# HANDLE ATTACHING IN WRAPPED FLOER HOMOLOGY AND BRAKE ORBITS IN CLASSICAL HAMILTONIAN SYSTEMS

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Abstract. The objective of this note is to prove the existence result for brake orbits in classical Hamiltonian systems (which is first proved by Bolotin) by using Floer theory. To this end, we compute an open string analogue of symplectic homology (so called wrapped Floer homology) of some domains in cotangent bundles, which appear naturally in study of classical Hamiltonian systems. The main part of the computations is to show invariance of wrapped Floer homology under certain handle attaching to domains.

## 1. Introduction

First we recall the definition of classical Hamiltonian systems. Let $N$ be a $n$-dimensional manifold. Then, $T^*N$ carries a symplectic form $\omega_N := \sum_{1 \le i \le n} dp_i \wedge dq_i$ where $(q_1, \ldots, q_n)$ is a local coordinate in $N$, and $(p_1, \ldots, p_n)$ is an associated coordinate on fibers.

Assume that $N$ carries a Riemannian metric. Then, for $V \in C^\infty(T^*N)$, we define $H_V \in C^\infty(T^*N)$ by $H_V(q,p) = V(q) + |p|^2/2$. A pair of symplectic manifold $(T^*N, \omega_N)$ and $H_V \in C^\infty(T^*N)$ is called *classical Hamiltonian system*. Its *Hamiltonian vector field* is defined by $i_{X_{H_V}}\omega_N = -dH_V$. As is well-known, $X_{H_V}$ describes free motion of a particle on $N$ under potential energy given by $V$.

Following theorem is first proved by Bolotin [B].

**Theorem 1.1.** *Let $N$ be a Riemannian manifold, and $V \in C^\infty(N)$. If $S_h := H_V^{-1}(h)$ is a compact and regular hypersurface in $T^*N$, then there exists a closed orbit of $X_{H_V}$ on $S_h$.*

When $S_h \cap N = \emptyset$, theorem 1.1 is easily obtained by existence of closed geodisics on compact Riemannian manifolds, using Maupertuis-Jacobi principle. So difficulty arises when $S_h \cap N \neq \emptyset$. In this case, theorem 1.1 is obtained by following result ([B]):

**Theorem 1.2.** *Let $N$ and $V$ are as in theorem 1.1. If $S_h \cap N \neq \emptyset$, there exists a non-trivial orbit of $X_{H_V}$ on $S_h$ , which starts from and ends at $S_h \cap N$.*

Define $I : T^*N \to T^*N$ by $I(q,p) = (q, -p)$. If $x : [0,l] \to S_h$ satisfies $\dot{x} = X_{H_V}(x)$ and $x(0), x(l) \in N$, then $\overline{x} : [0, 2l] \to S_h$ defined by

$$\overline{x}(t) = \begin{cases} x(t) & (0 \le t \le l) \\ I\bigl(x(2l-t)\bigr) & (l \le t \le 2l) \end{cases}$$

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is a closed orbit of $X_{H_V}$ (closed orbits of $X_{H_V}$ obtained in this way are so-called *brake orbits*). Hence theorem 1.2 implies theorem 1.1.

In this paper, we deduce theorem 1.2 from computations of certain Floer-theoric invariant. The invariant we use is an open string analogue of symplectic homology, and often called *wrapped Floer homology*. Foundations of wrapped Floer homology can be found in [AS] (they also construct an $A^\infty$-algebra structure on the chain complex underlyng the homology). Roughly speaking, wrapped Floer homology is defined for a pair $(M, L)$, where $M$ is a compact symplectic manifold with contact type boundary, and $L$ is a Lagrangian of $M$ (in fact, we need more data and additional conditions. see section 2 for details). Let us denote the wrapped Floer homology for $(M, L)$ by $\mathrm{WFH}_*(M, L)$.

We explain our main theorem briefly. Let $N$ be a Riemannian manifold, and $V \in C^\infty(N)$. Assume that $S_h = H_V^{-1}(h)$ is compact. Then, setting $D_h := H_V^{-1}\big((-\infty, h]\big)$, $(D_h, \omega_N)$ is a compact symplectic manifold with contact type boundary, and we can define wrapped Floer homology for $(D_h, D_h \cap N)$ (for details, see section 4). Our main theorem is theorem 4.2, which asserts that if $S_h \cap N \neq \emptyset$ and $D_h$ is connected, then $\mathrm{WFH}_*(D_h, D_h \cap N) = 0$.

Combined with basic results of wrapped Floer homology, theorem 4.2 implies theorem 1.2 (Details are explained in section 4). Theorem 4.2 is proved as follows. By "deformation invariance" of wrapped Floer homology (proposition 2.7), it is easy to show that $\mathrm{WFH}_*(D_h, D_h \cap N)$ depends only on diffeomorphism type of $D_h \cap N$. When $D_h \cap N$ is diffeormophic to the disk, $\mathrm{WFH}_*(D_h, D_h \cap N) = 0$ can be checked by simple argument. Hence all we have to show is invariance of $\mathrm{WFH}_*(D_h, D_h \cap N)$ under surgery on $D_h \cap N$ by attaching handles (lemma 4.10). This is proved by aruguments which are similar to Cieliebak's arguments in [Ci], where he proves invariance of symplectic homology under subcritical handle attaching.

We explain the structure of this paper. In section 2, we recall basics of wrapped Floer homology. We treat somewhat broader class of Hamiltonians than usually considered in Floer theory for manifolds with boundary, because this is needed to carry out arguments in section 5. For this reason, establishing $C^0$ estimate for Floer trajectories becomes harder than usual. The precious statement of the $C^0$ estimate is stated in section 2 (theorem 2.5), and proved in section 3. The proof given in section 3 is based on [FH]. In section 4, we explain basics of classical Hamiltonian systems, and state the main theorem (theorem 4.2). We also reduce theorem 4.2 to lemma 4.10 in section 4. Lemma 4.10 is proved in section 5.

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## 2. Wrapped Floer homology

In this section, we recall basics of wrapped Floer homology, which we will use in the following of this paper.

2.1. **Liouville quadruple.** First we define *Liouville quadruple*, for which we define wrapped Floer homology.

**Definition 2.1.** Let $(M, \omega)$ be a $2n$ dimensional compact symplectic manifold, $X \in \mathscr{X}(M)$, and $L$ be a Lagrangian of $M$. *Liouville quadruple* is a quadruple $(M, \omega, X, L)$ with following properties:

(1) $L_X \omega = \omega$.
(2) $X$ points strictly outwards on $\partial M$.
(3) $X_q \in T_q L$ for any $q \in L$.
(4) $L$ is transverse to $\partial M$, and $\partial L = L \cap \partial M$.

For Liouville quadruple $(M, \omega, X, L)$, let $\lambda := i_X \omega$. Then, $\lambda|_L = 0$. $(\partial M, \lambda)$ is a contact manifold, and $\partial L$ is a Legendrean of $(\partial M, \lambda)$. Recall that the *Reeb vector field* $R$ on $(\partial M, \lambda)$ is characterized by $i_R \omega = 0$, $\lambda(R) = 1$. Let $\mathscr{C}(\partial M, \lambda, \partial L)$ be the set of all *Reeb chords* of $\partial L$ in $(\partial M, \lambda)$, i.e.

$$\mathscr{C}(\partial M, \lambda, \partial L) := \big\{ x \colon [0, l] \to \partial M \bigm| l > 0, x(0), x(l) \in \partial L,\ \dot{x} = R(x) \big\}.$$

For $x \in \mathscr{C}(\partial M, \lambda, \partial L)$, let $\mathscr{A}(x) := \displaystyle\int_0^l x^* \lambda$. Define the *action spectrum* of $\partial L$

$$\mathscr{A}(\partial M, \lambda, \partial L) := \big\{ \mathscr{A}(x) \bigm| x \in \mathscr{C}(\partial M, \lambda, \partial L) \big\}.$$

It is easly verified that $\inf \mathscr{A}(\partial M, \lambda, \partial L) > 0$.

Let $\hat{M} := M \cup \partial M \times [1, \infty)$. We extend $X \in \mathscr{X}(M)$ to $\hat{X} \in \mathscr{X}(\hat{M})$ by $\hat{X} = \rho \partial_\rho$ on $\partial M \times [1, \infty)$, where $\rho$ stands for coordinate on $[1, \infty)$. Moreover, we extend $\lambda$ to $\hat{\lambda}$ by $\hat{\lambda} := \rho \lambda$ on $\partial M \times [1, \infty)$, and $\omega$ to $\hat{\omega} := d\hat{\lambda}$. Then, $\hat{L} := L \cup \partial L \times [1, \infty)$ is a Lagrangian of $(\hat{M}, \hat{\omega})$. We call $(\hat{M}, \hat{\omega}, \hat{X}, \hat{L})$ the *completion* of Liouville quadruple $(M, \omega, X, L)$.

Define $\Phi \colon \partial M \times (0, \infty) \to \hat{M}$ by

$$\Phi(z, 1) = z, \qquad \partial_\rho \Phi(z, \rho) = \rho^{-1} \hat{X}\big(\Phi(z, \rho)\big).$$

Then, $\Phi^* \hat{\lambda} = \rho \lambda$. We call $\mathrm{Im}\,(\Phi)$ *cylindrical part* of $\hat{M}$, and denote it by $\mathrm{Cyl}(\hat{M})$. We often identify $\mathrm{Cyl}(\hat{M})$ with $\partial M \times (0, \infty)$ via $\Phi$. For any $\rho \in (0, \infty)$, we define $M(\rho)$ to be the domain in $\hat{M}$, which is bounded by the hypersurface $\partial M \times \{\rho\}$. i.e.

$$M(\rho) := \begin{cases} M \cup \partial M \times (1, \rho] & (\rho \geq 1) \\ M \setminus \partial M \times (\rho, 1] & (\rho < 1). \end{cases}$$

2.2. **Chords and indexes.** For $H \in C^\infty(\hat{M})$, let

$$\mathscr{C}(H) := \big\{ x \colon [0, 1] \to \hat{M} \bigm| x(0), x(1) \in \hat{L},\ \dot{x} = X_H(x) \big\},$$

where $X_H$ is the *Hamiltonian vector field* of $H$, defined by $dH = -i_{X_H} \hat{\omega}$.

For $x \in \mathscr{C}(H)$ and $0 \leq t \leq 1$, let $\Phi_t \colon T_{x(0)} \hat{M} \to T_{x(t)} \hat{M}$ be the Poincare map of the flow generated by $X_H$. $x \in \mathscr{C}(H)$ is called *nondegenerate* if $\Phi_1 \colon T_{x(0)} \hat{M} \to T_{x(1)} \hat{M}$ satisfies $\Phi_1(T_{x(0)} \hat{L}) \cap T_{x(1)} \hat{L} = 0$.

For nondegenerate $x \in \mathscr{C}(H)$, we define its index $\mathrm{ind}(x)$. In the following of this paper, we assume that all Liouville quadruple $(M, \omega, X, L)$ satisfies

$$\pi_1(M, L) = \pi_2(M, L) = 0.$$

This is quite strong assumption, but it is enough to consider this case for our objective.

Consider $\mathbb{R}^{2n}$ with coordinate $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and the *standard symplectic form* $\omega_{\mathrm{st}} := \sum_{1 \le i \le n} dp_i \wedge dq_i$. Let $\mathscr{L}(n)$ be the space of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$. Note that $\{p = 0\} \in \mathscr{L}(n)$.

Let $x \in \mathscr{C}(H)$ and assume that $x$ is nondegenerate. Let $D^+ \colon = \{z \in \mathbb{C} \mid |z| \le 1, \mathrm{Im}\, z \ge 0\}$ and take $\overline{x} \colon D^+ \to \hat{M}$ such that $\overline{x}(e^{i\pi\theta}) = x(\theta)\,(0 \le \theta \le 1)$ and $\overline{x}(D^+ \cap \mathbb{R}) \subset \hat{L}$ (such $\overline{x}$ exists since $\pi_1(\hat{M}, \hat{L}) = 0$). Take arbitrary isomorphism of vector bundle $F : \overline{x}^* T\hat{M} \to (\mathbb{R}^{2n}, \omega_{\mathrm{st}}) \times D^+$ over $D^+$, such that $F_z : T_{\overline{x}(z)}\hat{M} \to \mathbb{R}^{2n}$ preserves symplectic form for any $z \in D^+$, and $F_z(T_{\overline{x}(z)}\hat{L}) = \{p = 0\}$ for any $z \in D^+ \cap \mathbb{R}$. Define $\Lambda : [0, 1] \to \mathscr{L}(n)$ by $\Lambda(\theta) := F_{e^{i\pi\theta}}\big(\Phi_\theta(T_{x(0)}\hat{L})\big)$, and let

$$\mathrm{ind}(x) := \frac{n}{2} + \mu_{\mathrm{RS}}\big(\Lambda, \{p = 0\}\big),$$

where $\mu_{\mathrm{RS}}$ is Robbin-Salamon index introduced in [RS]. Note that this definition is independent of the choice of $\overline{x}$ since $\pi_2(\hat{M}, \hat{L}) = 0$.

2.3. **Hamiltonians.** Let $K$ be a compact set in $\hat{M}$ which contains $M$. Then, $H \in C^\infty(\hat{M})$ is of *contact type* on $\hat{M} \setminus K$, if and only if there exists a smooth positive function $a$ on $\partial M$ and $b \in \mathbb{R}$ such that

$$(z, \rho) \in \hat{M} \setminus K \implies H(z, \rho) = a(z)\rho + b.$$

$a$ and $b$ are uniquely determined by $H$, and denoted by $a_H$, $b_H$. The set of all $H \in C^\infty(\hat{M})$ which are of contact type on $\hat{M} \setminus K$ is denoted by $\mathscr{H}_K(\hat{M})$. $H \in \mathscr{H}_K(\hat{M})$ is called *admissible* if $1 \notin \mathscr{A}(\partial M, a_H^{-1}\lambda, \partial L)$ and all elements of $\mathscr{C}(H)$ are nondegenerate. The set of all admissible elements of $\mathscr{H}_K(\hat{M})$ is denoted by $\mathscr{H}_{K,\mathrm{ad}}(\hat{M})$. Let $\mathscr{H}(\hat{M}) := \bigcup_K \mathscr{H}_K(\hat{M})$ and $\mathscr{H}_{\mathrm{ad}}(\hat{M}) := \bigcup_K \mathscr{H}_{K,\mathrm{ad}}(\hat{M})$, where $K$ runs over all compact sets in $\hat{M}$ which contain $M$. It is easily verified that if $H \in \mathscr{H}_{\mathrm{ad}}(\hat{M})$, then $\mathscr{C}(H)$ is a finite set.

Let $H, H' \in \mathscr{H}_{\mathrm{ad}}(\hat{M})$. $(H^s)_{s \in \mathbb{R}}$, a smooth family of elements of $\mathscr{H}(\hat{M})$, is called *monotone homotopy* from $H$ to $H'$, if it satisfies following conditions:

(1) There exists a compact set $K$ such that $H^s \in \mathscr{H}_K(M)$ for any $s$.
(2) There exists $s_0 > 0$ such that:
   (a) $H^s = \begin{cases} H & (s \le -s_0) \\ H' & (s \ge s_0) \end{cases}$.
   (b) For any $s \in (-s_0, s_0)$, $\partial_s H^s > 0$ on $\hat{M}$, and $\partial_s a_{H^s} > 0$ on $\partial M$.

2.4. **Almost complex structures.** Let $J$ be an almost complex structure on $\hat{M}$. $J$ is *compatible with* $\hat{\omega}$ if and only if

$$\langle\,\cdot\,,\,\cdot\,\rangle_J \colon T\hat{M} \times T\hat{M} \to \mathbb{R}; \quad (v,w) \mapsto \hat{\omega}(v, Jw)$$

is a Riemannian metric on $\hat{M}$. We denote the set of almost complex structures on $\hat{M}$ which is compatible with $\hat{\omega}$ by $\mathscr{J}(\hat{M},\hat{\omega})$. We often abbreviate it as $\mathscr{J}(\hat{M})$.

For smooth positive function $a$ on $\partial M$, define diffeomorphism

$$\Phi_a : \partial M \times (0,\infty) \to \mathrm{Cyl}(\hat{M}); \quad (z,\rho) \mapsto \big(z, a(z)^{-1}\rho\big).$$

Let $\lambda^a := a^{-1}\lambda \in \Omega^1(\partial M)$. Then, $(\Phi_a)^*(\hat{\lambda}) = \rho\lambda^a$. Let $\xi^a$ and $R^a$ be the contact distribution and the Reeb flow on the contact manifold $(\partial M, \lambda^a)$.

For $v \in T(\partial M)$, let

$$\overline{v} := (v,0) \in T(\partial M) \oplus \mathbb{R}\partial_\rho = T\big(\partial M \times (0,\infty)\big).$$

There is a natural decompositon

$$T\big(\partial M \times (0,\infty)\big) = \overline{\xi^a} \oplus \mathbb{R}\overline{R^a} \oplus \mathbb{R}\partial_\rho,$$

where $\overline{\xi^a} = \{\overline{v} \mid v \in \xi^a\}$.

**Definition 2.2.** Let $K$ be a compact set in $\hat{M}$ which contains $M$. Then, $J \in \mathscr{J}(\hat{M})$ is *of contact type* on $\hat{M} \setminus K$ with respect to $a$, if $\Phi_a^* J$ satisfies following:

(1) $\Phi_a^* J$ preserves $\overline{\xi^a}$ on $\Phi_a^{-1}(\hat{M} \setminus K)$.
(2) There exists $J^\infty$, an almost complex structure on $\xi^a$, such that $d\pi|_{\overline{\xi^a}} \circ \Phi_a^* J|_{\overline{\xi^a}} = J^\infty \circ d\pi|_{\overline{\xi^a}}$ on $\Phi_a^{-1}(\hat{M} \setminus K)$. ($\pi$ denotes the natural projection to $\partial M$.)
(3) There exists $c_J > 0$ such that $\Phi_a^* J(\partial_\rho) = \dfrac{1}{\rho c_J}\overline{R^a}$ on $\Phi_a^{-1}(\hat{M} \setminus K)$.

We denote the set of $J \in \mathscr{J}(\hat{M})$ which are of contact type on $\hat{M} \setminus K$ with respect to $a$, by $\mathscr{J}_{a,K}(\hat{M})$. Moreover, $\mathscr{J}_a(\hat{M}) := \bigcup_K \mathscr{J}_{a,K}(\hat{M})$ where $K$ runs over all compact sets in $\hat{M}$ which contain $M$. Clearly, for two positive functions $a$ and $a'$, if $a/a'$ is a constant function then $\mathscr{J}_{a,K}(\hat{M}) = \mathscr{J}_{a',K}(\hat{M})$.

Let $J \in \mathscr{J}_a(\hat{M})$, and $J^\infty$ be as in (2) in definition 2.2. Abbreviate the metric $\Phi_a^*\big(\langle\,\cdot\,,\,\cdot\,\rangle_J\big)$ on $\partial M \times (0,\infty)$ by $\langle\,\cdot\,,\,\cdot\,\rangle_{a,J}$. Moreover, define a metric $\langle\,\cdot\,,\,\cdot\,\rangle_{a,J,\partial M}$ on $\partial M$ by

- $\langle v, w\rangle_{a,J,\partial M} = (d\lambda^a)(v, J^\infty w)$ on $\xi^a$,
- $\langle v, R^a\rangle_{a,J,\partial M} = 0$ for any $v \in \xi^a$,
- $\big|R^a\big|_{a,J,\partial M} = c_J^{\frac{1}{2}}$.

Then, following properties are verified by simple calculation.

**Lemma 2.3.** (1) *On* $\Phi_a^{-1}(\hat{M}\setminus K)$, $\overline{\xi^a}$, $\overline{R^a}$, $\partial_\rho$ *are orthogonal to each other with respect to* $\langle\,\cdot\,,\,\cdot\,\rangle_{a,J}$.

(2) *For* $(z,\rho)\in\Phi_a^{-1}(\hat{M}\setminus K)$ *and* $v\in T(\partial M)$, $\bigl|\overline{v}(z,\rho)\bigr|_{a,J}=\rho^{\frac{1}{2}}|v|_{a,J,\partial M}$.
(3) *For* $(z,\rho)\in\Phi_a^{-1}(\hat{M}\setminus K)$, $\bigl|\partial_\rho(z,\rho)\bigr|_{a,J}=(\rho c_J)^{-1/2}$.

2.5. **Floer equation.** Let $H\in\mathscr{H}_{\mathrm{ad}}(\hat{M})$, and $(J_t)_{t\in[0,1]}$ be a smooth family of elements of $\mathscr{J}(\hat{M})$. For $x_-,x_+\in\mathscr{C}(H)$,

$$\begin{aligned}\hat{\mathscr{M}}_{H,(J_t)_t}(x_-,x_+):=\bigl\{u\colon \mathbb{R}\times[0,1]\to\hat{M}\bigm|\partial_s u-J_t\bigl(\partial_t u-X_H(u)\bigr)=0,\\ u(\mathbb{R}\times\{0,1\})\subset\hat{L},\ u(s)\to x_\pm\,(s\to\pm\infty)\bigr\}.\end{aligned}$$

$\hat{\mathscr{M}}_{H,(J_t)_t}$ admits a natural $\mathbb{R}$ action. We denote the quotient by $\mathscr{M}_{H,(J_t)_t}$.

We also consider the case where Hamiltonans are time-dependent. Let $H,H'\in\mathscr{H}_{\mathrm{ad}}(\hat{M})$ and $(H^s)_{s\in\mathbb{R}}$ be a monotone homotopy from $H$ to $H'$. Let $(J_t^s)_{s\in\mathbb{R},t\in[0,1]}$ be a smooth family of elements of $\mathscr{J}(\hat{M})$. For $x_-\in\mathscr{C}(H)$ and $x_+\in\mathscr{C}(H')$,

$$\begin{aligned}\hat{\mathscr{M}}_{(H^s,J_t^s)_{s,t}}(x_-,x_+):=\bigl\{u\colon \mathbb{R}\times[0,1]\to\hat{M}\bigm|\partial_s u-J_t^s\bigl(\partial_t u-X_{H^s}(u)\bigr)=0,\\ u(\mathbb{R}\times\{0,1\})\subset\hat{L},\ u(s)\to x_\pm\,(s\to\pm\infty)\bigr\}.\end{aligned}$$

For $x\in\mathscr{C}(H)$, we define its *action* by

$$\mathscr{A}_H(x):=\int_0^1 x^*\hat{\lambda}-H\bigl(x(t)\bigr)dt.$$

The following lemma can be proved by simple calculation.

**Lemma 2.4.** *For* $x_-\in\mathscr{C}(H)$, $x_+\in\mathscr{C}(H')$, *and* $u\in\hat{\mathscr{M}}_{(H^s,J_t^s)_{s,t}}(x_-,x_+)$,

$$-\partial_s\bigl(\mathscr{A}_{H^s}\bigl(u(s)\bigr)\bigr)=\int_0^1\bigl|\partial_s u(s,t)\bigr|^2_{J_t^s}+\partial_s H^s\bigl(u(s,t)\bigr)dt.$$

*In particular, if* $\hat{\mathscr{M}}_{(H^s,J_t^s)_{s,t}}(x_-,x_+)\neq\emptyset$, *then* $\mathscr{A}_H(x_-)>\mathscr{A}_{H'}(x_+)$.

We sometimes call elements of $\hat{\mathscr{M}}_{H,(J_t)_t}(x_-,x_+)$ and $\hat{\mathscr{M}}_{(H^s,J_t^s)_{s,t}}(x_-,x_+)$ *Floer trajectories* from $x_-$ to $x_+$. The next theorem asserts the existence of $C^0$ apriori estimate for Floer trajectories. This is proved in section 3.

**Theorem 2.5.** (1) *Let* $H\in\mathscr{H}_{\mathrm{ad}}(\hat{M})$ *and* $(J_t)_{0\le t\le 1}$ *be a family of elements of* $\mathscr{J}(\hat{M})$. *Assume that there exsits a compact set* $K$ *in* $\hat{M}$ *such that* $J_t\in\mathscr{J}_{a_H,K}(\hat{M})$ *for any* $t$. *Then, there exists a compact set* $B\subset\hat{M}$ *such that for any* $x_-,x_+\in\mathscr{C}(H)$ *and* $u\in\hat{\mathscr{M}}_{(H,J_t)_t}(x_-,x_+)$, $u\bigl(\mathbb{R}\times[0,1]\bigr)\subset B$.
(2) *Let* $H,H'\in\mathscr{H}_{\mathrm{ad}}(\hat{M})$ *and* $(H^s)_s$ *be a monotone homotopy from* $H$ *to* $H'$. *Let* $(J_t^s)_{s,t}$ *be a family of elements of* $\mathscr{J}(\hat{M})$ *such that for sufficiently large* $s_0>0$,

$$J_t^s=\begin{cases}J_t^{-s_0} & (s\le -s_0),\\ J_t^{s_0} & (s\ge s_0).\end{cases}$$

*Assume that there exists a compact set* $K$ *in* $\hat{M}$, *such that* $H^s\in\mathscr{H}_K(\hat{M})$ *and* $J_t^s\in\mathscr{J}_{a_{H^s},K}(\hat{M})$ *for any* $s,t$. *Then, there exists a compact set* $B\subset\hat{M}$, *such that for any* $x_-\in\mathscr{C}(H)$, $x_+\in\mathscr{C}(H')$ *and* $u\in\hat{\mathscr{M}}_{(H^s,J_t^s)_{s,t}}(x_-,x_+)$, $u\bigl(\mathbb{R}\times[0,1]\bigr)\subset B$.

Finally, we state transversality results.

**Lemma 2.6.** (1) *Let* $H \in \mathscr{H}_{\mathrm{ad}}(\hat{M})$*, and* $K$ *be a compact set in* $\hat{M}$ *which contains* $M$*. Assume that* $H \in \mathscr{H}_K(\hat{M})$ *and images of all elements of* $\mathscr{C}(H)$ *are contained in* $\mathrm{int}K$*. Then, for generic* $(J_t)_{t\in[0,1]}$*, where* $J_t \in \mathscr{J}_{a_H,K}(\hat{M})$*,* $\mathscr{M}_{H,(J_t)_t}(x_-,x_+)$ *is a* $\mathrm{ind}x_- - \mathrm{ind}x_+ - 1$ *dimensional smooth manifold for any* $x_-,x_+ \in \mathscr{C}(H)$*. We denote the set of such* $(J_t)_t$ *by* $\mathscr{J}_{H,K}(\hat{M})$*, and* $\mathscr{J}_H(\hat{M}) := \bigcup_K \mathscr{J}_{H,K}(\hat{M})$*, where* $K$ *runs over all compact sets in* $\hat{M}$ *with conditions as above.*

(2) *Let* $H, H' \in \mathscr{H}_{\mathrm{ad}}(\hat{M})$*,* $(H^s)_s$ *be a monotone homotopy from* $H$ *to* $H'$*, and* $K$ *be a compact set in* $\hat{M}$ *which contains* $M$*. Assume that* $H^s \in \mathscr{H}_K(\hat{M})$ *for any* $s$*, and images of all elements of* $\mathscr{C}(H), \mathscr{C}(H')$ *are contained in* $\mathrm{int}K$*. Then, for generic* $(J^s_t)_{s\in\mathbb{R},t\in[0,1]}$*, where* $J^s_t \in \mathscr{J}_{a_{H^s},K}(\hat{M})$*,* $\hat{\mathscr{M}}_{(H^s,J^s_t)_{s,t}}(x_-,x_+)$ *is a* $\mathrm{ind}x_- - \mathrm{ind}x_+$ *dimensional smooth manifold for any* $x_- \in \mathscr{C}(H), x_+ \in \mathscr{C}(H')$*. We denote the set of such* $(J^s_t)_{s,t}$ *by* $\mathscr{J}_{(H^s)_s,K}(\hat{M})$*, and* $\mathscr{J}_{(H^s)_s}(\hat{M}) := \bigcup_K \mathscr{J}_{(H^s)_s,K}(\hat{M})$*, where* $K$ *runs over all compact sets in* $\hat{M}$ *with conditions as above.*

***Proof***. First we prove (1). Let $(J_t)_t$ be a family of elements of $\mathscr{J}_{a_H,K}(\hat{M})$. Then, for any $x_-,x_+ \in \mathscr{C}(H)$ and $u \in \hat{\mathscr{M}}_{H,(J_t)_t}(x_-,x_+)$, $u^{-1}(\mathrm{int}K)$ is a non-empty open set in $\mathbb{R}\times[0,1]$, since both $x_-\big([0,1]\big)$ and $x_+\big([0,1]\big)$ are contained in $\mathrm{int}K$. By standard arguments (see [FHS]), one can perturb $(J_t)_t$ to achieve transversality conditions without violating the condition $J_t \in \mathscr{J}_{a_H,K}(\hat{M})$. This proves (1). (2) is proved by similar arguments. □

2.6. **Wrapped Floer homology.** In this subsection, we define wrapped Floer homology for Liouville quadruples. Once $C^0$ estimate for Floer trajectories is established (theorem 2.5), other arguments are parallel to Lagrangian Floer thoery for compact symplectic manifolds ([F]).

Let $H \in \mathscr{H}_{\mathrm{ad}}(\hat{M})$, and $k$ be an integer. Let

$$\mathscr{C}_k(H) := \big\{x \in \mathscr{C}(H) \bigm| \mathrm{ind}x = k\big\},$$

and $\mathrm{WFC}_k(H)$ be a free $\mathbb{Z}_2$ module generated over $\mathscr{C}_k(H)$.

Let $(J_t)_t \in \mathscr{J}_H(\hat{M})$. For each integer $k$, define $\partial^{H,(J_t)_t}_k \colon \mathrm{WFC}_k(H) \to \mathrm{WFC}_{k-1}(H)$ by

$$\partial^{H,(J_t)_t}_k[x] := \sum_{y\in\mathscr{C}_{k-1}(H)} \sharp\mathscr{M}_{H,(J_t)_t}(x,y)\cdot[y].$$

Then, $\big(\mathrm{WFC}_*(H), \partial^{H,(J_t)_t}_*\big)$ is a chain complex, and the resulting homology group does not depend on choice of $(J_t)_t$. We denote this homology group by $\mathrm{WFH}_*(H; M, \omega, X, L)$. We often abbreviate it as $\mathrm{WFH}_*(H)$.

Let $H, H' \in \mathscr{H}_{\mathrm{ad}}(\hat{M})$, and $(H^s)_s$ be a monotone homotopy from $H$ to $H'$, and $(J^s_t)_{s,t} \in \mathscr{J}_{(H^s)_s}(\hat{M})$. For each integer $k$, define $\varphi^{(H^s,J^s_t)_{s,t}}_k \colon \mathrm{WFC}_k(H) \to \mathrm{WFC}_k(H')$ by

$$\varphi^{(H^s,J^s_t)_{s,t}}_k[x] := \sum_{y\in\mathscr{C}_k(H')} \sharp\hat{\mathscr{M}}_{(H^s,J^s_t)_{s,t}}(x,y)\cdot[y].$$

$\left(\varphi_k^{(H^s,J^s_t)_{s,t}}\right)_k$ is a chain map, hence we can define a morphism $\mathrm{WFH}_*(H) \to \mathrm{WFH}_*(H')$.

We define relation $<$ on $\mathscr{H}_{\mathrm{ad}}(\hat{M})$ by

$$H < H' \iff H(x) < H'(x) \text{ for any } x \in \hat{M} \text{ and } a_H(z) < a_{H'}(z) \text{ for any } z \in \partial M.$$

If $H < H'$, then there exsits a monotone homotopy $(H^s)_s$ from $H$ to $H'$, and morphsim $\mathrm{WFH}_*(H) \to \mathrm{WFH}_*(H')$ obtained as above does not depend on choices of $(H^s, J^s_t)_{s,t}$. We call this morphism *monotone morphism.*

Finally, we define the wrapped Floer homology of $(M, \omega, X, L)$ by taking direct limit (though $\left(\mathscr{H}_{\mathrm{ad}}(\hat{M}), <\right)$ is not a pre-ordered set, we can define a direct limit):

$$\mathrm{WFH}_*(M, \omega, X, L) := \varinjlim_{H \in \mathscr{H}_{\mathrm{ad}}(\hat{M})} \mathrm{WFH}_*(H).$$

One of the important properties of wrapped Floer homology is its invariance under deformations. The next proposition is proved in section 3.5.

**Proposition 2.7.** *Let $(M, \omega^s, X^s, L)_{0 \le s \le 1}$ be a smooth family of Liouville quadruple. Then there exists a canonical isomorphism* $\mathrm{WFH}_*(M, \omega^0, X^0, L) \to \mathrm{WFH}_*(M, \omega^1, X^1, L)$.

If $(M, \omega, X, L)$ and $(M, \omega, X', L)$ are Liouville quadruples, then $(M, \omega, sX+(1-s)X', L)_{0 \le s \le 1}$ is a smooth family of Liouville quadruples. Hence, by proposition 2.7, $\mathrm{WFH}_*(M, \omega, X, L)$ does not depend on $X$. We often donote it by $\mathrm{WFH}_*(M, \omega, L)$.

Next corollary is easily obtained from propositon 2.7.

**Corollary 2.8.** *Let $(M, \omega, X, L)$ be a Liouville quadruple, and $M'$ be a compact submanifold of* $\mathrm{int} M$*, such that $(M', \omega|_{M'}, X|_{M'}, L \cap M')$ is also a Liouville quadruple. Assume that there exists $H \in C^\infty(M)$ such that $dH(X) > 0$ on $M \setminus \mathrm{int} M'$. Then* $\mathrm{WFH}_*(M, \omega, L) \cong \mathrm{WFH}_*(M', \omega|_{M'}, L \cap M')$.

***Proof.*** For any $x \in M \setminus M'$, an integral curve of $X$ through $x$ starts from $\partial M'$ and ends at $\partial M$. This is because $\inf_{M \setminus \mathrm{int} M'} dH(X) > 0$. Thus there exists a family $(M_t)_{0 \le t \le 1}$ of submanifolds of $M$ such that $(M_t, \omega|_{M_t}, X|_{M_t}, L \cap M_t)_{0 \le t \le 1}$ is a smooth family of Liouville quadruples and $M_0 = M'$, $M_1 = M$. Now claim follows from propositon 2.7. $\square$

We show an example of calculation of wrapped Floer homology. Consider $\mathbb{R}^{2n}$ with coordinate $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, and the standard symplectic form $\omega_{\mathrm{st}} = \sum_{1 \le i \le n} dp_i \wedge dq_i$. Let $D^{2n} := \left\{(q,p) \mid |q|^2 + |p|^2 \le 1\right\}$, $X := \frac{1}{2} \sum_{1 \le i \le n} q_i \partial_{q_i} + p_i \partial_{p_i}$. Then, $\left(D^{2n}, \omega_{\mathrm{st}}, X, D^{2n} \cap \{p = 0\}\right)$ is a Liouville quadruple.

**Proposition 2.9.** $\mathrm{WFH}_*\left(D^{2n}, \omega_{\mathrm{st}}, D^{2n} \cap \{p = 0\}\right) = 0$.

***Proof.*** Let $\lambda := i_X \omega_{\mathrm{st}}$. Take $(a_n)_n$, an increasing sequence of positive numbers such that $\lim_{n \to \infty} a_n = \infty$ and $a_n \notin \mathscr{A}\left(\partial D^{2n}, \lambda, \partial D^{2n} \cap \{p = 0\}\right)$ for each $n$.

We identify $\hat{D^{2n}}$ with $\mathbb{R}^{2n}$ using a flow generated by $X$, and define $H_n \in \mathscr{H}_{\mathrm{ad}}(\hat{D^{2n}})$ by $H_n(p,q) = n + a_n\big(|p|^2 + |q|^2\big)$. Then $H_1 < H_2 < \cdots$ and $(H_n)_n$ is a cofinal sequence in $\big(\mathscr{H}_{\mathrm{ad}}(\hat{D^{2n}}), <\big)$. Therefore $\mathrm{WFH}_*\big(D^{2n}, \omega_{\mathrm{st}}, D^{2n} \cap \{p=0\}\big) = \lim_{n\to\infty} \mathrm{WFH}_*(H_n)$. The only element of $\mathscr{C}(H_n)$ is a constant map to $(0,\ldots,0)$, and its index goes to $\infty$ as $n \to \infty$. Therefore, for any $k$, $\mathrm{WFH}_k(H_n) = 0$ for sufficiently large $n$. This completes the proof. □

We conclude this section with a remark on relation between wrapped Floer homology and Reeb chords. The following theorem can be proved by reduction to the finite dimensional Morse theory.

**Theorem 2.10.** *Let $(M,\omega,X,L)$ be a Liouville quadruple. If $\mathscr{C}(\partial M, \lambda, \partial L) = \emptyset$, then $\mathrm{WFH}_*(M,\omega,X,L) \cong H_*(L,\partial L)$.*

as a corllary, we get:

**Corollary 2.11.** *Let $(M,\omega,X,L)$ be a Liouville quadruple. If $\mathrm{WFH}_*(M,\omega,X,L) = 0$, then $\mathscr{C}(\partial M,\lambda,\partial L) \neq \emptyset$.*

**Remark 2.12.** The Reeb vector field on $(\partial M, \lambda)$ depends on $\lambda$, but "characteristic foliation" $\mathbb{R}R$ on $\partial M$ depends only on $\omega$. Since characteristic foliation determines Reeb chords up to parametrization, following assertion make sence: if $\mathrm{WFH}_*(M,\omega,L) = 0$, then $\mathscr{C}(\partial M, \partial L) \neq \emptyset$.

## 3. A $C^0$ estimate

The goal of this section is to prove theorem 2.5 and proposition 2.7. Theorem 2.5 is proved in section 3.1-3.4. We only prove (2), since proof of (1) is much simpler than that of (2). In 3.1, we reduce theorem 2.5 to three lemmas. These lemmas are proved in section 3.2, 3.3, 3.4. In 3.5, we prove proposition 2.7. The proof of proposition 2.7 is similar to the proof of invariance of symplectic homology under deformations (which can be found, for instance, in [S]). The crucial step in the proof of proposition 2.7 is $C^0$ estimate for Floer trajectories (lemma 3.9), and its proof is very similar to the proof of theorem 2.5. Hence in 3.5, we only mention few points which make difference.

3.1. **Reduction of the proof to three lemmas.** First, we introduce some abbreviations which we will use in the following of this section. We abbreviate $a_{H^s}$ by $a^s$, and $\Phi_{a^s}$, $\lambda^{a^s}$, $\xi^{a^s}$, $R^{a^s}$ by $\Phi_s$, $\lambda^s$, $\xi^s$, $R^s$. Moreover, we abbreviate $\langle\,\cdot\,,\,\cdot\,\rangle_{a^s,J^s_t}$ by $\langle\,\cdot\,,\,\cdot\,\rangle_{s,t}$, $\langle\,\cdot\,,\,\cdot\,\rangle_{a^s,J^s_t,\partial M}$ by $\langle\,\cdot\,,\,\cdot\,\rangle_{s,t,\partial M}$, and $c_{J^s_t}$ by $c_{s,t}$ (see section 2.4). Finally, we abbreviate an almost complex structure $(\Phi_s)^*(J^s_t)$ on $\partial M \times (0,\infty)$ by $\overline{J}^s_t$.

Take $\rho_0 > 0$ so large that $\Phi_s\big(\partial M \times [\rho_0,\infty)\big) \subset \hat{M} \setminus K$ for any $s$. Take smooth function $\varphi : (0,\infty) \to \mathbb{R}$ such that

$$\begin{aligned} \varphi''(\rho) &\geq 0, \\ \varphi'(\rho) &= 1 \quad (\rho \geq \rho_0 + 1), \\ \varphi(\rho) &= 0 \quad (\rho \leq \rho_0). \end{aligned}$$

Note that $\varphi(\rho) \geq \rho - (\rho_0 + 1)$ for any $\rho$.

For each $s \in \mathbb{R}$, we define $\varphi^s \colon \hat{M} \to \mathbb{R}$ by

$$\varphi^s(x) = \begin{cases} 0 & (x \in K) \\ \varphi(\rho) & \bigl(x = \Phi_s(z,\rho)\bigr). \end{cases}$$

By definition of $\rho_0$ and $\varphi$, it is easy to verify that each $\varphi^s$ is a smooth function on $\hat{M}$.

For $x_- \in \mathscr{C}(H)$, $x_+ \in \mathscr{C}(H')$ and $u \in \mathscr{M}_{(H^s,J^s_t)}(x_-,x_+)$, we define $\alpha^u : \mathbb{R} \times [0,1] \to \mathbb{R}$ by $\alpha^u(s,t) = \varphi^s\bigl(u(s,t)\bigr)$.

**Lemma 3.1.** *$\partial_t \alpha^u = 0$ on $\mathbb{R} \times \{0,1\}$.*

***Proof.*** If $u(s,t) \in K$, then $\alpha^u \equiv 0$ on some neighborhood of $(s,t)$, hence $\partial_t \alpha^u(s,t) = 0$. Therefore it is enough to consider the case $u(s,t) \notin K$. Let $D := \bigl\{(s,t) \in \mathbb{R} \times [0,1] \bigm| u(s,t) \notin K\bigr\}$. This is an open set in $\mathbb{R} \times [0,1]$. Define $v : D \to \partial M \times (0,\infty)$ by

$$v(s,t) := (\Phi_s)^{-1}\bigl(u(s,t)\bigr)$$

and $z : D \to \partial M$, $\rho : D \to (0,\infty)$ by

$$\bigl(z(s,t), \rho(s,t)\bigr) := v(s,t).$$

Since $u$ satisfies $\partial_s u - J^s_t\bigl(\partial_t u - X_{H^s}(u)\bigr) = 0$, by simple calculation we obtain:

$$\partial_s v - \overline{J}^s_t \partial_t v - \rho \cdot \bigl(c_{s,t} + \partial_s a^s(z) \cdot a^s(z)^{-1}\bigr)\partial_\rho = 0. \tag{1}$$

Since $\alpha^u(s,t) = \varphi\bigl(\rho(s,t)\bigr)$, it is enough to show $d\rho(\partial_t v) = 0$. By (1) in lemma 2.3, it is equivalent to $\langle \partial_t v, \partial_\rho \rangle_{s,t} = 0$. By (1), it is enough to check $\bigl\langle \overline{J}^s_t \partial_s v, \partial_\rho \bigr\rangle_{s,t} = \bigl\langle \overline{J}^s_t \partial_\rho, \partial_\rho \bigr\rangle_{s,t} = 0$. The latter is obvious. Since $u\bigl(\mathbb{R} \times \{0,1\}\bigr) \subset \hat{L}$, if $t \in \{0,1\}$ then

$$\partial_s v(s,t) \in T(\partial L) \oplus \mathbb{R}\partial_\rho \subset \xi^s \oplus \mathbb{R}\partial_\rho.$$

Hence we get $\overline{J}^s_t \partial_s v \in \xi^s \oplus \mathbb{R}R^s$ and $\overline{J}^s_t v_s$ is orthogonal to $\partial_\rho$. $\square$

Following three lemmas play crucial role in the proof of theorem 2.5. They are proved in section 3.2, 3.3, 3.4.

**Lemma 3.2.** *For any $x_- \in \mathscr{C}(H)$ and $x_+ \in \mathscr{C}(H')$, there exists $c_0(x_-,x_+), c_1(x_-,x_+) > 0$ such that $\Delta\alpha^u + c_0(x_-,x_+)\alpha^u + c_1(x_-,x_+) \geq 0$ for every $u \in \mathscr{M}_{(H^s,J^s_t)_{s,t}}(x_-,x_+)$.*

**Lemma 3.3.** *For any $x_- \in \mathscr{C}(H)$, $x_+ \in \mathscr{C}(H')$ and $\delta > 0$, there exists $c(x_-,x_+,\delta) > 0$ such that: for any $u \in \mathscr{M}_{(H^s,J^s_t)_{s,t}}(x_-,x_+)$, there exists a sequence $(s_k)_{k\in\mathbb{Z}}$ with following properties:*

1. *$0 < s_{k+1} - s_k < \delta$ for any $k$.*
2. *$\sup_{t\in[0,1]} \alpha^u(s_k,t) \leq c(x_-,x_+,\delta)$ for any $k$.*

**Lemma 3.4.** *Assume that $a, b, \lambda \geq 0$ and $\delta > 0$ are given such that $\delta^2\lambda < \pi^2$. Then, there exists $c(a,b,\lambda,\delta) > 0$ such that, if closed interval $I$ and smooth function $\alpha \colon I \times [0,1] \to \mathbb{R}$ satisfies $0 < |I| \leq \delta$ and*

1. *$\partial_t \alpha = 0$ on $I \times \{0,1\}$,*
2. *$\Delta\alpha + \lambda\alpha + a \geq 0$,*

(3) $\sup\{\alpha(s,t) \mid s \in \partial I\} \leq b$,

*then,* $\sup \alpha \leq c(a,b,\lambda,\delta)$.

We give a proof of theorem 2.5 (2) assuming those results. Since $\mathscr{C}(H)$ and $\mathscr{C}(H')$ are finite sets, it is enough to show that:

> For any $x_- \in \mathscr{C}(H)$ and $x_+ \in \mathscr{C}(H')$, there exists a compact set $B(x_-,x_+) \subset \hat{M}$ such that any $u \in \mathscr{M}_{(H^s,J^s_t)_{s,t}}(x_-,x_+)$ satisfies $u\big(\mathbb{R}\times[0,1]\big) \subset B(x_-,x_+)$.

Take $\delta > 0$ so small that $\delta^2 c_0 < \pi^2$. Then, for any $u \in \mathscr{M}_{(H^s,J^s_t)_{s,t}}(x_-,x_+)$, if we take $(s_k)_k$ as in lemma 3.3, $u|_{I\times[s_k,s_{k+1}]}$ satisfies assumptions of lemma 3.4 for each $k$, with $a = c_1, b = c(x_-,x_+,\delta), \lambda = c_0$. (it follows from lemma 3.1 and lemma 3.2). Hence $\sup \alpha_u \leq c\big(c_1, c(x_-,x_+,\delta), c_0, \delta\big)$. This proves the above claim.

3.2. **Proof of lemma 3.2.** Let $x_- \in \mathscr{C}(H)$ and $x_+ \in \mathscr{C}(H')$. Our goal is to show that there exist $c_0, c_1 > 0$, which are independent of $u \in \hat{\mathscr{M}}_{(H^s,J^s_t)_{s,t}}(x_-,x_+)$, such that

$$\Delta\alpha^u + c_0\alpha^u + c_1 \geq 0 \tag{2}$$

holds on $\mathbb{R}\times[0,1]$. In the following of this subsection, we fix $u$ and abbreviate $\alpha^u$ by $\alpha$.

If $u(s,t) \in K$, then $\alpha \equiv 0$ on some neighborhood of $(s,t)$, and (2) holds for any $c_0, c_1 > 0$. Therefore, it is enough to show (2) for $(s,t) \in D$ (we use notations $D, v, z, \rho$ which are introduced in the proof of lemma 3.1).

Since $\alpha|_D = \varphi\circ\rho$, we get

$$\Delta\alpha = \varphi''(\rho)\big((\partial_s\rho)^2 + (\partial_t\rho)^2\big) + \varphi'(\rho)\Delta\rho \geq \varphi'(\rho)\Delta\rho. \tag{3}$$

Assume for the moment that there exists $c_2 > 0$, which is independent of $u$ and

$$\Delta\rho + c_2\rho \geq 0 \text{ on } D. \tag{4}$$

Then, combining (3), (4) and $\varphi(\rho) \geq \rho - (\rho_0 + 1)$, we get

$$\begin{aligned}\Delta\alpha + c_2\alpha + c_2(\rho_0+1) &\geq \Delta\alpha + c_2\varphi'(\rho)(\alpha+\rho_0+1) \geq \Delta\alpha + c_2\varphi'(\rho)\rho \\ &\geq \varphi'(\rho)(\Delta\rho + c_2\rho) \geq 0.\end{aligned}$$

i.e. (2) holds for $c_0 = c_2$, $c_1 = c_2(\rho_0+1)$ on $D$. Hence our goal is to show the existence of $c_2 > 0$ such that (4) holds.

Applying $d\rho$ and $\lambda^s$ to (1), we get

$$\partial_s\rho + c_{s,t}(\rho\lambda^s)(\partial_t v) - \rho\cdot\big(c_{s,t} + \partial_s a^s(z)\cdot a^s(z)^{-1}\big) = 0, \tag{5}$$

$$c_{s,t}(\rho\lambda^s)(\partial_s v) - \partial_t\rho = 0. \tag{6}$$

By these two equations, we get

$$\begin{aligned}\Delta\rho &= c_{s,t}d(\rho\lambda^s)(\partial_t v, \partial_s v) + \partial_s\rho\cdot\big(c_{s,t} + \partial_s a^s(z)\cdot a^s(z)^{-1}\big) \\ &\quad + \rho\cdot\Big(\partial_s\big(c_{s,t} + \partial_s a^s(z)\cdot a^s(z)^{-1}\big) - c_{s,t}\cdot\partial_s\lambda^s(\partial_t z) + \partial_t c_{s,t}\cdot\lambda^s(\partial_s z) - \partial_s c_{s,t}\cdot\lambda^s(\partial_t z)\Big).\end{aligned}$$

On the other hand, by (1),

$$d(\rho\lambda^s)(\partial_t v, \partial_s v) = |\partial_s v|^2_{s,t} - c^{-1}_{s,t}\cdot\partial_s\rho\cdot\big(c_{s,t} + \partial_s a^s(z)\cdot a^s(z)^{-1}\big).$$

Then, we get

$$\Delta\rho = c_{s,t}|\partial_s v|^2_{s,t} + \rho\cdot\Big(\partial_s\big(c_{s,t}+\partial_s a^s(z)\cdot a^s(z)^{-1}\big) - c_{s,t}\cdot\partial_s\lambda^s(\partial_t z) + \partial_t c_{s,t}\cdot\lambda^s(\partial_s z) - \partial_s c_{s,t}\cdot\lambda^s(\partial_t z)\Big).$$

For $V \in T\big(\partial M \times (0,\infty)\big)$, we denote its $T(\partial M)$-part by $(V)_{\partial M}$. On $\Phi_s^{-1}(\hat{M}\setminus K)$, $T(\partial M)$ and $\partial_\rho$ are orthogonal to each other with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{s,t}$. Hence $|(V)_{\partial M}|_{s,t} \le |V|_{s,t}$ for any $V$. Then, we get (recall lemma 2.3):

$$|\partial_s z|_{s,t,\partial M} = \rho^{-1/2}\big|(\partial_s v)_{\partial M}\big|_{s,t} \le \rho^{-1/2}|\partial_s v|_{s,t},$$

$$|\partial_t z|_{s,t,\partial M} = \rho^{-1/2}\big|(\partial_t v)_{\partial M}\big|_{s,t} \le \rho^{-1/2}|\partial_t v|_{s,t} \le \rho^{-1/2}|\partial_s v|_{s,t} + c_{s,t}^{-1/2}\big(c_{s,t} + \partial_s a^s(z)\cdot a^s(z)^{-1}\big).$$

In the last inequality, we use (1) and $\big|\partial_\rho\big|_{s,t} = (\rho c_{s,t})^{-1/2}$. On the otherhand, there exist constants $c_3, c_4, c_5 > 0$, which are independent of $u$ and satisfy

$$\big|\partial_s\big(\partial_s a^s(z)\cdot a^s(z)^{-1}\big)\big| \le c_3|\partial_s z|_{s,t,\partial M} + c_4, \qquad \big|\partial_s\lambda^s(\partial_t z)\big| \le c_5|\partial_t z|_{s,t,\partial M}.$$

Hence there exists constants $c_6, c_7 > 0$, which are independent of $u$ and satisfy

$$\Delta\rho \ge c_{s,t}|\partial_s v|^2_{s,t} - c_6\rho^{1/2}|\partial_s v|_{s,t} - c_7\rho.$$

Therefore

$$\Delta\rho \ge c_{s,t}|\partial_s v|^2_{s,t} - \left(\frac{c_{s,t}|\partial_s v|^2_{s,t}}{2} + \frac{c_{s,t}^{-1}c_6^2\rho}{2}\right) - c_7\rho \ge -\left(\frac{c_{s,t}^{-1}c_6^2}{2} + c_7\right)\rho.$$

Hence (4) holds when $c_2 \ge \dfrac{\sup_{s,t} c_{s,t}^{-1}\cdot c_6^2}{2} + c_7$. This completes the proof of lemma 3.2.

3.3. **Proof of lemma 3.3.** Let $u \in \mathscr{M}_{(H^s,J^s_t)}(x_-,x_+)$. Recall lemma 2.4:

$$\partial_s\big(\mathscr{A}_{H^s}\big(u(s)\big)\big) = -\int_0^1 \big|\partial_s u(s,t)\big|^2_{J^s_t} + \partial_s H^s\big(u(s,t)\big)dt < 0.$$

In particular,

$$\mathscr{A}_{H'}(x_+) < \mathscr{A}_{H^s}\big(u(s)\big) < \mathscr{A}_H(x_-)$$

for any $s$. Hence, for any interval $I \subset \mathbb{R}$, there exists $s \in I$ such that

$$|I|\cdot\int_0^1 \big|\partial_s u(s,t)\big|^2_{J^s_t} + \partial_s H^s\big(u(s,t)\big)dt < \mathscr{A}_H(x_-) - \mathscr{A}_{H'}(x_+).$$

Hence, we can conclude:

**Lemma 3.5.** *For any $\delta > 0$ and $u \in \hat{\mathscr{M}}_{(H^s,J^s_t)}(x_-,x_+)$, there exists a sequence $(s_k)_{k\in\mathbb{Z}}$ with following properties:*

(1) $0 < s_{k+1} - s_k < \delta$ *for any* $k$.

(2) $\displaystyle\int_0^1 \big|\partial_s u(s,t)\big|^2_{J^s_t} + \partial_s H^s\big(u(s,t)\big)dt < \frac{2\big(\mathscr{A}_H(x_-) - \mathscr{A}_{H'}(x_+)\big)}{\delta}$ *for any* $k$.

Note that $|\partial_s u|_{J^s_t} = |\partial_t u - X_{H^s}\circ u|_{J^s_t}$. Therefore, to prove lemma 3.3, it is sufficient to prove the following:

**Lemma 3.6.** *For any $c > 0$, there exists $M(c) > 0$ such that: if $s \in \mathbb{R}$ and $x\colon [0,1] \to \hat{M}$ satisfy $x(0), x(1) \in \hat{L}$ and*

$$\int_0^1 \left|\partial_t x - X_{H^s}\big(x(t)\big)\right|^2_{J^s_t} + \partial_s H^s\big(x(t)\big) dt \le c,$$

*then* $\sup_{0\le t\le 1} \varphi^s\big(x(t)\big) \le M(c)$.

***Proof.*** If this lemma does not hold, there exist sequences $(s_k)_k$ and $(x_k)_k$ such that

$$\int_0^1 \left|\partial_t x_k - X_{H^{s_k}}\big(x_k(t)\big)\right|^2_{J^{s_k}_t} + \partial_s H^s\big(s_k, x_k(t)\big) dt \le c, \tag{7}$$

$$\lim_{k\to\infty} \sup_{0\le t\le 1} \varphi^{s_k}\big(x_k(t)\big) = \infty. \tag{8}$$

Recall that in statement of theorem 2.5, we take $s_0 > 0$ such that

$$J^s_t = \begin{cases} J^{-s_0}_t & (s \le -s_0), \\ J^{s_0}_t & (s \ge s_0). \end{cases}$$

By replacing $s_0$ if necessary, we may assume that $s_0$ also satisfies $H^s = \begin{cases} H & (s \le -s_0) \\ H' & (s \ge s_0) \end{cases}$. Then, we may assume that $s_k \in [-s_0, s_0]$ for all $k$. Note that (7) implies

$$\int_0^1 \left|\partial_t x_k - X_{H^{s_k}}\big(x_k(t)\big)\right|^2_{J^{s_k}_t} dt \le c, \tag{9}$$

$$\int_0^1 \partial_s H^s\big(s_k, x_k(t)\big) dt \le c. \tag{10}$$

First we show that $\lim_{k\to\infty} \inf_{0\le t\le 1} \varphi^{s_k}\big(x_k(t)\big) = \infty$. If this does not hold, by replacing $(s_k)_k$ and $(x_k)_k$ to their subsequences, we may assume that $\sup_k \inf_{0\le t\le 1} \varphi^{s_k}\big(x_k(t)\big) < \infty$. Then, for sufficiently large $k$, there exist $a_k, b_k \in [0,1]$ such that

$$\begin{aligned} &\sup_k \varphi^{s_k}\big(x_k(a_k)\big) < \infty, \\ &\lim_{k\to\infty} \varphi^{s_k}\big(x_k(b_k)\big) = \infty, \\ &0 \le \theta \le 1 \implies x_k\big(\theta a_k + (1-\theta) b_k\big) \subset \hat{M} \setminus K. \end{aligned}$$

Without loss of generality, we may assume that $a_k \le b_k$. Define $y_k : [a_k, b_k] \to \partial M \times (0,\infty)$, $z_k : [a_k, b_k] \to \partial M$, $\rho_k : [a_k, b_k] \to (0,\infty)$ by

$$y_k(t) := (\Phi_{s_k})^{-1}(x_k(t)), \qquad \big(z_k(t), \rho_k(t)\big) := y_k(t).$$

Then

$$\int_{a_k}^{b_k} \left|\partial_t x_k - X_{H^{s_k}}\big(x_k(t)\big)\right|^2_{J^{s_k}_t} dt = \int_{a_k}^{b_k} \left|\partial_t y_k - \overline{R^{s_k}}(y_k(t))\right|^2_{s_k,t} dt \ge \int_{a_k}^{b_k} \left|\big(\partial_t y_k\big)_{\partial_\rho}\right|^2_{s_k,t} dt$$

$$\ge \inf_{s,t} c_{s,t}^{-1} \int_{a_k}^{b_k} \big(\rho_k(t)^{-\frac{1}{2}} \cdot \partial_t \rho_k\big)^2 dt \ge \inf_{s,t} c_{s,t}^{-1} \cdot 4\big(\rho_k(b_k)^{\frac{1}{2}} - \rho_k(a_k)^{\frac{1}{2}}\big)^2 \cdot (b_k - a_k)^{-1}.$$

Since $\rho_k(a_k)$ is bounded and $\lim_{k\to\infty} \rho_k(b_k) = \infty$, we get

$$\lim_{k\to\infty} \int_{a_k}^{b_k} \left|\partial_t x_k - X_{H^{s_k}}\big(x_k(t)\big)\right|^2_{J_t^{s_k}} dt = \infty.$$

This contradicts (9), and we have shown that $\lim_{k\to\infty} \inf_{0\le t\le 1} \varphi^{s_k}\big(x_k(t)\big) = \infty$. In particular, $x_k\big([0,1]\big) \subset \hat{M} \setminus K$ for sufficiently large $k$. For such $k$, define $y_k : [0,1] \to \partial M \times (0,\infty)$, $z_k : [0,1] \to \partial M$, $\rho_k \colon [0,1] \to (0,\infty)$ by

$$y_k(t) := (\Phi_{s_k})^{-1}\big(x_k(t)\big), \qquad \big(z_k(t), \rho_k(t)\big) := y_k(t).$$

Then, by (9) and (10), $y_k$ satisfies

$$\int_0^1 \left|\partial_t y_k - \overline{R^{s_k}}\big(y_k(t)\big)\right|^2_{s_k,t} dt \le c, \tag{11}$$

$$\int_0^1 \partial_s a^s\big(s_k, z_k(t)\big) \cdot a^{s_k}\big(z_k(t)\big)^{-1} \cdot \rho_k(t) dt + \partial_s b(s_k) \le c. \tag{12}$$

Since $\lim_{k\to\infty} \inf_{0\le t\le 1} \varphi^{s_k}\big(x_k(t)\big) = \infty$, we get $\lim_{k\to\infty} \inf_{0\le t\le 1} \rho_k(t) = \infty$.

By replacing $(x_k)_k$ and $(s_k)_k$ to their subsequences, we may assume that $(s_k)_k$ converges to some $s_\infty \in [-s_0, s_0]$. Since

$$\left|\partial_t y_k - \overline{R^{s_k}}\big(y_k(t)\big)\right|^2_{s_k,t} \ge \left|\big(\partial_t y_k - \overline{R^{s_k}}\big(y_k(t)\big)\big)_{\partial M}\right|^2_{s_k,t} = \left|\partial_t z_k - R^{s_k}\big(z_k(t)\big)\right|^2_{s_k,t,\partial M} \cdot \rho_k(t),$$

we get from (11) and $\lim_{k\to\infty} \inf_{0\le t\le 1} \rho_k(t) = \infty$ that

$$\lim_{k\to\infty} \int_0^1 \left|\partial_t z_k - R^{s_k}\big(z_k(t)\big)\right|^2_{s_k,t,\partial M} dt = 0.$$

Then, by taking limit of certain subsequence of $(z_k)_k$, we get $z_\infty : [0,1] \to \partial M$ such that

$$z_\infty(0), z_\infty(1) \in \partial L, \qquad \partial_t z_\infty(t) = R^{s_\infty}\big(z_\infty(t)\big).$$

Therefore $1 \in \mathscr{A}(\partial M, \lambda^{s_\infty}, \partial L)$, hence $s_\infty \in (-s_0, s_0)$. By the definition of monotone homotopy, $\inf_{z\in\partial M} \partial_s a^s(s_\infty, z) > 0$. Hence, there exists $\varepsilon > 0$ such that $\inf_{z\in\partial M} \partial_s a^s(s_k, z) \ge \varepsilon$ for sufficiently large $k$. Let $A := \sup_{(s,z)\in\mathbb{R}\times\partial M} a^s(z)$. Then,

$$\int_0^1 \partial_s a^s\big(s_k, z_k(t)\big) \cdot a^{s_k}\big(z_k(t)\big)^{-1} \cdot \rho_k(t) dt \ge \varepsilon A^{-1} \int_0^1 \rho_k(t) dt$$

for sufficiently large $k$. Since $\lim_{k\to\infty} \inf_{0\le t\le 1} \rho_k(t) = \infty$, the right hand side of the above inequality goes to $\infty$ as $k \to \infty$. Hence the left hand side of the above inequality also goes to $\infty$ as $k \to \infty$. This contradicts (12). This completes the proof of lemma 3.6. □

3.4. **Proof of lemma 3.4.** We use following result, which is exactly the same as proposition 8 in [FH].

**Lemma 3.7.** *Assume that $a, b, \lambda \geq 0$ and $\delta > 0$ are given such that $\delta^2 \lambda < \pi^2$. Then, there exists $C(a, b, \lambda, \delta) > 0$ such that, if closed interval $I$ and smooth function $\alpha\colon I \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ satisfy $0 < |I| \leq \delta$ and*

$$\Delta\alpha + \lambda\alpha + a \geq 0,$$
$$\sup\bigl\{\alpha(s,t) \bigm| s \in \partial I\bigr\} \leq b,$$

*then,* $\sup \alpha \leq C(a, b, \lambda, \delta)$.

**Remark 3.8.** For any $\tau > 0$, lemma 3.7 holds if we replace $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}/\tau\mathbb{Z}$ in the statment.

***proof of lemma 3.4****:* For any $\varepsilon > 0$, there exists $\delta > 0$ and $\beta\colon I \times [0,1] \to \mathbb{R}$ such that:

$$\sup_{I\times[0,1]} |\alpha - \beta|, \sup_{I\times[0,1]} \bigl|\Delta(\alpha - \beta)\bigr| \leq \varepsilon,$$
$$1 - \delta \leq t \leq 1 \implies \beta(s,t) = \alpha(s,1) + \partial_t^2\alpha(s,1) \cdot \frac{(t-1)^2}{2},$$
$$0 \leq t \leq \delta \implies \beta(s,t) = \alpha(s,0) + \partial_t^2\alpha(s,0) \cdot \frac{t^2}{2}.$$

Define $\overline{\beta}\colon I \times \mathbb{R}/2\mathbb{Z} \to \mathbb{R}$ by

$$\overline{\beta}(s,t) = \begin{cases} \beta(s,t) & (0 \leq t \leq 1), \\ \beta(s, 2-t) & (1 \leq t \leq 2). \end{cases}$$

Then, $\overline{\beta} \in C^\infty(I \times \mathbb{R}/2\mathbb{Z})$. Moreover, $\overline{\beta}$ satisfies

$$\Delta\overline{\beta} + \lambda\overline{\beta} + \bigl(a + (1+\lambda)\varepsilon\bigr) \geq 0, \qquad \sup\bigl\{\overline{\beta}(s,t) \bigm| s \in \partial I\bigr\} \leq b + \varepsilon.$$

Then, if we take $C = C(a + (1+\lambda)\varepsilon, b + \varepsilon, \lambda, \delta)$ as in lemma 3.7, $\sup \beta = \sup \overline{\beta} \leq C$. Hence $\sup \alpha \leq C + \varepsilon$. □

3.5. **Proof of proposition 2.7.** First, we may assume that $(\omega_s, X_s) = (\omega_0, X_0)$ if $s$ is sufficiently close to 0, and $(\omega_s, X_s) = (\omega_1, X_1)$ if $s$ is sufficiently close to 1. Then, extend $(\omega_s, X_s)_{0\leq s\leq 1}$ to $(\omega_s, X_s)_{s\in\mathbb{R}}$ by

$$(\omega_s, X_s) = \begin{cases} (\omega_0, X_0) & (s \leq 0) \\ (\omega_1, X_1) & (s \geq 1). \end{cases}$$

The crucial step in the proof of proposition 2.7 is:

**Lemma 3.9.** *Let $H, H' \in \mathscr{H}_{\mathrm{ad}}(\hat{M})$ and $(H^s)_s$ be a monotone homotopy from $H$ to $H'$, such that $a_{H^s}$ is a constant function on $\partial M$ for any $s$ ($a_{H^s} \equiv: a(s)$). Let $(J_t^s)_{s,t}$ be a family of almost complex structures on $\hat{M}$ such that*

$$J_t^s = \begin{cases} J_t^0 & (s \leq 0) \\ J_t^1 & (s \geq 1) \end{cases}.$$

*Assume that there exsits a compact set $K$ in $\hat{M}$, which contains $M$ and $H^s \in \mathscr{H}_K(\hat{M})$, $J_t^s \in \mathscr{J}_{1,K}(\hat{M}; \hat{\omega^s})$ for any $s$ and $t$ (here $1$ denotes the constant function on $\partial M$). Then, there exsit constants $c_0, c_1 > 0$, which depend only on $(\omega^s, X^s)_s$ and $(J_t^s)_{s,t}$, with following* *property: if* $c_0 a(s) + c_1 \le a'(s)$ *for* $s \in [0,1]$*, there exsits a compact set* $B \subset \hat{M}$ *such that* $u\big(\mathbb{R}\times[0,1]\big) \subset B$ *for any* $x_- \in \mathscr{C}(H)$*,* $x_+ \in \mathscr{C}(H')$*,* $u \in \hat{\mathscr{M}}_{(H^s,J^s_t)}(x_-,x_+)$*.*

Once lemma 3.9 is established, we can define a chain map $\varphi_k^{(H^s,J^s_t)_{s,t}} : \mathrm{WFC}_k(H;M,\omega^0,X^0,L) \to \mathrm{WFC}_k(H';M,\omega^1,X^1,L)$ by

$$\varphi_k^{(H^s,J^s_t)_{s,t}}[x] = \sum_{y\in\mathscr{C}_k(H')} \sharp\hat{\mathscr{M}}_{(H^s,J^s_t)}(x,y)\cdot[y],$$

given a monotone homotopy $(H^s)_s$ with conditions as in lemma 3.9. Hence we get a morphism $\mathrm{WFH}_k(H;M,\omega^0,X^0,L) \to \mathrm{WFH}_k(H';M,\omega^1,X^1,L)$. By taking direct limit, we obtain a morphism

$$\mathrm{WFH}_*(M,\omega^0,X^0,L) \to \mathrm{WFH}_*(M,\omega^1,X^1,L).$$

We can also obtain a morphism in invert direction, and show that they are inverse to each other. This completes the proof of propositon 2.7. Hence all we have to show is lemma 3.9.

The proof of lemma 3.9 is very similar to the proof of theorem 2.5. First we take $\rho_0 > 1$ so that $K \subset \mathrm{int}M(\rho_0)$, take smooth function $\varphi : [1,\infty) \to \mathbb{R}$ such that

$$\begin{aligned} \varphi''(\rho) &\ge 0, \\ \varphi'(\rho) &= 1 \quad (\rho \ge \rho_0 + 1), \\ \varphi(\rho) &= 0 \quad (\rho \le \rho_0), \end{aligned}$$

and define $\alpha_u \in C^\infty\big(\mathbb{R}\times[0,1]\big)$ for $u \in \hat{\mathscr{M}}_{(H^s,J^s_t)}(x_-,x_+)$ by

$$\alpha_u(s,t) = \begin{cases} 0 & \big(u(s,t)\in K\big) \\ \varphi\big(\rho(s,t)\big) & \big(u(s,t)\in \hat{M}\setminus K\big). \end{cases}$$

Once we establish properties which correspond to lemma 3.1, lemma 3.2, lemma 3.3 for $\alpha_u$, the proof completes. The first two properties can be proved in completely same way. But to establish the property which corresponds to lemma 3.3, we need somewhat different arguments. In the following, we prove the property which corresponds to lemma 3.3. First we spell out what we have to prove.

**Lemma 3.10.** *Let* $H$*,* $H'$*,* $(H^s)_s$ *and* $(J^s_t)_{s,t}$ *are as in lemma 3.9. Then, there exsit constants* $c_0, c_1 > 0$*, which depend only on* $(\omega^s, X^s)_s$ *and* $(J^s_t)_{s,t}$*, with following property:*

> *Assume* $c_0 a(s) + c_1 \le a'(s)$ *for* $s \in [0,1]$*. Then, for any* $x_- \in \mathscr{C}(H)$*,* $x_+ \in \mathscr{C}(H')$ *and* $\delta > 0$*, there exsits* $c(x_-,x_+,\delta) > 0$ *such that for any* $u \in \hat{\mathscr{M}}_{(H^s,J^s_t)}(x_-,x_+)$*, there exsits a sequence* $(s_k)_{k\in\mathbb{Z}}$ *with :*
> (1) $0 < s_{k+1} - s_k < \delta$ *for any* $k$*.*
> (2) $\sup_{0\le t\le 1} \alpha_u(s_k,t) \le c(x_-,x_+,\delta)$ *for any* $k$*.*

***Proof*.** Let $\hat{\lambda}^s := i_{\hat{X}^s}\hat{\omega}^s$. By simple calculation, we get

$$(13) \qquad -\frac{\partial}{\partial s}\mathscr{A}_{H^s}(u(s)) = \int_0^1 \big|\partial_s u(s,t)\big|^2_{J^s_t} + \frac{\partial H^s}{\partial s}\big(u(s,t)\big) - \frac{\partial\hat{\lambda}^s}{\partial s}\big(\partial_t u(s,t)\big)dt.$$

existence of the third term in integrand requires more arguments than proof of lemma 3.3. In the following, we prove that: there exsits $c_0, c_1 > 0$ such that, if $c_0 a(s) + c_1 \le a'(s)$ holds for $s \in [0,1]$, then there exsits $c_2 > 0$ (which may depend on $(H^s)_s$) such that

$$(14)\quad \big|\partial_s u(s,t)\big|^2_{J^s_t} + \frac{\partial H^s}{\partial s}\big(u(s,t)\big) - \frac{\partial \hat{\lambda}^s}{\partial s}\big(\partial_t u(s,t)\big) + c_2 \ge \frac{1}{2}\biggl(\big|\partial_s u(s,t)\big|^2_{J^s_t} + \frac{\partial H^s}{\partial s}\big(u(s,t)\big)\biggr).$$

Once this is establised, lemma 3.10 is proved by same arguments as proof of lemma 3.3.

Since $K$ is compact, to prove (14) it is enough to show that there exsits $c_3 > 0$ such that

$$(15)\qquad u(s,t) \in \hat{M} \setminus K \implies \left|\frac{\partial \hat{\lambda}^s}{\partial s}\big(\partial_t u(s,t)\big)\right| \le c_3 + \frac{1}{2}\biggl(\big|\partial_s u(s,t)\big|^2_{J^s_t} + \frac{\partial H^s}{\partial s}\big(u(s,t)\big)\biggr).$$

First notice that, since $J^s_t \in \mathscr{J}_{1,K}(\hat{M}; \hat{\omega^s}, \hat{X^s})$, $\langle\,\cdot\,,\,\cdot\,\rangle_{J^s_t}$ satisfies following properties (see lemma 2.3):

(1) On $\hat{M} \setminus K$, a natural decomposition $T\hat{M} = T(\partial M) \oplus \mathbb{R}\frac{\partial}{\partial\rho}$ is an orthgonal decomposition with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{J^s_t}$.
(2) There exsits a metric $\langle\,\cdot\,,\,\cdot\,\rangle_{J^s_t,\partial M}$ on $\partial M$ such that $\big|\overline{v}(z,\rho)\big|_{J^s_t} = \rho^{\frac{1}{2}}\big|v(z)\big|_{J^s_t,\partial M}$ for any $v \in T(\partial M)$ and $(z,\rho) \in \hat{M} \setminus K$.
(3) There exists $c_{s,t}$ such that $\big|\frac{\partial}{\partial\rho}(z,\rho)\big|_{J^s_t} = (\rho c_{s,t})^{-\frac{1}{2}}$ on $\hat{M} \setminus K$.

We return to the proof of (15). Since $\hat{M} \setminus K \subset \partial M \times [1,\infty)$, we can write $u(s,t) = \big(z(s,t), \rho(s,t)\big)$. Let $c_4 := \sup\limits_{z,s,t} \left|\dfrac{\partial \lambda^s}{\partial s}(z)\right|_{J^s_t,\partial M}$. Then,

$$\left|\frac{\partial \hat{\lambda}^s}{\partial s}(\partial_t u)\right| = \rho(s,t)\left|\frac{\partial \lambda^s}{\partial s}(\partial_t z)\right| \le c_4 \rho(s,t)|\partial_t z|_{J^s_t,\partial M} \le c_4 \rho(s,t)^{\frac{1}{2}}|\partial_t u|_{J^s_t}.$$

Since $|\partial_t u|_{J^s_t} \le |\partial_s u|_{J^s_t} + \big|\nabla^s_t H^s\big|_{J^s_t}$ and $\big|\nabla H^s(z,\rho)\big|_{J^s_t} \le \sup\limits_{s,t} c_{s,t} \cdot a(s)\rho^{\frac{1}{2}}$, There exsit $c_5, c_6 > 0$ such that

$$\left|\frac{\partial \hat{\lambda}^s}{\partial s}(\partial_t u)\right| \le \frac{1}{2}|\partial_s u|^2_{J^s_t} + \big(c_5 a(s) + c_6\big)\rho(s,t).$$

On the other hand, $\dfrac{\partial H^s}{\partial s}(z,\rho) = a'(s)\rho + b'(s)$ on $\hat{M} \setminus K$. Hence, if $2c_5 a(s) + 2c_6 \le a'(s)$ and $0 \le 2c_3 + b'(s)$ on $s \in [0,1]$, (15) holds on $s \in [0,1]$. When $s \notin [0,1]$, the left hand side of (15) is zero. Hence, if $c_3 + \inf\limits_{s\in\mathbb{R}} b'(s) \ge 0$, (15) holds for $s \notin [0,1]$. This completes the proof of lemma 3.10. □

## 4. Classical Hamiltonian systems

First we recall notations which are introduced in section 1. Let $N$ be a $n$-dimensional manifold. Then, $T^*N$ carries a natural symplectic form $\omega_N := \sum\limits_{1\le i\le n} dp_i \wedge dq_i$.

Assume that $N$ carries a Riemannian metric. Then, for $V \in C^\infty(N)$, we define $H_V \in C^\infty(T^*N)$ by $H_V(q,p) = V(q) + |p|^2/2$. Note that $\mathrm{Crit}(H_V) = \mathrm{Crit}(V)$.

For $\xi \in \mathfrak{X}(N)$, We define $F_\xi \in C^\infty(T^*N)$ and $\tilde{\xi} \in \mathfrak{X}(T^*N)$ by $F_\xi(q,p) := p(\xi_q)$ and $\tilde{\xi} := X_{F_\xi}$. Then, $L_{\tilde{\xi}}\omega = 0$ and $\tilde{\xi}_{(q,0)} = \xi_q$. For $a \in \mathbb{R}$, define $Y_a \in \mathfrak{X}(T^*N)$ by $Y_a := \vec{r} + a\widetilde{\nabla V}$, where $\vec{r} := \sum_{1 \le i \le n} p_i \partial_{p_i}$.

**Lemma 4.1.** *Let $K$ be a compact set in $T^*N$ such that $K \cap \mathrm{Crit}(V) = \emptyset$. Then, $dH_V(Y_a) > 0$ on $K$ for sufficiently small $a > 0$.*

***Proof*.** Since $\widetilde{\nabla V}(q,0) = \nabla V(q)$ and $K \cap \mathrm{Crit}(V) = \emptyset$, $dH_V(\widetilde{\nabla V}) > 0$ on $K \cap N$. Let $K_0$ be a subset of $K$ defined by $dH_V(\widetilde{\nabla V}) \le 0$. Since $K_0$ is compact and disjoint from $N$, $m := \min_{K_0} |p|^2$ is positive. Take $M > 0$ so that $M > \max_{K_0} -dH_V(\widetilde{\nabla V})$, and take $0 < a < m/M$. Then, $dH_V(Y_a) > m - aM > 0$ on $K_0$. On the other hand, since $dH_V(\widetilde{\nabla V}) > 0$ and $dH_V(\vec{r}) \ge 0$ on $K \setminus K_0$, $dH_V(Y_a) > 0$ on $K \setminus K_0$ for any $a > 0$. ☐

As in section 1, we abbreviate $H_V^{-1}\big((-\infty,h]\big)$ by $D_h$, and $H_V^{-1}(h)$ by $S_h$. If $h$ is a regular value of $H_V$ and $S_h$ is compact, then $(D_h, \omega_N, Y_a, D_h \cap N)$ is a Liouville quadruple for sufficiently small $a > 0$. This is verified by applying lemma 4.1 for $K = S_h$. The main result of this paper is the following:

**Theorem 4.2.** *Let $N$ be a Riemannian manifold, and $V \in C^\infty(N)$. Assume that $h$ is a regular value of $V$, and $S_h$ is compact. If $S_h \cap N \neq \emptyset$ and $D_h$ is connected, then $\mathrm{WFH}_*(D_h, \omega_N, D_h \cap N) = 0$.*

By remark 2.12, theorem 4.2 implies:

**Corollary 4.3.** *Let $N$ and $V$ are as in theorem 4.2. Then, $\mathscr{C}(S_h, S_h \cap N) \neq \emptyset$.*

Since elements of $\mathscr{C}(S_h, S_h \cap N)$ correspond to orbits of $X_{H_V}$ on $S_h$ which start from and end at $S_h \cap N$, corollary 4.3 implies theorem 1.2.

In the remainder of this section, we reduce theorem 4.2 to lemma 4.10. First, we prove the following lemma:

**Lemma 4.4.** *$\mathrm{WFH}_*(D_h, \omega_N, D_h \cap N)$ depends only on diffeomorphism type of $D_h \cap N$.*

***Proof*.** Let $K := D_h \cap N$ and $\overline{K} := K \cup \partial K \times [0,1]$. Take any Riemannian metric $g$ on $\overline{K}$ and $W \in C^\infty(\overline{K})$ so that 0 is a regular value of $W$ and $K = W^{-1}((-\infty,0])$. For such $(g,W)$, define $H_{g,W} \in C^\infty(T^*\overline{K})$ by $H_{g,W}(q,p) = |p|_g^2/2 + W(q)$, and let $D_{g,W} := H_{g,W}^{-1}\big((-\infty,0]\big)$. For $a \in \mathbb{R}$, let $Y_{g,W,a} := \vec{r} + a\widetilde{\nabla_g W}$. Then, $(D_{g,W}, \omega_{\overline{K}}, Y_{g,W,a}, K)$ is a Liouville quadruple for sufficiently small $a > 0$.

We claim that $\mathrm{WFH}_*(D_{g,W}, \omega_{\overline{K}}, K)$ does not depend on choice of $g$ and $W$. In paticular, $\mathrm{WFH}_*(D_{g,W}, \omega_{\overline{K}}, K)$ depends only on diffeomorphism type of $K$. This is proved as follows. Take two choices $(g_0, W_0)$ and $(g_1, W_1)$. Let $g_t := tg_1 + (1-t)g_0$ and $W_t := tW_1 + (1-t)W_0$. Then, when we take $a > 0$ sufficiently small, $(D_{g_t,W_t}, \omega_{\overline{K}}, Y_{g_t,W_t,a}, K)$ is a smooth family of Liouville quadruples. Then, the claim follows from proposition 2.7.

Extend inclusion map $i\colon K \to N$ to an embedding $\overline{i}\colon \overline{K} \to N$. Let $g$ be the pullback of the Riemannian metric on $N$ by $\overline{i}$, and $W := V \circ \overline{i} - h$. Then, $(D_h, \omega_N, D_h \cap N)$ in theorem 4.2 can be identified with $(D_{g,W}, \omega_{\overline{K}}, K)$. So, the above claim proves the lemma. $\square$

We return to the proof of theorem 4.2. We may assume $h = 0$, and by lemma 4.4, we may assume that $V$ is Morse. Then, $\mathrm{Crit}(V) \cap V^{-1}\big((-\infty, 0]\big)$ consists of finitely many points. We denote it as $\{P_1, \dots, P_l\}$. Moreover, we may assume the following.

(1) $V(P_1) < \cdots < V(P_l) < 0$.
(2) $1 \le \mathrm{ind} P_m \le n-1$ for $2 \le m \le l$ and $\mathrm{ind} P_1 = 0$.

Note that we can eliminate critical points of index $n$, since $D_h \cap N$ is connected and its boundary is non-empty.

If $h \in \big(V(P_1), V(P_2)\big)$, $D_h \cap N$ is diffeomorphic to $D^n$. Hence, by lemma 4.4 and propositon 2.9, $\mathrm{WFH}_*(D_h, \omega_N, D_h \cap N) = 0$.

By lemma 4.4, if $[h, h']$ contains no critical value of $V$, then $\mathrm{WFH}_*(D_h, \omega_N, D_h \cap N) \cong \mathrm{WFH}_*(D_{h'}, \omega_N, D_{h'} \cap N)$. Therefore, if we prove the following theorem 4.5, we can prove theorem 4.2 by applying theorem 4.5 to each critical points $P_2, \dots, P_m$.

**Theorem 4.5.** *Let $N$ be a $n$-dimensional Riemannian manifold, $V$ be a Morse function on $N$, and $P \in \mathrm{Crit}(V)$ with $1 \le \mathrm{ind} P \le n-1$. Assume that there exists $\varepsilon > 0$ such that $\mathrm{Crit}(V) \cap V^{-1}\big([V(P) - \varepsilon, V(P) + \varepsilon]\big) = \{P\}$, and $D_{V(P)+\varepsilon}$ is compact. Then,*

$$\mathrm{WFH}_*(D_{V(P)-\varepsilon}, \omega_N, D_{V(P)-\varepsilon} \cap N) \cong \mathrm{WFH}_*(D_{V(P)+\varepsilon}, \omega_N, D_{V(P)+\varepsilon} \cap N).$$

In the remainder of this section, we reduce theorem 4.5 to lemma 4.10. By Morse lemma, there exists a coordinate neighborhood $U$ around $P$ and local chart $(q_1, \dots, q_n)$ on $U$ such that $P$ corresponds to $(0, \dots, 0)$ and

$$V(q) = V(P) + \big\{-(q_1^2 + \ldots + q_k^2) + (q_{k+1}^2 + \ldots + q_n^2)\big\}/2.$$

Here $k = \mathrm{ind} P$. Denote by $\pi_N$ the natural projection $T^*N \to N$. In the following of this paper, we often consider $\pi_N^{-1}(U)$ as a subset of $\mathbb{R}^{2n}$ using the coordinate $(q, p)$.

We introduce some notations which we use in the following of this paper. First, we abbreviate $(q_1, \dots, q_n)$ by $q$, $(p_1, \dots, p_n)$ by $p$, and $(p_1, \dots, p_k)$, $(p_{k+1}, \dots, p_n)$, $(q_1, \dots, q_k)$, $(q_{k+1}, \dots, q_n)$ by $p_-, p_+, q_-, q_+$. Moreover, we set

$$D\big([a,b]\big) := \big\{(q,p) \bigm| p = 0,\ a \le |q|^2 \le b\big\},$$
$$D_-\big([a,b]\big) := \big\{(q,p) \bigm| p = 0,\ q_+ = 0,\ a \le |q_-|^2 \le b\big\}.$$

$D\big((a,b]\big)$ etc. are defined in the same manner.

By lemma 4.4, we may assume that Riemannian metric on $U$ is $\sum_{1 \le i \le n} dq_i^2$. Take $b > 0$ sufficiently small so that $D\big([0, 2b]\big) \subset U$ and $\mathrm{Crit}(V) \cap V^{-1}\big([V(P) - b, V(P) + b]\big) = \{P\}$.

**Lemma 4.6.** *For sufficiently small $a > 0$, $dH_V(Y_a) > 0$ on $H_V^{-1}\big([V(P) - b, V(P) + b]\big) \setminus \{P\}$.*

***Proof.*** On $\pi_N^{-1}(U)$, we can write explicitly:

$$H_V(q,p) = \frac{|p|^2 - |q_-|^2 + |q_+|^2}{2}, \quad dH_V(q,p) = pdp - q_- dq_- + q_+ dq_+,$$
$$Y_a(q,p) = -aq_-\partial_{q_-} + (1+a)p_-\partial_{p_-} + aq_+\partial_{q_+} + (1-a)p_+\partial_{p_+}.$$

Then,

$$dH_V(Y_a) = (1-a)|p_+|^2 + (1+a)|p_-|^2 + a|q|^2.$$

Hence if $a \in (0,1)$, $dH_V(Y_a) > 0$ on $\pi_N^{-1}(U) \setminus \{P\}$. Therefore, to prove the claim, it is enough to show that $dH_V(Y_a) > 0$ on $H_V^{-1}\big([V(P)-b, V(P)+b]\big) \setminus \pi_N^{-1}(U)$ for sufficiently small $a > 0$. This follows from lemma 4.1, since $H_V^{-1}\big([V(P)-b, V(P)+b]\big) \setminus \pi_N^{-1}(U)$ is compact and disjoint from $\mathrm{Crit}(V)$. □

For $H \in C^\infty(T^*N)$, let $S(H)$ be the set of $x : I \to T^*N$ with $|I| > 0$, $\dot{x} = X_H(x)$, $x(\partial I) \subset N$ and $x(\partial I) \cap D_-\big((0,b)\big) \neq \emptyset$.

We will show that for generic $H$, which is obtained by perturbing $H_V$, $S(H)$ is a countable set. To put it more rigorously, we first explain the setting for perturbation. Let $\mathscr{H}$ be an affine space consits of $H \in C^\infty(T^*N)$ such that $\mathrm{supp}(H - H_V) \subset \big\{|p|^2 \leq 2b\big\} \setminus \pi_N^{-1}\big(D\big([0,2b)\big)\big)$. We equip $\mathscr{H}$ with usual $C^\infty$ topology, i.e. the topology induced by distance

$$d_{C^\infty}(H,H') := \sum_{m=0}^{\infty} 2^{-m} \frac{|H-H'|_{C^m}}{1+|H-H'|_{C^m}}.$$

Then, the following lemma holds. The proof is postponed until the end of this section.

**Lemma 4.7.** *There exists $\mathscr{H}' \subset \mathscr{H}$, such that $\mathscr{H}'$ is of second category in $\mathscr{H}$ and $S(H)$ is a countable set for any $H \in \mathscr{H}'$.*

Take $a > 0$ sufficiently small so that $dH_V(Y_a) > 0$ on $H_V^{-1}\big([V(P)-b, V(P)+b]\big) \setminus \{P\}$. Then, there exists $c > 0$ such that if $H \in \mathscr{H}_U$ satisfies $d_{C^\infty}(H, H_V) \leq c$, then $dH(Y_a) > 0$ on $H^{-1}\big([V(P)-b, V(P)+b]\big) \setminus \{P\}$.

By lemma 4.7, there exists $H \in \mathscr{H}_U$ such that $d_{C^\infty}(H,H_V) \leq c$ and $S(H)$ is countable. Moreover, there exists $\varepsilon \in (0, b/2)$ such that $H(x) \neq V(P) - \varepsilon$ for any $x \in S(H)$, since $S(H)$ is a countable set.

Let $D_\pm := H^{-1}(V(P) \pm \varepsilon)$, and

$$\Sigma := \partial D_- \cap D_-\big((0,b)\big) = \big\{(q,p) \bigm| p = q_+ = 0,\ |q_-|^2 = 2\varepsilon\big\}.$$

We summerize their properties:

**Lemma 4.8.** (1) $(D_\pm, \omega_N, Y_a, D_\pm \cap N)$ *are Liouville quadruples.*

(2) $\mathrm{WFH}_*(D_\pm, \omega_N, D_\pm \cap N) \cong \mathrm{WFH}_*(D_{V(P)\pm\varepsilon}, \omega_N, D_{V(P)\pm\varepsilon} \cap N)$.

(3) *For any $x \colon I \to \partial D_-$ in $\mathscr{C}(\partial D_-, \partial D_- \cap N)$, $x(\partial I) \cap \Sigma = \emptyset$.*

***Proof.*** Since $dH(Y_a) > 0$ on $H^{-1}\big([V(P)-b, V(P)+b]\big) \setminus \{P\}$, $Y_a$ points outwards on $\partial D_\pm$. This proves (1). To prove (2), consider $H^t := (1-t)H + tH_V$ for $0 \leq t \leq 1$ and $D^t_\pm := (H^t)^{-1}\big(V(P) \pm \varepsilon\big)$. Since $H^t \in \mathscr{H}_U$ and $d_{C^\infty}(H, H_t) \leq c$, same arguments as in (1) shows that $(D^t_\pm, \omega_N, Y_a, D^t_\pm \cap N)_{0 \leq t \leq 1}$ is a smooth family of Liouville quadruples. Hence

(2) follows from proposition 2.7. Finally we prove (3). If there exists $x : I \to \partial D_-$ in $\mathscr{C}(\partial D_-, \partial D_- \cap N)$ such that $x(\partial I) \cap \Sigma \neq \emptyset$, by reparametrizing $x$ we get an element of $S(H)$. This contradicts the choice of $\varepsilon$. $\square$

By (1) and (2) in lemma 4.8, to prove theorem 4.5 it is enough to show

$$\mathrm{WFH}_*(D_+, \omega_N, D_+ \cap N) \cong \mathrm{WFH}_*(D_-, \omega_N, D_- \cap N). \tag{16}$$

Take $\mu \in C^\infty(\mathbb{R})$ such that

(1) $\mu'(t) \geq 0$.
(2) $\mu(t) = \begin{cases} 0 & (t \leq 0) \\ t - 1/2 & (t \geq 1) \end{cases}$.

For $\delta > 0$, define $\mu_\delta \in C^\infty(\mathbb{R})$ by $\mu_\delta(t) = \dfrac{\delta}{2} + \delta \cdot \mu\left(\dfrac{t - 2\varepsilon}{\delta}\right)$, and let

$$D_\delta := D_- \cup \bigl\{(q,p) \bigm| |q_-|^2 - 2\varepsilon \leq |q_+|^2 + |p|^2 \leq \mu_\delta(|q_-|^2)\bigr\}.$$

Then, $D_- \subset D_\delta \subset D_+$ for sufficiently small $\delta > 0$.

**Lemma 4.9.** *For sufficiently small $a > 0$, $(D_\delta, \omega_N, Y_a, D_\delta \cap N)$ is a Liouville quadruple. Moreover,* $\mathrm{WFH}_*(D_\delta, \omega_N, D_\delta \cap N) \cong \mathrm{WFH}_*(D_+, \omega_N, D_+ \cap N)$.

***Proof.*** To prove the first assertion, it is enough to show that $Y_a$ points strictly outwards on $\partial D_\delta$. On $\pi_N^{-1}(U)$,

$$Y_a(q,p) = -aq_-\partial_{q_-} + (1+a)p_-\partial_{p_-} + aq_+\partial_{q_+} + (1-a)p_+\partial_{p_+}.$$

If $a \in (0,1)$, then $-a < 0$ and $1 + a, a, 1 - a > 0$. Therefore $Y_a$ points strictly outwards on $\partial D_\delta \cap \pi_N^{-1}(U)$, since $\mu'_\delta(t) \geq 0$. On the other hand, since $\partial D_\delta \setminus \pi_N^{-1}(U) = \partial D_- \setminus \pi_N^{-1}(U)$, $Y_a$ points outwards on $\partial D_\delta \setminus \pi_N^{-1}(U)$ for sufficiently small $a > 0$.

The latter assertion follows from corllary 2.8, since $dH(Y_a) > 0$ on $D_+ \setminus D_\delta$ for sufficiently small $a > 0$. $\square$

By lemma 4.9, (16) is reduced to:

**Lemma 4.10.** $\mathrm{WFH}_*(D_-, \omega_N, D_- \cap N) \cong \mathrm{WFH}_*(D_\delta, \omega_N, D_\delta \cap N)$.

Lemma 4.10 is proved in the next section. In the remainder of this section, we prove lemma 4.7.

***Proof.*** Define $S^-(H)$ and $S^+(H)$ by

$$S^-(H) = \bigl\{x : [0,l] \to T^*N \bigm| l > 0, \dot{x} = X_H(x),\ x(0) \in D_-\bigl((0,b)\bigr),\ x(l) \in N\bigr\},$$
$$S^+(H) = \bigl\{x : [0,l] \to T^*N \bigm| l > 0, \dot{x} = X_H(x),\ x(0) \in N,\ x(l) \in D_-\bigl((0,b)\bigr)\bigr\}.$$

In the following, we prove that there exists $\mathscr{H}^- \subset \mathscr{H}$ which is of second category in $\mathscr{H}$ and for any $H \in \mathscr{H}^-$, $S^-(H)$ is countable. By parallel arguments, we can also show that there exists $\mathscr{H}^+ \subset \mathscr{H}$ which is of second category in $\mathscr{H}$ and for any $H \in \mathscr{H}^+$, $S^+(H)$ is countable. Then, $\mathscr{H}' := \mathscr{H}^- \cap \mathscr{H}^+$ satisfies the requirements of lemma 4.7.

In the following, we prove that there exists $\mathscr{H}^-$ as above. The proof consists of 9 steps.

**Step1:** By definition of $\mathscr{H}$, any $H \in \mathscr{H}$ satisfes $H \equiv H_V$ on $\pi_N^{-1}\big(D([0,2b])\big)$. Hence, following (1), (2) holds for any $H \in \mathscr{H}$.

(1) If $x : [0,t] \to \pi_N^{-1}\big(D([0,2b])\big)$ satisfies $t > 0$, $\dot{x} = X_H(x)$ and $x(0) \in D_-\big((0,2b]\big)$, then $x(t) \notin N$.
(2) There exists $c > 0$, which is independent of $H$ and such that: if $x : \mathbb{R} \to T^*N$ satisfies $\dot{x} = X_H(x)$ and $x(0) \in D_-\big([0,b]\big)$ then $x\big([0,c]\big) \subset \pi_N^{-1}\big(D([0,2b])\big)$.

**Step2:** Let $\mathscr{B}$ be the set of $(l,x)$ where $l > 0$ and $x \in L^{1,2}\big([0,1], T^*N\big)$, such that:

(1) $x(0) \in D_-\big((0,b)\big)$, $x(1) \in N$.
(2) If $\dfrac{1}{2} \le t \le 1 - \dfrac{c}{l}$, then $x(t) \neq x(0)$.

It is easily verified that $\mathscr{B}$ is a Banach submanifold of $(0,\infty) \times L^{1,2}\big([0,1], T^*N\big)$. Let $\mathscr{E}$ be a Banach vector bundle over $\mathscr{B}$ defined by $\mathscr{E}_{(x,l)} = L^2\big(x^*T(T^*N)\big)$. For $H \in \mathscr{H}$, define $s_H \in \Gamma(\mathscr{E})$ by $s_H(x,l) = \dot{x}(t) - l \cdot X_H\big(x(t)\big)$. If $s_H(x,l) = 0$, then $x$ satisfies following conditions:

(a) $x\big([0,1]\big) \cap \big\{|p|^2 < 2b\big\} \setminus \pi_N^{-1}\big(D([0,2b])\big) \neq \emptyset$.
(b) $x|_{[0,1)}$ is injective.

By (1) in step 1 and $x(1) \in N$, $x\big([0,1]\big)$ is not contained in $\pi_N^{-1}\big(D([0,2b])\big)$. Moreover, if $x(t) = \big(q(t), p(t)\big) \in \pi_N^{-1}\big(D([0,2b])\big)$,

$$\big|p(t)\big|^2 = 2\Big(H\big(x(t)\big) - V\big(q(t)\big)\Big) = 2\Big(V\big(q(0)\big) - V\big(q(t)\big)\Big) \le 2b - \big|q(0)\big|^2.$$

(a) follows form this at once.

To prove (b), first notice that if there exists $1 - \dfrac{c}{l} < t < 1$ with $x(t) = x(0)$, then $x(1) \notin N$ by (1), (2) in step 1. Hence $x(t) \neq x(0)$ for $1 - \dfrac{c}{l} < t < 1$. Hence, if $x|_{[0,1)}$ is not injective, there exists largest $0 < t < 1$ such that $x(t) = x(0)$, and $t \le 1 - \dfrac{c}{l}$. Moreover, if $t < \dfrac{1}{2}$, then $x(2t) = x(0)$ but this contradicts maximality of $t$. Hence $\dfrac{1}{2} \le t \le 1 - \dfrac{c}{l}$, but this contradicts (2) in definition of $\mathscr{B}$.

**Step3:** Take any almost complex structure $J$ on $T^*N$, which is compatible with $\omega_N$. $J$ induces associated metric and its Levi-Civita connection on $T^*N$, and also on $\mathscr{E} \to \mathscr{B}$. Then, $(\nabla s_H)_{(x,l)} : T_{(x,l)}\mathscr{B} \to \mathscr{E}_{(x,l)}$ is a Fredholm operator. In particular, $\mathrm{Coker}\nabla s_H = (\mathrm{Im}\,\nabla s_H)^\perp \subset \mathscr{E}_{(x,l)}$ is finite dimensional. Note that the index of this operator is

$$\dim D_-\big((0,b)\big) + \dim N + 1 - \dim T^*N = k + 1 - n.$$

Let $\zeta \in \mathrm{Coker}\nabla s_H$, i.e. $\zeta$ is orthogonal to

$$\nabla_\xi(s_H) = \partial_t\xi - l(\nabla_\xi J \cdot \nabla H + J \cdot \nabla_\xi(\nabla H)) =: \partial_t\xi - lA(t) \cdot \xi(t),$$

for any $\xi \in L^{1,2}\big(x^*(T(T^*N))\big)$ with $\xi(0) \in T_{x(0)}D_-\big((0,b)\big)$ and $\xi(1) \in T_{x(1)}N$. Hence we obtain ($A^*(t)$ is the adjoint operator of $A(t)$):

$$\big(\partial_t + lA^*(t)\big)\zeta(t) = 0, \qquad \zeta(0) \in \big(T_{x(0)}D_-\big((0,b)\big)\big)^\perp, \qquad \zeta(1) \in \big(T_{x(1)}N\big)^\perp.$$

**Step4:** We claim that if $s_H(x,l) = 0$, then $\big(\partial_t + lA^*(t)\big)(\nabla H \circ x) = 0$ . This is verified as follows. If $(y,l) \in \mathscr{B}$ satisfies $y(0) = x(0)$ and $y(1) = x(1)$, then $(\nabla H \circ y) \cdot s_H(y,l) = H\big(y(1)\big) - H\big(y(0)\big) = (\nabla H \circ x) \cdot s_H(x,l)$. Hence, if $\xi \in L^{1,2}\big(x^*(T(T^*N))\big)$ satisfies $\xi(0) = 0$ and $\xi(1) = 0$, then $\nabla_\xi(\nabla H \cdot s_H) = 0$ at $(x,l)$. Since $s_H(x,l) = 0$, it follows that $\nabla H \cdot \nabla_\xi(s_H) = 0$. Since this holds for any $\xi \in L^{1,2}\big(x^*(T(T^*N))\big)$ such that $\xi(0) = 0$ and $\xi(1) = 0$, the claim follows.

**Step5:** Let $m \in \mathbb{Z}_{\geq 2}$, and let $\mathscr{H}^m$ be an affine space consists of $H \in C^m(T^*N)$ such that $\mathrm{supp}(H - H_V) \subset \big\{|p|^2 \leq 2b\big\} \setminus \pi_N^{-1}\big(D\big([0,2b)\big)\big)$. $\mathscr{H}^m$ is an affine Banach space with $C^m$ norm. Consider Banach vector bundle $\mathscr{H}^m \times \mathscr{E} \to \mathscr{H}^m \times \mathscr{B}$, and define a section of this bundle $s\colon (H,x,l) \mapsto s_H(x,l)$. $X_H$ is $C^{m-1}$ class vector field, hence $s$ is a $C^{m-1}$ class section. We prove that if $s(H,x,l) = 0$, then $\nabla s$ is surjective at $(H,x,l)$. If this is not true, there exists $\zeta \in \mathrm{Coker}\nabla s_H(x,l)$, such that $\zeta \neq 0$ and $\zeta \cdot (J\nabla h) \circ x = 0$ for any $h \in \mathscr{H}^m - H_V$. By (a) in step 2, there exists $0 < t_0 < t_1 < 1$ such that $x\big([t_0,t_1]\big) \subset \big\{|p|^2 < 2b\big\} \setminus \pi^{-1}\big(D\big([0,2b]\big)\big)$. Moreover, $x|_{[t_0,t_1]}$ is embedding by (b). If a section $\eta$ of $x^*\big(T(T^*N)\big)|_{[t_0,t_1]}$ satisfies $\displaystyle\int_{t_0}^{t_1} \eta(t) \cdot \dot{x}(t)dt = 0$ and $\mathrm{supp}\eta \subset (t_0,t_1)$, there exists $h \in \mathscr{H}^m - H_V$ such that $\eta(t) = \nabla h\big(x(t)\big)$. Hence $\zeta = a\nabla H \circ x$ on $(t_0,t_1)$ for some constant $a$. Since $\zeta$ and $\nabla H \circ x$ both vanishes by the differential operator $\partial_t + lA^*(t)$, $\zeta = a\nabla H \circ x$ on $[0,1]$. In particular, $\zeta(0) = a\nabla H(x(0))$. Hence $a\nabla H(x(0)) \in \big(T_{x(0)}D_-\big((0,b)\big)\big)^\perp$. On the other hand, $dH|_{T_{x(0)}D_-((0,b))} \neq 0$. Hence we obtain $a = 0$, contradicting $\zeta \neq 0$.

**Step6:** By step 4, $s^{-1}(0)$ is a $C^{m-1}$ class Banach submanifold of $\mathscr{H}^m \times \mathscr{B}$. Consider $\pi_{\mathscr{H}^m} : s^{-1}(0) \to \mathscr{H}^m; (H,x,l) \mapsto H$. This is a $C^{m-1}$ class Fredholm map of index $k + 1 - n \leq 0$ (recall $k \leq n-1$). Hence by Sard-Smale theorem, the set of regular value of $\pi_{\mathscr{H}^m}$ (denote by $\mathscr{H}^m_{\mathrm{reg}}$) is of second category in $\mathscr{H}^m$. Note that $H \in \mathscr{H}^m_{\mathrm{reg}}$ if and only if $s_H : \mathscr{B} \to \mathscr{E}$ is transversal to 0.

**Step7:** For any $\delta > 0$, let

$$\mathscr{B}(\delta) : = \big\{(x,l) \in \mathscr{B} \bigm| x(0) \in D_-\big([\delta, b-\delta]\big), \delta \leq l \leq \tfrac{1}{\delta}\big\},$$
$$\mathscr{H}^m_{\mathrm{reg},\delta} : = \mathscr{H}^m \setminus \pi_{\mathscr{H}^m}\big(\mathrm{Crit}(\pi_{\mathscr{H}^m}) \cap \mathscr{B}(\delta)\big).$$

Obviously, $\mathscr{H}^m_{\mathrm{reg}} = \bigcap_{\delta>0} \mathscr{H}^m_{\mathrm{reg},\delta}$. We show that $\mathscr{H}^m_{\mathrm{reg},\delta}$ is open in $\mathscr{H}^m$. If $(H_n,x_n,l_n)_n$ is a sequence on $\mathrm{Crit}(\pi_{\mathscr{H}^m}) \cap \mathscr{B}(\delta)$ and $(H_n)_n$ converges to some $H_\infty$ in $\mathscr{H}^m$, then certain subsequence of $(x_n,l_n)$ converges to some $(x_\infty,l_\infty)$, hence $(H_\infty,x_\infty,l_\infty) \in \mathrm{Crit}(\pi_{\mathscr{H}^m}) \cap \mathscr{B}(\delta)$. Therefore $\mathscr{H}^m \setminus \mathscr{H}^m_{\mathrm{reg},\delta}$ is closed in $\mathscr{H}^m$.

**Step8:** For any $\delta > 0$, let $\mathscr{H}_{\mathrm{reg},\delta} := \mathscr{H}^m_{\mathrm{reg},\delta} \cap \mathscr{H}$ (this does not depend on $m$). We show that $\mathscr{H}_{\mathrm{reg},\delta}$ is open dence set in $\mathscr{H}$. Openness is clear since $\mathscr{H}^m_{\mathrm{reg},\delta}$ is open in $\mathscr{H}^m$ and the inclusion $\mathscr{H} \to \mathscr{H}^m$ is continuous.

To show that $\mathscr{H}_{\mathrm{reg},\delta}$ is dence in $\mathscr{H}$, first notice that $\mathscr{H}^m_{\mathrm{reg},\delta}$ is dence in $\mathscr{H}^m$ by step 6. Hence for any $H \in \mathscr{H}$, there exists $H_m \in \mathscr{H}^m_{\mathrm{reg},\delta}$ such that $|H - H_m|_{C^m} \le 2^{-m}$. Since $\mathscr{H}^m_{\mathrm{reg},\delta}$ is open in $\mathscr{H}^m$, there exists $0 < c < 2^{-m}$ such that $c$-neighborhood of $H_k$ with respect to $|\cdot|_{C^m}$ is contained in $\mathscr{H}^m_{\mathrm{reg},\delta}$. Then, take $H'_m \in \mathscr{H}$ so that $|H_m - H'_m|_{C^m} < c$, then $H'_m \in \mathscr{H}_{\mathrm{reg},\delta}$ and $|H - H'_m|_{C^m} < 2^{1-m}$, hence $\lim_{m\to\infty} H'_m = H$ in $\mathscr{H}$. This shows that $\mathscr{H}_{\mathrm{reg},\delta}$ is dence in $\mathscr{H}$.

**Step9:** Let $\mathscr{H}_{\mathrm{reg}} := \bigcap_{\delta>0} \mathscr{H}_{\mathrm{reg},\delta}$. $\mathscr{H}_{\mathrm{reg}}$ is of second category in $\mathscr{H}$ by step 8. Note that $H \in \mathscr{H}_{\mathrm{reg}}$ if and only if $s_H : \mathscr{B} \to \mathscr{E}$ is transversal to 0. Since virtual dimension of $s_H^{-1}(0)$ is $1 + k - n \le 0$, $s_H^{-1}(0)$ is a countable set for any $H \in \mathscr{H}_{\mathrm{reg}}$. Therfore it is enough to show that if $s_H^{-1}(0)$ is countble, then $S^-(H)$ is countable.

Let $S^-_0(H) := \{x \in S^-(H) \mid x \text{ is injective}\}$, and $S^-_1(H) := S^-(H) \setminus S^-_0(H)$. $S^-_0(H)$ is countable, since there exists injection $S^-_0(H) \to s_H^{-1}(0)$ which maps $x : [0,l] \to T^*N$ to $[0,1] \to T^*N; t \mapsto x(tl)$. Hence it is enough to show that $S^-_1(H)$ is countable. Take $x \in S^-_1(H)$. Since $x$ is not constant, there exists smallest $0 < t < l$ such that $x(t) = x(0)$. Then $(y,t) \in s_H^{-1}(0)$ where $y\colon [0,1] \to T^*N; \tau \mapsto x(t\tau)$. Moreover, there are only countably many $\theta > 0$ such that $x(\theta) \in N$. Hence we obtain map $S^-_1(H) \to s_H^{-1}(0)$, such that preimage of each element of $s_H^{-1}(0)$ is countable. Therefore, $S^-_1(H)$ is countable. This completes the proof. □

## 5. Handle attaching

In this section, we prove lemma 4.10. In 5.1, we prove a preliminary lemma on Floer trajectories (lemma 5.1). In 5.2, we give a proof of 4.10.

### 5.1. Lemma on Floer trajectories.

**Lemma 5.1.** *Let $(M, \omega, X, L)$ be a Liouville quadruple, and $\lambda := i_X\omega$. Let $M^{\mathrm{in}}$ be a compact submanifold of $M$ such that $(M^{\mathrm{in}}, \omega|_{M^{\mathrm{in}}}, X|_{M^{\mathrm{in}}}, L \cap M^{\mathrm{in}})$ is a Liouville quadruple. We denote the Reeb vector field and the contact distribution on $(\partial M^{\mathrm{in}}, \lambda)$ by $R^{\mathrm{in}}$, $\xi^{\mathrm{in}}$.*

*Let $H, H' \in \mathscr{H}_{\mathrm{ad}}(\hat{M})$, $(H^s)_s$ be a monotone homotopy in $\mathscr{H}(\hat{M})$ from $H$ to $H'$, and $(J^s_t)_{s,t}$ be a family of elements of $\mathscr{J}(\hat{M})$. Assume that there exists $a \in C^\infty(\mathbb{R})$ and $0 < \nu < 1$ with following properties:*

1. *There exists $s_0 > 0$ such that: $a(s) = \begin{cases} a(-s_0) & (s \le -s_0) \\ a(s_0) & (s \ge s_0) \end{cases}$. Moreover, $a(-s_0), a(s_0) \notin \mathscr{A}(\partial M^{\mathrm{in}}, \lambda^{\mathrm{in}}, \partial L^{\mathrm{in}})$.*
2. *$H^s(z,\rho) = a(s)(\rho - \nu)$ on $M^{\mathrm{in}} \setminus M^{\mathrm{in}}(\nu^{\frac{1}{2}})$.*
3. *For any $s \in \mathbb{R}$ and $t \in [0,1]$, $J^s_t$ preserves $\overline{\xi^{\mathrm{in}}}$ and $J^s_t(\partial_\rho) = \rho^{-1}\overline{R^{\mathrm{in}}}$ on $M^{\mathrm{in}} \setminus M^{\mathrm{in}}(\nu^{\frac{1}{2}})$.*

*Assume that $x_- \in \mathscr{C}(H)$, $x_+ \in \mathscr{C}(H')$ satisfy $x_-\big([0,1]\big), x_+\big([0,1]\big) \subset M^{\mathrm{in}}$. Then, for any $u \in \hat{\mathscr{M}}_{(H^s, J^s_t)_{s,t}}(x_-, x_+)$, $u\big(\mathbb{R} \times [0,1]\big) \subset M^{\mathrm{in}}$.*

Following proof is based on [AS], section 7.

***Proof***. By $a(-s_0), a(s_0) \notin \mathscr{A}(\partial M^{\mathrm{in}}, \lambda^{\mathrm{in}}, \partial L^{\mathrm{in}})$ and assumption (2), $x_-\big([0,1]\big), x_+\big([0,1]\big) \subset M^{\mathrm{in}}(\nu^{\frac{1}{2}})$. We claim that $u\big(\mathbb{R}\times[0,1]\big) \subset M^{\mathrm{in}}(\nu^{\frac{1}{2}})$ for any $u \in \hat{\mathscr{M}}_{(H^s,J^s_t)_{s,t}}(x_-,x_+)$. First notice that for any $\rho \in (\nu^{\frac{1}{2}}, 1]$, $D_\rho := \mathbb{R}\times[0,1] \setminus u^{-1}\big(\mathrm{int}M^{\mathrm{in}}(\rho)\big)$ is a compact set. If the claim is not true, there exsits $\rho \in (\nu^{\frac{1}{2}}, 1]$ such that $D_\rho \neq \emptyset$. For generic $\rho$, $u$ and $u|_{\mathbb{R}\times\{0,1\}}$is transverse to $\partial M^{\mathrm{in}} \times \{\rho\}$, hence we may assume that $D_\rho$ is a compact surface with boundaries and corners.

Let

$$\partial_H D_\rho := \partial D_\rho \cap \mathbb{R}\times\{0,1\}, \qquad \partial_V D_\rho := \partial D_\rho \cap \mathbb{R}\times(0,1).$$

It is easily verified that $u_s$ is not constantly 0 on $D_\rho$. This implies

$$\int_{D_\rho} |\partial_s u|^2_{J^s_t}\, dsdt > 0.$$

Since $u$ satisfies the Floer equation $\partial_s u - J^s_t \partial_t u - \nabla^s_t H^s = 0$,

$$\int_{D_\rho} |\partial_s u|^2_{J^s_t} + \partial_s H^s\big(u(s,t)\big) dsdt = \int_{D_\rho} \hat{\omega}(\partial_t u, \partial_s u) + dH^s(\partial_s u) + \partial_s H^s\big(u(s,t)\big) dsdt$$

$$= \int_{\partial D_\rho} -u^*\hat{\lambda} + H^s\big(u(s,t)\big) dt.$$

We calculate the last term. First we calculate the integration on $\partial_H D_\rho$:

$$\int_{\partial_H D_\rho} -u^*\hat{\lambda} + H^s\big(u(s,t)\big) dt = \int_{\partial_H D_\rho} -u^*\hat{\lambda} = 0.$$

The first equality follows from $dt|_{\partial_H D^\rho} = 0$, and the second equality follows from $u(\partial_H D_\rho) \subset \hat{L}$ and $\hat{\lambda}|_{\hat{L}} = 0$. On the other hand, since $u(\partial_V D_\rho) \subset \partial M^{\mathrm{in}} \times \{\rho\}$, we get

$$(s,t) \in \partial_V D_\rho \implies H^s\big(u(s,t)\big) = a(s)\big(\rho - \nu\big), \quad \hat{\lambda}\big(X_{H^s}(u(s,t))\big) = a(s)\rho.$$

Therefore

$$\int_{\partial_V D_\rho} -u^*\hat{\lambda} + H^s\big(u(s,t)\big) dt = \int_{\partial_V D_\rho} \hat{\lambda}(X_{H^s} \otimes dt - du) - \nu \int_{\partial_V D_\rho} a(s) dt.$$

On the other hand, Floer equation is equivalent to

$$J^s_t \circ (X_{H^s} \otimes dt - du) = (du - X_{H^s} \otimes dt) \circ j,$$

where $j$ is a complex structure on $\mathbb{R}\times[0,1]$, defined by $j(\partial_s) = \partial_t$. Therefore

$$\int_{\partial_V D_\rho} \hat{\lambda}(X_{H^s} \otimes dt - du) = -\int_{\partial_V D_\rho} \hat{\lambda}\big(J^s_t \circ (du - X_{H^s} \otimes dt) \circ j\big).$$

$\hat{\lambda}(J^s_t \circ X_{H^s}) = -\hat{\lambda}(\nabla^s_t H^s) = 0$ on $\partial M^{\mathrm{in}} \times \{\rho\}$. Moreover, if $V$ is a vector tangent to $\partial_V D_\rho$, and positive with respect to the boundary orientation, then $jV$ points inwards, hence $d\rho(jV) \geq 0$. Hence $\hat{\lambda}\big(J^s_t \circ du \circ j\big)(V) \geq 0$. Therefore,

$$\int_{\partial_V D_\rho} \hat{\lambda}(X_{H^s} \otimes dt - du) \leq 0.$$

Finally,

$$\int_{D_\rho} |\partial_s u|^2_{J^s_t} + \partial_s H^s\big(u(s,t)\big)\, dsdt \leq -\nu \int_{\partial_V D_\rho} a(s)dt = -\nu \int_{D_\rho} \partial_s a(s,t)\, dsdt.$$

Since $\partial_s H^s \geq 0$ and $\partial_s a \geq 0$(this follows from (2) and $\partial_s H^s \geq 0$), this implies

$$\int_{D_\rho} |\partial_s u|^2_{J^s_t}\, dsdt \leq 0.$$

This is a contradiction. □

5.2. **Handle attaching.** In this subsection, we give a proof of lemma 4.10. First we prove the following lemma, which is easily proved using Moser's trick.

**Lemma 5.2.** *Let $X$ be a manifold (not assumed to be compact) and $Y$ be a submanifold of $X$. Let $(\lambda_t)_{0\leq t\leq 1}$ be a smooth family of contact forms on $X$ such that $\lambda_t|_Y = 0$ and $d\lambda_t = d\lambda_0$ for any $t$.*

*Then, for any compact set $K$ in $Y$, there exists $V$, a neighborhood of $K$ in $X$ , and $(\psi_t)_{0\leq t\leq 1}$, a smooth family of embeddings from $V$ to $X$ with following properties:*

1. *$\psi_0$ is an inclusion $V \hookrightarrow X$.*
2. *$\psi_t^*\lambda_0 = \lambda_t$.*
3. *$\psi_t^{-1}(Y) = V \cap Y$.*
4. *$\psi_t|_{V\cap Y}$ is an inclusion $V \cap Y \hookrightarrow X$.*

***Proof.*** First we show that there exists $W$, a neighborhood of $K$ in $X$, and $(\xi_t)_t$, a family of vector fields on $W$ such that $L_{\xi_t}\lambda_t + \partial_t\lambda_t = 0$ and $\xi_t = 0$ on $W \cap Y$.

Take $W$, a neighborhood of $K$ in $X$ so that the restriction morphism $H^*_{\mathrm{dR}}(W) \to H^*_{\mathrm{dR}}(W\cap Y)$ is an isomorphism. Since $d\lambda_t = d\lambda_0$ for any $t$, $\partial_t\lambda_t$ is a closed form. Moreover, $\partial_t\lambda_t\big|_Y = 0$ since $\lambda_t|_Y = 0$ for any $t$. Hence $(\partial_t\lambda_t)_t$ is a smooth family of exact one forms on $W$. Hence there exists $(f_t)_t$, a family of $C^\infty$ functions on $W$ such that $df_t = \partial_t\lambda_t$. We may assume that $f_t$ vanishes on $Y$, since $\partial_t\lambda_t$ vanishes on $Y$ and $H^0_{\mathrm{dR}}(W) \to H^0_{\mathrm{dR}}(W\cap Y)$ is an isomorphism.

Let $R_t$ be the Reeb vector field of $(X, \lambda_t)$ and $\xi_t := -f_t R_t$. Then, $\xi_t$ vanishes on $Y$ and

$$L_{\xi_t}\lambda_t = i_{\xi_t}(d\lambda_t) + d(i_{\xi_t}\lambda_t) = -df_t = -\partial_t\lambda_t.$$

Integrating $(\xi_t)_t$, we obtain $(\varphi_t)_t$, a family of embeddings from certain neighborhood of $K$ to $X$. Then, $\varphi_t^*\lambda_t = \lambda_0$. Finally, if we take $V$ sufficiently small, $\psi_t := (\varphi_t)^{-1}|_V$ can be defined for all $0 \leq t \leq 1$ and satisfies the condition of the lemma. □

We explain some definitions, which are used in the following of this paper. For sufficiently small $\delta > 0$, we define subsets of $\pi_N^{-1}(U)$, $A_\delta^-, A_\delta^+, B_\delta, C_\delta$ by

$$\begin{aligned} A_\delta^- &= \{(q,p) \mid |p|^2 + |q_+|^2 = |q_-|^2 - 2\varepsilon < \delta\}, \\ A_\delta^+ &= \{(q,p) \mid |p|^2 + |q_+|^2 = \mu_\delta(|q_-|^2) < \delta\}, \\ B_\delta &= \{(q,p) \mid |p|^2 + |q_+|^2 = |q_-|^2 - 2\varepsilon = \delta\}, \\ C_\delta &= \{(q,p) \mid |q_-|^2 - 2\varepsilon \leq |p|^2 + |q_+|^2 < \mu_\delta(|q_-|^2)\} \cup A_\delta^+. \end{aligned}$$

Recall that we have considered $\pi_N^{-1}(U)$ as a subset of $\mathbb{R}^{2n}$ using coordinate $(q,p)$. Hence we consider these sets also as subsets of $\mathbb{R}^{2n}$.

We have shown in lemma 4.9 that $(D_\delta, \omega_N, Y_a, D_\delta \cap N)$ is a Liouville quadruple for sufficiently small $a$. In the following of this paper, we fix such $a$ and denote it by $a_0$.

Take arbitrary smooth function $a$ on $[0,1]$ such that $a(0) = a_0$ and $a(1) = \frac{1}{2}$. By lemma 5.2, there exists $V$, a neighborhood of $\Sigma$ in $\partial D_-$, and $(\psi_t)_t$, a family of embeddings from $V$ to $\partial D_-$ with following properties:

(1) $\psi_0$ is an inclusion $V \hookrightarrow \partial D_-$.
(2) $\psi_t^* \lambda_{a_0} = \lambda_{a(t)}$. ($\lambda_a$ denotes $i_{Y_a}\omega_N$.)
(3) $\psi_t^{-1}(\partial D_- \cap N) = V \cap N$.
(4) $\psi_t|_{V \cap N}$ is an inclusion $V \cap N \hookrightarrow \partial D_-$.

Since $\bigcap_{\delta > 0} A_\delta^- = \Sigma$, $A_\delta^- \subset V$ for sufficiently small $\delta > 0$. If $A_\delta^- \subset V$, $\big(C_\delta, \omega_{\mathrm{st}}, Y_{a(t)}, C_\delta \cap N\big)$ can be glued to $(D_-, \omega_N, Y_{a_0}, D_- \cap N)$ by $\psi_t|_{C_\delta \cap \partial D_-}$. As a result, we get a Liouville quadruple. We denote it by $\big(C_\delta \cup_{\psi_t} D_-, \omega_t, Z_t, L_t\big)$.

We make two remarks which are clear from constructions:

**Remark 5.3.** (1) $\partial(C_\delta \cup_{\psi_t} D_-) = \big(\partial D_- \setminus \psi_t(A_\delta^-)\big) \cup A_\delta^+$.
(2) For any $\delta, \delta' > 0$, $\widehat{C_\delta \cup_{\psi_t} D_-}$ and $\widehat{C_{\delta'} \cup_{\psi_t} D_-}$ can be identified naturally.

It is clear from construction that $\big(C_\delta \cup_{\psi_0} D_-, \omega_0, Z_0, L_0\big)$ is isomorphic to $(D_\delta, \omega_N, Y_{a_0}, D_\delta \cap N)$ as Liouville quadruple. Hence, by proposition 2.7, to prove lemma 4.10 it is enough to show that

$$\mathrm{WFH}_*\big(C_\delta \cup_{\psi_1} D_-, \omega_1, L_1\big) \cong \mathrm{WFH}_*(D_-, \omega_N, D_- \cap N). \tag{17}$$

Let $(\alpha_i)_i$ be an increasing sequence of positive numbers, such that $\lim_{i\to\infty} \alpha_i = \infty$ and $\alpha_i \notin \mathscr{A}(\partial D_-, \lambda_{a_0}, \partial D_- \cap N)$. Let $\nu \in (0,1)$, and take $F_i \in \mathscr{H}_{\mathrm{ad}}(\hat{D}_-)$ such that:

- $F_1 < F_2 < \cdots$.
- $F_i(z,\rho) = \alpha_i(\rho - \nu)$ on $\partial D_- \times [\nu^{\frac{1}{2}}, \infty)$.

Since $(F_i)_i$ is cofinal in $\big(\mathscr{H}_{\mathrm{ad}}(\hat{D}_-), <\big)$,

$$\mathrm{WFH}_*(D_-, \omega_N, D_- \cap N) = \lim_{i\to\infty} \mathrm{WFH}_*(F_i) \tag{18}$$

Hence to prove (17), it is enough to show

$$\mathrm{WFH}_{\le m}\big(C_\delta \cup_{\psi_1} D_-, \omega_1, L_1\big) \cong \lim_{i\to\infty} \mathrm{WFH}_{\le m}(F_i) \tag{19}$$

for each positive integer $m$. In the following, we fix $\delta$ and denote it by $\delta_0$.

Denote the Reeb vector field on $(\partial D_-, \lambda_{a_0})$ by $R$.

**Lemma 5.4.** *For any $\alpha > 0$, there exists $\delta(\alpha) > 0$ such that any $\delta \in \big(0, \delta(\alpha)\big)$ satisfies following:*

> *Assume that $x : I \to \partial D_-$ satisfies $\dot{x} = R(x)$, $x(\partial I) \subset \psi_1(B_\delta) \cup (\partial D_- \cap N)$, $x(\partial I) \cap \psi_1(B_\delta) \neq \emptyset$ and $x(I)$ is not contained in $\psi_1(A^-_{\delta_0})$. Then, $|I| > \alpha$.*

***Proof.*** Assume that this lemma is not true. Then, there exists $y : J \to \partial D_-$ such that $\dot{y} = R(y)$, $y(\partial J) \subset \bigcap_{\delta>0} \psi_1\Big(\overline{A^-_\delta}\Big) \cup (\partial D_- \cap N)$, $y(\partial J) \cap \bigcap_{\delta>0} \psi_1\Big(\overline{A^-_\delta}\Big) \neq \emptyset$, and $y(J)$ is not contained in $\psi_1(A^-_{\delta_0})$. Note that

$$\bigcap_{\delta>0} \psi_1\Big(\overline{A^-_\delta}\Big) = \psi_1\Bigg(\bigcap_{\delta>0} \overline{A^-_\delta}\Bigg) = \psi_1(\Sigma) = \Sigma.$$

In the last equality, we use property (4) of $\psi_t$. Hence $y(\partial J) \subset \partial D_- \cap N$, and $y(\partial J) \cap \Sigma \neq \emptyset$. Since $\Sigma \subset \psi_1(A^-_{\delta_0})$, $y$ is not constant and $|J| > 0$. Hence $y \in \mathscr{C}(\partial D_-, \lambda_{a_0}, \partial D_- \cap N)$. But this contradicts (3) in lemma 4.8. □

We can take sequences $(\delta_i)_i$ and $(G_i)_i$, where $\delta_i \in \mathbb{R}_{>0}$ and $G_i \in \mathscr{H}_{\mathrm{ad}}(\widehat{C_{\delta_0} \cup_{\psi_1} D_-})$, such that $(\delta_i)_i$ satisfies

- $\delta$-(1): $0 < \delta_i < \min\big\{\delta_0, \delta(\alpha_i)\big\}$.
- $\delta$-(2): $\delta_1 \ge \delta_2 \ge \cdots$.
- $\delta$-(3): $\lim_{i\to\infty} \delta_i = 0$.

and $(G_i)_i$ satisfies

- $G$-(1): $G_i|_{D_-} = F_i|_{D_-}$.
- $G$-(2): $(G_i)_i$ is a cofinal sequence in $\big(\mathscr{H}_{\mathrm{ad}}(\widehat{C_{\delta_0} \cup_{\psi_1} D_-}), <\big)$.
- $G$-(3): There exsits a sequence $i_1 < i_2 < \cdots$ such that $G_{i_1} < G_{i_2} < \cdots$.
- $G$-(4): $G_i(z,\rho) = \alpha_i(\rho - \nu)$ for $(z,\rho) \in \big(\partial D_- \setminus \psi_1(A^-_{\delta_i})\big) \times [1,\infty)$.
- $G$-(5): If $x \in \mathscr{C}(G_i)$ satisfies $x\big([0,1]\big) \subset C_{\delta_0} \cup A^+_{\delta_0} \times (1,\infty)$, then $x$ is a constant map to $(0,\dots,0)$ and $\mathrm{ind} x > m$.

In $G$-(4), we consider $\big(\partial D_- \setminus \psi_1(A^-_{\delta_i})\big) \times [1,\infty)$ to be subset of $\widehat{C_{\delta_0} \cup_{\psi_1} D_-}$ (see remark 5.3).

By $G$-(1), $D_-$ is an invariant set of $X_{G_i}$. Hence $\mathscr{C}(G_i)$ is divided into two subsets:

$$\mathscr{C}_-(G_i) = \big\{x \in \mathscr{C}(G_i) \;\big|\; x\big([0,1]\big) \subset D_-\big\}, \quad \mathscr{C}_+(G_i) = \big\{x \in \mathscr{C}(G_i) \;\big|\; x\big([0,1]\big) \cap D_- = \emptyset\big\}.$$

By $G$-(1), $\mathscr{C}_-(G_i)$ can be naturally identified with $\mathscr{C}(F_i)$.

**Lemma 5.5.** *If $x \in \mathscr{C}_+(G_i)$, then $x\big([0,1]\big) \subset C_{\delta_0} \cup A^+_{\delta_0} \times (1,\infty)$.*

***Proof***. Assume that there exists $t \in [0,1]$ such that $x(t) \notin C_{\delta_0} \cup A^+_{\delta_0} \times (1,\infty)$, hence $x(t) \in \big(\partial D_- \setminus \psi_1(A^-_{\delta_0})\big) \times [1,\infty)$. Let $I$ be the largest closed interval which contains $t$ and $x(I) \subset \big(\partial D_- \setminus \psi_1(A^-_{\delta_i})\big) \times [1,\infty)$. Then $|I| > 0$, and $x(\partial I)$ is contained in $\big(\psi_1(B_{\delta_i}) \cup (\partial D_- \cap N)\big) \times [1,\infty)$.

By $G$-(4), $X_{G_i} = \alpha_i \cdot (R,0)$ on $\big(\partial D_- \setminus \psi_1(A^-_{\delta_i})\big) \times [1,\infty)$. Define $y : I \to \partial D_-$ by $y(t) = \pi \circ x$, where $\pi$ is a projection to $\partial D_-$. Then $\dot{y} = \alpha_i R(y)$, $y(\partial I) \subset \psi_1(B_{\delta_i}) \cup (\partial D_- \cap N)$ and $y(t) \notin \psi_1(A^-_{\delta_0})$. Since $\delta_i < \delta(\alpha_i)$, $y(\partial I) \cap \psi_1(B_{\delta_i}) = \emptyset$. Hence $y(\partial I) \subset \partial D_- \cap N$, but this contradicts $\alpha_i \notin \mathscr{A}(\partial D_-, \lambda_{a_0}, \partial D_- \cap N)$. □

By lemma 5.5 and $G$-(5), $\mathscr{C}_+(G_i)$ consists only of constant map to $(0,\dots,0)$ and its index is larger than $m$. Hence $\mathrm{WFC}_{\le m}(G_i)$ is generated by elements of $\mathscr{C}_-(G_i)$. On the otherhand, since $\mathscr{C}_-(G_i)$ can be identified with $\mathscr{C}(F_i)$, there is an isomorphism of $\mathbb{Z}_2$ module $\mathrm{WFC}_{\le m}(F_i) \to \mathrm{WFC}_{\le m}(G_i)$. By lemma 5.1, if almost complex structures (which are used to define differential on $\mathrm{WFC}_*(F_i)$ and $\mathrm{WFC}_*(G_i)$) satisfy assuption (3) in lemma 5.1 with $M^{\mathrm{in}} = D_-$, this is an isomorphism of chain complexes. Denote this isomorphism by $\Phi_i$.

Take $(i_k)_k$ as in $G$-(3), and consider following diagram:

$$\begin{array}{ccc} \mathrm{WFC}_{\le m}\big(F_{i_k}\big) & \longrightarrow & \mathrm{WFC}_{\le m}(F_{i_{k+1}}) \\ \Phi_{i_k}\big\downarrow & & \big\downarrow\Phi_{i_{k+1}} \\ \mathrm{WFC}_{\le m}\big(G_{i_k}\big) & \longrightarrow & \mathrm{WFC}_{\le m}(G_{i_{k+1}}). \end{array}$$

Horizontal arrows are monotone morphisms induced by strong monotone homotopies. Again by lemma 5.1, if almost complex structures (which are used to define monotone morphisms) satify assumption (3) in lemma 5.1 with $M^{\mathrm{in}} = D_-$, the above diagram commutes. Taking homology of this diagram and letting $i \to \infty$, we get (last equality follows from $G$-(2))

$$\lim_{i\to\infty} \mathrm{WFH}_{\le m}(F_i) \cong \lim_{i\to\infty} \mathrm{WFH}_{\le m}(G_i) = \mathrm{WFH}_{\le m}\big(C_\delta \cup_{\psi_1} D_-, \omega_1, L_1\big)$$

Hence we have proved (19).

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