SEMI-STABLE MINIMAL MODEL PROGRAM FOR VARIETIES WITH TRIVIAL CANONICAL DIVISOR

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Abstract. We discuss a semi-stable minimal model program for varieties with (numerically) trivial canonical divisor.

Contents

1. Introduction

We prove the following theorem, which is a semi-stable minimal model program for varieties with trivial canonical divisor. It was inspired by Yoshinori Gongyo's paper [G2] and Daisuke Matsushita's seminar talk on May 21, 2010 in Kyoto.

Theorem 1.1 (Semi-stable minimal model program for varieties with trivial canonical divisor)**.** *Let $f : X \to Y$ be a proper surjective morphism from a smooth quasi-projective variety X to a smooth quasi-projective curve Y with connected fibers. Let $P \in Y$ be a point. Assume that f^*P is a reduced simple normal crossing divisor on X and f is smooth over $Y \setminus P$. We further assume that $K_{f^{-1}Q} \sim 0$ for every $Q \in Y \setminus P$. Then there exists a sequence of flips and divisorial contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow \cdots \dashrightarrow X_m$$

over Y such that $K_{X_m} \sim_Y 0$. We note that X_m has only $\mathbb{Q}$-factorial terminal singularities. Moreover, the special fiber $S = f_m^{-1}P = f_m^*P$ of $f_m : X_m \to Y$ is Gorenstein, semi divisorial log terminal, and $K_S \sim 0$.

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For the definition of *semi divisorial log terminal*, see [F1, Definition 1.1]. Theorem 1.1 can be applied for semi-stable degenerations of Abelian varieties, Calabi-Yau varieties, and so on. From the minimal model theoretic viewpoint, the following theorem is a natural formulation of uniruled degenerations of varieties with numerically trivial canonical divisor (cf. [T, Theorem 1.1]).

Theorem 1.2 (Semi-stable minimal model program for varieties with numerically trivial canonical divisor)**.** *Let $f : X \to Y$ be a proper surjective morphism from a smooth quasi-projective variety X to a smooth quasi-projective curve Y with connected fibers. Let $P \in Y$ be a point. Assume that f^*P is a reduced simple normal crossing divisor on X and f is smooth over $Y \setminus P$. We further assume that $K_{f^{-1}Q} \equiv 0$, equivalently, $K_{f^{-1}Q} \sim_{\mathbb{Q}} 0$, for every $Q \in Y \setminus P$. Then there exists a sequence of flips and divisorial contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow \cdots \dashrightarrow X_m$$

over Y such that $K_{X_m} \sim_{\mathbb{Q},Y} 0$. We note that X_m has only $\mathbb{Q}$-factorial terminal singularities. Moreover, the special fiber $S = f_m^{-1}P = f_m^*P$ of $f_m : X_m \to Y$ is semi divisorial log terminal and $K_S \sim_{\mathbb{Q}} 0$. Therefore, if S is reducible, then every irreducible component of S is uniruled. If S is irreducible, then S is uniruled if and only if S is not canonical.

In this paper, we prove Theorem 1.1 and Theorem 1.2 as applications of the following theorem.

Theorem 1.3. *Let (X, Δ) be a $\mathbb{Q}$-factorial quasi-projective divisorial log terminal pair and let $f : X \to Y$ be a proper surjective morphism onto a smooth quasi-projective curve Y with connected fibers. Assume that $(K_X + \Delta)|_F \sim_{\mathbb{Q}} 0$ for a general fiber F of f. Then there exists a sequence of flips and divisorial contractions*

$$\begin{aligned}(X, \Delta) = (X_0, \Delta_0) &\dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \\ &\dashrightarrow (X_k, \Delta_k) \dashrightarrow \cdots \dashrightarrow (X_m, \Delta_m)\end{aligned}$$

over Y such that $K_{X_m} + \Delta_m \sim_{\mathbb{Q},Y} 0$ where Δ_k is the pushforward of Δ on X_k for every k.

Remark 1.4. It is known that $(K_X + \Delta)|_F \sim_{\mathbb{Q}} 0$ if and only if $(K_X + \Delta)|_F \equiv 0$. See, for example, [G2, Theorem 1.2].

We can also prove the following theorem as an application of Theorem 1.3. We recommend the reader to compare it with Kodaira's classification of elliptic fibrations (cf. [BPV, V. Examples]).

Theorem 1.5 (cf. [T, Theorem 1.1])**.** *Let $f : X \to Y$ be a proper surjective morphism from a smooth quasi-projective variety X to a smooth quasi-projective curve Y with connected fibers. Let $P \in Y$ be a point. Assume that* $\mathrm{Supp} f^*P$ *is a simple normal crossing divisor on X and f is smooth over $Y \setminus P$. We further assume that $K_{f^{-1}Q} \equiv 0$, equivalently, $K_{f^{-1}Q} \sim_{\mathbb{Q}} 0$, for every $Q \in Y \setminus P$. Then there exists a sequence of flips and divisorial contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow \cdots \dashrightarrow X_m$$

over Y such that X_m has only $\mathbb{Q}$-factorial terminal singularities and $K_{X_m} \sim_{\mathbb{Q},Y} 0$. Let $S = \mathrm{Supp} f_m^*P$ be the special fiber of $f_m : X_m \to Y$. If S is reducible, then every irreducible component of S is uniruled. If S is irreducible, then S is normal and has only canonical singularities if and only if S is not uniruled. We note that $K_S \sim_{\mathbb{Q}} 0$ when S is irreducible and has only canonical singularities.

By combining Theorem 1.3 with [L, Proposition 2.7], we obtain the following result.

Corollary 1.6. *Let $f : X \to Y$ be a projective surjective morphism from a smooth quasi-projective variety X onto a smooth quasi-projective curve Y with connected fibers. Assume that the general fiber F of f has a good minimal model and $\kappa(F) = 0$, where $\kappa(F)$ is the Kodaira dimension of F. Then there exists a sequence of flips and divisorial contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow \cdots \dashrightarrow X_m$$

over Y such that $K_{X_m} \sim_{\mathbb{Q},Y} 0$.

Remark 1.7. By [D, Corollaire 3.4], F has a good minimal model with $\kappa(F) = 0$ if and only if $\kappa_\sigma(F) = 0$, where $\kappa_\sigma(F)$ is the numerical Kodaira dimension in the sense of Nakayama. See also [G2, Theorem 1.2].

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Notation. Let $f : X \to Y$ be a proper morphism of normal algebraic varieties. Two $\mathbb{Q}$-divisors D_1 and D_2 on X are *$\mathbb{Q}$-linearly equivalent over Y*, denoted by $D_1 \sim_{\mathbb{Q},Y} D_2$, if their difference is a $\mathbb{Q}$-linear combination of principal divisors and a $\mathbb{Q}$-Cartier divisor pulled back from Y.

We work over $\mathbb{C}$, the complex number field, throughout this paper. We freely use the standard terminology on the log minimal model program in [BCHM] and [KM].

2. Proofs

For the proof of Theorem 1.3, we use the minimal model program with scaling (cf. [BCHM, 3.10] and [B, Definition 3.2]). The crucial point for the termination of flips is essentially Zariski's lemma (cf. [BPV, III. (8.2) Lemma]).

Proof of Theorem 1.3. Before we run the minimal model program with scaling, we note the following easy observation.

Step 1 (cf. [FM, Proposition 4.2])**.** There exist a $\mathbb{Q}$-divisor D on Y and an effective $\mathbb{Q}$-divisor B on X, which is vertical with respect to f, such that

$$K_X + \Delta \sim_{\mathbb{Q}} f^*D + B$$

with the following properties:

$$f_*\mathcal{O}_X(\llcorner mB \lrcorner) \simeq \mathcal{O}_Y$$

for every nonnegative integer m.

Let us run the minimal model program with scaling (cf. [BCHM, 3.10], [B, Definition 3.2], and [F2, Theorem 18.9]).

Step 2 (Minimal model program with scaling)**.** Let H be an effective $\mathbb{Q}$-divisor on X such that $(X, \Delta + H)$ is divisorial log terminal, $K_X + \Delta + H$ is f-nef, and the relative augmented base locus $\mathbf{B}_+(H/Y)$ (cf. [BCHM, Definition 3.5.1]) contains no lc centers of (X, Δ). We run the $(K_X + \Delta)$-minimal model program with scaling of H over Y. We obtain a sequence of divisorial contractions and flips

$$(X, \Delta) = (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_k, \Delta_k) \dashrightarrow \cdots$$

over Y. We note that

$$\lambda_i = \inf\{t \in \mathbb{R} \,|\, K_{X_i} + \Delta_i + tH_i \text{ is nef over } Y\},$$

where H_i (resp. Δ_i) is the pushforward of H (resp. Δ) on X_i for every i. By the definition, $0 \leq \lambda_i \leq 1$ and $\lambda_i \in \mathbb{Q}$ for every i and

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_k \geq \cdots .$$

We also note that the relative augmented base locus $\mathbf{B}_+(H_i/Y)$ contains no lc centers of (X_i, Δ_i) for every i (cf. [BCHM, Lemma 3.10.11]).

Step 3. We see that there are no infinite sequence of flips in the above $(K_X+\Delta)$-minimal model program with scaling of H over Y. By renumbering i, we assume that

$$(X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_k, \Delta_k) \dashrightarrow \cdots .$$

is an infinite sequence of flips.

Step 4. We further assume that

$$\lambda = \lim_{i \to \infty} \lambda_i > 0.$$

In this case, the sequence of flips we consider is a sequence of $(K_X + \Delta + \frac{1}{2}\lambda H)$-flips. We note that there exists an effective $\mathbb{Q}$-divisor B on X such that $\Delta + \frac{1}{2}\lambda H \sim_{\mathbb{Q}} B$, (X, B) is klt, $K_X + B + (1 - \frac{1}{2}\lambda)H$ is f-nef, $(X, B + (1 - \frac{1}{2}\lambda)H)$ is klt, and B is big over Y (cf. [BCHM, Lemma 3.7.3] and [G2, Lemma 5.1]). Therefore there are no infinite sequences of flips by [BCHM, Corollary 1.4.2]. It is a contradiction. Thus we can assume that $\lambda = 0$.

Step 5. Under the assumption that $\lambda = 0$, we see that $K_X + \Delta$ is a limit of movable $\mathbb{Q}$-divisors in $N^1(X/Y)$. Let G_i be a relative ample $\mathbb{Q}$-divisor on X_i such that $G_{iX} \to 0$ in $N^1(X/Y)$ for $i \to \infty$ where G_{iX} is the strict transform of G_i on X. We note that $K_{X_i} + \Delta_i + \lambda_i H_i + G_i$ is ample over Y for every i. Therefore the strict transform $K_X + \Delta + \lambda_i H + G_{iX}$ is movable on X for every i. Thus $K_X + \Delta$ is a limit of movable $\mathbb{Q}$-divisor in $N^1(X/Y)$.

Step 6. Since $K_X + \Delta$ is not f-nef, we have $B \not\sim_{\mathbb{Q},Y} 0$ (cf. [BPV, III. (8.2) Lemma]). Then we can find an irreducible component E of $\mathrm{Supp} B$ such that

$$B \cdot A^{n-2} \cdot E < 0.$$

where $n = \dim X$ and A is an f-ample Cartier divisor on X. It is essentially Zariski's lemma (cf. [BPV, III. (8.2) Lemma]). Thus

$$(K_X + \Delta) \cdot A^{n-2} \cdot E < 0.$$

On the other hand,

$$(K_X + \Delta + \lambda_i H + G_{iX}) \cdot A^{n-2} \cdot E \geq 0$$

for every i. Thus we obtain

$$(K_X + \Delta) \cdot A^{n-2} \cdot E = \lim_{i \to \infty} (K_X + \Delta + \lambda_i H + G_{iX}) \cdot A^{n-2} \cdot E \geq 0.$$

It is a contradiction. Anyway, there are no infinite sequences of flips.

Step 7. On the output X_m of the minimal model program, $K_{X_m}+\Delta_m \sim_{\mathbb{Q},Y} B_m$ where B_m is the pushforward of B on X_m. Since B_m is nef over Y, $B_m \sim_{\mathbb{Q},Y} 0$ (cf. [BPV, III. (8.2) Lemma]). Therefore, $K_{X_m}+\Delta_m \sim_{\mathbb{Q},Y} 0$.

We complete the proof of Theorem 1.3. □

Let us prove Theorems 1.1, 1.2, 1.5, and Corollary 1.6.

Proof of Theorem 1.1. By the assumptions, $f : X \to Y$ is a dlt morphism (cf. [KM, Definition 7.1]). By applying Theorem 1.3, we obtain a relative minimal model $f_m : X_m \to Y$ of $f : X \to Y$. We see that $f_m : X_m \to Y$ is automatically a dlt morphism. We note that X_m is $\mathbb{Q}$-factorial and has only terminal singularities. By adjunction,

$$(K_{X_m}+S)|_S = K_S$$

and S is semi divisorial log terminal because (X_m, S) is dlt (cf. [F1, Remark 1.2 (3)]). By the upper semicontinuity, $h^0(S, \mathcal{O}_S(K_S)) \geq 1$. Thus $K_S \sim 0$ because $K_{X_m} \sim_{\mathbb{Q},Y} 0$. Therefore, $K_{X_m} \sim_Y 0$ by the base change theorem. Note that S is Cohen–Macaulay because S is Cartier and X_m is Cohen–Macaulay. □

Proof of Theorem 1.2. The proof of Theorem 1.1 works in this setting. The only nontrivial part is that every irreducible component of S is uniruled if S is reducible. If S is reducible, semi divisorial log terminal, and $K_S \sim_{\mathbb{Q}} 0$, then it is easy to see that every irreducible component of S is uniruled. □

Proof of Theorem 1.5. The former part follows from Theorem 1.3. For the latter part, let E be any irreducible component of S when S is reducible, and let ε be a sufficiently small positive rational number. Apply Theorem 1.3 for $(X, \varepsilon E)$ over Y. Then it is easy to see that the divisor E must be contracted in this minimal model program. Therefore E is uniruled by [KMM, Proposition 5-1-8]. The final statement easily follows from adjunction. □

Proof of Corollary 1.6. We run the minimal model program with scaling over Y. Then, by [L, Proposition 2.7], we can assume that the general fiber of $f : X \to Y$ is a good minimal model. By Theorem 1.3, this minimal model program terminates after finitely many steps. □

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