AN OPEN FOUR-MANIFOLD HAVING NO INSTANTON

MASAKI TSUKAMOTO

Abstract. Taubes proved that all compact oriented four-manifolds admit non-flat instantons. We show that there exists a non-compact oriented four-manifold having no non-flat instanton.

1. introduction

Taubes [15] proved that all compact oriented Riemannian 4-manifolds admit non-flat instantons. To be precise, if X is a compact oriented Riemannian 4-manifold then there exists a principal $SU(2)$-bundle E on X which admits a non-flat anti-self-dual (ASD) connection. (Taubes [15] considered self-dual connections. But recently people usually study anti-self-dual ones. So we consider anti-self-dual connections in this paper.) The purpose of this paper is to show that an analogue of this striking existence theorem does *not* hold for general non-compact 4-manifolds.

Let $(\mathbb{C}P^2)^{\sharp\mathbb{Z}}$ be the connected sum of the infinite copies of the complex projective plane $\mathbb{C}P^2$ indexed by integers. (The precise definition of this infinite connected sum will be given in Section 2.1.) $(\mathbb{C}P^2)^{\sharp\mathbb{Z}}$ is a non-compact oriented 4-manifold.

Theorem 1.1. *There exists a complete Riemannian metric g on $(\mathbb{C}P^2)^{\sharp\mathbb{Z}}$ satisfying the following. If A is a g-ASD connection on a principal $SU(2)$-bundle over $(\mathbb{C}P^2)^{\sharp\mathbb{Z}}$ satisfying*

$$\int_X |F_A|_g^2 d\mathrm{vol}_g < +\infty, \tag{1}$$

then A is flat. Here F_A is the curvature of A. $|\cdot|_g$ and $d\mathrm{vol}_g$ are the norm and the volume form with respect to the metric g. A connection A is said to be g-ASD if it satisfies $*_g F_A = -F_A$ where $*_g$ is the Hodge star with respect to g.

For a more general and precise statement, see Theorem 2.1. As far as I know, this is the first example of oriented Riemannian 4-manifolds which cannot admit any non-flat instanton.

Remark 1.2. I think that the following question is still open: Is there an oriented Riemannian 4-manifold which does not have any non-flat ASD connection (not necessarily

Date: April 21, 2010.

2000 *Mathematics Subject Classification.* 53C07.

Key words and phrases. Yang-Mills theory, instanton, open four-manifold, infinite connected sum.

satisfying the finite energy condition (1))? We studied infinite energy ASD connections and their infinite dimensional moduli spaces in [11], [17], [18].

A naive idea toward the proof of Theorem 1.1 is as follows. Let g be a Riemannian metric on $(\mathbb{C}P^2)^{\sharp\mathbb{Z}}$. For each integer $n \geq 0$, let $M(n,g)$ be the moduli space of $SU(2)$ g-ASD connections on $(\mathbb{C}P^2)^{\sharp\mathbb{Z}}$ satisfying $\int_{(\mathbb{C}P^2)^{\sharp\mathbb{Z}}} |F_A|_g^2 d\mathrm{vol}_g = 8\pi^2 n$. We have $b_1((\mathbb{C}P^2)^{\sharp\mathbb{Z}}) = 0$ and, formally, $b_+((\mathbb{C}P^2)^{\sharp\mathbb{Z}}) = +\infty$. Therefore, if we formally apply the usual virtual dimension formula [5, Section 4.2.5] to $M(n,g)$, we get

$$\dim M(n,g) = 8n - 3(1 - b_1((\mathbb{C}P^2)^{\sharp\mathbb{Z}}) + b_+((\mathbb{C}P^2)^{\sharp\mathbb{Z}})) = 8n - \infty = -\infty.$$

This suggests the following observation: If we can achieve the transversality of the moduli spaces $M(n,g)$ by choosing the metric g sufficiently generic, then all $M(n,g)$ $(n \geq 1)$ become empty. ($M(0,g)$ is the moduli space of flat $SU(2)$ connections, and it does not depend on the choice of a Riemannian metric.)

Acknowledgement. I wish to thank Professor Kenji Fukaya most sincerely for his help and encouragement. I was supported by Grant-in-Aid for Young Scientists (B) (21740048).

2. Infinite connected sum

2.1. **Construction.** Let Y be a simply-connected compact oriented 4-manifold. Let $x_1, x_2 \in Y$ be two distinct points, and set $\hat{Y} := Y \setminus \{x_1, x_2\}$. Choose a Riemannian metric h on $\hat{Y}$ which becomes a tubular metric on the end (i.e. around x_1 and x_2). This means that there is a compact set $K \subset \hat{Y}$ such that $\hat{Y} \setminus K = Y_- \sqcup Y_+$ with $Y_- = (-\infty, -1) \times S^3$ and $Y_+ = (1, +\infty) \times S^3$. Here "=" means that they are isomorphic as oriented Riemannian manifolds. ($S^3 = S^3(1) = \{x \in \mathbb{R}^4 | \, |x| = 1\}$ is endowed with the Riemannian metric induced by the standard Euclidean metric on $\mathbb{R}^4$.) We can suppose that there is a smooth function $p : \hat{Y} \to \mathbb{R}$ satisfying the following conditions: $p(K) = [-1,1]$. p is equal to the projection to $(-\infty, -1)$ on $Y_- = (-\infty, -1) \times S^3$, and p is equal to the projection to $(1, +\infty)$ on $Y_+ = (1, +\infty) \times S^3$. For $T > 2$, we set $Y_T := p^{-1}(-T+1, T-1) = (-T+1, -1) \times S^3 \cup K \cup (1, T-1) \times S^3$. (Later we will choose T large.)

Let $Y^{(n)}$ be the copies of Y indexed by integers $n \in \mathbb{Z}$. We denote $K^{(n)}$, $Y_-^{(n)}$, $Y_+^{(n)}$, $p^{(n)}$, $Y_T^{(n)}$ as the copies of K, Y_-, Y_+, p, Y_T. ($K^{(n)}, Y_-^{(n)}, Y_+^{(n)}, Y_T^{(n)} \subset Y^{(n)}$ and $p^{(n)} : Y^{(n)} \to \mathbb{R}$.) We define $X = Y^{\sharp\mathbb{Z}}$ by

$$X := \bigsqcup_{n \in \mathbb{Z}} Y_T^{(n)} / \sim,$$

where we identify $Y_T^{(n)} \cap Y_+^{(n)}$ with $Y_T^{(n+1)} \cap Y_-^{(n+1)}$ by

$$\begin{aligned} Y_T^{(n)} \cap Y_+^{(n)} = (1, T-1) \times S^3 \ni (t, \theta) \\ \sim (t - T, \theta) \in (-T+1, -1) \times S^3 = Y_T^{(n+1)} \cap Y_-^{(n+1)}. \end{aligned} \tag{2}$$

We define $q : X \to \mathbb{R}$ by setting $q(x) := nT + p^{(n)}(x)$ on $Y_T^{(n)}$. This is compatible with the above identification (2). The identification (2) is an orientation preserving isometry. Hence X has an orientation and a Riemannian metric which coincide with the given ones over $Y_T^{(n)}$. We denote the Riemannian metric on X (given by this procedure) by g_0. g_0 depends on the Riemannian metric h on Y and the parameter T.

Since Y is simply-connected, X is also simply-connected. The homology groups of X are given as follows:

$$H_0(X) = \mathbb{Z}, \quad H_1(X) = 0, \quad H_2(X) \cong H_2(Y)^{\oplus \mathbb{Z}}, \quad H_3(X) = \mathbb{Z}, \quad H_4(X) = 0.$$

$H_2(X)$ is of infinite rank if $b_2(Y) \geq 1$. For every $n \in \mathbb{Z}$, the inclusion $Y_T^{(n)} \cap Y_+^{(n)} \subset X$ induces an isomorphism $H_3(Y_T^{(n)} \cap Y_+^{(n)}) \cong H_3(X)$. The fundamental class of the cross-section $S^3 \subset Y_T^{(n)} \cap Y_+^{(n)} = (1, T-1) \times S^3$ becomes a generator of $H_3(X)$.

2.2. **Statement of the main theorem.** Theorem 1.1 in Section 1 follows from the following theorem.

Theorem 2.1. *Suppose $b_-(Y) = 0$ and $b_+(Y) \geq 1$. If T is sufficiently large, then there exists a complete Riemannian metric g on $X = Y^{\sharp \mathbb{Z}}$ satisfying the following conditions (a) and (b).*
(a) g is equal to the periodic metric g_0 (defined in Section 2.1) outside a compact set.
(b) If A is a g-ASD connection on a principal $SU(2)$ bundle E on X satisfying

$$\int_X |F_A|_g^2 d\mathrm{vol}_g < \infty, \tag{3}$$

then A is flat.

The proof of this theorem will be given in Section 9.

Remark 2.2. (i) If a Riemannian metric g on X satisfies the condition (a), then it is complete.
(ii) From the condition (a), the above (3) is equivalent to

$$\int_X |F_A|_{g_0}^2 d\mathrm{vol}_{g_0} < \infty.$$

(iii) Since X is non-compact, all principal $SU(2)$-bundles on it are isomorphic to the product bundle $X \times SU(2)$. Hence we can assume that the principal $SU(2)$-bundle E in the condition (b) is equal to the product bundle $X \times SU(2)$.

2.3. **Ideas of the proof of Theorem 2.1.** In this subsection we explain the ideas of the proof of Theorem 2.1. Here we ignore several technical issues. Hence the real proof is different from the following argument in many points.

Let g be a Riemannian metric on X which is equal to g_0 outside a compact set. Let $E = X \times SU(2)$ be the product principal $SU(2)$-bundle over X. If a g-ASD connection

A on E satisfies $\int_X |F_A|_g^2 d\mathrm{vol}_g < \infty$, then we can show that $\frac{1}{8\pi^2}\int_X |F_A|_g^2 d\mathrm{vol}_g$ is a non-negative integer. For each integer $n \geq 0$, we define $M(n, g)$ as the moduli space of g-ASD connections A on E satisfying $\frac{1}{8\pi^2}\int_X |F_A|_g^2 d\mathrm{vol}_g = n$. Take $[A] \in M(n, g)$. We want to study a local structure of $M(n, g)$ around $[A]$.

Set $D_A := -d_A^* + d_A^{+_g} : \Omega^1(\mathrm{ad}E) \to (\Omega^0 \oplus \Omega^{+_g})(\mathrm{ad}E)$. Here d_A^* is the formal adjoint of $d_A : \Omega^0(\mathrm{ad}E) \to \Omega^1(\mathrm{ad}E)$ with respect to g_0, and $d_A^{+_g}$ is the g-self-dual part of $d_A : \Omega^1(\mathrm{ad}E) \to \Omega^2(\mathrm{ad}E)$. (Indeed we need to use appropriate weighted Sobolev spaces, and the definition of D_A should be modified with the weight. But here we ignore these points.) The equation $d_A^* a = 0$ for $a \in \Omega^1(\mathrm{ad}E)$ is the Coulomb gauge condition, and the equation $d_A^{+_g} a = 0$ is the linearization of the ASD equation $F^{+_g}(A + a) = 0$. Therefore we expect that we can get an information on the local structure of $M(n, g)$ from the study of the operator D_A. The most important point of the proof is to show the following three properties of D_A. (In other words, we need to choose an appropriate functional analysis setup in order to establish these properties.)
(i) The kernel of D_A is finite dimensional.
(ii) The image of D_A is closed in $(\Omega^0 \oplus \Omega^{+_g})(\mathrm{ad}E)$.
(iii) The cokernel of D_A is infinite dimensional.

Then the local model (i.e. the Kuranishi description) of $M(n, g)$ around $[A]$ is given by the zero set of a map

$$f : \mathrm{Ker}D_A \to \mathrm{Coker}D_A.$$

(Rigorously speaking, the map f is defined only in a small neighborhood of the origin.) From the conditions (i) and (iii), this is a map from the finite dimensional space to the infinite dimensional one. Therefore (we can hope that) if we perturb the map f appropriately, then the zero set disappear. The parameter g gives sufficient perturbation, and we can prove that $M(n, g)$ is empty for $n \geq 1$ and generic g.

Organization of the paper: In Section 3.1, we review the basic facts on anti-self-duality and conformal structure. In the above arguments we considered the moduli spaces $M(n, g)$ parametrized by Riemannian metrics g. But ASD equation depends only on conformal structures, and hence technically it is better to parameterize ASD moduli spaces by conformal structures. Section 3.1 is a preparation for this consideration. In Section 3.2, we prepare some estimates relating to the Laplacians.

In Section 4 we study the decay behavior of instantons over X, and show that they decay "sufficiently fast". This is important in showing that all instantons can be "captured" by the functional analysis setups constructed in Sections 6 and 8.3.

Section 5 is a preparation for Section 6. In Section 6 we study a (modified version of) operator $D_A = -d_A^* + d_A^{+_d}$ and establish the above mentioned properties (i), (ii), (iii).

In Section 7 we show that there is no non-flat reducible instantons on E. Here the condition $b_-(Y) = 0$ is essentially used.

Sections 8.1 and 8.2 are preparations for the perturbation argument in Section 8.3. In Section 8.3 we establish a transversality by using Freed-Uhlenbeck's metric perturbation. Here we use the results established in Sections 6 and 7. Combining the results in Sections 4 and 8.3, we prove Theorem 2.1 in Section 9.

3. Some preliminaries

3.1. **Anti-self-duality and conformal structure.** In this subsection we review some well-known facts on the relation between anti-self-duality and conformal structure. Specialists of the gauge theory don't need to read the details of the arguments in this subsection. The references are Donaldson-Sullivan [6, pp. 185-187] and Donaldson-Kronheimer [5, pp. 7-8].

We start with a linear algebra. Let V be an oriented real 4-dimensional linear space. We fix an inner product g_0 on V. The orientation and inner product give a natural isomorphism $\Lambda^4(V) \cong \mathbb{R}$, and we define a quadratic form $Q : \Lambda^2(V) \times \Lambda^2(V) \to \mathbb{R}$ by $Q(\xi, \eta) := \xi \wedge \eta \in \Lambda^4(V) \cong \mathbb{R}$. The dimensions of maximal positive subspaces and maximal negative subspaces with respect to Q are both 3

Let g and g' be two inner products on V. They are said to be conformally equivalent if there is $c > 0$ such that $g_2 = cg_1$. Let $\mathrm{Conf}(V)$ be the set of all conformal equivalence classes of inner-products on V. $\mathrm{Conf}(V)$ naturally admits a smooth manifold structure.

We define $\mathrm{Conf}'(V)$ as the set of all 3-dimensional subspaces $U \subset \Lambda^2(V)$ satisfying $Q(\omega, \omega) < 0$ for all non-zero $\omega \in U$. $\mathrm{Conf}'(V)$ depends on the orientation of V, but it is independent of the choice of the inner product g_0. $\mathrm{Conf}'(V)$ is an open set of the Grassmann manifold $Gr_3(\Lambda^2(V))$, and hence it is also a smooth manifold.

Let Λ^+ be the space of $\omega \in \Lambda^2(V)$ which is self-dual with respect to g_0, and Λ^- be the space of $\omega \in \Lambda^2(V)$ which is anti-self-dual with respect to g_0. We define $\mathrm{Conf}''(V)$ be the set of linear map $\mu : \Lambda^- \to \Lambda^+$ satisfying $|\mu| < 1$ (i.e. $|\mu(\omega)| < |\omega|$ for all non-zero $\omega \in \Lambda^-$ where the norm $|\cdot|$ is defined by g_0). This is also a smooth manifold as an open set of $\mathrm{Hom}(\Lambda^-, \Lambda^+)$. The map

$$\mathrm{Conf}''(V) \to \mathrm{Conf}'(V), \quad \mu \mapsto \{\omega + \mu(\omega) | \, \omega \in \Lambda^-\}$$

is a diffeomorphism. Hence $\mathrm{Conf}'(V)$ is contractible. (In particular it is connected.)

Lemma 3.1. *The map*

$$\mathrm{Conf}(V) \to \mathrm{Conf}'(V), \quad [g] \mapsto \{\omega \in \Lambda^2(V) | \, \omega \text{ is anti-self-dual with respect to } g\}, \tag{4}$$

is a diffeomorphism.

Proof. For $A \in SL(V)$ and $[g] \in \mathrm{Conf}(V)$ we define $[Ag] \in \mathrm{Conf}(V)$ by setting $(Ag)(u, v) := g(A^{-1}u, A^{-1}v)$. In this manner $SL(V)$ transitively acts on $\mathrm{Conf}(V)$, and the isotropy subgroup at $[g_0]$ is equal to $SO(V) = SO(V, g_0)$. Hence $\mathrm{Conf}(V) \cong SL(V)/SO(V)$. On the other hand, the Lie group $SO(\Lambda^2(V), Q)$ ($\cong SO(3, 3)$) naturally acts on $\mathrm{Conf}'(V)$. This action is transitive. (For $U \in \mathrm{Conf}'(V)$ set $U' := \{\omega \in \Lambda^2(V) | Q(\omega,\eta) = 0\ (\forall \eta \in U)\}$. $\Lambda^2(V) = U \oplus U'$. Q is negative definite on U and positive definite on U'. By choosing orthonormal bases on U and U' with respect to Q, we can construct $A \in SO(\Lambda^2(V), Q)$ satisfying $A(\Lambda^-) = U$.)

Let $SO(\Lambda^2(V), Q)_0$ be the identity component of $SO(\Lambda^2(V), Q)$. Since $\mathrm{Conf}'(V)$ is connected, $SO(\Lambda^2(V), Q)_0$ also transitively acts on $\mathrm{Conf}'(V)$. The isotropy group of this action at $\Lambda^- \in \mathrm{Conf}'(V)$ is equal to $SO(\Lambda^+) \times SO(\Lambda^-)$. (It is easy to see that if $A \in SO(\Lambda^2(V), Q)_0$ fixes Λ^- then it also fixes Λ^+. Hence $A \in O(\Lambda^+) \times O(\Lambda^-)$. Since $\mathrm{Conf}'(V)$ is contractible, the isotropy subgroup must be connected. Therefore $A \in SO(\Lambda^+) \times SO(\Lambda^-)$.) Thus $\mathrm{Conf}'(V) \cong SO(\Lambda^2(V), Q)_0 / SO(\Lambda^+) \times SO(\Lambda^-)$

$SL(V)$ naturally acts on $\Lambda^2(V)$, and it preserves the quadratic form Q. Hence we have a homomorphism $f : SL(V) \to SO(\Lambda^2(V), Q)_0$. A direct calculation shows that it induces an isomorphism between their Lie algebras. Hence the homomorphism $f : SL(V) \to SO(\Lambda^2(V), Q)_0$ is a (surjective) covering map. $f^{-1}(SO(\Lambda^+) \times SO(\Lambda^-))$ is equal to $SO(V)$. (It is easy to see that $SO(V) \subset f^{-1}(SO(\Lambda^+) \times SO(\Lambda^-))$ and their dimensions are both 6. $SL(V)$ is connected and $SL(V)/f^{-1}(SO(\Lambda^+) \times SO(\Lambda^-)) \cong SO(\Lambda^2(V), Q)_0/SO(\Lambda^+) \times SO(\Lambda^-) \cong \mathrm{Conf}'(V)$ is contractible. Hence $f^{-1}(SO(\Lambda^+) \times SO(\Lambda^-))$ must be connected. Therefore it is equal to $SO(V)$.) Thus $SL(V)/SO(V) \cong SO(\Lambda^2(V), Q)_0/SO(\Lambda^+) \times SO(\Lambda^-)$. This gives a diffeomorphism $\mathrm{Conf}(V) \cong \mathrm{Conf}'(V)$, and this diffeomorphism coincides with the above map (4). $\square$

Let M be an oriented 4-manifold (not necessarily compact), and g_0 be a smooth Riemannian metric on M. Two Riemannian metrics g and g' on M are said to be conformally equivalent if there is a positive function $\varphi : M \to \mathbb{R}$ satisfying $g' = \varphi g$. Let $\mathrm{Conf}(M)$ be the set of all conformal equivalence classes of $\mathcal{C}^\infty$-Riemannian metrics on M.

Let Λ^+ and Λ^- be the sub-bundles of $\Lambda^2 := \Lambda^2(T^*M)$ consisting of self-dual and anti-self-dual 2-forms with respect to g_0. For $[g] \in \mathrm{Conf}(M)$ we define a sub-bundle $\Lambda^-_g \subset \Lambda^2$ as the set of anti-self-dual 2-forms with respect to g. There is a $\mathcal{C}^\infty$-bundle map $\mu_g : \Lambda^- \to \Lambda^+$ such that $|(\mu_g)_x| < 1$ $(x \in M)$ and that Λ^-_g is equal to the graph $\{\omega + \mu_g(\omega) | \omega \in \Lambda^-\}$. Here $|(\mu_g)_x| < 1$ $(x \in M)$ means that $|\mu_g(\omega)| < |\omega|$ for all non-zero $\omega \in \Lambda^-$. ($|\cdot|$ is the norm defined by g_0.) From the previous argument, we get the following result.

Corollary 3.2. *The map*

$$\mathrm{Conf}(M) \to \{\mu : \Lambda^- \to \Lambda^+ : \mathcal{C}^\infty\text{-bundle map} | \, |\mu_x| < 1\ (x \in M)\}, \quad [g] \mapsto \mu_g,$$

is bijective.

3.2. **Eigenvalues of the Laplacians on differential forms over S^3.** We will sometimes need estimates relating to lower bounds on the eigenvalues of the Laplacians on S^3. Here the 3-sphere S^3 is endowed with the Riemannian metric induced by the inclusion

$S^3 = \{x \in \mathbb{R}^4 | \, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \subset \mathbb{R}^4$. ($\mathbb{R}^4$ has the standard Euclidean metric.) The formal adjoint of $d : \Omega^i \to \Omega^{i+1}$ is denoted by $d^* : \Omega^{i+1} \to \Omega^i$.

Lemma 3.3. *The first non-zero eigenvalue of the Laplacian $\Delta = d^*d$ acting on functions over S^3 is* 3.

Proof. See Sakai [13, p. 272, Proposition 3.13]. $\square$

Lemma 3.4. *Let* $\mathrm{Ker}(d^*) \subset \Omega^1$ *be the space of* 1*-forms a over S^3 satisfying $d^*a = 0$. Then the first eigenvalue of the Laplacian $\Delta = d^*d + dd^*$ acting on* $\mathrm{Ker}(d^*)$ *is* 4.

Proof. See Donaldson-Kronheimer [5, p. 310, Lemma (7.3.4)]. $\square$

As a corollary we get the following. (This is given in [5, p. 310, Lemma (7.3.4)].)

Corollary 3.5. *(i) Let a be a smooth* 1*-form over S^3 satisfying $d^*a = 0$. Then*

$$\int_{S^3} |a|^2 d\mathrm{vol} \le \frac{1}{4} \int_{S^3} |da|^2 d\mathrm{vol}.$$

(ii) For any smooth 1*-form a on S^3, we have*

$$\left| \int_{S^3} a \wedge da \right| \le \frac{1}{2} \int_{S^3} |da|^2 d\mathrm{vol}.$$

Proof. (i) $\int |da|^2 = \int \langle a, \Delta a \rangle \ge 4 \int |a|^2$.
(ii) There is a smooth function f on S^3 such that $b := a - df$ satisfies $d^*b = 0$. Then $\left|\int a \wedge da\right| = \left|\int b \wedge db\right| \le \sqrt{\int |b|^2}\sqrt{\int |db|^2} \le \frac{1}{2} \int |db|^2 = \frac{1}{2} \int |da|^2$. $\square$

4. Decay estimate of instantons

4.1. **Classification of adapted connections.** Let us go back to the situation of Section 2.1. Y is a simply connected, compact oriented 4-manifold, and $X = Y^{\sharp\mathbb{Z}}$ is the connected sum of the infinite copies of Y indexed by $\mathbb{Z}$. Since X is non-compact, every principal $SU(2)$-bundle on it is isomorphic to the product bundle $E = X \times SU(2)$. Following Donaldson [4, Definition 3.5], we make the following definition.

Definition 4.1. An adapted connection A on E is a connection on E which is flat outside a compact set. (That is, there is a compact set $L \subset X$ such that $F_A = 0$ over $X \setminus L$.) Two adapted connections A_1 and A_2 on E are said to be equivalent as adapted connections if there is a gauge transformation $u : E \to E$ such that $u(A_1)$ is equal to A_2 outside a compact set.

For $m \in \mathbb{Z}$, let $u_m : X \to SU(2)$ be a smooth map such that $(\rho_m)_* : H_3(X) \to H_3(SU(2))$ satisfies $(\rho_m)_*([S^3]) = m[SU(2)]$. (Here $[S^3]$ is the fundamental class of the cross-section $S^3 \subset Y_T^{(n)} \cap Y_+^{(n)}$, and it is a generator of $H_3(X) \cong \mathbb{Z}$. See Remark 4.2 below.) This means that the restriction of u_m to the cross-section $S^3 \subset Y_T^{(n)} \cap Y_+^{(n)}$ becomes a map of degree m from S^3 to $SU(2)$ (for every $n \in \mathbb{Z}$).

Remark 4.2. The cross-section $S^3 \subset Y_T^{(n)} \cap Y_+^{(n)}$ is endowed with the orientation so that the identification $Y_T^{(n)} \cap Y_+^{(n)} = (1, T-1) \times S^3$ is orientation preserving. (The interval $(1, T-1)$ has the standard orientation.) The orientation on the Lie group $SU(2)$ is chosen as follows: Let $\theta \in \Omega^1 \otimes su(2)$ be the left invariant 1-form (on $SU(2)$) valued in the Lie algebra $su(2)$ satisfying $\theta(X) = X$ for all $X \in su(2) = T_1 SU(2)$. (In the standard notation, we can write $\theta = g^{-1}dg$ for $g \in SU(2)$.) We choose the orientation on $SU(2)$ so that

$$\frac{1}{8\pi^2}\int_{SU(2)} tr\left(\theta \wedge d\theta + \frac{2}{3}\theta^3\right) = \frac{-1}{24\pi^2}\int_{SU(2)} tr(\theta^3) = 1. \tag{5}$$

Since E is the product bundle, u_m becomes a gauge transformation of E. Let ρ be the product flat connection on $E = X \times SU(2)$, and set $\rho_m := u_m^{-1}(\rho)$. Let $A(m)$ be a connection on E which is equal to ρ over $q^{-1}(-\infty, -1)$ and equal to ρ_m over $q^{-1}(1, +\infty)$. $A(m)$ is an adapted connection on E. For $t > 1$ we have

$$\frac{1}{8\pi^2}\int_X tr(F(A(m))^2) = \frac{1}{8\pi^2}\int_{q^{-1}(t)} u_m^*\left(tr(\theta \wedge d\theta + \frac{2}{3}\theta^3)\right) = m. \tag{6}$$

Here we have used (5) and $\deg(u_m|_{q^{-1}(t)} : q^{-1}(t) \to SU(2)) = m$.

Proposition 4.3. *For $m_1 \neq m_2$, $A(m_1)$ and $A(m_2)$ are not equivalent as adapted connections. If A is an adapted connection on E, then A is equivalent to $A(m)$ as an adapted connection where*

$$m = \frac{1}{8\pi^2}\int_X trF_A^2.$$

(An important point for us is that there are only countably many equivalence classes of adapted connections.)

Proof. The first statement follows from the equation (6).

Let A be an adapted connection on E. There is $M > 0$ such that A is flat on $q^{-1}(-\infty, -M]$ and $q^{-1}[M, \infty)$. We choose $M > 1$ so that $q^{-1}(M) = S^3 \subset Y_T^{(n)} \cap Y_+^{(n)}$ and $q^{-1}(-M) = S^3 \subset Y_T^{(-n)} \cap Y_-^{(-n)}$ for some $n > 0$. Since $q^{-1}(-\infty, -M]$ and $q^{-1}[M, \infty)$ are simply connected, there are gauge transformations u on $q^{-1}(-\infty, -M]$ and u' on $q^{-1}[M, \infty)$ such that $u(A) = \rho$ and $u'(A) = \rho$. We can extend u all over X. Hence we can suppose that $u = 1$ and that A is equal to ρ over $q^{-1}(-\infty, -M]$. Set $m := \deg(u'|_{q^{-1}(M)} : q^{-1}(M) \to SU(2))$. The degree of the map $(u_m^{-1}u')|_{q^{-1}(M)} : q^{-1}(M) \to SU(2)$ is zero. Then there is a gauge transformation u'' of E such that $u'' = u_m^{-1}u'$ on $q^{-1}[M, +\infty)$ and $u'' = 1$ on $q^{-1}(-\infty, M-1)$. Then $u''(A)$ is equal to ρ over $q^{-1}(-\infty, -M]$ and equal to $u_m^{-1}(\rho)$ over $q^{-1}[M, +\infty)$. Hence $u''(A)$ is equal to $A(m)$ outside a compact set. We have

$$m = \frac{1}{8\pi^2}\int_X trF(A(m))^2 = \frac{1}{8\pi^2}\int_X trF_A^2.$$

□

4.2. **Preliminaries for the decay estimate.** We need the following. (This is a special case of [9, Proposition 3.1, Remark 3.2].)

Proposition 4.4. *Let Z be a simply-connected compact Riemannian 4-manifold with (or without) boundary, and $W \subset Z$ be a compact subset with $W \cap \partial Z = \emptyset$. Then there are positive numbers $\varepsilon_1(W, Z)$ and $C_{1,k}(W, Z)$ $(k \geq 0)$ satisfying the following: Let A be an ASD connection on the product principal $SU(2)$-bundle over Z satisfying $\|F_A\|_{L^2(Z)} \leq \varepsilon_1(W, Z)$. Then A can be represented by a connection matrix $\tilde{A}$ over a neighborhood of W satisfying*

$$\left\|\tilde{A}\right\|_{\mathcal{C}^k(W)} \leq C_{1,k} \left\|F_A\right\|_{L^2(Z)},$$

for all $k \geq 0$.

Proof. See Fukaya [9, Proposition 3.1, Remark 3.2]. □

Lemma 4.5. *Let $L > 2$. There exist positive numbers ε_2 and $C_{2,k}$ $(k \geq 0)$ independent of L satisfying the following. If A is an ASD connection on the product principal $SU(2)$-bundle G over $(0, L) \times S^3$ satisfying $\|F_A\|_{L^2((0,L)\times S^3)} \leq \varepsilon_2$, then A can be represented by a connection matrix $\tilde{A}$ over a neighborhood of $[1, L-1] \times S^3$ satisfying*

$$|\nabla^k \tilde{A}(t, \theta)| \leq C_{2,k} \left\|F_A\right\|_{L^2((t-1,t+1)\times S^3)}, \tag{7}$$

for $(t, \theta) \in [1, L-1] \times S^3$ and $k \geq 0$.

Proof. Proposition 4.4 implies the following. There exist positive numbers ε_2' and $C_{2,k}'$ $(k \geq 0)$ such that if B is an ASD connection on the product principal $SU(2)$-bundle over $[0, 1] \times S^3$ satisfying $\|F_B\|_{L^2([0,1]\times S^3)} \leq \varepsilon_2'$ then B can be represented by a connection matrix $\tilde{B}$ over a neighborhood of $[1/4, 3/4] \times S^3$ satisfying

$$|\nabla^k \tilde{B}(x)| \leq C_{2,k}' \left\|F_B\right\|_{L^2([0,1]\times S^3)} \quad (x \in [1/4, 3/4] \times S^3,\ k \geq 0).$$

Let ε_2 be a small positive number with $\varepsilon_2 < \varepsilon_2'$. We will fix ε_2 later. Suppose that A is an ASD connection on the product principal $SU(2)$-bundle G over $(0, L) \times S^3$ satisfying $\|F_A\|_{L^2((0,L)\times S^3)} \leq \varepsilon_2$. For $2 \leq n \leq [4L - 4]$, set $I_n := [n/4, n/4 + 1/2]$ and $J_n := [n/4 - 1/4, n/4 + 3/4]$. We have $I_n \subset J_n$. For each n, there is a local trivialization h_n of G over a neighborhood of $I_n \times S^3$ such that the connection matrix $A_n := h_n(A)$ satisfies

$$|\nabla^k A_n(x)| \leq C_{2,k}' \left\|F_A\right\|_{L^2(J_n\times S^3)} \quad (x \in I_n \times S^3,\ k \geq 0).$$

Set $g_n := h_{n+1}h_n^{-1} : (I_n \cap I_{n+1}) \times S^3 \to SU(2)$. Then $g_n(A_n) = A_{n+1}$ (i.e. $dg_n = g_nA_n - A_{n+1}g_n$). In particular $|dg_n| \leq 4C_{2,0}'\varepsilon_2$. Fix a reference point $x_0 \in S^3$. By multiplying some constant gauge transformations on h_n's, we can assume that $g_n(n/4 + 1/4, x_0) = 1$. Then $|g_n - 1| \leq \mathrm{const}\cdot\varepsilon_2$ over $(I_n \cap I_{n+1}) \times S^3$ where const is independent of L, n. Since the exponential map $\exp : su(2) \to SU(2)$ is locally diffeomorphic around $0 \in su(2)$, if ε_2 is sufficiently small (but independent of L, n), we have $u_n := (\exp)^{-1} g_n : (I_n \cap I_{n+1}) \times S^3 \to su(2)$. (Here we have fixed $\varepsilon_2 > 0$.) Then $g_n = e^{u_n}$ over $(I_n \cap I_{n+1}) \times S^3$ with

$$|\nabla^k u_n(x)| \leq C''_{2,k} \|F_A\|_{L^2((J_n \cup J_{n+1}) \times S^3)} \quad (x \in (I_n \cap I_{n+1}) \times S^3,\ k \geq 0).$$

Let φ be a smooth function in $\mathbb{R}$ such that $\mathrm{supp}(d\varphi) \subset (1/4, 1/2)$, $\varphi(t) = 0$ for $t \leq 1/4$ and $\varphi = 1$ for $t \geq 1/2$. Set $\varphi_n(t) := \varphi(t - n/4)$. ($\mathrm{supp}(d\varphi_n) \subset \mathrm{Interior}(I_n \cap I_{n+1})$.) We define a trivialization h of G over the union of $(I_n \cap I_{n+1}) \times S^3$ $(2 \leq n \leq [4L-4])$ by setting $h := e^{\varphi_n u_n} \circ h_n$ on $(I_n \cap I_{n+1}) \times S^3$. Then h is smoothly defined over a neighborhood of $[1, L-1] \times S^3$, and the connection matrix $\tilde{A} := h(A)$ satisfies (7). □

Let us go back to the given manifolds Y and $Y_T = p^{-1}(-T+1, T-1)$.

Lemma 4.6. *Let $T > 4$. There exist positive numbers ε_3 and $C_{3,k}$ $(k \geq 0)$ independent of T satisfying the following. If A is an ASD connection on the product principal $SU(2)$-bundle over Y_T satisfying $\|F_A\|_{L^2(Y_T)} \leq \varepsilon_3$, then A can be represented by a connection matrix $\tilde{A}$ over Y_{T-1} such that*

$$|\nabla^k \tilde{A}(x)| \leq C_{3,k} \|F_A\|_{L^2(p^{-1}(t-6, t+6) \cap Y_T)} \quad (t = p(x)),$$

for $x \in Y_{T-1}$ and $k \geq 0$.

Proof. Set $Z := p^{-1}[-3, 3]$ and $W := p^{-1}[-5/2, 5/2] \subset Z$. We apply Proposition 4.4 to these Z and W: There is $\varepsilon'_3 > 0$ (depending only on Z, W and hence independent of T) such that if $\|F_A\|_{L^2(Z)} \leq \varepsilon'_3$ then A can be represented by a connection matrix A_1 over a neighborhood of W such that

$$|\nabla^k A_1(x)| \leq \mathrm{const}_k \|F_A\|_{L^2(Z)} \quad (x \in W,\ k \geq 0).$$

On the other hand, by applying Lemma 4.5 to the tubes $p^{-1}(-T+1, -1) = (-T+1, -1) \times S^3$ and $p^{-1}(1, T-1) = (1, T-1) \times S^3$, if $\|F_A\|_{L^2(Y_T)} \leq \varepsilon_2$ (the positive constant introduced in Lemma 4.5) then A can be represented by a connection matrix A_2 over a neighborhood of $p^{-1}[-T+2, -2] \sqcup p^{-1}[2, T-2] = [-T+2, -2] \times S^3 \sqcup [2, T-2] \times S^3$ such that

$$|\nabla^k A_2(t, \theta)| \leq C_{2,k} \|F_A\|_{L^2((t-1, t+1) \times S^3)} \quad ((t, \theta) \in [-T+2, -2] \times S^3 \sqcup [2, T-2] \times S^3,\ k \geq 0).$$

Then by patching A_1 and A_2 over $p^{-1}(-5/2, -2)$ and $p^{-1}(2, 5/2)$ as in the proof of Lemma 4.5, we get the desired connection matrix $\tilde{A}$. □

4.3. **Exponential decay.** In this subsection we study a decay estimate of instantons on the product principal $SU(2)$-bundle $E = X \times SU(2)$. The results in this section will be used in Section 9. Our method is based on the arguments of Donaldson [4, Section 4.1] and Donaldson-Kronheimer [5, Section 7.3]. In this subsection we always suppose $T > 4$. Let g be a Riemannian metric on X which is equal to g_0 (the Riemannian metric given in Section 2.1) outside a compact set. Let A be a g-ASD connection on E satisfying

$$\int_X |F_A|_g^2 d\mathrm{vol}_g < \infty.$$

For $t \in \mathbb{R}$, set

$$J(t) := \int_{q^{-1}(t,+\infty)} |F_A|_g^2 d\mathrm{vol}_g.$$

For $t \gg 1$ we have $J(t) = \int_{q^{-1}(t,+\infty)} |F_A|^2 d\mathrm{vol}$ where $|\cdot|$ and $d\mathrm{vol}$ are the norm and volume form with respect to the periodic metric g_0. Recall that for each integer n we have $q^{-1}(nT+1,(n+1)T-1) = Y_T^{(n)} \cap Y_+^{(n)} = (1, T-1) \times S^3$.

Lemma 4.7. *There is $n_0(A) > 0$ such that for $n \geq n_0(A)$*

$$J'(t) \leq -2J(t) \quad (nT+2 \leq t \leq (n+1)T-2).$$

(The value -2 is not optimal.)

Proof. In this proof we always suppose $nT+2 \leq t \leq (n+1)T-2$ and $n \gg 1$. We have

$$J'(t) = -\int_{q^{-1}(t)} |F_A|^2 d\mathrm{vol} = -2\left\|F(A_t)\right\|_{L^2(S^3)}^2 \quad (A_t := A|_{q^{-1}(t)}).$$

Here we have used the fact $|F_A|^2 = 2|F(A_t)|^2$. This is the consequence of the ASD condition. From Lemma 4.5, we can assume that, for $n \gg 1$, a connection matrix of A over $q^{-1}[nT+2,(n+1)T-2]$ is as small as we want with respect to the $\mathcal{C}^1$-norm (or any other $\mathcal{C}^k$-norm). In particular we have $\|F(A_t)\|_{L^2} \ll 1$ for $n \gg 1$. Then, by using [5, Proposition 4.4.11], we can suppose that A_t is represented by a connection matrix satisfying

$$\|A_t\|_{L_1^2(S^3)} \leq \mathrm{const}\, \|F(A_t)\|_{L^2(S^3)}. \tag{8}$$

Then we can prove

Sublemma 4.8.

$$J(t) = -\int_{S^3} \mathrm{tr}(A_t \wedge dA_t + \frac{2}{3}A_t^3) \quad (=: -\theta(A_t)).$$

Proof. For $m > n \gg 1$ and $mT+2 \leq s \leq (m+1)T-2$,

$$\int_{q^{-1}[t,s]} |F_A|^2 d\mathrm{vol} \equiv \theta(A_s) - \theta(A_t) \mod 8\pi^2\mathbb{Z}. \tag{9}$$

We can suppose that the connection matrix A_s also satisfies (8). Then both of the left and right hand sides of the above equation (9) are sufficiently small. Hence

$$\int_{q^{-1}[t,s]} |F_A|^2 d\mathrm{vol} = \theta(A_s) - \theta(A_t).$$

We have $\theta(A_s) \to 0$ as $m \to +\infty$. Then we get the above result. □

From Corollary 3.5 (ii),

$$\left|\int_{S^3} \mathrm{tr}(A_t \wedge dA_t)\right| \leq \frac{1}{2}\int_{S^3} |dA_t|^2.$$

Since $dA_t = F(A_t) - A_t^2$, $\|dA_t\|_{L^2(S^3)}^2 \leq \|F(A_t)\|_{L^2}^2 + 2\|F(A_t)\|_{L^2}\|A_t^2\|_{L^2} + \|A_t^2\|_{L^2}^2$. We have $L_1^2(S^3) \hookrightarrow L^6(S^3)$. Hence $\|A_t^2\|_{L^2} \leq \mathrm{const}\,\|A_t\|_{L_1^2}^2 \leq \mathrm{const}\,\|F(A_t)\|_{L^2}^2$ by (8). Hence $\|dA_t\|_{L^2}^2 \leq (1 + \mathrm{const}\,\|F(A_t)\|_{L^2})\,\|F(A_t)\|_{L^2}^2$. In a similar way, we have

$$\left|\int_{S^3} \mathrm{tr}(A_t^3)\right| \leq \mathrm{const}\,\|A_t\|_{L^3}^3 \leq \mathrm{const}\,\|F(A_t)\|_{L^2}^3 .$$

Thus we have

$$J(t) = -\theta(A_t) \leq \left(\frac{1}{2} + \mathrm{const}\,\|F(A_t)\|_{L^2}\right)\|F(A_t)\|_{L^2}^2 .$$

Since $J'(t) = -2\|F(A_t)\|_{L^2}^2$ and $\|F(A_t)\|_{L^2} \ll 1$, we have $J(t) \leq \|F(A_t)\|_{L^2}^2 = -\frac{1}{2}J'(t)$. Hence $J'(t) \leq -2J$. $\square$

Corollary 4.9. *For $t \geq n_0(A)T + 2$,*

$$J(t) \leq \mathrm{const}_{A,T} \cdot e^{-2(1-4/T)t}.$$

Here $\mathrm{const}_{A,T}$ is a positive constant depending on A and T.

Proof. First note that $J(t)$ is monotone non-increasing. For $nT + 2 \leq t \leq (n+1)T - 2$ ($n \geq n_0(A) =: n_0$), we have $J(t) \leq e^{-2(t-nT-2)}J(nT+2)$ by Lemma 4.7.

Set $a_n := J(nT+2)$ $(n \geq n_0)$. $a_{n+1} \leq J((n+1)T-2) \leq e^{-2(T-4)}a_n$. Hence $a_n \leq e^{-2(T-4)(n-n_0)}a_{n_0}$.

For $nT+2 \leq t \leq (n+1)T-2$, $J(t) \leq e^{-2(t-nT-2)}a_n \leq e^{-2(t-4n)}e^{4+2n_0T-8n_0}a_{n_0}$. Since $t \geq nT+2$, we have $t - 4n \geq (1-4/T)t + 8/T$. Hence $J(t) \leq \mathrm{const}_{A,T}e^{-2(1-4/T)t}$.

For $(n+1)T - 2 < t < (n+1)T + 2$, $J(t) \leq J((n+1)T-2) \leq \mathrm{const}'_{A,T}e^{-2(1-4/T)t}$. $\square$

In the same way we can prove the following.

Lemma 4.10. *For $t \gg 1$ we have*

$$\int_{q^{-1}(-\infty,-t)} |F_A|^2 d\mathrm{vol} \leq \mathrm{const}_{A,T} \cdot e^{-2(1-4/T)t}.$$

Corollary 4.11. *There exists an adapted connection A_0 on E satisfying*

$$|\nabla_{A_0}^k(A(x) - A_0(x))| \leq \mathrm{const}_{k,A,T} \cdot e^{-(1-4/T)|t|} \quad (t = q(x)),$$

for all integers $k \geq 0$.

Proof. For $|n| \gg 1$, we have $\|F_A\|_{L^2(Y_T^{(n)})} \leq \varepsilon_3$. ($\varepsilon_3$ is a positive constant introduced in Lemma 4.6.) Then by Lemma 4.6, Corollary 4.9 and Lemma 4.10, A can be represented by a connection matrix A_n on $Y_{T-1}^{(n)}$ ($|n| \gg 1$) such that

$$|\nabla^k A_n(x)| \leq \mathrm{const}_{k,A,T} \cdot e^{-(1-4/T)|t|} \quad (x \in Y_{T-1}^{(n)},\ t = q(x),\ k \geq 0).$$

By patching these connection matrices over $Y_{T-1}^{(n)} \cap Y_{T-1}^{(n+1)}$ ($|n| \gg 1$) as in the proof of Lemma 4.5, A can be represented by a connection matrix $\tilde{A}$ on $\{|t| \gg 1\}$ such that

$$|\nabla^k \tilde{A}(x)| \leq \mathrm{const}_{k,A,T} \cdot e^{-(1-4/T)|t|} \quad (|t| \gg 1,\ k \geq 0).$$

To be more precise, there are $t_0 \gg 1$ and a trivialization $h : E|_{\{|t|>t_0\}} \to \{|t| > t_0\} \times SU(2)$ such that $h(A)$ satisfies

$$|\nabla_\rho^k (h(A) - \rho)| \le \mathrm{const}_{k,A,T} \cdot e^{-(1-4/T)|t|} \quad (|t| > t_0,\, k \ge 0),$$

where ρ is the product connection. This means that

$$|\nabla_{h^{-1}(\rho)}^k (A - h^{-1}(\rho))| \le \mathrm{const}_{k,A,T} \cdot e^{-(1-4/T)|t|} \quad (|t| > t_0,\, k \ge 0).$$

Take a connection A_0 on E which is equal to $h^{-1}(\rho)$ over $\{|t| \ge t_0 + 1\}$. Then A_0 is an adapted connection satisfying the desired property. □

5. Preliminaries for linear theory

In this section, we study differential operators over X. The results in this section will be used in Section 6. All arguments in Sections 5.1 and 5.2 are essentially given in Donaldson [4, Chapters 3 and 4].

5.1. **Preliminary estimates over the tube.** Let α be a real number with $0 < |\alpha| < 1$. In this subsection we study some differential operators over $\mathbb{R} \times S^3$. We denote t as the parameter of the $\mathbb{R}$-factor (i.e. the natural projection $t : \mathbb{R} \times S^3 \to \mathbb{R}$). Let $d^* : \Omega^1_{\mathbb{R}\times S^3} \to \Omega^0_{\mathbb{R}\times S^3}$ be the formal adjoint of the derivative $d : \Omega^0_{\mathbb{R}\times S^3} \to \Omega^1_{\mathbb{R}\times S^3}$ over $\mathbb{R} \times S^3$. We have $d^* = - * d *$ where $*$ is the Hodge star over $\mathbb{R} \times S^3$. We define a differential operator $d^{*,\alpha} : \Omega^1_{\mathbb{R}\times S^3} \to \Omega^0_{\mathbb{R}\times S^3}$ by setting $d^{*,\alpha} b := e^{-2\alpha t} d^*(e^{2\alpha t} b)$ $(b \in \Omega^1_{\mathbb{R}\times S^3})$. Then

$$d^{*,\alpha} b = d^* b - 2\alpha * (dt \wedge *b).$$

If $f \in \Omega^0_{\mathbb{R}\times S^3}$ and $b \in \Omega^1_{\mathbb{R}\times S^3}$ have compact supports, then

$$\int_{\mathbb{R}\times S^3} e^{2\alpha t} \langle df, b\rangle d\mathrm{vol} = \int_{\mathbb{R}\times S^3} e^{2\alpha t} \langle f, d^{*,\alpha} b\rangle d\mathrm{vol}.$$

Consider $d^+ := \frac{1}{2}(1 + *)d : \Omega^1_{\mathbb{R}\times S^3} \to \Omega^+_{\mathbb{R}\times S^3}$, and set $D^\alpha := -d^{*,\alpha} + d^+ : \Omega^1_{\mathbb{R}\times S^3} \to \Omega^0_{\mathbb{R}\times S^3} \oplus \Omega^+_{\mathbb{R}\times S^3}$.

Let $\Lambda^i_{S^3}$ $(i \ge 0)$ be the bundle of i-forms over S^3. Consider the pull-back of $\Lambda^i_{S^3}$ by the projection $\mathbb{R} \times S^3 \to S^3$, and we also denote it as $\Lambda^i_{S^3}$ for simplicity. We can identify the bundle $\Lambda^1_{\mathbb{R}\times S^3}$ of 1-forms on $\mathbb{R} \times S^3$ with the bundle $\Lambda^0_{S^3} \oplus \Lambda^1_{S^3}$ by

$$\Lambda^0_{S^3} \oplus \Lambda^1_{S^3} \ni (b_0, \beta) \longleftrightarrow b_0 dt + \beta \in \Lambda^1_{\mathbb{R}\times S^3}.$$

We also naturally identify the bundle $\Lambda^0_{\mathbb{R}\times S^3}$ with $\Lambda^0_{S^3}$. The bundle $\Lambda^+_{\mathbb{R}\times S^3}$ of self-dual forms can be identified with the bundle $\Lambda^1_{S^3}$ by

$$\Lambda^1_{S^3} \ni \beta \longleftrightarrow \frac{1}{2}(dt \wedge \beta + *_3 \beta) \in \Lambda^+_{\mathbb{R}\times S^3} \quad (*_3\text{: the Hodge star on } S^3).$$

We define $L : \Gamma(\Lambda^0_{S^3} \oplus \Lambda^1_{S^3}) \to \Gamma(\Lambda^0_{S^3} \oplus \Lambda^1_{S^3})$ by setting

$$L \begin{pmatrix} b_0 \\ \beta \end{pmatrix} := \begin{pmatrix} 0 & -d_3^* \\ -d_3 & *_3 d_3 \end{pmatrix} \begin{pmatrix} b_0 \\ \beta \end{pmatrix},$$

where d_3 is the exterior derivative on S^3 and $d_3^* = - *_3 d_3 *_3$. Let $b = (b_0, \beta) \in \Gamma(\Lambda^0_{S^3} \oplus \Lambda^1_{S^3}) = \Omega^1_{\mathbb{R}\times S^3}$ (i.e. $b = b_0 dt + \beta$). Then $D^\alpha b \in \Omega^0_{\mathbb{R}\times S^3} \oplus \Omega^1_{\mathbb{R}\times S^3} = \Gamma(\Lambda^0_{S^3} \oplus \Lambda^1_{S^3})$ is given by

$$D^\alpha b = \frac{\partial}{\partial t}\begin{pmatrix} b_0 \\ \beta \end{pmatrix} + \left(L + \begin{pmatrix} 2\alpha & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} b_0 \\ \beta \end{pmatrix}.$$

For $u \in \Omega^i_{\mathbb{R}\times S^3}$ $(i \geq 0)$, we define the Sobolev norm $\|u\|_{L^2_k}$ $(k \geq 0)$ by

$$\|u\|^2_{L^2_k} := \sum_{j=0}^{k} \int_{\mathbb{R}\times S^3} |\nabla^j u|^2 d\mathrm{vol}. \tag{10}$$

We define the weighted Sobolev norm $\|u\|_{L^{2,\alpha}_k}$ by

$$\|u\|_{L^{2,\alpha}_k} := \left\|e^{\alpha t} u\right\|_{L^2_k}. \tag{11}$$

The map $L^{2,\alpha}_k(\mathbb{R}\times S^3, \Lambda^i_{\mathbb{R}\times S^3}) \ni u \mapsto e^{\alpha t} u \in L^2_k(\mathbb{R}\times S^3, \Lambda^i_{\mathbb{R}\times S^3})$ is an isometry.

D^α becomes a bounded linear map from $L^{2,\alpha}_{k+1}(\mathbb{R}\times S^3, \Lambda^1_{\mathbb{R}\times S^3})$ to $L^{2,\alpha}_k(\mathbb{R}\times S^3, \Lambda^0_{\mathbb{R}\times S^3} \oplus \Lambda^+_{\mathbb{R}\times S^3})$. For $b = b_0 dt + \beta \in \Omega^1_{\mathbb{R}\times S^3}$ as above, we have

$$e^{\alpha t} D^\alpha (e^{-\alpha t} b) = \frac{\partial}{\partial t}\begin{pmatrix} b_0 \\ \beta \end{pmatrix} + \left(L + \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}\right)\begin{pmatrix} b_0 \\ \beta \end{pmatrix}. \tag{12}$$

Set

$$L^\alpha := L + \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}.$$

Recall that we have assumed $0 < |\alpha| < 1$.

Lemma 5.1. *Consider L^α as an essentially self-adjoint elliptic differential operator acting on $\Omega^0_{S^3} \oplus \Omega^1_{S^3}$ over S^3. If λ is an eigenvalue of L^α, then $|\lambda| \geq |\alpha|$. Moreover if $\lambda \neq \alpha$, then $|\lambda| > 1$.*

Proof. We have

$$\Omega^0_{S^3} \oplus \Omega^1_{S^3} = (\Omega^0_{S^3} \oplus d_3(\Omega^0_{S^3})) \oplus \ker d_3^*,$$

where $d_3^* = - *_3 d_3 *_3 : \Omega^1_{S^3} \to \Omega^0_{S^3}$. The subspaces $\Omega^0_{S^3} \oplus d_3(\Omega^0_{S^3})$ and $\ker d_3^*$ are both L^α-invariant.

For $\beta \in \ker d_3^*$, $L^\alpha(0,\beta) = (0, *_3 d_3\beta - \alpha\beta)$. Suppose that $L^\alpha(0,\beta) = \lambda(0,\beta)$ and β is not zero. Since $d_3^*\beta = 0$ and $H^1(S^3) = 0$, we have $d_3\beta \neq 0$. Then $*_3 d_3 \beta = (\lambda + \alpha)\beta$ and $\lambda + \alpha \neq 0$. Since we have (Corollary 3.5 (ii))

$$\left|\int_{S^3} \beta \wedge d_3\beta\right| \leq \frac{1}{2}\int_{S^3} |d_3\beta|^2 d\mathrm{vol}$$

and $(\lambda+\alpha)\beta \wedge d_3\beta = |d_3\beta|^2 d\mathrm{vol}$, we have

$$2 \leq |\lambda + \alpha|.$$

Then $|\lambda| \geq 2 - |\alpha| > 1 > |\alpha|$.

For $(f, d_3 g) \in \Omega^0_{S^3} \oplus d_3(\Omega^0_{S^3})$ (f and g are smooth functions on S^3),

$$L^\alpha \begin{pmatrix} f \\ d_3 g \end{pmatrix} = \begin{pmatrix} \alpha f - \Delta_3 g \\ -d_3 f - \alpha d_3 g \end{pmatrix}, \quad (\Delta_3 = d_3^* d_3 \text{ is the Laplacian on functions over } S^3).$$

Suppose that $L^\alpha(f, d_3 g) = \lambda(f, d_3 g)$ and $(f, d_3 g)$ is not zero. Then

$$\Delta_3 g = (\alpha - \lambda) f, \quad d_3 f = -(\alpha + \lambda) d_3 g.$$

Case 1: Suppose $\alpha + \lambda = 0$. Then f is a constant, and

$$0 = \int_{S^3} \Delta_3 g \, d\mathrm{vol} = 2\alpha \int_{S^3} f d\mathrm{vol}.$$

Hence $f \equiv 0$. This implies $\Delta_3 g \equiv 0$ and hence $d_3 g \equiv 0$. This is a contradiction.
Case 2: Suppose $\alpha + \lambda \neq 0$. Then $\Delta_3 f = (\lambda^2 - \alpha^2) f$. Since the first non-zero eigenvalue of the Laplacian Δ_3 is 3 (Lemma 3.3), $\lambda^2 - \alpha^2 = 0$ or $\lambda^2 - \alpha^2 \geq 3$. Since $\lambda \neq -\alpha$, we have

$$\lambda = \alpha \quad \text{or} \quad |\lambda| \geq \sqrt{3 + \alpha^2} \geq \sqrt{3}.$$

□

Lemma 5.2. *For $a \in L^{2,\alpha}_1(\mathbb{R} \times S^3, \Lambda^1_{\mathbb{R}\times S^3})$, we have $\|a\|_{L^{2,\alpha}} \leq \sqrt{2}|\alpha|^{-1} \|D^\alpha a\|_{L^{2,\alpha}}$. Moreover $\|a\|_{L^{2,\alpha}_1} \leq \mathrm{const}_\alpha \|D^\alpha a\|_{L^{2,\alpha}}$.*

Proof. We can suppose that a is smooth and compact supported. Set $b := e^{\alpha t} a = b_0 dt + \beta$ where $(b_0, \beta) \in \Gamma(\mathbb{R} \times S^3, \Lambda^0_{S^3} \oplus \Lambda^1_{S^3})$. Let $\{\varphi_\lambda\}_\lambda$ be a complete orthonormal basis of $L^2(S^3, \Lambda^0_{S^3} \oplus \Lambda^1_{S^3})$ consisting of eigen-functions of L^α over S^3 with $L^\alpha \varphi_\lambda = \lambda \varphi_\lambda$ where λ runs over all eigenvalues of L^α. From Lemma 5.1, we have $|\lambda| \geq |\alpha|$. Decompose (b_0, β) by $\{\varphi_\lambda\}$ as

$$(b_0(t, \theta), \beta(t, \theta)) = \sum_\lambda c_\lambda(t) \varphi_\lambda(\theta).$$

Since a is compact supported, the functions c_λ are also compact supported. $(\partial/\partial t + L^\alpha)(b_0, \beta) = \sum_\lambda (c'_\lambda(t) + \lambda c_\lambda(t)) \varphi_\lambda$. If $(\partial/\partial t + L^\alpha)(b_0, \beta) = (b_1, \gamma)$, then $e^{\alpha t} D^\alpha(e^{-\alpha t} b) = (b_1, \frac{1}{2}(dt \wedge \gamma + *_3 \gamma))$. Hence $|e^{\alpha t} D^\alpha (e^{-\alpha t} b)| = \sqrt{|b_1|^2 + |\gamma|^2/2} \geq |(\partial/\partial t + L^\alpha)(b_0, \beta)|/\sqrt{2}$. Therefore

$$\int_{\mathbb{R}\times S^3} |e^{\alpha t} D^\alpha (e^{-\alpha t} b)|^2 d\mathrm{vol} \geq \frac{1}{2} \sum_\lambda \int_{-\infty}^{\infty} |c'_\lambda + \lambda c_\lambda|^2 dt.$$

$|c'_\lambda + \lambda c_\lambda|^2 = |c'_\lambda|^2 + \lambda (c_\lambda^2)' + \lambda^2 c_\lambda^2$. Since $|\lambda| \geq |\alpha|$ and the functions c_λ are compact supported,

$$\sum_\lambda \int_{-\infty}^{\infty} |c'_\lambda + \lambda c_\lambda|^2 dt \geq \alpha^2 \sum_\lambda \int_{-\infty}^{\infty} |c_\lambda|^2 dt = \alpha^2 \|b\|^2_{L^2}.$$

Then

$$\|D^\alpha a\|_{L^{2,\alpha}} = \left\| e^{\alpha t} D^\alpha (e^{-\alpha t} b) \right\|_{L^2} \geq \frac{|\alpha|}{\sqrt{2}} \|b\|_{L^2} = \frac{|\alpha|}{\sqrt{2}} \|a\|_{L^{2,\alpha}}.$$

Since $e^{\alpha t}D^\alpha e^{-\alpha t} = \frac{\partial}{\partial t} + L^\alpha$ is a translation invariant elliptic differential operator, for every $n \in \mathbb{Z}$ we have

$$\|b\|^2_{L^2_1((n,n+1)\times S^3)} \leq \mathrm{const}_\alpha \left(\|b\|^2_{L^2((n-1,n+2)\times S^3)} + \left\|e^{\alpha t}D^\alpha(e^{-\alpha t}b)\right\|^2_{L^2((n-1,n+2)\times S^3)}\right).$$

Here const_α is independent of n. By summing up this estimate over $n \in \mathbb{Z}$, we get

$$\|b\|_{L^2_1(\mathbb{R}\times S^3)} \leq \mathrm{const}_\alpha \left(\|b\|_{L^2(\mathbb{R}\times S^3)} + \left\|e^{\alpha t}D^\alpha(e^{-\alpha t}b)\right\|_{L^2(\mathbb{R}\times S^3)}\right).$$

This shows $\|a\|_{L^{2,\alpha}_1} \leq \mathrm{const}_\alpha(\|a\|_{L^{2,\alpha}} + \|D^\alpha a\|_{L^{2,\alpha}}) \leq \mathrm{const}'_\alpha \|D^\alpha a\|_{L^{2,\alpha}}$. □

Lemma 5.3. *(i) Suppose $\alpha > 0$. Let a be a smooth 1-form over the negative half tube $(-\infty, 0)\times S^3$ satisfying $\int_{(-\infty,0)\times S^3} e^{2\alpha t}|a|^2 d\mathrm{vol} < +\infty$. Suppose $D^\alpha a = 0$. Then*

$$|a|, |\nabla a| \leq \mathrm{const}_{a,\alpha} e^{(1-\alpha)t} \quad (t < -2).$$

(ii) Suppose $\alpha < 0$. Let a be a smooth 1-form over the positive half tube $(0, +\infty)\times S^3$ satisfying $\int_{(0,+\infty)\times S^3} e^{2\alpha t}|a|^2 d\mathrm{vol} < +\infty$, and suppose $D^\alpha a = 0$. Then

$$|a|, |\nabla a| \leq \mathrm{const}_{a,\alpha} e^{-(1+\alpha)t} \quad (t > 2).$$

Proof. We give the proof of the case (i) ($\alpha > 0$). The case (ii) can be proved in the same way. Set $b := e^{\alpha t}a = b_0 dt + \beta$ where $(b_0, \beta) \in \Gamma(\mathbb{R}\times S^3, \Lambda^0_{S^3}\oplus\Lambda^1_{S^3})$. Then $e^{\alpha t}D^\alpha(e^{-\alpha t}b) = 0$. Choose $\{\varphi_\lambda\}_\lambda$ as in the proof of Lemma 5.2. Decompose (b_0, β) by $\{\varphi_\lambda\}$ as $(b_0(t,\theta), \beta(t,\theta)) = \sum_\lambda c_\lambda(t)\varphi_\lambda(\theta)$. Since $(\partial/\partial t + L^\alpha)(b_0,\beta) = \sum(c'_\lambda(t) + \lambda c_\lambda(t))\varphi_\lambda = 0$, we have $c_\lambda(t) = d_\lambda e^{-\lambda t}$ where d_λ is a constant. For $t < 0$,

$$\int_{\{t\}\times S^3} |b|^2 d\mathrm{vol}_3 = \sum_\lambda |c_\lambda|^2 = \sum_\lambda |d_\lambda|^2 e^{-2\lambda t} \geq |d_\lambda|^2 e^{-2\lambda t}.$$

Since the L^2-norm of b over $(-\infty, 0)\times S^3$ is finite, we have $d_\lambda = 0$ for $\lambda \geq 0$. Set

$$B := e^2 \int_{\{-1\}\times S^3} |b|^2 d\mathrm{vol}_3 = e^2 \sum_{\lambda<0} |d_\lambda e^\lambda|^2 < \infty.$$

From Lemma 5.1, negative eigenvalues λ satisfy $\lambda < -1$. Hence for $t < -1$

$$\int_{\{t\}\times S^3} |b|^2 d\mathrm{vol}_3 = \sum_{\lambda<0} |d_\lambda e^\lambda|^2 e^{-2\lambda(t+1)} \leq \sum_{\lambda<0} |d_\lambda e^\lambda|^2 e^{2(t+1)} = Be^{2t}.$$

Then for $t < -2$,

$$\int_{(t-1,t+1)\times S^3} |b|^2 d\mathrm{vol} \leq B\int_{t-1}^{t+1} e^{2s} ds \leq Be^{2(t+1)}.$$

Since $e^{\alpha t}D^\alpha(e^{-\alpha t}b) = 0$ (and this is a translation invariant equation), the elliptic regularity implies

$$|b|, |\nabla b| \leq \mathrm{const}_\alpha \sqrt{B}\cdot e^t \quad (t < -2).$$

(Indeed we can choose const_α independent of α. But it is unimportant for us.) Since $a = e^{-\alpha t}b$, we have

$$|a|, |\nabla a| \leq \mathrm{const}'_{a,\alpha} e^{(1-\alpha)t} \quad (t < -2).$$

□

5.2. **Preliminary results over $\hat{Y}$.** Recall that Y is a simply connected closed oriented 4-manifold and that $\hat{Y} = Y \setminus \{x_1, x_2\}$. $\hat{Y}$ has cylinderical ends, and we have $p : \hat{Y} \to \mathbb{R}$. For a section of u of Λ^i $(i \geq 0)$ over $\hat{Y}$, we define the Sobolev norm $\|u\|_{L^2_k}$ $(k \geq 0)$ as in (10). We define the weighted Sobolev norm by $\|u\|_{L^{2,\alpha}_k} := \|e^{\alpha t} u\|_{L^{2,\alpha}}$ where $t = p(x)$ $(x \in \hat{Y})$. Recall $0 < |\alpha| < 1$.

For a 1-form a over $\hat{Y}$ we set $D^\alpha a := -d^{*,\alpha} a + d^+ a = -e^{-2\alpha t} d^*(e^{2\alpha t} a) + d^+ a$.

Lemma 5.4. *Let a be a 1-form over $\hat{Y}$ with $\|a\|_{L^{2,\alpha}_1} < \infty$. If $D^\alpha a = 0$, then $a = 0$.*

Proof. We give the proof of the case $\alpha > 0$. The case $\alpha < 0$ can be proved in the same way. We divide the proof into three steps.

Step 1: We will show that the above assumption implies $da = 0$. First we want to show $a, da \in L^2$. We have

$$\int_{t>0} |a|^2 d\mathrm{vol} \leq \int_{t>0} e^{2\alpha t} |a|^2 d\mathrm{vol} < \infty, \quad \int_{t>0} |da|^2 d\mathrm{vol} \leq \int_{t>0} e^{2\alpha t} |da|^2 d\mathrm{vol} < \infty.$$

Lemma 5.3 implies that the L^2-norms of a and da over $Y_- = (-\infty, -1) \times S^3$ are finite. Hence $a, da \in L^2$. For $R > 1$, let β_R be a smooth function over $\hat{Y}$ such that $\beta_R = 1$ over $p^{-1}(-R, R)$, $\beta_R = 0$ over $p^{-1}(-\infty, -2R) \cup p^{-1}(2R, \infty)$ and $|d\beta_R| \leq 2/R$.

$$0 = \int d(\beta_R a \wedge da) = \int \beta_R da \wedge da + \int d\beta_R \wedge a \wedge da.$$

Since $d^+ a = 0$, we have $da \wedge da = -|da|^2 d\mathrm{vol}$ and hence

$$\int \beta_R |da|^2 d\mathrm{vol} = \int d\beta_R \wedge a \wedge da \leq \frac{2}{R} \|a\|_{L^2} \|da\|_{L^2}.$$

Let $R \to +\infty$. Then $\int |da|^2 d\mathrm{vol} = 0$. Hence $da = 0$.

Step 2: We have

$$|a| \leq \mathrm{const}_{a,\alpha} e^{-\alpha t} \ (t > 1), \quad |a| \leq \mathrm{const}_{a,\alpha} e^{(1-\alpha)t} \ (t < -1).$$

The latter estimate comes from Lemma 5.3. The former one comes from the elliptic regularity and the following estimate: For $t > 1$,

$$\int_{p^{-1}(t,+\infty)} |a|^2 d\mathrm{vol} \leq e^{-2\alpha t} \int_{p^{-1}(t,+\infty)} e^{2\alpha p(x)} |a(x)|^2 d\mathrm{vol}(x) \leq \|a\|^2_{L^{2,\alpha}} e^{-2\alpha t}.$$

Step 3: From Step 1 and $H^1_{dR}(\hat{Y}) = 0$, there is a smooth function f on $\hat{Y}$ satisfying $a = df$. From Step 2, the limits $f(+\infty) := \lim_{t\to+\infty} f(t,\theta)$ and $f(-\infty) := \lim_{t\to-\infty} f(t,\theta)$ exist and independent of $\theta \in S^3$. In particular f is bounded. We can assume $f(+\infty) = 0$. Then for $t > 1$

$$f(t,\theta) = -\int_t^\infty \frac{\partial f}{\partial s}(s,\theta) ds.$$

Since $|\partial f/\partial s| \leq |a| \leq \mathrm{const}_{a,\alpha} e^{-\alpha t}$ for $t > 1$ (Step 2),

$$|f| \leq \mathrm{const}_{a,\alpha} \cdot e^{-\alpha t} \quad (t > 1). \tag{13}$$

Let β_R be the cut-off function used in Step 1. Since $e^{-2\alpha t} d^*(e^{2\alpha t} a) = d^{*,\alpha} a = 0$,

$$\begin{aligned} 0 = \int e^{2\alpha t} \langle \beta_R f, d^{*,\alpha} a \rangle d\mathrm{vol} &= \int e^{2\alpha t} \langle d(\beta_R f), a \rangle d\mathrm{vol} \\ &= \int e^{2\alpha t} f \langle d\beta_R, a \rangle d\mathrm{vol} + \int e^{2\alpha t} \beta_R |a|^2 d\mathrm{vol}. \end{aligned}$$

Hence

$$\int e^{2\alpha t} \beta_R |a|^2 d\mathrm{vol} \leq \frac{2}{R} \int_{\mathrm{supp}(d\beta_R)} e^{2\alpha t} |f||a| d\mathrm{vol}. \tag{14}$$

We have $\mathrm{supp}(d\beta_R) \subset p^{-1}(-2R, -R) \cup p^{-1}(R, 2R)$. Since $|f|$ and $|a|$ are bounded,

$$\int_{p^{-1}(-2R,-R)} e^{2\alpha t} |f||a| d\mathrm{vol} \to 0 \quad (R \to +\infty).$$

On the other hand, by the above (13)

$$\begin{aligned} &\frac{2}{R} \int_{p^{-1}(R,2R)} e^{2\alpha t} |f||a| d\mathrm{vol} \leq \frac{\mathrm{const}_{a,\alpha}}{R} \int_{p^{-1}(R,2R)} e^{\alpha t} |a| d\mathrm{vol} \\ &\leq \frac{\mathrm{const}_{a,\alpha}}{R} \sqrt{\mathrm{vol}((R, 2R) \times S^3)} \sqrt{\int_{p^{-1}(R,2R)} e^{2\alpha t} |a|^2 d\mathrm{vol}} \leq \mathrm{const}_{a,\alpha} \|a\|_{L^{2,\alpha}} / \sqrt{R}. \end{aligned}$$

This goes to 0 as $R \to +\infty$. From (14),

$$\int e^{2\alpha t} |a|^2 = 0.$$

Thus $a = 0$. □

Lemma 5.5. *For $a \in L^{2,\alpha}_1(\hat{Y}, \Lambda^1)$,*

$$\|a\|_{L^{2,\alpha}_1(\hat{Y})} \leq \mathrm{const}_\alpha \|D^\alpha a\|_{L^{2,\alpha}(\hat{Y})}.$$

Proof. Set $U := p^{-1}(-2, 2) \subset \hat{Y}$. By using Lemma 5.2, for all $a \in L^{2,\alpha}_1(\hat{Y})$

$$\|a\|_{L^{2,\alpha}_1(\hat{Y})} \leq \mathrm{const}_\alpha (\|a\|_{L^2(U)} + \|D^\alpha a\|_{L^{2,\alpha}(\hat{Y})}). \tag{15}$$

We want to show $\|a\|_{L^2(U)} \leq \mathrm{const}_\alpha \|D^\alpha a\|_{L^{2,\alpha}(\hat{Y})}$. Suppose on the contrary there exist a sequence a_n $(n \geq 1)$ in $L^{2,\alpha}_1(\hat{Y}, \Lambda^1)$ such that

$$1 = \|a_n\|_{L^2(U)} > n \|D^\alpha a_n\|_{L^{2,\alpha}(\hat{Y})}.$$

From the above (15), $\{a_n\}$ is bounded in $L^{2,\alpha}_1(\hat{Y})$. Hence, if we take a subsequence (also denoted by a_n), the sequence a_n weakly converges to some a in $L^{2,\alpha}_1(\hat{Y})$. We have $D^\alpha a = 0$. Hence Lemma 5.4 implies $a = 0$. By Rellich's lemma, a_n strongly converges to 0 in $L^2(U)$. (Note that U is pre-compact.) This contradicts $\|a_n\|_{L^2(U)} = 1$. □

5.3. **Preliminary results over $X = Y^{\sharp\mathbb{Z}}$.** Recall that $X = Y^{\sharp\mathbb{Z}}$ has the periodic metric g_0 which is compatible with the given metric h over every $Y^{(n)}$ $(n \in \mathbb{Z})$, and that g_0 depends on the parameter $T > 2$. We define the Sobolev norm $\|\cdot\|_{L^2_k}$ over X as in (10) by using the metric g_0 and its Levi-Civita connection. We define the weighted Sobolev norm by $\|u\|_{L^{2,\alpha}_k} := \|e^{\alpha t}u\|_{L^{2,\alpha}_k}$ where $t = q(x)$ $(x \in X)$. For a 1-form a over X we set $D^\alpha a := -d^{*,\alpha}a + d^+a = -e^{-2\alpha t}d^*(e^{2\alpha t}a) + d^+a$.

Lemma 5.6. *There exists $T_\alpha > 2$ such that if $T \geq T_\alpha$ then for any $a \in L^{2,\alpha}_1(X, \Lambda^1)$ we have*

$$\|a\|_{L^{2,\alpha}_1(X)} \leq \mathrm{const}_\alpha \|D^\alpha a\|_{L^{2,\alpha}(X)}.$$

The important point is that T_α depends only on α.

Proof. Let $\beta^{(n)}$ be a smooth function on X such that $0 \leq \beta^{(n)} \leq 1$, $\mathrm{supp}\beta^{(n)} \subset Y^{(n)}_T = q^{-1}((n-1)T+1, (n+1)T-1)$, $\beta^{(n)} = 1$ over $q^{-1}((n-1/2)T, (n+1/2)T)$ and $|d\beta^{(n)}| \leq 3/T$. Since $t = q(x) = p^{(n)}(x) + nT$ over $Y^{(n)}_T$, by applying Lemma 5.5 to $\beta^{(n)}a$, we get

$$\begin{aligned}
\left\|\beta^{(n)}a\right\|_{L^{2,\alpha}_1(X)} &= e^{\alpha nT}\left\|e^{\alpha p^{(n)}(x)}\beta^{(n)}a\right\|_{L^2_1(Y^{(n)}_T)} \\
&\leq \mathrm{const}_\alpha \cdot e^{\alpha nT}\left\|e^{\alpha p^{(n)}(x)}D^\alpha(\beta^{(n)}a)\right\|_{L^2(Y^{(n)}_T)} \\
&= \mathrm{const}_\alpha \left\|D^\alpha(\beta^{(n)}a)\right\|_{L^{2,\alpha}(X)} \\
&\leq \frac{\mathrm{const}_\alpha}{T}\|a\|_{L^{2,\alpha}(Y^{(n)}_T)} + \mathrm{const}_\alpha \|D^\alpha a\|_{L^{2,\alpha}(Y^{(n)}_T)}.
\end{aligned}$$

Then

$$\begin{aligned}
\|a\|^2_{L^{2,\alpha}_1(X)} &\leq \sum_{n\in\mathbb{Z}}\left\|\beta^{(n)}a\right\|^2_{L^{2,\alpha}_1(X)} \\
&\leq \frac{\mathrm{const}_\alpha}{T^2}\sum_{n\in\mathbb{Z}}\|a\|^2_{L^{2,\alpha}(Y^{(n)}_T)} + \mathrm{const}_\alpha\sum_{n\in\mathbb{Z}}\|D^\alpha a\|^2_{L^{2,\alpha}(Y^{(n)}_T)} \\
&\leq \frac{\mathrm{const}_\alpha}{T^2}\|a\|^2_{L^{2,\alpha}(X)} + \mathrm{const}_\alpha\|D^\alpha a\|^2_{L^{2,\alpha}(X)}.
\end{aligned}$$

If $T \gg 1$, then

$$\|a\|^2_{L^{2,\alpha}_1(X)} \leq \mathrm{const}_\alpha \|D^\alpha a\|^2_{L^{2,\alpha}(X)}.$$

□

For a 1-form a on X we set $\mathcal{D}a := -d^*a + d^+a$. Its formal adjoint $\mathcal{D}^*$ is given by $\mathcal{D}^*(u,\xi) = -du + d^*\xi = -du - *d\xi$ for $(u,\xi) \in \Omega^0 \oplus \Omega^+$. We consider $\mathcal{D}$ as an unbounded operator from $L^2(X, \Lambda^1)$ to $L^2(X, \Lambda^0 \oplus \Lambda^+)$.

The additive Lie group $\mathbb{Z}$ naturally acts on $X = Y^{\sharp\mathbb{Z}}$. Set $Y^+ := X/\mathbb{Z}$. We have $b_1(Y^+) = 1$ and $b_+(Y^+) = b_+(Y)$. The operator $\mathcal{D}$ is preserved by the $\mathbb{Z}$-action, and its quotient is equal to the operator $-d^* + d^+ : \Omega^1_{Y^+} \to \Omega^0_{Y^+} \oplus \Omega^+_{Y^+}$ on Y^+. Then we can apply Atiyah's Γ-index theorem (Atiyah [3], Roe [12, Chapter 13]) to $\mathcal{D}$ and get

$$\mathrm{ind}_{\mathbb{Z}}\mathcal{D} = \mathrm{ind}(-d^* + d^+ : \Omega^1_{Y^+} \to \Omega^0_{Y^+} \oplus \Omega^+_{Y^+}) = -1 + b_1(Y^+) - b_+(Y^+) = -b_+(Y).$$

Here $\mathrm{ind}_{\mathbb{Z}}\mathcal{D}$ is the Γ-index of $\mathcal{D}$ ($\Gamma = \mathbb{Z}$).

The above implies that if $b_+(Y) \geq 1$ then $\mathrm{Ker}\mathcal{D}^* \subset L^2(X, \Lambda^0 \oplus \Lambda^+)$ is infinite dimensional. Suppose $\rho = (u, \xi) \in L^2(X, \Lambda^0 \oplus \Lambda^+)$ satisfies $\mathcal{D}^*\rho = -du + d^*\xi = 0$ as a distribution. By the elliptic regularity, ρ is smooth, and for each $n \in \mathbb{Z}$

$$\|\rho\|_{L^2_1(q^{-1}(-(n-1/2)T,(n+1/2)T))} \leq \mathrm{const}_T \|\rho\|_{L^2(Y^{(n)}_T)}.$$

Here const_T is independent of $n \in \mathbb{Z}$. Hence $\|\rho\|_{L^2_1(X)} \leq \mathrm{const}_T \|\rho\|_{L^2(X)} < +\infty$, and $\rho \in L^2_1(X)$. In particular $u, \xi \in L^2_1(X)$ and hence $\langle du, d^*\xi\rangle_{L^2} = 0$. Then

$$0 = \langle \mathcal{D}^*\rho, du\rangle_{L^2} = -\|du\|_{L^2}.$$

So $du = 0$. This means that u is constant. But $u \in L^2$. Hence $u = 0$. Therefore $d^*\xi = 0$. Thus we get the following result.

Lemma 5.7. *Suppose $b_+(Y) \geq 1$. The space of $\xi \in L^2_1(X, \Lambda^+)$ satisfying $d^*\xi = 0$ is infinite dimensional.*

Take and fix a smooth function $|\cdot|' : \mathbb{R} \to \mathbb{R}$ satisfying $|t|' = |t|$ for $|t| \geq 1$. For $0 < |\alpha| < 1$, set $W(x) := e^{\alpha|q(x)|'}$ for $x \in X$. Hence W is a positive smooth function on X satisfying $W(x) = e^{\alpha|q(x)|}$ for $|q(x)| \geq 1$. For a section η of Λ^i ($i \geq 0$) we set $\|\eta\|_{L^{2,W}_k(X)} := \|W\eta\|_{L^2_k(X)}$. For a self-dual form η over X, we set $d^{*,W}\eta := -W^{-2} * d(W^2\eta)$. If $a \in \Omega^1_X$ and $\eta \in \Omega^+_X$ have compact supports, then $\int_X W^2\langle da, \eta\rangle d\mathrm{vol} = \int_X W^2\langle a, d^{*,W}\eta\rangle d\mathrm{vol}$.

Lemma 5.8. *Suppose $b_+(Y) \geq 1$ and $\alpha > 0$. Then the space of $\eta \in L^{2,W}_1(X, \Lambda^+)$ satisfying $d^{*,W}\eta = 0$ is infinite dimensional. Moreover it is closed in $L^{2,W}(X, \Lambda^+)$.*

Proof. Suppose that $\xi \in L^2_1(X, \Lambda^+)$ satisfies $d^*\xi = 0$. Set $\eta := W^{-2}\xi$. Then $d^{*,W}\eta = 0$ and $\|\eta\|_{L^{2,W}_1(X)} = \|W^{-1}\xi\|_{L^2_1(X)} < \infty$ from $\alpha > 0$. Thus Lemma 5.7 implies the first statement.

In order to prove the closedness of $\mathrm{Ker}(d^{*,W}) \subset L^{2,W}_1(X, \Lambda^+)$ in $L^{2,W}(X, \Lambda^+)$, it is enough to show that if $\eta \in L^{2,W}(X, \Lambda^+)$ satisfies $d^{*,W}\eta = 0$ (as a distribution) then $\eta \in L^{2,W}_1(X, \Lambda^+)$. η is smooth by the (local) elliptic regularity. The differential operator $d^{*,W}$ on $Y^{(n)}_T$ ($n > 0$) are naturally isomorphic to each other. The same statement also hold for $n < 0$. Hence, by the elliptic regularity,

$$\begin{aligned}\|W\eta\|_{L^2_1(q^{-1}((n-1/2)T,(n+1/2)T))} &\leq \mathrm{const}_{T,\alpha} \cdot (\|W\eta\|_{L^2(Y^{(n)}_T)} + \|d^{*,W}(W\eta)\|_{L^2(Y^{(n)}_T)}) \\ &\leq \mathrm{const}_{T,\alpha} \|W\eta\|_{L^2(Y^{(n)}_T)}.\end{aligned}$$

Here $\mathrm{const}_{T,\alpha}$ are independent of $n \in \mathbb{Z}$. Thus $\|\eta\|_{L^{2,W}_1(X)} \leq \mathrm{const}_{T,\alpha} \|\eta\|_{L^{2,W}(X)} < \infty$. $\square$

Lemma 5.9. *Suppose $b_+(Y) \geq 1$ and $\alpha > 0$. For any $\varepsilon > 0$ and any pre-compact open set $U \subset X$, there is $\eta \in L^{2,W}_1(X, \Lambda^+)$ such that $\eta = 0$ over U and*

$$\left\| d^{*,W}\eta \right\|_{L^{2,W}(X)} < \varepsilon \left\| \eta \right\|_{L^{2,W}(X)}.$$

Proof. First we prove the following statement: For any $\varepsilon > 0$ and any pre-compact open set $U \subset X$ there exists $\eta \in L^{2,W}_1(X, \Lambda^+)$ satisfying $d^{*,W}\eta = 0$ and $\|\eta\|_{L^{2,W}(U)} < \varepsilon \|\eta\|_{L^{2,W}(X)}$. Suppose that this statement does not hold. Then there are $\varepsilon > 0$ and a pre-compact open set $U \subset X$ such that all $\eta \in \mathrm{Ker}(d^{*,W}) \subset L^{2,W}_1(X, \Lambda^+)$ satisfies

$$\|\eta\|_{L^{2,W}(U)} \geq \varepsilon \|\eta\|_{L^{2,W}(X)}.$$

$\mathrm{Ker}(d^{*,W})$ is an infinite dimensional closed subspace in $L^{2,W}(X, \Lambda^+)$ (Lemma 5.8). Let $\{\eta_n\}_{n\geq 1}$ be a complete orthonormal basis of $\mathrm{Ker}(d^{*,W})$ with respect to the inner product of $L^{2,W}(X, \Lambda^+)$. They satisfies

$$\|\eta_n\|_{L^{2,W}(U)} \geq \varepsilon.$$

The sequence η_n weakly converges to 0 in $L^{2,W}(X)$, and hence $\eta_n|_U$ weakly converges to 0 in $L^{2,W}(U)$. Then, by the elliptic regularity and Rellich's lemma, a subsequence of $\eta_n|_U$ strongly converges to 0 in $L^{2,W}(U)$. But this contradicts $\|\eta_n\|_{L^{2,W}(U)} \geq \varepsilon$.

Next take a pre-compact open set $V \subset X$ satisfying the following: $U \subset V$ and there exists a smooth function β such that $0 \leq \beta \leq 1$, $\beta = 0$ on U, $\beta = 1$ on $X \setminus V$, $\mathrm{supp}(d\beta) \subset V$, and $|d\beta| \leq \varepsilon$. By the previous argument there exists $\eta \in L^{2,W}_1(X, \Lambda^+)$ satisfying $d^{*,W}\eta = 0$ and $\|\eta\|_{L^{2,W}(V)} < (1/3) \|\eta\|_{L^{2,W}(X)}$. Then $\|\beta\eta\|_{L^{2,W}(X)} > (2/3) \|\eta\|_{L^{2,W}(X)}$. Since $d^{*,W}(\beta\eta) = - * (d\beta \wedge \eta)$ is supported in V,

$$\left\| d^{*,W}(\beta\eta) \right\|_{L^{2,W}(X)} \leq \varepsilon \|\eta\|_{L^{2,W}(V)} < (\varepsilon/3) \|\eta\|_{L^{2,W}(X)} < (\varepsilon/2) \|\beta\eta\|_{L^{2,W}(X)}.$$

Hence $\beta\eta \in L^{2,W}_1(X, \Lambda^+)$ satisfies $\beta\eta = 0$ over U and $\left\| d^{*,W}(\beta\eta) \right\|_{L^{2,W}(X)} < \varepsilon \|\beta\eta\|_{L^{2,W}(X)}$. □

6. Linear theory

In this section we always assume $0 < \alpha < 1$ and

$$T \geq \max(T_\alpha, T_{-\alpha}). \tag{16}$$

Here T_α and $T_{-\alpha}$ are the positive constants introduced in Lemma 5.6. (Recall that they depend only on α.) The purpose of this section is to prove several basic properties of the linear operators D_A and D'_A introduced below. The constants introduced in this section often depend on several parameters (α, T, A_0, A, μ). But we usually don't explicitly write their dependence on parameters unless it causes a confusion.

6.1. **The image of D_A is closed.** Let $E = X \times SU(2)$ be the product principal $SU(2)$-bundle over X, and A_0 be an adapted connection on E (see Definition 4.1). Let $W = e^{\alpha|q(x)|'}$ be the weight function on X introduced in Section 5.3. For a section u of $\Lambda^i(\mathrm{ad}E)$ ($i \geq 0$), we define the Sobolev norm $\|u\|_{L^2_k}$ by using the periodic metric g_0 and the connection A_0. We define the weighted Sobolev norm by $\|u\|_{L^{2,W}_k} := \|Wu\|_{L^2_k}$.

Let Λ^+ and Λ^- be the bundles of self-dual and anti-self-dual forms (with respect to the metric g_0) on X, and $\mu : \Lambda^- \to \Lambda^+$ be a smooth bundle map. We assume $|\mu_x| < 1$ for all $x \in X$ (i.e. $|\mu(\omega)| < |\omega|$ for all non-zero $\omega \in \Lambda^-$ where the norm $|\cdot|$ is defined by the metric g_0). Moreover we assume that μ is compact supported. Hence μ corresponds to a conformal structure on X which coincides with $[g_0]$ outside a compact set (see Section 3.1).

We define $\mathcal{A} = \mathcal{A}_{A_0}$ as the space of $L^{2,W}_3$-connections (with respect to A_0) on E:

$$\mathcal{A} := \{A_0 + a |\, a \in L^{2,W}_3(X, \Lambda^1(\mathrm{ad}E))\}.$$

(Recall that the connection A_0 is used in the definition of the weighted Sobolev space $L^{2,W}_3(X, \Lambda^1(\mathrm{ad}E))$.) We will need the following multiplication rule: If $k \geq 3$ and $k \geq l$, then $L^{2,W}_k \times L^{2,W}_l \to L^{2,W}_l$, i.e. for $f_1 \in L^{2,W}_k$ and $f_2 \in L^{2,W}_l$ ($k \geq 3$, $k \geq l \geq 0$)

$$\|f_1 f_2\|_{L^{2,W}_l} \leq \mathrm{const}\, \|f_1\|_{L^{2,W}_k} \|f_2\|_{L^{2,W}_l}\,. \tag{17}$$

In particular, for $A = A_0 + a \in \mathcal{A}$, we have $F(A) = F(A_0) + d_{A_0}a + a \wedge a \in L^{2,W}_2$. For $b \in \Omega^1(\mathrm{ad}E)$ over X, we set

$$D_A b := -d_A^{*,W} b + (d_A^+ - \mu d_A^-)b = -W^{-2} d_A^*(W^2 b) + (d_A^+ - \mu d_A^-)b.$$

Here $d_A^* b = -*d_A(*b)$ and $d_A^\pm = \frac{1}{2}(1 \pm *)d_A$. ($*$ is the Hodge star defined by the metric g_0.) D_A is an elliptic differential operator since we assume $|\mu_x| < 1$ for all $x \in X$. Rigorously speaking, we should use the notation $D_A^{\mu,W}$ instead of D_A. But here we use the above notation for simplicity. We have

$$D_A b = D_{A_0} b + *[a \wedge *b] + [a \wedge b]^+ - \mu([a \wedge b]^-). \tag{18}$$

From this and the above (17), the map $D_A : L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E)) \to L^{2,W}_k(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ ($0 \leq k \leq 3$) becomes a bounded linear map.

Let r be a positive integer such that $q^{-1}(-rT, rT)$ contains the supports of $F(A_0)$ and μ. Set $U := q^{-1}(-(r+5/2)T, (r+5/2)T)$.

Lemma 6.1. *(i) For any $b \in L^{2,w}_{k+1}(X, \Lambda^1(\mathrm{ad}E))$ ($k \geq 0$) we have*

$$\|b\|_{L^{2,W}_{k+1}(X)} \leq \mathrm{const}\, (\|b\|_{L^2(U)} + \|D_{A_0} b\|_{L^{2,W}_k(X)}). \tag{19}$$

Here const *is a positive constant independent of b. (We will usually omit this kind of obvious remark below.)*

(ii) For any $A = A_0 + a \in \mathcal{A}$, there is a pre-compact open set $U_A \subset X$ (which depends on μ, α, T, A_0, A) such that for any $b \in L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E))$ $(0 \leq k \leq 3)$

$$\|b\|_{L^{2,W}_{k+1}(X)} \leq \mathrm{const}\,(\|b\|_{L^2(U_A)} + \|D_A b\|_{L^{2,W}_k(X)}). \tag{20}$$

Proof. (i) We first consider the case $k = 0$. From Lemma 5.6 and the condition (16), for any $b_1 \in L^{2,\alpha}_1(X, \Lambda^1)$ and $b_2 \in L^{2,-\alpha}_1(X, \Lambda^1)$

$$\|b_1\|_{L^{2,\alpha}_1} \leq \mathrm{const}\,\|D^\alpha b_1\|_{L^{2,\alpha}}\,, \tag{21}$$

$$\|b_2\|_{L^{2,-\alpha}_1} \leq \mathrm{const}\,\left\|D^{-\alpha} b_2\right\|_{L^{2,-\alpha}}\,. \tag{22}$$

Let $b \in L^{2,W}_1(X, \Lambda^1(\mathrm{ad}E))$. Let β be a smooth function on X such that $\beta = 0$ on $t \leq (r + 1/2)T$ and $\beta = 1$ on $t \geq (r + 1)T$ $(t = q(x))$. Recall that $\mathrm{supp}(\mu)$ and $\mathrm{supp}(F_{A_0})$ are contained in $q^{-1}(-rT, rT)$ and that $W = e^{\alpha t}$ for $t \geq 1$. By applying the above (21) to βa, we get

$$\|\beta b\|_{L^{2,W}_1(X)} \leq \mathrm{const}\,\|D_{A_0}(\beta b)\|_{L^{2,W}(X)} \leq \mathrm{const}\,(\|b\|_{L^2(U)} + \|D_{A_0} b\|_{L^{2,W}(X)}). \tag{23}$$

Let β' be a smooth function on X such that $\beta' = 0$ on $t \geq -(r + 1/2)T$ and $\beta' = 1$ on $t \leq -(r + 1)T$. By applying (22) to $\beta' b$, we get

$$\|\beta' b\|_{L^{2,W}_1(X)} \leq \mathrm{const}\,\|D_{A_0}(\beta' b)\|_{L^{2,W}(X)} \leq \mathrm{const}\,(\|b\|_{L^2(U)} + \|D_{A_0} b\|_{L^{2,W}(X)}). \tag{24}$$

From the elliptic regularity,

$$\|b\|_{L^{2,W}_1(q^{-1}(-(r+3/2)T,(r+3/2)T))} \leq \mathrm{const}\,(\|b\|_{L^2(U)} + \|D_{A_0} b\|_{L^2(U)}).$$

This estimate and the above (23) and (24) imply

$$\|b\|_{L^{2,W}_1(X)} \leq \mathrm{const}\,(\|b\|_{L^2(U)} + \|D_{A_0} b\|_{L^{2,W}(X)}). \tag{25}$$

Next let $b \in L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E))$. From the elliptic regularity, for any $n \in \mathbb{Z}$

$$\begin{aligned}\|b\|_{L^{2,W}_{k+1}(q^{-1}((n-1/2)T,\,(n+1/2)T))} &= \|Wb\|_{L^2_{k+1}(q^{-1}((n-1/2)T,\,(n+1/2)T))} \\ &\leq \mathrm{const}\,(\|Wb\|_{L^2_k(Y_T^{(n)})} + \|D_{A_0}(Wb)\|_{L^2_k(Y_T^{(n)})}) \\ &\leq \mathrm{const}\,(\|b\|_{L^{2,W}_k(Y_T^{(n)})} + \|D_{A_0} b\|_{L^{2,W}_k(Y_T^{(n)})}).\end{aligned}$$

The above two "const" are independent of $n \in \mathbb{Z}$. Therefore

$$\|b\|_{L^{2,W}_{k+1}(X)} \leq \mathrm{const}\,(\|b\|_{L^{2,W}_k(X)} + \|D_{A_0} b\|_{L^{2,W}_k(X)}). \tag{26}$$

By using this estimate and the above (25), we can inductively prove (19).

(ii) From (i)

$$\|b\|_{L^{2,W}_{k+1}(X)} \leq C(\|b\|_{L^2(U)} + \|D_{A_0} b\|_{L^{2,W}_k(X)}), \tag{27}$$

where the positive constant C depends on μ, α, T, A_0. Take $\varepsilon > 0$ so that $C\varepsilon < 1$. From (17), (18) and $a \in L^{2,W}_3$, there is a positive integer $r_A > r$ ($U_A := q^{-1}(-(r_A + 5/2)T, (r_A + 5/2)T) \supset U$) such that

$$\|D_A b - D_{A_0} b\|_{L^{2,W}_k(X\setminus\overline{U_A})} \leq \varepsilon \|b\|_{L^{2,W}_k(X)} \quad (0 \leq k \leq 3).$$

On the other hand

$$\|D_A b - D_{A_0} b\|_{L^{2,W}_k(U_A)} \leq \mathrm{const}\, \|b\|_{L^2_k(U_A)}.$$

Therefore, from (27),

$$\|b\|_{L^{2,W}_{k+1}(X)} \leq \mathrm{const}\,(\|b\|_{L^2_k(U_A)} + \|D_A b\|_{L^{2,W}_k(X)}) + C\varepsilon \|b\|_{L^{2,W}_k(X)}.$$

Since $C\varepsilon < 1$, we get

$$\|b\|_{L^{2,W}_{k+1}(X)} \leq \mathrm{const}\,(\|b\|_{L^2_k(U_A)} + \|D_A b\|_{L^{2,W}_k(X)}).$$

By the induction on k, we get (20). $\square$

Proposition 6.2. *Let $A \in \mathcal{A}$. If $b \in L^{2,W}(X, \Lambda^1(\mathrm{ad}E))$ satisfies $D_A b = 0$ as a distribution, then $b \in L^{2,W}_4(X)$. Let $0 \leq k \leq 3$. The kernel of the map $D_A : L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E)) \to L^{2,W}_k(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ is of finite dimension, and the image $D_A(L^{2,W}_{k+1}(X, \Lambda^1(adE)))$ is closed in $L^{2,W}_k(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$.*

Proof. The first regularity statement ($D_A b = 0 \Rightarrow b \in L^{2,W}_4$) follows from Lemma 6.1 (ii). Let $\mathrm{Ker} D_A$ be the space of $b \in L^{2,W}_4(X, \Lambda^1(\mathrm{ad}E))$ satisfying $D_A b = 0$. For any $b \in \ker D_A$, $\|b\|_{L^{2,W}_4(X)} \leq \mathrm{const}\, \|b\|_{L^2(U_A)}$ by Lemma 6.1 (ii). Here U_A is a pre-compact open set. Then the standard argument using Rellich's lemma shows the finite dimensionality of $\ker D_A$.

Sublemma 6.3. *If $b \in L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E))$ $(0 \leq k \leq 3)$ is $L^{2,W}$-orthogonal to $\mathrm{Ker} D_A$ (i.e. $\int_X W^2 \langle b, \beta\rangle d\mathrm{vol} = 0$ for all $\beta \in \mathrm{Ker} D_A$) then*

$$\|b\|_{L^{2,W}_{k+1}(X)} \leq \mathrm{const}\, \|D_A b\|_{L^{2,W}_k(X)}.$$

Proof. It is enough to prove $\|b\|_{L^2(U_A)} \leq \mathrm{const}\, \|D_A b\|_{L^{2,W}(X)}$. Since U_A is pre-compact, this follows from the standard argument using Lemma 6.1 (ii) and Rellich's lemma. $\square$

Let $H \subset L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E))$ be the $L^{2,W}$-orthogonal complement of $\ker D_A$. Then Sublemma 6.3 shows that $\mathrm{image}(D_A) = D_A(H)$ is a closed subspace in $L^{2,W}_k(X, \Lambda^1(\mathrm{ad}E))$. $\square$

6.2. **The kernel of D'_A is infinite dimensional.** For $\mu : \Lambda^- \to \Lambda^+$ we define $\mu^* : \Lambda^+ \to \Lambda^-$ by

$$\mu(\xi) \wedge \eta = \xi \wedge \mu^*(\eta) \quad (\xi \in \Lambda^-, \eta \in \Lambda^+).$$

Let $A = A_0 + a \in \mathcal{A}$. For $\omega \in \Omega^2(\mathrm{ad}E)$, we set $d^{*,W}_A \omega = -W^{-2} * d_A(*W^2\omega)$. If $b \in \Omega^1(\mathrm{ad}E)$ and $\omega \in \Omega^2(\mathrm{ad}E)$ have compact supports, then $\int_X W^2\langle d_A b, \omega\rangle d\mathrm{vol} = \int_X W^2 \langle b, d^{*,W}_A \omega\rangle d\mathrm{vol}$. For $\rho = (u, \eta) \in \Omega^0(\mathrm{ad}E) \oplus \Omega^+(\mathrm{ad}E)$, we set

$$D'_A \rho := -d_A u + d^{*,W}_A (1 + \mu^*)\eta = -d_A u - W^{-2} * d_A(W^2(1 - \mu^*)\eta).$$

D'_A is an elliptic differential operator. If $b \in \Omega^1(\mathrm{ad}E)$ and $\rho \in \Omega^0(\mathrm{ad}E) \oplus \Omega^+(\mathrm{ad}E)$ have compact supports, then $\int_X W^2 \langle D_A b, \rho \rangle d\mathrm{vol} = \int_X W^2 \langle b, D'_A \rho \rangle d\mathrm{vol}$. We have

$$D'_A(u,\eta) = D'_{A_0}(u,\eta) - [a,u] - *[a \wedge (1-\mu^*)\eta]. \tag{28}$$

From the multiplication rule (17), D'_A defines a bounded linear map $D'_A : L^{2,W}_{k+1}(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E)) \to L^{2,W}_k(X, \Lambda^1(\mathrm{ad}E))$ for $0 \le k \le 3$.

Lemma 6.4. *For any* $\rho \in L^{2,W}_{k+1}(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ $(0 \le k \le 3)$,

$$\|\rho\|_{L^{2,W}_{k+1}(X)} \le \mathrm{const}\,(\|\rho\|_{L^{2,W}(X)} + \|D'_A\rho\|_{L^{2,W}_k(X)}).$$

Hence if $\rho \in L^{2,W}(X)$ *satisfies* $D'_A\rho = 0$ *as a distribution, then* $\rho \in L^{2,W}_4(X)$.

Proof. In the same way as in the proof of the estimate (26), we get

$$\|\rho\|_{L^{2,W}_{k+1}(X)} \le \mathrm{const}\,(\|\rho\|_{L^{2,W}(X)} + \left\|D'_{A_0}\rho\right\|_{L^{2,W}_k(X)}).$$

By using the multiplication rule (17), we get the desired estimate. The regularity statement easily follows from the above estimate. □

Let $\mathrm{Ker}D_A$ be the space of $b \in L^{2,W}_4(X, \Lambda^1(\mathrm{ad}E))$ satisfying $D_A b = 0$, and $\mathrm{Ker}D'_A$ be the space of $\rho \in L^{2,W}_4(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ satisfying $D'_A\rho = 0$.

Lemma 6.5. *Let* $A \in \mathcal{A}$ *and* $0 \le k \le 3$.
(i) We have the following $L^{2,W}$*-orthogonal decomposition:*

$$L^{2,W}_k(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E)) = D_A(L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E))) \oplus \mathrm{Ker}D'_A.$$

(ii) If $\rho \in L^{2,W}_{k+1}(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ *is* $L^{2,W}$*-orthogonal to the space* $\mathrm{Ker}D'_A$*, then*

$$\|\rho\|_{L^{2,W}_{k+1}(X)} \le \mathrm{const}\,\|D'_A\rho\|_{L^{2,W}_k(X)}.$$

Hence $D'_A(L^{2,W}_{k+1}(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E)))$ *is a closed subspace in* $L^{2,W}_k(X, \Lambda^1(\mathrm{ad}E))$.
(iii) We have the following $L^{2,W}$*-orthogonal decomposition:*

$$L^{2,W}_k(X, \Lambda^1(\mathrm{ad}E)) = D'_A(L^{2,W}_{k+1}(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))) \oplus \mathrm{Ker}D_A.$$

Proof. (i) $\mathrm{Ker}D'_A$ is closed in $L^{2,W}$. From Proposition 6.2, $D_A(L^{2,W}_{k+1})$ is closed in $L^{2,W}_k$, and it is $L^{2,W}$-orthogonal to $\mathrm{Ker}D'_A$. If $\rho \in L^{2,W}((\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ is $L^{2,W}$-orthogonal to the space $D_A(L^{2,W}_1)$, then $D'_A\rho = 0$ as a distribution. Hence $L^{2,W} = D_A(L^{2,W}_1) \oplus \mathrm{Ker}D'_A$. By this decomposition, for $\rho \in L^{2,W}_k((\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$, there are $b \in L^{2,W}_1$ and $\rho' \in \mathrm{Ker}D'_A$ satisfying $\rho = D_A b + \rho'$. By Lemma 6.4, $\rho' \in L^{2,W}_4$ and hence $D_A b = \rho - \rho' \in L^{2,W}_k$. Then by Lemma 6.1 (ii), $b \in L^{2,W}_{k+1}$. This shows $L^{2,W}_k = D_A(L^{2,W}_{k+1}) \oplus \mathrm{Ker}D'_A$.

(ii) By (i), there is $b \in L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E))$ satisfying $\rho = D_A b$. We can choose b so that it is $L^{2,W}$-orthogonal to $\mathrm{Ker}D_A$ and that $\|b\|_{L^{2,W}} \le \mathrm{const}\,\|D_A b\|_{L^{2,W}} = \mathrm{const}\,\|\rho\|_{L^{2,W}}$ (by Sublemma 6.3). Then

$$\|\rho\|^2_{L^{2,W}} = \langle \rho, D_A b\rangle_{L^{2,W}} = \langle D'_A\rho, b\rangle_{L^{2,W}} \le \|D'_A\rho\|_{L^{2,W}}\,\|b\|_{L^{2,W}} \le \mathrm{const}\,\|D'_A\rho\|_{L^{2,W}}\,\|\rho\|_{L^{2,W}}.$$

Thus $\|\rho\|_{L^{2,W}} \leq \mathrm{const}\,\|D'_A\rho\|_{L^{2,W}}$. Then by using Lemma 6.4, we get the desired estimate.

(iii) $D'_A(L^{2,W}_{k+1})$ is $L^{2,W}$-orthogonal to $\mathrm{Ker}D_A$. If $b \in L^{2,W}(X, \Lambda^1(\mathrm{ad}E))$ is $L^{2,W}$-orthogonal to the space $D'_A(L^{2,W}_1)$, then $D_Ab = 0$ as a distribution. Hence $L^{2,W} = D'_A(L^{2,W}_1) \oplus \mathrm{Ker}D_A$. From this result, for any $b \in L^{2,W}_k(X, \Lambda^1(\mathrm{ad}E))$, there are $\rho \in L^{2,W}_1(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ and $\beta \in \mathrm{Ker}D_A$ satisfying $b = D'_A\rho + \beta$. From Lemma 6.2, $\beta \in L^{2,W}_4$. Thus $D'_A\rho \in L^{2,W}_k$, and hence (Lemma 6.4) $\rho \in L^{2,W}_{k+1}$. □

Lemma 6.6. *Let $0 \leq k \leq 3$, and $A \in \mathcal{A}$ be a μ-ASD connection (i.e. $F^+_A = \mu(F^-_A)$). Let $\mathrm{Ker}d^{*,W}_A \cap L^{2,W}_k$ be the space of $b \in L^{2,W}_k(X, \Lambda^1(\mathrm{ad}E))$ satisfying $d^{*,W}_A b = W^{-2}d^*_A(W^2b) = 0$ as a distribution. Then the following map is isomorphic:*

$$L^{2,W}_{k+1}(X, \Lambda^0(\mathrm{ad}E)) \oplus (\mathrm{Ker}d^{*,W}_A \cap L^{2,W}_k) \to L^{2,W}_k(X, \Lambda^1(\mathrm{ad}E)), \quad (u, b) \mapsto -d_Au + b. \tag{29}$$

Proof. The above (29) is a bounded linear map. If $(u, b) \in L^{2,W}_{k+1} \oplus (\mathrm{Ker}d^{*,W}_A \cap L^{2,W}_k)$ satisfies $-d_Au + b = 0$, then $\|b\|^2_{L^{2,W}} = \langle b, d_Au\rangle_{L^{2,W}} = 0$ by $d^{*,W}_A b = 0$ Hence $b = d_Au = 0$. $d_Au = 0$ implies that $|u|$ is constant. But $u \in L^{2,W}$. Hence $u = 0$. Therefore the map (29) is injective.

Let $b \in L^{2,W}_k(X, \Lambda^1(\mathrm{ad}E))$. By Lemma 6.5 (iii), there exists $(u, \eta) \in L^{2,W}_{k+1}(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ and $\beta \in \mathrm{Ker}D_A$ satisfying

$$b = D'_A(u, \eta) + \beta = -d_Au + d^{*,W}_A(1 + \mu^*)\eta + \beta.$$

Since $D_A\beta = 0$, we have $d^{*,W}_A\beta = 0$. Since A is μ-ASD, we have $d^{*,W}_A d^{*,W}_A(1 + \mu^*)\eta = 0$. Thus $d^{*,W}_A(d^{*,W}_A(1 + \mu^*)\eta + \beta) = 0$. This argument shows that the map (29) is surjective and hence isomorphic. □

Proposition 6.7. *Suppose $b_+(Y) \geq 1$. For any $A = A_0 + a \in \mathcal{A}$, the space $\mathrm{Ker}D'_A \subset L^{2,W}_4(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$ is infinite dimensional.*

Proof. Suppose that $\mathrm{Ker}D'_A$ is finite dimensional. Then there is a pre-compact open set $V \subset X$ such that for any non-zero $\rho \in \mathrm{Ker}D'_A$ we have $\rho|_V \not\equiv 0$. Then

$$\|\rho\|_{L^{2,W}(X)} \leq \mathrm{const}\,\|\rho\|_{L^2(V)} \quad (\rho \in \mathrm{Ker}D'_A).$$

We want to prove the following: There exists a positive constant C depending on μ, α, T, A_0, A such that for any $\rho \in L^{2,W}_1(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$

$$\|\rho\|_{L^{2,W}(X)} \leq C(\|\rho\|_{L^2(V)} + \|D'_A\rho\|_{L^{2,W}(X)}). \tag{30}$$

Let $\rho \in L^{2,W}_1(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$, and $\rho = \rho_0 + \rho_1$ be a decomposition such that $\rho_0 \in \mathrm{Ker}D'_A$ and that $\rho_1 \in L^{2,W}_1(X)$ is $L^{2,W}$-orthogonal to $\mathrm{Ker}D'_A$. Then

$$\begin{aligned} \|\rho\|_{L^{2,W}(X)} &\leq \|\rho_0\|_{L^{2,W}(X)} + \|\rho_1\|_{L^{2,W}(X)} \leq \mathrm{const}\,\|\rho_0\|_{L^2(V)} + \|\rho_1\|_{L^{2,W}(X)}, \\ &\leq \mathrm{const}(\|\rho\|_{L^2(V)} + \|\rho_1\|_{L^2(V)}) + \|\rho_1\|_{L^{2,W}(X)} \\ &\leq \mathrm{const}\,\|\rho\|_{L^2(V)} + (1 + \mathrm{const})\,\|\rho_1\|_{L^{2,W}(X)}. \end{aligned} \tag{31}$$

From Lemma 6.5 (ii)

$$\|\rho_1\|_{L^{2,W}(X)} \leq \mathrm{const}\,\|D'_A\rho_1\|_{L^{2,W}(X)} = \mathrm{const}\,\|D'_A\rho\|_{L^{2,W}(X)}.$$

From this and the above (31), we get (30).

Take $\varepsilon > 0$ satisfying $2\varepsilon C < 1$ (C is a constant in (30)). For this ε, we can choose a positive integer R such that $V' := q^{-1}(-(R+1/2)T, (R+1/2)T)$ contains $V \cup \mathrm{supp}(\mu) \cup \mathrm{supp}(F_{A_0})$ and that for any $\rho \in (\Omega^0 \oplus \Omega^+)(\mathrm{ad}E)$ we have (see (28))

$$|D'_{A_0}\rho(x) - D'_A\rho(x)| \leq \varepsilon|\rho(x)| \quad (x \in X \setminus V').$$

From Lemma 5.9, there is $\eta \in L^{2,W}_1(X, \Lambda^+(\mathrm{ad}E))$ such that $\eta = 0$ over V' and $\left\|D'_{A_0}\eta\right\|_{L^{2,W}(X)} < \varepsilon\,\|\eta\|_{L^{2,W}(X)}$. (Here $D'_{A_0}\eta := D'_{A_0}(0,\eta)$.) Then $\|D'_A\eta\|_{L^{2,W}(X)} < 2\varepsilon\,\|\eta\|_{L^{2,W}(X)}$. But the above (30) implies

$$\|\eta\|_{L^{2,W}(X)} \leq C\,\|D'_A\eta\|_{L^{2,W}(X)} < 2C\varepsilon\,\|\eta\|_{L^{2,W}(X)}.$$

Since we choose $2C\varepsilon < 1$, this is a contradiction. □

Let $\mathrm{Ker}(d^{*,W}_A(1+\mu^*))$ be the space of $\eta \in L^{2,W}(X, \Lambda^+(\mathrm{ad}E))$ satisfying $d^{*,W}_A(1+\mu^*)\eta = 0$ as a distribution. If $\eta \in \mathrm{Ker}(d^{*,W}_A(1+\mu^*))$, then $D'_A(0,\eta) = 0$. Hence $\eta \in L^{2,W}_4(X)$ by Lemma 6.4. The space $\mathrm{Ker}(d^{*,W}_A(1+\mu^*))$ is closed in $L^{2,W}(X, \Lambda^+(\mathrm{ad}E))$, and hence it is closed in $L^{2,W}_k(X, \Lambda^+(\mathrm{ad}E))$ for all $0 \leq k \leq 4$. The following proposition is the conclusion of this section.

Proposition 6.8. *Suppose that $A \in \mathcal{A}$ is a μ-ASD connection.*
(i) Let $(u,\eta) \in L^{2,W}(X, (\Lambda^0 \oplus \Lambda^+)(\mathrm{ad}E))$. We have $D'_A(u,\eta) = 0$ if and only if $u = 0$ and $d^{*,W}_A(1+\mu^*)\eta = 0$. Hence $\mathrm{Ker}D'_A = \mathrm{Ker}(d^{*,W}_A(1+\mu^*))$. Moreover if $b_+(Y) \geq 1$ then the space $\mathrm{Ker}(d^{*,W}_A(1+\mu^*))$ is infinite dimensional.
(ii) Let $0 \leq k \leq 3$. Let $\mathrm{Ker}d^{*,W}_A \cap L^{2,W}_{k+1}$ be the space of $b \in L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E))$ satisfying $d^{*,W}_A b = 0$. Then the space $(d^+_A - \mu d^-_A)(\mathrm{Ker}d^{*,W}_A \cap L^{2,W}_{k+1})$ is closed in $L^{2,W}_k(X, \Lambda^+(\mathrm{ad}E))$, and we have the following $L^{2,W}$-orthogonal decomposition:

$$L^{2,W}_k(X, \Lambda^+(\mathrm{ad}E)) = \mathrm{Ker}(d^{*,W}_A(1+\mu^*)) \oplus (d^+_A - \mu d^-_A)(\mathrm{Ker}d^{*,W}_A \cap L^{2,W}_{k+1}). \tag{32}$$

Proof. (i) Suppose $D'_A(u,\eta) = -d_A u + d^{*,W}_A(1+\mu^*)\eta = 0$. Then $(u,\eta) \in L^{2,W}_4(X)$ by Lemma 6.4. Since A is μ-ASD, $d_A u$ and $d^{*,W}_A(1+\mu^*)\eta$ are $L^{2,W}$-orthogonal to each other. Hence $d_A u = d^{*,W}_A(1+\mu^*)\eta = 0$. Then $u = 0$ and $d^{*,W}_A(1+\mu^*)\eta = 0$. Therefore $\mathrm{Ker}D'_A = \mathrm{Ker}(d^{*,W}_A(1+\mu^*))$. If $b_+(Y) \geq 1$, then $\mathrm{Ker}(d^{*,W}_A(1+\mu^*)) = \mathrm{Ker}D'_A$ is infinite dimensional by Proposition 6.7.

(ii) By Lemma 6.5 (i), $\eta \in L^{2,W}_k(X, \Lambda^+(\mathrm{ad}E))$ is $L^{2,W}$-orthogonal to $\mathrm{Ker}(d^{*,W}_A(1+\mu^*))$ if and only if there exists $b \in L^{2,W}_{k+1}(X, \Lambda^1(\mathrm{ad}E))$ satisfying $(0,\eta) = D_A b$ (i.e. $d^{*,W}_A b = 0$ and $(d^+_A - \mu d^-_A)b = \eta$). This shows that $(d^+_A - \mu d^-_A)(\mathrm{Ker}d^{*,W}_A \cap L^{2,W}_{k+1})$ is closed in $L^{2,W}_k(X, \Lambda^+(\mathrm{ad}E))$ and that we have the decomposition (32) (the factors of the decomposition are $L^{2,W}$-orthogonal to each other). □

7. Non-existence of reducible instantons

Lemma 7.1. *Let $I \subset \mathbb{R}$ be an open interval. Let ω be a smooth anti-self-dual 2-form on $I\times S^3$ satisfying $d\omega = 0$. Then there exists a smooth 1-form a on $I\times S^3$ satisfying $da = \omega$ and $\|a\|_{L^2(I\times S^3)} \leq (1/\sqrt{8})\,\|\omega\|_{L^2(I\times S^3)}$.*

Proof. Since ω is ASD, it can be written as:

$$\omega = dt\wedge\phi - *_3\phi,$$

where $\phi\in\Gamma(I\times S^3,\Lambda^1_{S^3})$ (cf. Section 5.1). Then $d\omega=0$ is equivalent to

$$\frac{\partial\phi}{\partial t} = - *_3 d_3\phi, \quad d_3^*\phi = 0.$$

Let $\mathrm{Ker}(d_3^*)\subset\Omega^1_{S^3}$ be the space of co-closed 1-forms in S^3, and consider the operator $*_3 d_3 : \mathrm{Ker}(d_3^*)\to\mathrm{Ker}(d_3^*)$. This is an isomorphism by $H^1_{dR}(S^3)=0$, and its inverse is given by $*_3d_3\Delta_3^{-1} : \mathrm{Ker}(d_3^*)\to\mathrm{Ker}(d_3^*)$. We set $a := - *_3 d_3\Delta_3^{-1}\phi\in\Gamma(I\times S^3,\Lambda^1_{S^3})$. a satisfies $d_3^*a = 0$ and $*_3d_3a = -\phi$. Then

$$*_3d_3\left(\frac{\partial a}{\partial t}\right) = -\frac{\partial\phi}{\partial t} = *_3d_3\phi.$$

Since $\partial a/\partial t$ and ϕ are both contained in $\mathrm{Ker}(d_3^*)$, we have $\partial a/\partial t = \phi$. Then we have $da=\omega$. Moreover (Corollary 3.5 (i))

$$\int_{\{t\}\times S^3}|\phi|^2d\mathrm{vol}_3 = \int_{\{t\}\times S^3}|d_3a|^2d\mathrm{vol}_3 \geq 4\int_{\{t\}\times S^3}|a|^2d\mathrm{vol}_3.$$

Since $|\omega|^2 = 2|\phi|^2$, we get $\|\omega\|_{L^2(I\times S^3)}\geq\sqrt{8}\,\|a\|_{L^2(I\times S^3)}$. □

Let $\mu:\Lambda^-\to\Lambda^+$ be a compact-supported smooth bundle map satisfying $|\mu_x|<1$ for all $x\in X$. A 2-form ω on X is said to be μ-ASD if it satisfies $\omega^+ = \mu(\omega^-)$ where ω^+ and ω^- are the self-dual and anti-self-dual parts of ω with respect to the periodic metric g_0. ω is μ-ASD if and only if ω is ASD with respect to the conformal structure corresponding to μ. (See Corollary 3.2.)

Proposition 7.2. *Suppose $b_-(Y)=0$. If ω is a smooth μ-ASD 2-form on X satisfying $d\omega=0$ and $\|\omega\|_{L^2(X)}<\infty$, then $\omega=0$. (Indeed, if $\omega\in L^2(X,\Lambda^2)$ is μ-ASD and satisfies $d\omega=0$ as a distribution, then ω is smooth by the elliptic regularity. Hence the assumption of the smoothness of ω can be weakened.)*

Proof. Suppose $\omega\neq 0$. We can assume $\|\omega\|_{L^2(X)}=1$. We have $\int_X\omega\wedge\omega = \int_X(|\mu(\omega^-)|^2 - |\omega^-|^2)d\mathrm{vol}<0$. So we can take $\delta>0$ so that $\int_X\omega\wedge\omega<-\delta$. Let $\varepsilon>0$ be a positive number satisfying

$$(2+\varepsilon)\varepsilon\leq\delta/2. \tag{33}$$

Let $N > 0$ be a large integer such that $U := q^{-1}(-NT, NT)$ satisfies $U \supset \mathrm{supp}(\mu)$ and

$$\frac{2T-3}{T-2} \|\omega\|_{L^2(X\setminus U)} \le \varepsilon. \tag{34}$$

(Recall $T > 2$.) Set $V := q^{-1}(-(N+1)T+1, -NT-1) \sqcup q^{-1}(NT+1, (N+1)T-1)$. V is isometric to the disjoint union of the two copies of $(1, T-1) \times S^3$, and we have $V \subset X \setminus U$. From Lemma 7.1, there exists a 1-form a on V satisfying $da = \omega$ and $\|a\|_{L^2(V)} \le (1/\sqrt{8}) \|\omega\|_{L^2(V)}$. Let β be a smooth function on X such that $0 \le \beta \le 1$, $\mathrm{supp}(d\beta) \subset V$, $\beta = 0$ over $|t| \ge (N+1)T-1$, $\beta = 1$ over $|t| \le NT+1$ and $|d\beta| \le 2/(T-2)$. (Here $t = q(x)$.) We define a compact-supported 2-form ω' by

$$\omega' := \begin{cases} \omega & \text{on } |t| \le NT+1 \\ d(\beta a) & \text{on } V \\ 0 & \text{on } |t| \ge (N+1)T-1. \end{cases}$$

ω' is a closed 2-form ($d\omega' = 0$).

$$\|d(\beta a)\|_{L^2(V)} \le \frac{2}{T-2}\|a\|_{L^2(V)} + \|\omega\|_{L^2(V)} \le \frac{1}{T-2}\|\omega\|_{L^2(V)} + \|\omega\|_{L^2(V)} \le \frac{T-1}{T-2}\|\omega\|_{L^2(V)}.$$

Then, by (34), $\|\omega' - \omega\|_{L^2(X)} \le \frac{2T-3}{T-2}\|\omega\|_{L^2(X\setminus U)} \le \varepsilon$ and $\|\omega'\|_{L^2(X)} \le 1 + \varepsilon$.

$$\left|\int_X \omega\wedge\omega - \int_X \omega'\wedge\omega'\right| \le \left(\|\omega\|_{L^2(X)} + \|\omega'\|_{L^2(X)}\right)\|\omega - \omega'\|_{L^2(X)} \le (2+\varepsilon)\varepsilon \le \delta/2.$$

Here we have used (33). Since we have $\int_X \omega\wedge\omega < -\delta$,

$$\int_X \omega'\wedge\omega' \le -\delta/2.$$

On the other hand, since ω' is closed and compact-supported ($\mathrm{supp}(\omega') \subset q^{-1}(-(N+1)T+1, (N+1)T-1)$), ω' can be considered as a closed 2-form defined on $Y^{\sharp(2N+1)}$ (the connected sum of the $(2N+1)$-copies of Y). Since $b_-(Y) = 0$, the intersection form of $Y^{\sharp(2N+1)}$ is positive definite. Hence

$$0 \le \int_{Y^{\sharp(2N+1)}} \omega'\wedge\omega' = \int_X \omega'\wedge\omega' \le -\delta/2.$$

Here δ is positive. This is a contradiction. □

Recall that $E = X \times SU(2)$ is the product principal $SU(2)$-bundle over X.

Corollary 7.3. *Suppose $b_-(Y) = 0$. If A is a reducible μ-ASD connection on E satisfying*

$$\int_X |F_A|^2 d\mathrm{vol} < +\infty,$$

then A is flat.

8. Moduli theory

8.1. **Sard-Smale's theorem.** In this subsection we review a variant of Sard-Smale's theorem [14] which will be used later. Let M_1, M_2, M_3 be Banach manifolds. We assume that they are all second countable. Let $f : M_1 \times M_2 \to M_3$ be a $\mathcal{C}^\infty$-map. Let $(x_0, y_0) \in M_1 \times M_2$ and set $z_0 := f(x_0, y_0) \in M_3$. Suppose that the following two conditions hold.
(i) The derivative $df_{(x_0,y_0)} : T_{x_0}M_1 \oplus T_{y_0}M_2 \to T_{z_0}M_3$ is surjective.
(ii) The partial derivative $d_1f_{(x_0,y_0)} : T_{x_0}M_1 \to T_{z_0}M_3$ with respect to M_1-direction is a Fredholm operator with $\dim \mathrm{Ker}(d_1f_{(x_0,y_0)}) < \dim \mathrm{Coker}(d_1f_{(x_0,y_0)})$.

Under these conditions we want to prove the following proposition. (Recall that a subset of a topological space is said to be of first category if it is a countable union of nowhere-dense subsets.)

Proposition 8.1. *There exists an open neighborhood $U \times U' \subset M_1 \times M_2$ of (x_0, y_0) such that the set $\{y \in U' | \, \exists x \in U : f(x, y) = z_0\}$ is of first category in M_2.*

I believe that this is a standard result. But for the completeness of the argument we will give its brief proof below.

Lemma 8.2. *There is a bounded linear map $Q : T_{z_0}M_3 \to T_{x_0}M_1 \oplus T_{y_0}M_2$ which is a right inverse of $df_{(x_0,y_0)}$, i.e. $df_{(x_0,y_0)} \circ Q = 1$.*

Proof. Set $D := d_1f_{(x_0,y_0)} : T_{x_0}M_1 \to T_{z_0}M_3$. Since D is Fredholm, we have decompositions: $T_{x_0}M_1 = \mathrm{Ker}D \oplus V$ and $T_{z_0}M_3 = \mathrm{Im}D \oplus W$ where V and W are closed subspaces and moreover W is finite dimensional. The restriction $D|_V : V \to \mathrm{Im}D$ is an isomorphism. Since $df_{(x_0,y_0)}$ is surjective and W is finite dimensional, there is a bound linear map $T : W \to T_{x_0}M_1 \oplus T_{y_0}M_2$ satisfying $df_{(x_0,y_0)} \circ T = 1$. Then the map

$$Q : T_{z_0}M_3 = \mathrm{Im}D \oplus W \to T_{x_0}M_1 \oplus T_{y_0}M_2, \quad (u, v) \mapsto (D|_V)^{-1}(u) + T(v),$$

gives a right inverse of $df_{(x_0,y_0)}$. □

By the implicit function theorem, there is an open neighborhood $U \times U' \subset M_1 \times M_2$ of (x_0, y_0) such that

$$M := \{(x, y) \in U \times U' | \, f(x, y) = z_0\}$$

is a smooth submanifold of $M_1 \times M_2$, and that for any $(x, y) \in U \times U'$ the derivative $df_{(x,y)} : T_xM_1 \oplus T_yM_2 \to T_{f(x,y)}M_3$ is surjective. Let $\pi : M \to M_2$ be the natural projection.

The set of Fredholm operators is open in the space of bounded operators, and the index is locally constant on it. Hence we can choose U and U' so small that for any $(x, y) \in U \times U'$ the map $d_1f_{(x,y)} : T_xM_1 \to T_{f(x,y)}M_3$ is Fredholm and satisfies $\dim \mathrm{Ker}(d_1f_{(x,y)}) < \dim \mathrm{Coker}(d_1f_{(x,y)})$. For $(x, y) \in M$ we have

$$T_{(x,y)}M = \{(u, v) \in T_xM_1 \oplus T_yM_2 | \, d_1f_{(x,y)}u + d_2f_{(x,y)}v = 0\}.$$

Then it is easy to see that $\pi : M \to M_2$ is a Fredholm map with $\mathrm{Ker}(d\pi_{(x,y)}) \cong \mathrm{Ker}(d_1 f_{(x,y)})$ and $\mathrm{Coker}(d\pi_{(x,y)}) \cong \mathrm{Coker}(d_1 f_{(x,y)})$ for $(x, y) \in M$. (The maps $\mathrm{Ker}(d_1 f_{(x,y)}) \ni u \mapsto (u, 0) \in \mathrm{Ker}(d\pi_{(x,y)})$ and $\mathrm{Coker}(d\pi_{(x,y)}) \ni [v] \mapsto [d_2 f_{(x,y)}(v)] \in \mathrm{Coker}(d_1 f_{(x,y)})$ give isomorphisms.) In particular $\mathrm{Index}(d\pi_{(x,y)}) < 0$ for $(x, y) \in M$. Then a point $y \in M_2$ is regular for π if and only if $\pi^{-1}(y)$ is empty. We apply Sard-Smale's theorem to the map π and conclude that $\pi(M)$ is of first category in M_2. This proves Proposition 8.1.

8.2. **Review of Floer's function space.** Here we review a function space introduced by Floer [7]. Let $\vec{\tau} = (\tau_0, \tau_1, \tau_2, \cdots)$ be a sequence of positive real numbers indexed by $\mathbb{Z}_{\geq 0}$. (We will choose a special $\vec{\tau}$ below.) Let $\mathcal{C}^\infty(\mathbb{R}^n)$ be the set of all $\mathcal{C}^\infty$-functions in $\mathbb{R}^n$. (We will need only the case $n = 4$.) For $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ we set $|\nabla^k f(x)| := \max_{|\alpha|=k} |\partial^\alpha f(x)|$ $(k \geq 0)$ where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We define the norm $\|f\|_{\vec{\tau}}$ by

$$\|f\|_{\vec{\tau}} := \sum_{k \geq 0} \tau_k \sup_{x \in \mathbb{R}^n} |\nabla^k f(x)|.$$

We define $\mathcal{C}^{\vec{\tau}}(\mathbb{R}^n)$ as the set of all $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfying $\|f\|_{\vec{\tau}} < \infty$. $(\mathcal{C}^{\vec{\tau}}(\mathbb{R}^n), \|\cdot\|_{\vec{\tau}})$ becomes a Banach space. For an open set $U \subset \mathbb{R}^n$ we define $\mathcal{C}^{\vec{\tau}}_0(U)$ as the space of all $f \in \mathcal{C}^{\vec{\tau}}_0(\mathbb{R}^n)$ satisfying $f(x) = 0$ for all $x \in \mathbb{R}^n \setminus U$. $\mathcal{C}^{\vec{\tau}}_0(U)$ is a closed subspace in $\mathcal{C}^{\vec{\tau}}(\mathbb{R}^n)$.

Lemma 8.3. *For any bounded open set $U \subset \mathbb{R}^n$, $\mathcal{C}^{\vec{\tau}}_0(U)$ is separable.*

Proof. Let $\mathcal{C}_0(\mathbb{R}^n)$ be the Banach space of all continuous functions f in $\mathbb{R}^n$ which vanish at infinity (i.e. for any $\varepsilon > 0$ there is a compact set $K \subset \mathbb{R}^n$ such that $|f(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n \setminus K$). Let B be the set of all sequences $\vec{f} = (f_\alpha)_{\alpha \in \mathbb{Z}^n_{\geq 0}}$ with $f_\alpha \in \mathcal{C}_0(\mathbb{R}^n)$ satisfying

$$\left\|\vec{f}\right\|_B := \sum_{k \geq 0} \tau_k \max_{|\alpha|=k} \|f_\alpha\|_{\mathcal{C}_0(\mathbb{R}^n)} < \infty, \quad (\|f_\alpha\|_{\mathcal{C}_0(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |f_\alpha(x)|).$$

$(B, \|\cdot\|_B)$ is a Banach space. Since $\mathcal{C}_0(\mathbb{R}^n)$ is separable, B is also separable. The map

$$\mathcal{C}^{\vec{\tau}}_0(U) \to B, \quad f \mapsto (\partial^\alpha f)_{\alpha \in \mathbb{Z}^n_{\geq 0}},$$

is an isometric embedding. (Note that $\partial^\alpha f$ vanishes at infinity because $f = 0$ outside U and U is bounded.) Hence $\mathcal{C}^{\vec{\tau}}_0(U)$ is separable. □

Let $\beta : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function satisfying $\beta(x) = 0$ for $x \leq 1/3$ and $\beta(x) = 1$ for $x \geq 2/3$. We define positive numbers a_k $(k \geq 0)$ by setting

$$a_k := \max_{x \in \mathbb{R}} |\beta^{(k)}(x)| + a_{k-1} \quad (a_{-1} := 0).$$

Here $\beta^{(k)}$ is the k-th derivative of β. We set $\tau_k := (a_k^n k^k)^{-1}$ $(k \geq 1)$ and $\tau_0 := 1$.

For $0 < \delta < L$, we define a $\mathcal{C}^\infty$-function $\beta_{\delta,L} : \mathbb{R} \to \mathbb{R}$ by

$$\beta_{\delta,L}(x) := \begin{cases} \beta(\frac{x+L}{\delta}) & (x \leq 0) \\ \beta(\frac{-x+L}{\delta}) & (x \geq 0). \end{cases}$$

$\beta_{\delta,L}$ approximates the characteristic function of the interval $[-L,L]$ as $\delta \to 0$. We have

$$|\beta_{\delta,L}^{(k)}(x)| \leq \delta^{-k} a_k \quad (k \geq 0). \tag{35}$$

Note that the right-hand-side is independent of L. For $y \in \mathbb{R}^n$ and $0 < \delta < L$ we set

$$f_{y,\delta,L}(x) := \prod_{i=1}^{n} \beta_{\delta,L}(x_i - y_i).$$

$f_{y,\delta,L}$ is supported in the open cube $K_{y,L} := (y_1 - L, y_1 + L) \times \cdots \times (y_n - L, y_n + L)$, and $\lim_{\delta\to 0} f_{y,\delta,L} = 1_{K_{y,L}}$ (the characteristic function of $K_{y,L}$) in $L^r(\mathbb{R}^n)$ for $1 \leq r < \infty$.

Lemma 8.4. *$f_{y,\delta,L}$ is contained in $\mathcal{C}_0^{\vec{\tau}}(K_{y,L})$.*

Proof. For $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\partial^\alpha f_{y,\delta,L}(x) = \prod_{i=1}^n \beta_{\delta,L}^{(\alpha_i)}(x_i - y_i)$. By using (35),

$$|\partial^\alpha f_{y,\delta,L}(x)| \leq \prod_{i=1}^{n} (\delta^{-\alpha_i} a_{\alpha_i}) \leq \delta^{-|\alpha|} a_{|\alpha|}^n.$$

Hence $|\nabla^k f_{y,\delta,L}(x)| = \max_{|\alpha|=k} |\partial^\alpha f_{y,\delta,L}(x)| \leq \delta^{-k} a_k^n$. Therefore

$$\sum_{k\geq 0} \tau_k \sup_{x\in\mathbb{R}^n} |\nabla^k f_{y,\delta,L}(x)| \leq 1 + \sum_{k\geq 1} (a_k^n k^k)^{-1} \delta^{-k} a_k^n = 1 + \sum_{k\geq 1} (k\delta)^{-k} < \infty.$$

Thus $\|f_{y,\delta,L}\|_{\vec{\tau}} < \infty$. $\square$

Lemma 8.5. *For any open set $U \subset \mathbb{R}^n$ and $1 \leq r < \infty$, the space $\mathcal{C}_0^{\vec{\tau}}(U)$ is dense in $L^r(U)$.*

Proof. It is enough to prove that for any $\varepsilon > 0$ and any measurable set $E \subset U$ with $\mathrm{vol}(E) < \infty$ there exists $f \in \mathcal{C}_0^{\vec{\tau}}(U)$ satisfying $\|f - 1_E\|_{L^r(\mathbb{R}^n)} < \varepsilon$.

There is an open set $V \subset U$ satisfying $E \subset V$ and $\mathrm{vol}(V \setminus E) < (\varepsilon/4)^r$. By Vitali's covering theorem, there are open cubes $K_i = K_{y_i,L_i} \subset V$ $(i = 1, 2, \cdots, N)$ such that $K_i \cap K_j = \emptyset$ $(i \neq j)$ and $\mathrm{vol}(E \setminus \bigsqcup_{i=1}^N K_i) < (\varepsilon/4)^r$. Then

$$\left\| 1_E - \sum_{i=1}^{N} 1_{K_i} \right\|_{L^r} \leq \left(\mathrm{vol}(E \setminus \bigsqcup_{i=1}^{N} K_i) + \mathrm{vol}(V \setminus E) \right)^{1/r} \leq \varepsilon/2.$$

From Lemma 8.4, there are $f_i \in \mathcal{C}_0^{\vec{\tau}}(K_i) \subset \mathcal{C}_0^{\vec{\tau}}(U)$ $(i = 1, \cdots, N)$ satisfying $\|1_{K_i} - f_i\|_{L^r} < \varepsilon/2^{i+1}$. Then

$$\left\| 1_E - \sum_{i=1}^{N} f_i \right\|_{L^r} \leq \left\| 1_E - \sum_{i=1}^{N} 1_{K_i} \right\|_{L^r} + \sum_{i=1}^{N} \|1_{K_i} - f_i\|_{L^r} < \varepsilon.$$

$\square$

Let us go back to our infinite connected sum space $X = Y^{\sharp\mathbb{Z}}$. Take two non-empty pre-compact open sets U and V in X such that $\overline{U} \subset V$ and V is diffeomorphic to $\mathbb{R}^4$. We fix a diffeomorphism between V and $\mathbb{R}^4$ (i.e. a coordinate chart on V). Moreover we fix bundle trivializations of Λ^+ and Λ^- over V. Here Λ^+ and Λ^- are the vector bundles of self-dual and anti-self-dual 2-forms with respect to g_0. Then a bundle map $\mu : \Lambda^-|_V \to \Lambda^+|_V$ over V can be identified with a matrix-valued function in $\mathbb{R}^4$ by using the coordinate chart on V and the bundle trivializations of $\Lambda^+|_V$ and $\Lambda^-|_V$. So we can consider its norm $\|\mu\|_{\vec{\tau}}$. We define a function space $\mathcal{C}_0^{\vec{\tau}}(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ as the set of all $\mathcal{C}^\infty$-bundle maps $\mu : \Lambda^-|_V \to \Lambda^+|_V$ satisfying $\|\mu\|_{\vec{\tau}} < \infty$ and $\mu_x = 0$ for all $x \in V \setminus U$. From Lemmas 8.3 and 8.5, we get the following.

Lemma 8.6. *The Banach space $\mathcal{C}_0^{\vec{\tau}}(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ is separable, and it is dense in $L^r(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ $(1 \le r < \infty)$.*

Since $\mu \in \mathcal{C}_0^{\vec{\tau}}(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ vanishes outside of U and $\overline{U} \subset V$, μ can be smoothly extended all over X by zero. By this extension, we consider that all $\mu \in \mathcal{C}_0^{\vec{\tau}}(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ are defined over X.

8.3. **Metric perturbation.** Recall that $X = Y^{\sharp\mathbb{Z}}$ is the infinite connected sum space with the periodic metric g_0 and the weight function $W = e^{\alpha|q(x)|'}$, and that $E = X \times SU(2)$ is the product principal $SU(2)$-bundle on X. In this subsection we suppose that $0 < \alpha < 1$ and the condition (16) in Section 6 holds. Therefore we can use the results proved in Section 6.

Let A_0 be an adapted connection on E. We define $\mathcal{A} = \mathcal{A}_{A_0}$ as the set of connections $A = A_0 + a$ with $a \in L_3^{2,W}(X, \Lambda^1(\mathrm{ad}E))$ (Section 6.1). Note that the definition of the Sobolev space $L_3^{2,W}(X, \Lambda^1(\mathrm{ad}E))$ uses the connection A_0. Let $\mathcal{C} \subset \mathcal{C}_0^{\vec{\tau}}(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ be the set of all $\mu \in \mathcal{C}_0^{\vec{\tau}}(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ satisfying $|\mu_x| < 1$ for all $x \in X$. Here the norm $|\mu_x|$ is defined by using the metric g_0. $\mathcal{C}$ is an open set in $\mathcal{C}_0^{\vec{\tau}}(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$. Each $\mu \in \mathcal{C}$ defines a conformal structure which coincides with $[g_0]$ outside U (see Corollary 3.2). A connection A on E is said to be μ-ASD if it satisfies $F_A^+ = \mu(F_A^-)$ where F_A^+ and F_A^- are the self-dual and anti-self-dual parts of F_A with respect to g_0.

Lemma 8.7. *(i) For any $A \in \mathcal{A}$ we have $\int_X tr(F_A \wedge F_A) = \int_X tr(F_{A_0} \wedge F_{A_0})$.*
(ii) If A_0 is not equivalent to a flat connection as an adapted connection, then any $A \in \mathcal{A}$ is not flat.
(iii) If A_0 is equivalent to a flat connection as an adapted connection and if $A \in \mathcal{A}$ is μ-ASD for some $\mu \in \mathcal{C}$, then A is flat.
(iv) If $\int_X tr(F_{A_0} \wedge F_{A_0}) < 0$, then for any $\mu \in \mathcal{C}$ there is no μ-ASD connection in $\mathcal{A}$.

Proof. (i) It is enough to prove that for any compact-supported smooth $a \in \Omega^1(\mathrm{ad}E)$ we have $\int_X tr(F(A_0 + a)^2) = \int_X tr(F(A_0)^2)$. Since we have $tr(F(A_0 + a)^2) - tr(F(A_0)^2) = d\,tr(2a \wedge F(A_0) + a \wedge d_{A_0} a + \frac{2}{3} a^3)$, it follows from Stokes' theorem.
(ii) Since A_0 is not equivalent to the flat connection as an adapted connection, the integral $\int_X tr(F(A_0)^2)$ is not equal to zero. (See Proposition 4.3.) Hence the result follows from (i).

(iii) If $A \in \mathcal{A}$ is μ-ASD, then $tr(F_A^2) = (|F_A^-|^2 - |\mu(F_A^-)|^2) d\mathrm{vol}$ ($d\mathrm{vol}$ is the volume form with respect to g_0). We have $|F_A^-|^2 - |\mu(F_A^-)|^2 \geq 0$, and moreover if F_A is not zero at $x \in X$ then $|F_A^-|^2 - |\mu(F_A^-)|^2 > 0$ at $x \in X$. If A_0 is equivalent to the flat connection, then $\int_X tr(F(A_0)^2) = 0$. Hence if $A \in \mathcal{A}$ is μ-ASD then

$$\int_X (|F_A^-|^2 - |\mu(F_A^-)|^2) d\mathrm{vol} = 0.$$

Therefore $F_A = 0$ all over X. We can prove (iv) by a similar argument. □

We define $\mathcal{M} \subset \mathcal{A} \times \mathcal{C}$ by

$$\mathcal{M} := \{(A, \mu) \in \mathcal{A} \times \mathcal{C} |\, A \text{ is } \mu\text{-ASD}\}.$$

Let $\pi : \mathcal{M} \to \mathcal{C}$ be the projection. The main purpose of this subsection is to prove the following proposition by using the metric perturbation technique originally due to Freed-Uhlenbeck [8].

Proposition 8.8. *Suppose that $b_+(Y) \geq 1$ and $b_-(Y) = 0$ and that A_0 is not equivalent to a flat connection as an adapted connection. Then $\pi(\mathcal{M})$ is of first category in $\mathcal{C}$.*

In the rest of this subsection we always assume that $b_+(Y) \geq 1$ and $b_-(Y) = 0$ and that A_0 is not equivalent to a flat connection as an adapted connection.

Fix $(A, \mu) \in \mathcal{M}$. Let $\mathrm{Ker} d_A^{*,W} \cap L_3^{2,W}$ be the space of $b \in L_3^{2,W}(X, \Lambda^1(\mathrm{ad}E))$ satisfying $d_A^{*,W} b = 0$. Let $\mathrm{Ker} D_A$ be the space of $b \in L_4^{2,W}(X, \Lambda^1(\mathrm{ad}E))$ satisfying $D_A b = -d_A^{*,W} b + (d_A^+ - \mu d_A^-) b = 0$, and $\mathrm{Ker}(d_A^{*,W}(1+\mu^*))$ be the space of $\eta \in L_4^{2,W}(X, \Lambda^+(\mathrm{ad}E))$ satisfying $d_A^{*,W}(1+\mu^*)\eta = 0$. $\mathrm{Ker} D_A$ is finite dimensional (Proposition 6.2), and $\mathrm{Ker}(d_A^{*,W}(1+\mu^*))$ is infinite dimensional (Proposition 6.8). Hence we can take a finite dimensional sub-vector space $H \subset \mathrm{Ker}(d_A^{*,W}(1+\mu^*))$ satisfying $\dim H > \dim \mathrm{Ker} D_A$. Let $H' \subset \mathrm{Ker}(d_A^{*,W}(1+\mu^*))$ be the $L^{2,W}$-orthogonal complement of H in $\mathrm{Ker}(d_A^{*,W}(1+\mu^*))$. Since $\mathrm{Ker}(d_A^{*,W}(1+\mu^*))$ is closed in $L^{2,W}(X, \Lambda^+(\mathrm{ad}E))$, H' is a closed subspace in $L^{2,W}(X, \Lambda^+(\mathrm{ad}E))$.

The spaces $(d_A^+ - \mu d_A^-)(\mathrm{Ker} d_A^{*,W} \cap L_3^{2,W})$, H and H' are closed subspaces in $L_2^{2,W}(X, \Lambda^+(\mathrm{ad}E))$, and they are $L^{2,W}$-orthogonal to each other (Proposition 6.8 (ii)). Moreover, from Proposition 6.8 (ii),

$$L_2^{2,W}(X, \Lambda^+(\mathrm{ad}E)) = (d_A^+ - \mu d_A^-)(\mathrm{Ker} d_A^{*,W} \cap L_3^{2,W}) \oplus H \oplus H'. \tag{36}$$

Let $\Pi : L_2^{2,W}(X, \Lambda^+(\mathrm{ad}E)) \to (d_A^+ - \mu d_A^-)(\mathrm{Ker} d_A^{*,W} \cap L_3^{2,W}) \oplus H$ be the projection with respect to this decomposition. We define

$$\begin{aligned} f : (\mathrm{Ker} d_A^{*,W} \cap L_3^{2,W}) \times \mathcal{C} &\to (d_A^+ - \mu d_A^-)(\mathrm{Ker} d_A^{*,W} \cap L_3^{2,W}) \oplus H, \\ (b, \nu) &\mapsto \Pi\{F^+(A+b) - \nu(F^-(A+b))\}. \end{aligned} \tag{37}$$

We have $f(0,\mu)=0$. The derivative of f at $(0,\mu)$ is given by

(38)
$$df_{(0,\mu)} : (\mathrm{Ker}d_A^{*,W}\cap L_3^{2,W})\oplus \mathcal{C}_0^{\vec{\tau}}(U,\mathrm{Hom}(\Lambda^-,\Lambda^+)) \to (d_A^+-\mu d_A^-)(\mathrm{Ker}d_A^{*,W}\cap L_3^{2,W})\oplus H,$$
$$(b,\nu)\mapsto (d_A^+-\mu d_A^-)b-\Pi(\nu(F_A^-)).$$

Lemma 8.9. *(i) The map (38) is surjective.*
(ii) The partial derivative $d_1f_{(0,\nu)} : \mathrm{Ker}d_A^{*,W}\cap L_3^{2,W}\to (d_A^+-\mu d_A^-)(\mathrm{Ker}d_A^{*,W}\cap L_3^{2,W})\oplus H$, $b\mapsto (d_A^+-\mu d_A^-)b$, *with respect to* $(\mathrm{Ker}d_A^{*,W}\cap L_3^{2,W})$*-direction is a Fredholm operator with its index* <0.

Proof. The statement (ii) is obvious because $\mathrm{Ker}D_A$ and H are both finite dimensional and satisfy $\dim\mathrm{Ker}D_A<\dim H$.

Next we will show (i) by using the argument of Donaldson-Kronheimer [5, p. 154]. Let $\Pi_H : L_2^{2,W}(X,\Lambda^+(\mathrm{ad}E))\to H$ be projection to H with respect to the decomposition (36). It is enough for the proof of (i) to show that the map $\Pi_H\circ df_{(0,\mu)} : (\mathrm{Ker}d_A^{*,W}\cap L_3^{2,W})\oplus \mathcal{C}_0^{\vec{\tau}}(U,\mathrm{Hom}(\Lambda^-,\Lambda^+))\to H$ is surjective. Here we have $\Pi_H\circ df_{(0,\mu)}(b,\nu)=-\Pi_H(\nu(F_A^-))$.

Suppose that it is not surjective. Since H is finite dimensional, this implies that there exists a non-zero $\eta\in H$ satisfying $\langle\eta,\nu(F_A^-)\rangle_{L^{2,W}}=0$ for all $\nu\in\mathcal{C}_0^{\vec{\tau}}(U,\mathrm{Hom}(\Lambda^-,\Lambda^+))$. (Here $\langle\cdot,\cdot\rangle_{L^{2,W}}$ is the $L^{2,W}$-inner product.) This is equivalent to $\langle F_A^-\cdot\eta,\nu\rangle_{L^{2,W}}=0$ for $\nu\in\mathcal{C}_0^{\vec{\tau}}(U,\mathrm{Hom}(\Lambda^-,\Lambda^+))$. Here $F_A^-\cdot\eta\in\Gamma(\Lambda^-\otimes\Lambda^+)$ is the contraction of $F_A^-\otimes\eta\in\Gamma(\Lambda^-(\mathrm{ad}E)\otimes\Lambda^+(\mathrm{ad}E))$ by the inner product of $\mathrm{ad}E$, and we identify $\Lambda^-\otimes\Lambda^+$ with $\mathrm{Hom}(\Lambda^-,\Lambda^+)$ by the metric g_0. Since $\mathcal{C}_0^{\vec{\tau}}(U,\mathrm{Hom}(\Lambda^-,\Lambda^+))$ is dense in $L^2(U,\mathrm{Hom}(\Lambda^-,\Lambda^+))$ (Lemma 8.6), the above means that $F_A^-\cdot\eta=0$ over U. Then for every point $x\in U$, the images of the maps

$$(F_A^-)_x : (\Lambda^-)^*_x\to(\mathrm{ad}E)_x,\quad \eta_x : (\Lambda^+)^*_x\to(\mathrm{ad}E)_x,$$

are orthogonal to each other. Since the rank of $\mathrm{ad}E$ is equal to $\dim su(2)=3$, this implies that $\min(\mathrm{rank}(F_A^-)_x,\mathrm{rank}(\eta_x))\le 1$ for every $x\in U$. Then we use the following sublemma. This is [5, Lemma (4.3.25)].

Sublemma 8.10. *Let $\mathcal{O}\subset X$ be an non-empty open set. Suppose that one of the following conditions (i), (ii) is satisfied. Then A is reducible over X.*
(i) There is $\phi\in\Gamma(\mathcal{O},\Lambda^-(\mathrm{ad}E))$ such that ϕ has rank 1 over $\mathcal{O}$ (as a map from $(\Lambda^-)^*$ to $\mathrm{ad}E$) and $d_A(1+\mu)\phi=0$ over $\mathcal{O}$.
(ii) There is $\phi\in\Gamma(\mathcal{O},\Lambda^+(\mathrm{ad}E))$ such that ϕ has rank 1 over $\mathcal{O}$ (as a map from $(\Lambda^+)^*$ to $\mathrm{ad}E$) and $d_A(1-\mu^*)\phi=0$ over $\mathcal{O}$.

Proof. We assume the condition (i). The case (ii) can be proved in the same way. By making $\mathcal{O}$ smaller, we can assume that $\phi=s\otimes\omega$ where $s\in\Gamma(\mathcal{O},\mathrm{ad}E)$ and $\omega\in\Gamma(\mathcal{O},\Lambda^-)$ with $|s|=1$. Here ω is not zero at any point of $\mathcal{O}$. $d_A(1+\mu)\phi=d_A(s\otimes(1+\mu)\omega)=0$

implies

$$d_A s \wedge (1+\mu)\omega + s \otimes d(1+\mu)\omega = 0.$$

Since $|s| = 1$, we have $0 = d(s,s) = 2(d_A s, s)$. From this and the above equation, we get $d_A s \wedge (1+\mu)\omega = 0$. Since $\omega \in \Omega^-$ and $\mu(\omega) \in \Omega^+$,

$$|d_A s \wedge \omega| = \frac{1}{\sqrt{2}}|d_A s||\omega|, \quad |d_A s \wedge \mu(\omega)| = \frac{1}{\sqrt{2}}|d_A s||\mu(\omega)|.$$

Since $|\mu(\omega)| < |\omega|$, $d_A s \wedge (1+\mu)\omega = 0$ implies $d_A s = 0$. This shows that A is reducible over $\mathcal{O}$. Since X is simply-connected and A is μ-ASD, the unique continuation principle ([5, Lemma (4.3.21)]) implies that A is reducible over X. $\square$

We have $d_A(1+\mu)F_A^- = d_A F_A = 0$ and $d_A((1-\mu^*)W^2\eta) = 0$ since $\eta \in H \subset \mathrm{Ker}(d_A^{*,W}(1+\mu^*))$. If F_A^- is zero on some non-empty open set, then A is flat on it. Then the unique continuation principle ([5, pp. 150-152], [1], [2, p. 248, Remark 3]) implies that A is flat all over X. But this contradicts Lemma 8.7 (ii) because A_0 is not equivalent to a flat connection as an adapted connection. Therefore F_A^- cannot vanish on any non-empty open set. The unique continuation principle also implies that η cannot vanish on any non-empty open set. (Note that $(1-\mu^*)W^2\eta$ is self-dual with respect to the conformal structure corresponding to μ.)

Since we have $\min(\mathrm{rank}(F_A^-)_x, \mathrm{rank}(\eta_x)) \leq 1$ for every $x \in U$, there is a non-empty open set $\mathcal{O} \subset U$ such that one of F_A^-, η has rank 1 over $\mathcal{O}$. Then one of the conditions (i), (ii) in Sublemma 8.10 is satisfied. Thus A is reducible on X. Then, from Corollary 7.3, A is flat over X. But this contradicts Lemma 8.7 (ii). $\square$

$\mathrm{Ker}d_A^{*,W} \cap L_3^{2,W}$ and $\mathcal{C} \subset \mathcal{C}^{\vec{r}}(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ are both separable (see Lemma 8.6) and hence second countable. Therefore we can apply Proposition 8.1 to the map f in (37) and conclude that there exists an open neighborhood $\mathcal{U} \times \mathcal{U}'$ of $(0,\mu)$ in $(\mathrm{Ker}d_A^{*,W} \cap L_3^{2,W}) \times \mathcal{C}$ such that the set $\{\nu \in \mathcal{U}' | \, \exists b \in \mathcal{U} : f(b,\nu) = 0\}$ is of first category in $\mathcal{C}$.

Lemma 8.11. *There exists an open neighborhood $\mathcal{V}$ of (A,μ) in $\mathcal{M}$ such that $\pi(\mathcal{V})$ is of first category in $\mathcal{C}$.*

Proof. Consider the following map (Coulomb gauge):

$$L_4^{2,W}(X, \Lambda^0(\mathrm{ad}E)) \times (\mathrm{Ker}d_A^{*,W} \cap L_3^{2,W}) \to \mathcal{A}, \quad (u,b) \mapsto e^u(A+b).$$

The derivative of this map at $(0,0)$ is given by

$$L_4^{2,W}(X, \Lambda^0(\mathrm{ad}E)) \oplus (\mathrm{Ker}d_A^{*,W} \cap L_3^{2,W}) \to L_3^{2,W}(X, \Lambda^1(\mathrm{ad}E)), \quad (u,b) \mapsto -d_A u + b.$$

This is isomorphic (Lemma 6.6). Therefore, by the inverse mapping theorem, there is an open neighborhood $\mathcal{W}$ of A in $\mathcal{A}$ such that for any $B \in \mathcal{W}$ there are $u \in L_4^{2,W}(X, \Lambda^0(\mathrm{ad}E))$ and $b \in \mathcal{U} \subset (\mathrm{Ker}d_A^{*,W} \cap L_3^{2,W})$ satisfying $B = e^u(A+b)$. Set $\mathcal{V} := (\mathcal{W} \times \mathcal{U}') \cap \mathcal{M}$. Then $\pi(\mathcal{V})$ is contained in the set $\{\nu \in \mathcal{U}' | \, \exists b \in \mathcal{U} : f(b,\nu) = 0\}$, which is of first category in $\mathcal{C}$. $\square$

Since $\mathcal{M} \subset \mathcal{A} \times \mathcal{C}$ is second countable, Lemma 8.11 implies Proposition 8.8.

9. Proof of Theorem 2.1

We will prove Theorem 2.1 in this section. So we assume $b_-(Y) = 0$ and $b_+(Y) \geq 1$. We fix $0 < \alpha < 1$. (For example, $\alpha = 1/2$ will do.) We choose a positive parameter T so that

$$T > \max\left(T_\alpha, T_{-\alpha}, \frac{4}{1-\alpha}\right).$$

This implies

$$T > 4, \quad T \geq \max(T_\alpha, T_{-\alpha}), \quad 1 - 4/T > \alpha.$$

Recall that we assumed $T > 4$ in Section 4.3 and $T \geq \max(T_\alpha, T_{-\alpha})$ in Sections 6 and 8.3. The condition $1 - 4/T > \alpha$ is related to Corollary 4.11. We will show that there is a complete Riemannian metric g on X satisfying the conditions (a) and (b) in Theorem 2.1.

Let $A(m)$ ($m \in \mathbb{Z}$) be adapted connections on E introduced in Section 4.1. They satisfy $\int_X tr(F(A(m))^2) = 8\pi^2 m$. $A(0)$ is equivalent to a flat connection as an adapted connection. $\{A(m) |\, m \in \mathbb{Z}\}$ becomes a complete system of representatives of equivalence classes of adapted connections on E. (See Proposition 4.3.) We define $\mathcal{A}_m$ as the set of all connections $A(m) + a$ such that $a \in L^2_{3,loc}(X, \Lambda^1(\mathrm{ad}E))$ satisfies $\nabla^k_{A(m)} a \in L^{2,W}$ for $0 \leq k \leq 3$. We set

$$\mathcal{M}_m := \{(A, \mu) \in \mathcal{A}_m \times \mathcal{C} |\, A \text{ is } \mu\text{-ASD}\}.$$

Here $\mathcal{C}$ is the space of $\mu \in \mathcal{C}^{\vec{r}}_0(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$ satisfying $|\mu_x| < 1$ ($x \in X$) as in Section 8.3. If $m < 0$, then $\mathcal{M}_m$ is empty by Lemma 8.7 (iv). $(A, \mu) \in \mathcal{M}_0$ if and only if A is flat by Lemma 8.7 (iii).

Let $\pi_m : \mathcal{M}_m \to \mathcal{C}$ be the natural projection. Then $\bigcup_{m \geq 1} \pi_m(\mathcal{M}_m)$ is of first category in $\mathcal{C}$ by Proposition 8.8. $\mathcal{C}$ is an open set in the Banach space $\mathcal{C}^{\vec{r}}_0(U, \mathrm{Hom}(\Lambda^-, \Lambda^+))$. Thus, by Baire's category theorem, there exists $\mu \in \mathcal{C} \setminus \left(\bigcup_{m\geq 1} \pi_m(\mathcal{M}_m)\right)$. Let g be a Riemannian metric on X whose conformal equivalence class corresponds to μ. (See Corollary 3.2.) Since μ is zero outside U (a pre-compact open set in X), we can choose g so that it is equal to g_0 outside a compact set. In particular it is a complete metric.

We want to prove that there is no non-flat instanton with respect to the metric g. Suppose, on the contrary, that there exists a non-flat g-ASD connection A on E satisfying $\int_X |F_A|^2 d\mathrm{vol} < \infty$. Then by Corollary 4.11 and the condition $1 - 4/T > \alpha$, there is a gauge transformation $u : E \to E$ such that $u(A)$ is contained in some $\mathcal{A}_m$. This means that $\mu \in \pi_m(\mathcal{M}_m)$. Since A is not flat, we have $m \geq 1$. This contradicts the choice of μ.

We have completed all the proofs of Theorem 2.1.

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Masaki Tsukamoto
Department of Mathematics, Faculty of Science
Kyoto University
Kyoto 606-8502
Japan
E-mail address: `tukamoto@math.kyoto-u.ac.jp`