ON THE LOWER CENTRAL SERIES OF A FREE ABELIAN BY POLYNILPOTENT GROUP

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Abstract. Let F_n be a free group of rank n, and $\Gamma_n(k)$ the k-th term of the lower central series of F_n. For $l \geq 1$, let K'_l be the $(l+2)$-nd term of the lower central series of $\Gamma_n(2)$. We denote by $F_n^{N_l}$ the quotient group of F_n by the subgroup $[\Gamma_n(3), \Gamma_n(3)]K'_l$. In this paper, we show that each of the graded quotients of the lower central series of the group $F_n^{N_l}$ for any $l \geq 1$ is a free abelian group, and give a basis of it by using a generalized Chen's integration in free groups.

1. Introduction

Let G be a group, and $\Gamma_G(k)$ the k-th term of the lower central series of G defined by

$$\Gamma_G(1) := G, \quad \Gamma_G(k) := [\Gamma_G(k-1), G], \ \ k \geq 2.$$

For each $k \geq 1$, we denote by $\mathcal{L}_G(k) := \Gamma_G(k)/\Gamma_G(k+1)$ the graded quotient of the lower central series of G, and by

$$\mathcal{L}_G := \bigoplus_{k \geq 1} \mathcal{L}_G(k)$$

its graded sum. Then $\mathcal{L}_G$ has a graded Lie algebra structure induced from the commutator bracket on G. The Lie algebra $\mathcal{L}_G$ is called the associated graded Lie algebra of a group G.

In general, $\mathcal{L}_G$ reflects and supplies much useful information about a group G. It is often appeared in a study in topology as well as group theory. Now there is a broad range of results for the lower central series and the associated graded Lie algebra of a group. Especially, in our research, they are powerful tools to investigate a deep structure of the Johnson filtration of the mapping class group of a surface and the automorphism group of a free group. (For example, see [9] and [5] for the mapping class group, and [1], [3] and [12] for the automorphism group of a free group respectively.)

To clarify the Lie algebra structure of $\mathcal{L}_G$, it is important to determine the structure of each of $\mathcal{L}_G(k)$ as an abelian group. If a group G is finitely generated, it is easily seen that $\mathcal{L}_G(k)$ is finitely generated abelian group. It is, however, quite difficult problem in general to determine the structure of $\mathcal{L}_G(k)$ for $k \geq 1$ even in the case where G is

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finitely generated. Hence, to establish a method to clarify the structure of $\mathcal{L}_G(k)$ for a given group G is an important problem.

Classically, in combinatorial group theory, the case where G is a free group of finite rank is the most important. Let F_n be a free group with basis $x_1, \ldots, x_n$. For simplicity, we write $\Gamma_n(k)$ and $\mathcal{L}_n(k)$ for $\Gamma_{F_n}(k)$ and $\mathcal{L}_{F_n}(k)$ respectively. By a pioneer work due to Magnus, it is well known that each of $\mathcal{L}_n(k)$ is a finitely generated free abelian group. (See [8], for example.) It is also well known due to Witt [13] that the rank of $\mathcal{L}_n(k)$ is completely determined. (See (1) in Subsection 2.3.) Furthermore, P. Hall [6] introduced basic commutators of F_n, and showed that the coset classes of those of weight k form a basis of $\mathcal{L}_n(k)$. Namely, for $G = F_n$ the group structure of $\mathcal{L}_G(k)$ is completely determined.

In addition to the above, the structures of $\mathcal{L}_G(k)$ has been studied in the case where G is a quotient group of F_n by a certain commutator subgroup. Let F_n^M be a quotient group of F_n by the second derived subgroup $[\Gamma_n(2), \Gamma_n(2)]$. The group F_n^M is called a free metabelian group. For simplicity, we write $\mathcal{L}_n^M(k)$ for $\mathcal{L}_{F_n^M}(k)$. By a remarkable work of Chen [2], it is known that each of $\mathcal{L}_n^M(k)$ is a free abelian group of finite rank. He [2] also gave the rank of $\mathcal{L}_n^M(k)$. (See (2) in Subsection 2.4.) In particular, he [2] introduced the integration in free groups which is used to detect linearly independent elements in $\mathcal{L}_n^M(k)$.

On the other hand, the result for the free metabelian group were generalized to that for a free abelian by nilpotent group by Gaglione and Spellman [4]. Let $F_n^{AN_c}$ be a quotient group of F_n by a subgroup $[\Gamma_n(c), \Gamma_n(c)]$ for $c \geq 2$. Clearly $F_n^{AN_2} = F_n^M$. In [4], they showed that for each $k \geq 1$, the graded quotient $\mathcal{L}_n^{AN_c}(k)$ is a free abelian group, and gave the rank of it.

In this paper, we consider some "intermediate" groups between $F_n^{AN_3}$ and F_n^M. For any $l \geq 1$, let

$$K_l' := [[\cdots[\Gamma_n(2), \Gamma_n(2)], \ldots, \Gamma_n(2)], \Gamma_n(2)]$$

be the $(l+2)$-nd term of the lower central series of $\Gamma_n(2)$, and set $K_l := [\Gamma_n(3), \Gamma_n(3)]K_l'$. Then we define $F_n^{N_l}$ to be the quotient group of F_n by K_l. For simplicity, we write $\mathcal{L}_n^{N_l}(k)$ for $\mathcal{L}_{F_n^{N_l}}(k)$. In our previous paper [11], in order to study the cokernel of the Johnson homomorphism of the automorphism group of a free group, we determine the structure of $\mathcal{L}_n^{N_1}(k)$. In particular, we showed that $\mathcal{L}_n^{N_1}(k)$ is a free abelian group of finite rank, and obtained a basis of it for $k \geq 1$ by using a generalized Chen's integration in free groups which we introduced in [11].

Main goal of the paper is to show that we can also apply the generalized Chen's integration to determine the structure of $\mathcal{L}_n^{N_l}(k)$ for each $l \geq 2$ and $k \geq 1$. Let $\mathfrak{C}_n^l(k)$ be the set of commutators

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_{k-2l}}, [x_{j_1}, x_{j_2}], \ldots, [x_{j_{2l-1}}, x_{j_{2l}}]]$$

such that $i_1 > i_2 \leq \cdots \leq i_{k-2l}$, $j_{2s-1} > j_{2s}$ and $(j_1, j_2) \leq \cdots \leq (j_{2l-1}, j_{2l})$. Here $(p, q) < (r, s)$ is a usual lexicographic order. Namely, $(p, q) < (r, s)$ if and only if $p < r$, or $p = r$ and $q < s$. Then our main theorem is

Theorem 1. *(= Theorem 4.1.) For any $l \geq 1$ and $k \geq 2l+4$, $\mathcal{L}_n^{N_l}(k)$ is a free abelian group with basis*
$$\mathfrak{C}_n^0(k) \cup \mathfrak{C}_n^1(k) \cup \cdots \cup \mathfrak{C}_n^l(k).$$

We prove this theorem by induction on $l \geq 1$. More precisely, we determine the structure of the kernel $\mathcal{K}_n^l(k)$ of a natural surjective homomorphism $\mathcal{L}_n^{N_l}(k) \to \mathcal{L}_n^{N_{l-1}}(k)$ induced from $F_n^{N_l} \to F_n^{N_{l-1}}$ step by step. In particular, we prove that $\mathfrak{C}_n^l(k)$ is a basis of $\mathcal{K}_n^l(k)$. In order to show that $\mathfrak{C}_n^l(k)$ is linearly independent in $\mathcal{K}_n^l(k)$, we use the generalized Chen's integration in free groups.

This paper consists of five sections. In Section 2, we recall the associated graded Lie algebra of a group. In Section 3, we recall the generalized Chen's integration in free groups. In Section 4, we give a proof of our main theorem.

Contents

2. Preliminaries

In this section, we recall the definition and some properties of the associated graded Lie algebra of a group G.

2.1. Notation and conventions.

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- For an element $g \in G$, we also denote the coset class of g in G/N by g if there is no confusion.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. Associated graded Lie algebra of a group.

In this subsection we recall the associated graded Lie algebra of a group G. Let G be a group, and $\Gamma_G(k)$ the k-th term of the lower central series of G defined by
$$\Gamma_G(1) := G, \quad \Gamma_G(k) := [\Gamma_G(k-1), G], \ \ k \geq 2.$$

For each $k \geq 1$, set $\mathcal{L}_G(k) := \Gamma_G(k)/\Gamma_G(k+1)$ and

$$\mathcal{L}_G := \bigoplus_{k \geq 1} \mathcal{L}_G(k).$$

Then $\mathcal{L}_G$ has a graded Lie algebra structure induced from the commutator bracket on G. We call $\mathcal{L}_G$ the associated Lie algebra of a group G. Clearly, the correspondence from G to $\mathcal{L}_G$ is a covariant functor from the category of groups to that of graded Lie algebras. In particular, if $f : G_1 \to G_2$ be a surjective group homomorphism, the induced homomorphism $f_* : \mathcal{L}_{G_1} \to \mathcal{L}_{G_2}$ is also surjective.

For any $g_1, \ldots, g_k \in G$, a commutator of weight k among the components $g_1, \ldots, g_k$ of the type

$$[[\cdots[[g_1, g_2], g_3], \cdots], g_k]$$

with all of its brackets to the left of all the elements occurring is called a simple k-fold commutator, denoted by $[g_1, g_2, \cdots, g_k]$. In general, if G is generated by $g_1, \ldots, g_n$ then for each $k \geq 1$, $\mathcal{L}_G(k)$ is generated by (the coset classes of) the simple k-fold commutators

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}], \quad i_j \in \{1, \ldots, n\}.$$

(For details, see [8] for example.)

2.3. **Free groups.**

Here we consider the case where G is a free group F_n with basis $x_1, \ldots, x_n$. For simplicity, if $G = F_n$, we write $\Gamma_n(k)$, $\mathcal{L}_n(k)$ and $\mathcal{L}_n$ for $\Gamma_G(k)$, $\mathcal{L}_G(k)$ and $\mathcal{L}_G$ respectively. Let H be the abelianization of F_n. In general, the associated graded Lie algebra $\mathcal{L}_n$ is isomorphic to the free Lie algebra generated by H. (See [10] for basic materials concerning the free Lie algebra, for example.) It is classically well known due to Magnus that for each $k \geq 1$, the graded quotient $\mathcal{L}_n(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of finite rank. Furthermore, Witt [13] determined the rank of $\mathcal{L}_n(k)$ by

$$r_n(k) := \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}} \tag{1}$$

where μ is the Möbius function.

Next, we briefly recall some properties of the Hall basis of $\mathcal{L}_n(k)$ for each $k \geq 1$. In [6], P. Hall introduced basic commutators of F_n, and showed that the coset classes of those of weight k form a basis of $\mathcal{L}_n(k)$. These are called the Hall basis of $\mathcal{L}_n(k)$. (For details for the basic commutators, see [7] and [10] for example.) In this paper , we consider a fixed sequence of basic commutators of F_n beginning with

$$x_1 < x_2 < \cdots < x_n < [x_2, x_1] < [x_3, x_1] < [x_3, x_2] < \cdots < [x_n, x_{n-1}] < \cdots$$

where the ordering among $[x_i, x_j]$ is defined by the usual lexicographic order. Namely, $[x_i, x_j] < [x_k, x_l]$ if $i < k$, or $i = k$ and $j < l$.

For each $k \geq 1$, let $c_{k,1} < \ldots < c_{k,m_k}$ be all the basic commutators of weight k. Then P. Hall showed that for any $k \geq 1$, any element $w \in F_n$ is uniquely written as a form

$$w \equiv c_{1,1}^{e_{1,1}} \cdots c_{1,n}^{e_{1,n}} \cdots c_{k,1}^{e_{k,1}} \cdots c_{k,m_k}^{e_{k,m_k}} \pmod{\Gamma_n(k+1)}$$

for some $e_{i,m_i} \in \mathbf{Z}$. We call it the mod-$\Gamma_n(k+1)$ normal form of w. By the Hall's correcting process, if any $w \in F_n$ is given, we can rewrite w as its mod-$\Gamma_n(k+1)$ normal form with finitely many steps for any $k \geq 1$. (For details of the Hall's correcting process, see [7].)

2.4. **Some quotient groups of a free group.**

In this subsection, we consider some quotient groups of F_n, and its associated graded Lie algebras.

Free metabelian group. First, we recall a free metabelian group. Let F_n^M be the quotient group of F_n by a subgroup $[\Gamma_n(2), \Gamma_n(2)]$. The group F_n^M is called a free metabelian group of rank n. For simplicity, we write $\mathcal{L}_n^M(k)$ and $\mathcal{L}_n^M$ for $\mathcal{L}_{F_n^M}(k)$ and $\mathcal{L}_{F_n^M}$ respectively. The associated graded Lie algebra $\mathcal{L}_n^M$ is called the free metabelian Lie algebra generated by H, or the Chen Lie algebra. By a remarkable pioneer work by Chen [2], it is known that for each $k \geq 1$ the graded quotient $\mathcal{L}_n^M(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank

$$r_n^M(k) := (k-1)\binom{n+k-2}{k} \tag{2}$$

with basis

$$\{[x_{i_1}, x_{i_2}, \ldots, x_{i_k}] \in \mathcal{L}_n^M(k) \,|\, i_1 > i_2 \leq i_3 \leq \cdots \leq i_k,\ \ 1 \leq i_j \leq n\}.$$

In particular, we see that for any $k \geq 2$, the basic commutators which do not belong to $[\Gamma_n(2), \Gamma_n(2)]$ are $[x_{i_1}, x_{i_2}, \ldots, x_{i_k}] \in F_n$ for $i_1 > i_2 \leq i_3 \leq \cdots \leq i_k$.

Free abelian by nilpotent group. For any $c \geq 2$, let $F_n^{AN_c}$ be the quotient group of F_n by a subgroup $[\Gamma_n(c), \Gamma_n(c)]$. The group $F_n^{AN_c}$ is called a free abelian by nilpotent group of class $c-1$. For $c = 2$, $F_n^{AN_2}$ is exactly the free metabelian group F_n^M. We write $\mathcal{L}_n^{AN_c}(k)$ for $\mathcal{L}_{F_n^{AN_c}}(k)$ for simplicity. Gaglione and Spellman [4] showed that each of $\mathcal{L}_n^{AN_c}(k)$ is a free abelian group and determined the rank of it.

In this paper we consider the case where $c = 3$. Set

$$\begin{aligned} X(k) := \{[x_{i_1}, x_{i_2}, \ldots, x_{i_p}, [x_{j_1}, x_{j_2}], \ldots, [x_{j_{2q-1}}, x_{j_{2q}}]] \in \mathcal{L}_n^{AN_3}(k) \\ |\, p + 2q = k,\ \ 3 \leq p \leq k,\ \ i_1 > i_2 \leq \cdots \leq i_p,\ \ j_{2s-1} > j_{2s}, \\ (j_1, j_2) \leq \cdots \leq (j_{2q-1}, j_{2q})\} \end{aligned}$$

for any k, and

$$\begin{aligned} Y(k) := \{[[x_{j_1}, x_{j_2}], \ldots, [x_{j_{2q-1}}, x_{j_{2q}}]] \in \mathcal{L}_n^{AN_3}(k) \\ |\, j_{2s-1} > j_{2s},\ \ (j_1, j_2) > (j_3, j_4) \leq (j_5, j_6) \leq \cdots \leq (j_{2q-1}, j_{2q})\} \end{aligned}$$

if $k = 2q$. Then, by the work due to Gaglione and Spellman [4], we see that $\mathcal{L}_n^{AN_3}(k)$ is a free abelian group with basis

$$Z(k) := \begin{cases} X(k) \cup Y(k), & \text{if } k \geq 6 \text{ and } k \text{ is even}, \\ X(k), & \text{if } k \geq 7 \text{ and } k \text{ is odd}. \end{cases}$$

We remark that for $1 \leq k \leq 5$, $\mathcal{L}_n^{AN_3}(k) \cong \mathcal{L}_n(k)$ as a $\mathrm{GL}(n, \mathbf{Z})$-module. In fact, since $\Gamma_n(k)[\Gamma_n(3), \Gamma_n(3)] = \Gamma_n(k)$ for $1 \leq k \leq 6$,

$$\mathcal{L}_n^{AN_3}(k) \cong \Gamma_n(k)[\Gamma_n(3), \Gamma_n(3)] \Big/ \Gamma_n(k+1)[\Gamma_n(3), \Gamma_n(3)] \cong \mathcal{L}_n(k)$$

for $1 \leq k \leq 5$.

Free abelian by polynilpotent group. Next, for any $l \geq 1$, let

$$K_l' := [[\cdots[\Gamma_n(2), \Gamma_n(2)], \ldots, \Gamma_n(2)], \Gamma_n(2)]$$

be the $(l+2)$-nd term of the lower central series of $\Gamma_n(2)$, and set $K_l := [\Gamma_n(3), \Gamma_n(3)]K_l'$. Then we define $F_n^{N_l}$ to be the quotient group of F_n by K_l. In general, the groups $F_n^{N_l}$ are free abelian by polynilpotent groups since $F_n^{N_l}$ has an abelian normal subgroup $\Gamma_{F_n^{N_l}}(3)$ such that the quotient group of $F_n^{N_l}$ by $\Gamma_{F_n^{N_l}}(3)$ is a polynilpotent group. For simplicity, we write $\Gamma_n^{N_l}(k)$ and $\mathcal{L}_n^{N_l}(k)$ for $\Gamma_{F_n^{N_l}}(k)$ and $\mathcal{L}_{F_n^{N_l}}(k)$ respectively. Then we have sequences of natural surjective homomorphisms

$$F_n^{AN_3} \to \cdots \to F_n^{N_l} \to F_n^{N_{l-1}} \to \cdots \to F_n^{N_1} \to F_n^M,$$

and

$$\mathcal{L}_n^{AN_3}(k) \to \cdots \to \mathcal{L}_n^{N_l}(k) \to \mathcal{L}_n^{N_{l-1}}(k) \to \cdots \to \mathcal{L}_n^{N_1}(k) \to \mathcal{L}_n^M(k)$$

for each $k \geq 1$.

In our previous paper [11], we showed that $\mathcal{L}_n^{N_1}(k)$ is a free abelian group, and obtained a basis of it for $k \geq 1$. In this paper, we determine the group structure of $\mathcal{L}_n^{N_l}(k)$ for each $l \geq 2$ and $k \geq 1$. More precisely, we show that each of $\mathcal{L}_n^{N_l}(k)$ is a free abelian group, and give its rank in Section 4.

3. A generalization of the Chen's integration in free groups

In order to determine the structure of $\mathcal{L}_n^{N_l}(k)$, we use a generalization of the Chen's integration in free groups, which was established in our previous paper [11]. In this section, we recall the definition and some properties of the generalized Chen's integration. (For details, see Section 3 in [11].)

Let F_n be the free group generated by $x_1, \ldots, x_n$ as above. Denote by $\mathbf{E}$ the vector space over the real field $\mathbf{R}$ with basis $x_1, \ldots, x_n$ and $[x_i, x_j]$ for $1 \leq j < i \leq n$. A Euclidean metric is introduced into $\mathbf{E}$ by taking $x_1, \ldots, x_n$ and $[x_i, x_j]$ as an orthonormal basis. Then $\mathbf{E}$ is a Euclidean $n(n+1)/2$-space. The orthonormal basis induces a Cartesian coordinate system in $\mathbf{E}$. We call the coordinates corresponding to x_i and $[x_i, x_j]$ the t_i-coordinates and the $t_{i,j}$-coordinates.

Let Ω_n be the set of words among the letters $x_1, \ldots, x_n$. For any word $w = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_m}^{e_m}$ with $e_k = \pm 1$, and any integers $a_1, \ldots, a_n \in \mathbf{Z}$, we define points $P_s \in \mathbf{E}$ for $0 \leq s \leq m$ by

$$\begin{aligned} P_0 &:= \mathbf{0}, \\ P_s &:= P_{s-1} + e_s t_{i_s} + \sum_{i_s < j} \Big\{ \Big(a_j + \sum_{\substack{1 \leq l \leq s-1 \\ i_l = j}} e_l \Big) e_s t_{j, i_s} \Big\} \end{aligned}$$

for $1 \leq s \leq m$. Let $\overline{P_sP_{s+1}}$ be the path from P_s to P_{s+1} defined by a segment, and $l_w(a_1, \ldots, a_n)$ the polygonal path which successive vertices are $P_0, P_1, \ldots, P_m$. Since the vertex P_m depends only on the integers $a_1, \ldots, a_n$ and the equivalence class of w in F_n, we denote P_m above by $P_w(a_1, \ldots, a_n)$ for $w \in F_n$. Then we have

Lemma 3.1. *As the notation above,*

(1) $P_1(a_1, \ldots, a_n) = \mathbf{0}$,
(2) *If* $w = x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}$ *in* $H_1(F_n, \mathbf{Z})$ *then* t_i*-coordinate of* $P_w(a_1, \ldots, a_n)$ *is* w_i *for* $1 \leq i \leq n$,
(3) *If* $w = x_{i_1}^{e_1} \cdots x_{i_m}^{e_m} \in \Gamma_n(2)$ *and*

$$w = [x_2, x_1]^{w_{2,1}} \cdots [x_n, x_{n-1}]^{w_{n,n-1}} \in \mathcal{L}_n(2),$$

the $t_{i,j}$*-coordinate of* $P_w(a_1, \ldots, a_n)$ *is* $w_{i,j}$.

Now, for any $w \in \Omega_n$, $a_1, \ldots, a_n \in \mathbf{Z}$ and continuous real-valued function $f : \mathbf{E} \to \mathbf{R}$, we define integrations by

$$I_j(f, w; a_1, \ldots, a_n) := \int_{l_w(a_1, \ldots, a_n)} f(t) dt_j$$

for each $1 \leq j \leq n$. Since the integration $I_j(f, w; a_1, \ldots, a_n)$ depends only on f, $a_1, \ldots, a_n$ and the equivalence class of w in F_n, we consider $I_j(f, w; a_1, \ldots, a_n)$ for $w \in F_n$. We remark that if $f : \mathbf{E} \to \mathbf{R}$ does not depend on the coordinates $t_{i,j}$ for any $1 \leq j < i \leq n$, the integration $I_j(f, w; a_1, \ldots, a_n)$ coincides with the Chen's original integration $I_j(\bar{f}, w)$ for each $1 \leq j \leq n$, where $\bar{f}$ is the restriction of f to the subspace $\mathbf{E}'$ of $\mathbf{E}$ generated by the basis $x_1, \ldots, x_n$. In the following, if there is no confusion, we always write f for $\bar{f}$ for simplicity.

For any $P \in \mathbf{E}$, the translation function on $\mathbf{E}$ defined by

$$t \mapsto t + P$$

is denoted by T_P. Here we recall some properties of $I_j(f, w; a_1, \ldots, a_n)$. (For details, see [11].)

Lemma 3.2. *For any* $a_1, \ldots, a_n \in \mathbf{Z}$, $u, v \in F_n$ *such that* $u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}, v = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} \in H_1(F_n, \mathbf{Z})$, *and real-valued functions* f, g *on* $\mathbf{E}$, *we have*

(1) $I_j(\alpha f + \beta g, w; a_1, \ldots, a_n) = \alpha I_j(f, w; a_1, \ldots, a_n) + \beta I_j(g, w; a_1, \ldots, a_n)$ *for any* $\alpha, \beta \in \mathbf{R}$.
(2) $I_j(f, 1; a_1, \ldots, a_n) = 0$.
(3) $I_j(f, uv; a_1, \ldots, a_n)$
$\quad = I_j(f, u; a_1, \ldots, a_n) + I_j(f \circ T_{P_u(a_1, \ldots, a_n)}, v; a_1 + u_1, \ldots, a_n + u_n)$.
(4) $I_j(f, u^{-1}; a_1, \ldots, a_n) = -I_j(f \circ T_{P_{u^{-1}}(a_1, \ldots, a_n)}, u; a_1 - u_1, \ldots, a_n - u_n)$.
(5) $I_j(f, [u, v]; a_1, \ldots, a_n)$
$\quad = I_j(f, u; a_1, \ldots, a_n) + I_j(f \circ T_{P_u(a_1, \ldots, a_n)}, v; a_1 + u_1, \ldots, a_n + u_n)$
$\quad - I_j(f \circ T_{P_{uvu^{-1}}(a_1, \ldots, a_n)}, u; a_1 + v_1, \ldots, a_n + v_n)$
$\quad - I_j(f \circ T_{P_{[u,v]}(a_1, \ldots, a_n)}, v; a_1, \ldots, a_n)$.

Let $\mathbf{R}[t]$ be the commutative polynomial ring over $\mathbf{R}$ among indeterminates t_i for $1 \le i \le n$ and $t_{i,j}$ for $1 \le j < i \le n$. Each element of $\mathbf{R}[t]$ is regarded as a real-valued function on $\mathbf{E}$ in a usual way. We consider the polynomial ring $\mathbf{R}[t_1, \ldots, t_n]$ as a subring of $\mathbf{R}[t]$. For any $f \in \mathbf{R}[t]$, we denote by $\deg(f)$, $\deg_1(f)$ and $\deg_2(f)$ the degree of f, that of f with respect to the indeterminates $t_1, \ldots, t_n$ and that of f with respect to the indeterminates $t_{2,1}, \ldots, t_{n,n-1}$ respectively. For example, for $f = t_1^2 t_3 t_{2,1} t_{3,1} + t_{2,1}^4$,

$$\deg(f) = 5, \quad \deg_1(f) = 3, \quad \deg_2(f) = 4.$$

Here we give a few examples of calculations of the integrations. Clearly, for any $w \in F_n$, $I_j(1, w; a_1, \ldots, a_n) = I_j(1, w)$ is the sum of the exponents of those x_j which occur in w.

Lemma 3.3. *For any $1 \le i, j \le n$, we have*

(1) *For any $p > q$,*

$$I_j(t_i, [x_p, x_q]; a_1, \ldots, a_n) = \begin{cases} \delta_{jq}, & i = p, \\ -\delta_{jp}, & i = q, \\ 0 \quad , & i \neq p, q. \end{cases}$$

(2) *For any $w \in \Gamma_n(3)$, $I_j(t_i, w; a_1, \ldots, a_n) = 0$.*

See [11] for the proof of Lemma 3.3. The following theorem is essentially due to Chen [2].

Theorem 3.4 (Chen [2])**.** *Let $k \ge 2$ and $f \in \mathbf{R}[t_1, \ldots, t_n]$.*

(1) *If $w \in [\Gamma_n(2), \Gamma_n(2)]$, $I_j(f, w; a_1, \ldots, a_n) = 0$.*

(2) *If $w = [x_{i_1}, x_{i_2}, \ldots, x_{i_k}]$ and $\deg(f) \le k - 1$,*

$$I_j(f, w; a_1, \ldots, a_n) = \begin{cases} (-1)^{k-1}\alpha_1, & j = i_1, \\ (-1)^k \alpha_2, & j = i_2, \\ 0, & j \neq i_1, i_2 \end{cases}$$

where

$$\alpha_1 = \frac{\partial^{k-1} f}{\partial t_{i_2} \partial t_{i_3} \cdots \partial t_{i_k}}, \quad \alpha_2 = \frac{\partial^{k-1} f}{\partial t_{i_1} \partial t_{i_3} \cdots \partial t_{i_k}}.$$

(3) *If $w \in \Gamma_n(k)$ and $\deg(f) \le k - 2$, then $I_j(f, w; a_1, \ldots, a_n) = 0$.*

We generalize the theorem above to

Proposition 3.5. *For $l \ge 1$, $k \ge 2l + 4$, take*

$$w = [x_{i_1}, x_{i_2}, \ldots, x_{i_{k-2l}}, [x_{j_1}, x_{j_2}], \ldots, [x_{j_{2l-1}}, x_{j_{2l}}]],$$

$j_{2s-1} > j_{2s}$, $(j_1, j_2) \le \cdots \le (j_{2l-1}, j_{2l})$*. Then for any $f \in \mathbf{R}[t]$ such that* $\deg(f) \le k - l - 1$ *and* $\deg_2(f) \le l$*, we have*

$$I_j(f, w; a_1, \ldots, a_n) = \begin{cases} (-1)^{k-l+1}\beta_1, & j = i_1, \\ (-1)^{k-l}\beta_2, & j = i_2, \\ 0, & j \neq i_1, i_2 \end{cases}$$

where

$$\beta_1 = \frac{\partial^{k-l-1} f}{\partial t_{i_2} \partial t_{i_3} \cdots \partial t_{i_{k-2l}} \partial t_{j_1, j_2} \cdots \partial t_{j_{2l-1}, j_{2l}}},$$
$$\beta_2 = \frac{\partial^{k-l-1} f}{\partial t_{i_1} \partial t_{i_3} \cdots \partial t_{i_{k-2l}} \partial t_{j_1, j_2} \cdots \partial t_{j_{2l-1}, j_{2l}}}.$$

Proof. Set

$$w' := [x_{i_1}, x_{i_2}, \dots, x_{i_{k-2l}}, [x_{j_1}, x_{j_2}], \dots, [x_{j_{2l-3}}, x_{j_{2l-2}}]].$$

Then for any $f \in \mathbf{R}[t]$ such that $\deg(f) \leq k-l-1$ and $\deg_2(f) \leq l$, using Lemma 3.2, we have

$$\begin{aligned}
&I_j(f, w; a_1, \dots, a_n) \\
&\quad = I_j(f, w'; a_1, \dots, a_n) + I_j(f, [x_{j_{2l-1}}, x_{j_{2l}}]; a_1, \dots, a_n) \\
&\qquad - I_j(f \circ T_{P_{[x_{j_{2l-1}}, x_{j_{2l}}]}(a_1, \dots, a_n)}, w'; a_1, \dots, a_n) \\
&\qquad - I_j(f, [x_{j_{2l-1}}, x_{j_{2l}}]; a_1, \dots, a_n), \\
&\quad = I_j(f - f \circ T_{P_{[x_{j_{2l-1}}, x_{j_{2l}}]}(a_1, \dots, a_n)}, w'; a_1, \dots, a_n), \\
&\quad = -I_j\Big(\frac{\partial f}{\partial t_{j_{2l-1}, j_{2l}}}, w'; a_1, \dots, a_n\Big).
\end{aligned}$$

By repeating this process, we have

$$I_j(f, w; a_1, \dots, a_n) = (-1)^l I_j\Big(\frac{\partial^l f}{\partial t_{j_1, j_2} \cdots \partial t_{j_{2l-1}, j_{2l}}}, w'; a_1, \dots, a_n\Big).$$

Then by the hypothesis, we see

$$\frac{\partial^l f}{\partial t_{j_1, j_2} \cdots \partial t_{j_{2l-1}, j_{2l}}} \in \mathbf{R}[t_1, \dots, t_n],$$

and hence, from Theorem 3.4, we obtain the required results. This completes the proof of Proposition 3.5. □

As a corollary, we obtain

Corollary 3.6. *Using the notation in Proposition 3.5, we have*

(1) *If* $\deg(f) \leq k-l-2$ *and* $\deg_2(f) \leq l$, $I_j(f, w; a_1, \dots, a_n) = 0$.
(2) $I_j(t_{p_1} t_{p_2} \cdots t_{p_{k-2l-1}} t_{q_1, q_2} \cdots t_{q_{2l-1}, q_{2l}}, w; a_1, \dots, a_n) \neq 0$ *if and only if*
   (i) $t_{q_1, q_2} \cdots t_{q_{2l-1}, q_{2l}} = t_{j_1, j_2} \cdots t_{j_{2l-1}, j_{2l}}$,
   (ii) $t_{p_1} \cdots t_{p_{k-2l-1}} t_j = t_{i_1} \cdots t_{i_{k-2l}}$,
   (iii) $j = i_1$ *or* $j = i_2$.

4. The structure of the graded quotients $\mathcal{L}_n^{N_l}(k)$

In this section, we determine the group structure of the graded quotient $\mathcal{L}_n^{N_l}(k)$ of the lower central series of $F_n^{N_l}$ for $l \geq 2$ and $k \geq 1$. In particular, we show that each of $\mathcal{L}_n^{N_l}(k)$ is a free abelian group, and give a basis of it by using a generalized Chen's integration in free groups. Since $K_l = \Gamma_n(k)$ for $1 \leq k \leq 6$, we have $\mathcal{L}_n^{N_l}(k) \cong \mathcal{L}_n(k)$ for

$1 \leq k \leq 5$. On the other hand, since $\Gamma_n(k)K_l = \Gamma_n(k)[\Gamma_n(3), \Gamma_n(3)]$ for $6 \leq k \leq 2l+3$, we see that

$$\begin{aligned} \mathcal{L}_n^{N_l}(k) &\cong \Gamma_n(k)K_l \Big/ \Gamma_n(k+1)K_l, \\ &\cong \Gamma_n(k)[\Gamma_n(3), \Gamma_n(3)] \Big/ \Gamma_n(k+1)[\Gamma_n(3), \Gamma_n(3)] \cong \mathcal{L}_n^{AN_3}(k). \end{aligned}$$

Hence there is nothing to do anymore in these cases. In the following, we consider the case where $l \geq 2$ and $k \geq 2l+4$.

Let $\mathfrak{C}_n^l(k)$ be the set of basic commutators

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_{k-2l}}, [x_{j_1}, x_{j_2}], \ldots, [x_{j_{2l-1}}, x_{j_{2l}}]]$$

of F_n of weight k such that $i_1 > i_2 \leq \cdots \leq i_{k-2l}$, $j_{2s-1} > j_{2s}$ and $(j_1, j_2) \leq \cdots \leq (j_{2l-1}, j_{2l})$. Our main theorem is

Theorem 4.1. *For any $l \geq 1$ and $k \geq 2l+4$, $\mathcal{L}_n^{N_l}(k)$ is a free abelian group with basis*

$$\mathfrak{C}_n^0(k) \cup \mathfrak{C}_n^1(k) \cup \cdots \cup \mathfrak{C}_n^l(k).$$

Before giving a proof of Theorem 4.1, we observe several facts and prepare some lemmas. First, for $l = 1$, we have already shown the theorem in our previous paper [11]. For $l \geq 2$ and $k \geq 1$, let

$$\iota_k^l : \mathcal{L}_n^{N_l}(k) \to \mathcal{L}_n^{N_{l-1}}(k)$$

be a natural surjective homomorphism induced from the natural map $F_n^{N_l} \to F_n^{N_{l-1}}$. Since $\mathcal{L}_n^{N_{l-1}}(k)$ is a free abelian group, if we denote by $\mathcal{K}_n^l(k)$ the kernel of ι_k^l, we have

$$\mathcal{L}_n^{N_l}(k) \cong \mathcal{K}_n^l(k) \oplus \mathcal{L}_n^{N_{l-1}}(k)$$

as a $\mathbf{Z}$-module.

Now, we have a natural isomorphism

$$\mathcal{L}_n^{N_m}(k) \cong \Gamma_n(k)K_m \Big/ \Gamma_n(k+1)K_m$$

for any $m \geq 1$. In general, for a group F and its normal subgroups G, H and K such that H is a subgroup of G, we have a natural isomorphism

$$GK/HK \cong G \Big/ H(G \cap K). \tag{3}$$

Using (3), we see

$$\mathcal{L}_n^{N_m}(k) \cong \Gamma_n(k) \Big/ \Gamma_n(k+1)(\Gamma_n(k) \cap K_m)$$

for $m \geq 1$. Hence, we verify that

$$\begin{aligned} \mathcal{K}_n^l(k) &\cong \Gamma_n(k+1)(\Gamma_n(k) \cap K_{l-1}) \Big/ \Gamma_n(k+1)(\Gamma_n(k) \cap K_l), \\ &\cong \Gamma_n(k) \cap K_{l-1} \Big/ (\Gamma_n(k) \cap K_l)(\Gamma_n(k+1) \cap K_{l-1}), \\ &\cong (\Gamma_n(k) \cap K_{l-1})K_l \Big/ (\Gamma_n(k+1) \cap K_{l-1})K_l \end{aligned}$$

by using (3).

To determine the structure of $\mathcal{K}_n^l(k)$, we prepare a descending series of subgroups of F_n. For $k \geq l+2$, denote by $\Theta_n^l(k)$ the subset of F_n which consists of elements w such that

$$I_j(f, w; a_1, \ldots, a_n) = 0, \quad 1 \leq j \leq n$$

for any $a_1, \ldots, a_n \in \mathbf{Z}$ and any $f \in \mathbf{R}[t]$ such that $\deg(f) \leq k-(l+2)$ and $\deg_2(f) \leq l$. Then we have

$$\Theta_n^l(l+2) \supset \Theta_n^l(l+3) \supset \Theta_n^l(l+4) \supset \cdots .$$

Since $I_j(1, w; a_1, \ldots, a_n) = I_j(1, w)$ is the sum of the exponents of those x_j which occur in w, we see $\Theta_n^l(l+2) = \Gamma_n(2)$. By (3) and (4) of Lemma 3.2, $\Theta_n^l(k)$ is a subgroup of F_n for each $k \geq l+2$. Furthermore, by (5) of Lemma 3.2, each of $\Theta_n^l(k)$ contains $[\Gamma_n(3), \Gamma_n(3)]$. Here we show each of $\Theta_n^l(k)$ is a normal subgroup of F_n. First, we consider

Lemma 4.2. $\Theta_n^l(l+3) \subset \Gamma_n(3)$.

Proof. For any $w \in \Theta_n^l(l+3)$, since $w \in \Gamma_n(2)$, we have

$$w = [x_2, x_1]^{w_{2,1}} \cdots [x_n, x_{n-1}]^{w_{n,n-1}} \gamma$$

for some $w_{i,j} \in \mathbf{Z}$ and $\gamma \in \Gamma_n(3)$. For any $1 \leq j < i \leq n$, using (3) of Lemma 3.2 and Lemma 3.3 , we see

$$\begin{aligned} I_j(t_i, w; a_1, \ldots, a_n) &= I_j(t_i, [x_2, x_1]^{w_{2,1}} \cdots [x_n, x_{n-1}]^{w_{n,n-1}}; a_1, \ldots, a_n), \\ &= \sum_{r>s} w_{r,s} I_j(t_i, [x_r, x_s]; a_1, \ldots, a_n), \\ &= w_{i,j} = 0. \end{aligned}$$

This shows $w = \gamma \in \Gamma_n(3)$. This completes the proof of Lemma 4.2. □

Now, for $k \geq l+3$, $w \in \Theta_n^l(k)$, $u \in F_n$ and $f \in \mathbf{R}[t]$ such that $\deg(f) \leq k-(l+2)$ and $\deg_2(f) \leq l$, we have

$$\begin{aligned} I_j(f, &uwu^{-1}; a_1, \ldots, a_n) \\ &= I_j(f, u; a_1, \ldots, a_n) + I_j(f \circ T_{P_u(a_1, \ldots, a_n)}, w; a_1 + u_1, \ldots, a_n + u_n) \\ &\quad - I_j(f \circ T_{P_{uwu^{-1}}(a_1, \ldots, a_n)}, u; a_1, \ldots, a_n), \\ &= 0 \end{aligned}$$

since $uwu^{-1} \in \Gamma_n(3)$. Therefore $\Theta_n^l(k)$ is a normal subgroup of F_n.

Next, we show

Lemma 4.3. *For $k \geq l+2$, $K_l' = [[\cdots[\Gamma_n(2), \Gamma_n(2)], \ldots], \Gamma_n(2)] \subset \Theta_n^l(k)$.*

Proof. Since K_l' is generated by

$$\{[y_1, y_2, \ldots, y_{l+2}] \,|\, y_i \in \Gamma_n(2)\},$$

it suffices to show $[y_1, y_2, \dots, y_{l+2}] \in \Theta_n^l(k)$ for $y_i \in \Gamma_n(2)$. For any $f \in \mathbf{R}[t]$ such that $\deg(f) \le k - (l+2)$ and $\deg_2(f) \le l$, using Lemma 3.2, we have

$$\begin{aligned}
&I_j(f, [y_1, y_2, \dots, y_{l+2}]; a_1, \dots, a_n) \\
&\qquad = I_j(f, y'; a_1, \dots, a_n) + I_j(f \circ T_{P_{y'}(a_1,\dots,a_n)}, y_{l+2}; a_1, \dots, a_n) \\
&\qquad\quad - I_j(f \circ T_{P_{y' y_{l+2} y'^{-1}}(a_1,\dots,a_n)}, y'; a_1, \dots, a_n) \\
&\qquad\quad - I_j(f \circ T_{P_{[y', y_{l+2}]}(a_1,\dots,a_n)}, y_{l+2}; a_1, \dots, a_n) \\
&\qquad = I_j(f - f \circ T_{P_{y' y_{l+2} y'^{-1}}(a_1,\dots,a_n)}, y'; a_1, \dots, a_n)
\end{aligned}$$

where $y' = [y_1, y_2, \dots, y_{l+1}]$. Hence, if

$$y_{l+2} = [x_2, x_1]^{z_{2,1}} \cdots [x_n, x_{n-1}]^{z_{n,n-1}} \in \mathcal{L}_n(3)$$

for $z_{i,j} \in \mathbf{Z}$, we see

$$P_{y' y_{l+2} y'^{-1}}(a_1, \dots, a_n) = z_{2,1} t_{2,1} + \cdots + z_{n,n-1} t_{n,n-1},$$

and if we set $g := f - f \circ T_{P_{y' y_{l+2} y'^{-1}}(a_1,\dots,a_n)}$ then $\deg(g) \le k - (l+2) - 1$ and $\deg_2(g) \le l - 1$. Repeating this process, we see that

$$I_j(f, [y_1, y_2, \dots, y_{l+2}]; a_1, \dots, a_n) = I_j(h, [y_1, y_2]; a_1, \dots, a_n)$$

for some $h \in \mathbf{R}[t_1, \dots, t_n]$. Then from (1) of Theorem 3.4, we have

$$I_j(f, [y_1, y_2, \dots, y_{l+2}]; a_1, \dots, a_n) = 0.$$

This completes the proof of Lemma 4.3. □

By Lemma 4.3, we see that $K_l \subset \Theta_n^l(k)$ for $k \ge l+2$. For any $l \ge 1$ and $k \ge 2l+4$, set

$$D_n^l(k) := [[\cdots[\Gamma_n(k-2l), \Gamma_n(2)], \cdots], \Gamma_n(2)] \subset \Gamma_n(k)$$

where $\Gamma_n(2)$ appears l times in the commutator above. Then we have

Lemma 4.4. *For $l \ge 1$, $k \ge 2l+4$, $D_n^l(k) \subset \Theta_n^l(k)$.*

Proof. Since $D_n^l(k)$ is generated by

$$\{[y_1, y_2, \dots, y_{l+1}] \mid y_1 \in \Gamma_n(k-2l), \;\; y_2, \dots, y_{l+1} \in \Gamma_n(2)\},$$

it suffices to show $[y_1, y_2, \dots, y_{l+1}] \in \Theta_n^l(k)$ for $y_1 \in \Gamma_n(k-2l)$ and $y_2, \dots, y_{l+1} \in \Gamma_n(2)$. For any $f \in \mathbf{R}[t]$ such that $\deg(f) \le k - (l+2)$ and $\deg_2(f) \le l$, using Lemma 3.2, we have

$$\begin{aligned}
&I_j(f, [y_1, y_2, \dots, y_{l+1}]; a_1, \dots, a_n) \\
&\qquad = I_j(f - f \circ T_{P_{y' y_{l+1} y'^{-1}}(a_1,\dots,a_n)}, y'; a_1, \dots, a_n)
\end{aligned}$$

where $y' = [y_1, y_2, \dots, y_l]$. Hence, if we set $g := f - f \circ T_{P_{y' y_{l+1} y'^{-1}}(a_1,\dots,a_n)}$ then $\deg(g) \le k - (l+2) - 1$ and $\deg_2(g) \le l - 1$. Repeating this process, we see that

$$I_j(f, [y_1, y_2, \dots, y_{l+1}]; a_1, \dots, a_n) = I_j(h, y_1; a_1, \dots, a_n)$$

for some $h \in \mathbf{R}[t_1, \dots, t_n]$ such that $\deg(h) = k - 2l - 2$. Since $y_1 \in \Gamma_n(k-2l)$, from (3) of Theorem 3.4, we obtain

$$I_j(f, [y_1, y_2, \dots, y_{l+1}]; a_1, \dots, a_n) = 0.$$

This completes the proof of Lemma 4.4. □

For $l \geq 1$, $k \geq 2l+4$ and $1 \leq m \leq l$, set

$$D_n^{l,m}(k) := [[\cdots[\Gamma_n(k-2l), \Gamma_n(2)], \cdots], \Gamma_n(2)] \subset \Gamma_n(k-2l+2m)$$

where $\Gamma_n(2)$ appears m times in the commutator above. Clearly, $D_n^{l,l}(k) = D_n^l(k)$. Then we have

Lemma 4.5. *For any $w \in K'_{l-1}$, w is written as*

$$w \equiv c_1^{e_1} \cdots c_s^{e_s} \pmod{D_n^{l,m}(k)}$$

where c_i are the basic commutators in $\Gamma_n(2)$.

Proof. We prove this lemma by the induction on $m \geq 1$. First, consider the case where $m = 1$. In general, for any a, $b \in \Gamma_n(2)$, there exist some $a', b' \in \Gamma_n(k-2l)$, and $d_{i,j}, d'_{i,j} \in \mathbf{Z}$ for $2 \leq i \leq k-1$ and $1 \leq j \leq m_i$ such that

$$a = c_{2,1}^{d_{2,1}} \cdots c_{k-1,m_{k-1}}^{d_{k-1,m_{k-1}}} a', \quad b = c_{2,1}^{d'_{2,1}} \cdots c_{k-1,m_{k-1}}^{d'_{k-1,m_{k-1}}} b'.$$

Hence,

$$[a,b] \equiv [c_{2,1}^{d_{2,1}} \cdots c_{k-1,m_{k-1}}^{d_{k-1,m_{k-1}}}, c_{2,1}^{d'_{2,1}} \cdots c_{k-1,m_{k-1}}^{d'_{k-1,m_{k-1}}}] \pmod{[\Gamma_n(k-2l), \Gamma_n(2)]}.$$

Since $[\Gamma_n(2), \Gamma_n(2)]$ is generated by $[a,b]$ for a, $b \in \Gamma_n(2)$, we see that any $w \in K'_{l-1} \subset [\Gamma_n(2), \Gamma_n(2)]$ is written as

$$w \equiv c_1^{e_1} \cdots c_s^{e_s} \pmod{D_n^{l,1}(k) = [\Gamma_n(k-2l), \Gamma_n(2)]}$$

where c_i are the basic commutators in $\Gamma_n(2)$.

Next, assume $m \geq 2$. Since K'_{l-1} is generated by $\{[y,z] \,|\, y \in K'_{l-2},\ z \in \Gamma_n(2)\}$, it suffices to show above for such $[y,z]$. By the inductive hypothesis, we can write

$$y = c_1^{e_1} \cdots c_r^{e_r} y'$$

for some basic commutators c_i in $\Gamma_n(2)$ and some $y' \in D_n^{l,m-1}(k)$. On the other hand, considering the mod-$\Gamma_n(k-2l)$ normal form of z, we see

$$z = d_1^{f_1} \cdots d_s^{f_s} z'$$

for some basic commutators d_j in $\Gamma_n(2)$ and some $z' \in \Gamma_n(k-2l)$. If we set $\bar{y} := c_1^{e_1} \cdots c_r^{e_r}$ and $\bar{z} := d_1^{f_1} \cdots d_s^{f_s}$, then

$$\begin{aligned}[y,z] &= [\bar{y}y', z] = [\bar{y}, [y', z]][y', z][\bar{y}, z], \\ &\equiv [\bar{y}, z] \pmod{D_n^{l,m}(k)}.\end{aligned}$$

Since $y \in K'_{l-2}$ and $y' \in K'_{m-2}$, we see $\bar{y} \in K'_{m-2}$. Hence $[\bar{y}, z'] \in [\Gamma_n(k-2l), K'_{m-2}] \subset D_n^{l,m}(k)$ by the following Lemma 4.6. Therefore we obtain

$$\begin{aligned}[\bar{y}, z] &= [\bar{y}, \bar{z}z'] = [\bar{y}, \bar{z}][\bar{y}, z'][[z', \bar{y}], \bar{z}], \\ &\equiv [\bar{y}, \bar{z}] \pmod{D_n^{l,m}(k)}.\end{aligned}$$

This completes the proof of Lemma 4.5. □

Lemma 4.6. *Let G be a group, A and B normal subgroups of G. For any $m \geq 1$, define normal subgroups $N_m(A, B)$ and $M_m(A, B)$ by*

$$N_m(A, B) := [A, [B, B, \dots, B]], \quad M_m(A, B) := [A, B, B, \dots, B]$$

where B appears m times in each of commutators. Then

$$N_m(A, B) \subset M_m(A, B).$$

Proof. We prove this lemma by the induction on $m \geq 1$. If $m = 1$, it is clear. Consider the case where $m = 2$. In general, for any subgroups X, Y and Z of G, a commutator subgroup $[X, [Y, Z]]$ is contained in the product of $[Y, [Z, X]]$ and $[Z, [X, Y]]$. (See Theorem 5.2 in [8], for example.) Using this fact, we see that

$$[A, [B, B]] \subset [B, [B, A]] \cdot [B, [A, B]] = [A, B, B].$$

Next, assume $m \geq 3$. Similarly, we have

$$\begin{aligned} N_m(A, B) &= [A, [B, B, \dots, B]] \subset [[B, \dots, B], [B, A]] \cdot [B, [A, [B, \dots, B]]], \\ &= N_{m-1}([A, B], B) \cdot [N_{m-1}(A, B), B]. \end{aligned}$$

By the inductive hypothesis, we see $N_m(A, B) \subset M_m(A, B)$. This completes the proof of Lemma 4.6. $\square$

Here we consider a mod-$D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]$ normal form of an element of K'_{l-1}.

Lemma 4.7. *Let $l \geq 1$ and $k \geq 2l + 4$. For any $w \in K'_{l-1}$, there exists some $r \geq 1$ and $e_1, \dots, e_r \in \mathbf{Z}$ such that*

$$w \equiv c_1^{e_1} \cdots c_r^{e_r} \pmod{D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]}$$

where $c_1 < \cdots < c_r$ are the basic commutators of F_n which belong to $[\Gamma_n(2), \Gamma_n(2)]$ but $D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]$.

Proof. Using Lemma 4.5 for $m = l$, we see that for any $w \in K'_{l-1}$, there exist some the basic commutators $\bar{c}_i$ in $\Gamma_n(2)$ such that

$$w \equiv \bar{c}_1^{e'_1} \cdots \bar{c}_s^{e'_s} \pmod{D_n^l(k)}.$$

By applying the Hall's correcting process to the element $w' := \bar{c}_1^{e'_1} \cdots \bar{c}_s^{e'_s}$ to obtain the mod-$\Gamma_n(k)$ normal form, we have

$$w' = c_1^{e_1} \cdots c_r^{e_r} \gamma$$

where $c_1 < \cdots < c_r$, all c_i belong to $[\Gamma_n(2), \Gamma_n(2)]$, and $\gamma \in \Gamma_n(k+1)$ is a product of commutators u among the components $c_i^{\pm 1}$. In fact, if there exists some c_i such that $c_i \in \Gamma_n(2) \setminus [\Gamma_n(2), \Gamma_n(2)]$ and $e_i \neq 0$, set

$$k' := \min\{\mathrm{wt}(c_i) \,|\, 1 \leq i \leq r, \ \ c_i \notin [\Gamma_n(2), \Gamma_n(2)], \ \ e_i \neq 0\} \leq k,$$

and let $\mathrm{wt}(c_j) = k'$. Then $c_j = [x_{j_1}, x_{j_2}, \dots, x_{j_{k'}}]$ for some $j_1 > j_2 \leq j_3 \leq \cdots \leq j_{k'}$. Hence by observing the image of w' by a natural homomorphism $\mathcal{L}_n(k') \to \mathcal{L}_n^M(k')$, we see that $e_j = 0$. This is a contradiction.

Now we show that the commutators u above belong to $D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]$. In fact, set $u := [u_1, u_2]$. If $\min\{\mathrm{wt}(u_1), \mathrm{wt}(u_2)\} \geq 3$, $u \in [\Gamma_n(3), \Gamma_n(3)]$. Assume $\min\{\mathrm{wt}(u_1), \mathrm{wt}(u_2)\} = 2$, without the loss of generality, it suffices to consider the case where $\mathrm{wt}(u_2) = 2$. Then, u_1 is of form $[u_{11}, u_{12}]$ where u_{11} and u_{12} are commutators among the components $c_i^{\pm 1}$. If $\min\{\mathrm{wt}(u_{11}), \mathrm{wt}(u_{12})\} \geq 3$, $u_1 \in [\Gamma_n(3), \Gamma_n(3)]$. If not, we may assume $\mathrm{wt}(u_{12}) = 2$. By repeating this process, finally we see $u \in [\Gamma_n(3), \Gamma_n(3)]$ or $u \in D_n^l(k)$. This completes the proof of Lemma 4.7. □

Proof of Theorem 4.1. Now we give a proof of Theorem 4.1. We prove the theorem by the induction on $l \geq 1$. If $l = 1$, it is given in our previous paper [11]. Assume $l \geq 2$, and for $1 \leq m \leq l-1$, $\mathcal{L}_n^{N_m}(k)$ is a free abelian group with basis

$$\mathfrak{C}_n^0(k) \cup \mathfrak{C}_n^1(k) \cup \cdots \cup \mathfrak{C}_n^m(k)$$

for each $k \geq 1$. In order to find a basis of $\mathcal{K}_n^l(k)$, we use the following lemma.

Lemma 4.8. *For $l \geq 2$ and $k \geq 2l+4$, $\Gamma_n(k) \cap K_{l-1} \subset D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]$.*

Proof. For any $w \in \Gamma_n(k) \cap K_{l-1} \subset K_{l-1} = K'_{l-1}[\Gamma_n(3), \Gamma_n(3)]$, there exist some $w' \in K'_{l-1}$ and $w'' \in [\Gamma_n(3), \Gamma_n(3)]$ such that $w = w'w''$. On the other hand, we have

$$w' \equiv c_1^{e_1} \cdots c_r^{e_r} \pmod{D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]}$$

for basic commutators c_i of F_n such that $c_1 < \cdots < c_r$ and each of c_i belongs to $[\Gamma_n(2), \Gamma_n(2)]$ but $D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]$ from Lemma 4.7. Here, we claim that $\mathrm{wt}(c_i) \geq k$ for any $1 \leq i \leq r$. In fact, if there exists some c_i such that $\mathrm{wt}(c_i) < k$ and $e_i \neq 0$, set

$$k' := \min\{\mathrm{wt}(c_i) \mid 1 \leq i \leq r,\ \ e_i \neq 0\},$$

and let $\mathrm{wt}(c_j) = k'$. Then $c_j \in Z(k')$, and hence by observing the image of w by a natural homomorphism $\mathcal{L}_n(k') \to \mathcal{L}_n^{AN_3}(k')$, we see that $e_j = 0$. This is a contradiction. Therefore $\mathrm{wt}(c_i) \geq k$ for any $1 \leq i \leq r$.

On the other hand, by observing the image of w by the natural homomorphism $\mathcal{L}_n^{N_l}(k) \to \mathcal{L}_n^{N_m}(k)$ for each $0 \leq m \leq l-1$, we see that the index e_i of the basic commutator c_i such that $c_i \in \mathfrak{C}_n^0(k) \cup \mathfrak{C}_n^1(k) \cup \cdots \cup \mathfrak{C}_n^{l-1}(k)$ is zero. Since a basic commutator $c \notin \mathfrak{C}_n^0(k) \cup \cdots \cup \mathfrak{C}_n^{l-1}(k)$ satisfies $c \in D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]$, we obtain $w \equiv w' \equiv 0 \pmod{D_n^l(k)[\Gamma_n(3), \Gamma_n(3)]}$. This completes the proof of Lemma 4.8. □

From Lemmas 4.4 and 4.8, we see that for each $k \geq 2l+4$,

$$\Gamma_n(k) \cap K_{l-1} \subset \Theta_n^l(k).$$

Using this, we can show that $\mathfrak{C}_n^l$ is a basis of $\mathcal{K}_n^l(k)$ for $k \geq 2l+4$. First, we show $\mathfrak{C}_n^l$ generates $\mathcal{K}_n^l(k)$. For any $x \in \Gamma_n(k) \cap K_{l-1}$, as mentioned above, we can write

$$x = c_1^{e_1} \cdots c_r^{e_r} x'$$

for some $x' \in \Gamma_n(k+1)$ and basic commutators $c_1 < \cdots < c_r$ of weight k such that $c_i \in D_n^l(k)[\Gamma_n(3), \Gamma_n(3)] \subset K_{l-1}$ for $1 \leq i \leq r$. Since $x \in K_{l-1}$, this shows $x' \in K_{l-1}$, and hence $x' = 0 \in \mathcal{K}_n^l(k)$. Now each of c_i belongs to K_l or $\mathfrak{C}_n^l$ since $k \geq 2l+4$. This shows that $\mathfrak{C}_n^l$ generates $\mathcal{K}_n^l(k)$.

Next we show $\mathfrak{C}_n^l$ is linearly independent. Set

$$v := \prod [x_{i_1}, x_{i_2}, \ldots, x_{i_{k-2l}}, [x_{j_1}, x_{j_2}], \ldots, [x_{j_{2l-1}}, x_{j_{2l}}]]^{b_{i_1,\ldots,j_{2l}}} \in \Gamma_n(k) \cap K_{l-1}$$

for $b_{i_1,\ldots,j_{2l}} \in \mathbf{Z}$ where the product runs over $i_1 > i_2 \leq \cdots \leq i_{k-2l}$, $j_{2s-1} > j_{2s}$ and $(j_1, j_2) \leq \cdots \leq (j_{2l-1}, j_{2l})$. Suppose $v = 1 \in \mathcal{K}_n^l(k)$. For any $i'_1 > i'_2 \leq \cdots \leq i'_{k-2l}$, $j'_{2s-1} > j'_{2s}$ and $(j'_1, j'_2) \leq \cdots \leq (j'_{2l-1}, j'_{2l})$, consider

$$g := t_{i'_2} \cdots t_{i'_{k-2l}} t_{j'_1, j'_2} \cdots t_{j'_{2l-1}, j'_{2l}} \in \mathbf{R}[t].$$

Since $\deg(g) = k - l - 1$, $\deg_2(g) = l$ and $v \in \Theta_n^l(k+1)$, for any $a_1, \ldots, a_n$, we have

$$\begin{aligned} 0 &= I_{i'_1}(g, v; a_1, \ldots, a_n), \\ &= (-1)^{k-l} b_{i'_1,\ldots,j'_{2l}} \frac{\partial^{k-l-1} f}{\partial t_{i'_2} \partial t_{i'_3} \cdots \partial t_{i'_{k-2l}} \partial t_{j'_1,j'_2} \cdots \partial t_{j'_{2l-1},j'_{2l}}}. \end{aligned}$$

from Proposition 3.5. Since

$$\frac{\partial^{k-l-1} f}{\partial t_{i'_2} \partial t_{i'_3} \cdots \partial t_{i'_{k-2l}} \partial t_{j'_1,j'_2} \cdots \partial t_{j'_{2l-1},j'_{2l}}} \neq 0,$$

we obtain $b_{i'_1,\ldots,j'_{2l}} = 0$. This shows that $\mathfrak{E}_n^l$ is linearly independent. This completes the proof of Theorem 4.1. $\square$

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