GEOMETRIC BOGOMOLOV CONJECTURE FOR ABELIAN VARIETIES AND SOME RESULTS FOR THOSE WITH SOME DEGENERATION (WITH AN APPENDIX BY WALTER GUBLER: THE MINIMAL DIMENSION OF A CANONICAL MEASURE)

KAZUHIKO YAMAKI

Introduction

Let K be a number field, or a function field of a curve over a base field k. We fix an algebraic closure $\overline{K}$ of K. Let A be an abelian variety over $\overline{K}$ and let L be an ample line bundle on A, and assume it is even, i.e., $[-1]^*L = L$. Then the canonical height function $\hat{h}_L$ associated with L, also called the Néron-Tate height, is a semi-positive definite quadratic form on $A\left(\overline{K}\right)$. It is well-known that $\hat{h}_L(x) = 0$ if x is a torsion point.

Let X be a closed subvariety of A. We put

$$X(\epsilon; L) := \left\{ x \in X\left(\overline{K}\right) \middle| \hat{h}_L(x) \leq \epsilon \right\}$$

for a positive real number $\epsilon > 0$. Then the Bogomolov conjecture for abelian varieties insists that there should be $\epsilon > 0$ such that $X(\epsilon; L)$ is not Zariski dense in X, unless X is a kind of "exceptional" closed subvarieties, such as torsion subvarieties for example.

In the case where K is a number field, namely, in the arithmetic case, this conjecture was solved more than ten years ago, known as a theorem of Zhang:

Theorem 0.1 (Corollary 3 of [21], arithmetic version of Bogomolov conjecture for abelian varieties)**.** Let K be a number field. If X is not a torsion subvariety, then there is $\epsilon > 0$ such that $X(\epsilon; L)$ is not Zariski dense in X.

The Bogomolov conjecture is originally a statement concerning the jacobian of a curve and an embedding of the curve, that is, A is a jacobian and X is an embedded curve. This is called the Bogomolov conjecture for curves, which is proved by Ullmo in [17] in case that K is a number field at the same time when Zhang proved Theorem 0.1. The ideas of Ullmo and Zhang are same — based on the equidistribution theory, which will be recalled in this introduction.

When K is a finitely generated field over $\mathbb{Q}$, a kind of arithmetic height functions can be defined, after a choice of polarizations of K, due to Moriwaki [16]. It is still an arithmetic setting namely, and the Bogomolov conjecture for abelian varieties with respect to the height associated with a big polarization has been proved by Moriwaki himself. The classical geometric height is also a kind of Moriwaki's arithmetic height, but it does not arise from a big polarization — rather a degenerate one. Hence we cannot say anything about the geometric version of the conjecture with Moriwaki's theory.

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How about the geometric case, namely the case where K is a function field over an algebraically closed field k and the height is the classical geometric height? In this case, we cannot expect the same statement as Theorem 0.1 because a subvariety defined over the constant field can have dense small points. Accordingly, we have to reformulate the conjecture, or have to consider it in a restricted situation.

The Bogomolov conjecture for curves is one of the problems studied for a long time. In characteristic 0, Cinkir proved this conjecture in [7] recently. In positive characteristic, the conjecture for curves is still open, but there are some partial answers such as in [15] by Moriwaki and in [18, 19] by the author. In this case, the exceptional X's are the isotrivial curves. Another important result is the one due to Gubler. He proved in [9] the following theorem:

Theorem 0.2 (Theorem 1.1 of [9]). Assume that there is a place v at which the abelian variety A is totally degenerate. Then $X(\epsilon; L)$ is not Zariski dense in X for some $\epsilon > 0$ unless X is a torsion subvariety.

The exceptional X's are the torsion subvarieties in this theorem as in the arithmetic case, because there do not appear constant subvarieties in the totally degenerate case.

There are two things to do in this paper. One is to give the precise statement of our geometric Bogomolov conjecture for abelian varieties, which is a reformulation of Zhang's theorem after consideration of subvarieties defined over k. The other is to give partial answers to it for abelian varieties with some kind of degeneration.

We would like to give the statement of our conjecture now. Let G_X be the stabilizer of X of a closed subvariety of an abelian variety A, and put $B := A/G_X$ and $Y := X/G_X$. We call X a *special* subvariety of A if Y is the image of a closed subvariety of the $\overline{K}/k$-trace of B defined over k, up to the translation by a torsion point of B (cf. § 2.2). Note that if there is a place v at which A is totally degenerate, then the notion of special subvarieties coincides with that of torsion subvarieties since the $\overline{K}/k$-trace is trivial. In §1, we will see that the special subvariety has dense small points (cf. Corollary 2.8). Our geometric Bogomolov conjecture insists that the converse should hold true:

Conjecture 0.3 (cf. Conjecture 2.9 and Remark 5.4). Let K be a function field. Let all A, L and X be as above. Then there exists $\epsilon > 0$ such that $X(\epsilon; L)$ is not Zariski dense in X unless X is a special subvariety.

For an irreducible closed subvariety $X \subset A$ and a place v of $\overline{K}$, we can define an integer $b(X_v)$, which we need in proposing our main result. Let us give rough description of it although we refer to § 4.3 for the precise definition in terms of Raynaud extension. For simplicity, we assume that A and X are defined over K and v is a place K. Further assume that A has a model $\mathcal{A}$ over the ring of v-integers such that the reduction $\tilde{\mathcal{A}}$ is a semi-abelian variety. Then we have a surjective homomorphism $\tilde{q}'$ from $\tilde{\mathcal{A}}$ to an abelian variety B over k such that $\operatorname{Ker} \tilde{q}'$ is an algebraic torus. Let $\mathcal{X}$ be the closure of X in $\mathcal{A}$. Then the reduction $\tilde{\mathcal{X}}$ is a closed subset of $\tilde{\mathcal{A}}$ and our $b(X_v)$ coincides with $\dim \tilde{q}'(\tilde{\mathcal{X}})$.

We can see that if there is a place v with $\dim(X/G_X) > b((X/G_X)_v)$, then X is not a special subvariety (cf. Proposition 5.1). Hence if our conjecture holds true, then such X should not have dense small points. In fact, we will show the following result, which is our main theorem of this paper:

Theorem 0.4 (cf. Theorem 5.2 and Remark 5.4)**.** *Assume that there exists a place v such that* $\dim(X/G_X) > b((X/G_X)_v)$. *Then $X(\epsilon; L)$ is not Zariski dense for some $\epsilon > 0$.*

This theorem roughly says that a non-special subvariety of "relatively large" dimension in some sense cannot have dense small points. By virtue of Theorem 0.4 together with Cinkir's theorem, we can see that the geometric Bogomolov conjecture almost holds for an abelian variety A having a place v with $b(A_v) = 1$ in characteristic 0. (cf. Theorem 5.3). Note that the above theorem itself holds true in the case that K is a higher dimensional function field as well as Theorem 0.2 (cf. Remark 5.4).

In the rest of this introduction, we would like to describe the idea of our proof. Before that, let us recall the proof of Theorem 0.1 and that of Theorem 0.2, which gives us a basic strategy.

First we recall the admissible metric. Let A be an abelian variety over $\mathbb{C}$. Let L be an even ample line bundle on X. It is well known that there is a canonical hermitian metric h_{can} on L, called the canonical metric, such that $[n]^* c_1(L, h_{can}) = n^2 c_1(L, h_{can})$ and that the curvature form $c_1(L, h_{can})$ is smooth and positive. For a closed subvariety $X \subset A$ of dimension d, put

$$\mu_{X,L} := \frac{1}{\deg_L(X)} c_1(L, h_{can})^d.$$

It has the total volume 1 and is *smooth* and *positive* on X.

Now let K be a number field. We recall what the equidistribution theorem says. Let X be a closed subvariety of A. Let $(x_l)_{l\in\mathbb{N}}$ be a generic sequence of small points. Let σ be an archimedean place, X_σ the complex analytic space of X over σ, and let L_σ be the restriction of L to X_σ. Roughly speaking, the equidistribution theorem says that the Galois orbit of $(x_l)_l$, approximatively as $l \to \infty$, are equidistributed in X_σ with respect to μ_{X_σ, L_σ}.

Let us recall the proof in the arithmetic case due to Ullmo and Zhang. It is done by contradiction. Suppose we have a counterexample X for the Bogomolov conjecture. Then taking the quotient if necessary, we can easily reduce ourselves to the case where the stabilizer is trivial and $d := \dim X > 0$. Consider, for $N \in \mathbb{N}$, a morphism

$$\alpha : X^N \to A^{N-1}, \quad \alpha(x_1, \ldots, x_N) = (x_2 - x_1, \ldots, x_N - x_{N-1}).$$

For large N, we can see that α gives a birational morphism $X^N \to \alpha(X^N)$. We fix such an N, writing $X' := X^N$ and $Y := \alpha(X')$ for simplicity. Then it induces an isomorphism between some Zariski-dense open subsets $U \subset X'$ and $V \subset Y$. Let L' and M be even ample line bundles on X' and Y respectively. Then we can see that X' is again a counterexample for the Bogomolov conjecture with respect to the line bundle L'. That implies that we can find a generic sequence of small points $(x_l)_{l\in\mathbb{N}}$, and we may assume they sit in U. Moreover, we can see that the image $(\alpha(x_l))_{l\in\mathbb{N}}$ is also a generic sequence of small points. By virtue of the equidistribution theorem, $(x_l)_{l\in\mathbb{N}}$ and $(\alpha(x_l))_{l\in\mathbb{N}}$ are equidistributed in X' and Y with respect to $\mu_{X'_\sigma, L'_\sigma}$ and μ_{Y_σ, M_σ} respectively, for an archimedean place σ. Furthermore since α gives an isomorphism between U and V, we can conclude

$$\mu_{X'_\sigma, L'_\sigma}|_U = \alpha^*(\mu_{Y_\sigma, M_\sigma}|_V).$$

Since both $\mu_{X'_\sigma, L'_\sigma}$ and $\alpha^*(\mu_{Y_\sigma, M_\sigma})$ are smooth forms, we have

$$\mu_{X'_\sigma, L'_\sigma} = \alpha^*(\mu_{Y_\sigma, M_\sigma})$$

on X_σ. The right-hand side however cannot be positive over the diagonal of $X' = X^N$. It is a contradiction since the left-hand side is positive.

How about the case of Gubler? In contrast to the arithmetic case, there are no archimedean places in the geometric case. That fact had prevented us from enjoying an analogous proof of the arithmetic case. To overcome that difficulty, Gubler used non-archimedean analytic spaces over a non-archimedean place and their tropicalizations.

Let $X \subset A$ be a closed subvariety of dimension d. To a place v of K, it is well-known that the Berkovich spaces $X_v \subset A_v$ can be associated. Gubler defined the canonical Chambert-Loir measure μ_{X_v,L_v} on X_v. Suppose here that A_v is totally degenerate. Then Gubler defined the tropicalization X_v^{trop}, which is denoted by $\overline{\mathrm{val}}(X_v)$ in his article, of X_v and showed that it is a "d-dimensional polytope". This plays the role of a counterpart of the complex space over an archimedean place. Furthermore he investigated in detail the push-out $\mu^{\mathrm{trop}}_{X_v,L_v}$ to the tropicalization of μ_{X_v,L_v}, describing it very concretely. In fact he showed that it is a d-dimensional positive Lebesgue measure on the equi-d-dimesional polytope X_v^{trop}.

Now the idea of Ullmo and Zhang can be applied to this situation. If there is a counterexample to the Bogomolov conjecture, we can make the similar situation $\alpha : X' \to Y$ to that of the arithmetic case, where X' and Y are some closed subvarieties of abelian varieties. Recall that X' is also a counterexample of dimension $d' > 0$ and that it has a generic net of small points. We should note also that α is a generically finite morphism and the image of the diagonal by α is one point. Tropicalizing them, we have

$$\alpha^{\mathrm{trop}} : (X'_v)^{\mathrm{trop}} \to Y_v^{\mathrm{trop}},$$

which is a morphism of polytopes. Since the subset corresponding to the diagonal contracts to a point, there is a d'-dimensional face E such that $F := \alpha^{\mathrm{trop}}(E)$ is a lower dimensional face. Using the equidistribution theorem of himself to a generic net of small points, we can obtain

$$\alpha^{\mathrm{trop}}_*(\mu^{\mathrm{trop}}_{X_v,L'_v}) = \mu^{\mathrm{trop}}_{Y_v,M_v}$$

as well, where L' and M respectively are even ample line bundles as before. It is impossible: the left-hand side has a positive measure at a lower dimensional F, but the right one is the d'-dimensional usual Lebesgue measure as mentioned. Thus a contradiction comes out.

Then how about the non-totally degenerate case? The basic strategy for the proof is same—the equidistribution method. It is known that the canonical measure $\mu_{X_v,L}$ exists on the Berkovich space X_v. Gubler defined in [11] the tropicalization X_v^{trop} and studied the push-out $\mu^{\mathrm{trop}}_{X_v,L}$ of the canonical measure. He actually proved that X_v^{trop} has the structure of a simplicial set and that $\mu^{\mathrm{trop}}_{X_v,L}$ can be described as

$$\mu^{\mathrm{trop}}_{X_v,L} = \sum_{i=1}^{N} r_i \delta_{\Delta_i}, \tag{0.4.0}$$

where Δ_i runs through faces and δ_{Δ_i} is a usual relative Lebesgue measure on the simplex Δ_i. On the other hand, he also proved in [10] that the equidistribution theorem holds true in this situation. Thus we seem to have everything we need for the Bogomolov conjecture, but we do not in fact. Then what we need more?

When we obtain the contradiction by using the equidistribution theorem, it was important that the canonical form, or the canonical measure, is a "regular" one. If the canonical form was not smooth or positive in the arithmetic case, a contradiction would not come out. In Gubler's case also, it was the key that the tropicalization of the canonical measure is the Lebesgue measure on the equi-d'-dimensional polytope. In the general case however, lower dimensional Δ_i's often appear in (0.4.0), and that is troublesome. In fact we can make the same situation as before, that is, we have a morphism $\alpha^{\mathrm{trop}} : (X'_v)^{\mathrm{trop}} \to Y_v^{\mathrm{trop}}$ and $\alpha_*^{\mathrm{trop}}\left(\mu^{\mathrm{trop}}_{X'_v,L'_v}\right) = \mu^{\mathrm{trop}}_{Y_v,M_v}$ if we have a counterexample, but it is not sufficient to reach a contradiction because $\mu^{\mathrm{trop}}_{Y_v,M_v}$ may contain a relative Lebesgue measure with a lower dimensional support. That problem requires us more detailed analysis on the canonical measure. We will show in the proof that $\mu^{\mathrm{trop}}_{Y_v,M_v}$ does not have a component with lower dimensional support than we expect.

This article is organized as follows. We will give some remarks on the trace of an abelian variety in § 1. Those who are familiar with the trace will not have to read this section. In § 2, we will formulate the geometric Bogomolov conjecture for abelian varieties. We will recall in § 3 some results concerning our conjecture and will prove some immediate consequences. We will describe in § 4 some basic properties on Berkovich spaces and their tropicalization as far as we will need later. Our main results will be stated in § 5. In § 6, we will note some properties on the canonical measures and will complete the proof of our main result. The appendix is due to W. Gubler. In communicating with the author on the previous version of this paper, he found a proof of the fact that the minimal dimension of the support of the components of $\mu^{\mathrm{trop}}_{X_v,L}$ for ample L is exactly $\dim X - b(X_v)$. Although we do not need this detailed information in the proof of our main theorem, it is quite interesting and will often play an important role when you use the canonical measures.

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Conventions and terminology. Throughout of this paper, let k be a fixed algebraically closed field, and let K be the function field of a reduced irreducible smooth curve over k. We fix an algebraic closure $\overline{K}$.

Let K'/k be a field extension. For a scheme X over k, we write $X_{K'} := X \times_{\operatorname{Spec} k} \operatorname{Spec} K'$. If $\phi : X \to Y$ is a morphism of schemes over k, we write $\phi_{K'} : X_{K'} \to Y_{K'}$ for the base extension to K'.

When we say "height", it means an absolute logarithmic height, for $\mathbf{F} = K$ with the notation of [14, Chapter 3 §1].

For a finite extension K' of K, let $M_{K'}$ denote the set of places of K'. If K'' is a finite extension of K', then there is a natural surjective map $M_{K''} \to M_{K'}$. We put $M_{\overline{K}} := \varprojlim_{K'} M_{K'}$, where the K' runs through all the finite extension of K in $\overline{K}$, and call an element of $M_{\overline{K}}$ a *place* of $\overline{K}$. A place can be naturally regarded as a valuation of height 1. For a place $v \in M_{\overline{K}}$, let $\overline{K}_v$ be the completion of $\overline{K}$ with respect to v.

1. Descent of the base field of abelian varieties

We let L/k be any field extension, keeping our assumption $k = \overline{k}$. We will discuss in this section when an abelian variety over L can be defined over k, and will give remark on the trace of an abelian variety.

We begin with a lemma:

Lemma 1.1. *Let A and B be abelian varieties over L and k respectively. If $\phi : B_L \to A$ is an étale isogeny, then A and ϕ are defined over k: precisely, there exists a subgroup scheme G of B over k such that $\operatorname{Ker}\phi = G \otimes_k L$ and hence $A = (B/G)_L$.*

Proof. Let N be the degree of ϕ. Let $(B_L)[N]$ be the kernel of the N-times homomorphism. Then $\operatorname{Ker}\phi \subset (B_L)[N]_{\mathrm{red}}$ since $\operatorname{Ker}\phi$ is reduced by our assumption. Taking account that the field extension L/k is regular, we have

$$\operatorname{Ker}\phi \subset (B_L)[N]_{\mathrm{red}} = ((B[N])_L)_{\mathrm{red}} = ((B[N])_{\mathrm{red}})_L ,$$

which tells us that $\operatorname{Ker}\phi$ is defined over k, namely, there exists a subgroup scheme G of B over k such that $\operatorname{Ker}\phi = G \otimes_k L$. □

We recall a quite fundamental theorem due to Chow here:

Theorem 1.2 (cf. II §1 Theorem 5 of [13])**.** Let A be an abelian variety over k and let B be an abelian subvariety of A_L. Then there exists an abelian subvariety $B' \subset A$ with $B'_L = B$.

We can now show the following slight generalization of Theorem 1.2:

Proposition 1.3. *Let A be an abelian variety over k and let G be a reduced closed subgroup of A_L. Then there exists a closed subgroup G' of A with $(G')_L = G$.*

Proof. By Theorem 1.2, there exists an abelian subvariety $G^\circ \subset A$ such that G°_L is the identity component of G. Consider the natural homomorphism $\phi' : (A/G^\circ)_L \to (A_L)/G$. It is an étale isogeny since G is reduced, so by Lemma 1.1, there exist an abelian variety H over k and a homomorphism $\psi : A/G^\circ \to H$ such that ψ_L coincides with ϕ'. Now let G' be the kernel of the composition $A \to A/G^\circ \to H$. Then we immediately find $G'_L = G$. □

Let B be an abelian variety over k and let $\phi : B_L \to A$ be a smooth homomorphism between abelian varieties over L. Then, as a corollary of Proposition 1.3, we can take an abelian variety A' over k and a homomorphism $\phi' : B \to A'$ such that $A = A'_L$ and $\phi'_L = \phi$. In fact, there exists a reduced closed subgroup G' of B with $\operatorname{Ker}\phi = G'_L$ by Proposition 1.3. Then $A' := B/G'$ suffices our requirement.

Next we will give remark on the Chow trace. Let F/L be a field extension. Let A be an abelian variety over L. Recall that a pair $\left(A^{F/k}, \operatorname{Tr}_A^{F/k}\right)$ of an abelian variety $A^{F/k}$ over k and a homomorphism $\operatorname{Tr}_A^{F/k} : (A^{F/k})_F \to A \times_{\operatorname{Spec}_L} \operatorname{Spec} F$ over F is called a *F/k-trace*, or *Chow trace*, if it satisfies the following universal property: for any abelian variety B over k and for any homomorphism $\phi : B_F \to A \times_{\operatorname{Spec}_L} \operatorname{Spec} F$, there exists a unique homomorphism $\phi' : B \to A^{F/k}$ over k such that $\operatorname{Tr}_A^{F/k} \circ \phi'_F = \phi$ (cf. [13] and [14]).

Lemma 1.4. $\operatorname{Tr}_A^{F/k}$ *is finite and purely inseparable.*

Proof. By virtue of Proposition 1.3, we can take a closed subgroup $G' \subset A^{F/k}$ such that $G'_F = \left(\operatorname{Ker}\operatorname{Tr}_A^{F/k}\right)_{\mathrm{red}}$. Let $\pi : A^{F/k} \to A^{F/k}/G' =: B$ be the quotient by G'. Then we have naturally a homomorphism $\phi : B_F \to A \times_{\operatorname{Spec}_L} \operatorname{Spec} F$. By the universal property, we obtain the factorization $\phi' : B \to A^{F/k}$ over k, and the universality also says that $\phi' \circ \pi = id_{A^{F/k}}$. That concludes π is an isomorphism and hence $\left(\operatorname{Ker}\operatorname{Tr}_A^{F/k}\right)_{\mathrm{red}} = 0$, namely, $\operatorname{Tr}_A^{F/k}$ is finite and purely inseparable. □

The uniqueness of the F/k-trace is immediate from the definition. We can find in [13] a proof for the existence, but we should note one thing: in the definition of F/k-trace of [13, VIII §8], Lang assumed that $\operatorname{Tr}_A^{F/k}$ is finite. This assumption is not necessary since it follows from the definition automatically by virtue of Lemma 1.4 (cf. [14, the last line in p.138]).

Finally, we give remark on the homomorphism between the F/k-traces in $\operatorname{char}(k) = 0$, although it will not be needed in the sequel. Let A and B be abelian varieties over L and let $\phi : A \to B$ be a homomorphism. Then ϕ induces a unique homomorphism $\operatorname{Tr}(\phi) : A^{F/k} \to B^{F/k}$ by the universal property.

Proposition 1.5 ($\operatorname{char}(k) = 0$)**.** *Suppose that ϕ is surjective. Then $\operatorname{Tr}(\phi) : A^{F/k} \to B^{F/k}$ is surjective.*

Proof. Let us take an abelian subvariety $A' \subset A$ finite and surjective over B. Then we have a composition of homomorphism $(A')^{F/k} \to A^{F/k} \to B^{F/k}$ by the universality, and hence we may assume that A is finite from the beginning, namely, ϕ is an isogeny. Let G be the identity component of

$$\left(B^{F/k}\right)_F \times_{(B \times_{\operatorname{Spec} L} \operatorname{Spec} F)} (A \times_{\operatorname{Spec} L} \operatorname{Spec} F).$$

Then it is an abelian variety over F and we have a natural homomorphism $\psi : G \to \left(B^{F/k}\right)_F$. It is also an isogeny, and its dual isogeny $\hat{\psi}$ is an étale isogeny since $\operatorname{char}(k) = 0$. By virtue of Lemma 1.1, we can take an isogeny $\psi' : G' \to B^{F/k}$ such that $\psi'_F = \psi$. Applying the universality of the F/k-trace to the natural homomorphism $G'_F = G \to A \times_{\operatorname{Spec} L} \operatorname{Spec} F$, we see that ψ' factors as

$$G' \longrightarrow A^{F/k} \xrightarrow{\operatorname{Tr}(\phi)} B^{F/k}.$$

Consequently, the induced homomorphism $\operatorname{Tr}(\phi)$ is surjective since so is ψ. □

2. Geometric Bogomolov conjecture

2.1. **Small points.** Let A be an abelian variety over $\overline{K}$. For an even ample line bundle L on A, let us consider the canonical height function $\hat{h}_L$. It is known to be a semi-positive quadratic form on $A(\overline{K})$. Let X be a closed subvariety of A. For each $\epsilon > 0$, we put

$$X(\epsilon; L) := \left\{ x \in X(\overline{K}) \,\middle|\, \hat{h}_L(x) \leq \epsilon \right\}.$$

Lemma 2.1. *Let A and X be as above. Let $\phi : A \to B$ be a surjective homomorphism of abelian varieties over $\overline{K}$ and put $Y := \phi(X)$. Let L and M be even ample line bundles on A and B respectively. Then if $X(\epsilon; L)$ is Zariski-dense in X for any $\epsilon > 0$, then $Y(\epsilon'; M)$ is Zariski dense in Y for any $\epsilon' > 0$.*

Proof. Since L is ample, we can find a positive integer n such that $L^{\otimes n} \otimes \phi^*(M)^{-1}$ is ample. Then we have $n\hat{h}_L \geq \phi^*\hat{h}_M$, and hence

$$Y(\epsilon; M) = \phi\left(\left\{x \in X(\overline{K}) \,\middle|\, \hat{h}_M(\phi(x)) \leq \epsilon\right\}\right) \supset \phi\left(\left\{x \in X(\overline{K}) \,\middle|\, n\hat{h}_L(x) \leq \epsilon\right\}\right) = \phi(X(\epsilon/n; L)).$$

The right-hand side is Zariski dense in Y by our assumption, and thus we have our assertion. □

Let L_1 and L_2 be even ample line bundles on A. Then $X(\epsilon; L_1)$ is Zariski dense for any $\epsilon > 0$ if and only if so is $X(\epsilon'; L_2)$ for any $\epsilon' > 0$, by virtue of the above lemma. Accordingly, the following definition makes sense:

Definition 2.2. We say that X *has dense small points* if $X(\epsilon; L)$ is Zariski dense in $X(\overline{K})$ for any $\epsilon > 0$ and for some, hence any, even ample line bundle L on A.

We end this subsection with the following two basic lemmas on small points, which will be used later:

Lemma 2.3. *Let $\phi : A \to B$ be a homomorphism of abelian varieties over $\overline{K}$. Let $X \subset A$ be a closed subvariety and put $Y := \phi(X)$. Suppose that ϕ is a finite morphism.*

(1) *X has dense small points if Y has dense small points.*
(2) *Suppose further that ϕ is surjective. Then X has dense small points if and only if Y has dense small points.*

Proof. Let M be an even ample line bundle on B. Then $L := \phi^*M$ is also even and ample and we have $\phi(X(\epsilon; L)) = Y(\epsilon; M)$. Then if $X(\epsilon; L)$ is not Zariski dense for any $\epsilon > 0$, then neither is not $Y(\epsilon; M)$ since ϕ is finite. This proves (1).

The assertion (2) follows immediately from (1) and Lemma 2.1. □

Lemma 2.4. *Let A and B be abelian varieties over $\overline{K}$ and let $X \subset A$ and $Y \subset B$ be closed subvarieties. If X and Y has dense small points, then the closed subvariety $X \times Y \subset A \times B$ has also dense small points.*

Proof. Let $p : A \times B \to B$ and $q : A \times B \to B$ be the canonical projections. For even ample line bundles L and M on A and B respectively, we write $L \boxtimes M := p^*L \otimes q^*M$. It is even ample and we have $\hat{h}_{L\boxtimes M} = p^*\hat{h}_L + q^*\hat{h}_M$. Accordingly we have

$$(X \times Y)(2\epsilon; L \boxtimes M) \supset X(\epsilon; L) \times Y(\epsilon; M),$$

and hence we obtain our assertion. □

2.2. **Special subvarieties and the conjecture.** First of all, we would like to define the notion of special subvarieties. For an abelian variety A over $\overline{K}$, let $\left(A^{\overline{K}/k}, \mathrm{Tr}_A^{\overline{K}/k}\right)$ denote the $\overline{K}/k$-trace of A. We refer to [14, Chapter 6] or 1 for its definition. Since $A^{\overline{K}/k}$ is defined over k, we have the notion of k-points. We note $A^{\overline{K}/k}(k) \subset A^{\overline{K}/k}\left(\overline{K}\right)$ naturally.

Definition 2.5. Let A be an abelian variety over $\overline{K}$.

(1) Let $X \subset A$ be an irreducible closed subvariety. Put $B := A/G_X$ and $Y := X/G_X \subset B$, where G_X is the stabilizer of X. We call X a *special subvariety* if there exist a torsion point $\tau \in B(\overline{K})_{tors}$, and a closed subvariety $Y' \subset B^{\overline{K}/k}$ over k such that

$$Y = \mathrm{Tr}_B^{\overline{K}/k}(Y'_{\overline{K}}) + \tau.$$

(2) A point $\sigma \in A(\overline{K})$ is called a *special point* of A if the closed subvariety $\{\sigma\}$ is a special subvariety. We denote by A_{sp} the set of special points of A.

Let L be an even ample line bundle. Then we have

$$A_{sp} = A(\overline{K})_{tors} + \mathrm{Tr}_A^{\overline{K}/k}\left(A^{\overline{K}/k}(k)\right) = \left\{x \in A(\overline{K}) \,\middle|\, \hat{h}_L(x) = 0\right\}. \tag{2.5.1}$$

In fact, the first equality is immediate from the definition. The second one follows from [14, Theorem 4.5 and 5.4.2]. In particular, a special point is a point of height 0.

Lemma 2.6. *Let $\phi : A \to B$ be a surjective homomorphism of abelian varieties over $\overline{K}$. Then it induces a surjective homomorphism $A_{sp} \to B_{sp}$.*

Proof. The inclusion $\phi\left(A\left(\overline{K}\right)_{tors}\right) \subset B\left(\overline{K}\right)_{tors}$ is obvious. Moreover, the homomorphism ϕ induces a homomorphism $\mathrm{Tr}^{\overline{K}/k}(\phi) : A^{\overline{K}/k} \to B^{\overline{K}/k}$ (cf. 1). Now it is clear that $\phi(A_{sp}) \subset B_{sp}$.

Let us show the other inclusion. We can take an abelian subvariety $J \subset A$ such that $\phi|_J$ is a finite surjective homomorphism. Since a point $x \in J\left(\overline{K}\right)$ is of height 0 if and only if so is $\phi(x)$. Therefore the induced map $J_{sp} \to B_{sp}$ is surjective by (2.5.1). Since $J_{sp} \subset A_{sp}$, we thus obtain our assertion. □

Here is a remark. Suppose that ϕ is surjective. If $\mathrm{char}\, k = 0$, we see that the induced homomorphism $\mathrm{Tr}^{\overline{K}/k}(\phi)$ in the above proof is also surjective (cf. Proposition 1.5), but the author does not know whether it holds in positive characteristic or not.

The following assertion says that the special subvarieties have dense small points:

Proposition 2.7. *If X is a special subvariety of A, then $X \cap A_{sp}$ is dense in X.*

Proof. First let us consider the case where $G_X = 0$. Since our assertion is independent of the translation by a torsion point, we may assume $\tau = 0$ and hence $\mathrm{Tr}_A^{\overline{K}/k}(Y'_{\overline{K}}) = X$ for some $Y' \subset A^{\overline{K}/k}$. Then our assertion is trivial since $\mathrm{Tr}_A^{\overline{K}/k}(Y'(k)) \subset A_{sp}$ and $Y'(k)$ is dense in $Y'_{\overline{K}}$.

Next let us consider the general case. Let $\phi : A \to B := A/G_X$ be the quotient, and put $Y := X/G_X = \phi(X)$. We have $X = \phi^{-1}(Y)$ since G_X is the stabilizer of X. From the surjectivity of ϕ, we see

$$\phi\left(X\left(\overline{K}\right) \cap A_{sp}\right) = Y\left(\overline{K}\right) \cap B_{sp} \tag{2.7.2}$$

by Lemma 2.6. Since Y is a special subvariety of B and is stabilizer-free, we see that $Y\left(\overline{K}\right) \cap B_{sp}$ is dense in Y as shown above. By (2.7.2), we thus conclude that $\phi\left(X\left(\overline{K}\right) \cap A_{sp}\right)$ is dense in $\phi(X)$. On the other hand, take any $x \in X\left(\overline{K}\right) \cap A_{sp}$ with $y = \phi(x)$ for each $y \in \phi\left(X\left(\overline{K}\right) \cap A_{sp}\right)$. Since $\phi^{-1}(Y) = X$, we have

$$x + (G_X)_{sp} \subset \phi^{-1}(y) \cap A_{sp}.$$

Since $x + (G_X)_{sp}$ is dense in $x + G_X = \phi^{-1}(y)$, we find therefore $\phi^{-1}(y) \cap A_{sp}$ is dense in $\phi^{-1}(y)$. That says that the set of special points in the fiber of $\phi|_X : X \to Y$ over y is also dense in the fiber. Together with the fact that (2.7.2) is dense in Y, we can conclude that $X\left(\overline{K}\right) \cap A_{sp}$ is dense in X. □

In particular, we have the following:

Corollary 2.8. *A special subvariety has dense small points.*

Now let us propose the statement of our geometric Bogomolov conjecture for abelian varieties, which insists that the converse of Corollary 2.8 should hold true:

Conjecture 2.9 (Geometric Bogomolov conjecture for abelian varieties (cf. Remark 5.4))**.** *X should not have dense small points unless it is a special subvariety.*

We end this section with the following characterization of the special subvarieties.

Proposition 2.10. *Let $X \subset A$ be a closed subvariety and let G_X be the stabilizer of X. Put $B := A/G_X$ and $Y := X/G_X$. Then the following statements are equivalent to each other:*

(a) *X is a special subvariety of A.*
(b) *Y is a special subvariety of B.*
(c) *There exist an abelian variety C over k, a homomorphism $\phi : C_{\overline{K}} \to B$, a closed subvariety $Z' \subset C$, and a special point $\sigma \in Y$ such that $Y = \phi(Z'_{\overline{K}}) + \sigma$.*
(d) *There exist a variety W' over k, a k-point $w_0 \in W'(k)$, a special point $\sigma \in Y(\overline{K})$ and a surjective morphism $\psi : W'_{\overline{K}} \to Y$ such that $\psi(w_0) = \sigma$.*

Proof. The equivalence between the first and the second statements is trivial from the definition. The implication from (b) to (c) and that from (c) to (d) are also trivial. Let us show that (d) implies (b).

Let W', w_0, σ and ψ be as in (d). For a fixed $y \in B(\overline{K})$, we define $T_y : B \to B$ by $T_y(x) = x + y$. First note that we can write $\sigma = \mathrm{Tr}_B^{\overline{K}/k}(t) + \tau$ with some $t \in B^{\overline{K}/k}(k)$ and $\tau \in B(\overline{K})_{tors}$ by (2.5.1). Then, by considering $T_{-\tau}(Y)$ and $T_{-\tau} \circ \psi$ instead of Y and ψ respectively, we may assume that $\sigma = \mathrm{Tr}_B^{\overline{K}/k}(t)$. Further, taking an alteration of W' if necessary, we may and do assume that W' is nonsingular.

Let us consider the albanese morphism

$$\alpha'_{w_0} : W' \to \mathrm{Alb}(W')$$

with respect to the base point w_0. Then

$$\alpha_{w_0} := (\alpha'_{w_0})_{\overline{K}} : W'_{\overline{K}} \to \mathrm{Alb}(W')_{\overline{K}} = \mathrm{Alb}(W'_{\overline{K}})$$

is the albanese morphism of $W'_{\overline{K}}$ with respect to w_0. By applying the universal property of α_{w_0} to the morphism $T_{-\sigma} \circ \psi$, we obtain a homomorphism

$$\phi : \mathrm{Alb}(W'_{\overline{K}}) \to B$$

with $\phi \circ \alpha_{w_0} = T_{-\sigma} \circ \psi$. Then by the universal property of the $\overline{K}/k$-trace, ϕ factors through the $\overline{K}/k$-trace, that is, there is a homomorphism $\phi' : \mathrm{Alb}(W') \to B^{\overline{K}/k}$ such that

$$\mathrm{Tr}_B^{\overline{K}/k} \circ (\phi'_{\overline{K}}) = \phi.$$

We now consider a closed subvariety $Y' := \phi'(\alpha'_{w_0}(W')) + t$ of $A^{\overline{K}/k}$. Then we have

$$\mathrm{Tr}_B^{\overline{K}/k}(Y'_{\overline{K}}) = \phi(\alpha_{w_0}(W'_{\overline{K}})) + \mathrm{Tr}_B^{\overline{K}/k}(t) = (T_{-\sigma} \circ \psi)(W'_{\overline{K}}) + \sigma = (Y - \sigma) + \sigma = Y$$

as required. □

3. Known results on the conjecture

In this section, we recall some known results concerning the geometric Bogomolov conjecture and give remarks on their consequences.

3.1. **Totally degenerate case.** There are few abelian varieties for which the geometric Bogomolov conjecture is proved. The following beautiful theorem due to Gubler, which is a restatement of Theorem 0.2, is the only case proved with full generality on X.

Theorem 3.1 (Theorem 1.1 of [9])**.** Let A be an abelian variety over $\overline{K}$. The geometric Bogomolov conjecture holds true for A if there is a place v at which A is totally degenerate.

3.2. 0**-dimensional case.** It is easy to obtain the result when $\dim X/G_X = 0$:

Lemma 3.2. *Let A be an abelian variety over $\overline{K}$ and let $X \subset A$ be a irreducible closed subvariety such that* $\dim X/G_X = 0$*. If X is not a special subvariety, then it does not have dense small points.*

Proof. We can write $X/G_X = \{\sigma\}$. If X has dense small points, then so does X/G_X by Lemma 2.1 and hence σ is a special point. That implies that X/G_X and hence X are special. □

3.3. **Jacobian case and its immediate consequence.** Although we do not know other abelian varieties for which Conjecture 2.9 holds true, we still have some partial answers in the case where the subvariety X in consideration is a curve and the abelian variety is its jacobian variety: Let C be a curve over $\overline{K}$, and let J_C be the Jacobian variety of C. For each divisor on C of degree 1, let $j_D : C \to J_C$ be the embedding defined by $j_D(x) = D - x$. For each $\sigma \in J_C$, we note $j_D(x) + \sigma = j_{D+\sigma}(x)$. The following assertion is an immediate consequence of the theorem of Zhang and that of Cinkir. We recall here that a curve C over $\overline{K}$ is *isotrivial* if it is a base extension to $\overline{K}$ of a curve over k.

Proposition 3.3. *Fix $c_0 \in C\left(\overline{K}\right)$. For each $\sigma \in J_C\left(\overline{K}\right)$, we put $X^{\pm}_{c_0,\sigma} := [\pm 1](j_{c_0}(C) + \sigma)$, where $[\pm 1]$ is the ± 1-multiplication on J_C.*

1. *Suppose that C is isotrivial. Let $\psi : Z'_{\overline{K}} \cong C$ be an isomorphism, where Z' is a curve over k. We assume further that $c_0 \in \psi(Z'(k))$. Then $X^{\pm}_{c_0,\sigma}$ has dense small points if and only if σ is a special point.*
2. *Assume* $\mathrm{char}\, k = 0$*. If C is non-isotrivial, then $X^{\pm}_{c_0,\sigma}$ does not have dense small points.*

Proof. It is enough to consider $X_{c_0,\sigma} := X^{+}_{c_0,\sigma}$ only. Taking a finite extension of K if necessary, we may assume C is a curve defined over K with stable reduction at any place, and $c_0 \in C(K)$. Then the assertion (2) is immediate from [7, Theorem 2.12] and [20, Theorem 5.6].

To see the assertion (1), we first note that the admissible pairing (ω_a, ω_a) vanishes in this case. By virtue of [20, Theorem 5.6], we find that $X_{c_0,\sigma}$ has dense small points if and only if the canonical height of the point corresponding to the divisor class $(2g-2)(c_0+\sigma)-\omega_C$ in the jacobian vanishes. That is equivalent to σ being special in this case by (2.5.1). Thus we obtain our assertion. □

In the rest of this subsection, we will see what follows from Proposition 3.3. Let us prepare a technical lemma:

Lemma 3.4. *Let X be a closed subvariety of A, and let $H \subset A$ be an abelian subvariety. Suppose that there exists x_0 with $X - x_0 \subset H$ and that X has dense small points. Then there exists a special point σ of A such that $X - \sigma \subset H$. Moreover, $X - \sigma$ has dense small points.*

Proof. The last statement follows form [14, Theorem 4.5 and 5.4.2] since X has dense small points. To complete the proof, we may assume $H \subsetneq A$. Let $\phi : A \to A/H$ be the quotient. Since $X - x_0 \subset H$, we have $\phi(X) = \phi(x_0)$. Since X has dense small points, $\phi(x_0)$ is a special point by Lemma 2.1. By virtue of Lemma 2.6, there exists $\sigma \in A_{sp}$ with $\phi(\sigma) = \phi(x_0)$. Then we have $X - \sigma \subset H$. □

Now we can show the following assertion, which is a partial answer to the geometric Bogomolov conjecture when the closed subvariety X is a curve:

Proposition 3.5 ($\mathrm{char}\, k = 0$)**.** *Let X be an irreducible closed subvariety of A of dimension 1, and let $\nu : Y \to X$ be the normalization. Let J_Y be the jacobian variety of Y. Suppose that J_Y is simple. Then X does not have dense small points unless it is a special subvariety.*

Proof. For a fixed $y_0 \in Y(\overline{K})$, we put $x_0 := \nu(y_0)$ and $X_0 := X - x_0$. Then $0 \in X_0(\overline{K})$ and we have naturally $\nu_0 : Y \to X_0$ with $\nu(y_0) = 0$, by composing the translation by $-x_0$ to ν. Then we can draw a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{j_{y_0}} & J_Y \\ {\scriptstyle \nu_0}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ X_0 & \longrightarrow & A, \end{array}$$

in which $X_0 \to A$ is the inclusion.

We require an additional condition on y_0 in case that Y is an isotrivial curve: we can take a variety Y' over k and an isomorphism $\psi : Y'_{\overline{K}} \cong Y$, and our requirement is $y_0 \in \psi(Y'(k))$.

Under the setting above, we will show it by contradiction. Suppose that X is not a special subvariety but it has dense small points. Let H be the image of the homomorphism ϕ. Then by Lemma 3.4, there is $\sigma \in A_{sp}$ such that $X_1 := X - \sigma \subset H$, and moreover X_1 has dense small points. We put $z := \sigma - x_0$. Then we have $X_1 = X_0 - z$ and $z \in H$. We take $w \in J_Y$ with $\phi(w) = z$ and consider $Y_1 := Y - w$. Note $\phi(Y_1) = X_1$. The homomorphism ϕ is finite since J_Y is simple by our assumption. Therefore, we see that Y_1 has dense small points by Lemma 2.3.

Here we divide ourselves into two cases. The first one is the case where Y is non-isotrivial. Then Y_1 cannot have dense small points by Proposition 3.3 (2), hence the contradiction immediately comes out.

Let us consider the other case, namely, the case where Y is isotrivial. Since Y_1 has dense small points and $Y_1 = j_{y_0}(Y) - w$, we see that w is a special point by Proposition 3.3 (1). That says that $z = \phi(w)$ is a special point, which implies X_1 a special subvariety by

Proposition 2.10. Accordingly $X = X_1 + \sigma$ is also a special subvariety by Proposition 2.10, which contradicts our assumption. Thus we have proved our assertion. $\square$

4. Preliminary

We fix our conventions and terminology. When we write $\mathbb{K}$, it is a field which is complete with respect to a non-archimedean absolute value $|\cdot| : \mathbb{K}^\times \to \mathbb{R}$. Our $\overline{K}_v$ is a typical example of $\mathbb{K}$. We put

$$\mathbb{K}^\circ := \{a \in K \mid |a| \leq 1\},$$

the ring of integers of $\mathbb{K}$, and put

$$\mathbb{K}^{\circ\circ} := \{a \in K \mid |a| < 1\},$$

the maximal ideal of the valuation ring $\mathbb{K}^\circ$. Further we write $\tilde{\mathbb{K}} := \mathbb{K}^\circ/\mathbb{K}^{\circ\circ}$. When we say $\mathscr{X}$ is a formal scheme, *it always means an admissible formal scheme* over $\mathbb{K}^\circ$, that is, $\mathscr{X}$ is locally the Spf of an admissible algebra over $\mathbb{K}^\circ$ (cf. [8, 11]). For a formal scheme $\mathscr{X}$, we write $\tilde{\mathscr{X}} := \mathscr{X} \times_{\mathrm{Spf}\,\mathbb{K}^\circ} \mathrm{Spec}\,\tilde{\mathbb{K}}$.

4.1. **Berkovich spaces.** In the theory of rigid analytic geometry, there are several kind of "visualization" or in other words, some kind of spaces that realize the theory of rigid analytic geometry. In this article, we adopt the spaces introduced by Berkovich which are called *Berkovich spaces*. Here let us recall some properties of Berkovich spaces associated to algebraic varieties or formal schemes, as far as we use later. For details, we refer to his original papers [1, 2, 3, 4], or to Gubler's expositions in his papers [8, 11], which would be good reviews to that theory.

First recall that we can associate a Berkovich space to an algebraic variety. We denote by X^{an} the Berkovich space associate to an algebraic variety X over $\mathbb{K}$. Note that we have naturally $X(\mathbb{K}) \subset X^{\mathrm{an}}$.

Next let $\mathscr{X}$ be a formal scheme over $\mathbb{K}^\circ$. Then we can also associate a Berkovich space $\mathscr{X}^{\mathrm{an}}$, and we have $\mathscr{X}(\mathbb{K}^\circ) \subset \mathscr{X}^{\mathrm{an}}$. There is a reduction map $\mathrm{red}_{\mathscr{X}} : \mathscr{X}^{\mathrm{an}} \to \tilde{\mathscr{X}}$. Let Z an irreducible component of $\tilde{\mathscr{X}}$ with the generic point ξ_Z. Then there is a unique point $\eta_Z \in \mathscr{X}^{\mathrm{an}}$ with $\mathrm{red}_{\mathscr{X}}(\eta_Z) = \xi_Z$. Thus we can naturally regard the generic point of each irreducible component of the special fiber as a point of the associated Berkovich spaces.

Let us compare the Berkovich space associated to an algebraic variety and that done to a formal scheme. Let X be an algebraic scheme over $\mathbb{K}$. Let $\mathcal{X}$ be a model of X, that is, $\mathcal{X}$ is a scheme flat and of finite type over $\mathbb{K}^\circ$ with the generic fiber X. Let $\hat{\mathcal{X}}$ be the completion with respect to a nontrivial principal open ideal of $\mathbb{K}^\circ$. Then it is an admissible formal scheme and $\hat{\mathcal{X}}^{\mathrm{an}}$ is an analytic subdomain of X^{an}. Moreover if $\mathcal{X}$ is proper over $\mathbb{K}^\circ$, then $\hat{\mathcal{X}}^{\mathrm{an}} = X^{\mathrm{an}}$.

Finally, let X be a proper algebraic variety over $\mathbb{K}$ and let Y be its closed subvariety. We can take a formal scheme $\mathscr{X}$ such that $\mathscr{X}^{\mathrm{an}} = X^{\mathrm{an}}$. Then there is a unique formal subscheme $\mathscr{Y} \subset \mathscr{X}$ with $\mathscr{Y}^{\mathrm{an}} = Y^{\mathrm{an}}$. We call this $\mathscr{Y}$ the *closure* of Y in $\mathscr{X}$.

Let X be an algebraic scheme over $\overline{K}$, and let v be a place of $\overline{K}$. Then we have a Berkovich space associated to $X \times_{\overline{K}} \mathrm{Spec}\,\overline{K}_v$. We denote it by X_v. It is a typical Berkovich space that we will mainly deal with in the sequel.

4.2. **Raynaud extension, tropicalization and Mumford models.** For simplicity, we assume further that $\mathbb{K}$ is algebraically closed. We recall here the Raynaud extension, the tropicalization, and the Mumford models, as far as need in the sequel. See [5, §1] and [11, §4] for details.

Let A be an abelian variety over $\mathbb{K}$. According to [5, Theorem 1.1], there exists a unique subgroup $A^\circ \subset A^{\mathrm{an}}$, that is, a unique analytic subdomain with the subgroup space structure, such that there is a formal group scheme $\mathscr{A}^\circ$ with the following properties:

- $(\mathscr{A}^\circ)^{\mathrm{an}} \cong A^\circ$.
- There is a short exact sequence
$$1 \longrightarrow \mathscr{T}^\circ \longrightarrow \mathscr{A}^\circ \longrightarrow \mathscr{B} \longrightarrow 0,$$
where $\mathscr{T}^\circ$ is a formal torus and $\mathscr{B}$ is a formal abelian variety.

Consider the associated exact sequence of rigid analytic ones, that is, an exact sequence
$$1 \longrightarrow T^\circ \longrightarrow A^\circ \xrightarrow{q^\circ} B \longrightarrow 0$$
of analytic groups, where $B := \mathscr{B}^{\mathrm{an}}$ and $T^\circ := (\mathscr{T}^\circ)^{\mathrm{an}}$. Naturally T° is a quasi-compact open subgroup of the rigid torus T, and hence we can obtain the push-out of the above extension:
$$1 \longrightarrow T \longrightarrow E \xrightarrow{q^{\mathrm{an}}} B \longrightarrow 0. \tag{4.0.3}$$
Note that there is a natural inclusion $A^\circ \to E$. Now [5, Theorem 1.2] says that the homomorphism $T^\circ \hookrightarrow A^{\mathrm{an}}$ extends uniquely to a homomorphism $T \to A^{\mathrm{an}}$ and hence to $p^{\mathrm{an}} : E \to A^{\mathrm{an}}$. This p^{an} is called the *Raynaud extension* of A. It is known that p^{an} is a surjective homomorphism and moreover $M := \operatorname{Ker} p^{\mathrm{an}}$ is a lattice. Thus A^{an} can be described as a quotient of E by a lattice M. The dimension of T is called the *torus rank* of A.

We recall the tropicalization next. Taking into account that the transition function of the T-torsor (4.0.3) can be valued in T°, we can define a continuous map
$$\mathrm{val} : E \to \mathbb{R}^n,$$
as in [5]. In fact, we can take an analytic subdomain $V \subset B$ and a trivialization
$$(q^{\mathrm{an}})^{-1}(V) \cong V \times T \tag{4.0.4}$$
such that its restriction induces a trivialization
$$(q^\circ)^{-1}(V) \cong V \times T^\circ.$$
Let us consider the composition $r_V : (q^{\mathrm{an}})^{-1}(V) \cong V \times T \to T$ of (4.0.4) and the second projection. We see that if $x \in (q^{\mathrm{an}})^{-1}(V)$, then $\mathrm{val}(x) = (v(r_V(x)_1), \ldots, v(r_V(x)_n))$, where $r_V(x)_j$ is the j-th coordinate of $r_V(x)$. Moreover, the lattice M is mapped by val to a lattice $\Lambda \subset \mathbb{R}^n$ and we have a diagram
$$\begin{array}{ccc} E & \xrightarrow{\mathrm{val}} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ A^{\mathrm{an}} & \xrightarrow{\overline{\mathrm{val}}} & \mathbb{R}^n/\Lambda \end{array} \tag{4.0.5}$$

that commutes. If X is a closed subvariety of A, then the image $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is a closed subset of $\mathbb{R}^n/\Lambda$. We put

$$X^{\mathrm{trop}} := \overline{\mathrm{val}}(X^{\mathrm{an}}),$$

calling it the *tropicalization* of X. It is well-known that X^{trop} has the structure of a polytopal set (cf. [11, Theorem 1.1]).

The following assertion will be used in the proof of our main result:

Lemma 4.1. *Let A_1 and A_2 be abelian varieties over $\mathbb{K}$ and let $X_1 \subset A_1$ and $X_2 \subset A_2$ be closed subvarieties. Then we have naturally*

$$(X_1 \times X_2)^{\mathrm{trop}} = {X_1}^{\mathrm{trop}} \times {X_2}^{\mathrm{trop}}.$$

Proof. From the definition of the tropicalization, we immediately see

$${A_1}^{\mathrm{trop}} \times {A_2}^{\mathrm{trop}} = \mathbb{R}^{n_1}/\Lambda_1 \times \mathbb{R}^{n_2}/\Lambda_2 = \mathbb{R}^{n_1+n_2}/\Lambda_1 \times \Lambda_2 = (A_1 \times A_2)^{\mathrm{trop}},$$

where n_i is the torus rank of A_i and Λ_i is the lattice as in (4.0.5) for A_i. Both $(X_1 \times X_2)^{\mathrm{trop}}$ and ${X_1}^{\mathrm{trop}} \times {X_2}^{\mathrm{trop}}$ are subsets of the above real torus. On the other hand, we have the natural surjective map $(X_1 \times X_2)^{\mathrm{trop}} \to {X_1}^{\mathrm{trop}} \times {X_2}^{\mathrm{trop}}$ associated to the natural surjective continuous map

$$|(X_1 \times X_2)^{\mathrm{an}}| \to |(X_1)^{\mathrm{an}}| \times |(X_2)^{\mathrm{an}}|,$$

where $|X^{\mathrm{an}}|$ stands for the underlying topological space of a Berkovich space X^{an}. Thus we conclude $(X_1 \times X_2)^{\mathrm{trop}} = {X_1}^{\mathrm{trop}} \times {X_2}^{\mathrm{trop}}$. □

Let $\mathscr{C}$ be a Λ-periodic polytopal decomposition of $\mathbb{R}^n$ (cf. [8, §6.1]). Taking the quotient by Λ, we have a polytopal decomposition of $\mathbb{R}^n/\Lambda$. Gubler constructed the *Mumford model* $p = p_{\mathscr{C}} : \mathscr{E} \to \mathscr{A}$ associated to $\mathscr{C}$. We also call $\mathscr{A}$ the Mumford model of A. We refer to [11, §4] for details, and recall some properties that will be needed:

- The surjection $q^{\mathrm{an}} : E \to B$ extends to $q : \mathscr{E} \to \mathscr{B}$ uniquely. If $\mathscr{T}$ denote the closure of T in $\mathscr{E}$, then

$$q : \mathscr{E} \to \mathscr{B} \tag{4.1.6}$$

  is a fiber bundle with the fiber $\mathscr{T}$.
- The lattice M acts freely on $\mathscr{E}$ and $\mathscr{E}/M = \mathscr{A}$. In particular p is locally isomorphic.
- If $\mathscr{C}'$ is a polytopal decomposition of $\mathbb{R}^n$ finer than $\mathscr{C}$, and if $\mathscr{E}' \to \mathscr{A}'$ is the Mumford model associated to $\mathscr{C}'$, then there is a natural morphism $\mathscr{E}' \to \mathscr{E}$ and $\mathscr{A}' \to \mathscr{A}$.

4.3. **The dimension of the abelian part of a closed subvariety.** In this subsection, let A be an abelian variety over $\mathbb{K}$ and let $X \subset A$ be an irreducible closed subvariety.

Lemma 4.2. *For $i = 0, 1$, let $p_i : \mathscr{E}_i \to \mathscr{A}_i$ be a Mumford model of the Raynaud extension of A and let $q_i : \mathscr{E}_i \to \mathscr{B}$ be the morphism as (4.1.6). Let $\mathscr{X}_i$ be the closure of X in $\mathscr{A}_i$ and let $\mathscr{Y}_i$ be a quasicompact open subscheme of $p^{-1}(\mathscr{X}_i)$ such that $p(\mathscr{Y}_i) = \mathscr{X}_i$. Then we have $\dim \tilde{q}_1(\tilde{\mathscr{Y}}_1) = \dim \tilde{q}_2(\tilde{\mathscr{Y}}_2)$.*

Proof. We can take a Mumford model $p : \mathscr{E} \to \mathscr{A}$ such that $\mathscr{A}$ dominates both $\mathscr{A}_0$ and $\mathscr{A}_1$. Let $q : \mathscr{E} \to \mathscr{B}$ be the morphism as (4.1.6). We also have a dominant morphism $\mathscr{X} \to \mathscr{X}_i$ for $i = 0, 1$, where $\mathscr{X}$ is the closure of X in $\mathscr{A}$. Set $\mathscr{Y}_i' := \mathscr{X} \times_{\mathscr{X}_i} \mathscr{Y}_i$. Then $\mathscr{Y}_i'$ is a quasicompact open formal subscheme of $p^{-1}(\mathscr{X})$ such that $\mathscr{Y}_i' \to \mathscr{X}$ is surjective. Moreover $\tilde{\mathscr{Y}}_i' \to \tilde{\mathscr{Y}}_i$ is surjective, and hence $\dim(\tilde{q}(\tilde{\mathscr{Y}}_i'))=\dim(\tilde{q}_i(\tilde{\mathscr{Y}}_i))$. Accordingly, by pulling everything back to $\mathscr{A}$, we may assume $\mathscr{A}_1=\mathscr{A}_0=\mathscr{A}$ and hence $\mathscr{X}_1=\mathscr{X}_0=\mathscr{X}$, $\mathscr{E}_1=\mathscr{E}_0=:\mathscr{E}$ and $\mathscr{Y}_0,\mathscr{Y}_1\subset p^{-1}(\mathscr{X})$.

Let us fix an irreducible component W of $\tilde{\mathscr{X}}$. There are irreducible components Z_0 and Z_1 of $\tilde{\mathscr{Y}}_0$ and $\tilde{\mathscr{Y}}_1$ lying over W. Since $\mathscr{A}=\mathscr{E}/M$, there exist $m\in M$ such that $Z_1\cap(Z_0+m)$ is a Zariski dense open subset of both Z_1 and Z_0+m. Accordingly,

$$\dim\tilde{q}(Z_1)=\dim\tilde{q}(Z_1\cap(Z_0+m))=\dim\tilde{q}(Z_0+m)=\dim\tilde{q}(Z_0).$$

Consequently, the number $\dim\tilde{q}(Z)$, where Z is an irreducible component over W, depends only on W. If we write $\alpha(W)$ for this number, we see, for each i, that

$$\dim\tilde{q}(\tilde{\mathscr{Y}}_i)=\max_W\alpha(W),$$

where W runs through the irreducible components of the quasicompact scheme $\tilde{\mathscr{X}}$. Thus our assertion follows. □

By virtue of the above lemma, we can make the following definition:

Definition 4.3. For an an irreducible closed subvariety X of A, we define $b(X)$ to be the number $\dim\tilde{q}_1(\tilde{\mathscr{Y}}_1)=\dim\tilde{q}_2(\tilde{\mathscr{Y}}_2)$ in Lemma 4.2. We call it *the dimension of the abelian part of* X.

Note $b(A)=0$ if and only if A is totally degenerate.

Remark 4.4. Suppose that $\mathcal{A}$ is a semi-abelian model over $\mathbb{K}^\circ$ of A. Let $\tilde{\mathcal{A}}$ denote the reduction. Then we have a surjective homomorphism $\tilde{q}'$ from $\tilde{\mathcal{A}}$ to an abelian variety over $\tilde{\mathbb{K}}$ such that $\operatorname{Ker}\tilde{q}'$ is an algebraic torus over $\tilde{\mathbb{K}}$. Let $X\subset A$ be an irreducible closed subvariety and let $\mathcal{X}$ be the closure of X in $\mathcal{A}$. Then the reduction $\tilde{\mathcal{X}}$ is a closed subset of $\tilde{\mathcal{A}}$, and it follows from the definition that $b(X)=\dim\tilde{q}'(\tilde{\mathcal{X}})$.

Lemma 4.5. *Let $X\subset A$ be an irreducible closed subvariety and let G_X be the stabilizer of X. Then $b(X)\geq b(X/G_X)$.*

Proof. Let $\phi:A\to A/G_X$ be the quotient homomorphism. Then ϕ lifts to a homomorphism between the Raynaud extensions of A and A/G_X by [5, Theorem 1.2]. Therefore, if $\mathscr{B}$ and $\mathscr{C}$ are the formal abelian varieties such that $\mathscr{B}^{\mathrm{an}}$ and $\mathscr{C}^{\mathrm{an}}$ are the abelian parts of the Raynaud extensions of A and A/G_X respectively, then we have an induced homomorphism $\mathscr{B}\to\mathscr{C}$. Now our assertion follows immediately from the definition of $b(X)$. □

5. Partial answers to the conjecture

We recall the following notation: for a projective variety Y over $\overline{K}$, let Y_v denote the Berkovich space associated with $Y\times_{\operatorname{Spec}\overline{K}}\operatorname{Spec}\overline{K}_v$ over $v\in M_{\overline{K}}$ (cf. § 4.1).

5.1. **Special subvariety and the dimension of the abelian part.** Here we note the following assertion.

Proposition 5.1. *Let X be an irreducible closed subvariety of A.*

(1) *If there is a place v with $\dim X/G_X>b((X/G_X)_v)$, then X is not a special subvariety.*

(2) *If X is a special subvariety and* $\dim X/G_X \geq b((A/G_X)_v)$*, then there is a special point σ such that $X = G_X + \sigma$, namely, X is an abelian subvariety up to a special point.*

Proof. Suppose that X is a special subvariety. Taking the quotient by G_X, we may assume $G_X = 0$, and further, taking the translation of X by a torsion point if necessary, we may assume that there is a closed subvariety $Y' \subset A^{\overline{K}/k}$, such that

$$\mathrm{Tr}_A^{\overline{K}/k}(Y'_{\overline{K}}) = X.$$

We write $\mathbb{K} := \overline{K}_v$. By the existence of the Néron model and the semistable reduction theorem, we can take a semi-abelian scheme $\mathcal{A}$ over $\mathbb{K}^\circ$ and a homomorphism

$$\tau : A^{\overline{K}/k} \times_{\operatorname{Spec} k} \operatorname{Spec} \mathbb{K}^\circ \to \mathcal{A}$$

extending $\mathrm{Tr}_A^{\overline{K}/k}$. Note that τ is finite by Lemma 1.4. Let $\tilde{\mathcal{A}}$ be the special fiber. It is a semi-abelian variety, and we have a surjective homomorphism $\tilde{q}'$ from $\tilde{\mathcal{A}}$ to an abelian variety over k such that $\operatorname{Ker} \tilde{q}'$ is an algebraic torus. Let $\tilde{\tau}$ be the reduction of τ. Since $\tilde{\tau}\left(A^{\overline{K}/k}\right)$ is proper over k and $\operatorname{Ker} \tilde{q}'$ is affine, we see $\tilde{q}'|_{\tilde{\tau}\left(A^{\overline{K}/k}\right)}$ is finite. Taking account that τ is finite, we conclude $\tilde{q}' \circ \tilde{\tau}$ is also finite. Accordingly we have $b(X_v) = \dim \tilde{q}'(\tilde{\tau}(Y')) = \dim Y'$ and $b(A_v) \geq \dim A^{\overline{K}/k}$ (cf. Remark 4.4).

Now (1) follows immediately since

$$\dim Y' = \dim X \geq b(X_v) = \dim Y'.$$

To show (2), we suppose further $\dim X \geq b(A_v)$. Then we have

$$\dim Y' = \dim X \geq b(A_v) \geq \dim A^{\overline{K}/k} \geq \dim Y',$$

which says $Y' = A^{\overline{K}/k}$ and X is an abelian subvariety. Since $G_X = 0$, we have $X = 0$, as required. □

Accordingly to the above proposition and Lemma 4.5, any special subvariety is a torsion subvariety if there is a place at which A is totally degenerate (cf. Theorem 3.1). If there is a place v with $b(A_v) = 1$, then a special subvariety is an abelian subvariety up to the translation by a special point (cf. Theorem 5.3).

5.2. **Main results.** According to Proposition 5.1, an irreducible closed subvariety with $\dim(X/G_X) > b((X/G_X)_v)$ for some v is not a special subvariety. If the geometric Bogomolov conjecture holds true, then such a closed subvariety should not have dense small points. In fact, it is our main assertion:

Theorem 5.2 (cf. Theorem 0.4)**.** *Let A be an abelian variety over $\overline{K}$ and let X be an irreducible closed subvariety of A. Let $G_X \subset A$ be the stabilizer of X. Suppose* $\dim(X/G_X) > b((X/G_X)_v)$ *for some place v. Then X does not have dense small points.*

Roughly speaking, we see from the above theorem that non-special closed subvariety of "relatively large" dimension cannot have dense small points. The proof will be delivered in the next section.

As a consequence of Proposition 3.5 and Theorem 5.2 together with the facts in § 3, we obtain the following theorem.

Theorem 5.3 (char $k=0$)**.** *Let A be an abelian variety such that $b(A_v)=1$ for some place $v \in M_{\overline{K}}$. Let $X \subset A$ be an irreducible closed subvariety. Suppose X is not special, and further suppose one of the following holds:*

(a) $\dim X/G_X \neq 1$.
(b) $b((X/G_X)_v)=0$.
(c) $\dim X/G_X = 1$, *and the jacobian variety of the normalization of X/G_X is simple.*

Then X does not have dense small points.

Proof. We may assume $\dim X/G_X > 0$ by Lemma 3.2. Then if (a) or (b) holds, then we have $\dim X/G_X > b((X/G_X)_v)$ and hence obtain our assertion by Theorem 5.2.

If we assume (c), then this theorem follows from Proposition 3.5 immediately. □

Theorem 5.3 says that if there exists a place v of K with $b(A_v)=1$, then the geometric Bogomolov conjecture almost holds true for A. The only case we have not yet know the validity is the case where $\dim X/G_X = b((X/G_X)_v) = 1$ and the jacobian variety of the normalization of X/G_X is not simple.

Remark 5.4. We can consider Conjecture 2.9 also in the case where K is the function field of a higher dimensional normal projective variety. Theorem 5.2 holds true in such a case as well as Theorem 0.2, because our proof will be given by a local methods which hold for any discrete valuation. However, Theorem 5.3 needs the assumption that K is the function field of a curve since we have applied Cinkir's result.

6. Proof of Theorem 5.2

The basic strategy of the proof will be same as that of the totally degenerate case due to Gubler, but we need technically more information on the canonical Chambert-Loir measures and their tropicalization.

6.1. **Chambert-Loir measures.** The purpose of this subsection is to give a remark on the product of them. We refer to [11, §3] for all the notions such as admissible metric and Chambert-Loir measures.

Let X be a projective variety over $\overline{K}$. To an admissibly metrized line bundle $\overline{L}$ on X (cf. [11, §3.5]), we can associate a Borel measure

$$\mu_{X_v,\overline{L}} := \frac{1}{\deg_L X} c_1(\overline{L})^d$$

on $|X_v|$ (cf. [11, Proposition 3.8]), where we emphasize with $|\cdot|$ that $|X_v|$ is the underlying topological space of the Berkovich space X_v.

The following formula is the one mentioned in [6, §2.8] essentially, but we restate it with a proof for readers' convenience.

Proposition 6.1 (§2.8 of [6])**.** *Let X and Y be projective varieties over $\overline{K}$ and let $\overline{L}$ and $\overline{M}$ be admissibly metrized line bundles on X and Y respectively. Let p and q be the canonical projections from $X \times Y$ to X and Y respectively, and let $r : |X_v \times Y_v| \to |X_v| \times |Y_v|$ be the canonical continuous map induced from the projections. Then we have*

$$\mu_{X_v,\overline{L}} \times \mu_{Y_v,\overline{M}} = r_* \left(\mu_{X_v \times Y_v, \overline{L} \boxtimes \overline{M}}\right),$$

where $\overline{L} \boxtimes \overline{M} = p^*\overline{L} \otimes q^*\overline{M}$ and $\mu_{X_v,\overline{L}} \times \mu_{Y_v,\overline{M}}$ is the product measure on $|X_v| \times |Y_v|$.

Proof. First let us consider the case where the admissible metric on L and M are the formal metrics arising from models $(\mathscr{X}, \mathscr{L})$ and $(\mathscr{Y}, \mathscr{M})$ respectively (cf. [8, §3]). Then $\overline{L} \boxtimes \overline{M}$ is the formally metrized line bundles arising from the model $(\mathscr{X} \times \mathscr{Y}, \mathscr{L} \boxtimes \mathscr{M})$. By virtue of [8, Proposition 3.11], we have explicit formulas

$$\mu_{\overline{L}} = \frac{1}{\deg_L X} \sum_{A \in \operatorname{Irr}\left(\tilde{\mathscr{X}}\right)} (\deg_{\mathscr{L}} A)\, \delta_{\eta_A}$$

$$\mu_{\overline{M}} = \frac{1}{\deg_L X} \sum_{B \in \operatorname{Irr}\left(\tilde{\mathscr{Y}}\right)} (\deg_{\mathscr{M}} B)\, \delta_{\eta_B}$$

$$\mu_{\overline{M}} = \frac{1}{\deg_{L \boxtimes M} X \times Y} \sum_{C \in \operatorname{Irr}\left(\widetilde{\mathscr{X} \times \mathscr{Y}}\right)} (\deg_{\mathscr{L} \boxtimes \mathscr{M}} C)\, \delta_{\eta_C},$$

where "Irr" means the set of irreducible components and η_A denotes the point of the Berkovich space corresponding to A (cf. § 4.1). Since $\widetilde{\mathscr{X} \times \mathscr{Y}} = \tilde{\mathscr{X}} \times \tilde{\mathscr{Y}}$, we have naturally $\operatorname{Irr}\left(\widetilde{\mathscr{X} \times \mathscr{Y}}\right) = \operatorname{Irr}\left(\tilde{\mathscr{X}}\right) \times \operatorname{Irr}\left(\tilde{\mathscr{Y}}\right)$. If $C = A \times B$, then it is easy to see

$$\deg_{\mathscr{L} \boxtimes \mathscr{M}} C = \binom{d+e}{d} (\deg_{\mathscr{L}} A) \cdot (\deg_{\mathscr{M}} B)$$

and $r_* \delta_{\eta_C} = \delta_{\eta_A} \times \delta_{\eta_B}$, where $d := \dim X$ and $e := \dim Y$. Accordingly, we have

$$r_* \left((\deg_{\mathscr{L} \boxtimes \mathscr{M}} C)\, \delta_{\eta_C} \right) = \binom{d+e}{d} \left((\deg_{\mathscr{L}} A)\, \delta_{\eta_A} \right) \times \left((\deg_{\mathscr{M}} B)\, \delta_{\eta_B} \right).$$

Since

$$\deg_{L \boxtimes M} X \times Y = \binom{d+e}{d} (\deg_L X) \cdot (\deg_M Y),$$

we thus have our formula in this case.

Now let us consider the general case. Let $(\mathscr{X}_n, \mathscr{L}_n)$ and $(\mathscr{Y}_n, \mathscr{M}_n)$ be approximating sequences of models of $\overline{L}$ and $\overline{M}$ respectively. Then $(\mathscr{X}_n \times \mathscr{Y}_n, \mathscr{L}_n \boxtimes \mathscr{M}_n)$ is an approximating sequence of $\overline{L} \boxtimes \overline{M}$, and we have

$$r_* \left(\mu_{\mathscr{L}_n \boxtimes \mathscr{M}_n} \right) = \mu_{\mathscr{L}_n} \times \mu_{\mathscr{M}_n}$$

as we have shown. Taking the limit as $n \to +\infty$, we obtain our assertion. □

6.2. **Non-degenerate strata.** We recall the notion of non-degenerate strata here. First of all, let us recall the notion of stratification of a variety (cf. [1], [11]). Let Z be a reduced scheme of finite type over a field k. Put $Z^{(0)} := Z$. For $r \in \mathbb{Z}_{\geq 0}$, define $Z^{(r+1)} \subset Z^{(r)}$ to be the compliment of the set of normal points of $Z^{(r)}$. Then $Z^{(r+1)}$ is a proper closed subset of $Z^{(r)}$, and we obtain a chain of closed subsets

$$Z = Z^{(0)} \supsetneq Z^{(1)} \supsetneq \cdots \supsetneq Z^{(s-1)} \supsetneq Z^{(s)} = \emptyset$$

for some $s \in \mathbb{N}$. The irreducible component of $Z^{(r)} \setminus Z^{(r+1)}$ for any $r \in \mathbb{Z}_{\geq 0}$ is called a *stratum* of Z, and the set of the strata of Z is denoted by $\operatorname{str}(Z)$.

We use here the same notations and conventions as those in § 4. Let $\mathscr{X}'$ be a strictly semistable formal scheme (cf. [11, 5.1]). Berkovich defined in [4, §5] the *skeleton* $S(\mathscr{X}')$. It is a closed subset of $(\mathscr{X}')^{\mathrm{an}}$, with important properties:

- There is a continuous map $\mathrm{Val} : (\mathscr{X}')^{\mathrm{an}} \to S(\mathscr{X}')$ that restricts to the identity on $S(\mathscr{X}')$.
- $S(\mathscr{X}')$ has a canonical structure of metrized simplicial set: there is a family of metrized simplicial sets $\{\Delta_S\}_{S \in \mathrm{str}(\tilde{\mathscr{X}}')}$ which covers $S(\mathscr{X}')$.

Let us describe Δ_S above a little more. By the definition of strict semistability, we can take $\pi \in \mathbb{K}^{\circ\circ} \setminus \{0\}$ and an open subset $\mathscr{U}' \subset \mathscr{X}'$ with an étale morphism

$$\phi : \mathscr{U}' \to \mathrm{Spf}\, \mathbb{K}^{\circ}\langle\langle x'_0, \ldots, x'_d \rangle\rangle / (x'_0 \ldots x'_r - \pi)$$

such that $S \cap \tilde{\mathscr{U}}'$ dominates $x'_0 \ldots x'_r = 0$. Then we have an identification

$$\left\{ \mathbf{u}' \in \mathbb{R}^{r+1}_{\geq 0} \,\middle|\, u'_0 + \cdots + u'_r = v(\pi) \right\} \cong \Delta_S.$$

Let A be an abelian variety over $\mathbb{K}$. Recall that we have a continuous map $\overline{\mathrm{val}} : A^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$, where $n = \dim A - b(A)$ and Λ is a lattice (cf. (4.0.5)). Let $\mathscr{A}$ be a Mumford model of A, $\mathscr{X}'$ a quasicompact strictly semistable formal scheme, and let $f : \mathscr{X}' \to \mathscr{A}$ be a morphism. Gubler found in [11, Proposition 5.11] a unique continuous map $\overline{f}_{\mathrm{aff}} : S(\mathscr{X}') \to \mathbb{R}^n/\Lambda$ such that $\overline{f}_{\mathrm{aff}} \circ \mathrm{Val} = \overline{\mathrm{val}} \circ f$. Let S be a stratum of $\tilde{\mathscr{X}}'$. We consider the cartesian diagram:

$$\begin{array}{ccc} \mathscr{Y}' & \xrightarrow{g} & \mathscr{E} \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathscr{X}' & \xrightarrow{f} & \mathscr{A}. \end{array}$$

Since p' is locally isomorphic, we can take an irreducible locally closed subset $T \subset \tilde{\mathscr{Y}}'$ such that $\tilde{p'}|_T : T \to S$ is an isomorphism. With this notation, we say S is *non-degenerate* with respect to f if $\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim(\Delta_S)$ and $\dim(\tilde{q} \circ \tilde{g}(T)) = \dim S$. We also say Δ_S is *non-degenerate* with respect to f if S is non-degenerate with respect to f, following Gubler's terminology (cf. [11, § 6.3]).

6.3. **Minimum of the dimension of the components of the canonical measure.** Let A be an abelian variety over $\overline{K}$ and let $X \subset A$ be an irreducible closed subvariety of dimension d. From now on, we consider only canonical metrics on line bundles on abelian varieties, hence when we write $\overline{L}$, it always means a line bundle L with a canonical metric.

Let v be a place of $\overline{K}$ and put $n := \dim A - b(A_v)$. Since we have a continuous map $\overline{\mathrm{val}} : A_v \to \mathbb{R}^n/\Lambda$, we can consider the tropicalization

$$\mu^{\mathrm{trop}}_{X_v,\overline{L}} := \overline{\mathrm{val}}_*(\mu_{X_v,\overline{L}})$$

of the canonical measure, which we call the *tropical canonical measure*. The measures $\mu_{X_v,\overline{L}}$ and $\mu^{\mathrm{trop}}_{X_v,\overline{L}}$ were studied in [11]. We first recall the explicit description obtained there:

Theorem 6.2 (The case of $L_1 = \cdots = L_d = L$ in Theorem 1.1 of [11])**.** With the notation above, suppose that L is ample. Then there are rational simplexes $\overline{\Delta}_1, \ldots, \overline{\Delta}_N$ in $\mathbb{R}^n/\Lambda$ with the following properties:

(a) $d - b(A_v) \leq \dim \overline{\Delta}_j \leq d$ for all $j = 1, \ldots, N$.
(b) $X_v^{\mathrm{trop}} = \bigcup_{j=1}^N \overline{\Delta}_j$.
(c) There are $r_1, \ldots, r_N > 0$ such that

$$\mu^{\mathrm{trop}}_{X_v, \overline{L}} = \sum_{j=1}^N r_j \delta_{\overline{\Delta}_j},$$

where $\delta_{\overline{\Delta}_j}$ is the pushforward to $\mathbb{R}^n/\Lambda$ of the canonical Lebesgue measure on $\overline{\Delta}_j$.

In general, let μ be a measure on a polytopal subset of $\mathbb{R}^n/\Lambda$ of form

$$\mu = \sum_{i=1}^N r_i \delta_{\overline{\Delta}_i} \quad (r_i > 0).$$

Then we define $\sigma(\mu)$ by

$$\sigma(\mu) := \min_i \{\dim \overline{\Delta}_i\}.$$

Let $\mathscr{X}$ be the closure of X_v in a Mumford model of A_v. We can take a semistable alteration $f : \mathscr{X}' \to \mathscr{X}$ of a model $\mathscr{X}$ of X_v by virtue of [12, Theorem 6.5]. We can write

$$c_1(f^*\overline{L})^d = \sum_S r_S \delta_{\Delta_S}$$

by [11, Corollary 6.9], where S ranges over all the non-degenerate strata of $\tilde{\mathscr{X}}'$ with respect to f, and r_S is positive. By [11, Propositions 3.9 and 5.11], we have

$$\overline{\mathrm{val}}_* \left(c_1(\overline{L})^d\right) = (\deg f)(\overline{f}_{\mathrm{aff}})_*(c_1(f^*\overline{L})^d).$$

Therefore we can write

$$\mu^{\mathrm{trop}}_{X_v, \overline{L}} = \sum_S r'_S \delta_{\overline{f}_{\mathrm{aff}}(\Delta_S)}$$

for some $r'_S > 0$. Since Δ_S is non-degenerate, we have $\dim \Delta_S = \dim f_{\mathrm{aff}}(\Delta_S)$. Taking account of $\dim \Delta_S = d - \dim S$, we thus obtain

(6.2.7)
$$\sigma\left(\mu^{\mathrm{trop}}_{X_v, \overline{L}}\right) = d - \max\left\{\dim S \,\middle|\, S \in \mathrm{str}\left(\tilde{\mathscr{X}}'\right) \text{ is non-degenerate with respect to } f\right\}.$$

6.4. **Proof.** Let us start the proof of Theorem 5.2. We argue by contradiction. Suppose there exists a counterexample X to Theorem 5.2. Then, the closed subvariety $X/G_X \subset A/G_X$ has dense small points by Lemma 2.1. That tells us that X/G_X is again a counterexample. Accordingly we may assume $G_X = 0$ and our assumption in the theorem says $d := \dim X > b(X_v)$. Since $G_X = 0$, there exists an integer $N > 0$ such that

$$\alpha : X^N \to A^{N-1}, \quad (x_1, \ldots, x_N) \mapsto (x_2 - x_1, \ldots, x_N - x_{N-1})$$

is generically finite. We put $X' := X^N$ and $Y := \alpha(X')$. The closed subvariety $X' \subset A^N$ also has dense small points by Lemma 2.4.

Let L and M be even ample line bundles on X and Y respectively. Then the line bundle $L' := L^{\boxtimes N}$ of A^N is even and ample. Let μ and ν be the tropical canonical measures on $(X'_v)^{\mathrm{trop}} = (X_v^{\mathrm{trop}})^N$ and Y_v^{trop} arising from L' and M respectively. We simply write $\hat{h}_{X'}$ and

$\hat{h}_Y$ for the canonical heights on them associated with L' and M respectively. Since X' has dense small points, we can find a generic net $(P_m)_{m\in I}$, where I is a directed set, such that $\lim_m \hat{h}_{X'}(P_m) = 0$. The image $(\alpha(P_m))_{m\in I}$ is also a generic net of Y with $\lim_m \hat{h}_Y(\alpha(P_m)) = 0$. Then by using the equidistribution theorem [10, Theorem 1.2], we can obtain

$$(\overline{\alpha}^{\mathrm{trop}})_*\mu = \nu$$

in the usual way (cf. [9, Proof of Theorem 1.1]), where $\overline{\alpha}^{\mathrm{trop}} : (X'_v)^{\mathrm{trop}} \to Y_v^{\mathrm{trop}}$ is the map between tropical varieties associated to α. (In the Gubler's article, it is denoted by $\overline{\alpha}_{\mathrm{val}}$.)

Let us take Mumford models $\mathscr{A}_1$ of A_v^N and $\mathscr{A}_2$ of A_v^{N-1} such that $\alpha : X'_v \to Y_v$ extends to the morphism of models $h : \mathscr{X}' \to \mathscr{Y}$, where $\mathscr{X}'$ is the closure of X'_v in $\mathscr{A}_1$ and $\mathscr{Y}$ is that of Y in $\mathscr{A}_2$. Let $f : \mathscr{X}'' \to \mathscr{X}'$ be a strictly semistable alteration. Then $g := h \circ f$ is also a strictly semistable alteration for $\mathscr{Y}$ since h is a generically finite surjective morphism. Let S be a stratum of $\tilde{\mathscr{X}}''$. Then, we immediately see from the definition of non-degeneracy that S is non-degenerate with respect to f if so is S with respect to g. In particular we have

$$\max\{\dim S \mid S \text{ is non-degenerate with respect to } f\} \geq \max\{\dim S \mid S \text{ is non-degenerate with respect to } g\},$$

and by (6.2.7), we find

$$\sigma(\mu) \leq \sigma(\nu). \tag{6.2.8}$$

Let us write

$$\mu^{\mathrm{trop}}_{X_v,\overline{L}} = \sum_{j=1}^N r_j \delta_{\overline{\Delta}_j}$$

as in Theorem 6.2. Renumbering them if necessary, we may assume $\dim \overline{\Delta}_1 = \sigma\left(\mu^{\mathrm{trop}}_{X_v,\overline{L}}\right)$. Since $d > b(X_v)$, which is our assumption, we have $\dim S < d$ for non-degenerate S and hence $\dim \overline{\Delta}_1 > 0$ by (6.2.7). Taking account of Lemma 4.1 and Proposition 6.1, we can write

$$\mu = \sum_{j_1,\dots,j_N} r_{j_1}\dots r_{j_N}\left(\delta_{\overline{\Delta}_{j_1}} \times \dots \times \delta_{\overline{\Delta}_{j_N}}\right) = \sum_{j_1,\dots,j_N} r_{j_1}\dots r_{j_N}\left(\delta_{\overline{\Delta}_{j_1}\times\dots\times\overline{\Delta}_{j_N}}\right).$$

The coefficients in the summation are all positive, and we have

$$\dim \overline{\Delta}_1^{\,N} = \sigma(\mu) = N\sigma\left(\mu^{\mathrm{trop}}_{X_v,\overline{L}}\right) > 0.$$

Since α contracts the diagonal of X' to the origin of A^{N-1}, we see $\overline{\alpha}^{\mathrm{trop}}$ also contracts that of $\overline{\Delta}_1^{\,N}$ to $\overline{\mathbf{0}}$. Therefore, there exists a $\sigma(\mu)$-dimensional simplex $\overline{\Delta} \subset \overline{\Delta}_1^{\,N}$ such that $\dim \overline{\alpha}^{\mathrm{trop}}(\overline{\Delta}) < \sigma(\mu)$. On the other hand, we have $\nu(\overline{\tau}) = 0$ for any simplex τ of dimension less than $\sigma(\mu)$ by (6.2.8), which says $\nu(\overline{\alpha}^{\mathrm{trop}}(\overline{\Delta})) = 0$ in particular. Accordingly, for any $\epsilon > 0$, there exists a continuous function f on Y_v^{trop} with $0 \leq f \leq 1$, $f|_{\overline{\alpha}^{\mathrm{trop}}(\overline{\Delta})} = 1$ and $\int_{Y_v^{\mathrm{trop}}} f d\nu < \epsilon$. Then we have

$$\mu(\overline{\Delta}) \leq \int_{(X'_v)^{\mathrm{trop}}} f \circ \overline{\alpha}^{\mathrm{trop}} d\mu = \int_{Y_v^{\mathrm{trop}}} f d\nu < \epsilon,$$

which says $\mu(\overline{\Delta}) = 0$. That is a contradiction, and thus we complete the proof.

Appendix by Walter Gubler. THE MINIMAL DIMENSION OF A CANONICAL MEASURE

Let K be a field with a discrete valuation v and let $\mathbb{K} = \mathbb{C}_K$ be a minimal algebraically closed field which is complete with respect to a valuation extending v. The valuation ring of $\mathbb{K}$ is denoted by $\mathbb{K}^\circ$.

We consider an irreducible d-dimensional closed subvariety X of an abelian variety A defined over $\overline{K}$. We will recall in A.4 that the Berkovich analytic space X^{an} over $\mathbb{K}$ associated to X has a canonical piecewise linear subspace T which is the support of every canonical measure on X. Let $b(X)$ be the dimension of the abelian part of X (see § 4.3). We will also use the uniformization $p : E \to A^{\mathrm{an}} = E/M$ from the Raynaud extension and the corresponding tropicalization maps $\mathrm{val} : E \to \mathbb{R}^n$ and $\overline{\mathrm{val}} : A^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$ (see § 4.2).

The goal of this appendix is to show the following result.

Theorem A.1. *There are rational simplices $\Delta_1, \dots, \Delta_N$ in T with the following five properties:*

(a) *For $j = 1, \dots, N$, we have $\dim(\Delta_j) \leq d$.*
(b) *$T = \bigcup_{j=1}^N \Delta_j$.*
(c) *The restriction of $\overline{\mathrm{val}}$ to Δ_j induces a linear isomorphism onto a simplex $\overline{\Delta}_j$ of $\mathbb{R}^n/\Lambda$.*
(d) *For canonically metrized line bundles $\overline{L_1}, \dots, \overline{L_d}$ on A, there are $r_j \in \mathbb{R}$ with*

$$c_1(\overline{L_1}|_X) \wedge \cdots \wedge c_1(\overline{L_d}|_X) = \sum_{j=1}^N r_j \cdot \delta_{\Delta_j},$$

where δ_{Δ_j} is the pushforward of the Lebesgue measure on Δ_j normalized by $\delta_{\Delta_j}(\Delta_j) = (\dim(\Delta_j)!)^{-1}$.
(e) *If all line bundles in (d) are ample, then $r_j > 0$ for all $j \in \{1, \dots, N\}$.*

For any such covering of T, we have $\min\{\dim(\Delta_j) \mid j = 1, \dots, N\} = d - b(X)$.

The proof will be given in A.6.

Corollary A.2. *Let $\overline{\Delta}_1, \dots, \Delta_N$ be the components of the tropical canonical measure $\mu^{\mathrm{trop}}_{X^{\mathrm{an}}, \overline{L}}$ considered in Theorem 6.2. Then we have*

$$\min_{j=1,\dots,N} \dim(\overline{\Delta}_j) = d - b(X).$$

Proof. The tropical canonical measure satisfies

$$\mu^{\mathrm{trop}}_{X^{\mathrm{an}}, \overline{L}} = \frac{1}{\deg_L X} \overline{\mathrm{val}}_* \left(c_1(\overline{L}|_X)^{\wedge d} \right)$$

and hence Corollary A.2 follows from Theorem A.1. □

A.3. Let $\mathscr{A}_0$ be the Mumford model of A over $\mathbb{K}^\circ$ associated to a rational polytopal decomposition $\overline{\mathscr{C}_0}$ of $\mathbb{R}^n/\Lambda$. We denote the closure of X^{an} in $\mathscr{A}_0$ by $\mathscr{X}_0$ which is a formal $\mathbb{K}^\circ$-model of X^{an}. It follows easily from de Jong's alteration theorem that there is a proper surjective morphism $\varphi_0 : \mathscr{X}' \to \mathscr{X}_0$ from a strictly semistable formal scheme $\mathscr{X}'$ over $\mathbb{K}^\circ$ whose generic fibre is an irreducible d-dimensional proper algebraic variety X' (see [11, 6.2]). The generic fibre of φ_0 is denoted by f.

A.4. The *canonical subset* T of X^{an} is defined as the support of a canonical measure $c_1(\overline{L_1}|_X) \wedge \cdots \wedge c_1(\overline{L_d}|_X)$. Similarly as in [11, Remark 6.11], the definition of T does not depend on the choice of the canonically metrized ample line bundles $\overline{L_1}, \ldots, \overline{L_d}$ of A. By [11, Theorem 6.12] T is a rational piecewise linear space. The piecewise linear structure is characterized by the fact that the restriction of f to the union of all canonical simplices which are non-degenerate with respect to f induces a piecewise linear map onto T with finite fibres. This structure does not depend on the choice of $\mathscr{A}_0$ and f in (A.3).

Theorem A.5. *Let $\varphi : \mathscr{X}' \to \mathscr{X}_0$ be a strictly semistable alteration as in A.3 with generic fibre $f : (X')^{\mathrm{an}} \to X^{\mathrm{an}}$. Then there is a $b(X)$-dimensional stratum S of $\tilde{\mathscr{X}}'$ such that the canonical simplex Δ_S of $S(\mathscr{X}')$ is non-degenerate with respect to f.*

Proof. We use the same method as in the proofs of Theorem 6.7 and Lemma 7.1 in [11]. Let $\overline{\Sigma}$ be the collection of simplices of $X^{\mathrm{trop}} = \overline{\mathrm{val}}(X^{\mathrm{an}})$ given by $\overline{f}_{\mathrm{aff}}(\Delta_S)$ together with all their closed faces where S ranges over all strata of $\tilde{\mathscr{X}}'$. There is a rational polytopal decomposition $\overline{\mathscr{C}_1}$ of $\mathbb{R}^n/\Lambda$ which is transversal to $\overline{\Sigma}$, i.e. $\overline{\Delta} \cap \overline{\sigma}$ is either empty or of dimension $\dim(\overline{\Delta}) + \dim(\overline{\sigma}) - n$ for all $\overline{\Delta} \in \overline{\mathscr{C}_1}$ and $\overline{\sigma} \in \overline{\Sigma}$. Note that the existence of such a transversal $\overline{\mathscr{C}_1}$ is much easier than the construction in [11, Lemma 6.5], and no extension of the base field is needed here.

We consider the polytopal decomposition $\overline{\mathscr{C}} := \{\overline{\Delta}_0 \cap \overline{\Delta}_1 \mid \overline{\Delta}_0 \in \overline{\mathscr{C}}_0, \overline{\Delta}_1 \in \overline{\mathscr{C}}_1\}$ which is the coarsest refinement of $\overline{\mathscr{C}_0}$ and $\overline{\mathscr{C}_1}$. Let $\mathscr{A}_1, \mathscr{A}$ be the Mumford models associated to $\overline{\mathscr{C}_1}$ and $\overline{\mathscr{C}}$. Then we get the following commutative diagram of canonical morphisms of formal schemes over $\mathbb{K}^\circ$:

$$\begin{array}{ccccc} \mathscr{X}'' & \xrightarrow{\varphi} & \mathscr{A} & \xrightarrow{\iota_1} & \mathscr{A}_1 \\ \Big\downarrow \iota' & & \Big\downarrow \iota_0 & & \\ \mathscr{X}' & \xrightarrow{\varphi_0} & \mathscr{A}_0 & & \end{array}$$

Here the formal scheme $\mathscr{X}''$ with reduced special fibre is determined by the fact that the rectangle is cartesian on the level of formal analytic varieties (see [11, 5.17]).

Let $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}$ be the $\mathbb{K}^\circ$-models of the uniformization E associated to $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}$ (see § 4.2). For $i = 1, 2$, let $\iota_i' : \mathscr{E} \to \mathscr{E}_i$ be the unique morphism extending the identity on the generic fibre. By construction, we have $\mathscr{A}_i := \mathscr{E}_i/M$ and $\mathscr{A} = \mathscr{E}/M$ with quotient morphisms p_i and p. The homomorphism $q : E \to B$ from the Raynaud extension is the generic fibre of unique morphisms $q_i : \mathscr{E} \to \mathscr{B}$ and $q : \mathscr{E} \to \mathscr{B}$. Let $\mathscr{X}_1$ (resp. $\mathscr{X}$) be the closure of X in $\mathscr{A}_1$ (resp. $\mathscr{A}$) and let $\mathscr{Y}_1 := p_1^{-1}(\mathscr{X}_1)$, $\mathscr{Y} := p^{-1}(\mathscr{X})$. By definition of $b(X)$, there is an irreducible component W_1 of $\tilde{\mathscr{Y}}_1$ with

$$\dim \tilde{q}_1(W_1) = b(X). \tag{A.5.9}$$

Since $\mathscr{Y}_1 = \iota_1'(\mathscr{Y})$, there is an irreducible component W of $\tilde{\mathscr{Y}}$ with $W_1 = \tilde{\iota}_1'(W)$. By [11, Propositions 5.7 and 5.13], there is a bijective correspondence between the vertices of the polytopal subdivision

$$\mathscr{D} := \{\Delta_S \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\Delta}) \mid S \text{ stratum of } \tilde{\mathscr{X}}' \text{ and } \overline{\Delta} \in \overline{\mathscr{C}}\}$$

of the skeleton $S(\mathscr{X}')$ and the d-dimensional strata of $\tilde{\mathscr{X}}''$. Since $\tilde{p}$ is a local isomorphism, it is clear that $\tilde{p}(W)$ is an irreducible component of $\tilde{\mathscr{X}}$. Using the fact that $\tilde{\varphi}$ is a proper surjective morphism onto $\tilde{\mathscr{X}}$, there is a d-dimensional stratum R of $\tilde{\mathscr{X}}'$ with $\tilde{\varphi}(R)$ dense in $\tilde{p}(W)$. Let u' be the vertex of $\mathscr{D}$ corresponding to R and let S be the unique stratum of $\tilde{\mathscr{X}}'$ with u' contained in the relative interior $\operatorname{relint}(\Delta_S)$.

By [11, Lemma 5.15], we have $\tilde{\iota}'(R) = S$, the map $\tilde{\varphi}_0 : S \to \tilde{\mathscr{A}_0} = \tilde{\mathscr{E}_0}/M$ has a lift $\tilde{\Phi}_0 : S \to \tilde{\mathscr{E}_0}$ and there is a unique lift $\tilde{\Phi} : R \to \tilde{\mathscr{E}}$ of $\tilde{\varphi} : R \to \tilde{\mathscr{A}} = \tilde{\mathscr{E}}/M$ with $\tilde{\Phi}_0 \circ \tilde{\iota}' = \tilde{\iota}_0' \circ \tilde{\Phi}$ on R. The lift $\tilde{\Phi}_0$ is unique up to M-translation and hence we may fix it by requiring that $\tilde{\Phi}(R)$ is dense in W. It follows that

$$\tilde{q}_0(\tilde{\Phi}_0(S)) = \tilde{q}_0 \circ \tilde{\Phi}_0 \circ \tilde{\iota}'(R) = \tilde{q}_0 \circ \tilde{\iota}_0' \circ \tilde{\Phi}(R) = \tilde{q}_1 \circ \tilde{\iota}_1' \circ \tilde{\Phi}(R) \tag{A.5.10}$$

is dense in $\tilde{q}_1(W_1)$. By (A.5.9), we get

$$\dim \tilde{q}_0(\tilde{\Phi}_0(S)) = b(X). \tag{A.5.11}$$

Since u' is a vertex of $\mathscr{D}$ contained in $\operatorname{relint}(\Delta_S)$, it is clear that

$$\dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S \tag{A.5.12}$$

(see also the argument after (25) in [11, Remark 5.17]). There is a unique $\overline{\Delta}_1 \in \overline{\mathscr{C}_1}$ with $\overline{f}_{\mathrm{aff}}(u') \in \operatorname{relint}(\overline{\Delta}_1)$. Since $\overline{f}_{\mathrm{aff}}(u')$ is also contained in $\overline{f}_{\mathrm{aff}}(\Delta_S) \in \overline{\Sigma}$, the transversality of $\overline{\mathscr{C}_1}$ and $\overline{\Sigma}$ yields

$$\operatorname{codim} \overline{\Delta}_1 \leq \dim \overline{f}_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S. \tag{A.5.13}$$

By [11, Proposition 5.14], $\tilde{\iota}_1 \circ \tilde{\varphi}(R)$ is contained in the stratum of $\tilde{\mathscr{A}_1}$ corresponding to $\operatorname{relint}(\overline{\Delta}_1)$. This correspondence is described in [11, Proposition 4.8], showing also that $W_1^\circ := \tilde{\iota}_1' \circ \tilde{\Phi}(R)$ is contained in the stratum Z_{Δ_1} of $\tilde{\mathscr{E}_1}$ corresponding to $\operatorname{relint}(\Delta_1)$ for a suitable polytope Δ_1 of $\mathbb{R}^n$ with image $\overline{\Delta}_1$ in $\mathbb{R}^n/\Lambda$. By [11, Remark 4.9], this stratum is a torsor $\tilde{q}_1 : Z_{\Delta_1} \to \tilde{\mathscr{B}}$ with fibres isomorphic to a torus of dimension equal to $\operatorname{codim}(\Delta_1)$. Since $\tilde{\Phi}(R)$ is dense in W, it follows that W_1° is dense in W_1. We conclude that W_1° is contained in a fibre bundle over $\tilde{q}_1(W_1^\circ)$ with $\operatorname{codim}(\Delta_1)$-dimensional fibres. This and (A.5.10) yield

$$\dim S \geq \dim \tilde{q}_0(\tilde{\Phi}_0(S)) = \dim \tilde{q}_1(W_1^\circ) \geq \dim W_1 - \operatorname{codim} \Delta_1. \tag{A.5.14}$$

Since W_1 is an irreducible component of $\tilde{\mathscr{Y}_1}$, we have $\dim W_1 = d$. By (A.5.13), we get

$$\dim W_1 - \operatorname{codim} \Delta_1 \geq d - \dim \Delta_S = \dim S.$$

We conclude that equality occurs everywhere in (A.5.14) proving

$$\dim S = \dim \tilde{q}_0(\tilde{\Phi}_0(S)). \tag{A.5.15}$$

By (A.5.11) and (A.5.15), the canonical simplex Δ_S is non-degenerate with respect to f. Using (A.5.11) and (A.5.15), we conclude that S is a $b(X)$-dimensional stratum of $\tilde{\mathscr{X}}'$. □

A.6. It remains to proof Theorem A.1. We choose a strictly semistable alteration $\varphi_0 : \mathscr{X}' \to \mathscr{X}_0$ as in A.3 with generic fibre $f : (X')^{\mathrm{an}} \to X^{\mathrm{an}}$. Moreover, we may assume that the restriction of f to Δ_S is a linear isomorphism onto a rational simplex of the canonical subset T for all canonical simplices Δ_S of $S(\mathscr{X}')$ which are non-degenerate with respect to f (see the proof of [11, Theorem 6.12]). We number these simplices of T by $\Delta_1, \ldots, \Delta_N$. By projection formula ([11, Proposition 3.8]), we have

$$f_* \left(c_1(f^*\overline{L_1}|_X) \wedge \cdots \wedge c_1(f^*\overline{L_d}|_X) \right) = \deg(f) c_1(\overline{L_1}|_X) \wedge \cdots \wedge c_1(\overline{L_d}|_X).$$

By [11, Theorem 6.7 and Remark 6.8], there are numbers r_S with

$$c_1(f^*\overline{L_1}|_X) \wedge \cdots \wedge c_1(f^*\overline{L_d}|_X) = \sum_S r_S \delta_{\Delta_S}$$

where S ranges over all strata of $\tilde{\mathscr{X}}'$ such that the canonical simplex Δ_S of the skeleton $S(\mathscr{X}')$ is non-degenerate with respect to f. Note that the numbers r_S are positive if all line bundles are ample. This yields already properties (a)–(e) in Theorem A.1 and the last claim follows from Theorem A.5. □

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Department of Mechanical Engineering and Science, Graduate School of Engineering, Kyoto University, Kyoto, 606-8501, Japan

E-mail address: yamaki@math.kyoto-u.ac.jp