ON THE JOHNSON FILTRATION OF THE BASIS-CONJUGATING AUTOMORPHISM GROUP OF A FREE GROUP

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ABSTRACT. In this paper, we show that the abelianization of each subgroup of the lower central series of the basis-conjugating automorphism group of a free group, except for the first term, contains a free abelian group with infinite rank. As a corollary, we also show that each subgroup of the Johnson filtration of the basis-conjugating automorphism group of a free group has the same property.

1. INTRODUCTION

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, x_2, \ldots, x_n$, and $F_n = \Gamma_n(1)$, $\Gamma_n(2), \ldots$ its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n which induce the identity on the nilpotent quotient group $F_n/\Gamma_n(k+1)$. The group $\mathcal{A}_n(1)$ is called the IA-automorphism group and also denoted by IA_n. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, called the Johnson filtration of $\mathrm{Aut}\, F_n$.

The Johnson filtration of $\mathrm{Aut}\, F_n$ was originally introduced in 1963 through the remarkable pioneer work by Andreadakis [1] who showed that $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, $\ldots$ is a descending central series of $\mathcal{A}_n(1)$, and that for each $k \geq 1$ the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. Andreadakis [1] also computed the rank of $\mathrm{gr}^1(\mathcal{A}_n)$. Recently, by independent works of Cohen-Pakianathan [2, 3], Farb [5] and Kawazumi [6], it is known that $\mathrm{gr}^1(\mathcal{A}_n)$ is isomorphic to the abelianization of IA_n. For $k = 2$ and 3, the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n) \otimes_{\mathbf{Z}} \mathbf{Q}$ is determined by Pettet [14] and Satoh [15] respectively. For $k \geq 4$, however, there are few results for the structure of $\mathrm{gr}^k(\mathcal{A}_n)$.

When we study the Johnson filtration, we often face a problem of how to find a generating set of $\mathcal{A}_n(k)$. Although each of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ is a finitely generated free abelian group, it is still not determined whether each of $\mathcal{A}_n(k)$ is finitely generated or not for $k \geq 2$. Furthermore, it is also not known whether the abelianization $\mathcal{A}_n(k)^{\mathrm{ab}}$ of $\mathcal{A}_n(k)$ is finitely generated or not for $k \geq 2$.

In the study of the Johnson filtration of $\mathrm{Aut}\, F_n$, it is also interesting to determine whether $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, $\ldots$ coincides with the lower central series $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, $\ldots$

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of $\mathcal{A}_n(1)$ or not. Andreadakis [1] showed that $\mathcal{A}_2(k) = \mathcal{A}'_2(k)$ and $\mathcal{A}_3(3) = \mathcal{A}'_3(3)$. From the results due to Cohen-Pakianathan [2, 3], Farb [5] and Kawazumi [6], we have $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ for $n \geq 3$. Furthermore, Pettet [14] obtained that $\mathcal{A}'_n(3)$ has finite index in $\mathcal{A}_n(3)$. Now it is conjectured by Andreadakis that $\mathcal{A}_n(k) = \mathcal{A}'_n(k)$ for any $n \geq 3$ and $k \geq 3$.

In this paper, we consider a few problems as mentioned above for a certain subgroup of $\mathrm{Aut}\, F_n$. An automorphism σ of F_n such that x_i^σ is conjugate to x_i for each $1 \leq i \leq n$ is called a basis-conjugating automorphism of F_n. Let $\mathrm{P}\Sigma_n$ be the subgroup of $\mathrm{Aut}\, F_n$ consisting of the basis-conjugating automorphisms. The group $\mathrm{P}\Sigma_n$ is called the basis-conjugating automorphism group of F_n or the McCool group. It is easily checked that $\mathrm{P}\Sigma_n \subset \mathrm{IA}_n$. In general, by a work due to McCool [9], it is known that $\mathrm{P}\Sigma_n$ has a finite presentation. (See Subsection 2.4.)

In our previous paper [16], we study the image of the Johnson filtration of IA_n by the Burau representation τ_B. In particular, we are interested in some problems for $\tau_B(\mathrm{IA}_n)$ which correspond to the open problems for IA_n as mentioned above. For example, we consider whether $\tau_B(\mathrm{IA}_n)$ is finitely presentable or not, whether $\tau_B(\mathcal{A}_n(k))$ are finitely generated or not, and so on. In general, however, these problems are still difficult to handle. On the other hand, one of notable points is that the image of IA_n by τ_B coincides with that of $P\Sigma_n$. Hence it is meaningful to study the structure of the Johnson filtration of $P\Sigma_n$ from the view point of the research of the Burau representation.

Now, set $\mathcal{P}_n(k) := \mathrm{P}\Sigma_n \cap \mathcal{A}_n(k)$ for each $k \geq 1$. Then we have a descending central filtration

$$\mathrm{P}\Sigma_n = \mathcal{P}_n(1) \supset \mathcal{P}_n(1) \supset \mathcal{P}_n(2) \supset \cdots$$

of $\mathrm{P}\Sigma_n$. We call it the Johnson filtration of $\mathrm{P}\Sigma_n$. On the other hand, let $\mathcal{P}'_n(1) \supset \mathcal{P}'_n(2) \supset \cdots$ be the lower central series of $\mathrm{P}\Sigma_n$. Then $\mathcal{P}'_n(1) = \mathcal{P}_n(1)$ by definition. Since the Johnson filtration is central, we see $\mathcal{P}'_n(k) \subset \mathcal{P}_n(k)$ for each $k \geq 1$.

In this paper, we concentrate ourselves on the subgroups $\mathcal{P}'_n(k)$ and $\mathcal{P}_n(k)$ for $k \geq 1$. By a recent remarkable work by Cohen, Pakianathan, Vershinin and Wu [4], some properties of the associated graded Lie algebra and the (co)homological structure of $P\Sigma_n$ are studied. In particular, from one of their results, we see that the abelianization of $P\Sigma_n$ is a free abelian group of rank $n(n-1)$, and that Magnus generators type of $K_{i,j}$ form a basis of $P\Sigma_n^{\mathrm{ab}}$. As a corollary, we obtain $\mathcal{P}'_n(2) = \mathcal{P}_n(2)$. (See Corollary 2.2.) Then by determining the image of the second Johnson homomorphism restricted to $P\Sigma_n$, we obtain:

Lemma 1. *(= Corollary 2.4.) For $n \geq 3$, $\mathcal{P}'_n(3) = \mathcal{P}_n(3)$.*

From this, it immediately follows that $\tau_B(\mathcal{A}'_n(3)) = \tau_B(\mathcal{A}_n(3))$ which has already obtained in [16].

Next, using the Reidemeister-Schreier method, we obtain an infinite presentation for $\mathcal{P}'_n(2) = \mathcal{P}_n(2)$. Then we construct a certain surjective homomorphism

$$\Psi : \mathcal{P}'_n(2) \to A$$

which target is a free abelian group with infinite rank. (See Subsection 3.2.) Observing Ψ and its restriction to the subgroups $\mathcal{P}'_n(k)$ for $k \geq 3$, we obtain our main result of the paper:

Theorem 1. *(= Theorems 3.1 and 3.5.) For $n \geq 3$ and $k \geq 2$, $\mathcal{P}'_n(k)^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

As a corollary, we have:

Corollary 1. *(= Corollaries 3.2 and 3.7.) For $n \geq 3$ and $k \geq 2$, $\mathcal{P}_n(k)^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

We also remark that these results shows that $\mathcal{P}'_n(k)$ and $\mathcal{P}_n(k)$ are not finitely generated for $k \geq 2$.

This paper consists of four sections. In Section 2, we recall the IA-automorphism group, the Johnson filtration of $\mathrm{Aut}\, F_n$, and of the basis-conjugating automorphism group of a free group. In Section 3, we give an infinite presentation for $\mathcal{P}'_n(2)$, and detect infinitely many linearly independent elements in $\mathcal{P}_n(k)^{\mathrm{ab}}$ for $k \geq 2$.

Contents

2. Preliminaries

In this section, after fixing notation and conventions, we briefly recall the definition and some properties of the IA-automorphism group, the Johnson filtration of $\mathrm{Aut}\, F_n$, and of the basis-conjugating automorphism group of a free group.

2.1. Notation and conventions.

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group $\mathrm{Aut}\, G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

- For elements $g_1, \ldots, g_k \in G$, a commutator of weight k of the type
$$[[\cdots[[g_1, g_2], g_3], \cdots], g_k]$$
with all of its brackets to the left of all the elements occurring is called a simple k-fold commutator, and is denoted by $[g_{i_1}, g_{i_2}, \cdots, g_{i_k}]$.

2.2. IA-automorphism group.

In this paper, we fix a basis $x_1, \ldots, x_n$ of a free group F_n of rank n. Let $H := F_n^{\mathrm{ab}}$ be the abelianization of F_n and $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ the natural homomorphism induced from the abelianization of F_n. In the following, we identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of ρ is called the IA-automorphism group of F_n. It is clear that the inner automorphism group $\mathrm{Inn}\, F_n$ of F_n is contained in IA_n. In general, however, IA_n for $n \geq 3$ is much larger than $\mathrm{Inn}\, F_n$. In fact, Magnus [8] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms
$$K_{i,j} : x_t \mapsto \begin{cases} x_j{}^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$
for distinct i, $j \in \{1, 2, \ldots, n\}$ and
$$K_{i,j,l} : x_t \mapsto \begin{cases} x_i [x_j, x_l], & t = i, \\ x_t, & t \neq i \end{cases}$$
for distinct i, j, $l \in \{1, 2, \ldots, n\}$ such that $j < l$.

Recently, Cohen-Pakianathan [2, 3], Farb [5] and Kawazumi [6] independently showed
$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$
as a $\mathrm{GL}(n, \mathbf{Z})$-module where $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ is the $\mathbf{Z}$-linear dual group of H. In particular, from their result, we see that $\mathrm{IA}_n^{\mathrm{ab}}$ is a free abelian group with basis the coset classes of the Magnus generators $K_{i,j}$ and $K_{i,j,l}$.

2.3. Johnson filtration.

In this subsection, we recall the Johnson filtration of $\mathrm{Aut}\, F_n$. Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by the rule
$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2.$$
We denote by $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ the graded quotient of the lower central series of F_n for each $k \geq 1$. It is classically well known due to Witt [19] that each $\mathcal{L}_n(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank
$$\frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}} \tag{2}$$
where μ is the Möbius function. For example, $\mathcal{L}_n(1) = H$ and $\mathcal{L}_n(2) \cong \Lambda^2 H$, the exterior product of H of degree 2.

For each $k \geq 0$, the action of $\mathrm{Aut}\, F_n$ on the nilpotent quotient group $F_n/\Gamma_n(k+1)$ of F_n induces a homomorphism

$$\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

We denote its kernel by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, with $\mathcal{A}_n(1) = \mathrm{IA}_n$. (See [1] for details.) It is called the Johnson filtration of $\mathrm{Aut}\, F_n$. Set $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ for each $k \geq 1$. Then the graded sum $\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k\geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ has a graded Lie algebra structure induced from the commutator bracket on IA_n. The subgroups $\mathcal{A}_n(k)$ and the graded quotients $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ play an important role on the various study of the IA-automorphism group IA_n. It is known that each of $\mathrm{gr}^k(\mathcal{A}_n)$ is a free abelian group of finite rank due to Andreadakis [1]. Its rank is, however, not determined yet in general.

In order to study the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$, the Johnson homomorphisms are often used. For each $k \geq 1$, define a homomorphism $\tilde{\tau}_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \ \mapsto \ (x \mapsto x^{-1}x^{\sigma}), \quad x \in H.$$

Then the kernel of this homomorphism is exactly $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)).$$

The homomorphism τ_k is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. Hence to study the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ is equivalent to study the images of the Johnson homomorphisms. For the Magnus generators of IA_n, their images by τ_1 are given by

$$\tau_1(K_{i,j}) = x_i^* \otimes [x_i, x_j], \quad \tau_1(K_{i,j,l}) = x_i^* \otimes [x_j, x_l] \tag{3}$$

where $x_1^*, \ldots, x_n^* \in H^*$ is the dual basis of $x_1, \ldots, x_n \in H$. We remark that τ_1 is an isomorphism and nothing but the abelianization of IA_n. (See [2, 3, 5, 6].)

Let $\mathrm{Der}\,(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree k part of $\mathrm{Der}\,(\mathcal{L}_n)$ is considered as $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, and we identify them in this paper. Then the sum of the Johnson homomorphisms

$$\tau := \bigoplus_{k \geq 1} \tau_k : \mathrm{gr}(\mathcal{A}_n) \to \mathrm{Der}\,(\mathcal{L}_n)$$

is a graded Lie algebra homomorphism. In fact, if we denote by $\partial\xi$ the element of $\mathrm{Der}\,(\mathcal{L}_n)$ corresponding to an element $\xi \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$, and write the action of $\partial\xi$ on $X \in \mathcal{L}_n$ as $X^{\partial\xi}$ then we have

$$\tau_{k+l}([\sigma, \sigma']) = \tau_k(\sigma)^{\partial\tau_l(\sigma')} - \tau_l(\sigma')^{\partial\tau_k(\sigma)}.$$

for any $\sigma \in \mathcal{A}_n(k)$ and $\sigma' \in \mathcal{A}_n(l)$. This formula is very useful to calculate the image of the Johnson homomorphism inductively.

Now, for $1 \leq k \leq 3$, the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n) \otimes_{\mathbf{Z}} \mathbf{Q}$ is completely determined by Andreadakis [1], Pettet [14] and Satoh [15] for $k = 1, 2$ and 3 respectively. It seems, however, to study $\mathrm{gr}^k(\mathcal{A}_n)$ for $k \geq 4$ is quite difficult problem in general since even its generating set has not been obtained yet.

Let $\mathcal{A}'_n(k)$ be the lower central series of IA_n with $\mathcal{A}'_n(1) = \mathrm{IA}_n$. Since the Johnson filtration is central, $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for each $k \geq 1$. It is conjectured that $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis who showed $\mathcal{A}'_2(k) = \mathcal{A}_2(k)$ for each $k \geq 1$ and $\mathcal{A}'_3(3) = \mathcal{A}_3(3)$ in [1]. Now, it is known that $\mathcal{A}'_n(2) = \mathcal{A}_n(2)$ due to Cohen-Pakianathan [2, 3], Farb [5] and Kawazumi [6], and that $\mathcal{A}'_n(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [14]. Since IA_n is finitely generated, so does each of the graded quotient $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. Hence $\mathrm{gr}^k(\mathcal{A}'_n)$ is easier to handle rather than $\mathrm{gr}^k(\mathcal{A}_n)$ in general.

A restriction of $\tilde{\tau}_k$ to $\mathcal{A}'_n(k)$ induces a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

We also call it the Johnson homomorphism by an abuse of language. In our previous papers [15], [17] and [18], we have studied the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbf{Q}} = \tau'_k \otimes \mathrm{id}_{\mathbf{Q}}$. Although the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}'_n)$ is considerably clarified today through these works, the group structure of each of $\mathcal{A}'_n(k)$ is still not well-understood. For example, it is not determined whether each of $\mathcal{A}'_n(k)$ is finitely generated or not.

2.4. **Basis-conjugating automorphism group.**

In this subsection, we recall the basis-conjugating automorphisms of F_n. In general, an automorphism σ of F_n such that x_i^σ is conjugate to x_i for each $1 \leq i \leq n$ is called a basis-conjugating automorphism of F_n. Let $\mathrm{P}\Sigma_n$ be the subgroup of $\mathrm{Aut}\, F_n$ consisting of the basis-conjugating automorphisms. The group $\mathrm{P}\Sigma_n$ is called the basis-conjugating automorphism group of F_n or the McCool group. It is easily checked that $\mathrm{P}\Sigma_n \subset \mathrm{IA}_n$ and an IA-automorphism $K_{i,j}$ for $i \neq j$ belongs to $\mathrm{P}\Sigma_n$. McCool obtained a finite presentation for $\mathrm{P}\Sigma_n$ as follows:

Theorem 2.1 (McCool [9]). *The group $\mathrm{P}\Sigma_n$ has a finite presentation with generators $K_{i,j}$ for $1 \leq i \neq j \leq n$ subject to relations:*
(R1): $[K_{i,j}, K_{k,j}] = 1$ *for* $i < k$,
(R2): $[K_{i,j}, K_{k,l}] = 1$ *for* $i < k$,
(R3): $[K_{i,k}, K_{i,j}K_{k,j}] = 1$
where the subscripts i, j, k, l are distinct.

In this paper we consider the restriction of the Johnson filtration to $\mathrm{P}\Sigma_n$. Namely, set $\mathcal{P}_n(k) := \mathrm{P}\Sigma_n \cap \mathcal{A}_n(k)$ for each $k \geq 1$. Then we have a descending central filtration

$$\mathrm{P}\Sigma_n = \mathcal{P}_n(1) \supset \mathcal{P}_n(1) \supset \mathcal{P}_n(2) \supset \cdots$$

of $\mathrm{P}\Sigma_n$. We call it the Johnson filtration of $\mathrm{P}\Sigma_n$. Then each of the graded quotients $\mathrm{gr}^k(\mathcal{P}_n) := \mathcal{P}_n(k)/\mathcal{P}_n(k+1)$ is a $\mathbf{Z}$-submodule of $\mathrm{gr}^k(\mathcal{A}_n)$ for $k \geq 1$. We denote by $\tau_{P,k}$ the k-th Johnson homomorphism τ_k restricted to $\mathrm{gr}^k(\mathcal{P}_n)$ for $k \geq 1$.

Let $\mathcal{P}'_n(1) \supset \mathcal{P}'_n(2) \supset \cdots$ be the lower central series of $\mathrm{P}\Sigma_n$. Then $\mathcal{P}'_n(1) = \mathcal{P}_n(1)$ by definition. Since the Johnson filtration is central, we see $\mathcal{P}'_n(k) \subset \mathcal{P}_n(k)$ for each $k \geq 1$. Set $\mathrm{gr}^k(\mathcal{P}'_n) := \mathcal{P}'_n(k)/\mathcal{P}'_n(k+1)$ for $k \geq 1$. There is a natural homomorphism $\iota_k : \mathrm{gr}^k(\mathcal{P}'_n) \to \mathrm{gr}^k(\mathcal{P}_n)$ for each $k \geq 1$ induced from the inclusion $\mathcal{P}'_n(k) \hookrightarrow \mathcal{P}_n(k)$. Then we define a homomorphism $\tau'_{P,k}$ to be the composition of ι_k and the Johnson homomorphism $\tau_{P,k}$:

$$\tau'_{P,k} = \tau_{P,k} \circ \iota_k : \mathrm{gr}^k(\mathcal{P}'_n) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)).$$

According to the Andreadakis's conjecture as mentioned before, it would be seemed $\mathcal{P}'_n(k) = \mathcal{P}_n(k)$ for each $k \geq 1$. It is, however, still an open problem in general. Here we observe $\mathcal{P}'_n(2) = \mathcal{P}_n(2)$, and show $\mathcal{P}'_n(3) = \mathcal{P}_n(3)$ by using the second Johnson homomorphism. First, we consider the abelianization of $P\Sigma_n = \mathcal{P}_n(1)$. By a work of Cohen, Pakianathan, Vershinin and Wu [4], we have

$$\mathcal{P}_n(1)^{\mathrm{ab}} \cong \mathbf{Z}^{\oplus n(n-1)} \tag{4}$$

for $n \geq 3$. This isomorphism is given by the first Johnson homomorphism $\tau_{P,1}$. In particular, we see that the Magnus generators $K_{i,j}$ for $1 \leq i \neq j \leq n$ form a basis of $\mathcal{P}_n(1)^{\mathrm{ab}}$. Hence, as a corollary, we have

Corollary 2.2. *For $n \geq 3$, $\mathcal{P}'_n(2) = \mathcal{P}_n(2)$.*

Proof. Since $\tau_{P,1}$ is the abelianization of $\mathcal{P}_n(1)$, the natural homomorphism $\iota_1 : \mathrm{gr}^1(\mathcal{P}'_n) \to \mathrm{gr}^1(\mathcal{P}_n)$ must be injective. Hence $\mathcal{P}'_n(2) = \mathcal{P}_n(2)$. This completes the proof of Corollary 2.2. $\square$

Next, we consider $\mathcal{P}'_n(3)$. First, we determine the image of the Johsnon homomorphism $\tau_{P,2}$.

Lemma 2.3. *For $n \geq 3$, $\mathrm{Im}(\tau_{P,2}) \cong \mathbf{Z}^{\oplus n(n-1)^2/2}$.*

Proof. From Corollary 2.2, we see that $\mathrm{Im}(\tau_{P,2}) = \mathrm{Im}(\tau'_{P,2})$. We show that $\mathrm{gr}^2(\mathcal{A}'_n)$ is generated by $n(n-1)^2/2$ elements:

$$S' := \{[K_{i,j}, K_{i,q}] \,|\, 1 \leq i \leq n,\ \ 1 \leq j < q \leq n,\ \ j, q \neq i\} \\ \cup \{[K_{i,j}, K_{j,i}] \,|\, 1 \leq i < j \leq n\}.$$

In general, $\mathrm{gr}^2(\mathcal{P}'_n)$ is generated by commutators $[K_{i,j}, K_{p,q}]$. Set $N := \sharp\{i, j, p, q\}$ as above. If N=4, or $N = 3$ and $q = j$, we have $[K_{i,j}, K_{p,q}] = 1 \in \mathcal{P}_n(2)$ by the relation **(R1)** and **(R2)**. It is also clear that if $N = 3$ and $q = i$, $[K_{i,j}, K_{p,i}] = [K_{p,i}, K_{i,j}]^{-1} \in \mathcal{P}_n(2)$, and that if $N = 3$, $p = i$ and $j > q$, $[K_{i,j}, K_{i,q}] = [K_{i,q}, K_{i,j}]^{-1}$. On the other hand, using the relation **(R3)**, we have

$$[K_{i,j}, K_{i,q}K_{j,q}] = [K_{i,j}, K_{i,q}] + [K_{i,j}, K_{j,q}] = 0 \in \mathrm{gr}^2(\mathcal{A}'_n).$$

Hence we can reduce the generators type of $[K_{i,j}, K_{j,q}]$. Then we see that S' generates $\mathrm{gr}^2(\mathcal{A}'_n)$.

This shows that $\mathrm{Im}(\tau_{P,2})$ is generated by

$$S := \{\tau_{P,2}([K_{i,j}, K_{i,q}]) \,|\, 1 \leq i \leq n,\ \ 1 \leq j < q \leq n,\ \ j, q \neq i\} \\ \cup \{\tau_{P,2}([K_{i,j}, K_{j,i}]) \,|\, 1 \leq i < j \leq n\}.$$

Here we have

$$\tau_{P,2}([K_{i,j}, K_{i,q}]) = x_i^* \otimes [x_i, x_q, x_j] - x_i^* \otimes [x_i, x_j, x_q]$$

and

$$\tau_{P,2}([K_{i,j}, K_{j,i}]) = x_i^* \otimes [x_i, x_j, x_i] - x_j^* \otimes [x_j, x_i, x_j].$$

In order to show S is linearly independent in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3)$, set

$$\sum{}' a_{i,j,q}\tau_{P,2}([K_{i,j}, K_{i,q}]) + \sum{}'' b_{i,j}\tau_{P,2}([K_{i,j}, K_{j,i}]) = 0$$

where the first sum runs over $1 \leq i \leq n$, and $1 \leq j < q \leq n$ such that $j, q \neq i$, and the second one $1 \leq i < j \leq n$. Then for any $1 \leq i_0 \leq n$, we have

$$\begin{aligned}(5)\qquad \sum_{j<q}^{j,q\neq i_0} a_{i_0,j,q}\tau_{P,2}([K_{i_0,j}, K_{i_0,q}]) + \sum_{i_0<j} b_{i_0,j} x_{i_0}^* \otimes [x_{i_0}, x_j, x_{i_0}] \\ - \sum_{j<i_0} b_{j,i_0} x_{i_0}^* \otimes [x_{i_0}, x_j, x_{i_0}] = 0.\end{aligned}$$

Then consider the composition of homomorphisms

$$\Phi : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to H^* \otimes_{\mathbf{Z}} H^{\otimes 3} \to H^{\otimes 2}$$

where the first homomorphism is induced from the natural embedding $\mathcal{L}_n(3) \hookrightarrow H^{\otimes 3}$ defined by $[X, Y] \mapsto X \otimes Y - Y \otimes X$, and the second one is the contraction map

$$x_i^* \otimes x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \mapsto x_i^*(x_{i_1}) \cdot x_{i_2} \otimes x_{i_3}.$$

Observing the image of (5) by Φ, we obtain

$$\begin{aligned}\sum_{j<q}^{j,q\neq i_0} a_{i_0,j,q}(x_q \otimes x_j - x_j \otimes x_q) + \sum_{i_0<j} b_{i_0,j}(x_j \otimes x_{i_0} - x_{i_0} \otimes x_j) \\ - \sum_{j<i_0} b_{j,i_0}(x_j \otimes x_{i_0} - x_{i_0} \otimes x_j) = 0 \in H^{\otimes 2}.\end{aligned}$$

This shows that $a_{i_0,j,q} = b_{i_0,j} = b_{j,i_0} = 0$ for $1 \leq j < q \leq n$ and $j, q \neq i_0$. Hence S is linearly independent in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3)$. Therefore we obtain the required result. This completes the proof of Lemma 2.3. □

Corollary 2.4. *For $n \geq 3$, $\mathcal{P}_n'(3) = \mathcal{P}_n(3)$.*

Proof. Since the image of the generating set S' of $\mathrm{gr}^2(\mathcal{A}_n')$ by $\tau_{P,2}'$ is a basis of $\mathrm{Im}(\tau_{P,2}') = \mathrm{Im}(\tau_{P,2})$, it turns out that $\mathrm{gr}^2(\mathcal{A}_n')$ is a free abelian group with basis S', and that

$$\tau_{P,2}' : \mathrm{gr}^2(\mathcal{A}_n') \xrightarrow{\iota_2} \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_{P,2}} \mathrm{Im}(\tau_{P,2})$$

is an isomorphism. Hence ι_2 is also an isomorphism. This shows $\mathcal{P}_n'(3) = \mathcal{P}_n(3)$. This completes the proof of Corollary 2.4. □

Observing the proofs of Lemma 2.3 and of Corollary 2.4, we see that for each $k \geq 3$ if we determine the rank $r_n(k) := \mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{A}_n))$ of $\mathrm{gr}^k(\mathcal{A}_n)$, and show that $\mathrm{gr}^k(\mathcal{A}_n')$ is generated by $r_n(k)$ elements, then we can show $\mathcal{P}_n'(k) = \mathcal{P}_n(k)$ inductively. It seems, however, quite complicated since we have to consider too many types of commutators among $K_{i,j}$ in $\mathrm{gr}^k(\mathcal{A}_n')$ for large k.

3. On the abelianization of $\mathcal{P}'_n(k)$ and $\mathcal{P}_n(k)$ for $k \geq 2$

In this section, we show that the abelianization of $\mathcal{P}'_n(k)$ contains a free abelian group of infinite rank for $n \geq 2$ and $k \geq 2$. Then we prove that $\mathcal{P}_n(k)$ also has the same property. In order to show this, we obtain a presentation for the commutator subgroup $\mathcal{P}'_n(2)$ of $P\Sigma_n$ by the Reidemeister-Schreier method. Then we construct a homomorphism from $\mathcal{P}'_n(2)$ to some free abelian group which detects an infinitely generated free abelian subgroup in $\mathcal{P}'_n(k)$ for $k \geq 2$.

3.1. A presentation for $\mathcal{P}'_n(2)$.

In this subsection, we obtain a presentation for $\mathcal{P}'_n(2)$ by applying the Reidemeister-Schreier method. (For the details for the Reidemeister-Schreier method, see Proposition 4.1 in Chapter II in [7] for example.)

Let F be a free group with basis $\{K_{i,j} \,|\, 1 \leq i \neq j \leq n\}$, and $\varphi : F \to P\Sigma_n$ the canonical map. Set $N = \varphi^{-1}(\mathcal{P}'_n(2))$. Then a subset

$$T := \{K_{1,2}^{e_{1,2}} K_{1,3}^{e_{1,3}} \cdots K_{n,n-1}^{e_{n,n-1}} \,|\, e_{i,j} \in \mathbf{Z}\} \subset F$$

is a Schreier transversal for N of F since $P\Sigma_n^{\mathrm{ab}}$ is a free abelian group with basis $\{K_{i,j} \,|\, 1 \leq i \neq j \leq n\}$. Here the order of the product among $K_{i,j}^{e_{i,j}}$ in $K_{1,2}^{e_{1,2}} K_{1,3}^{e_{1,3}} \cdots K_{n,n-1}^{e_{n,n-1}}$ is the usual lexicographic order with respect to the index set

$$I := \{(i,j) \,|\, 1 \leq i \neq j \leq n\}.$$

Namely, for any (i,j) and $(p,q) \in I$, $(i,j) < (p,q)$ if and only if $i < p$, or $i = p$ and $j < q$.

Set

$$\gamma_{i,j}(e_{1,2}, \ldots, e_{n,n-1}) := (K_{1,2}^{e_{1,2}} \cdots K_{n,n-1}^{e_{n,n-1}}) K_{i,j} (K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1} \in F.$$

Then by applying the Reidemeister-Schreier method to the McCool's presentation for $P\Sigma_n$ and the Schreier transversal T for N of F, we see that $\mathcal{P}'_n(2)$ is generated by

$$\begin{aligned} \mathfrak{E} := \{\gamma_{i,j}(e_{1,2}, \ldots, e_{n,n-1}) \,|\, (1,2) \leq (i,j) \leq (n, n-2),\ \ & e_{p,q} \neq 0 \ \text{ for some} \\ & (i, j+1) \leq (p,q) \leq (n, n-1)\}, \end{aligned}$$

subject to relators

$$\tau(trt^{-1}) \ \text{ for } \ t \in T \ \text{ and } \ r = (\mathbf{R1}), (\mathbf{R2}), (\mathbf{R3})$$

where τ denotes the "rewriting function". Namely, for any word $w \in N$ among $K_{i,j}$s, $\tau(w)$ $(= w$ in $F)$ is a word among $\gamma_{i,j}(e_{1,2}, \ldots, e_{n,n-1})$s. In the following, for any $t = K_{1,2}^{e_{1,2}} \cdots K_{n,n-1}^{e_{n,n-1}} \in T$, we write down $\tau(trt^{-1})$ explicitly.

Case (I). $r = [K_{i,j}, K_{k,j}]$ for distinct $1 \leq i, j, k \leq n$ and $i < k$.

In this case, we have

$$\begin{aligned} trt^{-1} = & (K_{1,2}^{e_{1,2}} \cdots K_{n,n-1}^{e_{n,n-1}}) K_{i,j} (K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1} \\ & \cdot (K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}}) K_{k,j} (K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{k,j}^{e_{k,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1} \\ & \cdot (K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{k,j}^{e_{k,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}}) K_{i,j}^{-1} (K_{1,2}^{e_{1,2}} \cdots K_{k,j}^{e_{k,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1} \\ & \cdot (K_{1,2}^{e_{1,2}} \cdots K_{k,j}^{e_{k,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}}) K_{k,j}^{-1} (K_{1,2}^{e_{1,2}} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1}. \end{aligned}$$

(1) If $e_{p,q} \neq 0$ for some $(k, j+1) \leq (p,q) \leq (n, n-1)$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{n,n-1})\gamma_{k,j}(e_{1,2}, \ldots, e_{i,j}+1, \ldots, e_{n,n-1}) \cdot \gamma_{i,j}(e_{1,2}, \ldots, e_{k,j}+1, \ldots, e_{n,n-1})^{-1}\gamma_{k,j}(e_{1,2}, \ldots, e_{n,n-1})^{-1}$$

(2) If $\{(p,q) \in I \,|\, (k, j+1) \leq (p,q) \leq (n, n-1),\ \ e_{p,q} \neq 0\} = \phi$,

$$trt^{-1} = (K_{1,2}^{e_{1,2}} \cdots K_{k,j}^{e_{k,j}})K_{i,j}(K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{k,j}^{e_{k,j}})^{-1} \cdot (K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{k,j}^{e_{k,j}+1})K_{i,j}^{-1}(K_{1,2}^{e_{1,2}} \cdots K_{k,j}^{e_{k,j}+1})^{-1}.$$

Hence, we see:

(i) If $e_{p,q} \neq 0$ for some $(i, j+1) \leq (p,q) \leq (k, j-1)$, or $e_{k,j} \neq 0, -1$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{k,j}, 0, \ldots, 0)\gamma_{i,j}(e_{1,2}, \ldots, e_{k,j}+1, 0, \ldots, 0)^{-1}.$$

(ii) If $\{(p,q) \in I \,|\, (i, j+1) \leq (p,q) \leq (k, j-1),\ \ e_{p,q} \neq 0\} = \phi$ and $e_{k,j} = 0$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{i,j}, 0, \ldots, 1, \ldots, 0)^{-1}$$

where 1 appears in (k,j) entry.

(iii) If $\{(p,q) \in I \,|\, (i, j+1) \leq (p,q) \leq (k, j-1),\ \ e_{p,q} \neq 0\} = \phi$ and $e_{k,j} = -1$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{i,j}, 0, \ldots, -1, \ldots, 0)$$

where -1 appears in (k,j) entry.

Similarly, we have the cases (II) and (III) as follows.

Case (II). $r = [K_{i,j}, K_{k,l}]$ for distinct $1 \leq i,j,k,l \leq n$ and $i < k$.

In this case, we have

$$\begin{aligned} trt^{-1} =& (K_{1,2}^{e_{1,2}} \cdots K_{n,n-1}^{e_{n,n-1}})K_{i,j}(K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1} \\ &\cdot (K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{n,n-1}^{e_{n,n-1}})K_{k,l}(K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{k,l}^{e_{k,l}+1} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1} \\ &\cdot (K_{1,2}^{e_{1,2}} \cdots K_{i,j}^{e_{i,j}+1} \cdots K_{k,l}^{e_{k,l}+1} \cdots K_{n,n-1}^{e_{n,n-1}})K_{i,j}^{-1}(K_{1,2}^{e_{1,2}} \cdots K_{k,l}^{e_{k,l}+1} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1} \\ &\cdot (K_{1,2}^{e_{1,2}} \cdots K_{k,l}^{e_{k,l}+1} \cdots K_{n,n-1}^{e_{n,n-1}})K_{k,l}^{-1}(K_{1,2}^{e_{1,2}} \cdots K_{n,n-1}^{e_{n,n-1}})^{-1}. \end{aligned}$$

(1) If $e_{p,q} \neq 0$ for some $(k, l+1) \leq (p,q) \leq (n, n-1)$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{n,n-1})\gamma_{k,l}(e_{1,2}, \ldots, e_{i,j}+1, \ldots, e_{n,n-1}) \cdot \gamma_{i,j}(e_{1,2}, \ldots, e_{k,l}+1, \ldots, e_{n,n-1})^{-1}\gamma_{k,l}(e_{1,2}, \ldots, e_{n,n-1})^{-1}.$$

(2) If $\{(p,q) \in I \,|\, (k, l+1) \leq (p,q) \leq (n, n-1),\ \ e_{p,q} \neq 0\} = \phi$, we have:

(i) If $e_{p,q} \neq 0$ for some $(i, j+1) \leq (p,q) \leq (k, l-1)$, or $e_{k,l} \neq 0, -1$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{k,l}, 0, \ldots, 0)\gamma_{i,j}(e_{1,2}, \ldots, e_{k,l}+1, 0, \ldots, 0)^{-1}.$$

(ii) If $\{(p,q) \in I \,|\, (i, j+1) \leq (p,q) \leq (k, l-1),\ \ e_{p,q} \neq 0\} = \phi$ and $e_{k,l} = 0$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{i,j}, 0, \ldots, 1, \ldots, 0)^{-1}$$

where 1 appears in (k,l) entry.

(iii) If $\{(p,q) \in I \,|\, (i,j+1) \leq (p,q) \leq (k,l-1),\;\; e_{p,q} \neq 0\} = \phi$ and $e_{k,l} = -1$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2},\ldots,e_{i,j},0,\ldots,-1,\ldots,0)$$

where -1 appears in (k,l) entry.

Case (III). $r = [K_{i,k}, K_{i,j}K_{k,j}]$ for distinct $1 \leq i,j,k \leq n$.

(1) $i < k$ and $j < k$. Then $(i,j) < (i,k) < (k,j)$.

In this case, we have

$$\begin{aligned}
trt^{-1} =&(K_{1,2}^{e_{1,2}}\cdots K_{n,n-1}^{e_{n,n-1}})K_{i,k}(K_{1,2}^{e_{1,2}}\cdots K_{i,k}^{e_{i,k}+1}\cdots K_{n,n-1}^{e_{n,n-1}})^{-1}\\
&\cdot(K_{1,2}^{e_{1,2}}\cdots K_{i,k}^{e_{i,k}+1}\cdots K_{n,n-1}^{e_{n,n-1}})K_{i,j}(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{i,k}^{e_{i,k}+1}\cdots K_{n,n-1}^{e_{n,n-1}})^{-1}\\
&\cdot(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{i,k}^{e_{i,k}+1}\cdots K_{n,n-1}^{e_{n,n-1}})K_{k,j}\\
&\qquad(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}}\cdots K_{i,k}^{e_{i,k}+1}\cdots K_{k,j}^{e_{k,j}+1}\cdots K_{n,n-1}^{e_{n,n-1}})^{-1}\\
&\cdot(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{i,k}^{e_{i,k}+1}\cdots K_{k,j}^{e_{k,j}+1}\cdots K_{n,n-1}^{e_{n,n-1}})K_{i,k}^{-1}\\
&\qquad(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{k,j}^{e_{k,j}+1}\cdots K_{n,n-1}^{e_{n,n-1}})^{-1}\\
&\cdot(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{k,j}^{e_{k,j}+1}\cdots K_{n,n-1}^{e_{n,n-1}})K_{k,j}^{-1}(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{n,n-1}^{e_{n,n-1}})^{-1}\\
&\cdot(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{n,n-1}^{e_{n,n-1}})K_{ij}^{-1}(K_{1,2}^{e_{1,2}}\cdots K_{n,n-1}^{e_{n,n-1}})^{-1}.
\end{aligned}$$

(i) If $e_{p,q} \neq 0$ for some $(k,j+1) \leq (p,q) \leq (n,n-1)$,

$$\begin{aligned}
\tau(trt^{-1}) = &\gamma_{i,k}(e_{1,2},\ldots,e_{n,n-1})\gamma_{i,j}(e_{1,2},\ldots,e_{i,k}+1,\ldots,e_{n,n-1})\\
&\cdot\gamma_{k,j}(e_{1,2},\ldots,e_{i,j}+1,\ldots,e_{i,k}+1,\ldots,e_{n,n-1})\\
&\cdot\gamma_{i,k}(e_{1,2},\ldots,e_{i,j}+1,\ldots,e_{k,j}+1,\ldots e_{n,n-1})^{-1}\\
&\cdot\gamma_{k,j}(e_{1,2},\ldots,e_{i,j}+1,\ldots,e_{n,n-1})^{-1}\gamma_{i,j}(e_{1,2},\ldots e_{n,n-1})^{-1}.
\end{aligned}$$

(ii) If $\{(p,q) \in I \,|\, (k,j+1) \leq (p,q) \leq (n,n-1),\;\; e_{p,q} \neq 0\} = \phi$, we have:

① If $e_{p,q} \neq 0$ for some $(i,k+1) \leq (p,q) \leq (k,j-1)$, or $e_{k,j} \neq 0,-1$,

$$\begin{aligned}
\tau(trt^{-1}) = &\gamma_{i,k}(e_{1,2},\ldots,e_{k,j},0\ldots,0)\gamma_{i,j}(e_{1,2},\ldots,e_{i,k}+1,\ldots,e_{k,j},0\ldots,0)\\
&\cdot\gamma_{i,k}(e_{1,2},\ldots,e_{i,j}+1,\ldots,e_{k,j}+1,0\ldots0)^{-1}\\
&\cdot\gamma_{i,j}(e_{1,2},\ldots e_{k,j},0,\ldots,0)^{-1}.
\end{aligned}$$

② If $\{(p,q) \in I \,|\, (i,k+1) \leq (p,q) \leq (k,j-1),\;\; e_{p,q} \neq 0\} = \phi$ and $e_{k,j} = -1$,

$$\begin{aligned}
\tau(trt^{-1}) = &\gamma_{i,k}(e_{1,2},\ldots,e_{i,k},0\ldots,-1,\ldots,0)\gamma_{i,j}(e_{1,2},\ldots,e_{i,k}+1,0,\ldots,-1,\ldots,0)\\
&\cdot\gamma_{i,j}(e_{1,2},\ldots e_{i,k},0\ldots,-1,\ldots,0)^{-1}
\end{aligned}$$

where -1 appears in (k,j) entries in $\gamma_{i,k}$ and $\gamma_{i,j}$s.

③ If $\{(p,q)\in I\,|\,(i,k+1)\le(p,q)\le(k,j-1),\ \ e_{p,q}\neq 0\}=\phi$ and $e_{k,j}=0$,

$$\begin{aligned}trt^{-1}=&(K_{1,2}^{e_{1,2}}\cdots K_{i,k}^{e_{i,k}+1})K_{i,j}(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{i,k}^{e_{i,k}+1})^{-1}\\ &\cdot(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{i,k}^{e_{i,k}+1}K_{k,j})K_{i,k}^{-1}(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{i,k}^{e_{i,k}}K_{k,j})^{-1}\\ &\cdot(K_{1,2}^{e_{1,2}}\cdots K_{i,j}^{e_{i,j}+1}\cdots K_{i,k}^{e_{i,k}})K_{i,j}^{-1}(K_{1,2}^{e_{1,2}}\cdots K_{i,k}^{e_{i,k}})^{-1}.\end{aligned}$$

Hence, we have:

(a) If $e_{p,q}\neq 0$ for some $(i,j+1)\le(p,q)\le(i,k-1)$, or $e_{i,k}\neq 0,-1$,

$$\begin{aligned}\tau(trt^{-1})=&\gamma_{i,j}(e_{1,2},\ldots,e_{i,k}+1,0,\ldots,0)\\ &\cdot\gamma_{i,k}(e_{1,2},\ldots,e_{i,j}+1,\ldots,e_{i,k},0\ldots,1,\ldots,0)^{-1}\\ &\cdot\gamma_{i,j}(e_{1,2},\ldots e_{i,k},0\ldots,\ldots,0)^{-1}\end{aligned}$$

where 1 appears in (k,j) entry in $\gamma_{i,k}$.

(b) If $\{(p,q)\in I\,|\,(i,j+1)\le(p,q)\le(i,k-1),\ \ e_{p,q}\neq 0\}=\phi$ and $e_{i,k}=0$,

$$\begin{aligned}\tau(trt^{-1})=&\gamma_{i,j}(e_{1,2},\ldots,e_{i,j},0,\ldots,1,\ldots,0)\\ &\cdot\gamma_{i,k}(e_{1,2},\ldots,e_{i,j}+1,0,\ldots,1,\ldots,0)^{-1}\end{aligned}$$

where 1 appears in (i,k) and (k,j) entries in $\gamma_{i,j}$ and $\gamma_{i,k}$ respectively.

(c) If $\{(p,q)\in I\,|\,(i,j+1)\le(p,q)\le(i,k-1),\ \ e_{p,q}\neq 0\}=\phi$ and $e_{i,k}=-1$,

$$\begin{aligned}\tau(trt^{-1})=&\gamma_{i,k}(e_{1,2},\ldots,e_{i,j}+1,0,\ldots,-1,\ldots,1,\ldots,0)^{-1}\\ &\cdot\gamma_{i,j}(e_{1,2},\ldots,e_{i,j},0,\ldots,-1,\ldots,0)^{-1}\end{aligned}$$

where -1 appears in (i,k) entries in $\gamma_{i,k}$ and $\gamma_{i,j}$, and 1 appears (k,j) entry in $\gamma_{i,k}$.

Similarly, we can obtain the other three cases.

(2) $i<k$ and $k<j$. Then $(i,k)<(i,j)<(k,j)$.

(i) If $e_{p,q}\neq 0$ for some $(k,j+1)\le(p,q)\le(n,n-1)$,

$$\begin{aligned}\tau(trt^{-1})=&\gamma_{i,k}(e_{1,2},\ldots,e_{n,n-1})\gamma_{i,j}(e_{1,2},\ldots,e_{i,k}+1,\ldots,e_{n,n-1})\\ &\cdot\gamma_{k,j}(e_{1,2},\ldots,e_{i,k}+1,\ldots,e_{i,j}+1,\ldots,e_{n,n-1})\\ &\cdot\gamma_{i,k}(e_{1,2},\ldots,e_{i,j}+1,\ldots,e_{k,j}+1,\ldots e_{n,n-1})^{-1}\\ &\cdot\gamma_{k,j}(e_{1,2},\ldots,e_{i,j}+1,\ldots,e_{n,n-1})^{-1}\gamma_{i,j}(e_{1,2},\ldots e_{n,n-1})^{-1}.\end{aligned}$$

(ii) If $\{(p,q)\in I\,|\,(k,j+1)\le(p,q)\le(n,n-1),\ \ e_{p,q}\neq 0\}=\phi$, we have:

① If $e_{p,q}\neq 0$ for some $(i,j+1)\le(p,q)\le(k,j-1)$, or $e_{k,j}\neq 0,-1$,

$$\begin{aligned}\tau(trt^{-1})=&\gamma_{i,k}(e_{1,2},\ldots,e_{k,j},0\ldots,0)\gamma_{i,j}(e_{1,2},\ldots,e_{i,k}+1,\ldots,e_{k,j},0\ldots,0)\\ &\cdot\gamma_{i,k}(e_{1,2},\ldots,e_{i,j}+1,\ldots,e_{k,j}+1,0\ldots0)^{-1}\\ &\cdot\gamma_{i,j}(e_{1,2},\ldots e_{k,j},0,\ldots,0)^{-1}.\end{aligned}$$

② If $\{(p,q)\in I\,|\,(i,j+1)\le(p,q)\le(k,j-1),\ \ e_{p,q}\neq 0\}=\phi$ and $e_{k,j}=-1$, we see:

(a) If $e_{p,q} \neq 0$ for some $(i,k+1) \leq (p,q) \leq (i,j-1)$, or $e_{i,j} \neq -1$,

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{i,k}(e_{1,2},\dots,e_{i,j},0\dots,-1,\dots,0) \\ & \cdot \gamma_{i,j}(e_{1,2},\dots,e_{i,k}+1,\dots,e_{i,j},0,\dots,-1,\dots,0) \\ & \cdot \gamma_{i,k}(e_{1,2},\dots e_{i,j}+1,0,\dots,0)^{-1}\gamma_{i,j}(e_{1,2},\dots,e_{i,j},0,\dots,-1,\dots,0)^{-1}\end{aligned}$$

where -1 appears in (k,j) entries in $\gamma_{i,k}$ and $\gamma_{i,j}$s.

(b) If $\{(p,q) \in I \,|\, (i,k+1) \leq (p,q) \leq (i,j-1),\ \ e_{p,q} \neq 0\} = \phi$ and $e_{i,j} = -1$,

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{i,k}(e_{1,2},\dots,e_{i,k},0,\dots,-1,\dots,-1,\dots,0) \\ & \cdot \gamma_{i,j}(e_{1,2},\dots,e_{i,k}+1,0,\dots,-1,\dots,-1,\dots,0) \\ & \cdot \gamma_{i,j}(e_{1,2},\dots,\dots,e_{i,k},0,\dots,-1,\dots,-1,\dots,0)^{-1}\end{aligned}$$

where -1 appears in (i,j) and (k,j) entries in $\gamma_{i,k}$ and $\gamma_{i,j}$s.

③ If $\{(p,q) \in I \,|\, (i,j+1) \leq (p,q) \leq (k,j-1),\ \ e_{p,q} \neq 0\} = \phi$ and $e_{k,j} = 0$, we see

(a) If $e_{p,q} \neq 0$ for some $(i,k+1) \leq (p,q) \leq (i,j-1)$, or $e_{i,j} \neq 0,-1$,

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{i,k}(e_{1,2},\dots,e_{i,j},0,\dots,0) \\ & \cdot \gamma_{i,k}(e_{1,2},\dots,e_{i,j}+1,0\dots,1,\dots,0)^{-1}\end{aligned}$$

where 1 appears in (k,j) entry in $\gamma_{i,k}$.

(b) If $\{(p,q) \in I \,|\, (i,k+1) \leq (p,q) \leq (i,j-1),\ \ e_{p,q} \neq 0\} = \phi$ and $e_{i,j} = 0$,

$$\tau(trt^{-1}) = \gamma_{i,k}(e_{1,2},\dots,e_{i,k},0,\dots,1,\dots,1,\dots,0)^{-1}$$

where 1 appears in (i,j) and (k,j) entries.

(c) If $\{(p,q) \in I \,|\, (i,k+1) \leq (p,q) \leq (i,j-1),\ \ e_{p,q} \neq 0\} = \phi$ and $e_{i,j} = -1$,

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{i,k}(e_{1,2},\dots,e_{i,k},0,\dots,-1,\dots,0) \\ & \cdot \gamma_{i,k}(e_{1,2},\dots,e_{i,k},0,\dots,1,\dots,0)^{-1}\end{aligned}$$

where -1 appears in (i,j) entry in $\gamma_{i,k}$, and 1 appears in (k,j) entry in $\gamma_{i,k}$.

(3) $k < i$ and $j < k$. Then $(k,j) < (i,j) < (i,k)$.

(i) If $e_{p,q} \neq 0$ for some $(i,k+1) \leq (p,q) \leq (n,n-1)$,

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{i,k}(e_{1,2},\dots,e_{n,n-1})\gamma_{i,j}(e_{1,2},\dots,e_{i,k}+1,\dots,e_{n,n-1}) \\ & \cdot \gamma_{k,j}(e_{1,2},\dots,e_{i,j}+1,\dots,e_{i,k}+1,\dots,e_{n,n-1}) \\ & \cdot \gamma_{i,k}(e_{1,2},\dots,e_{k,j}+1,\dots,e_{i,j}+1,\dots e_{n,n-1})^{-1} \\ & \cdot \gamma_{k,j}(e_{1,2},\dots,e_{i,j}+1,\dots,e_{n,n-1})^{-1}\gamma_{i,j}(e_{1,2},\dots e_{n,n-1})^{-1}.\end{aligned}$$

(ii) If $\{(p,q) \in I \,|\, (i,k+1) \leq (p,q) \leq (n,n-1),\ \ e_{p,q} \neq 0\} = \phi$, we have:

① If $e_{p,q} \neq 0$ for some $(i,j+1) \leq (p,q) \leq (i,k-1)$, or $e_{i,k} \neq 0,-1$,

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{i,j}(e_{1,2},\dots,e_{i,k}+1,0\dots,0)\gamma_{k,j}(e_{1,2},\dots,e_{i,j}+1,\dots,e_{i,k}+1,0\dots,0) \\ & \cdot \gamma_{k,j}(e_{1,2},\dots,e_{i,j}+1,\dots,e_{i,k},0\dots 0)^{-1}\gamma_{i,j}(e_{1,2},\dots e_{i,k},0,\dots,0)^{-1}.\end{aligned}$$

② If $\{(p,q) \in I \,|\, (i,j+1) \leq (p,q) \leq (i,k-1),\ \ e_{p,q} \neq 0\} = \phi$ and $e_{i,k} = -1$,

(a) If $e_{p,q} \neq 0$ for some $(k, j+1) \leq (p,q) \leq (i, j-1)$, or $e_{i,j} \neq -1$,

$$\tau(trt^{-1}) = \gamma_{k,j}(e_{1,2}, \ldots e_{i,j}+1, 0, \ldots, 0)\gamma_{k,j}(e_{1,2}, \ldots, e_{i,j}+1, 0, \ldots, -1, \ldots, 0)^{-1} \cdot \gamma_{i,j}(e_{1,2}, \ldots, e_{i,j}, 0, \ldots, -1, \ldots, 0)^{-1}$$

where -1 appears (i,k) entries in $\gamma_{k,j}$ and $\gamma_{i,j}$.

(b) If $\{(p,q) \in I \,|\, (k, j+1) \leq (p,q) \leq (i, j-1), \;\; e_{p,q} \neq 0\} = \phi$ and $e_{i,j} = -1$,

$$\tau(trt^{-1}) = \gamma_{k,j}(e_{1,2}, \ldots, e_{k,j}, 0, \ldots, -1, \ldots, 0)^{-1} \cdot \gamma_{i,j}(e_{1,2}, \ldots, e_{k,j}, 0, \ldots, -1, \ldots, -1, \ldots, 0)^{-1}$$

where -1 appears in (i,j) and (i,k) entries in $\gamma_{i,j}$, and in (i,k) entry in $\gamma_{k,j}$.

③ If $\{(p,q) \in I \,|\, (i, j+1) \leq (p,q) \leq (i, k-1), \;\; e_{p,q} \neq 0\} = \phi$ and $e_{i,k} = 0$,

(a) If $e_{p,q} \neq 0$ for some $(k, j+1) \leq (p,q) \leq (i, j-1)$, or $e_{i,j} \neq -1$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{i,j}, 0, \ldots, 1, \ldots, 0)\gamma_{k,j}(e_{1,2}, \ldots, e_{i,j}+1, 0, \ldots, 1, \ldots, 0) \cdot \gamma_{k,j}(e_{1,2}, \ldots e_{i,j}+1, 0, \ldots, 0)^{-1}$$

where 1 appears (i,k) entries in $\gamma_{i,j}$ and $\gamma_{k,j}$.

(b) If $\{(p,q) \in I \,|\, (k, j+1) \leq (p,q) \leq (i, j-1), \;\; e_{p,q} \neq 0\} = \phi$ and $e_{i,j} = -1$,

$$\tau(trt^{-1}) = \gamma_{i,j}(e_{1,2}, \ldots, e_{k,j}, 0, \ldots, -1, \ldots, 1, \ldots, 0) \cdot \gamma_{k,j}(e_{1,2}, \ldots, e_{k,j}, 0, \ldots, 1, \ldots, 0)$$

where 1 appears in (i,k) entries in $\gamma_{i,j}$ and $\gamma_{k,j}$, -1 appears in (i,j) entry in $\gamma_{i,j}$.

(4) $k < i$ and $k < j$. Then $(k,j) < (i,k) < (i,j)$.

(i) If $e_{p,q} \neq 0$ for some $(i, j+1) \leq (p,q) \leq (n, n-1)$,

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{i,k}(e_{1,2}, \ldots, e_{n,n-1})\gamma_{i,j}(e_{1,2}, \ldots, e_{i,k}+1, \ldots, e_{n,n-1}) \\ & \cdot \gamma_{k,j}(e_{1,2}, \ldots, e_{i,k}+1, \ldots, e_{i,j}+1, \ldots, e_{n,n-1}) \\ & \cdot \gamma_{i,k}(e_{1,2}, \ldots, e_{k,j}+1, \ldots, e_{i,j}+1, \ldots e_{n,n-1})^{-1} \\ & \cdot \gamma_{k,j}(e_{1,2}, \ldots, e_{i,j}+1, \ldots, e_{n,n-1})^{-1}\gamma_{i,j}(e_{1,2}, \ldots e_{n,n-1})^{-1}.\end{aligned}$$

(ii) If $\{(p,q) \in I \,|\, (i, j+1) \leq (p,q) \leq (n, n-1), \;\; e_{p,q} \neq 0\} = \phi$, we have:

① If $e_{p,q} \neq 0$ for some $(i, k+1) \leq (p,q) \leq (i, j-1)$, or $e_{i,j} \neq 0, -1$,

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{i,k}(e_{1,2}, \ldots, e_{i,j}, 0 \ldots, 0)\gamma_{k,j}(e_{1,2}, \ldots, e_{i,k}+1, \ldots, e_{i,j}+1, 0 \ldots, 0) \\ & \cdot \gamma_{i,k}(e_{1,2}, \ldots, e_{k,j}+1, \ldots, e_{i,j}+1, 0 \ldots 0)^{-1} \\ & \cdot \gamma_{k,j}(e_{1,2}, \ldots e_{i,j}+1, 0, \ldots, 0)^{-1}.\end{aligned}$$

② If $\{(p,q) \in I \,|\, (i, k+1) \leq (p,q) \leq (i, j-1), \;\; e_{p,q} \neq 0\} = \phi$ and $e_{i,j} = -1$,

(a) If $e_{p,q} \neq 0$ for some $(k, j+1) \leq (p,q) \leq (i, k-1)$, or $e_{i,k} \neq 0, -1$,

$$\tau(trt^{-1}) = \gamma_{i,k}(e_{1,2}, \ldots e_{i,k}, 0, \ldots, -1, \ldots, 0)\gamma_{k,j}(e_{1,2}, \ldots, e_{i,k}+1, 0, \ldots, 0) \cdot \gamma_{k,j}(e_{1,2}, \ldots, e_{i,k}, 0, \ldots, 0)^{-1}$$

where -1 appears in (i,j) entry in $\gamma_{i,k}$.

(b) If $\{(p,q)\in I\,|\,(k,j+1)\le(p,q)\le(i,k-1),\;\; e_{p,q}\neq 0\}=\phi$ and $e_{i,k}=-1$,

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{i,k}&(e_{1,2},\dots,e_{k,j},0,\dots,-1,\dots,-1,\dots,0)\\ &\cdot\gamma_{k,j}(e_{1,2},\dots,e_{k,j},0,\dots,-1,\dots,0)^{-1}\end{aligned}$$

where -1 appears in (i,k) and (i,j) entries in $\gamma_{i,k}$, and in (i,k) entry in $\gamma_{k,j}$.

(c) If $\{(p,q)\in I\,|\,(k,j+1)\le(p,q)\le(i,k-1),\;\; e_{p,q}\neq 0\}=\phi$ and $e_{i,k}=0$,

$$\tau(trt^{-1})=\gamma_{i,k}(e_{1,2},\dots,e_{k,j},0,\dots,-1,\dots,0)\gamma_{k,j}(e_{1,2},\dots,e_{k,j},0,\dots,1,\dots,0)$$

where -1 appears in (i,j) entry in $\gamma_{i,k}$, and 1 appears in (i,k) entry in $\gamma_{k,j}$.

③ If $\{(p,q)\in I\,|\,(i,k+1)\le(p,q)\le(i,j-1),\;\; e_{p,q}\neq 0\}=\phi$ and $e_{i,j}=0$,

(a) If $e_{p,q}\neq 0$ for some $(k,j+1)\le(p,q)\le(i,k-1)$, or $e_{i,k}\neq -1$,

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{k,j}&(e_{1,2},\dots,e_{i,k}+1,0,\dots,1,\dots,0)\\ &\cdot\gamma_{i,k}(e_{1,2},\dots,e_{k,j}+1,\dots .e_{i,k},0,\dots,1,\dots,0)^{-1}\\ &\cdot\gamma_{k,j}(e_{1,2},\dots e_{i,k},0,\dots,1,\dots,0)^{-1}\end{aligned}$$

where 1 appears in (i,j) entries in $\gamma_{i,k}$ and $\gamma_{k,j}$s.

(b) If $\{(p,q)\in I\,|\,(k,j+1)\le(p,q)\le(i,k-1),\;\; e_{p,q}\neq 0\}=\phi$ and $e_{i,k}=-1$,

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{k,j}&(e_{1,2},\dots,e_{k,j},0,\dots,1,\dots,0)\\ &\cdot\gamma_{i,k}(e_{1,2},\dots,e_{k,j}+1,0,\dots,-1,\dots,1,\dots,0)^{-1}\\ &\cdot\gamma_{k,j}(e_{1,2},\dots,e_{k,j},0,\dots,-1,\dots,1,\dots,0)^{-1}\end{aligned}$$

where -1 appears in (i,k) entries in $\gamma_{i,k}$ and $\gamma_{k,j}$, and 1 appears in (i,j) entries in $\gamma_{i,k}$ and $\gamma_{k,j}$s.

3.2. **Infinitely many linearly independent elements in $\mathcal{P}_n'(2)^{\mathrm{ab}}$.**

In this subsection, we detect infinitely many linearly independent elements in $\mathcal{P}_n'(2)^{\mathrm{ab}}$ using the presentation for $\mathcal{P}_n'(2)$ which we have obtained in the previous subsection. Then we show that $\mathcal{P}_n'(k)^{\mathrm{ab}}$ for $k\ge 3$ also contains infinitely many linearly independent elements. Finally we confirm ourselves that for $k\ge 2$, the abelianization of each subgroups $\mathcal{P}_n(k)$ of the Johnson filtration of $P\Sigma_n$ has the same property.

To begin with, set

$$A:=\mathrm{Span}_{\mathbf{Z}}\{b_{i,j}(e,e')\,|\,1\le i<j\le n,\;\; e\in\mathbf{Z},\;\; e'\in\mathbf{Z}\setminus\{0\}\}.$$

Clearly, A is a free abelian group of infinite rank. Since $N=\varphi^{-1}(\mathcal{P}_n'(2))$ is a free group with basis $\mathfrak{E}$, we can define a surjective homomorphism $\Psi':N\to A$ by

$$\Psi'(\gamma_{ij}(e_{1,2},\dots,e_{n,n-1}))=\begin{cases}b_{i,j}(e_{i,j},e_{j,i}) & \text{if } i<j \text{ and } e_{j,i}\neq 0,\\ 0, & \text{if otherwise.}\end{cases}$$

Then it is easily checked that $\Psi'(\tau(trt^{-1}))=0$ for any $t\in T$ and $r=(\mathbf{R1}),(\mathbf{R2}),(\mathbf{R3})$ from the results obtained in the previous subsection. This shows that Ψ' induces a surjective homomorphism

$$\Psi:\mathcal{P}_n(2)\to A.$$

Since the target of Ψ is abelian, Ψ factors through the abelianization $\mathcal{P}_n(2)^{\mathrm{ab}}$ of $\mathcal{P}_n(2)$. Hence we obtain

Theorem 3.1. *For any $n \geq 3$, $\mathcal{P}_n'(2)^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

As a corollary, we see

Corollary 3.2. *For any $n \geq 3$, $\mathcal{P}_n'(2)$ is not finitely generated.*

Next, let us consider $\mathcal{P}_n'(k)$ for $k \geq 3$. For $1 \leq i < j \leq n$, $e \in \mathbf{Z}$ and $e' \in \mathbf{Z}_{\geq 1}$, set

$$\alpha_{i,j}(e,e') := K_{i,j}^{e}[K_{j,i},[K_{j,i},\ldots,[K_{j,i},K_{i,j}]]\cdots]K_{i,j}^{-e},$$

$$\beta_{i,j}(e,e') := \gamma_{i,j}(0,\ldots,e,\ldots,e',\ldots,0) = K_{i,j}^{e}K_{j,i}^{e'}K_{i,j}K_{j,i}^{-e'}K_{i,j}^{-(e+1)}$$

where $K_{j,i}$ appears e' times in the definition of $\alpha_{i,j}(e,e')$.

Here we prepare one lemma. In general, for any group G and x, $y \in G$, set

$$\theta_{e'}(x,y) := [x,[x,\ldots,[x,y]]\cdots] \in G$$

for $e' \geq 1$ where x appears e' times in the commutator above. Then we have

Lemma 3.3. *With the notation above, $[x^{e'},y] \in G$ is written as*

$$\theta_{e'}(x,y)\theta_{e_1'}(x,y)\cdots\theta_{e_p'}(x,y)$$

in G for some $1 \leq e_1',\ldots,e_p' \leq e'-1$.

Proof of Lemma 3.3. We show this lemma by the induction on e'. If $e'=1$, it is obvious that $[x,y]=\theta_1(x,y)$. Assume $e' \geq 2$. Using a commutator formula

$$[ab,c]=[a,[b,c]][b,c][a,c],$$

we see

$$[x^{e'},y]=[x^{e'-1},[x,y]]\,[x,y]\,[x^{e'-1},y].$$

Hence, by the inductive hypothesis, we have

$$\begin{aligned}[x^{e'},y] &= \theta_{e'-1}(x,[x,y])\theta_{e_1'}(x,[x,y])\cdots\theta_{e_p'}(x,[x,y])\theta_1(x,y)\\ &\quad \cdot\theta_{e'-1}(x,y)\theta_{e_1''}(x,y)\cdots\theta_{e_q''}(x,y)\end{aligned}$$

for some $1 \leq e_j', e_j'' \leq e'-2$. On the other hand, since $\theta_{e-1}(x,[x,y])=\theta_e(x,y)$ for any $e \geq 2$, we obtain the required result. This completes the proof of Lemma 3.3. $\square$

Now, we consider a relation between $\alpha_{i,j}(e,e')$ and $\beta_{i,j}(e,e')$.

Lemma 3.4. *For any $1 \leq i < j \leq n$, $e \in \mathbf{Z}$ and $e' \in \mathbf{Z}_{\geq 1}$, there exist some $1 \leq e_1',\ldots,e_p' \leq e'-1$ such that*

$$\alpha_{i,j}(e,e')=\beta_{i,j}(e,e')\beta_{i,j}(e,e_1')^{d_1}\cdots\beta_{i,j}(e,e_p')^{d_p} \text{ for } d_j=\pm 1.$$

Proof. We show this lemma by the induction on e'. If $e'=1$, it is clear that $\alpha_{i,j}(e,1)=\beta_{i,j}(e,1)$. Assume $e' \geq 2$. By using Lemma 3.3, we have

$$\begin{aligned}\beta_{i,j}(e,e') &= K_{i,j}^{e}\,[K_{j,i}^{e'},K_{i,j}]\,K_{i,j}^{-e},\\ &= K_{i,j}^{e}\,\theta_{e'}(K_{j,i},K_{i,j})\theta_{e_1'}(K_{j,i},K_{i,j})\cdots\theta_{e_p'}(K_{j,i},K_{i,j})\,K_{i,j}^{-e},\\ &= \alpha_{i,j}(e,e')\alpha_{i,j}(e,e_1')\cdots\alpha_{i,j}(e,e_p')\end{aligned}$$

for some $1 \leq e'_j \leq e' - 1$. Hence, using the inductive hypothesis, we obtain the required result. This completes the proof of Lemma 3.4. $\square$

Here we prove our main theorem.

Theorem 3.5. *For $n \geq 3$ and $k \geq 3$, $\mathcal{P}'_n(k)^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

Proof. For any $d \geq k$, consider $d - k + 2$ elements

$$\alpha_{i,j}(e, k-1), \alpha_{i,j}(e, k), \ldots, \alpha_{i,j}(e, d)$$

in $\mathcal{P}'_n(k)$ for any $e \in \mathbf{Z}$. We denote by

$$c_{i,j}(e, k-1), c_{i,j}(e, k), \ldots, c_{i,j}(e, d)$$

the images of $\alpha_{i,j}(e, k-1)$, $\alpha_{i,j}(e, k), \ldots, \alpha_{i,j}(e, d)$ by the composition map $\Psi'_k : \mathcal{P}'_n(k) \hookrightarrow \mathcal{P}'_n(2) \xrightarrow{\Psi} A$ respectively.

Here we show that $c_{i,j}(e, k-1)$, $\ldots$, $c_{i,j}(e, d)$ are linearly independent in A. Assume that

$$a_{k-1} c_{i,j}(e, k-1) + \cdots + a_d c_{i,j}(e, d) = 0$$

for a_{k-1}, $\ldots$, $a_d \in \mathbf{Z}$. Then using Lemma 3.4, we see that the coefficient of $b_{i,j}(e, d)$ is exactly a_d, and hence $a_d = 0$. This shows that

$$a_{k-1} c_{i,j}(e, k-1) + \cdots + a_{d-1} c_{i,j}(e, d-1) = 0.$$

By the same argument as the above, we can show that $a_{d-1} = a_{d-2} = \cdots = a_{k-1} = 0$ recursively. Therefore $c_{i,j}(e, k-1)$, $\ldots$, $c_{i,j}(e, d)$ are linearly independent in A.

Since Ψ'_k factors through the abelianization $\mathcal{P}'_n(k)^{\mathrm{ab}}$ of $\mathcal{P}'_n(k)$, we see that $\mathcal{P}'_n(k)^{\mathrm{ab}}$ contains a free abelian group with rank $d - k + 2$. On the other hand, since we can take $d \geq k$ arbitrarily, we conclude that $\mathcal{P}'_n(k)^{\mathrm{ab}}$ contains a free abelian group with infinite rank. This completes the proof of Theorem 3.5. $\square$

As a corollary, we have

Corollary 3.6. *For $n \geq 3$ and $k \geq 3$, $\mathcal{P}'_n(k)$ is not finitely generated.*

Finally, we consider the Johnson filtration of $P\Sigma_n$.

Corollary 3.7. *For $n \geq 3$ and $k \geq 3$, $\mathcal{P}_n(k)^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

Proof. Consider a homomorphism $\Psi : \mathcal{P}_n(k) \hookrightarrow \mathcal{P}'_n(2) \xrightarrow{\Psi} A$ which factors through the abelianization $\mathcal{P}_n(k)^{\mathrm{ab}}$ of $\mathcal{P}_n(k)$. Observing the proof of Theorem 3.5, we see that for any $d \geq k$ the images of $\alpha_{i,j}(e, k-1)$, $\ldots$, $\alpha_{i,j}(e, d) \in \mathcal{P}'_n(k) \subset \mathcal{P}_n(k)$ by Ψ_k are linearly independent in A. Hence we obtain the required result. This completes the proof of Corollary 3.7. $\square$

And hence, we obtain

Corollary 3.8. *For $n \geq 3$ and $k \geq 3$, $\mathcal{P}_n(k)$ is not finitely generated.*

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