REMARK ON DYNAMICAL MORSE INEQUALITY

MASAYUKI ASAOKA, TOMOHIRO FUKAYA AND MASAKI TSUKAMOTO

Abstract. We solve a transversality problem relating to Bertelson-Gromov's "dynamical Morse inequality".

1. introduction

Bertelson-Gromov proposed a study of "dynamical Morse inequality" in [2]. It is a new kind of Morse theory in (asymptotically) infinite dimensional situations. The authors think that the paper [2] opened a way to a fruitful new research area. The purpose of this paper is to give some complementary results which clarify the value of the paper [2].

Let X be a compact connected smooth manifold of dimension ≥ 1, and $f : X \times X \to \mathbb{R}$ be a smooth function. For $n = 1, 2, 3, \cdots$, we define $f_n : X^{n+1} \to \mathbb{R}$ by

$$f_n(x_0, x_1, \cdots, x_n) := \frac{1}{n}\sum_{i=0}^{n-1} f(x_i, x_{i+1}). \tag{1}$$

The study of this kind of functions was proposed by Bertelson-Gromov [2]. (See also Bertelson [1].) The "physical" meaning of f_n is as follows. Consider a "crystal" which consists of n "atoms" in a line. Suppose that the "internal state" of each atom is described by the manifold X and that each atom interacts with the next one by the "potential function" $f(x, y)$. Then the critical points of f_n correspond to the "stationary states" of the crystal.

Let c be a real number, and δ a positive real number. We define $N_n(c, \delta)$ as the number of critical points p of f_n with $c - \delta < f_n(p) < c + \delta$:

$$N_n(c, \delta) := \sharp\{p \in X^{n+1} |\, (df_n)_p = 0,\, c - \delta < f_n(p) < c + \delta\}.$$

We set

$$N(c, \delta) := \liminf_{n\to\infty} \frac{1}{n} \log N_n(c, \delta).$$

Date: December 17, 2009.

2000 *Mathematics Subject Classification.* 57R70.

Key words and phrases. critical point, Morse function, Betti-number entropy.

M. Asaoka and M. Tsukamoto were supported by Grant-in-Aid for Young Scientists (B) (19740085) and (21740048) respectively from the Ministry of Education, Culture, Sports, Science and Technology. T. Fukaya was supported by Grant-in-Aid for JSPS Fellows (19·3177) from Japan Society for the Promotion of Science.

We define $N(c)$ by

$$N(c) := \lim_{\delta \to 0} N(c, \delta).$$

Bertelson-Gromov [2] (essentially) proved the following "dynamical Morse inequality". (See [2, Remark 8.2].)

Theorem 1.1 (Bertelson-Gromov)**.** *Suppose the following:*

$$\text{All } f_n : X^{n+1} \to \mathbb{R} \ (n \geq 1) \text{ are Morse functions.} \tag{2}$$

Then for any $c \in \mathbb{R}$

$$N(c) \geq b(c). \tag{3}$$

Here $b(c)$ is the "Betti-number entropy" introduced in [2]*. We review its definition and basic properties in Section 3*

For the convenience of the readers we give a proof of this result in Section 3. Set $m_n := \min_{p \in X^{n+1}} f_n(p)$ and $M_n := \max_{p \in X^{n+1}} f_n(p)$. It can be easily seen that the following limits exist: $m_\infty := \lim_{n\to\infty} m_n = \sup_{n \geq 1} m_n$ and $M_\infty := \lim_{n \to \infty} M_n = \inf_{n \geq 1} M_n$ (see Section 3.2). The function $b(c)$ is concave and $b(c) > 0$ for $m_\infty < c < M_\infty$. Therefore, if $m_\infty \neq M_\infty$, the above dynamical Morse inequality (3) gives a non-trivial estimate for $N(c)$. (We have $m_\infty \neq M_\infty$ for generic $f \in \mathcal{C}^\infty(X \times X)$. See [3, Section 1].)

Theorem 1.1 rises the following natural question: How common is the condition (2) for smooth functions? The main issue of this note is to give an affirmative answer to this question. Notice that the answer is not apparent because of the symmetry of the function f_n. For example, the value

$$\begin{aligned} n f_n(x, \cdots, x, &y_1, \cdots, y_m, x, \cdots, x) \\ &= f(x, y_1) + f(y_m, x) + (n - m - 1) f(x, x) + \sum_{i=1}^{m-1} f(y_i, y_{i+1}) \end{aligned}$$

does not depend on the number of x's before the sequence of $y_1, \cdots, y_m$ appears. So the standard arguments to show the prevalence of Morse functions (e.g. Guillemin-Pollack [4, Chapter 1, Section 7], Hirsch [5, Chapter 6, Section 1]) do not work.

Let $\mathcal{C}^\infty(X \times X)$ be the space of all (real valued) $\mathcal{C}^\infty$-functions in $X \times X$. $\mathcal{C}^\infty(X \times X)$ is endowed with the topology of $\mathcal{C}^\infty$-convergence, and we can give this space a structure of infinite dimensional Fréchet space. Recall that a subset $U \subset \mathcal{C}^\infty(X \times X)$ is said to be of second category if it is the intersection of countably many open dense subsets. If $U \subset \mathcal{C}^\infty(X \times X)$ is of second category, then it is dense in $\mathcal{C}^\infty(X \times X)$ (Baire). The main result of this paper is the following.

Theorem 1.2. *The set of all functions $f \in \mathcal{C}^\infty(X \times X)$ satisfying the condition (2) is of second category in $\mathcal{C}^\infty(X \times X)$.*

Let $\mathcal{C}^\infty_s(X \times X)$ be the space of all $f \in \mathcal{C}^\infty(X \times X)$ satisfying the symmetric relation $f(x, y) = f(y, x)$ for all $x, y \in X$. This is a closed subspace in $\mathcal{C}^\infty(X \times X)$. If we consider X^{n+1} as the "configuration space of a crystal" as we explained before, then it is natural to suppose that the "potential function" f is symmetric. So we think that the following result is also interesting.

Theorem 1.3. *The set of all functions $f \in \mathcal{C}^\infty_s(X \times X)$ satisfying (2) is of second category in $\mathcal{C}^\infty_s(X \times X)$.*

2. Proof of Theorems 1.2 and 1.3

In this section we assume that the closed manifold X is smoothly embedded into the Euclidean space $\mathbb{R}^N$. For $n \geq 1$, we denote $\mathbb{P}_n$ as the set of all partitions of $\{0, 1, 2, \cdots, n\}$. For $\sigma = \{P_1, P_2, \cdots, P_l\} \in \mathbb{P}_n$, we set $|\sigma| = l$ and $\sigma(i) = P_j$ for $i \in P_j$, $(i = 0, 1, 2, \cdots, n)$. For example, if $\sigma = \{\{0\}, \{1, 3\}, \{2\}\} \in \mathbb{P}_3$, then $|\sigma| = 3$ and $\sigma(0) = \{0\}$, $\sigma(1) = \sigma(3) = \{1, 3\}$, $\sigma(2) = \{2\}$. For $\sigma, \tau \in \mathbb{P}_n$, we denote $\tau \geq \sigma$ if we have $\tau(i) \supset \sigma(i)$ for all $i = 0, 1, \cdots, n$. (This means that σ is a subdivision of τ.) The maximum partition with respect to this ordering is $\{\{0, 1, 2, \cdots, n\}\}$, and the minimum partition is $\{\{0\}, \{1\}, \cdots, \{n\}\}$.

For $\sigma \in \mathbb{P}_n$, we set

$$X_\sigma := \{(x_0, x_1, \cdots, x_n) \in X^{n+1} |\, x_i = x_j \text{ if } \sigma(i) = \sigma(j)\},$$

$$\mathbb{R}^N_\sigma := \{(v_0, v_1, \cdots, v_n) \in (\mathbb{R}^N)^{n+1} |\, v_i = v_j \text{ if } \sigma(i) = \sigma(j)\}.$$

We have $X_\sigma \subset \mathbb{R}^N_\sigma$. If $\tau \geq \sigma$, then $X_\tau \subset X_\sigma$ and $\mathbb{R}^N_\tau \subset \mathbb{R}^N_\sigma$. We set

$$\Sigma_\sigma := \bigcup_{\tau \gneqq \sigma} X_\tau.$$

Here τ runs over all partitions in $\mathbb{P}_n$ strictly greater than σ. We have $\Sigma_\sigma \subset X_\sigma$.

Remark 2.1. (i) For $\boldsymbol{x} = (x_0, x_1, \cdots, x_n) \in X^{n+1}$, we have $\boldsymbol{x} \in X_\sigma \setminus \Sigma_\sigma$ if and only if the following condition is satisfied: "$x_i = x_j \Leftrightarrow \sigma(i) = \sigma(j)$".
(ii) $X^{n+1} = \bigcup_{\sigma \in \mathbb{P}_n} (X_\sigma \setminus \Sigma_\sigma)$.
(iii) The pair $(\mathbb{R}^N_\sigma, X_\sigma)$ is diffeomorphic to the pair $((\mathbb{R}^N)^{|\sigma|}, X^{|\sigma|})$.

For $f \in \mathcal{C}^\infty(X \times X)$, we define $S_n(f) \in \mathcal{C}^\infty(X^{n+1})$ by

$$S_n(f) := \sum_{i=0}^{n-1} f(x_i, x_{i+1}).$$

We have $f_n = S_n(f)/n$ (see (1)).

For each (fixed) $n \geq 1$, the set $\{f \in \mathcal{C}^\infty(X \times X) |\, S_n(f) \text{ is a Morse function}\}$ is obviously open in $\mathcal{C}^\infty(X \times X)$. (A similar statement for $\mathcal{C}_s(X \times X)$ is also true.) Hence Theorems 1.2 and 1.3 follow from the following.

Theorem 2.2. *Fix $n \geq 1$. Every $f \in \mathcal{C}^\infty(X \times X)$ can be approximated arbitrarily well (in the $\mathcal{C}^\infty$-topology) by $g \in \mathcal{C}^\infty(X \times X)$ such that $S_n(g)$ is a Morse function. Moreover, if f is symmetric (i.e. $f(x,y) = f(y,x)$ for any $x, y \in X$), then we can choose a symmetric approximation g.*

For a while we will prepare some preliminary results for proving this theorem. In the rest of this section we fix $n \geq 1$. First recall the following well-known result (see Guillemin-Pollack [4, p. 43]).

Proposition 2.3. *Let M be a closed manifold embedded in $\mathbb{R}^m$ and $f : M \to \mathbb{R}$ a smooth function. Fix $x_0 \in \mathbb{R}^m$. Then for almost every $\alpha \in \mathbb{R}^m$, the function*

$$M \ni x \mapsto f(x) + \langle \alpha, x - x_0 \rangle \in \mathbb{R}$$

is a Morse function. Here $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^m$.

We will also need the following (well-known, we believe).

Lemma 2.4. *Let M be a closed manifold embedded in $\mathbb{R}^m$ and $f : M \to \mathbb{R}$ be a smooth function. Let $p = (p_1, \cdots, p_m) \in M$ be a critical point of f. Let $a_1, \cdots, a_m$ be positive numbers. Then for all but finitely many $c \in \mathbb{R}$, the point p is a non-degenerate critical point of the following function:*

$$g_c : M \ni x \mapsto f(x) + c \sum_{i=1}^{m} a_i |x_i - p_i|^2 \in \mathbb{R}.$$

Proof. First note that the following fact: Let A and B be two matrices of the same degree, and suppose B is regular. Then $A + cB$ is also regular for $c \gg 1$. Hence $\det(A + cB)$ is not identically zero as the polynomial of c. So it has only finitely many zeros. Then $A + cB$ is regular for all but finitely many $c \in \mathbb{R}$.

We can assume $p = 0$. Let $\varphi = (\varphi_1, \cdots, \varphi_m) : \mathbb{R}^k \to M \subset \mathbb{R}^m$ ($\varphi(0) = 0$) be a local coordinate around $0 \in M$. We have $g_c \circ \varphi(y) = f \circ \varphi(y) + c \sum_{i=1}^m a_i(\varphi_i(y))^2$. Then

$$\frac{\partial^2 g_c \circ \varphi}{\partial y_\alpha \partial y_\beta}(0) = \frac{\partial^2 f \circ \varphi}{\partial y_\alpha \partial y_\beta}(0) + 2c \sum_{i=1}^{m} a_i \frac{\partial \varphi_i}{\partial y_\alpha}(0) \frac{\partial \varphi_i}{\partial y_\beta}(0).$$

It is easy to see that the symmetric matrix $(\sum_i a_i \cdot \partial\varphi_i(0)/\partial y_\alpha \cdot \partial\varphi_i(0)/\partial y_\beta)_{\alpha,\beta}$ is positive definite and hence regular. Hence the desired result follows from the above remark. $\square$

For $\boldsymbol{p} = (p_0, p_1, \cdots, p_n) \in X^{n+1}$, we put

$$r(\boldsymbol{p}) := \min\{|p_i - p_j| \,|\, p_i \neq p_j\}, \quad U_{\boldsymbol{p}} := \{\boldsymbol{x} \in X^{n+1} |\, |\boldsymbol{x} - \boldsymbol{p}| < r(\boldsymbol{p})/3\}.$$

When $p_0 = p_1 = \cdots = p_n$, we set $r(\boldsymbol{p}) = +\infty$ and $U_{\boldsymbol{p}} = X^{n+1}$. Let χ be a $\mathcal{C}^\infty$-function on $\mathbb{R}$ such that $\chi = 1$ on $[0, 1/3]$ and $\chi = 0$ on $[2/3, +\infty)$. For $\boldsymbol{p} = (p_0, p_1, \cdots, p_n) \in X^{n+1}$ and $j = 0, 1, \cdots, n$, we set $\chi_{\boldsymbol{p},j}(x) := \chi(|x - p_j|/r(\boldsymbol{p}))$ for $x \in X$. If $p_0 = p_1 = \cdots = p_n$, then we set $\chi_{\boldsymbol{p},j} \equiv 1$.

Lemma 2.5. *Let $\sigma \in \mathbb{P}_n$. For $\boldsymbol{p} \in X_\sigma \setminus \Sigma_\sigma$ and $\boldsymbol{x} = (x_0, x_1, \cdots, x_n) \in U_{\boldsymbol{p}}$,*

$$\chi_{\boldsymbol{p},j}(x_i) = \begin{cases} 1 & \text{if } \sigma(i) = \sigma(j), \\ 0 & \text{if } \sigma(i) \neq \sigma(j). \end{cases}$$

Proof. If $\sigma(i) = \sigma(j)$, then $p_i = p_j$. So $|x_i - p_j| = |x_i - p_i| \leq |\boldsymbol{x} - \boldsymbol{p}| < r(\boldsymbol{p})/3$. If $\sigma(i) \neq \sigma(j)$, then $p_i \neq p_j$. So $|x_i - p_j| \geq |p_i - p_j| - |x_i - p_i| \geq 2r(\boldsymbol{p})/3$. □

For $i = 0, 1, 2, \cdots, n$, we set

$$\mu(i) = \begin{cases} 1 & i = 0, n, \\ 2 & \text{otherwise.} \end{cases}$$

Lemma 2.6. *Let $\sigma \in \mathbb{P}_n$, $\boldsymbol{p} = (p_0, p_1, \cdots, p_n) \in X_\sigma \setminus \Sigma_\sigma$ and $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in \mathbb{R}^N_\sigma$. Then there is symmetric $f_{\boldsymbol{p},\boldsymbol{\alpha}} \in \mathcal{C}^\infty(X \times X)$ such that $S_n(f_{\boldsymbol{p},\boldsymbol{\alpha}})(\boldsymbol{x}) = \langle \boldsymbol{\alpha}, \boldsymbol{x} - \boldsymbol{p} \rangle$ for all $\boldsymbol{x} \in U_{\boldsymbol{p}} \cap X_\sigma$.*

Proof. We define $h \in \mathcal{C}^\infty(X)$ by

$$h(x) := \sum_{j=0}^{n} \left(\sum_{k \in \sigma(j)} \mu(k) \right)^{-1} \langle \alpha_j, \chi_{\boldsymbol{p},j}(x)(x - p_j) \rangle.$$

Put $f_{\boldsymbol{p},\boldsymbol{\alpha}}(x, y) := h(x) + h(y)$. For $\boldsymbol{x} = (x_0, x_1, \cdots, x_n) \in U_{\boldsymbol{p}} \cap X_\sigma$,

$$\begin{aligned} S_n(f_{\boldsymbol{p},\boldsymbol{\alpha}})(\boldsymbol{x}) &= \sum_{i=0}^{n-1} (h(x_i) + h(x_{i+1})) = \sum_{i=0}^{n} \mu(i) h(x_i) \\ &= \sum_{0 \leq i,j \leq n} \mu(i) \left(\sum_{k \in \sigma(j)} \mu(k) \right)^{-1} \langle \alpha_j, \chi_{\boldsymbol{p},j}(x_i)(x_i - p_j) \rangle \\ &= \sum_{j=0}^{n} \left(\sum_{k \in \sigma(j)} \mu(k) \right)^{-1} \left(\sum_{i \in \sigma(j)} \mu(i) \langle \alpha_j, x_j - p_j \rangle \right) \\ &\quad \text{(by Lemma 2.5 and } x_i = x_j \text{ for } i \in \sigma(j)) \\ &= \sum_{j=0}^{n} \langle \alpha_j, x_j - p_j \rangle = \langle \boldsymbol{\alpha}, \boldsymbol{x} - \boldsymbol{p} \rangle. \end{aligned}$$

□

Lemma 2.7. *For $\boldsymbol{p} = (p_0, p_1, \cdots, p_n) \in X^{n+1}$, there is symmetric $g_{\boldsymbol{p}} \in \mathcal{C}^\infty(X \times X)$ such that $S_n(g_{\boldsymbol{p}})(\boldsymbol{x}) = \sum_{i=0}^{n} \mu(i) |x_i - p_i|^2$ for all $\boldsymbol{x} = (x_0, \cdots, x_n) \in U_{\boldsymbol{p}}$.*

Proof. Choose $\sigma \in \mathbb{P}_n$ such that $\boldsymbol{p} \in X_\sigma \setminus \Sigma_\sigma$. We define $h \in \mathcal{C}^\infty(X)$ by

$$h(x) := \sum_{j=0}^{n} \frac{\chi_{\boldsymbol{p},j}(x)}{\sharp \sigma(j)} |x - p_j|^2.$$

Set $g_{\boldsymbol{p}}(x,y) := h(x) + h(y)$. For $\boldsymbol{x} = (x_0, x_1, \cdots, x_n) \in U_{\boldsymbol{p}}$,

$$\begin{aligned}
S_n(g_{\boldsymbol{p}})(\boldsymbol{x}) &= \sum_{i=0}^{n-1}(h(x_i) + h(x_{i+1})) = \sum_{i=0}^{n} \mu(i)h(x_i) \\
&= \sum_{0\le i,j\le n} \mu(i)\frac{\chi_{\boldsymbol{p},j}(x_i)}{\sharp\sigma(j)}|x_i - p_j|^2 \\
&= \sum_{i=0}^{n} \mu(i) \sum_{j\in\sigma(i)} \frac{1}{\sharp\sigma(i)}|x_i - p_j|^2 \quad \text{(by Lemma 2.5)} \\
&= \sum_{i=0}^{n} \mu(i)|x_i - p_i|^2 \quad (p_j = p_i \text{ for } j \in \sigma(i)).
\end{aligned}$$

□

Let M be a manifold, and $f : M \to \mathbb{R}$ be a smooth function. We denote $C(f)$ as the set of all critical points of f, and $C_*(f)$ as the set of all degenerate critical points of f.

Lemma 2.8. *Let $\sigma \in \mathbb{P}_n$ and $K \subset X_\sigma$ be a compact set. Let $f \in \mathcal{C}^\infty(X \times X)$. Suppose $C_*(S_n(f)|_{X_\sigma}) \cap K = \emptyset$. Then f can be approximated arbitrarily well by $g \in \mathcal{C}^\infty(X \times X)$ such that $C_*(S_n(g)) \cap K = \emptyset$. If f is symmetric, then we can choose a symmetric approximation g.*

Proof. All critical points of $S_n(f)|_{X_\sigma}$ in K are isolated in X_σ. In particular $C(S_n(f)|_{X_\sigma}) \cap K$ is a finite set. Since $C_*(S_n(f)) \cap K$ is contained in $C(S_n(f)|_{X_\sigma}) \cap K$, $C_*(S_n(f)) \cap K$ is also finite. We prove the lemma by the induction on $l := \sharp(C_*(S_n(f)) \cap K)$.

The case $l = 0$ is trivial. Suppose $C_*(S_n(f)) \cap K = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_l\}$. There are open subsets $V_1, \cdots, V_l$ of X_σ such that $\boldsymbol{p}_i \in V_i$, $C(S_n(f)|_{X_\sigma}) \cap \overline{V}_i = \{\boldsymbol{p}_i\}$ $(i = 1, \cdots, l)$ and $\overline{V}_i \cap \overline{V}_j = \emptyset$ $(i \neq j)$. Since non-degenerate critical points are persistent, there is a neighborhood $\mathcal{U}$ of f in $\mathcal{C}^\infty(X \times X)$ such that for all $g \in \mathcal{U}$

(i) $C_*(S_n(g)) \cap K \subset \bigcup_{i=1}^{l} V_i$,

(ii) $\sharp\,(C(S_n(g)|_{X_\sigma}) \cap V_i) = 1$ for $i = 1, \cdots, l$. (Then, $\sharp\,(C(S_n(g)) \cap V_i) \le 1$.)

Take $c > 0$ such that $f + cg_{\boldsymbol{p}_1} \in \mathcal{U}$ ($g_{\boldsymbol{p}_1}$ is the function given in Lemma 2.7.) and that $\boldsymbol{p}_1$ is a non-degenerate critical point of the following function:

$$X^{n+1} \to \mathbb{R}, \quad \boldsymbol{x} \mapsto S_n(f)(\boldsymbol{x}) + c\left(\sum_{j=0}^{n} \mu(j)|x_j - p_{1,j}|^2\right),$$

where $\boldsymbol{p}_1 = (p_{1,0}, p_{1,1}, \cdots, p_{1,n})$. (The latter condition is satisfied for all but finitely many $c \in \mathbb{R}$ by Lemma 2.4.) Put $g_1 := f + cg_{\boldsymbol{p}_1}$. From Lemma 2.7, for $\boldsymbol{x} = (x_0, \cdots, x_n) \in U_{\boldsymbol{p}_1}$,

$$S_n(g_1)(\boldsymbol{x}) = S_n(f) + c\left(\sum_{j=0}^{n} \mu(j)|x_j - p_{1,j}|^2\right).$$

By the choice of $g_1 \in \mathcal{U}$, $\boldsymbol{p}_1$ is the unique critical point of $S_n(g_1)$ in V_1 (see the above condition (ii)), and it is non-degenerate. Therefore we have $C_*(S_n(g_1)) \cap K \subset \bigcup_{i=2}^{l} V_i$. This implies $\sharp\,(C_*(S_n(g_1)) \cap K) \le l-1$. By the assumption of induction, g_1 can be approximated by $g \in \mathcal{C}^\infty(X \times X)$ such that $C_*(S_n(g)) \cap K = \emptyset$. If f is symmetric, then g_1 is also symmetric and we can choose a symmetric approximation g. $\square$

Proposition 2.9. *Let $\sigma \in \mathbb{P}_n$ and $f \in \mathcal{C}^\infty(X \times X)$. Suppose $C_*(S_n(f)) \cap \Sigma_\sigma = \emptyset$. Then f can be approximated arbitrarily well by $f' \in \mathcal{C}^\infty(X \times X)$ such that $C_*(S_n(f')) \cap X_\sigma = \emptyset$. If f is symmetric, then we can choose a symmetric approximation f'.*

Proof. Since Σ_σ is compact and $C_*(S_n(f)) \cap \Sigma_\sigma = \emptyset$, there is an open set $W_0 \subset X_\sigma$ such that $\Sigma_\sigma \subset W_0$ and $\overline{W}_0 \cap C_*(S_n(f)) = \emptyset$. Take $\boldsymbol{p}_1, \cdots, \boldsymbol{p}_k \in X_\sigma \setminus \Sigma_\sigma$ and open sets $V_1, \cdots, V_k \subset X_\sigma$ such that $\boldsymbol{p}_i \in V_i$, $\overline{V}_i \subset U_{\boldsymbol{p}_i}$ and $X_\sigma = W_0 \cup \bigcup_{i=1}^{k} V_i$. Put $f_0 := f$ and $W_i := W_0 \cup \bigcup_{j=1}^{i} V_j$ for $i = 1, \cdots, k$.

We will inductively show that if $f_i \in \mathcal{C}^\infty(X \times X)$ satisfies $C_*(S_n(f_i)) \cap \overline{W}_i = \emptyset$ then f_i can be approximated by $f_{i+1} \in \mathcal{C}^\infty(X \times X)$ satisfying $C_*(S_n(f_{i+1})) \cap \overline{W}_{i+1} = \emptyset$. (If f_i is symmetric, then we can choose f_{i+1} symmetric.) Since $C_*(S_n(f_0)) \cap \overline{W}_0 = \emptyset$ and $X_\sigma = W_0 \cup \bigcup_{i=1}^{k} V_i$, this will complete the proof.

Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m$ be the standard basis of $\mathbb{R}^N_\sigma$ ($m = N|\sigma|$). By Proposition 2.3, for a.e. $(c_1, \cdots, c_m) \in \mathbb{R}^m$, the following is a Morse function.

$$\mathbb{R}^N_\sigma \supset X_\sigma \ni \boldsymbol{x} \mapsto S_n(f_i)(\boldsymbol{x}) + \langle \sum_{j=1}^{m} c_j \boldsymbol{e}_j, \boldsymbol{x} - \boldsymbol{p}_{i+1} \rangle \in \mathbb{R}. \tag{4}$$

Take small $(c_1, \cdots, c_m) \in \mathbb{R}^m$ such that (4) is a Morse function. Put $g_i := f_i + \sum_{j=1}^{m} c_j f_{\boldsymbol{p}_{i+1}, \boldsymbol{e}_j}$. ($f_{\boldsymbol{p}_{i+1}, \boldsymbol{e}_j}$ is the function introduced in Lemma 2.6.) Then $S_n(g_i)(\boldsymbol{x}) = S_n(f_i)(\boldsymbol{x}) + \langle \sum_{j=1}^{m} c_j \boldsymbol{e}_j, \boldsymbol{x} - \boldsymbol{p}_{i+1} \rangle$ for $\boldsymbol{x} \in U_{\boldsymbol{p}_{i+1}} \cap X_\sigma$. This implies $C_*(S_n(g_i)|_{X_\sigma}) \cap \overline{V}_{i+1} = \emptyset$. By Lemma 2.8, g_i can be approximated by f_{i+1} satisfying $C_*(S_n(f_{i+1})) \cap \overline{V}_{i+1} = \emptyset$.

Since $C_*(S_n(f_i)) \cap \overline{W}_i = \emptyset$ by the assumption, if we choose $(c_1, \cdots, c_m)$ sufficiently small and f_{i+1} sufficiently close to g_i then $C_*(S_n(f_{i+1})) \cap \overline{W}_i = \emptyset$. Thus we have $C_*(S_n(f_{i+1})) \cap \overline{W}_{i+1} = \emptyset$. $\square$

Proof of Theorem 2.2. Set $f_0 := f$. We will inductively construct f_i below. Let $\mathbb{P}_n = \{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ ($m = |\mathbb{P}_n|$), and we can assume that these are indexed so that $\sigma_i \ge \sigma_j \Rightarrow i \le j$. If $f_i \in \mathcal{C}^\infty(X \times X)$ satisfies $C_*(S_n(f_i)) \cap \left(\bigcup_{j \le i} X_{\sigma_j}\right) = \emptyset$, then by Proposition 2.9, f_i can be approximated by $f_{i+1} \in \mathcal{C}^\infty(X \times X)$ satisfying $C_*(S_n(f_{i+1})) \cap X_{\sigma_{i+1}} = \emptyset$. We can choose f_{i+1} sufficiently close to f_i so that $C_*(S_n(f_{i+1})) \cap \left(\bigcup_{j \le i} X_{\sigma_j}\right) = \emptyset$. Hence $C_*(S_n(f_{i+1})) \cap \left(\bigcup_{j \le i+1} X_{\sigma_j}\right) = \emptyset$. By induction $f = f_0$ can be approximated by $f_m \in \mathcal{C}^\infty(X \times X)$ satisfying $C_*(S_n(f_m)) = \emptyset$. If f is symmetric, then we can choose all f_i symmetric. $\square$

3. Review of Betti-number entropy

This section is independent of Section 2. This section contains no essentially new ideas. But, perhaps, this section will be useful for some readers.

3.1. **Preliminaries.** This subsection is a preparation for introducing "Betti-number entropy" in the next subsection. The main task here is to prove Proposition 3.4 ([2, Proposition 8.1 (b)]). Some details of the proof of [2, Proposition 8.1 (b)] were not given. Here, we give a detailed proof with a slightly different argument.

Let M be a compact connected smooth manifold, and set $n := \dim M$. If M is oriented, then we use cohomology over $\mathbb{R}$. And if M is unoriented, then we use cohomology over $\mathbb{Z}/2\mathbb{Z}$. Let $a \in H^*(M) := \oplus_{k\geq 0} H^k(M)$, and $U \subset M$ be an open subset. We write $\operatorname{supp} a \subset U$ if there exists an open subset $V \subset M$ such that $M = U \cup V$ and $a|_V = 0$ in $H^*(V)$ ([2, Notation 4.1]). When we use the de Rham cohomology, $\operatorname{supp} a \subset U$ means that there exists a smooth differential form α with $d\alpha = 0$ such that $\operatorname{supp} \alpha \subset U$ and $a = [\alpha]$ in $H^*(M)$. An important property of this notion is the following: Let U, V be open sets in M, and $a, b \in H^*(M)$. If $\operatorname{supp} a \subset U$ and $\operatorname{supp} b \subset V$, then $\operatorname{supp}(a \cup b) \subset U \cap V$. (For the proof, see [3, Section 2].)

Example 3.1. Let $a \in H^n(M)$ (recall: $n = \dim M$). Then for any non-empty open set $U \subset M$ we have $\operatorname{supp} a \subset U$.

Lemma 3.2. *Let $U \subset M$ be an open set, and $A \subset H^*(M)$ be a subvector space. Suppose that all $a \in A$ satisfy $\operatorname{supp} a \subset U$. Then there exists an open set $V \subset M$ such that $\overline{V} \subset U$ and that all $a \in A$ satisfy $\operatorname{supp} a \subset V$.*

Proof. Let $a_1, \cdots, a_N$ be a basis of A. For each a_i there is an open set $V_i \subset M$ such that $M = U \cup V_i$ and $a_i|_{V_i} = 0$ in $H^*(V_i)$. There is an open set $V \subset M$ satisfying $\cup_{i=1}^N (M \setminus V_i) \subset V$ and $\overline{V} \subset U$. Then every a_i satisfies $\operatorname{supp} a_i \subset V$. Then all $a \in A$ satisfy $\operatorname{supp} a \subset V$. □

For $a \in H^q(M)$, we denote $\mathrm{PD}(a) := a \cap [M] \in H_{n-q}(M)$ as the Poincaré dual of a.

Lemma 3.3. *Let $a \in H^q(M)$, and $U \subset M$ be an open subset. We have $\operatorname{supp} a \subset U$ if and only if $\mathrm{PD}(a)$ is contained in the image of the map $H_{n-q}(U) \to H_{n-q}(M)$. (This statement is partly used in the proof of [2, Proposition 10.2].)*

Proof. Suppose $\operatorname{supp} a \subset U$. Then there exists an open set $V \subset M$ such that $M = U \cup V$ and $a|_V = 0$ in $H^q(V)$. Set $K := M \setminus U$. K is a closed set and $K \subset V$. We have the following commutative diagram:

$$\begin{array}{ccc} H^q(M) & \xrightarrow{\mathrm{PD}} & H_{n-q}(M) \\ \downarrow & & \downarrow \\ H^q(V) & \xrightarrow{\mathrm{PD}} & H_{n-q}(M, M \setminus K) \end{array} \tag{5}$$

The map $H^q(M) \to H_{n-q}(M)$ is the usual Poincaré dual. The map $H^q(V) \to H_{n-q}(M, M \setminus K)$ is defined as follows. Let $[M]_K \in H_n(M, M \setminus K)$ be the image of the fundamental class $[M] \in H_n(M)$ under the map $H_n(M) \to H_n(M, M \setminus K)$. Since we have the excision isomorphism $i_* : H_n(V, V \setminus K) \cong H_n(M, M \setminus K)$ $(i : V \subset M)$, there exists $w \in H^q(V, V \setminus K)$ satisfying $i_*(w) = [M]_K$. Then we define $H^q(V) \to H_{n-q}(M, M \setminus K)$ by $a \mapsto i_*(a \cap w)$. Here $a \cap w$ is the cap product $H^q(V) \times H_n(V, V \setminus K) \to H_{n-q}(V, V \setminus K)$.

Since $a|_V = 0$ in $H^q(V)$, we have $[\mathrm{PD}(a)] = 0$ in $H_{n-q}(M, M \setminus K) = H_{n-q}(M, U)$. We have the exact sequence $H_*(U) \to H_*(M) \to H_*(M, U)$. Hence $\mathrm{PD}(a)$ is contained in the image of the map $H_{n-q}(U) \to H_{n-q}(M)$.

Next suppose that $\mathrm{PD}(a)$ is contained in the image of the map $H_{n-q}(U) \to H_{n-q}(M)$. Set $K := M \setminus U$. From the assumption, $[\mathrm{PD}(a)] = 0$ in $H_{n-q}(M, U) = H_{n-q}(M, M \setminus K)$. By taking the direct limit of the above diagram (5), we have the following commutative diagram:

$$\begin{array}{ccc} H^q(M) & \xrightarrow{\mathrm{PD}} & H_{n-q}(M) \\ \downarrow & & \downarrow \\ \varinjlim_{K \subset V} H^q(V) & \xrightarrow{\mathrm{PD}} & H_{n-q}(M, M \setminus K) \end{array}$$

Here $\varinjlim_{K \subset V} H^q(V)$ is the direct limit of $H^q(V)$ over the set of open sets V with $V \supset K$ $(V_1 \leq V_2 \Leftrightarrow V_1 \supset V_2)$. The first and second horizontal lines are both isomorphisms. Since $[\mathrm{PD}(a)] = 0$ in $H_{n-q}(M, M \setminus K)$, we have $[a] = 0$ in $\varinjlim_{K \subset V} H^q(V)$. This means that there exists an open set $V \subset M$ with $K \subset V$ (i.e. $M = U \cup V$) and $a|_V = 0$ in $H^q(V)$. Therefore $\operatorname{supp} a \subset U$. $\square$

Proposition 3.4. *Let $\varphi : M \to \mathbb{R}$ be a Morse function, and $c < d$ be two real numbers. Let $A \subset H^*(M)$ be a subvector space satisfying the following conditions: All $a \in A$ satisfy* $\operatorname{supp} a \subset \varphi^{-1}(-\infty, d)$, *and moreover for all $a \in A$ with $a \neq 0$, there exists $b \in H^*(M)$ satisfying* $\operatorname{supp} b \subset \varphi^{-1}(c, +\infty)$ *and $a \cup b \neq 0$. Then*

$$\sharp\{p \in \varphi^{-1}(c, d) | (d\varphi)_p = 0\} \geq \dim A.$$

Proof. From Lemma 3.2, we can assume that both c and d are regular values of φ without loss of generality.

Step 1. First we assume that the open interval (c, d) contains at most one critical value of φ. This step corresponds to [2, Proposition 8.1 (b)]. Their argument uses certain piecewise smooth cycles which they did not explain how to construct. Here we give a slightly different argument.

By considering the handle decomposition by φ, we have

$$\sharp\{p \in \varphi^{-1}(c, d) | (d\varphi)_p = 0\} = \dim H_*(\varphi^{-1}(-\infty, d), \varphi^{-1}(-\infty, c)).$$

From Lemma 3.3, there exists a subvector space $A' \subset H_*(\varphi^{-1}(-\infty, d))$ with $\dim A' = \dim A$ such that for every $w \in A'$ with $w \neq 0$ there exists $a \in A$ satisfying $a \neq 0$ and $[w] =$

$\mathrm{PD}(a)$ in $H_*(M)$. We want to show that A' injects into $H_*(\varphi^{-1}(-\infty,d),\varphi^{-1}(-\infty,c))$ by the map $H_*(\varphi^{-1}(-\infty,d)) \to H_*(\varphi^{-1}(-\infty,d),\varphi^{-1}(-\infty,c))$. Suppose that there exists $w \in A'$ satisfying $w \neq 0$ and $[w]=0$ in $H_*(\varphi^{-1}(-\infty,d),\varphi^{-1}(-\infty,c))$. Since we have the exact sequence $H_*(\varphi^{-1}(-\infty,c)) \to H_*(\varphi^{-1}(-\infty,d)) \to H_*(\varphi^{-1}(-\infty,d),\varphi^{-1}(-\infty,c))$, there is $w' \in H_*(\varphi^{-1}(-\infty,c))$ satisfying $w=[w']$. By the definition of A', there is $a \in A$ with $a \neq 0$ satisfying $\mathrm{PD}(a)=[w]=[w']$ in $H_*(M)$. From Lemma 3.3, we have $\operatorname{supp} a \subset \varphi^{-1}(-\infty,c)$. From the assumption on A, there is $b \in H^*(M)$ satisfying $\operatorname{supp} b \subset \varphi^{-1}(c,+\infty)$ and $a \cup b \neq 0$. But $\operatorname{supp}(a\cup b) \subset \varphi^{-1}(-\infty,c)\cap\varphi^{-1}(c,+\infty)=\emptyset$. This implies $a \cup b = 0$. This is a contradiction. Hence $A' \hookrightarrow H_*(\varphi^{-1}(-\infty,d),\varphi^{-1}(-\infty,c))$. Thus $\sharp\{p \in \varphi^{-1}(c,d) | (d\varphi)_p = 0\} = \dim H_*(\varphi^{-1}(-\infty,d),\varphi^{-1}(-\infty,c)) \geq \dim A' = \dim A$.

Step 2: The general case. Here we use the method of [2, Proposition 8.1 (a)]. Let $c=c_0<c_1<c_2<\cdots<c_N=d$ be a division of the interval (c,d) satisfying the following conditions: All c_k are regular values of φ, and each open interval (c_k,c_{k+1}) contains at most one critical value of φ.

We will repeatedly use the following fact below: Since c_k is a regular value, the gradient flow of φ defines an ambient isotopy which maps $\varphi^{-1}(-\infty,c_k)$ to $\varphi^{-1}(-\infty,c_k+\varepsilon)$ ($|\varepsilon| \ll 1$). (See Milnor [6, Chapter 3].) Hence $\operatorname{supp} a \subset \varphi^{-1}(-\infty,c_k)$ is equivalent to $\operatorname{supp} a \subset \varphi^{-1}(-\infty,c_k+\varepsilon)$. A similar statement for $\varphi^{-1}(c_k,+\infty)$ is also true.

We define a decomposition $A=A_1\oplus A_2\oplus\cdots\oplus A_N$ as follows. First note that all $a \in A$ with $a \neq 0$ satisfy $a|_{\varphi^{-1}(c_0,+\infty)} \neq 0$ in $H^*(\varphi^{-1}(c_0,+\infty))$. We define $A_1 \subset A$ as the space of $a \in A$ satisfying $a|_{\varphi^{-1}(c_1,+\infty)}=0$ in $H^*(\varphi^{-1}(c_1,+\infty))$. Let $B_1 \subset A$ be a complement of A_1 in A: $A=A_1\oplus B_1$. We define $A_2 \subset B_1$ as the space of $a \in B_1$ satisfying $a|_{\varphi^{-1}(c_2,+\infty)}=0$ in $H^*(\varphi^{-1}(c_2,+\infty))$. Let $B_2 \subset B_1$ be a complement of A_1 in B_1: $B_1=A_2\oplus B_2$. Inductively, we define $A_k \subset B_{k-1}$ as the space of $a \in B_{k-1}$ satisfying $a|_{\varphi^{-1}(c_k,+\infty)}=0$, and let $B_k \subset B_{k-1}$ be a complement of A_k in B_{k-1}: $B_{k-1}=A_k\oplus B_k$. All $a\in A$ satisfy $a|_{\varphi^{-1}(c_N,+\infty)}=0$ in $H^*(\varphi^{-1}(c_N,+\infty))$. Hence $B_{N-1}=A_N$ and $B_N=0$. Then we get $A=A_1\oplus A_2\oplus\cdots\oplus A_N$.

For each $k=1,2,\cdots,N$, all $a\in A_k$ satisfy $\operatorname{supp} a \subset \varphi^{-1}(-\infty,c_k)$. If $a \in A_k$ is not zero, then $a|_{\varphi^{-1}(c_{k-1},+\infty)} \neq 0$ in $H^*(\varphi^{-1}(c_{k-1},+\infty))$. Then there is $w \in H_*(\varphi^{-1}(c_{k-1},+\infty))$ satisfying $\langle a|_{\varphi^{-1}(c_{k-1},+\infty)}, w\rangle = \langle a,[w]\rangle \neq 0$. (Here $[w]\in H_*(M)$ is the image of w by the map $H_*(\varphi^{-1}(c_{k-1},+\infty)) \to H_*(M)$.) Then $a\cup \mathrm{PD}^{-1}([w]) \neq 0$. $\mathrm{PD}^{-1}([w]) \in H^*(M)$ satisfies $\operatorname{supp}\mathrm{PD}^{-1}([w]) \subset \varphi^{-1}(c_{k-1},+\infty)$ by Lemma 3.3. Thus all $a \in A_k$ satisfy $\operatorname{supp} a \subset \varphi^{-1}(-\infty,c_k)$, and for all $a\in A_k$ with $a\neq 0$ there is $b \in H^*(M)$ satisfying $\operatorname{supp} b \subset \varphi^{-1}(c_{k-1},+\infty)$ and $a\cup b\neq 0$. By applying Step 1 to A_k, we get

$$\sharp\{p\in\varphi^{-1}(c_{k-1},c_k) |\, (d\varphi)_p=0\} \geq \dim A_k,$$

for each $k=1,2,\cdots,N$. By summing up this estimate over $k=1,\cdots,N$, we get the desired result. $\square$

3.2. **Betti-number entropy.** All results in this subsection are contained in Bertelson-Gromov [2]. Let X be a compact connected smooth manifold of dimension ≥ 1. Let $f : X \times X \to \mathbb{R}$ be a smooth function. We define $f_n : X^{n+1} \to \mathbb{R}$ by setting $f_n(x_0, \cdots, x_n) := \frac{1}{n}\sum_{i=0}^{n-1} f(x_i, x_{i+1})$.

Set $\pi_n : X^{n+1} \to X^n$, $(x_0, \cdots, x_n) \mapsto (x_0, \cdots, x_{n-1})$. For an open set $U \subset X^{n+1}$ we define a subvector space $A_n(U) \subset H^*(X^{n+1})$ as the set of $a \in \pi_n^*(H^*(X^n))$ satisfying $\operatorname{supp} a \subset U$. For $c \in \mathbb{R}$ and $\delta > 0$, consider the following linear map:

$$A_n(f_n^{-1}(-\infty, c+\delta)) \to \mathrm{Hom}(A_n(f_n^{-1}(c-\delta, +\infty)), A_n(f_n^{-1}(c-\delta, c+\delta)),$$
$$a \mapsto (b \mapsto a \cup b).$$

We define $b_n(c, \delta)$ as the rank of this linear map. If $(c-\delta, c+\delta) \subset (c'-\delta', c'+\delta')$, then $b_n(c, \delta) \leq b_n(c', \delta')$.

Lemma 3.5. *For $c, c' \in \mathbb{R}$ and $\delta > 0$,*

$$b_{n+m}(\alpha c + (1-\alpha)c', \delta) \geq b_n(c, \delta) b_m(c', \delta), \quad \left(\alpha = \frac{n}{n+m}\right).$$

Proof. See [2, Lemma 5.1]. □

Set $m_n := \min f_n$ and $M_n := \max f_n$. nm_n is super-additive ($(n+k)m_{n+k} \geq nm_n + km_k$), and nM_n is sub-additive. Hence we have the limits $m_\infty := \lim_{n\to\infty} m_n = \sup_n m_n$ and $M_\infty := \lim_{n\to\infty} M_n = \inf_n M_n$. We have $m_\infty \leq M_\infty$.

Lemma 3.6. *For $m_\infty \leq c \leq M_\infty$ and $\delta > 0$, we have $b_n(c, \delta) \geq 1$ $(n \gg 1)$. On the other hand, if $c < m_\infty$ or $c > M_\infty$, then there is $\delta_0 > 0$ such that, for $\delta \leq \delta_0$, we have $b_n(c, \delta) = 0$ $(n \gg 1)$.*

Proof. The following argument is suggested by [2, Example 10.7, Remark 10.8]. First we prove that there is $n_0(\delta) > 0$ such that $b_n(m_\infty, \delta) \geq 1$ and $b_n(M_\infty, \delta) \geq 1$ for $n \geq n_0(\delta)$. We suppose $f(X \times X) \subset [0, 1]$ for simplicity. Let $\delta > 0$. We have $(1 - 1/n)(m_{n-1} + 1/n) + 1/n < m_\infty + \delta$ for $n \gg 1$ Let $a \in H^{n \dim X}(X^n)$ be a non-zero cohomology class of top-degree, and set $a' := \pi_n^* a$. Let $U \subset X^n$ be a non-empty open subset such that $f_{n-1} < m_{n-1} + 1/n$ on U. Then $f_n \leq (1-1/n)(m_{n-1} + 1/n) + 1/n < m_\infty + \delta$ on $\pi_n^{-1}(U)$ for $n \gg 1$. We have $\operatorname{supp} a \subset U$ (Example 3.1), and hence $\operatorname{supp} a' \subset \pi_n^{-1}(U)$. Hence $a' \in A_n(f_n^{-1}(-\infty, m_\infty + \delta))$. On the other hand, $m_\infty - \delta < m_n$ for $n \gg 1$ and hence $f_n^{-1}(m_\infty - \delta, +\infty) = X^{n+1}$. Therefore $1 \in A_n(f_n^{-1}(m_\infty - \delta, +\infty))$ for $n \gg 1$. We have $a' \cup 1 = a' \neq 0$. Hence $b_n(m_\infty, \delta) \geq 1$ for $n \gg 1$. The statement for M_∞ can be proved in the same way.

Next suppose $m_\infty < c < M_\infty$. There is $N_0(\delta) > 0$ such that if $N \geq N_0(\delta)$ then N has a decomposition $N = n + m$ satisfying $n, m \geq n_0(\delta/2)$ and $|\alpha m_\infty + (1-\alpha)M_\infty - c| < \delta/2$, $(\alpha = n/N)$. From Lemma 3.5, we have $b_N(\alpha m_\infty + (1-\alpha)M_\infty, \delta/2) \geq$

$b_n(m_\infty, \delta/2) b_m(M_\infty, \delta/2) \geq 1$. Since $(\alpha m_\infty + (1-\alpha)M_\infty - \delta/2, \alpha m_\infty + (1-\alpha)M_\infty + \delta/2) \subset (c-\delta, c+\delta)$, we have $b_N(c,\delta) \geq 1$ for $N \geq N_0(\delta)$.

Finally suppose $c < m_\infty$ (The case $c > M_\infty$ can be proved in the same way.) Let $0 < \delta \leq (m_\infty - c)/2 =: \delta_0$. We have $c + \delta < m_n$ for $n \gg 1$, and hence $f_n^{-1}(-\infty, c+\delta) = \emptyset$. Thus $b_n(c,\delta) = 0$ for $n \gg 1$. $\square$

Then we can define the Betti-number entropy $b(c)$ by

$$b(c) := \lim_{\delta \to 0} \left(\lim_{n \to \infty} \frac{1}{n} \log b_n(c,\delta) \right).$$

Proof of Theorem 1.1. From Proposition 3.4, we have $N_n(c,\delta) \geq b_n(c,\delta)$ for $c \in \mathbb{R}$ and $\delta > 0$. Hence $N(c) \geq b(c)$. $\square$

We gather some basic properties of $b(c)$ below.

Lemma 3.7. *If $m_\infty \leq c \leq M_\infty$, then $b(c) \geq 0$. If $c < m_\infty$ or $c > M_\infty$, then $b(c) = -\infty$.*

Proof. This follows from Lemma 3.6. $\square$

Lemma 3.8. *$b(c)$ is concave: For $c, c' \in \mathbb{R}$ and $0 \leq \alpha \leq 1$, $b(\alpha c + (1-\alpha)c') \geq \alpha b(c) + (1-\alpha) b(c')$*

Proof. See [2, Proposition 9.2]. $\square$

Proposition 3.9. *There exists $c \in \mathbb{R}$ satisfying $b(c) > 0$.*

Proof. See [2, Proposition 10.1]. $\square$

From Lemmas 3.7 and 3.8 and Proposition 3.9, we get the following:

Corollary 3.10. *If $m_\infty \neq M_\infty$, then the Betti-number entropy $b(c)$ is positive over the open interval $c \in (m_\infty, M_\infty)$.*

References

[1] M. Bertelson, Topological invariant for discrete group actions, Lett. Math. Phys. **62** (2004), 147-156

[2] M. Bertelson, M. Gromov, Dynamical Morse entropy, in: Modern dynamical systems and applications, 27-44, Cambridge Univ. Press, Cambridge, 2004

[3] T. Fukaya, M. Tsukamoto, Asymptotic distribution of critical values, Geom. Dedicata, DOI 10.1007/s10711-009-9372-3

[4] V. Guillemin, A. Pollack, Differential topology, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974

[5] M.W. Hirsch, Differential topology, Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York, 1976

[6] J. Milnor, Morse Theory, Annals of Mathematics Studies, **51**, Princeton University Press, Princeton, 1963

Masayuki Asaoka
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `asaoka@math.kyoto-u.ac.jp`

Tomohiro Fukaya
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `tomo_xi@math.kyoto-u.ac.jp`

Masaki Tsukamoto
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `tukamoto@math.kyoto-u.ac.jp`