MINIMAL MODEL THEORY FOR LOG SURFACES

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Abstract. We discuss the log minimal model theory for log surfaces. We show that the log minimal model program, the finite generation of log canonical rings, and the log abundance theorem for log surfaces hold true under much weaker assumptions than everybody expected.

Contents

1. Introduction

We explain the log minimal model theory for log surfaces. This paper completes Fujita's results on the semi-ampleness of semi-positive parts of Zariski decompositions of log canonical divisors and the finite generation of log canonical rings for smooth projective log surfaces in [Ft] and the log minimal model program for projective log canonical surfaces discussed by Kollár and Kovács in [KK]. The main purpose of this paper is to show that the log minimal model program for surfaces works and the log abundance theorem and the finite generation of log

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canonical rings for surfaces hold true under much weaker assumptions than we expected (cf. Theorems 3.3, 4.3, and 6.1).

It is obvious that the log minimal model program works for $\mathbb{Q}$-factorial log surfaces and log canonical surfaces by our new cone and contraction theorem for log varieties (cf. [F3, Theorem 1.1]), which is the culmination of the works of several authors. By our log minimal model program for log surfaces, Fujita's results in [Ft] are clarified and generalized. In [Ft], Fujita treated a pair (X, Δ) where X is a smooth projective surface and Δ is a boundary $\mathbb{Q}$-divisor on X without any assumptions on singularities of the pair (X, Δ). We note that our log minimal model program explained in this paper works for such pairs (cf. Theorem 3.3). It is not necessary to assume that (X, Δ) is log canonical.

Roughly speaking, we will prove the following theorem in this paper. We think that nobody expected the case (A) in Theorem 1.1.

Theorem 1.1 (cf. Theorems 3.3 and 6.1)**.** *Let X be a normal projective surface defined over $\mathbb{C}$ and Δ an effective $\mathbb{Q}$-divisor on X such that every coefficient of Δ is less than or equal to one. Assume that one of the following conditions holds*:

(A) *X is $\mathbb{Q}$-factorial, or*
(B) *(X, Δ) is log canonical.*

Then we can run the log minimal model program with respect to $K_X+\Delta$ and obtain a sequence of extremal contractions

$$(X, \Delta) = (X_0, \Delta_0) \overset{\varphi_0}{\to} (X_1, \Delta_1) \overset{\varphi_1}{\to} \cdots \overset{\varphi_{k-1}}{\to} (X_k, \Delta_k) = (X^*, \Delta^*)$$

such that

(1) (Minimal model) *$K_{X^*} + \Delta^*$ is semi-ample if $K_X + \Delta$ is pseudo-effective, and*
(2) (Mori fiber space) *there is a morphism $g : X^* \to C$ such that $-(K_{X^*} + \Delta^*)$ is g-ample,* $\dim C < 2$, *and the relative Picard number $\rho(X^*/C) = 1$, if $K_X + \Delta$ is not pseudo-effective.*

Note that (X, Δ) is not necessarily log canonical in the case (A).

The following result is an obvious corollary of Theorem 1.1.

Corollary 1.2 (cf. Corollary 4.4)**.** *Let X be a projective surface with only rational singularities. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

We note that the general classification theory of algebraic surfaces is due essentially to the Italian school, and has been worked out in detail by Kodaira, in Shafarevich's seminar, and so on. The theory of log surfaces was studied by Iitaka, Kawamata, Miyanishi, Sakai, Fujita, and many others. See, for example, [M] and [S2]. Our viewpoint seems to be much more minimal model theoretic than any other works. We do not use the notion of *Zariski decomposition* in this paper (see Remark 3.9).

We summarize the contents of this paper. Section 2 collects some preliminary results. In Section 3, we discuss the log minimal model program for log surfaces. It is a direct consequence of the cone and contraction theorem for log varieties (cf. [F3, Theorem 1.1]). In Section 4, we show the finite generation of log canonical rings for log surfaces. More precisely, we prove a special case of the log abundance theorem for log surfaces. In Section 5, we prove the non-vanishing theorem for log surfaces. It is an important step of the log abundance theorem for log surfaces. In Section 6, we prove the log abundance theorem for log surfaces. It is a generalization of Fujita's main result in [Ft]. Section 7 is a supplementary section. We will prove the finite generation of log canonical rings and the log abundance theorem for log surfaces in the relative setting. Consequently, Theorem 1.1 also holds in the relative setting. In Section 8: Appendix, we prove the base point free theorem for log surfaces in full generality (cf. Theorem 8.1). It completely generalizes Fukuda's base point free theorem for log canonical surfaces (cf. [Fk, Main Theorem]). It is not necessary for the log minimal model theory for log surfaces discussed in this paper. The proof given there is different from Fukuda's and depends on the theory of *quasi-log varieties* (cf. [A], [F4], and [F7]).

We will work over $\mathbb{C}$, the complex number field, throughout this paper. But we note that by using the Lefschetz principle, we can extend everything to the case where the base field is an algebraically closed field of characteristic zero. Our arguments heavily depend on the Kodaira type vanishing theorem (cf. [F3]). So, we can not directly apply them in characteristic p. We also note that [Ft] and [KK] treat algebraic surfaces defined over an algebraically closed field of *any* characteristic.

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2. Preliminaries

We collect some basic definitions and results. We will freely use the notation and terminology in [KM] and [F3] throughout this paper.

2.1 ($\mathbb{Q}$-divisors and $\mathbb{R}$-divisors)**.** Let X be a normal variety. For an $\mathbb{R}$-divisor $D=\sum_{j=1}^{r} d_j D_j$ on X such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$ (resp. *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$), where for every real number x, $\llcorner x \lrcorner$ (resp. $\ulcorner x \urcorner$) is the integer defined by $x-1<\llcorner x \lrcorner \leq x$ (resp. $\ulcorner x \urcorner = -\llcorner -x \lrcorner$). The *fractional part* $\{D\}$ of D denotes $D-\llcorner D \lrcorner$. We define

$$D^{>a}=\sum_{d_j>a} d_j D_j, \quad D^{<a}=\sum_{d_j<a} d_j D_j$$

and

$$D^{=a}=\sum_{d_j=a} d_j D_j = a\sum_{d_j=a} D_j$$

for any real number a. We call D a *boundary* $\mathbb{R}$-divisor if $0 \leq d_j \leq 1$ for every j. We note that $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-*linear equivalence* of $\mathbb{Q}$-divisors. Of course, $\sim$ (resp. $\equiv$) denotes the usual *linear equivalence* (resp. *numerical equivalence*) of divisors.

Let $f: X \to Y$ be a morphism and B a Cartier divisor on X. We say that B is *linearly f-trivial* (denoted by $B \sim_f 0$) if and only if there is a Cartier divisor B' on Y such that $B \sim f^*B'$. Two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors B_1 and B_2 on X are called *numerically f-equivalent* (denoted by $B_1 \equiv_f B_2$) if and only if $B_1 \cdot C = B_2 \cdot C$ for every curve C such that $f(C)$ is a point.

We say that X is $\mathbb{Q}$-*factorial* if every prime Weil divisor on X is $\mathbb{Q}$-Cartier. The following lemma is well known.

Lemma 2.2 (Projectivity)**.** *Let X be a complete normal $\mathbb{Q}$-factorial algebraic surface. Then X is projective.*

Proof. Let $f: Y \to X$ be a projective birational morphism from a smooth projective surface Y. Let H be an effective general ample Cartier divisor on Y. We consider the effective $\mathbb{Q}$-Cartier Weil divisor $A=f_*H$ on X. Then $A \cdot C = H \cdot f^*C > 0$ for every curve C on X. Therefore, A is ample by Nakai's criterion. Thus, X is projective. □

2.3 (Singularities of pairs)**.** Let X be a normal variety and Δ an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $f: Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$ has a simple normal crossing support, where $\mathrm{Exc}(f)$ is the *exceptional locus* of f and $f_*^{-1}\Delta$ is the *strict transform* of Δ on Y. We can write

$$K_Y = f^*(K_X+\Delta)+\sum_i a_i E_i.$$

We say that (X,Δ) is *log canonical* (*lc*, for short) if $a_i \geq -1$ for every i. We say that (X,Δ) is *Kawamata log terminal* (*klt*, for short) if $a_i > -1$ for every i. We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of E_i with respect to (X,Δ). We note that $\mathrm{Nklt}(X,\Delta)$ (resp. $\mathrm{Nlc}(X,\Delta)$) denotes the image of $\sum_{a_i\leq -1} E_i$ (resp. $\sum_{a_i<-1} E_i$) and is called the *non-klt locus* (resp. *non-lc locus*) of (X,Δ). If there exist a resolution $f: Y \to X$ and a divisor E on Y such that $a(E,X,\Delta)=-1$ and that $f(E)\not\subset \mathrm{Nlc}(X,\Delta)$, then $f(E)$ is called a *log canonical center* (*lc center*, for short) with respect to (X,Δ). If there exist a resolution $f: Y\to X$ and a divisor E on Y such that $a(E,X,\Delta)\leq -1$, then $f(E)$ is called a *non-klt center* with respect to (X,Δ).

When X is a surface, the notion of *numerically log canonical* and *numerically dlt* is sometimes useful. See [KM, Notation 4.1] and Proposition 3.4 below.

2.4 (Kodaira dimension and numerical Kodaira dimension)**.** We note that κ (resp. ν) denotes the *Iitaka–Kodaira dimension* (resp. *numerical Kodaira dimension*).

Let X be a normal projective variety, D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, and n a positive integer such that nD is Cartier. By definition, $\kappa(X,D)=-\infty$ if and only if $h^0(X,\mathcal{O}_X(mnD))=0$ for every $m>0$, and $\kappa(X,D)=k>-\infty$ if and only if

$$0<\limsup_{m>0}\frac{h^0(X,\mathcal{O}_X(mnD))}{m^k}<\infty.$$

We see that $\kappa(X,D)\in\{-\infty,0,1,\cdots,\dim X\}$. If D is nef, then

$$\nu(X,D)=\max\{e\in\mathbb{Z}_{\geq 0}\,|\,D^e \text{ is not numerically zero}\}.$$

We say that D is *abundant* if $\nu(X,D)=\kappa(X,D)$.

Let Y be a projective irreducible variety and B a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. We say that B is *big* if ν^*B is *big*, that is, $\kappa(Z,\nu^*B)=\dim Z$, where $\nu: Z\to Y$ is the normalization of Y,

2.5 (Nef dimension)**.** Let L be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety X. Then $n(X,L)$ denotes the *nef dimension* of L. It is well known that

$$\kappa(X,L)\leq\nu(X,L)\leq n(X,L).$$

For details, see [8]. We will use the *reduction map* associated to L in Section 6.

Let us quickly recall the reduction map and the nef dimension in [8]. By [8, Theorem 2.1], for a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on X, we can construct an almost holomorphic, dominant rational map $f : X \dashrightarrow Y$ with connected fibers, called a *reduction map* associated to L such that

(i) L is numerically trivial on all compact fibers F of f with $\dim F = \dim X - \dim Y$, and
(ii) for every general point $x \in X$ and every irreducible curve C passing through x with $\dim f(C) > 0$, we have $L \cdot C > 0$.

The map f is unique up to birational equivalence of Y. We define the *nef dimension* of L as follows (cf. [8, Definition 2.7]):

$$n(X, L) := \dim Y.$$

2.6 (Non-lc ideal sheaves)**.** The ideal sheaf $\mathcal{J}_{NLC}(X, \Delta)$ denotes the *non-lc ideal sheaf* associated to the pair (X, Δ). More precisely, let X be a normal variety and Δ an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$ and that $\mathrm{Supp}\Delta_Y$ is simple normal crossing. Then we have

$$\mathcal{J}_{NLC}(X, \Delta) = f_*\mathcal{O}_Y(-\llcorner \Delta_Y \lrcorner + \Delta_Y^{=1}) \subset \mathcal{O}_X.$$

For details, see, for example, [F3, Section 7] or [F8]. We note that

$$\mathcal{J}(X, \Delta) = f_*\mathcal{O}_Y(-\llcorner \Delta_Y \lrcorner) \subset \mathcal{O}_X$$

is the *multiplier ideal sheaf* associated to the pair (X, Δ).

2.7 (Kodaira type vanishing theorem)**.** Let $f : X \to Y$ be a birational morphism from a smooth projective variety X to a normal projective variety Y. Let Δ be a boundary $\mathbb{Q}$-divisor on X such that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor and L a Cartier divisor on X. Assume that

$$L - (K_X + \Delta) \sim_{\mathbb{Q}} f^*H,$$

where H is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y such that $H|_{f(C)}$ is big for every lc center C of the pair (X, Δ). Then we obtain

$$H^i(Y, R^j f_*\mathcal{O}_X(L)) = 0$$

for every $i > 0$ and $j \geq 0$. It is a special case of [F4, Theorem 2.47], which is the culmination of the works of several authors. We recommend [F6] as an introduction to new vanishing theorems.

2.8. Let Λ be a linear system. Then $\mathrm{Bs}\Lambda$ denotes the *base locus* of Λ.

3. Minimal model program for log surfaces

Let us introduce the notion of *log surfaces*.

Definition 3.1 (Log surfaces)**.** Let X be a normal algebraic surface and Δ a boundary $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair (X, Δ) is called a *log surface*. We note that a *boundary* $\mathbb{R}$-divisor means an effective $\mathbb{R}$-divisor whose coefficients are less than or equal to one.

We note that we assume nothing on singularities of (X, Δ).

From now on, we discuss the log minimal model program for log surfaces. The following cone and contraction theorem is a special case of [F3, Theorem 1.1], which is the culmination of the works of several authors. For details, see [F3].

Theorem 3.2 (cf. [F3, Theorem 1.1])**.** *Let (X, Δ) be a log surface and $\pi : X \to S$ a projective morphism onto an algebraic variety S. Then we have*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+\Delta\geq 0} + \sum R_j$$

with the following properties.

1. *R_j is a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X/S)$ for every j.*
2. *Let H be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many R_j's included in $(K_X + \Delta + H)_{<0}$. In particular, the R_j's are discrete in the half-space $(K_X + \Delta)_{<0}$.*
3. *Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X/S)$. Then there exists a contraction morphism $\varphi_R : X \to Y$ over S with the following properties.*
   - (i) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_R(C)$ is a point if and only if $[C] \in R$.*
   - (ii) $\mathcal{O}_Y \simeq (\varphi_R)_*\mathcal{O}_X$.
   - (iii) *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in R$. Then there exists a line bundle L_Y on Y such that $L \simeq \varphi_R^* L_Y$.*

A key point is that the non-lc locus of a log surface (X, Δ) is zero-dimensional. So, there are no curves contained in the non-lc locus of (X, Δ). We will prove that R_j in Theorem 3.2 (1) is spanned by a rational curve C_j with $-(K_X + \Delta) \cdot C_j \leq 3$ in Proposition 3.7 below.

By Theorem 3.2, we can run the log minimal model program for log surfaces under some mild assumptions.

Theorem 3.3 (Minimal model program for log surfaces)**.** *Let (X, Δ) be a log surface and $\pi : X \to S$ a projective morphism onto an algebraic variety S. We assume one of the following conditions*:

(A) *X is $\mathbb{Q}$-factorial.*
(B) *(X, Δ) is log canonical.*

Then, by Theorem 3.2*, we can run the log minimal model program over S with respect to $K_X + \Delta$. So, there is a sequence of at most $\rho(X/S) - 1$ contractions*

$$(X, \Delta) = (X_0, \Delta_0) \overset{\varphi_0}{\to} (X_1, \Delta_1) \overset{\varphi_1}{\to} \cdots \overset{\varphi_{k-1}}{\to} (X_k, \Delta_k) = (X^*, \Delta^*)$$

over S such that one of the following holds:

(1) (Minimal model) *$K_{X^*} + \Delta^*$ is nef over S. In this case, (X^*, Δ^*) is called a* minimal model *of (X, Δ).*
(2) (Mori fiber space) *There is a morphism $g : X^* \to C$ over S such that $-(K_{X^*} + \Delta^*)$ is g-ample,* $\dim C < 2$, *and $\rho(X^*/C) = 1$. We sometimes call $g : (X^*, \Delta^*) \to C$ a* Mori fiber space.

We note that X_i is $\mathbb{Q}$-factorial (*resp.* *(X_i, Δ_i) is lc*) *for every i in the case* (A) (*resp.* (B))*.*

Proof. It is obvious by Theorem 3.2. In the case (B), we have to check that (X_i, Δ_i) is lc for $\Delta_i = \varphi_{i-1*}\Delta_{i-1}$. Since $-(K_{X_{i-1}} + \Delta_{i-1})$ is φ_{i-1}-ample, it is easy to see that (X_i, Δ_i) is numerically lc (cf. [KM, Notation 4.1]) by the negativity lemma. By Proposition 3.4 below, the pair (X_i, Δ_i) is log canonical. In particular, $K_{X_i} + \Delta_i$ is $\mathbb{R}$-Cartier. In the case (A), we can easily check that X_i is $\mathbb{Q}$-factorial for every i by the usual method (cf. [KM, Proposition 3.36]). □

Let us contain [KM, Proposition 4.11] for the reader's convenience. The statement (2) in the following proposition is missing in the English edition of [KM]. For definitions, see [KM, Notation 4.1].

Proposition 3.4 (cf. [KM, Proposition 4.11])**.** *We have the following two statements.*

(1) *Let (X, Δ) be a numerically dlt pair. Then every Weil divisor on X is $\mathbb{Q}$-Cartier, that is, X is $\mathbb{Q}$-factorial.*
(2) *Let (X, Δ) be a numerically lc pair. Then it is lc.*

Proof. In both cases, if $\Delta \neq 0$, then $(X, 0)$ is numerically dlt by [KM, Corollary 4.2] and we can reduce the problem to the case (1) with $\Delta = 0$. Therefore, we can assume that $\Delta = 0$ when we prove this proposition. Let $f : Y \to X$ be a minimal resolution and Δ_Y the f-exceptional $\mathbb{Q}$-divisor on Y such that $K_Y + \Delta_Y \equiv_f 0$. Then $\Delta_Y \geq 0$ by [KM, Corollary 4.3].

(1) See the proof of [KM, Proposition 4.11].

(2) We can assume that $(X, 0)$ is not numerically dlt, that is, $\llcorner \Delta_Y \lrcorner \neq 0$. By [KM, Theorem 4.7], $\{\Delta_Y\}$ is a simple normal crossing divisor. Since $-\llcorner \Delta_Y \lrcorner \equiv_f K_Y + \{\Delta_Y\}$, we have

$$R^1 f_* \mathcal{O}_Y(n(K_Y + \Delta_Y) - \llcorner \Delta_Y \lrcorner) = 0$$

by the Kawamata–Viehweg vanishing theorem for $n \in \mathbb{Z}_{>0}$ such that $n\Delta_Y$ is a Weil divisor. Therefore, we obtain a surjection

$$f_* \mathcal{O}_Y(n(K_Y + \Delta_Y)) \twoheadrightarrow f_* \mathcal{O}_{\llcorner \Delta_Y \lrcorner}(n(K_Y + \Delta_Y)).$$

Therefore, if we check

$$n(K_Y + \Delta_Y)|_{\llcorner \Delta_Y \lrcorner} \sim 0,$$

then we obtain $n(K_Y + \Delta_Y) \sim_f 0$ and $nK_X = f_*(n(K_Y + \Delta_Y))$ is a Cartier divisor. This statement can be checked by [KM, Theorem 4.7] as follows. By the classification, $\llcorner \Delta_Y \lrcorner$ is a cycle and $\Delta_Y = \llcorner \Delta_Y \lrcorner$ (cf. [KM, Definition 4.6]), or $\llcorner \Delta_Y \lrcorner$ is a simple normal crossing divisor consisting of rational curves and the dual graph is a tree. In the former case, we have $K_{\Delta_Y} \sim 0$. So, $n = 1$ is sufficient. In the latter case, since $H^1(\mathcal{O}_{\llcorner \Delta_Y \lrcorner}) = 0$, $n(K_Y + \Delta_Y)|_{\llcorner \Delta_Y \lrcorner} \sim 0$ if we choose $n > 0$ such that $n(K_Y + \Delta_Y)$ is a numerically trivial Cartier divisor (cf. [KM, Theorem 4.13]). $\square$

We give an important remark on rational singularities.

Remark 3.5. Let X be an algebraic surface. If X has only rational singularities, then it is well known that X is $\mathbb{Q}$-factorial. Therefore, we can apply the log minimal model program in Theorem 3.3 for pairs of surfaces with only rational singularities and boundary $\mathbb{R}$-divisors on them. We note that there are many two-dimensional rational singularities which are not lc.

We take a rational non-lc surface singularity $P \in X$. Let $\pi : Z \to X$ be the index one cover of X. In this case, Z is not log canonical nor rational.

We note that our log minimal model program works inside the class of surfaces with only rational singularities by the next proposition. It is very similar to [KM, Proposition 2.71]. It is mysterious that [KM, Proposition 2.71] is also missing in the English edition of [KM].

Proposition 3.6. *Let (X, Δ) be a log surface and $f : X \to Y$ a projective surjective morphism onto a normal surface Y. Assume that $-(K_X + \Delta)$ is f-ample. Then $R^i f_* \mathcal{O}_X = 0$ for every $i > 0$. Therefore, if X has only rational singularities, then Y also has only rational singularities.*

Proof. We consider the short exact sequence

$$0 \to \mathcal{J}_{NLC}(X, \Delta) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}_{NLC}(X, \Delta) \to 0,$$

where $\mathcal{J}_{NLC}(X, \Delta)$ is the non-lc ideal sheaf associated to the pair (X, Δ). By the vanishing theorem (cf. [F3, Theorem 8.1]), we know $R^i f_* \mathcal{J}_{NLC}(X, \Delta) = 0$ for every $i > 0$. Since Δ is a boundary $\mathbb{R}$-divisor, we have $\dim_{\mathbb{C}} \operatorname{Supp}(\mathcal{O}_X/\mathcal{J}_{NLC}(X, \Delta)) = 0$. So, we obtain $R^i f_*(\mathcal{O}_X/\mathcal{J}_{NLC}(X, \Delta)) = 0$ for every $i > 0$. Thus, $R^i f_* \mathcal{O}_X = 0$ for all $i > 0$. □

As a corollary, we can check the following result.

Proposition 3.7 (Extremal rational curves)**.** *Let (X, Δ) be a log surface and $\pi : X \to S$ a projective surjective morphism onto a variety S. Let R be a $(K_X + \Delta)$-negative extremal ray. Then R is spanned by a rational curve C on X such that $-(K_X + \Delta) \cdot C \leq 3$. Moreover, if $X \not\simeq \mathbb{P}^2$, then we can choose C with $-(K_X + \Delta) \cdot C \leq 2$.*

Proof. We consider the extremal contraction $\varphi_R : X \to Y$ over S associated to R. Let $f : Z \to X$ be the minimal resolution such that $K_Z + \Delta_Z = f^*(K_X + \Delta)$. Note that Δ_Z is effective. First, we assume that Y is a point. Let D be a general curve on Z. Then $D \cdot (K_Z + \Delta_Z) = D \cdot f^*(K_X + \Delta) < 0$. Therefore, $\kappa(Z, K_Z) = -\infty$. If $X \simeq \mathbb{P}^2$, then the statement is obvious. So, we can assume that $X \not\simeq \mathbb{P}^2$. In this case, there exists a morphism $g : Z \to B$ onto a smooth curve B. Let D be a general fiber of g. Then $D \simeq \mathbb{P}^1$ and $-(K_Z + \Delta_Z) \cdot D = -f^*(K_X + \Delta) \cdot D \leq 2$. Thus, $C = f(D) \subset X$ has the desired properties. Next, we assume that Y is a curve. In this case, we take a general fiber of $\varphi_R \circ f : Z \to X \to Y$. Then, it gives a desired curve as in the previous case. Finally, we assume that $\varphi_R : X \to Y$ is birational. Let E be an irreducible component of the exceptional locus of φ_R. We consider the short exact sequence

$$0 \to \mathcal{I}_E \to \mathcal{O}_X \to \mathcal{O}_E \to 0,$$

where $\mathcal{I}_E$ is the defining ideal sheaf of E on X. By Proposition 3.6, $R^1 \varphi_{R*} \mathcal{O}_X = 0$. Therefore, $R^1 \varphi_{R*} \mathcal{O}_E = H^1(E, \mathcal{O}_E) = 0$. Thus, $E \simeq \mathbb{P}^1$. Let F be the strict transform of E on Z. Then the coefficient of F in Δ_Z is ≤ 1 and $F^2 < 0$. Therefore, $-f^*(K_X + \Delta) \cdot F = -(K_Z + \Delta_Z) \cdot F \leq 2$. This means that $-(K_X + \Delta) \cdot E \leq 2$ and E spans R. □

We note the following easy result.

Proposition 3.8 (Uniqueness)**.** *Let (X, Δ) be a log surface and $\pi : X \to S$ a projective morphism onto a variety S as in* Theorem 3.3.

Let (X^*, Δ^*) and $(X^\dagger, \Delta^\dagger)$ be minimal models of (X, Δ) over S. Then $(X^*, \Delta^*) \simeq (X^\dagger, \Delta^\dagger)$ over S.

Proof. We consider
$$K_X + \Delta = f^*(K_{X^*} + \Delta^*) + E,$$
and
$$K_X + \Delta = g^*(K_{X^\dagger} + \Delta^\dagger) + F,$$
where $f : X \to X^*$ and $g : X \to X^\dagger$. We note that $\mathrm{Supp}E = \mathrm{Exc}(f)$ and $\mathrm{Supp}F = \mathrm{Exc}(g)$. By the negativity lemma, we obtain $E = F$. Therefore, $(X^*, \Delta^*) \simeq (X^\dagger, \Delta^\dagger)$ over S. □

We close this section with a remark on the *Zariski decomposition*.

Remark 3.9. Let (X, Δ) be a projective log surface such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and pseudo-effective. Assume that (X, Δ) is log canonical or X is $\mathbb{Q}$-factorial. Then there exists the unique minimal model (X^*, Δ^*) of (X, Δ) by Theorem 3.3 and Proposition 3.8. Let $f : X \to X^*$ be the natural morphism. Then we can write
$$K_X + \Delta = f^*(K_{X^*} + \Delta^*) + E,$$
where E is an effective $\mathbb{Q}$-divisor such that $\mathrm{Supp}E = \mathrm{Exc}(f)$. It is easy to see that $f^*(K_{X^*} + \Delta^*)$ (resp. E) is the semi-positive (resp. negative) part of the Zariski decomposition of $K_X + \Delta$. By Theorem 6.1 below, the semi-positive part $f^*(K_{X^*} + \Delta^*)$ of the Zariski decomposition of $K_X + \Delta$ is semi-ample.

4. Finite generation of log canonical rings

In this section, we prove that the log canonical ring of a $\mathbb{Q}$-factorial projective log surface is finitely generated.

First, we prove a special case of the log abundance conjecture for log surfaces. Our proof heavily depends on the Kodaira type vanishing theorem.

Theorem 4.1 (Semi-ampleness)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial projective log surface. Assume that $K_X + \Delta$ is nef and big and that Δ is a $\mathbb{Q}$-divisor. Then $K_X + \Delta$ is semi-ample.*

Proof. We divide the proof into several steps.

Step 0. Let $\llcorner \Delta \lrcorner = \sum_i C_i$ be the irreducible decomposition. We put
$$A = \sum_{C_i \cdot (K_X + \Delta) = 0} C_i \quad \text{and} \quad B = \sum_{C_i \cdot (K_X + \Delta) > 0} C_i.$$

Then $\llcorner \Delta \lrcorner = A + B$. We note that $(C_i)^2 < 0$ if $C_i \cdot (K_X + \Delta) = 0$ by the Hodge index theorem. We can decompose A into the connected components as follows:

$$A = \sum_j A_j.$$

First, let us recall the following well-known easy result. Strictly speaking, Step 1 is redundant by more sophisticated arguments in Step 5 and Step 6.

Step 1. Let P be an isolated point of $\mathrm{Nklt}(X, \Delta)$. Then $P \notin \mathrm{Bs}|n(K_X + \Delta)|$, where n is a divisible positive integer.

Proof of Step 1. Let $\mathcal{J}(X, \Delta)$ be the multiplier ideal sheaf associated to (X, Δ). Then we have

$$H^i(X, \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathcal{J}(X, \Delta)) = 0$$

for every $i > 0$ by the Kawamata–Viehweg–Nadel vanishing theorem (cf. 2.7). Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(n(K_X + \Delta))) \to H^0(X, \mathcal{O}_X(n(K_X + \Delta))/\mathcal{J}(X, \Delta))$$

is surjective. By assumption, the evaluation map

$$H^0(X, \mathcal{O}_X(n(K_X + \Delta))) \to \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathbb{C}(P)$$

at P is surjective. This implies that $P \notin \mathrm{Bs}|n(K_X + \Delta)|$. $\square$

Next, we will check that $\mathrm{Bs}|n(K_X + \Delta)|$ contains no non-klt centers for a divisible positive integer n from Step 2 to Step 7 (cf. [F3, Theorem 12.1] and [F5, Theorem 1.1]).

Step 2. We consider A_j such that $\mathrm{Nlc}(X, \Delta) \cap A_j \neq \emptyset$. Let $A_j = \sum_i D_i$ be the irreducible decomposition. We can easily check that D_i is rational for every i and that there exists a point $P \in \mathrm{Nlc}(X, \Delta)$ such that $P \in D_i$ for every i by calculating differents (see, for example, [F3, Section 14]). We can also see that $D_k \cap D_l = P$ for $k \neq l$ and that D_i is smooth outside P for every i. If $D_i \cap (\Delta - D_i) \neq \emptyset$, then D_i spans a $(K_X + D_i)$-negative extremal ray. So, we can contract D_i in order to prove the semi-ampleness of $K_X + \Delta$. We note that $(K_X + \Delta) \cdot D_i = 0$. Therefore, by replacing X with its contraction, we can assume that A_j is irreducible. We can further assume that A_j is isolated in $\mathrm{Supp}\Delta$. It is because we can contract A_j if A_j is not isolated in $\mathrm{Supp}\Delta$.

If A_j is $\mathbb{P}^1$, then it is easy to see that $\mathcal{O}_{A_j}(n(K_X + \Delta)) \simeq \mathcal{O}_{A_j}$ since $A_j \cdot (K_X + \Delta) = 0$.

If A_j is singular, then we obtain $H^1(A_j, \mathcal{O}_{A_j}) \neq 0$. Therefore, by Serre duality, we obtain $H^0(A_j, \omega_{A_j}) \neq 0$, where ω_{A_j} is the dualizing sheaf of A_j. We note that

$$0 \to \mathcal{T} \to \mathcal{O}_X(K_X + A_j) \otimes \mathcal{O}_{A_j} \to \omega_{A_j} \to 0$$

is exact, where $\mathcal{T}$ is the torsion part of $\mathcal{O}_X(K_X + A_j) \otimes \mathcal{O}_{A_j}$. See Lemma 4.2 below. Since A_j is a curve, $\mathcal{T}$ is a skyscraper sheaf on A_j. So, $H^0(A_j, \omega_{A_j}) \neq 0$ implies

$$\mathrm{Hom}(\mathcal{O}_{A_j}, \mathcal{O}_X(K_X + A_j) \otimes \mathcal{O}_{A_j}) \simeq H^0(A_j, \mathcal{O}_X(K_X + A_j) \otimes \mathcal{O}_{A_j}) \neq 0.$$

Therefore, we obtain an inclusion map

$$\mathcal{O}_{A_j} \to \mathcal{O}_X(n(K_X + A_j)) \otimes \mathcal{O}_{A_j} \simeq \mathcal{O}_{A_j}(n(K_X + \Delta))$$

for a divisible positive integer n. Since $A_j \cdot (K_X + \Delta) = 0$, we see that $\mathcal{O}_{A_j}(n(K_X + \Delta)) \simeq \mathcal{O}_{A_j}$.

Step 3. If $\mathrm{Nlc}(X, \Delta) \cap A_j = \emptyset$, then $\mathcal{O}_{A_j}(n(K_X + \Delta)) \simeq \mathcal{O}_{A_j}$ for some divisible positive integer n by the abundance theorem for semi log canonical curves (cf. [F1]).

Anyway, we obtain $\mathcal{O}_A(n(K_X + \Delta)) \simeq \mathcal{O}_A$ for a divisible positive integer n.

Step 4. We have $A \cap \mathrm{Bs}|n(K_X + \Delta)| = \emptyset$.

Proof of Step 4. Let $f : Y \to X$ be a resolution such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$. We can assume that

(1) $f^{-1}(A)$ has a simple normal crossing support, and
(2) $\mathrm{Supp} f_*^{-1}\Delta \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor on Y.

Let W_1 be the union of the irreducible components of $\Delta_Y^{=1}$ which are mapped into A by f. We write $\Delta_Y^{=1} = W_1 + W_2$. Then

$$-W_1 - \llcorner \Delta_Y^{>1} \lrcorner + \ulcorner -(\Delta_Y^{<1}) \urcorner - (K_Y + \{\Delta_Y\} + W_2) \sim_{\mathbb{Q}} -f^*(K_X + \Delta).$$

We put

$$\mathcal{J}_1 = f_*\mathcal{O}_Y(-W_1 - \llcorner \Delta_Y^{>1} \lrcorner + \ulcorner -(\Delta_Y^{<1}) \urcorner) \subset \mathcal{O}_X.$$

Then we can easily check that

$$0 \to \mathcal{J}_1 \to \mathcal{O}_X(-A) \to \delta \to 0$$

is exact, where δ is a skyscraper sheaf, and

$$H^i(X, \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathcal{J}_1) = 0$$

for every $i > 0$ by 2.7, where n is a divisible positive integer. By the above exact sequence, we obtain

$$H^i(X, \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathcal{O}_X(-A)) = 0$$

for $i > 0$. By this vanishing theorem, we see that the restriction map

$$H^0(X, \mathcal{O}_X(n(K_X + \Delta))) \to H^0(A, \mathcal{O}_A(n(K_X + \Delta)))$$

is surjective. Since $\mathcal{O}_A(n(K_X+\Delta)) \simeq \mathcal{O}_A$, we have $\mathrm{Bs}|n(K_X+\Delta)| \cap A = \emptyset$. $\square$

Step 5. Let P be a zero-dimensional lc center of (X, Δ). Then $P \notin \mathrm{Bs}|n(K_X + \Delta)|$, where n is a divisible positive integer.

Proof of Step 5. If $P \in A$, then it is obvious by Step 4. So, we can assume that $P \cap \mathrm{Supp} A = \emptyset$. Let $f : Y \to X$ be the resolution as in the proof of Step 4. We can further assume that

(3) $f^{-1}(P)$ has a simple normal crossing support.

Let W_3 be the union of the irreducible components of $\Delta_Y^{=1}$ which are mapped into $A \cup P$ by f. We put $\Delta_Y^{=1} = W_3 + W_4$. Then

$$-W_3 - \llcorner \Delta_Y^{>1} \lrcorner + \ulcorner -(\Delta_Y^{<1}) \urcorner - (K_Y + \{\Delta_Y\} + W_4) \sim_{\mathbb{Q}} -f^*(K_X + \Delta).$$

We put

$$\mathcal{J}_2 = f_*\mathcal{O}_Y(-W_3 - \llcorner \Delta_Y^{>1} \lrcorner + \ulcorner -(\Delta_Y^{<1}) \urcorner) \subset \mathcal{O}_X.$$

Then, we have

$$H^i(X, \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathcal{J}_2) = 0$$

for every $i > 0$ by 2.7, where n is a divisible positive integer. Thus, the restriction map

$$H^0(X, \mathcal{O}_X(n(K_X + \Delta))) \to H^0(X, \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathcal{O}_X/\mathcal{J}_2)$$

is surjective. Therefore, the evaluation map

$$H^0(X, \mathcal{O}_X(n(K_X + \Delta))) \to \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathbb{C}(P)$$

is surjective since $P \cap \mathrm{Supp} A = \emptyset$. So, we have $P \notin \mathrm{Bs}|n(K_X+\Delta)|$. $\square$

Step 6. Let $P \in \mathrm{Nlc}(X, \Delta)$. Then $P \notin \mathrm{Bs}|n(K_X + \Delta)|$.

Proof of Step 6. If $P \in A$, then it is obvious by Step 4. So, we can assume that $P \cap \mathrm{Supp} A = \emptyset$. By the proof of Step 4, we obtain that the restriction map

$$H^0(X, \mathcal{O}_X(n(K_X + \Delta))) \to H^0(X, \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathcal{O}_X/\mathcal{J}_1)$$

is surjective. Since $P \cap \mathrm{Supp} A = \emptyset$, we see that the evaluation map

$$H^0(X, \mathcal{O}_X(n(K_X + \Delta))) \to \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathbb{C}(P)$$

is surjective. So, we have $P \notin \mathrm{Bs}|n(K_X + \Delta)|$. $\square$

Step 7. We see that $E_i \not\subset \mathrm{Bs}|n(K_X + \Delta)|$, where E_i is any irreducible component of B and n is a divisible positive integer.

Proof of Step 7. We can assume that $E_i \cap A = \emptyset$ by Step 4 and (X, Δ) is log canonical in a neighborhood of E_i by Step 6. We note that $\mathcal{O}_{E_i}(n(K_X + \Delta))$ is ample. So, $\mathcal{O}_{E_i}(n(K_X + \Delta))$ is generated by global sections. Let $f : Y \to X$ be the resolution as in the proof of Step 4. We can further assume that

(4) $f^{-1}(E_i)$ has a simple normal crossing support.

Let W_5 be the union of the irreducible components of $\Delta_Y^{=1}$ which are mapped into $A \coprod E_i$ by f. We put $\Delta_Y^{=1} = W_5 + W_6$. Then

$$-W_5 - \llcorner \Delta_Y^{>1} \lrcorner + \ulcorner -(\Delta_Y^{<1}) \urcorner - (K_Y + \{\Delta_Y\} + W_6) \sim_{\mathbb{Q}} -f^*(K_X + \Delta).$$

We put

$$\mathcal{J}_3 = f_*\mathcal{O}_Y(-W_5 - \llcorner \Delta_Y^{>1} \lrcorner + \ulcorner -(\Delta_Y^{<1}) \urcorner) \subset \mathcal{O}_X.$$

Then, we have

$$H^i(X, \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathcal{J}_3) = 0$$

for every $i > 0$, where n is a divisible positive integer. We note that there exists a short exact sequence

$$0 \to \mathcal{J}_3 \to \mathcal{O}_X(-A - E_i) \to \delta' \to 0,$$

where δ' is a skyscraper sheaf on X. Thus,

$$H^i(X, \mathcal{O}_X(n(K_X + \Delta)) \otimes \mathcal{O}_X(-A - E_i)) = 0$$

for every $i > 0$, Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(n(K_X + \Delta))) \to H^0(E_i, \mathcal{O}_{E_i}(n(K_X + \Delta)))$$

is surjective since $\mathrm{Supp} E_i \cap \mathrm{Supp} A = \emptyset$.

This implies that $E_i \not\subset \mathrm{Bs}|n(K_X + \Delta)|$ for every irreducible component E_i of B. $\square$

Therefore, we have checked that $\mathrm{Bs}|n(K_X + \Delta)|$ contains no non-klt centers of (X, Δ).

Finally, we will prove that $K_X + \Delta$ is semi-ample.

Step 8. If $|n(K_X + \Delta)|$ is free, then there are nothing to prove. So, we assume that $\mathrm{Bs}|n(K_X + \Delta)| \neq \emptyset$. We take general members $\Xi_1, \Xi_2, \Xi_3 \in |n(K_X + \Delta)|$ and put $\Theta = \Xi_1 + \Xi_2 + \Xi_3$. Then Θ contains no non-klt centers of (X, Δ) and $K_X + \Delta + \Theta$ is not lc at the generic point of any irreducible component of $\mathrm{Bs}|n(K_X + \Delta)|$ (see, for example, [F3, Lemma 13.2]). We put

$$c = \max\{t \in \mathbb{R} \mid K_X + \Delta + t\Theta \text{ is lc outside } \mathrm{Nlc}(X, \Delta)\}.$$

Then we can easily check that $c \in \mathbb{Q}$ and $0 < c < 1$. In this case,

$$K_X + \Delta + c\Theta \sim_{\mathbb{Q}} (1 + cn)(K_X + \Delta)$$

and there exists an lc center $\mathcal{C}$ of $(X, \Delta + c\Theta)$ contained in $\mathrm{Bs}|(n(K_X + \Delta)|$. We take positive integer l and m such that

$$l(K_X + \Delta + c\Theta) \sim mn(K_X + \Delta).$$

Replace $n(K_X + \Delta)$ with $l(K_X + \Delta + c\Theta)$ and apply the previous arguments. Then, we obtain $\mathcal{C} \not\subset \mathrm{Bs}|kl(K_X + \Delta + c\Theta)|$ for some positive integer k. Therefore, we have

$$\mathrm{Bs}|kmn(K_X + \Delta)| \subsetneq \mathrm{Bs}|n(K_X + \Delta)|.$$

It is because there is an lc center $\mathcal{C}$ of $(X, \Delta + c\Theta)$ such that $\mathcal{C} \subset \mathrm{Bs}|n(K_X + \Delta)|$, and $l(K_X + \Delta + c\Theta) \sim mn(K_X + \Delta)$. By noetherian induction, we obtain that $(K_X + \Delta)$ is semi-ample.

We finish the proof of Theorem 4.1. □

We used the following lemma in the proof of Theorem 4.1.

Lemma 4.2 (Adjunction)**.** *Let X be a normal projective surface and D a pure one-dimensional reduced irreducible closed subscheme. Then we have the following short exact sequence*:

$$0 \to \mathcal{T} \to \omega_X(D) \otimes \mathcal{O}_D \to \omega_D \to 0,$$

where $\mathcal{T}$ is the torsion part of $\omega_X(D) \otimes \mathcal{O}_D$. In particular, $\mathcal{T}$ is a skyscraper sheaf on D.

Proof. We consider the following short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

By tensoring $\omega_X(D)$, where $\omega_X(D) = (\omega_X \otimes \mathcal{O}_X(D))^{**}$, we obtain

$$\omega_X(D) \otimes \mathcal{O}_X(-D) \to \omega_X \to \omega_X(D) \otimes \mathcal{O}_D \to 0.$$

On the other hand, by taking $\mathcal{E}xt^i_{\mathcal{O}_X}(\underline{\quad}, \omega_X)$, we obtain

$$0 \to \omega_X \to \omega_X(D) \to \omega_D \simeq \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_D, \omega_X) \to 0.$$

Note that $\omega_X(D) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-D), \omega_X)$. The natural homomorphism

$$\alpha : \omega_X(D) \otimes \mathcal{O}_X(-D) \to \omega_X \simeq (\omega_X(D) \otimes \mathcal{O}_X(-D))^{**}$$

induces the following commutative diagram.

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathcal{T} \arrow[d] & \\
 & \omega_X(D)\otimes\mathcal{O}_X(-D) \arrow[r] \arrow[d, "\alpha"] & \omega_X(D) \arrow[r] \arrow[d, equal] & \omega_X(D)\otimes\mathcal{O}_D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_X \arrow[r] \arrow[d] & \omega_X(D) \arrow[r] & \omega_D \arrow[r] \arrow[d] & 0 \\
 & \mathcal{T} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}
$$

It is easy to see that $\mathcal{T}$ is the torsion part of $\omega_X(D)\otimes\mathcal{O}_D$ and α is surjective in codimension one. $\square$

The next theorem is a generalization of Fujita's result in [Ft].

Theorem 4.3 (Finite generation of log canonical rings)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial projective log surface such that Δ is a $\mathbb{Q}$-divisor. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X+\Delta)\lrcorner))$$

is a finitely generated $\mathbb{C}$-algebra.

Proof. Without loss of generality, we can assume that $\kappa(X, K_X+\Delta)\geq 0$. By Theorem 3.3, we can further assume that $K_X+\Delta$ is nef. If $K_X+\Delta$ is big, then $K_X+\Delta$ is semi-ample by Theorem 4.1. Therefore, $R(X,\Delta)$ is finitely generated. If $\kappa(X, K_X+\Delta)=1$, then we can easily check that $K_X+\Delta$ is semi-ample (cf. [Ft, (4.1) Theorem]). So, $R(X,\Delta)$ is finitely generated. If $\kappa(X, K_X+\Delta)=0$, then it is obvious that $R(X,\Delta)$ is finitely generated. $\square$

As a corollary, we obtain the finite generation of canonical rings for projective surfaces with only rational singularities.

Corollary 4.4. *Let X be a projective surface with only rational singularities. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

Remark 4.5. In Theorems 4.1 and 4.3, the assumption that Δ is a boundary $\mathbb{Q}$-divisor is crucial. By Zariski's example, we can easily construct a smooth projective surface X and an effective $\mathbb{Q}$-divisor Δ on X such that $\mathrm{Supp}\Delta$ is simple normal crossing, $K_X + \Delta$ is nef and big, and

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner))$$

is not a finitely generated $\mathbb{C}$-algebra. Of course, $K_X + \Delta$ is not semi-ample. See, for example, [L, 2.3.A Zariski's Construction].

5. Non-vanishing theorem

In this section, we prove the following non-vanishing theorem.

Theorem 5.1 (Non-vanishing theorem)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial projective log surface such that Δ is a $\mathbb{Q}$-divisor. Assume that $K_X + \Delta$ is pseudo-effective. Then $\kappa(X, K_X + \Delta) \geq 0$.*

Proof. By Theorem 3.3, we can assume that $K_X + \Delta$ is nef. Let $f : Y \to X$ be the minimal resolution. We put $K_Y + \Delta_Y = f^*(K_X + \Delta)$. We note that Δ_Y is effective. If $\kappa(Y, K_Y) \geq 0$, then it is obvious that

$$\kappa(X, K_X + \Delta) = \kappa(Y, K_Y + \Delta_Y) \geq \kappa(Y, K_Y) \geq 0.$$

So, from now on, we assume $\kappa(Y, K_Y) = -\infty$. When Y is rational, we can easily check $\kappa(Y, K_Y + \Delta_Y) \geq 0$ by the Riemann–Roch formula (see, for example, the proof of [FM, 11.2.1 Lemma]). Therefore, we can assume that Y is an irrational ruled surface. Let $p : Y \to C$ be the Albanese fibration. We can write $K_Y + \Delta_Y = K_Y + \Delta_1 + \Delta_2$, where Δ_1 is an effective $\mathbb{Q}$-divisor on Y such that Δ_1 has no vertical components with respect to p, $0 \leq \Delta_1 \leq \Delta_Y$, $(K_Y + \Delta_1) \cdot F = 0$ for any general fiber F of p, and $\Delta_2 = \Delta_Y - \Delta_1 \geq 0$. When we prove $\kappa(Y, K_Y + \Delta_Y) \geq 0$, we can replace Δ_Y with Δ_1 because $\kappa(Y, K_Y + \Delta_Y) \geq \kappa(Y, K_Y + \Delta_1)$. Therefore, we can assume that $\Delta_Y = \Delta_1$. By taking blow-ups, we can further assume that $\mathrm{Supp}\Delta_Y$ is smooth. We note the following easy but important lemma.

Lemma 5.2. *Let B be any smooth irreducible curve on Y such that $p(B) = C$. Then B is not f-exceptional.*

Proof of Lemma 5.2. Let $\{E_i\}_{i\in I}$ be the set of all f-exceptional divisors. We consider the subgroup G of $\mathrm{Pic}(B)$ generated by $\{\mathcal{O}_B(E_i)\}_{i\in I}$. Let $\mathcal{L}=\mathcal{O}_C(D)$ be a sufficiently general member of $\mathrm{Pic}^0(C)$. We note that the genus $g(C)$ of C is positive. Then

$$(p|_B)^*\mathcal{L}\in \mathrm{Pic}^0(B)\otimes_{\mathbb{Z}}\mathbb{Q}\setminus G\otimes_{\mathbb{Z}}\mathbb{Q}.$$

Assume that B is f-exceptional. We consider $E=p^*D$ on Y. Since X is $\mathbb{Q}$-factorial,

$$E\sim_{\mathbb{Q}} f^*f_*E+\sum_{i\in I}a_iE_i$$

with $a_i\in\mathbb{Q}$ for every i. By restricting the above relation to C, we obtain $(p|_B)^*\mathcal{L}\in G\otimes_{\mathbb{Z}}\mathbb{Q}$. It is a contradiction. Therefore, B is not f-exceptional. □

Thus, every irreducible component B of Δ_Y is not f-exceptional. So, its coefficient in Δ_Y is not greater than one because Δ is a boundary $\mathbb{Q}$-divisor. By applying [Ft, (2.2) Theorem], we obtain that $\kappa(Y,K_Y+\Delta_Y)\geq 0$. We finish the proof. □

6. Abundance theorem for log surfaces

In this section, we prove the log abundance theorem for $\mathbb{Q}$-factorial projective log surfaces.

Theorem 6.1 (Abundance theorem)**.** *Let (X,Δ) be a $\mathbb{Q}$-factorial projective log surface such that Δ is a $\mathbb{Q}$-divisor. Assume that $K_X+\Delta$ is nef. Then $K_X+\Delta$ is semi-ample.*

Proof. By Theorem 5.1, we have $\kappa(X,K_X+\Delta)\geq 0$. If $\kappa(X,K_X+\Delta)=2$, then $K_X+\Delta$ is semi-ample by Theorem 4.1. If $\kappa(X,K_X+\Delta)=1$, then $\kappa(X,K_X+\Delta)=\nu(X,K_X+\Delta)=1$ and we can easily check that $K_X+\Delta$ is semi-ample (cf. [Ft, (4.1) Theorem]). Therefore, all we have to do is to prove $K_X+\Delta\sim_{\mathbb{Q}}0$ when $\kappa(X,K_X+\Delta)=0$. It is Theorem 6.2 below. □

The proof of the following theorem depends on the argument in [Ft, §5. The case $\kappa=0$] and Sakai's classification result in [S1].

Theorem 6.2. *Let (X,Δ) be a $\mathbb{Q}$-factorial projective log surface such that Δ is a $\mathbb{Q}$-divisor. Assume that $K_X+\Delta$ is nef and $\kappa(X,K_X+\Delta)=0$. Then $K_X+\Delta\sim_{\mathbb{Q}}0$.*

Proof. Let $f:V\to X$ be the minimal resolution. We put $K_V+\Delta_V=f^*(K_X+\Delta)$. We note that Δ_V is effective. It is sufficient to see that $K_V+\Delta_V\sim_{\mathbb{Q}}0$. Let

$$\varphi: V=:V_0\xrightarrow{\varphi_0}V_1\xrightarrow{\varphi_1}\cdots\xrightarrow{\varphi_{k-1}}V_k=:S$$

be a sequence of blow-downs such that

(1) φ_i is a blow-down of a (-1)-curve C_i on V_i,
(2) $\Delta_{V_{i+1}} = \varphi_{i*}\Delta_{V_i}$, and
(3) $(K_{V_i} + \Delta_{V_i}) \cdot C_i = 0$,

for every i. We can assume that there are no (-1)-curves C on S with $(K_S + \Delta_S) \cdot C = 0$. We note that $K_V + \Delta_V = \varphi^*(K_S + \Delta_S)$. It is sufficient to see that $K_S + \Delta_S \sim_{\mathbb{Q}} 0$. By assumption, there is a member Z of $|m(K_S + \Delta_S)|$ for some divisible positive integer m. Then, for every positive integer t, tZ is the unique member of $|tm(K_S + \Delta_S)|$. We can easily check the following lemma. See, for example, [Ft, (5.4)].

Lemma 6.3 (cf. [Ft, (5.5) Lemma])**.** *Let $Z = \sum_i \xi_i Z_i$ be the prime decomposition of Z. Then $K_S \cdot Z_i = \Delta_S \cdot Z_i = Z \cdot Z_i$ for every i.*

We will derive a contradiction assuming $Z \neq 0$, equivalently, $\nu(S, K_S + \Delta_S) = 1$. We can decompose Z into the connected components as follows:

$$Z = \sum_{i=1}^{r} \mu_i Y_i,$$

where $\mu_i Y_i$ is a connected component of Z such that μ_i is the greatest common divisor of the coefficients of prime components of Y_i in Z for every i, and $\mu_i Y_i \neq \mu_j Y_j$ for $i \neq j$. Then we obtain $\omega_{Y_i} \simeq \mathcal{O}_{Y_i}$ for every i. It is because Y_i is *indecomposable of canonical type* in the sense of Mumford by Lemma 6.3 (see, for example, [Ft, (5.6)]).

Step 1 (cf. [Ft, (5.7)])**.** We assume that $\kappa(S, K_S) \geq 0$. Since $0 \leq \kappa(S, K_S) \leq \kappa(S, K_S + \Delta_S) = 0$, we obtain $\kappa(S, K_S) = 0$. If S is not minimal, then we can find a (-1)-curve E on S such that $E \cdot (K_S + \Delta_S) = 0$. Therefore, S is minimal by the construction of (S, Δ_S). We show $\kappa(S, K_S + \Delta_S) = \kappa(S, Z) \geq 1$ in order to get a contradiction. By taking an étale cover, we can assume that S is an Abelian surface or a $K3$ surface. In this case, it is easy to see that $\kappa(S, K_S + \Delta_S) = \kappa(S, Z) \geq 1$ since $Z \neq 0$.

From now on, we assume that $\kappa(S, K_S) = -\infty$.

Step 2. We further assume that $H^1(S, \mathcal{O}_S) = 0$. If $n(S, K_S + \Delta_S) = 1$, then there is a surjective morphism $g : S \to T$ onto a smooth projective curve T and a nef $\mathbb{Q}$-divisor $A \neq 0$ on T such that $K_S + \Delta_S \equiv g^*A$ (cf. [8, Proposition 2.11]). Here, g is the reduction map associated to $K_S + \Delta_S$. Since $H^1(S, \mathcal{O}_S) = 0$, we obtain $K_S + \Delta_S \sim_{\mathbb{Q}} g^*A$. Therefore, $\kappa(S, K_S + \Delta_S) = 1$. It is a contradiction.

Step 3. We assume that $n(S, K_S+\Delta_S)=2$. By [S1, Proposition 4], we know $r=1$, that is, $Z=\mu_1 Y_1$. In this case, S is a degenerate del Pezzo surface, that is, nine times blow-ups of $\mathbb{P}^2$, and $Z\in|-nK_S|$ for some positive integer n (cf. [S1, Proposition 5]). Since $\kappa(S,-K_S)=0$ and $m(K_S+\Delta_S)\sim Z\sim -nK_S$, we obtain $m\Delta_S=(m+n)D$, where D is the unique member of $|-K_S|$. Thus,

$$\Delta_S=\frac{m+n}{m}D \quad \text{and} \quad Z=nD.$$

In particular, we obtain $\Delta_S=\Delta_S^{\geq 1}$. We will see that $\mathcal{O}_D(aD)\simeq\mathcal{O}_D$ for some positive integer a in Step 4. This implies that the normal bundle $\mathcal{N}_D=\mathcal{O}_D(D)$ is a torsion. It is a contradiction by [S1, Proposition 5].

Step 4. In this step, we will prove that $\mathcal{O}_D(aD)\simeq\mathcal{O}_D$ for some positive integer a. We put $D_k=D$ and construct D_i inductively. It is easy to see that $\varphi_i: V_i\to V_{i+1}$ is the blow-up at P_{i+1} with $\mathrm{mult}_{P_{i+1}}\Delta_{V_{i+1}}\geq 1$ for every i by calculating discrepancy coefficients since Δ_{V_i} is effective. If $\mathrm{mult}_{P_{i+1}}D_{i+1}=0$, then we put $D_i=\varphi_{i+1}^*D_{i+1}$. If $\mathrm{mult}_{P_{i+1}}D_{i+1}>0$, then we put $D_i=\varphi^*_{i+1}D_{i+1}-C_i$, where C_i is the exceptional curve of φ_i. We note that $\mathrm{mult}_P\Delta_{V_{i+1}}>\mathrm{mult}_PD_{i+1}$ for every $P\in V_{i+1}$ and $\mathrm{mult}_PD_{i+1}\in\mathbb{Z}$. Thus, we obtain D_0 on $V_0=V$. We can see that D_0 is effective and $\mathrm{Supp}D_0\subset\mathrm{Supp}\Delta_V^{\geq 1}$ by the above construction. We note that $\varphi_{i*}\mathcal{O}_{D_i}\simeq\mathcal{O}_{D_{i+1}}$ for every i. It is because $\varphi_{i*}\mathcal{O}_{V_i}(-D_i)\simeq\mathcal{O}_{V_{i+1}}(-D_{i+1})$ and $R^1\varphi_{i*}\mathcal{O}_{V_i}(-D_i)=0$ for every i. See the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{V_{i+1}}(-D_{i+1}) \arrow[r] \arrow[d, "\simeq"] & \mathcal{O}_{V_{i+1}} \arrow[r] \arrow[d, "\simeq"] & \mathcal{O}_{D_{i+1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varphi_{i*}\mathcal{O}_{V_i}(-D_i) \arrow[r] & \varphi_{i*}\mathcal{O}_{V_i} \arrow[r] & \varphi_{i*}\mathcal{O}_{D_i} \arrow[r] & R^1\varphi_{i*}\mathcal{O}_{V_i}(-D_i)=0
\end{tikzcd}$$

Therefore, we obtain $\varphi_*\mathcal{O}_{D_0}\simeq\mathcal{O}_D$. Since $\mathrm{Supp}D_0\subset\mathrm{Supp}\Delta_V^{\geq 1}$, we see that D_0 is f-exceptional. Since $K_V+\Delta_V=f^*(K_X+\Delta)$, we obtain $\mathcal{O}_{D_0}(b(K_V+\Delta_V))\simeq\mathcal{O}_{D_0}$ for some positive divisible integer b. Thus,

$$\mathcal{O}_D(b(K_S+\Delta_S))\simeq\varphi_*\mathcal{O}_{D_0}(b(K_V+\Delta_V))\simeq\mathcal{O}_D.$$

In particular, $\mathcal{O}_D(aD)\simeq\mathcal{O}_D$ for some positive integer a. It is because

$$b(K_S+\Delta_S)\sim\frac{bn}{m}D.$$

Step 5. Finally, we assume that S is an irrational ruled surface. Let $\alpha: S\to B$ be the Albanese fibration. In this case, every irreducible component of $\mathrm{Supp}\Delta_S^{\geq 1}$ is vertical with respect to α (cf. Lemma 5.2).

Therefore, [Ft, (5.9)] works without any changes. Thus, we get a contradiction.

We finish the proof of Theorem 6.2. □

We close this section with the following corollary.

Corollary 6.4 (Abundance theorem for log canonical surfaces)**.** *Let (X, Δ) be a complete log canonical surface such that Δ is a $\mathbb{Q}$-divisor. Assume that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semi-ample.*

Proof. Let $f : V \to X$ be the minimal resolution. We put $K_V + \Delta_V = f^*(K_X + \Delta)$. Since (X, Δ) is log canonical, Δ_V is a boundary $\mathbb{Q}$-divisor. Since V is smooth, V is automatically projective. Apply Theorem 6.1 to the pair (V, Δ_V). We obtain $K_V + \Delta_V$ is semi-ample. It implies that $K_X + \Delta$ is semi-ample. □

7. Relative setting

In this section, we discuss the finite generation of log canonical rings and the log abundance theorem in the relative setting.

Theorem 7.1 (Relative finite generation)**.** *Let (X, Δ) be a log surface such that Δ is a $\mathbb{Q}$-divisor. Let $\pi : X \to S$ be a proper surjective morphism onto a variety S. Assume that X is $\mathbb{Q}$-factorial or that (X, Δ) is log canonical. Then*

$$R(X/S, \Delta) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner)$$

is a finitely generated $\mathcal{O}_S$-algebra.

Proof. (cf. Proof of Theorem 1.1 in [F2]). When (X, Δ) is log canonical, we replace X with its minimal resolution. So, we can always assume that X is $\mathbb{Q}$-factorial. If $\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) = -\infty$, where η is the generic point of S, X_η is the generic fiber of π, and $\Delta_\eta = \Delta|_{X_\eta}$, then the statement is trivial. So, we assume that $\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) \geq 0$. We further assume that S is affine by shrinking $\pi : X \to S$. By compactifying $\pi : X \to S$, we can assume that S is projective. Since X is $\mathbb{Q}$-factorial, X is automatically projective (cf. Lemma 2.2). In particular, π is projective. Let H be a very ample divisor on S and G a general member of $|4H|$. We run the log minimal model program for $(X, \Delta + \pi^*G)$. By Proposition 3.7, this log minimal model program is a log minimal model program over S. It is because any $(K_X + \Delta + \pi^*G)$-negative extremal ray of $\overline{NE}(X)$ is a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X/S)$. When we prove this theorem, by Theorem 3.3, we can assume that $K_X + \Delta + \pi^*G$ is nef over S, equivalently, $K_X + \Delta + \pi^*G$

is nef. By Theorem 6.1, $K_X + \Delta + \pi^*G$ is semi-ample. In particular, $K_X + \Delta$ is π-semi-ample. Thus,

$$R(X/S, \Delta) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner)$$

is a finitely generated $\mathcal{O}_S$-algebra. □

Theorem 7.2 (Relative abundance theorem)**.** *Let (X, Δ) be a log surface such that Δ is a $\mathbb{Q}$-divisor. Let $\pi : X \to S$ be a proper surjective morphism onto a variety S. Assume that X is $\mathbb{Q}$-factorial or that (X, Δ) is log canonical. We further assume that $K_X + \Delta$ is π-nef. Then $K_X + \Delta$ is π-semi-ample.*

Proof. As in the proof of Theorem 7.1, we can always assume that X is $\mathbb{Q}$-factorial. By Theorem 6.1, we can assume that $\dim S \geq 1$. By Theorem 7.1, we have that

$$R(X/S, \Delta) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner)$$

is a finitely generated $\mathcal{O}_S$-algebra. It is easy to see that $K_{X_\eta} + \Delta_\eta$ is nef and abundant. Therefore, $K_X + \Delta$ is π-semi-ample (see, for example, [F2, Lemma 3.12]). □

We recommend the reader to see [F2, 3.1. Appendix] for related topics.

Anyway, we obtain the relative log minimal model program for log surfaces (cf. Theorem 3.3) and the relative log abundance theorem for log surfaces (cf. Theorem 7.2) in full generality. Therefore, we can freely use the log minimal model theory for log surfaces in the relative setting.

8. Appendix: Base point free theorem for log surfaces

In this appendix, we prove the base point free theorem for log surfaces in full generality. It completely generalizes Fukuda's base point free theorem for log canonical surfaces (cf. [Fk, Main Theorem]). Our proof is different from Fukuda's and depends on the theory of *quasi-log varieties*. We note that this result is not necessary for the minimal model theory for log surfaces discussed in this paper. We also note that a much more general result was stated in [A, Theorem 7.2] without any proofs (cf. [F4, Theorem 4.1]).

Theorem 8.1 (Base point free theorem for log surfaces)**.** *Let (X, Δ) be a log surface and $\pi : X \to S$ a proper surjective morphism onto a variety S. Let L be a π-nef Cartier divisor on X. Assume that*

$aL-(K_X+\Delta)$ is π-nef and π-big and that $(aL-(K_X+\Delta))|_C$ is π-big for every lc center C of the pair (X,Δ), where a is a positive number. Then there exists a positive integer m_0 such that $\mathcal{O}_X(mL)$ is π-generated for every $m\geq m_0$.

Remark 8.2. In Theorem 8.1, the condition that $(aL-(K_X+\Delta))|_C$ is π-big for every lc center C of the pair (X,Δ) is equivalent to the following condition: $(aL-(K_X+\Delta))\cdot C>0$ for every irreducible component C of $\llcorner \Delta \lrcorner$ such that $\pi(C)$ is a point.

Proof. Without loss of generality, we can assume that S is affine since the problem is local. We divide the proof into several steps.

Step 1 (Quasi-log structures)**.** Since (X,Δ) is a log surface, the pair $[X,\omega]$, where $\omega=K_X+\Delta$, has a natural quasi-log structure. It induces a quasi-log structure $[V,\omega']$ on $V=\mathrm{Nklt}(X,\Delta)$ with $\omega'=\omega|_V$. More precisely, let $f:Y\to X$ be a resolution such that $K_Y+\Delta_Y=f^*(K_X+\Delta)$ and that $\mathrm{Supp}\Delta_Y$ is a simple normal crossing divisor on Y. By the relative Kawamata–Viehweg vanishing theorem, we obtain the following short exact sequence

$$\begin{aligned}0\to f_*\mathcal{O}_Y(-\llcorner \Delta_Y\lrcorner)&\to f_*\mathcal{O}_Y(\ulcorner(-\Delta_Y^{<1})\urcorner-\llcorner \Delta_Y^{>1}\lrcorner)\\ &\to f_*\mathcal{O}_{\Delta_Y^{=1}}(\ulcorner(-\Delta_Y^{<1})\urcorner-\llcorner \Delta_Y^{>1}\lrcorner)\to 0.\end{aligned}$$

Note that

$$-\llcorner \Delta_Y\lrcorner=\ulcorner(-\Delta_Y^{<1})\urcorner-\llcorner \Delta_Y^{>1}\lrcorner-\Delta_Y^{=1}.$$

We also note that the scheme structure of V is defined by the multiplier ideal sheaf $\mathcal{J}(X,\Delta)=f_*\mathcal{O}_Y(-\llcorner \Delta_Y\lrcorner)$ of the pair (X,Δ) and that $X_{-\infty}$ (resp. $V_{-\infty}$) is defined by the ideal sheaf $f_*\mathcal{O}_Y(\ulcorner(-\Delta_Y^{<1})\urcorner-\llcorner \Delta_Y^{>1}\lrcorner)=:\mathcal{I}_{X_{-\infty}}$ (resp. $f_*\mathcal{O}_{\Delta_Y^{=1}}(\ulcorner(-\Delta_Y^{<1})\urcorner-\llcorner \Delta_Y^{>1}\lrcorner)=:\mathcal{I}_{V_{-\infty}}$). By construction, $X_{-\infty}\simeq V_{-\infty}$ and $X_{-\infty}=\mathrm{Nlc}(X,\Delta)$. We note the following commutative diagram.

$$\begin{tikzcd}0 \arrow[r] & \mathcal{J}(X,\Delta) \arrow[r] \arrow[d, equal] & \mathcal{I}_{X_{-\infty}} \arrow[r] \arrow[d] & \mathcal{I}_{V_{-\infty}} \arrow[r] \arrow[d] & 0\\ 0 \arrow[r] & \mathcal{J}(X,\Delta) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_V \arrow[r] & 0\end{tikzcd}$$

For details, see [A, Section 4], [F4, Section 3.2], and [F7].

Step 2 (Freeness on $\mathrm{Nklt}(X,\Delta)$)**.** By assumption, $aL|_V-\omega'$ is π-ample and $\mathcal{O}_{V_{-\infty}}(mL)$ is $\pi|_{V_{-\infty}}$-generated for every $m\geq 0$. We note that V is one-dimensional and $V_{-\infty}$ is zero-dimensional. Therefore, by [F4, Theorem 3.66], $\mathcal{O}_V(mL)$ is π-generated for every $m\gg 0$.

Step 3 (Lifting of sections)**.** We consider the following short exact sequence

$$0 \to \mathcal{J}(X, \Delta) \to \mathcal{O}_X \to \mathcal{O}_V \to 0,$$

where $\mathcal{J}(X, \Delta)$ is the multiplier ideal sheaf of (X, Δ). Then we obtain that the restriction map

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(V, \mathcal{O}_V(mL))$$

is surjective for every $m \geq a$ since

$$H^1(X, \mathcal{J}(X, \Delta) \otimes \mathcal{O}_X(mL)) = 0$$

for $m \geq a$ by the relative Kawamata–Viehweg–Nadel vanishing theorem. Thus, there exists a positive integer m_1 such that $\mathrm{Bs}|mL| \cap \mathrm{Nklt}(X, \Delta) = \emptyset$ for every $m \geq m_1$.

So, all we have to do is to prove that $|mL|$ is free for every $m \gg 0$ under the assumption that $\mathrm{Bs}|nL| \cap \mathrm{Nklt}(X, \Delta) = \emptyset$ for every $n \geq m_1$.

Step 4 (Kawamata's X-method)**.** Let $f : Y \to X$ be a resolution with a simple normal crossing divisor $F = \sum_j F_j$ on Y. We can assume the following conditions.

(a) $K_Y = f^*(K_X + \Delta) + \sum_j a_j F_j$ for some $a_j \in \mathbb{R}$.
(b) $f^*|p^l L| = |M| + \sum_j r_j F_j$, where $|M|$ is free, p is a prime number such that $p^l \geq m_1$, and $\sum_j r_j F_j$ is the fixed part of $f^*|p^l L|$ for some $r_j \in \mathbb{Z}$ with $r_j \geq 0$.
(c) $f^*(aL - (K_X + \Delta)) - \sum_j \delta_j F_j$ is π-ample for some $\delta_j \in \mathbb{R}$ with $0 < \delta_j \ll 1$.

We set

$$c = \min \left\{ \frac{a_j + 1 - \delta_j}{r_j} \right\}$$

where the minimum is taken for all the j such that $r_j \neq 0$. Then, we obtain $c > 0$. Here, we used the fact that $a_j > -1$ if $r_j > 0$. It is because $\mathrm{Bs}|p^l L| \cap \mathrm{Nklt}(X, \Delta) = \emptyset$. By a suitable choice of the δ_j, we can assume that the minimum is attained at exactly one value $j = j_0$. We put

$$A = \sum_j (-cr_j + a_j - \delta_j) F_j.$$

We consider

$$\begin{aligned} N &:= p^{ll'}f^*L - K_Y + \sum_j (-cr_j + a_j - \delta_j)F_j \\ &= (p^{ll'} - cp^l - a)f^*L \qquad (\pi\text{-nef if } p^{ll'} \geq cp^l + a) \\ &\quad + c(p^l f^*L - \sum_j r_jF_j) \qquad (\pi\text{-free}) \\ &\quad + f^*(aL - (K_X + \Delta)) - \sum_j \delta_jF_j \qquad (\pi\text{-ample}) \end{aligned}$$

for some positive integer l'. Then N is π-ample if $p^{ll'} \geq cp^l + a$. By the relative Kawamata–Viehweg vanishing theorem, we have

$$H^i(Y, \mathcal{O}_Y(K_Y + \ulcorner N \urcorner)) = 0$$

for every $i > 0$. We can write $\ulcorner A \urcorner = B - F - D$, where B is an effective f-exceptional Cartier divisor, $F = F_{j_0}$, D is an effective Cartier divisor such that $\mathrm{Supp} D \subset \mathrm{Supp} \sum_{a_j \leq -1} F_j$, and $\mathrm{Supp} B$, $\mathrm{Supp} F$, and $\mathrm{Supp} D$ have no common irreducible components one another by $\mathrm{Bs}|p^lL| \cap \mathrm{Nklt}(X, \Delta) = \emptyset$. We note that $K_Y + \ulcorner N \urcorner = p^{ll'}f^*L + \ulcorner A \urcorner$. Then the restriction map

$$\begin{aligned} &H^0(Y, \mathcal{O}_Y(p^{ll'}f^*L + B)) \\ &\to H^0(F, \mathcal{O}_F(p^{ll'}f^*L + B)) \oplus H^0(D, \mathcal{O}_D(p^{ll'}f^*L + B)) \end{aligned}$$

is surjective. Here, we used the fact that $\mathrm{Supp} F \cap \mathrm{Supp} D = \emptyset$. Thus we obtain that

$$H^0(X, \mathcal{O}_X(p^{ll'}L)) \simeq H^0(Y, \mathcal{O}_Y(p^{ll'}f^*L + B)) \to H^0(F, \mathcal{O}_F(p^{ll'}f^*L + B))$$

is surjective. We note that $H^0(F, \mathcal{O}_F(p^{ll'}f^*L + B)) \neq 0$ by Shokurov's non-vanishing theorem. Therefore, $\mathrm{Bs}|p^{ll'}L| \subsetneq \mathrm{Bs}|p^lL|$ since $f(F) \subset \mathrm{Bs}|p^lL|$. By noetherian induction, we obtain $\mathrm{Bs}|p^kL| = \emptyset$ for some positive integer k.

Let q be a prime number with $q \neq p$. Then we can find $k' > 0$ such that $\mathrm{Bs}|q^{k'}L| = \emptyset$ by the same argument as in Step 4. So, we can find a positive integer m_0 such that $\mathrm{Bs}|mL| = \emptyset$ for $m \geq m_0$. $\square$

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