases} K_{jii_{l-1}} - 1 & \text{if } j \ne i_{l-1}, \\ K_{ji}^{-1} - 1 & \text{if } j = i_{l-1}, \end{cases}$$

$$c := (K_{ii_{l-2}} - 1)(K_{ii_{l-3}} - 1) \cdots (K_{ii_3} - 1) \in \Delta^{l-4}(\mathrm{IA}_n^M)$$

and

$$d := \begin{cases} K_{ii_1i_2} - 1 & \text{if } i \ne i_1, \\ K_{ii_2} - 1 & \text{if } i = i_1. \end{cases}$$

Then we have
$$\mu_k(abcd) = x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{l-1}}, x_i, x_i, \dots, x_i] - 1).$$
On the other hand, $\mu_k(dbac) = 0$. Hence we have
$$\mu_k(abcd - dbac) = x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{l-1}}, x_i, x_i, \dots, x_i] - 1).$$
Therefore since $abcd - dbac \in \mathrm{Ker}(\pi_k^M)$, we conclude that $\mu_k|_{\mathrm{Ker}(\pi_k^M)}$ is surjective. This completes the proof of Proposition 5.2. □

Then we have

Theorem 5.3. *For $k \geq 2$ and $n \geq 4$, a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism*
$$\mu_k \oplus \pi_k : Q^k(\mathrm{IA}_n^M) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))) \bigoplus S^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$$
defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

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Department of Mathematics, Graduate School of Science, Kyoto University, Kitashirakawaoiwake cho, Sakyo-ku, Kyoto city 606-8502, Japan
E-mail address: takao@math.kyoto-u.ac.jp