

THE KERNEL OF THE MAGNUS REPRESENTATION OF THE AUTOMORPHISM GROUP OF A FREE GROUP IS NOT FINITELY GENERATED

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ABSTRACT. In this paper, we show that the abelianization of the kernel of the Magnus representation of the automorphism group of a free group is not finitely generated.

1. INTRODUCTION

Let F_n be a free group of rank $n \geq 2$, and $\text{Aut } F_n$ the automorphism group of F_n . Let denote $\rho : \text{Aut } F_n \rightarrow \text{Aut } H$ the natural homomorphism induced from the abelianization $F_n \rightarrow H$. The kernel of ρ is called the IA-automorphism group of F_n , denoted by IA_n . The subgroup IA_n reflects many richness and complexity of the structure of $\text{Aut } F_n$, and plays important roles on various studies of $\text{Aut } F_n$.

Although the study of the IA-automorphism group has a long history since its finitely many generators were obtained by Magnus [14] in 1935, the combinatorial group structure of IA_n is still quite complicated. For instance, any presentation for IA_n is not known in general. Nielsen [19] showed that IA_2 coincides with the inner automorphism group, hence, is a free group of rank 2. For $n \geq 3$, however, IA_n is much larger than the inner automorphism group $\text{Inn } F_n$. Krstić and McCool [11] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not.

In general, one of the most standard ways to study a group is to consider its representations. If a representation of the group is given, it is important to determine whether it is faithful or not. Furthermore, if not, it is also worth studying how far it is from faithful, namely, to determine its kernel. In this paper, we consider these matters for the Magnus representation

$$r_M : \text{IA}_n \rightarrow \text{GL}(n, \mathbf{Z}[H])$$

of IA_n . (See Subsection 2.3.)

Classically, the Magnus representation was used to study certain subgroups of IA_n . One of the most famous subgroup is the pure braid group P_n . The restriction of r_M to P_n is called the Gassner representation. (For a basic materials for the braid group and the pure braid group, see [4] for example.) It is known due to Magnus and Pelso [15] that $r_M|_{P_n}$ is faithful for $n = 3$. Although the faithfulness of $r_M|_{P_n}$ has been studied

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for a long time by many authors, it seems, however, still open problem to determine it for $n \geq 4$.

Another important subgroup is the Torelli $\mathcal{I}_{g,1}$ subgroup of the mapping class group of a surface. By a classical work of Dehn and Nielsen, the Torelli group $\mathcal{I}_{g,1}$ is considered as a subgroup of IA_{2g} . Suzuki [22] showed that the restriction $r_M|_{\mathcal{I}_{g,1}}$ to $\mathcal{I}_{g,1}$ is not faithful for $g \geq 2$. Furthermore, by a recent remarkable work of Church and Farb [7], it is known that the abelianization of the kernel of $r_M|_{\mathcal{I}_{g,1}}$ is not finitely generated for $g \geq 2$.

From the facts as mentioned above, it is immediately seen that r_M itself is not faithful. There are, however, few results for the study of the kernel of r_M . Let \mathcal{K}_n be the kernel of r_M . In our previous paper [21], we showed that the abelianization $\mathcal{K}_n^{\text{ab}}$ of \mathcal{K}_n contains a certain free abelian group of finite rank. In this paper, we show that r_M is far from faithful. That is,

Theorem 1. (= Theorem 4.1 and Proposition 5.2.) *For $n \geq 2$, $\mathcal{K}_n^{\text{ab}}$ is not finitely generated.*

In order to prove the theorem for $n \geq 3$, we consider embeddings of \mathcal{K}_n into the IA-automorphism group of a free group of higher rank. Then, taking advantage of the first Johnson homomorphisms, we detect infinitely many linearly independent elements in $\mathcal{K}_n^{\text{ab}}$. For $n = 2$, in particular, we show

Proposition 1. $\mathcal{K}_2 = [[F_2, F_2], [F_2, F_2]]$

by using a usual and classical argument in the combinatorial group theory.

This paper consists of six sections. In Section 2, we recall the IA-automorphism group and the Magnus representation of the automorphism group of a free group. In Section 3, we prove the main theorem for $n \geq 3$. In Section 5, we consider the case where $n = 2$.

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2. PRELIMINARIES

In this section, after fixing notation and conventions, we recall the IA-automorphism group and the Magnus representation of $\text{Aut } F_n$.

2.1. Notation and conventions.

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G .

- The abelianization of G is denoted by G^{ab} .
- For any group G , we denote by $\mathbf{Z}[G]$ the integral group ring of G .
- The group $\text{Aut } G$ of G acts on G from the right. For any $\sigma \in \text{Aut } G$ and $x \in G$, the action of σ on x is denoted by x^σ .
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For elements x and y of G , the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. IA-automorphism group.

In this paper, we fix a basis x_1, \dots, x_n of F_n . Let $H := F_n^{\text{ab}}$ be the abelianization of F_n and $\rho : \text{Aut } F_n \rightarrow \text{Aut } H$ the natural homomorphism induced from the abelianization of F_n . In the following, we identify $\text{Aut } H$ with the general linear group $\text{GL}(n, \mathbf{Z})$ by fixing the basis of H induced from the basis x_1, \dots, x_n of F_n . The kernel IA_n of ρ is called the IA-automorphism group of F_n . It is clear that the inner automorphism group $\text{Inn } F_n$ of F_n is contained in IA_n . In general, however, IA_n for $n \geq 3$ is much larger than $\text{Inn } F_n$. In fact, Magnus [14] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} x_j^{-1}x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i, j \in \{1, 2, \dots, n\}$ and

$$K_{ijl} : x_t \mapsto \begin{cases} x_i[x_j, x_l], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i, j, l \in \{1, 2, \dots, n\}$ such that $j < l$. In this paper, for the convenience, we also consider an automorphism K_{ijl} for $j \geq l$ defined as above. We remark that $K_{ijj} = 1$ and $K_{ilj} = K_{ijl}^{-1}$.

Recently, Cohen-Pakianathan [5, 6], Farb [8] and Kawazumi [10] independently showed

$$(1) \quad \text{IA}_n^{\text{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

as a $\text{GL}(n, \mathbf{Z})$ -module where $H^* := \text{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ is the \mathbf{Z} -linear dual group of H . This isomorphism is induced from the first Johnson homomorphism

$$\tau_1 : \text{IA}_n \rightarrow H^* \otimes_{\mathbf{Z}} \Lambda^2 H.$$

(For a basic materials for the Johnson homomorphisms, see [16], Kawazumi [10] and [20] for example.) Then we see that (the coset classes of) the Magnus generators K_{ij} s and K_{ijk} s form a basis of IA_n^{ab} as a free abelian group.

In Section 4, we use the following lemmas.

Lemma 2.1. *For $n \geq 4$, and distinct integers $1 \leq i, j, l, m \leq n$, let $\omega \in \text{IA}_n$ be an automorphism defined by*

$$x_t \mapsto \begin{cases} x_i x_j x_l^{-1} x_m x_j^{-1} x_l x_m^{-1}, & t = i, \\ x_t, & t \neq i. \end{cases}$$

Then, we have

$$\omega = K_{il} K_{iml} K_{ijm} K_{ilj} K_{il}^{-1} \in \text{IA}_n,$$

and in particular,

$$\omega = K_{iml} + K_{ijm} + K_{ilj} \in \text{IA}_n^{\text{ab}}.$$

Proof. Since $K_{il} K_{iml} K_{ijm} K_{ilj} K_{il}^{-1}$ fix x_t for $t \neq i$, it suffices to consider the image of x_i . Then we see

$$\begin{aligned} x_i &\xrightarrow{K_{il}} x_l^{-1} x_i x_l \xrightarrow{K_{iml}} x_l^{-1} x_i [x_m, x_l] x_l = x_l^{-1} x_i x_m x_l x_m^{-1} \xrightarrow{K_{ijm}} x_l^{-1} x_i [x_j, x_m] x_m x_l x_m^{-1} \\ &= x_l^{-1} x_i x_j x_m x_j^{-1} x_l x_m^{-1} \xrightarrow{K_{ilj}} x_l^{-1} x_i [x_l, x_j] x_j x_m x_j^{-1} x_l x_m^{-1} \\ &= x_l^{-1} x_i x_l x_j x_l^{-1} x_m x_j^{-1} x_l x_m^{-1} \xrightarrow{K_{il}^{-1}} x_i x_j x_l^{-1} x_m x_j^{-1} x_l x_m^{-1}. \end{aligned}$$

This completes the proof of Lemma 2.1. \square

Similarly, we obtain the following lemmas. Since the proofs are done by a straightforward calculation as above, we leave them to the reader as exercises.

Lemma 2.2. *For $n \geq 5$, and distinct integers $1 \leq i, j, l, m, p \leq n$, let $\omega \in \text{IA}_n$ be an automorphism defined by*

$$x_t \mapsto \begin{cases} x_i x_j x_l^{-1} x_p x_m x_p^{-1} x_j^{-1} x_l x_p x_m^{-1} x_p^{-1}, & t = i, \\ x_t, & t \neq i. \end{cases}$$

Then, we have

$$\omega = K_{mp} K_{il} K_{iml} K_{ijm} K_{ilj} K_{il}^{-1} K_{mp}^{-1} \in \text{IA}_n,$$

and in particular,

$$\omega = K_{iml} + K_{ijm} + K_{ilj} \in \text{IA}_n^{\text{ab}}.$$

Lemma 2.3. *For $n \geq 5$, and distinct integers $1 \leq i, j, l, m, p \leq n$, let $\omega \in \text{IA}_n$ be an automorphism defined by*

$$x_t \mapsto \begin{cases} x_i x_p x_j x_p^{-1} x_l^{-1} x_p x_m x_p^{-1} x_p x_j^{-1} x_p^{-1} x_l x_p x_m^{-1} x_p^{-1}, & t = i, \\ x_t, & t \neq i. \end{cases}$$

Then, we have

$$\omega = K_{jp} K_{mp} K_{il} K_{iml} K_{ijm} K_{ilj} K_{il}^{-1} K_{mp}^{-1} K_{jp}^{-1} \in \text{IA}_n,$$

and in particular,

$$\omega = K_{iml} + K_{ijm} + K_{ilj} \in \text{IA}_n^{\text{ab}}.$$

2.3. Magnus representation.

In this subsection we recall the Magnus representation of $\text{Aut } F_n$. (For details, see [4].) For each $1 \leq i \leq n$, let

$$\frac{\partial}{\partial x_i} : \mathbf{Z}[F_n] \rightarrow \mathbf{Z}[F_n]$$

be the Fox derivation defined by

$$\frac{\partial}{\partial x_i}(w) = \sum_{j=1}^r \epsilon_j \delta_{\mu_j, i} x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_j}^{\frac{1}{2}(\epsilon_j - 1)} \in \mathbf{Z}[F_n]$$

for any reduced word $w = x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_r}^{\epsilon_r} \in F_n$, $\epsilon_j = \pm 1$. Let $\mathbf{a} : F_n \rightarrow H$ be the abelianization of F_n . We also denote by \mathbf{a} the ring homomorphism $\mathbf{Z}[F_n] \rightarrow \mathbf{Z}[H]$ induced from \mathbf{a} . For any matrix $A = (a_{ij}) \in \text{GL}(n, \mathbf{Z}[F_n])$, set $A^{\mathbf{a}} = (a_{ij}^{\mathbf{a}}) \in \text{GL}(n, \mathbf{Z}[H])$. The Magnus representation $r_M : \text{Aut } F_n \rightarrow \text{GL}(n, \mathbf{Z}[H])$ of $\text{Aut } F_n$ is defined by

$$\sigma \mapsto \left(\frac{\partial x_i^\sigma}{\partial x_j} \right)^{\mathbf{a}}$$

for any $\sigma \in \text{Aut } F_n$. This map is not a homomorphism but a crossed homomorphism. Namely,

$$r_M(\sigma\tau) = r_M(\sigma)^{\tau^*} \cdot r_M(\tau)$$

where $r_M(\sigma)^{\tau^*}$ denotes the matrix obtained from $r_M(\sigma)$ by applying the automorphism $\tau^* : \mathbf{Z}[H] \rightarrow \mathbf{Z}[H]$ induced from $\rho(\tau) \in \text{Aut}(H)$ on each entry. Hence by restricting r_M to IA_n , we obtain a homomorphism $\text{IA}_n \rightarrow \text{GL}(n, \mathbf{Z}[H])$, also denote it by r_M .

Let \mathcal{K}_n be the kernel of the homomorphism $r_M : \text{IA}_n \rightarrow \text{GL}(n, \mathbf{Z}[H])$. The main purpose of the paper is to show that the abelianization $\mathcal{K}_n^{\text{ab}}$ of \mathcal{K}_n is not finitely generated. More precisely, we prove that it contains a free abelian group of infinite rank.

Here we consider the group \mathcal{K}_n from the view point of the metabelianization of F_n . Let F_n^M be the quotient group of F_n by the second derived subgroup $[[F_n, F_n], [F_n, F_n]]$ of F_n . The group F_n^M is called the free metabelian group of rank n . The abelianization of F_n^M is canonically isomorphic to $H = F_n^{\text{ab}}$. Let $\mu_n : \text{Aut } F_n \rightarrow \text{Aut } F_n^M$ be the induced homomorphism from the action of $\text{Aut } F_n$ on F_n^M . It is known that μ_n is surjective for $n \geq 2$ except for $n = 3$. (See [1] for $n = 2$, and see [3] for $n \geq 4$.) Let IA_n^M be the IA-automorphism group of F_n^M . Namely, the group IA_n^M consists of automorphisms of F_n^M which act on the abelianization of F_n^M trivially. Then the restriction of μ_n induces a homomorphism $\text{IA}_n \rightarrow \text{IA}_n^M$, also denoted by μ_n .

Now, in [1], Bachmuth constructed a faithful representation

$$r'_M : \text{IA}_n^M \rightarrow \text{GL}(n, \mathbf{Z}[H])$$

using the Magnus representation of F_n^M . Then we can easily see that $r_M = r'_M \circ \mu_n$ by observing the image of the Magnus generators of IA_n . (For details, see [1].) Therefore, the faithfulness of r'_M induces $\mathcal{K}_n = \text{Ker}(\mu_n)$. In particular, we have

$$\mathcal{K}_n = \{ \sigma \in \text{Aut } F_n \mid x^{-1}x^\sigma \in [[F_n, F_n], [F_n, F_n]], x \in F_n \} \subset \text{IA}_n.$$

3. EMBEDDINGS OF THE IA-AUTOMORPHISM GROUP

In this section, we consider certain finitely generated subgroups of the free group F_n , and its automorphism group.

3.1. Subgroups $W_{n,d}$ of F_n .

For any integer $d \geq 2$, let C_d be the cyclic group of order d generated by s . Let $\iota_{n,d} : F_n \rightarrow C_d$ be a homomorphism defined by

$$x_i^{\iota_{n,d}} := \begin{cases} s, & i = 1, \\ 1, & i \neq 1. \end{cases}$$

We denote by $W_{n,d}$ the kernel of $\iota_{n,d}$. Then, $W_{n,d}$ is also a free group, and its rank is given by $d(n-1) + 1$ since the index of $W_{n,d}$ in F_n is d . Furthermore, applying the Reidemeister-Schreier method to the following data:

$$\begin{aligned} X &:= \{x_1, x_2, \dots, x_n\}; \text{ the set of generators of } F_n, \\ T &:= \{1, x_1, \dots, x_1^{d-1}\}; \text{ a Schreier-transversal of } W_{n,d} \text{ of } F_n, \end{aligned}$$

we see that a set

$$(2) \quad \{(t, x) \in W_{n,d} \mid t \in T, x \in X, (t, x) \neq 1\}$$

form a basis of $W_{n,d}$, where $(t, x) = tx(\overline{tx})^{-1}$ and a map $\bar{\cdot} : F_n \rightarrow T$ is defined by the condition that

$$W_{n,d} y = W_{n,d} \bar{y} \subset F_n$$

for any $y \in F_n$. (For details for the Reidemeister-Schreier method, see [12] for example.)

Then the set (2) is written down as

$$(3) \quad \begin{aligned} &x_1^d, \\ &x_2, x_1 x_2 x_1^{-1}, \dots, x_1^{d-1} x_2 x_1^{-(d-1)}, \\ &\quad \quad \quad \vdots \\ &x_n, x_1 x_n x_1^{-1}, \dots, x_1^{d-1} x_n x_1^{-(d-1)}. \end{aligned}$$

In the following, we fit this set as a basis of $W_{n,d}$.

3.2. Action of IA_n on $W_{n,d}$.

Since each of $W_{n,d}$ contains the commutator subgroup $[F_n, F_n]$ of F_n , the IA-automorphism group

$$\text{IA}_n = \{\sigma \in \text{Aut } F_n \mid x^{-1}x^\sigma \in [F_n, F_n], x \in F_n\}$$

naturally acts on them. Let $\rho = \rho_{n,d} : \text{IA}_n \rightarrow \text{Aut } W_{n,d}$ be the homomorphism induced from the action. Then we have

Lemma 3.1. *For any $n \geq 2$ and $d \geq 2$, the homomorphism ρ is injective.*

Proof. For an element $\sigma \in \text{IA}_n$, assume $\rho(\sigma) = 1$. Since $x_i^\sigma = x_i$ for $2 \leq i \leq n$, in order to see $\sigma = 1$, it suffices to show $x_1^\sigma = x_1$.

Set $x_1^\sigma = x_1 v$ for some irreducible word v . Then we have

$$(4) \quad x_1^d = (x_1^d)^\sigma = (x_1^\sigma)^d = x_1 v x_1 v \cdots x_1 v,$$

In the right hand side of the equation above, no cancellation of words happen. In fact, if v is of type $x_1^{-1}w$ for some irreducible word w , then we have $x_1^d = w^d$. By a usual argument in the Combinatorial group theory, there exists some $z \in F_n$ such that $x_1 = z^a$ and $w = z^b$ for some $a, b \in \mathbf{Z}$. (See proposition 2.17. in [12].) From the former equation, we have $z = x_1$ and $a = 1$. Hence

$$x_1^d w^{-d} = x_1^{d-bd} = 1.$$

Since F_n is torsion free, $b = 1$. This shows that $w = x_1$. This is a contradiction to the irreducibility of $v = x_1^{-1}w$. On the other hand, If v is of type wx_1^{-1} for some irreducible word w , then we have

$$x_1^d = x_1^{-1} x_1^d x_1 = x_1^{-1} (x_1 w^d x_1^{-1}) x_1 = w^d.$$

By an argument similar to the above, we see that this is also contradiction.

Thus, observing the word length of both side of (4), we obtain that $v = 1$. This shows that $\sigma = 1$. \square

By this lemma, we can consider IA_n as a subgroup of each of $\text{Aut } W_{n,d}$. In the following, we identify IA_n with the image of it by ρ .

Now, let $U_{n,d}$ be the abelianization of $W_{n,d}$. We denote by $\text{IA}(W_{n,d})$ the IA-automorphism group of $W_{n,d}$. Namely, the group $\text{IA}(W_{n,d})$ consists of automorphisms of $W_{n,d}$ which act on $U_{n,d}$ trivially. Let us define an linear order among (3) by

$$x_1^d < x_1 x_2 x_1^{-1} < \dots < x_1^{d-1} x_2 x_1^{-(d-1)} < \dots < x_1 x_n x_1^{-1} < \dots < x_1^{d-1} x_n x_1^{-(d-1)}.$$

Then, by a result of Magnus [14] as mentioned in Subsection 2.2, the IA-automorphism group $\text{IA}(W_{n,d})$ is finitely generated by automorphisms

$$(5) \quad \begin{aligned} K_{\alpha,\beta} &: \alpha \mapsto \alpha^{-1}\beta\alpha, \quad \alpha < \beta, \\ K_{\alpha,\beta,\gamma} &: \alpha \mapsto \alpha[\beta,\gamma], \quad \alpha \neq \beta < \gamma \neq \alpha \end{aligned}$$

where α, β, γ are elements of the basis (3).

On the other hand, from (1), we have

$$\text{IA}(W_{n,d})^{\text{ab}} \cong U_{n,d}^* \otimes_{\mathbf{Z}} \Lambda^2 U_{n,d}$$

where $U_{n,d}^*$ denotes the \mathbf{Z} linear dual group $\text{Hom}_{\mathbf{Z}}(U_{n,d}, \mathbf{Z})$ of $U_{n,d}$. This isomorphism is induced from the first Johnson homomorphism

$$\tau_1 : \text{IA}(W_{n,d}) \rightarrow U_{n,d}^* \otimes_{\mathbf{Z}} \Lambda^2 U_{n,d}.$$

In particular, (the coset classes of) the elements $K_{\alpha,\beta}$ s and $K_{\alpha,\beta,\gamma}$ s form a basis of $\text{IA}(W_{n,d})^{\text{ab}}$ as a free abelian group.

In general, the induced action of IA_n on $U_{n,d}$ by ρ is not trivial. However, its restriction to the kernel

$$\mathcal{K}_n := \{ \sigma \in \text{Aut } F_n \mid x^{-1}x^\sigma \in [[F_n, F_n], [F_n, F_n]], \quad x \in F_n \}$$

of the Magnus representation is trivial. Namely, ρ induces an embedding

$$\rho|_{\mathcal{K}_n} : \mathcal{K}_n \hookrightarrow \text{IA}(W_{n,d}).$$

By this embedding, we consider \mathcal{K}_n as a subgroup of $\text{IA}(W_{n,d})$ for any $d \geq 2$.

4. LOWER BOUNDS ON $H_1(\mathcal{K}_n, \mathbf{Z})$

In the following, we always assume $n \geq 3$ and $d \geq 3$. For each $2 \leq m \leq d-1$, let $\sigma_m \in \text{Aut } F_n$ be an automorphism defined by

$$x_i \mapsto \begin{cases} x_2[[x_1, x_3], [x_1^m, x_3]], & i = 2, \\ x_i, & i \neq 2. \end{cases}$$

Clearly, we see $\sigma_m \in \mathcal{K}_n$. We show that $\sigma_2, \dots, \sigma_{d-1}$ are linearly independent in $\mathcal{K}_n^{\text{ab}}$. To do this, we consider the images of σ_m by the natural homomorphism

$$\pi_{n,d} : \mathcal{K}_n \hookrightarrow \text{IA}(W_{n,d}) \rightarrow \text{IA}(W_{n,d})^{\text{ab}}.$$

To begin with, in order to describe σ_m in $\text{IA}(W_{n,d})^{\text{ab}}$ with $K_{\alpha,\beta}$ and $K_{\alpha,\beta,\gamma}$, we consider the action of σ_m on the basis (3) of $W_{n,d}$. Observing

$$x_2^{\sigma_m} = x_2 \cdot x_1 x_3 x_1^{-1} \cdot x_3^{-1} \cdot x_1^m x_3 x_1^{-m} \cdot x_1 x_3^{-1} x_1^{-1} \cdot x_3 \cdot x_1^m x_3^{-1} x_1^{-m},$$

we see

$$\begin{aligned} (x_1^d)^{\sigma_m} &= x_1^d, \\ (x_1^k x_i x_1^{-k})^{\sigma_m} &= x_1^k x_i x_1^{-k}, \quad i \neq 2, \\ (x_1^k x_2 x_1^{-k})^{\sigma_m} &= x_1^k x_2 x_1^{-k} \cdot x_1^{k+1} x_3 x_1^{-(k+1)} \cdot x_1^k x_3^{-1} x_1^{-k} \cdot x_1^{m+k} x_3 x_1^{-(m+k)} \\ &\quad \cdot x_1^{k+1} x_3^{-1} x_1^{-(k+1)} \cdot x_1^k x_3 x_1^{-k} \cdot x_1^{m+k} x_3^{-1} x_1^{-(m+k)}. \end{aligned}$$

Hence, for any $0 \leq k \leq d-1$, if we denote by $\omega_{m,k}$ the automorphism of $W_{n,d}$ defined by

$$(6) \quad \begin{aligned} x_1^k x_2 x_1^{-k} \mapsto & x_1^k x_2 x_1^{-k} \cdot x_1^{k+1} x_3 x_1^{-(k+1)} \cdot x_1^k x_3^{-1} x_1^{-k} \cdot x_1^{m+k} x_3 x_1^{-(m+k)} \\ & \cdot x_1^{k+1} x_3^{-1} x_1^{-(k+1)} \cdot x_1^k x_3 x_1^{-k} \cdot x_1^{m+k} x_3^{-1} x_1^{-(m+k)}, \end{aligned}$$

then

$$\sigma_m = \omega_{m,0} \omega_{m,2} \cdots \omega_{m,d-1}.$$

Here we remark that for any $0 \leq k, l \leq d-1$, the automorphisms $\omega_{m,k}$ and $\omega_{m,l}$ are commutative in $\text{Aut } W_{n,d}$.

Let us describe $\omega_{m,k}$ in $\text{IA}(W_{n,d})^{\text{ab}}$ with $K_{\alpha,\beta}$ and $K_{\alpha,\beta,\gamma}$ for each $0 \leq k \leq d-1$.

Case 1. The case of $0 \leq k \leq d-1-m$.

In this case, we have $k+1 \leq d-1$ and $m+k \leq d-1$. Thus, by (6) and Lemma 2.1, we see

$$\begin{aligned} \omega_{m,k} &= K_{x_1^k x_2 x_1^{-k}, x_1^k x_3 x_1^{-k}} K_{x_1^k x_2 x_1^{-k}, x_1^{m+k} x_3 x_1^{-(m+k)}, x_1^k x_3 x_1^{-k}} \\ &\quad \cdot K_{x_1^k x_2 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}, x_1^{m+k} x_3 x_1^{-(m+k)}} \\ &\quad \cdot K_{x_1^k x_2 x_1^{-k}, x_1^k x_3 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}} K_{x_1^k x_2 x_1^{-k}, x_1^k x_3 x_1^{-k}}^{-1} \end{aligned}$$

in $\text{IA}(W_{n,d})$, and hence,

$$\begin{aligned} \omega_{m,k} &= K_{x_1^k x_2 x_1^{-k}, x_1^{m+k} x_3 x_1^{-(m+k)}, x_1^k x_3 x_1^{-k}} + K_{x_1^k x_2 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}, x_1^{m+k} x_3 x_1^{-(m+k)}} \\ &\quad + K_{x_1^k x_2 x_1^{-k}, x_1^k x_3 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}}. \end{aligned}$$

in $\text{IA}(W_{n,d})^{\text{ab}}$.

Case 2. The case of $d - m \leq k \leq d - 2$.

In this case, we have $k + 1 \leq d - 1 < m + k$. Hence (6) is written as

$$\begin{aligned} x_1^k x_2 x_1^{-k} \mapsto & x_1^k x_2 x_1^{-k} \cdot x_1^{k+1} x_3 x_1^{-(k+1)} \cdot x_1^k x_3^{-1} x_1^{-k} \cdot x_1^d \cdot x_1^{m+k-d} x_3 x_1^{-(m+k-d)} \cdot x_1^{-d} \\ & \cdot x_1^{k+1} x_3^{-1} x_1^{-(k+1)} \cdot x_1^k x_3 x_1^{-k} \cdot x_1^d \cdot x_1^{m+k-d} x_3^{-1} x_1^{-(m+k-d)} \cdot x_1^{-d}. \end{aligned}$$

Then using Lemma 2.2, we obtain

$$\begin{aligned} \omega_{m,k} = & K_{x_1^k x_2 x_1^{-k}, x_1^{m+k-d} x_3 x_1^{-(m+k-d)}, x_1^k x_3 x_1^{-k}} + K_{x_1^k x_2 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}, x_1^{m+k-d} x_3 x_1^{-(m+k-d)}} \\ & + K_{x_1^k x_2 x_1^{-k}, x_1^k x_3 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}}. \end{aligned}$$

in $\text{IA}(W_{n,d})^{\text{ab}}$.

Case 3. The case of $k = d - 1$.

In this case, we have $k + 1 = d$ and $d \leq m + k$. Hence (6) is written as

$$\begin{aligned} x_1^{d-1} x_2 x_1^{-(d-1)} \mapsto & x_1^{d-1} x_2 x_1^{-(d-1)} \\ & \cdot x_1^d \cdot x_3 \cdot x_1^{-d} \cdot x_1^{d-1} x_3^{-1} x_1^{-(d-1)} \cdot x_1^d \cdot x_1^{m-1} x_3 x_1^{-(m-1)} \cdot x_1^{-d} \\ & \cdot x_1^d \cdot x_3^{-1} \cdot x_1^{-d} \cdot x_1^{d-1} x_3 x_1^{-(d-1)} \cdot x_1^d \cdot x_1^{m-1} x_3^{-1} x_1^{-(m-1)} \cdot x_1^{-d}. \end{aligned}$$

Then using Lemma 2.3, we obtain

$$\begin{aligned} \omega_{m,d-1} = & K_{x_1^{d-1} x_2 x_1^{-(d-1)}, x_1^{m-1} x_3 x_1^{-(m-1)}, x_1^{d-1} x_3 x_1^{-(d-1)}} + K_{x_1^{d-1} x_2 x_1^{-(d-1)}, x_3, x_1^{m-1} x_3 x_1^{-(m-1)}} \\ & + K_{x_1^{d-1} x_2 x_1^{-(d-1)}, x_1^{d-1} x_3 x_1^{-(d-1)}, x_3}. \end{aligned}$$

in $\text{IA}(W_{n,d})^{\text{ab}}$.

Therefore for $2 \leq m \leq d - 1$, we obtain

$$\begin{aligned} \sigma_m = & \sum_{0 \leq k \leq d-1-m} \{ K_{x_1^k x_2 x_1^{-k}, x_1^{m+k} x_3 x_1^{-(m+k)}, x_1^k x_3 x_1^{-k}} \\ & + K_{x_1^k x_2 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}, x_1^{m+k} x_3 x_1^{-(m+k)}} + K_{x_1^k x_2 x_1^{-k}, x_1^k x_3 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}} \} \\ & + \sum_{d-m \leq k \leq d-1} \{ K_{x_1^k x_2 x_1^{-k}, x_1^{m+k-d} x_3 x_1^{-(m+k-d)}, x_1^k x_3 x_1^{-k}} \\ & + K_{x_1^k x_2 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}, x_1^{m+k-d} x_3 x_1^{-(m+k-d)}} + K_{x_1^k x_2 x_1^{-k}, x_1^k x_3 x_1^{-k}, x_1^{k+1} x_3 x_1^{-(k+1)}} \} \end{aligned}$$

in $\text{IA}(W_{n,d})^{\text{ab}}$. Using this, we obtain

Lemma 4.1. *For any $n \geq 3$ and $d \geq 3$, the elements $\sigma_2, \dots, \sigma_{d-1}$ are linearly independent in $\text{IA}(W_{n,d})^{\text{ab}}$.*

Proof. Suppose

$$a_2 \sigma_2 + \dots + a_d \sigma_d = 0 \in \text{IA}(W_{n,d})^{\text{ab}}.$$

Then, observing the coefficients of $K_{x_2, *, *}$ in the left hand side of the equation above, we see

$$\sum_{2 \leq m \leq d-1} a_m \{ K_{x_2, x_1^m x_3 x_1^{-m}, x_3} + K_{x_2, x_1 x_3 x_1^{-1}, x_1^m x_3 x_1^{-m}} + K_{x_2, x_3, x_1 x_3 x_1^{-1}} \} = 0.$$

Hence, we have $a_2 = \dots = a_{d-1} = 0$. \square

This Lemma induces our main theorem.

Theorem 4.1. *For $n \geq 3$, $\mathcal{K}_n^{\text{ab}}$ is not finitely generated.*

Proof. For any $d \geq 3$, the image of the induced map

$$\mathcal{K}_n^{\text{ab}} \rightarrow \text{IA}(W_{n,d})^{\text{ab}}$$

from the natural homomorphism $\pi_{n,d} : \mathcal{K}_n \rightarrow \text{IA}(W_{n,d})^{\text{ab}}$ contains a free abelian group of rank $d - 2$. Since we can take $d \geq 3$ arbitrarily, we obtain the required result. This completes the proof of Theorem 4.1. \square

As a Corollary, we have

Corollary 4.1. *For $n \geq 3$, \mathcal{K}_n is not finitely generated.*

5. THE CASE WHERE $n = 2$

In this section, we show that \mathcal{K}_2 is not finitely generated. More precisely, we show that \mathcal{K}_2 is isomorphic to the second derived subgroup $[[F_2, F_2], [F_2, F_2]]$ of F_2 .

To begin with, we remark that IA_2 is equal to the inner automorphism group $\text{Inn } F_2$ of F_2 due to Nielsen [19]. Furthermore, Bachmuth [1] showed that IA_2^M coincides with $\text{Inn } F_2^M$. Hence we see

$$\mathcal{K}_2 = \text{Ker}(\text{Inn } F_2 \xrightarrow{\mu_2} \text{Inn } F_2^M).$$

In general, for any group G , a homomorphism from G onto the inner automorphism group $\text{Inn } G$ of G defined by

$$g \mapsto \iota_g = (x \mapsto g^{-1}xg, \quad x \in G)$$

induces an isomorphism $\text{Inn } G \cong G/Z(G)$ where $Z(G)$ denotes the center of G . Since the center of F_2 is trivial, we have $\text{Inn } F_2 \cong F_2$.

Here we show that $Z(F_n^M)$ is trivial for any $n \geq 2$. By a result of Magnus [13], F_n^M is faithfully represented in the group of 2×2 matrices of the form

$$\begin{pmatrix} x & L \\ 0 & 1 \end{pmatrix}$$

where $x \in H$ and L is a linear form $v_1t_1 + \cdots + v_nt_n$ in the indeterminates t_1, \dots, t_n with coefficients in $\mathbf{Z}[H]$. Here we identify F_n^M with the image of it by the representation. The image of $x_i \in F_n^M$ by this representation is given by

$$\begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix}.$$

Let

$$\begin{pmatrix} x & v_1t_1 + \cdots + v_nt_n \\ 0 & 1 \end{pmatrix} \in Z(F_n^M).$$

Then for any $1 \leq i \leq n$, we have

$$\begin{aligned} \begin{pmatrix} x & v_1t_1 + \cdots + v_nt_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} xx_i & v_1t_1 + \cdots + (v_i + x)t_i + \cdots + v_nt_n \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & v_1t_1 + \cdots + v_nt_n \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} x_ix & (x_iv_1)t_1 + \cdots + (x_iv_i + 1)t_i + \cdots + (x_iv_n)t_n \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and hence,

$$\begin{aligned} x_i v_j &= v_j, & 1 \leq j \leq n, & j \neq i, \\ v_i + x &= x_i v_i + 1. \end{aligned}$$

Since $\mathbf{Z}[H]$ is a domain, we see $v_j = 0$ for $j \neq i$ from the former equation above. Since we can take $1 \leq i \leq n$ arbitrarily, we obtain $v_1 = \cdots = v_n = 0 \in \mathbf{Z}[H]$. Then $x = 1 \in H$. Therefore we see $Z(F_n^M)$ is trivial.

By the argument above, we verify that $\text{Inn } F_2^M \cong F_2^M$, and that \mathcal{K}_2 is the kernel of the natural projection $F_2 \rightarrow F_2^M$. That is,

Proposition 5.1. $\mathcal{K}_2 = [[F_2, F_2], [F_2, F_2]]$.

By an easy argument in the Combinatorial group theory, we see that $[[F_2, F_2], [F_2, F_2]]$ is a free group of infinite rank as follows. First, we consider

Lemma 5.1. *If F is a free group of countably infinite rank, so is its commutator subgroup $[F, F]$.*

Proof. Let $Y := \{y_1, y_2, \dots\}$ be the set of generators of F . Then, applying the Reidemeister-Schreier method to Y and a Schreier-transversal

$$S := \{y_1^{e_1} y_2^{e_2} \cdots \mid e_i \in \mathbf{Z}, \text{ all but finite } e_i\text{'s are } 0.\}$$

of $[F, F]$ of F , we see that $[F, F]$ is a free group with basis

$$\{(s, y) = sy(\overline{sy})^{-1} \mid s \in S, y \in Y, (s, y) \neq 1\}.$$

Hence, it suffices to show the set above contains infinitely many elements.

Let us consider $y = y_1$ and $s = y_1^{e_1} y_2$ for any \mathbf{Z} . Then we have

$$(s, y) = y_1^{e_1} y_2 y_1 y_2^{-1} y_1^{-(e_1+1)} \neq 1$$

and

$$y_1^{e_1} y_2 y_1 y_2^{-1} y_1^{-(e_1+1)} \neq y_1^{e'_1} y_2 y_1 y_2^{-1} y_1^{-(e'_1+1)}$$

if $e_1 \neq e'_1$. This completes the proof of Lemma 5.1. \square

In general, since the commutator subgroup $[F_n, F_n]$ of the free group F_n is a free group of countably infinite rank for $n \geq 2$, using Lemma 5.1, we conclude

Proposition 5.2. *The group \mathcal{K}_2 is a free group of countably infinite rank for $n \geq 2$.*

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