ON MAXIMAL ALBANESE DIMENSIONAL VARIETIES

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Abstract. We prove that every smooth projective variety with maximal Albanese dimension has a good minimal model. We also treat Ueno's problem on subvarieties of Abelian varieties.

Contents

1. Introduction

In this short paper, we prove that every smooth projective variety with maximal Albanese dimension has a good minimal model. It is an easy consequence of [BCHM]. This paper is also a supplement to [F1]. We will also treat Ueno's problem on subvarieties of Abelian varieties. A key point of this paper is the simple fact that there are no rational curves on an Abelian variety. The topics treated here seem to be easy exercises for experts.

Let us recall the following minimal model conjecture.

Conjecture 1.1 (Good minimal model conjecture (weak form))**.** *Let X be a smooth projective variety defined over $\mathbb{C}$. Assume that K_X is pseudo-effective. Then there exists a normal projective variety X' which satisfies the following conditions*:

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(i) *X' is birationally equivalent to X.*
(ii) *X' has only $\mathbb{Q}$-factorial terminal singularities.*
(iii) *$K_{X'}$ is semi-ample.*

In particular, the Kodaira dimension $\kappa(X) = \kappa(X, K_X)$ is non-negative. We sometimes call X' a good minimal model *of X.*

Remark 1.2. By [BCHM], we can replace (ii) with the following slightly weaker condition: (ii′) X' has at most canonical singularities.

The conjecture: Conjecture 1.1 was established in the following cases.
(A) $\dim X \leq 3$ (see, for example, [FA]).
(B) varieties of general type in any dimension (see, for example, [BCHM]).
(C) Δ-regular divisors on complete toric varieties in any dimension (see [I]).
(D) irregular fourfolds (see [F4]).

As stated above, in this paper, we will prove that the conjecture: Conjecture 1.1 holds for projective maximal Albanese dimensional varieties.

Notation. For a proper birational morphism $f : X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

We will freely use the basic notation and definitions in [KM] and [U].

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We will work over $\mathbb{C}$, the complex number field, throughout this paper.

2. Preliminaries

Let us recall the definition of maximal Albanese dimensional varieties.

Definition 2.1. Let X be a smooth projective variety. Let $\mathrm{Alb}(X)$ be the Albanese variety of X and $\alpha : X \to \mathrm{Alb}(X)$ the corresponding Albanese map. We say that X has *maximal Albanese dimension* if $\dim \alpha(X) = \dim X$.

Remark 2.2. A smooth projective variety X has maximal Albanese dimension if and only if the cotangent bundle of X is generically generated by its global sections, that is,

$$H^0(X, \Omega^1_X) \otimes \mathcal{O}_X \to \Omega^1_X$$

is surjective at the generic point of X. It can be checked without any difficulties.

We note that the notion of maximal Albanese dimension is birationally invariant. So, we can define the notion of maximal Albanese dimension for singular varieties as follows.

Definition 2.3. Let X be a projective variety. We say that X has *maximal Albanese dimension* if there is a resolution $\pi : \overline{X} \to X$ such that $\overline{X}$ has maximal Albanese dimension.

The following lemma is almost obvious by the definition of maximal Albanese dimensional varieties and the basic properties of Albanese mappings. We leave the proof for the reader's exercise.

Lemma 2.4. *Let X be a projective variety with maximal Albanese dimension. Let $\pi : \overline{X} \to X$ be a resolution and $\alpha : \overline{X} \to \mathrm{Alb}(\overline{X})$ the Albanese mapping. Let Y be a subvariety of X. Assume that $Y \not\subset \pi(\mathrm{Exc}(\pi) \cup \mathrm{Exc}(\beta))$, where $\beta : \overline{X} \to V$ is the Stein factorization of $\alpha : \overline{X} \to \mathrm{Alb}(\overline{X})$. Then Y has maximal Albanese dimension.*

Let us recall some basic definitions.

Definition 2.5 (Iitaka's D-dimension and numerical D-dimension)**.** Let X be a normal projective variety and D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Assume that mD is Cartier for a positive integer m. Let

$$\Phi_{|tmD|} : X \dashrightarrow \mathbb{P}^{\dim |tmD|}$$

be rational mappings given by linear systems $|tmD|$ for positive integers t. We define *Iitaka's D-dimension*

$$\kappa(X, D) = \begin{cases} \max\limits_{t>0} \dim \Phi_{|tmD|}(X), & \text{if } \ |tmD| \neq \emptyset \ \text{ for some } \ t, \\ -\infty, & \text{otherwise.} \end{cases}$$

In case D is nef, we can also define the *numerical D-dimension*

$$\nu(X, D) = \max\{\, e \mid D^e \not\equiv 0\},$$

where $\equiv$ denotes *numerical equivalence*. We note that $\nu(X, D) \geq \kappa(X, D)$ holds.

In this paper, we adopt the following definition of *log terminal models* for klt pairs.

Definition 2.6 (Log terminal models for klt pairs)**.** Let $f : X \to S$ be a projective morphism of normal quasi-projective varieties. Suppose that (X, B) is klt and let $\phi : X \dashrightarrow X'$ be a birational map of normal quasi-projective varieties over S, where X' is projective over S. We put $B' = \phi_* B$. In this case, (X', B') is a *log terminal model* of (X, B) over S if the following conditions hold.

(i) ϕ^{-1} contracts no divisors.
(ii) (X', B') is a $\mathbb{Q}$-factorial klt pair.
(iii) $K_{X'} + B'$ is nef over S.
(iv) $a(E, X, B) < a(E, X', B')$ for all ϕ-exceptional divisors $E \subset X$.

3. Main results

3.1. **Minimal model.** First, we recall the following elementary but very important lemma.

Lemma 3.1 (Negative rational curves)**.** *Let X be a projective variety and B an effective $\mathbb{R}$-divisor on X such that (X, B) is log canonical. Assume that $K_X + B$ is not nef. Then there exists a rational curve C on X such that $(K_X + B) \cdot C < 0$.*

Proof. It is obvious by the cone theorem for log canonical pairs. See, for example, [F3, Proposition 3.21] and [F5, Section 18]. We note that [K4] is sufficient when (X, B) is klt. □

By Lemma 3.1, we obtain the next lemma.

Lemma 3.2. *Let $f : X \to S$ be a proper surjective morphism between projective varieties. Let B be an effective $\mathbb{R}$-divisor on X such that (X, B) is log canonical. Assume that $K_X + B$ is f-nef and S contains no rational curves. Then $K_X + B$ is nef.*

Proof. If $K_X + B$ is not nef, then there exists a rational curve C on X such that $(K_X + B) \cdot C < 0$ by Lemma 3.1. Since $K_X + B$ is f-nef, $f(C)$ is not a point. On the other hand, S contains no rational curves by the assumption. It is a contradiction. Therefore, $K_X + B$ is nef. □

Therefore, the following lemma is obvious by Definition 2.6 and Lemma 3.2.

Lemma 3.3. *Let $f : X \to S$ be a proper surjective morphism between projective varieties. Let B be an effective $\mathbb{R}$-divisor on X such that (X, B) is klt. Let (X', B') be a log terminal model of (X, B) over S. Assume that S contains no rational curves. Then (X', B') is a log terminal model of (X, B).*

We give an easy consequence of the main theorem of [BCHM].

Theorem 3.4 (Existence of log terminal models)**.** *Let X be a normal projective variety and B an effective $\mathbb{R}$-divisor on X such that (X, B) is klt. Assume that X has maximal Albanese dimension. Then (X, B) has a log terminal model.*

Proof. Let $\pi : \overline{X} \to X$ be a resolution and $\alpha : \overline{X} \to \mathrm{Alb}(\overline{X})$ the Albanese mapping of $\overline{X}$. Since X has only rational singularities, $\overline{X} \to S = \alpha(\overline{X})$ decomposes as

$$\alpha : \overline{X} \xrightarrow{\pi} X \xrightarrow{f} S.$$

See, for example, [BS, Lemma 2.4.1]. Since X has maximal Albanese dimension, $f : X \to S$ is generically finite. By [BCHM, Theorem 1.2], there exists a log terminal model $f : (X', B') \to S$ of (X, B) over S. By Lemma 3.3, (X', B') is a log terminal model of (X, B). $\square$

The following corollary is obvious by Theorem 3.4.

Corollary 3.5 (Minimal models)**.** *Let X be a smooth projective variety with maximal Albanese dimension. Then X has a minimal model.*

3.2. **Abundance theorem.** Let us consider the abundance theorem for maximal Albanese dimensional varieties.

Theorem 3.6 (Abundance theorem)**.** *Let X be a projective variety with only canonical singularities. Assume that X has maximal Albanese dimension. If K_X is nef, then K_X is semi-ample.*

Proof. By Lemma 3.7, we have $\kappa(X, K_X) \geq 0$. Therefore, if $\nu(X, K_X) = 0$, then $\kappa(X, K_X) = \nu(X, K_X) = 0$. Thus, X is an Abelian variety by Proposition 3.8. In particular, K_X is semi-ample. By Lemma 3.9, it is sufficient to see that $\nu(X, K_X) > 0$ implies $\kappa(X, K_X) > 0$ because $\nu(X, K_X) = \kappa(X, K_X)$ means K_X is semi-ample by [K2, Theorem 1.1] and [F2, Corollary 2.5]. See also Remark 3.11 below. By Proposition 3.8, we know that $\nu(X, K_X) > 0$ implies $\kappa(X, K_X) > 0$. We finish the proof. $\square$

Lemma 3.7. *Let X be a projective variety with only canonical singularities. Assume that X has maximal Albanese dimension. Then $\kappa(X, K_X) \geq 0$.*

Proof. Since X has only canonical singularities, we can assume that X is smooth by replacing X with its resolution. Then this lemma is obvious by the basic properties of the Kodaira dimension (cf. [U, Theorem 6.10]). Note that every subvariety of an Abelian variety has non-negative Kodaira dimension (cf. [U, Lemma 10.1]). $\square$

Proposition 3.8. *Let X be a projective variety with only canonical singularities. Assume that X has maximal Albanese dimension, K_X is nef, and $\kappa(X, K_X) = 0$. Then X is an Abelian variety. In particular, X is smooth and $K_X \sim 0$.*

Proof. We know that $f : X \to S = \mathrm{Alb}(\overline{X})$ is birational by [K1, Theorem 1], where $\overline{X}$ is a resolution of X and $\mathrm{Alb}(\overline{X})$ is the Albanese variety of $\overline{X}$. We can write

$$K_X = f^*K_S + E$$

such that E is effective and $\mathrm{Supp}E = \mathrm{Exc}(f)$. Since K_X is nef and $K_S \sim 0$, we obtain $E = 0$. This means that $f : X \to S$ is an isomorphism. □

Lemma 3.9 (cf. [K2, Theorem 7.3]). *Let X be a projective variety with only canonical singularities. Assume that X has maximal Albanese dimension and that K_X is nef. If $\nu(X, K_X) > 0$ implies $\kappa(X, K_X) > 0$, then $\nu(X, K_X) = \kappa(X, K_X)$.*

Sketch of the proof. The proof of [K2, Theorem 7.3] works without any changes. We give some comments for the reader's convenience. We use the same notation as in the proof of [K2, Theorem 7.3]. By Lemma 2.4, W has maximal Albanese dimension. On the other hand, $\kappa(W) = 0$ by the construction. By Proposition 3.8, W_{min} is an Abelian variety. In particular, $K_{W_{min}} \sim 0$. □

Remark 3.10. In the above proof of the abundance theorem, we did not use [K3, Theorem 8.2]. It is because we have Proposition 3.8.

Remark 3.11. On the assumption that $\nu(X, K_X) = \kappa(X, K_X)$, K_X is semi-ample if and only if the canonical ring of X is finitely generated. Therefore, by [BCHM], we can check the semi-ampleness of K_X without appealing [K2, Theorem 1.1] and [F2, Corollary 2.5].

By Corollary 3.5 and Theorem 3.6, we obtain the main result of this short paper.

Corollary 3.12 (Good minimal models). *Let X be a smooth projective variety with maximal Albanese dimension. Then X has a good minimal model. This means that there is a normal projective variety X' with only $\mathbb{Q}$-factorial terminal singularities such that X' is birationally equivalent to X and $K_{X'}$ is semi-ample.*

3.3. **Iitaka–Viehweg's conjecture.** By combining Corollary 3.12 with the main theorem of [K3], we obtain the following theorem. We write it for the reader's convenience. For the details, see [K3].

Theorem 3.13 (cf. [K3, Theorem 1.1]). *Let $f : X \to S$ be a surjective morphism between smooth projective varieties with connected fibers and $\mathcal{L}$ a line bundle on S. Assume that the geometric generic fiber of f has maximal Albanese dimension. Then the following assertions hold*:

(i) *There exists a positive integer* n *such that*

$$\kappa(S, \widehat{\det}(f_*(\omega^n_{X/S}))) \geq \mathrm{Var}(f).$$

(ii) *If* $\kappa(S, \mathcal{L}) \geq 0$*, then*

$$\kappa(X, \omega_{X/S} \otimes f^*\mathcal{L}) \geq \kappa(X_\eta) + \max\{\kappa(S, \mathcal{L}), \mathrm{Var}(f)\},$$

where X_η *is the generic fiber of* f.

The next corollary is a special case of [K3, Corollary 1.2] (cf. [F1, Theorem 0.2].

Corollary 3.14. *Under the same assumptions as in* Theorem 3.13*,*

(i) $\kappa(X, \omega_{X/S}) \geq \kappa(X_\eta) + \mathrm{Var}(f)$*, and*
(ii) *if* $\kappa(S) \geq 0$*, then* $\kappa(X) \geq \kappa(X_\eta) + \max\{\kappa(S), \mathrm{Var}(f)\}$.

3.4. **Ueno's problem.** The final theorem is a supplement to [U, Remark 10.13]. It is an easy consequence of Lemma 3.1.

Theorem 3.15. *Let* X *be a projective variety and* B *an effective* $\mathbb{R}$*-divisor on* X *such that* (X, B) *is log canonical. Assume that there are no rational curves on* X*. Then* $K_X + B$ *is nef.*

Furthermore, we assume that (X, B) *is klt,* B *is an effective* $\mathbb{Q}$*-divisor, and* $K_X + B$ *is big. Then* $K_X + B$ *is ample.*

Proof. The first half of this theorem is obvious by Lemma 3.1. If (X, B) is klt, B is an effective $\mathbb{Q}$-divisor, and $K_X + B$ is big, then $K_X + B$ is semi-ample by the base point free theorem since $K_X + B$ is nef. Then there exists a birational morphism

$$f = \Phi_{|m(K_X+B)|} : X \to S$$

with $f_*\mathcal{O}_X \simeq \mathcal{O}_S$ for some large and divisible integer m. By the construction, there is an ample Cartier divisor H on S such that $m(K_X + B) \sim f^*H$. Assume that f is not an isomorphism. Let A be an f-ample Cartier divisor. Then there is an effective Cartier divisor D on X such that $D \sim -A + f^*lH$ for some large integer l. Therefore, $(X, B + \varepsilon D)$ is klt and $K_X + B + \varepsilon D$ is not nef for $0 < \varepsilon \ll 1$ since f is not an isomorphism. By Lemma 3.1, there exists a rational curve C on X such that $(K_X + B + \varepsilon D) \cdot C < 0$. It is a contradiction because there are no rational curves on X. Therefore, f is an isomorphism. Thus, $K_X + B$ is ample. □

Corollary 3.16 (cf. [U, Remark 10.13])**.** *Let* W *be a submanifold of a complex torus* T *with* $\kappa(W) = \dim W$*. Then* K_W *is ample.*

Proof. By [U, Lemma 10.8], there is an Abelian variety A which is a complex subtorus of T such that $W \subset A$. Thus, W is projective. Then, by Theorem 3.15, we obtain K_W is ample. □

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