Minimality of hyperplane arrangements and basis of local system cohomology

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Abstract

The purpose of this paper is applying minimality of hyperplane arrangements to local system cohomology groups. It is well known that twisted cohomology groups with coefficients in a generic rank one local system vanish except in the top degree, and bounded chambers form a basis of the remaining cohomology group. We determine precisely when this phenomenon happens for two dimensional arrangements.

1 Introduction

The purpose of this paper is applying minimality of hyperplane arrangements to local system cohomology groups. In §1.1 and §1.2, we will recall basic notions and results on these topics. In §1.3, we will give the plan of the paper.

1.1 Minimality of hyperplane arrangements

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in $\mathbb{C}^\ell$. Namely a finite set of affine hyperplanes. We assume each hyperplane $H_i = \{\alpha_i = 0\} \subset \mathbb{C}^\ell$ is defined by an affine linear equation α_i. We denote the complement of hyperplanes by $\mathsf{M}(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{i=1}^n H_i$.

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After the discovery of combinatorial description of the cohomology ring $H^*(\mathsf{M}(\mathcal{A}), \mathbb{Z})$ [OS] and $K(\pi, 1)$-property for simplicial arrangements [D], it has been revealed that the complement $\mathsf{M}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ has a very special homotopy type among other complex affine varieties. In particular, the following minimality seems one of the most peculiar properties to $\mathsf{M}(\mathcal{A})$ [DP, R, PS, F].

Theorem 1.1. (Minimality of arrangements.) The complement $\mathsf{M}(\mathcal{A})$ is homotopy equivalent to a finite minimal CW-complex X. Namely, X satisfies the following minimality: The number of k-dimensional cells $\sharp\{k\text{-}\dim\ \text{cells}\}$ is equal to the k-th Betti number $b_k(X)$.

The minimality is expected to be useful for computations of local system cohomology groups. An immediate corollary is the following upper bounds for dimensions of rank one local system cohomology groups, which were conjectured by Aomoto and first proved in [Co] by using another methods.

Corollary 1.2. Let $\mathcal{L}$ be a complex rank one local system on $\mathsf{M}(\mathcal{A})$. Then the dimension of $\mathcal{L}$-coefficients cohomology group is bounded by Betti number:

$$\dim H^k(\mathsf{M}(\mathcal{A}), \mathcal{L}) \leq b_k(\mathsf{M}(\mathcal{A})),$$

for $k = 0, 1, \ldots, \ell$.

For further applications of the minimality to computations of local system cohomology groups, the description of the minimal CW-complex X, in particular the attaching map of each cell, is needed. However Theorem 1.1 does not tell it. It should be noted that the proof of Theorem 1.1 is based on Morse theoretic arguments. The constructions of cells are relying on a transcendental methods, namely using gradient flows of a Morse function.

Recently two approaches appeared to the problem of describing attaching maps of minimal cells. Both are

- assuming $\mathcal{A}$ is defined over the real numbers $\mathbb{R}$, and
- describing attaching maps by using combinatorial structure of chambers.

However they used different methods.

- In [Y1], we studied Lefschetz's hyperplane plane section theorem for $\mathsf{M}(\mathcal{A})$, and described the attaching maps of the top cells.

- In [SS], Salvetti and Settepanella developed discrete Morse theory on the Salvetti complex, and then described the minimal cell complex by using discrete Morse flows.

See [Del, DS] for subsequent developments. Furthermore, in [GS], 2-dimensional algebraic minimal chain complex is described. The present article can be considered as a counterpart of [GS].

1.2 Non-resonant local systems

A nonempty intersection of elements of $\mathcal{A}$ is called an *edge*. We denote by $L(\mathcal{A})$ the set of edges. An edge $X \in L(\mathcal{A})$ is called a *dense edge* if the localization $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\}$ is indecomposable. We denote by $\mathsf{D}(\mathcal{A}) \subset L(\mathcal{A})$ the set of dense edges.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Then λ determines a rank one representation of $\pi_1(\mathsf{M}(\mathcal{A}))$ by $\pi_1(\mathsf{M}(\mathcal{A})) \ni \gamma \longmapsto \exp(\int_\gamma \sum_{i=1}^n \lambda_i d\log\alpha_i) \in \mathbb{C}^*$ and the associated local system $\mathcal{L} = \mathcal{L}_\lambda$. In other words, $\mathcal{L}$ is determined by the local monodromy $q_i = e^{2\pi\sqrt{-1}\lambda_i} \in \mathbb{C}^*$ around each hyperplane H_i. For an edge $X \in L(\mathcal{A})$, denote $q_X = \prod_{X \subset H_i} q_i$. We also denote the half twist by $q_i^{1/2} = e^{\pi\sqrt{-1}\lambda_i}$.

We can embed the affine space $\mathbb{C}^\ell$ in $\mathbb{CP}^\ell$ as $\mathbb{C}^\ell = \mathbb{CP}^\ell \setminus H_\infty$. We call $\mathcal{A}_\infty := \{\overline{H} \mid H \in \mathcal{A}\} \cup \{H_\infty\}$ the projective closure of $\mathcal{A}$. The monodromy of $\mathcal{L}_\lambda$ around the hyperplane at infinity H_∞ is $\prod_{i=1}^n q_i^{-1}$. It is natural to define $q_\infty = \prod_{i=1}^n q_i^{-1}$.

The the structure of the cohomology group $H^k(\mathsf{M}(\mathcal{A}), \mathcal{L})$ with local system coefficients has been studied well [A, ESV, K, STV]. In particular, it is known that if $\mathcal{L}$ is generic, then the cohomology vanishes except in $k = \ell$. Among others, let us recall two results in this direction. ([DT, L, CDO])

Theorem 1.3. ([DT]) Suppose that $\mathcal{A}$ is defined over $\mathbb{R}$ and the local system $\mathcal{L}_\lambda$ satisfies

$$q_X \neq 1, \text{ for } \forall X \in \mathsf{D}(\mathcal{A}_\infty). \tag{1}$$

Then

$$H^k(\mathsf{M}(\mathcal{A}), \mathcal{L}_\lambda) = \begin{cases} 0 & \text{for } k \neq \ell, \\ \bigoplus_{C \in \mathsf{bch}(\mathcal{A})} \mathbb{C} \cdot [C], & \text{for } k = \ell, \end{cases} \tag{2}$$

where $\mathsf{bch}(\mathcal{A})$ stands for the set of all bounded chambers. A chamber $[C]$ can be considered as a locally finite cycle, in other words, an element of Borel-Moore homology $[C] \in H_\ell^{BM}(\mathsf{M}(\mathcal{A}))$. In (2) we identify the chamber C with cohomology via the canonical isomorphism $H_\ell^{BM}(\mathsf{M}(\mathcal{A})) \simeq H^\ell(\mathsf{M}(\mathcal{A}))$.

Definition 1.4. $\mathsf{D}_\infty(\mathcal{A}_\infty) := \{X \in \mathsf{D}(\mathcal{A}_\infty) \mid X \subset H_\infty\}$.

Theorem 1.5. ([CDO]) Suppose that the local system $\mathcal{L}_\lambda$ satisfies

$$q_X \neq 1, \text{ for } \forall X \in \mathsf{D}_\infty(\mathcal{A}_\infty). \tag{3}$$

Then

$$H^k(\mathsf{M}(\mathcal{A}), \mathcal{L}_\lambda) \simeq \begin{cases} 0 & \text{for } k \neq \ell, \\ \mathbb{C}^{|\chi(\mathsf{M}(\mathcal{A}))|} & \text{for } k = \ell, \end{cases}$$

where $\chi(\mathsf{M}(\mathcal{A}))$ is the Euler characteristic of $\mathsf{M}(\mathcal{A})$.

1.3 Plan of the paper

The purpose of this paper is to refine vanishing results Theorem 1.3 and Theorem 1.5 for $\ell = 2$ by using minimal complex arising from minimal CW-decomposition of $\mathsf{M}(\mathcal{A})$. We will prove that the assertion (2) of Theorem 1.3 is true under the weaker assumption (3). Furthermore, if $\mathcal{A}$ is indecomposable, we also prove that the assumption can not be weakened any more. Our main result asserts that (3) and (2) are equivalent. (For $\ell = 2$.)

In §2, we treat combinatorial structures of chambers, which will play a crucial role in the study of minimal complex.

In §3, we will describe the minimal cochain complex arising from Lefschetz's hyperplane section theorem. Particularly, we treat the case $\ell = 2$ in details.

In §4, we prove the main result, that is, for an indecomposable two dimensional arrangement $\mathcal{A}$, conditions (3) and (2) are equivalent.

2 Chambers and flags

2.1 Involution on unbounded chambers

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^\ell$. We denote the set of chambers, bounded chambers, unbounded chambers by $\mathsf{ch}(\mathcal{A}), \mathsf{bch}(\mathcal{A}), \mathsf{uch}(\mathcal{A})$, respectively. Note that $\mathsf{ch}(\mathcal{A}) = \mathsf{bch}(\mathcal{A}) \sqcup \mathsf{uch}(\mathcal{A})$.

Let $C \in \mathsf{uch}(\mathcal{A})$ be an unbounded chamber. Then the closure $cl(C)$ in the projective space $\mathbb{RP}^\ell$ intersects the hyperplane H_∞ at infinity.

Definition 2.1. Let $C \in \mathsf{uch}(\mathcal{A})$. (i) Define $X(C)$ to be the smallest subspace of H_∞ which contains $cl(C) \cap H_\infty$. (ii) There exists a unique chamber which is the opposite with respect to $cl(C) \cap H_\infty$. We denote the opposite chamber by $C^\vee$ (see Figure 1). Obviously we have $C^{\vee\vee} = C$.

See Figure 1 for an example. In this figure, $X(C_1) = X(C_4) = H_\infty$, and $X(C_2) = X(C_3) = cl(C_2) \cap H_\infty$.

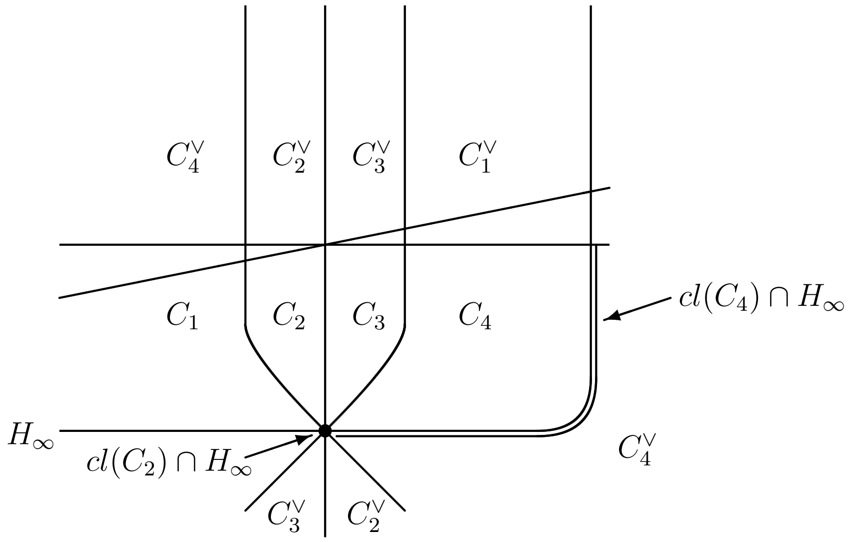

Figure 1: C and $C^\vee$

Definition 2.2. Define the involution ι by

$$\begin{array}{rcl} \iota : \mathsf{uch}(\mathcal{A}) & \longrightarrow & \mathsf{uch}(\mathcal{A}) \\ C & \longmapsto & C^\vee \end{array}$$

We now characterize dense edges contained in H_∞ by using $X(C)$. First we prove an easy lemma.

Lemma 2.3. Let $\mathcal{A}$ be an essential central arrangement in $\mathbb{R}^\ell$. Then the following are equivalent.

(1) $\mathcal{A}$ is indecomposable.

(2) There exist $H \in \mathcal{A}$ and $C \in \mathsf{ch}(\mathcal{A})$ such that $cl(C) \cap H = \{0\}$.

(3) For any $H \in \mathcal{A}$, there exists $C \in \mathsf{ch}(\mathcal{A})$ such that $cl(C) \cap H = \{0\}$.

Proof. Let $H \in \mathcal{A}$ and consider the deconing $\mathbf{d}_H\mathcal{A}$ with respect to H. Note that $\mathbf{d}_H\mathcal{A}$ is an affine arrangement of rank $(\ell - 1)$. Using [OT, §3.3], $\mathcal{A}$ is indecomposable if and only if the β-invariant of $\mathbf{d}_H\mathcal{A}$ is nonzero. By the famous result of Zaslavsky [Z], it is equivalent to the existence of bounded chambers of the deconing $\mathbf{d}_H\mathcal{A}$. Choose a bounded chamber of $\mathbf{d}_H\mathcal{A}$, and let C be it's cone. Then $cl(C) \cap H = \{0\}$. This proves (1) $\Rightarrow$ (3). Other implications can also be similarly proved. $\square$

Using the above lemma, we obtain the following.

Proposition 2.4. Let $\mathcal{A}$ be an affine arrangement in $\mathbb{R}^\ell$. An edge $X \in L(\mathcal{A}_\infty)$ satisfies $X \in \mathsf{D}_\infty(\mathcal{A}_\infty)$ if and only if $X = X(C)$ for some $C \in \mathsf{uch}(\mathcal{A})$.

2.2 Generic flags

Let $\mathcal{F}$ be a generic flag in $\mathbb{R}^\ell$

$$\mathcal{F} : \emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^\ell = \mathbb{R}^\ell,$$

where each $\mathcal{F}^q$ is a *generic* q-dimensional affine subspace, that is, $\dim \mathcal{F}^q \cap X = q + \dim X - \ell$ for $X \in L(\mathcal{A}_\infty)$. Let $\{h_1, \dots h_\ell\}$ be a system of defining equations of $\mathcal{F}$, that is,

$$\mathcal{F}^q = \{h_{q+1} = \cdots = h_\ell = 0\}, \text{ for } q = 0, 1, \dots, \ell - 1,$$

where each h_i is an affine linear form on $\mathbb{R}^\ell$. Using the flag $\mathcal{F}$, we decompose the set of chambers into several subsets.

Definition 2.5. Define

$$\mathsf{ch}^q(\mathcal{A}) = \{C \in \mathsf{ch}(\mathcal{A}) \mid C \cap \mathcal{F}^q \neq \emptyset \quad \text{and} \quad C \cap \mathcal{F}^{q-1} = \emptyset\},$$

for $q = 0, 1, \dots, \ell$.

Proposition 2.6. ([Y1]) $\sharp\mathsf{ch}^q(\mathcal{A}) = b_q(\mathsf{M}(\mathcal{A}))$. $\square$

Remark 2.7. The above proposition gives a refinement of Zaslavsky's formula $\sum_{i=0}^{\ell} b_i(\mathsf{M}(\mathcal{A})) = \sharp\mathsf{ch}(\mathcal{A})$, ([Z]).

We assume that $\mathcal{F}$ satisfies the following : For $q = 0, \ldots, \ell$, $\mathcal{F}^q_{>0}$ denotes

$$\{h_{q+1} = h_{q+2} = \cdots = h_\ell = 0, h_q > 0\}.$$

1. For an arbitrary chamber C, if belonging to $\mathsf{ch}^q(\mathcal{A})$, then $C \cap \mathcal{F}^q \subset \mathcal{F}^q_{>0}$.
2. For any two X, $X' \in L(\mathcal{A})$ with $\dim X = \dim X' = \ell - q$ (i.e. satisfying $X \cap \mathcal{F}^q = \{pt\}$ and $X' \cap \mathcal{F}^q = \{pt\}$), if $X \neq X'$,
$$h_q(X \cap \mathcal{F}^q) \neq h_q(X' \cap \mathcal{F}^q).$$

In the remainder of the paper we fix a generic flag $\mathcal{F}$ satisfying the above conditions. And also fix the orientation of $\mathcal{F}^q$ by the oriented basis $(\partial_{h_1}, \ldots, \partial_{h_q})$ of the tangent space $T_x\mathcal{F}^q$.

Next we further decompose $\mathsf{ch}^q(\mathcal{A})$ into two subsets.

Definition 2.8. Define subsets $\mathsf{bch}^q(\mathcal{A})$ and $\mathsf{uch}^q(\mathcal{A})$ of $\mathsf{ch}^q(\mathcal{A})$ by

$$\begin{aligned}\mathsf{bch}^q(\mathcal{A}) &= \{C \in \mathsf{ch}^q(\mathcal{A}) \mid C \cap \mathcal{F}^q \text{ is bounded}\},\\ \mathsf{uch}^q(\mathcal{A}) &= \{C \in \mathsf{ch}^q(\mathcal{A}) \mid C \cap \mathcal{F}^q \text{ is unbounded}\}.\end{aligned}$$

We note that $\mathsf{bch}^\ell(\mathcal{A}) = \mathsf{bch}(\mathcal{A})$.

Example 2.9. Let us consider the arrangement of four lines $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$ with a generic flag $\mathcal{F}$ as in Figure 2.

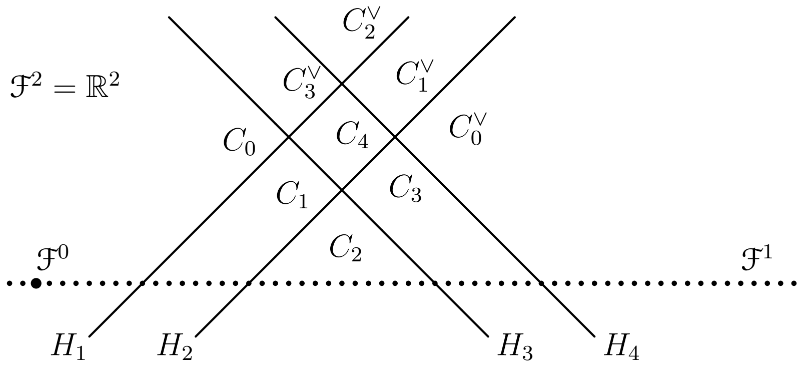

Figure 2: $\mathsf{bch}^q(\mathcal{A})$ and $\mathsf{uch}^q(\mathcal{A})$.

Then we have by definition

$\mathsf{ch}^0(\mathcal{A}) = \{C_0\}$, $\mathsf{ch}^1(\mathcal{A}) = \{C_1, C_2, C_3, C_0^\vee\}$, $\mathsf{ch}^2(\mathcal{A}) = \{C_1^\vee, C_2^\vee, C_3^\vee, C_4\}$,
$\mathsf{bch}^0(\mathcal{A}) = \{C_0\}$, $\mathsf{bch}^1(\mathcal{A}) = \{C_1, C_2, C_3\}$, $\mathsf{bch}^2(\mathcal{A}) = \{C_4\}$,
$\mathsf{uch}^0(\mathcal{A}) = \emptyset$, $\mathsf{uch}^1(\mathcal{A}) = \{C_0^\vee\}$, $\mathsf{uch}^2(\mathcal{A}) = \{C_1^\vee, C_2^\vee, C_3^\vee\}$.

Theorem 2.10. The involution ι induces a bijection

$$\iota : \mathsf{bch}^{q-1}(\mathcal{A}) \xrightarrow{\sim} \mathsf{uch}^{q}(\mathcal{A}).$$

Proof. Suppose $C \in \mathsf{bch}^{q-1}(\mathcal{A})$, that is, $C \cap \mathcal{F}^{q-1}$ is bounded. Then $C^{\vee} \cap \mathcal{F}^{q-1} = \emptyset$. By the assumption on the flag, $C \cap \mathcal{F}^{q}$ is unbounded. Since $\mathcal{F}^{q}$ is generic, $cl(\mathcal{F}^{q})$ intersects $cl(C) \cap H_{\infty}$ transversally. Hence $C^{\vee} \cap \mathcal{F}^{q} \neq \emptyset$ and unbounded. We have $C^{\vee} \in \mathsf{uch}^{q}(\mathcal{A})$. Conversely if $C^{\vee} \in \mathsf{uch}^{q}(\mathcal{A})$, then $C^{\vee\vee} = C$ intersects $\mathcal{F}^{q-1}$. Suppose $C \cap \mathcal{F}^{q-1}$ is unbounded. In this case, $C^{\vee}$ also intersects $\mathcal{F}^{q-1}$. This contradicts the fact $C^{\vee} \in \mathsf{uch}^{q}(\mathcal{A}) \subset \mathsf{ch}^{q}(\mathcal{A})$. □

Corollary 2.11. $\sharp\mathsf{bch}^{q-1}(\mathcal{A}) = \sharp\mathsf{uch}^{q}(\mathcal{A})$. □

Remark 2.12. (1) Corollary 2.11 together with Proposition 2.6 and $\sharp\mathsf{ch}^{q}(\mathcal{A}) = \sharp\mathsf{bch}^{q}(\mathcal{A}) + \sharp\mathsf{uch}^{q}(\mathcal{A})$, gives a "bijective proof" for Zaslavsky's formula $\sharp\mathsf{bch}(\mathcal{A}) = \sum_{i=0}^{\ell}(-1)^{\ell-i}b_i(\mathsf{M}(\mathcal{A}))$.

(2) The bijective correspondence (Theorem 2.10) plays a crucial role in §4.

3 Minimal complexes

Let $\mathcal{A}$ be an essential real arrangement and $\mathcal{F}$ be a generic flag as in the previous section. Set $F = \mathcal{F}^{\ell-1} \otimes \mathbb{C}$ the complexification of $\mathcal{F}^{\ell-1}$. Compare the complexified complement $\mathsf{M}(\mathcal{A})$ with the generic hyperplane section $\mathsf{M}(\mathcal{A}) \cap F$. Lefschetz's hyperplane section theorem [HL] tells us that $\mathsf{M}(\mathcal{A})$ is homotopy equivalent to the space obtained from $\mathsf{M}(\mathcal{A}) \cap F$ by attaching some ℓ-dimensional cells. Namely we have the following homotopy equivalence:

$$\mathsf{M}(\mathcal{A}) \approx (\mathsf{M}(\mathcal{A}) \cap F) \cup_{\varphi_i} \bigcup_i D^{\ell},$$

where $\varphi_i : \partial D^{\ell} \longrightarrow \mathsf{M}(\mathcal{A}) \cap F$ is the attaching map. In [Y1], we described the homotopy type of the attaching maps. The ℓ-dimensional cells are naturally encoded by the set $\mathsf{ch}^{\ell}(\mathcal{A})$ of chambers which do not intersect $\mathcal{F}^{\ell-1}$. By using the description of attaching maps, we constructed a cochain complex

$$(\mathbb{C}[\mathsf{ch}^{q}(\mathcal{A})], d_{\mathcal{L}})_{q=0}^{\ell} : \cdots \longrightarrow \mathbb{C}[\mathsf{ch}^{q}(\mathcal{A})] \xrightarrow{d_{\mathcal{L}}} \mathbb{C}[\mathsf{ch}^{q+1}(\mathcal{A})] \longrightarrow \cdots$$

which computes local system cohomology groups for arbitrary rank one local system $\mathcal{L}$. Namely, we have $H^*(\mathbb{C}[\mathsf{ch}^{\bullet}(\mathcal{A})], d_{\mathcal{L}}) \simeq H^*(\mathsf{M}(\mathcal{A}), \mathcal{L})$. In §3.1, we shall describe the cochain complex $(\mathbb{C}[\mathsf{ch}^{\bullet}(\mathcal{A})], d_{\mathcal{L}})$ based on [Y1], and in §3.2 we investigate the case $\ell = 2$ closely.

3.1 Minimal complex arising from Lefschetz's Theorem

Definition 3.1. (Separating hyperplanes) Let $C_1, C_2 \in \mathsf{ch}(\mathcal{A})$ be chambers. Define

$$\mathrm{Sep}(C_1, C_2) = \{H \in \mathcal{A} \mid H \text{ separates } C_1 \text{ and } C_2\}.$$

And also

$$q_{\mathrm{Sep}(C_1,C_2)}^{1/2} = \prod_{H_i \in \mathrm{Sep}(C_1,C_2)} q_i^{1/2}.$$

To describe the coboundary map $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^q(\mathcal{A})] \to \mathbb{C}[\mathsf{ch}^{q+1}(\mathcal{A})]$, we need the notion of degree map

$$\deg : \mathsf{ch}^q(\mathcal{A}) \times \mathsf{ch}^{q+1}(\mathcal{A}) \longrightarrow \mathbb{Z},$$

which we will define below.

Suppose $C \in \mathsf{ch}^q(\mathcal{A})$ and $C' \in \mathsf{ch}^{q+1}(\mathcal{A})$ are given. Let $D = D^q \subset \mathcal{F}^q$ be a q-dimensional ball with sufficiently large radius so that every 0-dimensional edge $x \in L(\mathcal{A} \cap \mathcal{F}^q)$ is in the interior of D^q. There exists a tangent vector field $U(x) \in T_x\mathcal{F}^q$ for $x \in D$ which satisfies the following properties:

- if $x \in \partial D$, then $U(x) \notin T_x(\partial D)$, and $U(x)$ directs inside of D,
- if $x \in H$ with $H \in \mathcal{A}$, then $U(x) \notin T_x(H \cap \mathcal{F}^q) \subset T_x\mathcal{F}^q$ and $U(x)$ directs the side in which C' is contained.

From the properties, we have $U(x) \neq 0$ for $x \in \partial(cl(C) \cap D)$, where $cl(C)$ is the closure of C in $\mathcal{F}^q$. Roughly speaking, the degree $\deg(C, C')$ is defined to be the degree of the Gauss map

$$\frac{U}{|U|} : \partial(cl(C) \cap D) \longrightarrow S^{q-1}.$$

Definition 3.2. Let $C \in \mathsf{ch}^q(\mathcal{A})$ and $C' \in \mathsf{ch}^{q+1}(\mathcal{A})$. Fix U as above. Then define $\deg(C, C')$ as follows.

(0) When $q = 0$, then $\deg(C, C') = 1$.

(1) When $q = 1$, then $cl(C) \cap D \simeq [-1, 1]$. In this case $S^0 \simeq \{\pm 1\}$. The degree of the Gauss maps $g := \frac{U}{|U|} : \{\pm 1\} \longrightarrow \{\pm 1\}$ is defined by

$$\deg(g) = \begin{cases} 0 & \text{if } g(\{\pm 1\}) = \{+1\} \text{ or } g(\{\pm 1\}) = \{-1\}, \\ 1 & \text{if } g(\pm 1) = \pm 1, \\ -1 & \text{if } g(\pm 1) = \mp 1. \end{cases}$$

(2) When $q \geq 2$,

$$\deg(C, C') = \deg\left(\frac{U}{|U|} : \partial(cl(C) \cap D) \longrightarrow S^{q-1}\right).$$

(It is easily seen that $\deg(C, C')$ does not depend on U.)

Now let us define the map

$$d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^q(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^{q+1}(\mathcal{A})]$$

by

$$\mathsf{ch}^q(\mathcal{A}) \ni [C] \longmapsto \sum_{C' \in \mathsf{ch}^{q+1}(\mathcal{A})} \deg(C, C') \cdot \left(q_{\mathrm{Sep}(C,C')}^{1/2} - q_{\mathrm{Sep}(C,C')}^{-1/2}\right) \cdot [C']. \quad (4)$$

Theorem 3.3. ([Y1, 6.4.1]) With notation as above, $(\mathbb{C}[\mathsf{ch}^\bullet(\mathcal{A})], d_{\mathcal{L}})$ is a cochain complex. Furthermore,

$$H^*(\mathbb{C}[\mathsf{ch}^\bullet(\mathcal{A})], d_{\mathcal{L}}) \simeq H^*(\mathsf{M}(\mathcal{A}), \mathcal{L}).$$

In the above formula (4), the degree $\deg(C, C') \in \mathbb{Z}$ is difficult to determine. The author wonders how to compute $\deg(C, C')$. Let us pose a problem which might be interesting from the view point of combinatorics of polytopes.

Problem 3.4. Let $P \subset \mathbb{R}^d$ be a bounded d-dimensional convex polytope. Let $\{F_e\}_{e \in E}$ be the set of facets (i.e., $(d-1)$-dimensional faces). Let $U(x) \in T_x\mathbb{R}^d$ be a vector field on $\mathbb{R}^d$. Suppose that U satisfies $U(x) \neq 0$ when $x \in \partial P$ and, furthermore, $U(x) \notin T_x F_e$ for any point $x \in F_e$ in a facet. We can associate a sign vector $X \in \{+1, -1\}^E$ by

$$X(e) = \begin{cases} +1 & \text{if } U \text{ directs outside of } P \text{ on } F_e, \\ -1 & \text{if } U \text{ directs inside of } P \text{ on } F_e. \end{cases}$$

Then how to compute the degree $\deg\left(\frac{U}{|U|} : \partial P \to S^{d-1}\right)$ of the Gauss map from the sign vector $X \in \{\pm 1\}^E$?

3.2 The case $\ell = 2$

In this section, we look at the minimal complex $(\mathbb{C}[\mathsf{ch}^\bullet(\mathcal{A})], d_{\mathcal{L}})$ for $\ell = 2$ more closely.

First note that $\mathsf{ch}^0(\mathcal{A}) = \{C_0\}$ consists of a chamber. The map $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^1(\mathcal{A})]$ is determined by $d_{\mathcal{L}}([C_0])$, which is

$$d([C_0]) = \sum_{C \in \mathsf{ch}^1(\mathcal{A})} \left(q_{\mathrm{Sep}(C_0,C)}^{1/2} - q_{\mathrm{Sep}(C_0,C)}^{-1/2} \right) \cdot [C].$$

As in §2.1, we decompose $\mathsf{ch}^1(\mathcal{A}) = \mathsf{bch}^1(\mathcal{A}) \sqcup \mathsf{uch}^1(\mathcal{A})$. Note that by Theorem 2.10, $\mathsf{uch}^1(\mathcal{A}) = \{C_0^\vee\}$ consists of a chamber which is the opposite one of C_0. The second coboundary map $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^1(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^2(\mathcal{A})]$ is given by the formula (4). The degree $\deg(C, C')$ behaves differently according as $C \in \mathsf{bch}^1(\mathcal{A})$ or $C \in \mathsf{uch}^1(\mathcal{A})$.

(i) Suppose $C \in \mathsf{bch}^1(\mathcal{A})$. Then $C \cap \mathcal{F}^1$ is a closed interval, the boundary (two points) can be expressed as $(H \cap \mathcal{F}^1) \cup (H' \cap \mathcal{F}^1)$ for $H, H' \in \mathcal{A}$. $\deg(C, C')$ can be computed as

$$\deg(C, C') = \begin{cases} 1 & \text{if } H, H' \in \mathrm{Sep}(C, C'), \\ -1 & \text{if } H, H' \notin \mathrm{Sep}(C, C'), \\ 0 & \text{others.} \end{cases}$$

(ii) Suppose $C \in \mathsf{uch}^1(\mathcal{A})$. Then $C \cap \mathcal{F}^1$ is an unbounded interval, the boundary (a point) can be expressed as $H \cap \mathcal{F}^1$. $\deg(C, C')$ can be computed as

$$\deg(C, C') = \begin{cases} -1 & \text{if } H \notin \mathrm{Sep}(C, C'), \\ 0 & \text{if } H \in \mathrm{Sep}(C, C'). \end{cases}$$

In particular, we have,

Lemma 3.5. Let $C \in \mathsf{bch}^1(\mathcal{A})$. The boundary of $C \cap \mathcal{F}^1$ is expressed as $(H \cap \mathcal{F}^1) \cup (H' \cap \mathcal{F}^1)$. Then

$$\deg(C, C^\vee) = \begin{cases} 1 & \text{if } H \text{ and } H' \text{ are not parallel,} \\ -1 & \text{if } H \text{ and } H' \text{ are parallel.} \end{cases}$$

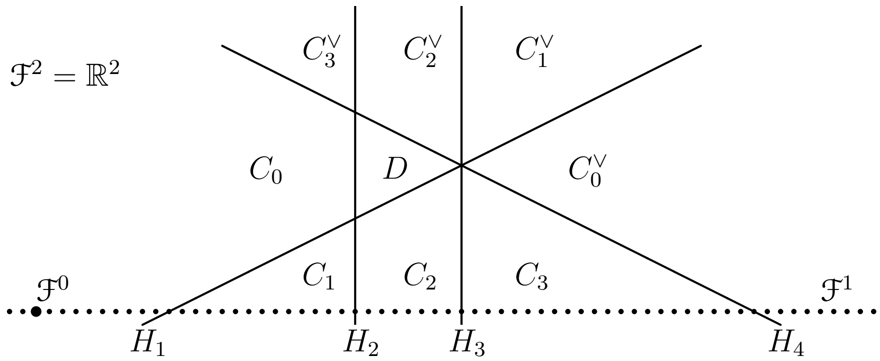

Figure 3: Example 3.6.

Example 3.6. Consider the arrangement of four lines $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$ in $\mathbb{R}^2$ and a generic flag $\mathcal{F}$ as in Figure 3. Then

$$\begin{array}{lll} \mathsf{bch}^0(\mathcal{A}) = \{C_0\} & \mathsf{bch}^1(\mathcal{A}) = \{C_1, C_2, C_3\} & \mathsf{bch}^2(\mathcal{A}) = \{D\} \\ \mathsf{uch}^0(\mathcal{A}) = \emptyset & \mathsf{uch}^1(\mathcal{A}) = \{C_0^\vee\} & \mathsf{uch}^2(\mathcal{A}) = \{C_1^\vee, C_2^\vee, C_3^\vee\}. \end{array}$$

The coboundary map $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0] \to \mathbb{C}[\mathsf{ch}^1]$ is determined by

$$d_{\mathcal{L}}([C_0]) = (q_1^{\frac{1}{2}} - q_1^{-\frac{1}{2}})[C_1] + (q_{12}^{\frac{1}{2}} - q_{12}^{-\frac{1}{2}})[C_2] + (q_{123}^{\frac{1}{2}} - q_{123}^{-\frac{1}{2}})[C_3] + (q_{1234}^{\frac{1}{2}} - q_{1234}^{-\frac{1}{2}})[C_0^\vee],$$

and $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^1] \to \mathbb{C}[\mathsf{ch}^2]$ is as follows.

$$\begin{array}{llll} d_{\mathcal{L}}([C_1]) = \underline{(q_{1234}^{\frac{1}{2}} - q_{1234}^{-\frac{1}{2}})}[C_1^\vee] & +(q_{124}^{\frac{1}{2}} - q_{124}^{-\frac{1}{2}})[C_2^\vee] & & +(q_{12}^{\frac{1}{2}} - q_{12}^{-\frac{1}{2}})[D] \\ d_{\mathcal{L}}([C_2]) = & -\underline{(q_{14}^{\frac{1}{2}} - q_{14}^{-\frac{1}{2}})}[C_2^\vee] & & -(q_1^{\frac{1}{2}} - q_1^{-\frac{1}{2}})[D] \\ d_{\mathcal{L}}([C_3]) = & +(q_{134}^{\frac{1}{2}} - q_{134}^{-\frac{1}{2}})[C_2^\vee] + \underline{(q_{1234}^{\frac{1}{2}} - q_{1234}^{-\frac{1}{2}})}[C_3^\vee] & & \\ d_{\mathcal{L}}([C_0^\vee]) = -(q_1^{\frac{1}{2}} - q_1^{-\frac{1}{2}})[C_1^\vee] & -(q_{13}^{\frac{1}{2}} - q_{13}^{-\frac{1}{2}})[C_2^\vee] - (q_{123}^{\frac{1}{2}} - q_{123}^{-\frac{1}{2}})[C_3^\vee] & & \end{array}$$

The coefficients of the diagonals have another expressions. Observe that $X(C_1) = X(C_3) = H_\infty$ and $X(C_2) = \overline{H_2} \cap \overline{H_3} \cap H_\infty$. Since $q_\infty = q_{1234}^{-1}$ and $q_{X(C_2)} = q_2 q_3 q_\infty = q_{14}^{-1}$, we have

$$\begin{array}{rcl} \underline{(q_{1234}^{\frac{1}{2}} - q_{1234}^{-\frac{1}{2}})} & = & -(q_{X(C_1)}^{\frac{1}{2}} - q_{X(C_1)}^{-\frac{1}{2}}) = -(q_{X(C_3)}^{\frac{1}{2}} - q_{X(C_3)}^{-\frac{1}{2}}), \\ -\underline{(q_{14}^{\frac{1}{2}} - q_{14}^{-\frac{1}{2}})} & = & q_{X(C_2)}^{\frac{1}{2}} - q_{X(C_2)}^{-\frac{1}{2}}. \end{array}$$

In general, we have

Proposition 3.7. Let $C \in \mathsf{bch}^1(\mathcal{A})$. Then the coefficient of $[C^\vee]$ in $d_{\mathfrak{L}}([C])$ is given by $\pm(q_{X(C)}^{1/2} - q_{X(C)}^{-1/2})$.

Proof. Let $H \in \mathcal{A}$. Then H separates C and $C^\vee$ if and only if $\overline{H}$ does go through $X(C) \in H_\infty$. Using $q_1 q_2, \dots q_n q_\infty = 1$, we have

$$q_{\mathrm{Sep}(C,C^\vee)} = q_{X(C)}^{-1}.$$

Hence $\pm(q_{\mathrm{Sep}(C,C^\vee)}^{1/2} - q_{\mathrm{Sep}(C,C^\vee)}^{-1/2}) = \mp(q_{X(C)}^{1/2} - q_{X(C)}^{-1/2})$. □

For use in the next section, we analyze the induced map

$$\mathbb{C}[\mathsf{bch}^1(\mathcal{A})] \hookrightarrow \mathbb{C}[\mathsf{ch}^1(\mathcal{A})] \xrightarrow{d_{\mathfrak{L}}} \mathbb{C}[\mathsf{ch}^2(\mathcal{A})] \twoheadrightarrow \mathbb{C}[\mathsf{uch}^2(\mathcal{A})].$$

We write the map $\overline{d_{\mathfrak{L}}} : \mathbb{C}[\mathsf{bch}^1(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{uch}^2(\mathcal{A})]$. As Theorem 2.10, the bases of the source and the target of $\overline{d_{\mathfrak{L}}}$ are naturally identified by the involution ι. Thus the determinant $\det(\overline{d_{\mathfrak{L}}}) \in \mathbb{C}$ makes sense. The matrix $\overline{d_{\mathfrak{L}}}$ is expressed by an upper triangular matrix, and the determinant can be computed.

Theorem 3.8. The determinant $\det(\overline{d_{\mathfrak{L}}})$ can be expressed as

$$\det(\overline{d_{\mathfrak{L}}}) = \pm \prod_{X \in \mathsf{D}_\infty(\mathcal{A}_\infty)} (q_X^{1/2} - q_X^{-1/2})^{n_X}, \tag{5}$$

where n_X is a positive integer.

Proof. First note that, for $C \in \mathsf{uch}(\mathcal{A})$, $X(C)$ is either 0-dimensional or equal to H_∞. We call an unbounded chamber $C \in \mathsf{uch}(\mathcal{A})$ *narrow* (resp. *wide*) if $X(C) \subset H_\infty$ is 0-dimensional (resp. $X(C) = H_\infty$). We decompose $\mathbb{C}[\mathsf{bch}^1(\mathcal{A})]$ and $\mathbb{C}[\mathsf{uch}^2(\mathcal{A})]$ into direct sum of subspaces. Set

$$N^1 = \mathbb{C}[\{C \in \mathsf{bch}^1(\mathcal{A}) \mid C : \text{ narrow}\}], \quad W^1 = \mathbb{C}[\{C \in \mathsf{bch}^1(\mathcal{A}) \mid C : \text{ wide}\}],$$
$$N^2 = \mathbb{C}[\{C \in \mathsf{uch}^2(\mathcal{A}) \mid C : \text{ narrow}\}], \quad W^2 = \mathbb{C}[\{C \in \mathsf{uch}^2(\mathcal{A}) \mid C : \text{ wide}\}].$$

Then clearly $\mathbb{C}[\mathsf{bch}^1(\mathcal{A})] = W^1 \oplus N^1$ and $\mathbb{C}[\mathsf{uch}^2(\mathcal{A})] = W^2 \oplus N^2$. The map $\overline{d_{\mathfrak{L}}}$ preserves N^i. Furthermore, the matrix presentation of $\overline{d_{\mathfrak{L}}}|_{N^1} : N^1 \to N^2$ is diagonal. Indeed suppose that $C \in \mathsf{bch}^1(\mathcal{A})$ is a narrow chamber with walls $H \cap \mathcal{F}^1$ and $H' \cap \mathcal{F}^1$. Then H and H' are parallel. By definition of degree map, $d_{\mathfrak{L}}([C])$ is a linear combination of chambers which are put between H

and H'. The opposite chamber $C^\vee$ is the unique such element in $\mathsf{uch}^2(\mathcal{A})$. By Proposition 3.7, we obtain the explicit formula

$$\overline{d_{\mathcal{L}}}([C]) = (q_{X(C)}^{1/2} - q_{X(C)}^{-1/2})[C^\vee]$$

for a narrow chamber $C \in \mathsf{bch}^1(\mathcal{A})$. Next we consider W^1 and W^2. Since $\mathbb{C}[\mathsf{bch}^1]/N^1 \simeq W^1$ and $\mathbb{C}[\mathsf{uch}^2]/N^2 \simeq W^2$, we have the induced map $\widetilde{d_{\mathcal{L}}} : W^1 \to W^2$. This map is again expressed by a diagonal matrix. Indeed, for a wide chamber $C \in \mathsf{bch}^1(\mathcal{A})$, we have

$$\begin{aligned}\widetilde{d_{\mathcal{L}}}([C]) &= -(q_\infty^{1/2} - q_\infty^{-1/2})[C^\vee] \\ &= -(q_{X(C)}^{1/2} - q_{X(C)}^{-1/2})[C^\vee].\end{aligned}$$

Thus

$$\begin{aligned}\det\left(\overline{d_{\mathcal{L}}} : \mathbb{C}[\mathsf{bch}^1] \to \mathbb{C}[\mathsf{uch}^2]\right) &= \det(\overline{d_{\mathcal{L}}}|_{N^1}) \cdot \det(\widetilde{d_{\mathcal{L}}}) \\ &= \pm \prod_{C \in \mathsf{bch}^1(\mathcal{A})} (q_{X(C)}^{1/2} - q_{X(C)}^{-1/2}).\end{aligned}$$

By Proposition 2.4, $X(C)$ in the above formulas runs all dense edges contained H_∞. Hence we obtain (5). □

Corollary 3.9. The map $\overline{d_{\mathcal{L}}} : \mathbb{C}[\mathsf{bch}^1(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{uch}^2(\mathcal{A})]$ is nondegenerate if and only if $q_X \neq 1$ for any dense edge $X \in \mathsf{D}_\infty(\mathcal{A}_\infty)$ in H_∞.

The decomposability of $\mathcal{A}$ is related to W^2 as follows. We omit the proof (cf. Figure 2 and Figure 3).

Proposition 3.10. For $\ell = 2$, $\mathcal{A}$ is decomposable if and only if $\dim W^2 = 1$.

4 An application

As we saw in the previous sections, the basis of our cochain complex is encoded by the set of chambers. There is also an involution ι among unbounded chambers. In this section, we prove that if the monodromies around dense edges at infinity are nontrivial, then the bases corresponding to unbounded chambers C and $C^\vee = \iota(C)$ are cancelled each other, and finally, only bounded chambers survive. This leads to a proof of the refined version of vanishing theorem.

Our main result is the following.

Theorem 4.1. Let $\mathcal{A}$ be an indecomposable line arrangement in $\mathbb{R}^2$. Let $\mathcal{L}$ be a rank one local system. Then the following are equivalent.

(i) $q_X \neq 1$ for any dense edge $X \in \mathsf{D}_\infty(\mathcal{A}_\infty)$ contained in H_∞.

(ii)
$$H^k(\mathsf{M}(\mathcal{A}), \mathcal{L}) = \begin{cases} 0 & \text{for } k = 0, 1, \\ \bigoplus_{C \in \mathsf{bch}(\mathcal{A})} \mathbb{C} \cdot [C] & \text{for } k = 2. \end{cases}$$

(iii) $H^2(\mathcal{A}, \mathcal{L})$ is generated by $\{[C] \mid C \in \mathsf{bch}(\mathcal{A})\}$.

Remark 4.2. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) holds for any arrangement $\mathcal{A}$ (without indecomposability). However (iii)$\Rightarrow$(i) requires the decomposability of $\mathcal{A}$. (See Remark 4.3.) For comments to the higher dimensional cases ($\ell \geq 3$) see the next §5.

Proof of Theorem 4.1. (i)$\Rightarrow$(ii): Let $C_0 \in \mathsf{ch}^0(\mathcal{A})$. Since

$$d_{\mathcal{L}}([C_0]) = -(q_\infty^{1/2} - q_\infty^{-1/2})[C_0^\vee] + \dots,$$

and $q_\infty \neq 1$, we have rank $\big(d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0(\mathcal{A})] \to \mathbb{C}[\mathsf{ch}^1(\mathcal{A})]\big) = 1$ (and in particular, $H^0(\mathbb{C}[\mathsf{ch}^\bullet(\mathcal{A})], d_{\mathcal{L}}) = \mathrm{Ker}(d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0] \to \mathbb{C}[\mathsf{ch}^1]) = 0$).

To show that

- $H^1(\mathbb{C}[\mathsf{ch}^\bullet(\mathcal{A})], d_{\mathcal{L}}) = 0$,
- $H^2(\mathbb{C}[\mathsf{ch}^\bullet(\mathcal{A})], d_{\mathcal{L}}) = \mathrm{Coker}(d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^1] \to \mathbb{C}[\mathsf{ch}^2])$ has $\{[C]\}_{C \in \mathsf{bch}(\mathcal{A})}$ as a basis,

it suffices to prove that the induced map

$$\overline{d_{\mathcal{L}}} : \mathbb{C}[\mathsf{bch}^1(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{uch}^2(\mathcal{A})]$$

is surjective (hence bijective). However this easily follows from Corollary 3.9.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i): Let us assume (iii). Since $H^2 = \mathrm{Coker}(d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^1] \to \mathbb{C}[\mathsf{ch}^2])$, the assumption implies that the induced map

$$\mathbb{C}[\mathsf{ch}^1(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{uch}^2(\mathcal{A})] \text{ is surjective.} \tag{6}$$

As in the proof of Theorem 3.8, $d_{\mathcal{L}}$ maps N^1 to N^2. Thus the induced map

$$\begin{array}{cccc} \widetilde{d_{\mathcal{L}}}: & W^1 \oplus \mathbb{C}\cdot[C_0^\vee] & \longrightarrow & W^2 \\ & \wr\| & & \wr\| \\ & \mathbb{C}[\mathsf{ch}^1]/N^1 & & \mathbb{C}[\mathsf{uch}^2]/N^2 \end{array}$$

is surjective. Now if $q_\infty = 1$, then $\widetilde{d_{\mathcal{L}}}$ is the zero map on W^1, and hence W^2 is at most one dimensional. This is a contradiction to the assumption $\mathcal{A}$ is indecomposable (see Proposition 3.10). Thus we have $q_\infty \neq 1$.

Set $\mathsf{bch}^1(\mathcal{A}) = \{C_1, \dots, C_k\}$. Then $\mathsf{ch}^1(\mathcal{A}) = \{C_0^\vee, C_1, \dots, C_k\}$. $d_{\mathcal{L}}([C_0])$ is expressed as

$$d_{\mathcal{L}}([C_0]) = a_0[C_0^\vee] + \sum_{i=1}^k a_i[C_i].$$

Note that $a_0 = -(q_\infty^{1/2} - q_\infty^{-1/2}) \neq 0$. Since $d_{\mathcal{L}}^2 = 0$, $d_{\mathcal{L}}([C_0^\vee]) \in \mathbb{C}[\mathsf{ch}^2]$ can be expressed as a linear combination of $d_{\mathcal{L}}([C_1]), \dots, d_{\mathcal{L}}([C_k])$. The assumption (6) implies that (recall that $\mathbb{C}[\mathsf{ch}^1(\mathcal{A})] = \mathbb{C}[\mathsf{bch}^1(\mathcal{A})] \oplus \mathbb{C}\cdot[C_0^\vee]$)

$$\mathbb{C}[\mathsf{bch}^1(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{uch}^2(\mathcal{A})] \text{ is surjective.} \tag{7}$$

Again by Theorem 3.8, we conclude that $q_X \neq 1$ for any dense edge $X \in \mathsf{D}_\infty(\mathcal{A}_\infty)$ in H_∞. □

Remark 4.3. The assumption "$\mathcal{A}$ is indecomposable" is necessary to prove (iii) ⇒ (i) in Theorem 4.1. Indeed, consider the arrangement in Figure 2, which is decomposable. Let $\mathcal{L}$ be a rank one local system such that $q_1, q_2, q_3 \in \mathbb{C}^*$ are generic and $q_4 = q_1^{-1}q_2^{-1}q_3^{-1}$. Then $q_\infty = 1$. The map $\widetilde{d_{\mathcal{L}}} : \mathbb{C}[\mathsf{ch}^1] \to \mathbb{C}[\mathsf{uch}^2]$ is computed as:

$$\begin{array}{rcllll}
\widetilde{d_{\mathcal{L}}}([C_1]) & = & -(q_{34}^{\frac{1}{2}} - q_{34}^{-\frac{1}{2}})[C_1^\vee] & & & \\
\widetilde{d_{\mathcal{L}}}([C_2]) & = & (q_{234}^{\frac{1}{2}} - q_{234}^{-\frac{1}{2}})[C_1^\vee] & & & -(q_{12}^{\frac{1}{2}} - q_{12}^{-\frac{1}{2}})[C_3^\vee] \\
\widetilde{d_{\mathcal{L}}}([C_3]) & = & & & & -(q_{123}^{\frac{1}{2}} - q_{123}^{-\frac{1}{2}})[C_3^\vee] \\
\widetilde{d_{\mathcal{L}}}([C_0^\vee]) & = & -(q_2^{\frac{1}{2}} - q_2^{-\frac{1}{2}})[C_1^\vee] & -(q_{12}^{\frac{1}{2}} - q_{12}^{-\frac{1}{2}})[C_2^\vee]. & &
\end{array}$$

Hence the map $\widetilde{d_{\mathcal{L}}} : \mathbb{C}[\mathsf{ch}^1] \to \mathbb{C}[\mathsf{uch}^2]$ has rank three. This implies that $H^2(\mathbb{C}[\mathsf{ch}^\bullet], d_{\mathcal{L}})$ is generated by $\mathsf{bch}^2 = \{C_4\}$. Thus (iii) is satisfied, however (i) is false ($q_\infty = 1$).

5 Remarks and conjectures

We conclude this paper with some remarks on higher dimensional cases $\ell \geq 3$. As in the case $\ell = 2$, it seems natural to focus on the induced map

$$\overline{d_{\mathcal{L}}} : \mathbb{C}[\mathsf{bch}^{q-1}] \longrightarrow \mathbb{C}[\mathsf{uch}^{q}]$$

defined by the composition $\mathbb{C}[\mathsf{bch}^{q-1}] \hookrightarrow \mathbb{C}[\mathsf{ch}^{q-1}] \xrightarrow{d_{\mathcal{L}}} \mathbb{C}[\mathsf{ch}^{q}] \twoheadrightarrow \mathbb{C}[\mathsf{uch}^{q}]$. Since the bases of two spaces $\mathbb{C}[\mathsf{bch}^{q-1}]$ and $\mathbb{C}[\mathsf{uch}^{q}]$ are naturally identified by the involution ι, it makes sense to consider the determinant of $\overline{d_{\mathcal{L}}}$.

Conjecture 5.1. The determinant $\det\left(\overline{d_{\mathcal{L}}} : \mathbb{C}[\mathsf{bch}^{q-1}] \to \mathbb{C}[\mathsf{uch}^{q}]\right)$ is expressed in the following form

$$\det(\overline{d_{\mathcal{L}}}) = \pm \prod_{X} (q_X^{1/2} - q_X^{-1/2})^{n_X},$$

where X runs all dense edge $X \in \mathsf{D}_{\infty}(\mathcal{A}_{\infty})$ with $\dim X \geq \ell - q$ and $n_X > 0$.

Once the above conjecture is established, it deduces the following.

Conjecture 5.2. Let $\mathcal{A}$ be an essential affine arrangement in $\mathbb{R}^{\ell}$. If the rank one local system $\mathcal{L}$ satisfies the condition (3), then (2) holds.

"Proof of 5.1 ⇒ 5.2." Since the composition $\overline{d_{\mathcal{L}}} : \mathbb{C}[\mathsf{bch}^{q-1}] \hookrightarrow \mathbb{C}[\mathsf{ch}^{q-1}] \xrightarrow{d_{\mathcal{L}}} \mathbb{C}[\mathsf{ch}^{q}] \twoheadrightarrow \mathbb{C}[\mathsf{uch}^{q}]$ is bijective, rank of the map $\mathbb{C}[\mathsf{ch}^{q-1}] \xrightarrow{d_{\mathcal{L}}} \mathbb{C}[\mathsf{ch}^{q}]$ is at least $\sharp\mathsf{bch}^{q-1} = \sharp\mathsf{uch}^{q}$. Hence,

$$\begin{aligned}
\dim \operatorname{Im}(d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^{q-1}] \to \mathbb{C}[\mathsf{ch}^{q}]) &\geq \sharp\mathsf{uch}^{q} = \sharp\mathsf{bch}^{q-1}, \\
\dim \operatorname{Ker}(d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^{q}] \to \mathbb{C}[\mathsf{ch}^{q+1}]) &\leq \sharp\mathsf{ch}^{q} - \sharp\mathsf{bch}^{q} = \sharp\mathsf{uch}^{q},
\end{aligned}$$

for $q \leq \ell - 1$. This implies $H^k(\mathbb{C}[\mathsf{ch}^{\bullet}], d_{\mathcal{L}}) = 0$ for $k \leq \ell - 1$. Also this deduces $H^{\ell}(\mathbb{C}[\mathsf{ch}^{\bullet}], d_{\mathcal{L}})$ is generated by $\mathsf{bch}^{\ell}(\mathcal{A}) = \mathsf{bch}(\mathcal{A})$. □

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