FUNDAMENTAL THEOREMS FOR THE LOG MINIMAL MODEL PROGRAM

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ABSTRACT. In this paper, we prove the cone theorem and the contraction theorem for pairs (X, B), where X is a normal variety and B is an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier.

CONTENTS

1. Introduction

The main purpose of this paper is to prove the following cone and contraction theorem. It is the culmination of the works of several authors: Ambro, Benveniste, Birkar, Kawamata, Kollár, Mori, Reid, Shokurov, and others. It is indispensable for the study of the log minimal model program for log canonical pairs.

Theorem 1.1 (cf. Theorems 16.3, 16.5, 18.2, and 18.8). *Let X be a normal variety defined over $\mathbb{C}$ and B an effective $\mathbb{R}$-divisor such that $K_X + B$ is $\mathbb{R}$-Cartier, and $\pi : X \to S$ a projective morphism onto a variety S. Then we have*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+B\geq0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} + \sum R_j$$

with the following properties.

(1) $\mathrm{Nlc}(X, B)$ *is the non-lc locus of* (X, B) *and*

$$\overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} = \mathrm{Im}(\overline{NE}(\mathrm{Nlc}(X, B)/S) \to \overline{NE}(X/S)).$$

(2) R_j *is a* $(K_X + B)$*-negative extremal ray of* $\overline{NE}(X/S)$ *such that* $R_j \cap \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} = \{0\}$ *for every* j.

(3) *Let A be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many R_j's included in $(K_X + B + A)_{<0}$. In particular, the R_j's are discrete in the half-space $(K_X + B)_{<0}$.*

(4) *Let F be a face of* $\overline{NE}(X/S)$ *such that*

$$F \cap (\overline{NE}(X/S)_{K_X+B\geq0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)}) = \{0\}.$$

Then there exists a contraction morphism $\varphi_F : X \to Y$ over S.

(i) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.*

(ii) $\mathcal{O}_Y \simeq (\varphi_F)_* \mathcal{O}_X$.

(iii) *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Y on Y such that $L \simeq \varphi_F^* L_Y$.*

From now on, we further assume that (X, B) is log canonical, that is, $\mathrm{Nlc}(X, B) = \emptyset$. Then we have the following properties.

(5) *Every (K_X+B)-negative extremal ray R is spanned by a rational curve C with $-(K_X + B) \cdot C \leq 2 \dim X$.*

(6) *Let H be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $K_X + B + H$ is π-nef and $(X, B + H)$ is log canonical. Then, either $K_X + B$ is also π-nef or there is a $(K_X + B)$-negative extremal ray R such that $(K_X + B + \lambda H) \cdot R = 0$, where*

$$\lambda := \inf\{t \geq 0 \,|\, K_X + B + tH \text{ is } \pi\text{-nef}\}.$$

Of course, $K_X + B + \lambda H$ is π-nef.

The first half of Theorem 1.1 is the main result of [A1]. His proof depends on the theory of *quasi-log varieties.* In this paper, we give a proof of the above cone and contraction theorem without using the notion of quasi-log varieties. Our approach is much more direct than Ambro's theory of quasi-log varieties. We note that the reader does not have to refer to [A1] nor the book [F9] in order to read this paper. The latter half of Theorem 1.1 is a slight generalization of the results obtained by Kawamata, Shokurov, and Birkar. It will play important roles in the log minimal model program with scaling. So, we include this part in our cone and contraction theorem.

Let us briefly recall the history of the cone and contraction theorem. In the epoch-making paper [Mo], Mori invented the cone theorem for smooth projective variety and the contraction theorem for smooth projective threefold by his ingenious ideas. See, for example, [KM, Theorems 1.24 and 1.32]. After Mori's pioneering works, the cone and contraction theorem was proved and generalized for singular varieties by the completely different method, which is now called X-method (cf. [Ka1], [Ko1], [R], and [S1]). In [A1], Ambro introduced the notion of quasi-log varieties and generalized the cone and contraction theorem. See, for example, [F9, Chapter 3]. For the details of the history of the cone and contraction theorem up to [KMM], we recommend the reader to see the introductions of Chapters 2, 3, and 4 of [KMM].

We summarize the contents of this paper. Section 2 is a warm-up. Here, we discuss the base point free theorem for projective log canonical surfaces to motivate the reader. In Section 3, we explain our philosophy on various vanishing theorems. This section helps the reader to understand the following sections on our new vanishing theorems. Section 4 collects the preliminary definitions and results. In Section 5, we explain the Hodge theoretic aspect of the injectivity theorem. It is an easy consequence of the mixed Hodge theory on compact support cohomology groups. Section 6 treats generalizations of Kollár's injectivity, torsion-free, and vanishing theorems. These results will play crucial roles in the following sections. They will replace the Kawamata–Viehweg vanishing theorem. In Section 7, we introduce the notion of non-lc ideal sheaves. It is an analogue of the well-known multiplier ideal sheaves. Section 8 contains a very important vanishing theorem. It is a generalization of the Nadel vanishing theorem. It is very useful for the study of log canonical pairs. In Section 9, we recall the basic properties of lc centers. Section 10 treats the dlt modifications following Hacon. Here, we need [BCHM]. In Section 11, we give a vanishing theorem for minimal lc

centers. By the dlt modifications obtained in Section 10, we can easily prove this very important vanishing theorem. Section 12 is devoted to the proof of the non-vanishing theorem. In Section 13, we prove the base point free theorem. It is a direct consequence of the non-vanishing theorem. In Section 14, we quickly recall Shokurov's differents. Section 15 is devoted to the rationality theorem. In Section 16, we obtain the cone theorem and contraction theorem by using the rationality theorem and base point free theorem. Section 17 is a supplement to the base point free theorem. In Section 18, we discuss the estimate of lengths of extremal rays. It is very important for the study of the log minimal model program with scaling. In Section 19, we quickly explain some results which were obtained by the theory of quasi-log varieties but can not be covered by our new approach. In the final section: Section 20, we briefly explain some related topics obtained by the author.

This paper is a very expanded version of [F14]. The result in Section 10 heavily depends on [BCHM]. We use it to prove a vanishing theorem for minimal lc centers in Section 11. We note that we can prove the result in Section 11 without applying [BCHM] if we discuss the mixed Hodge theory on reducible varieties. It was carried out in [F9, Chapter 2]. We note that [F9, Chapter 2] is independent of the log minimal model program for klt pairs. So, the non-vanishing theorem: Theorem 12.1, the base point free theorem: Theorem 13.1, the rationality theorem: Theorem 15.1, and the cone theorem: Theorem 16.5 do not depend on the corresponding results for klt pairs. Therefore, our proofs are new even for klt pairs. In Section 18, we need Theorem 10.4, which is a consequence of [BCHM], to prove Theorem 18.2. There are no proofs of Theorem 18.2 without using [BCHM]. However, Theorem 18.2 can be directly proved if we have an appropriate vanishing theorem for projective morphisms between analytic spaces. For the details, see [F9, Remark 3.22].

Our approach which will be described in this paper has some advantages over Kawamata's X-method.

(A) It can be applied to log canonical pairs.
(B) We do not need Hironaka's resolution theorem.
(C) We are released from perturbations of coefficients.

We recommend the reader to compare our arguments in Sections 12, 13, and 15 with Kawamata's traditional X-method.

picture of a Kleiman-Mori cone. He also thanks Takeshi Abe for useful discussions.

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

2. Warm-ups

In this section, we explain the base point free theorem for projective log canonical surfaces to motivate the reader. The following theorem is a very special case of Theorem 13.1.

Theorem 2.1 (Base point free theorem for lc surfaces). *Let (X, B) be a projective log canonical surface. Let L be a nef Cartier divisor on X such that $aL - (K_X + B)$ is ample for some $a > 0$. Then $|mL|$ is base point free for $m \gg 0$.*

It can not be proved by the traditional X-method. A key ingredient of this paper is the following generalization of Kollár's vanishing theorem. We will describe it in Section 8.

Theorem 2.2 (cf. Theorem 8.1). *Let (X, B) be a projective log canonical pair. Let D be a Cartier divisor on X such that $D - (K_X + B)$ is ample. Let C be an lc center of (X, B) with a reduced scheme structure. Then*

$$H^i(X, \mathcal{I}_C \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$, where $\mathcal{I}_C$ is the defining ideal sheaf of C. In particular, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(C, \mathcal{O}_C(D))$$

is surjective.

In Theorem 2.2, we do not assume that C is isolated in the non-klt locus of the pair (X, B), neither do we assume that there exists another boundary $\mathbb{R}$-divisor B' on X such that (X, B') is klt. Therefore, it can not be proved by the traditional arguments depending on the Kawamata–Viehweg–Nadel vanishing theorem.

The next theorem is a special case of Theorem 12.1. This formulation was first introduced in [F14]. We will see that it is equivalent to Theorem 2.1.

Theorem 2.3 (Non-vanishing theorem for lc surfaces). *Let X be a projective log canonical surface. Let L be a nef Cartier divisor on X such that $aL - (K_X + B)$ is ample for some $a > 0$. Then the base locus $\mathrm{Bs}|mL|$ of $|mL|$ contains no lc centers of (X, B) for $m \gg 0$.*

Proof. It is sufficient to check that Bs$|mL|$ contains no minimal lc centers of (X, B) for $m \gg 0$. Let C be a minimal lc center of (X, B). If C is a point P, then Bs$|mL|$ does not contain C for every $m \geq a$. It is because the evaluation map

$$H^0(X, \mathcal{O}_X(mL)) \to \mathbb{C}(P) \simeq H^0(P, \mathcal{O}_P(mL))$$

is surjective for every $m \geq a$ by Theorem 2.2. If C is a curve, then $C \subset \llcorner B \lrcorner$ and (X, B) is plt around C. Therefore,

$$K_C + B_C = (K_X + B)|_C$$

is klt by adjunction. Since $aL|_C - (K_C + B_C)$ is ample, there exists m_1 such that $|mL|_C|$ is base point free for every $m \geq m_1$. By Theorem 2.2, the restriction map

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(C, \mathcal{O}_C(mL))$$

is surjective for every $m \geq a$. Thus, Bs$|mL|$ does not contain C for $m \gg 0$. So, we finish the proof since there are only finitely many minimal lc centers. $\qquad\square$

In the above proof, C is a point or a divisor on X. So, there are no difficulties to investigate minimal lc centers. When $\dim X \geq 3$, we need a more powerful vanishing theorem (cf. Theorem 11.1) to study linear systems on minimal lc centers.

Let us go to the proof of Theorem 2.1.

Proof of Theorem 2.1. If (X, B) is klt, then the statement is well-known as the Kawamata–Shokurov base point free theorem (cf. [KM, Theorem 3.3]). So, we can assume that (X, B) is lc but not klt. By Theorem 2.3, we can take general members $D_1, D_2, D_3 \in |m_1 L|$ for some $m_1 > 0$. We note that $(X, B + D)$, where $D = D_1 + D_2 + D_3$, is log canonical outside Bs$|m_1 L|$ and that $(X, B + D)$ is not log canonical at the generic point of every irreducible component of Bs$|m_1 L|$. Let c be the log canonical threshold of (X, B) with respect to D. Then $0 < c$ by Theorem 2.3 and $c < 1$ because $(X, B + D)$ is not log canonical. By the construction, $(X, B + cD)$ is log canonical and there is an lc center C of $(X, B + cD)$ such that C is contained in Bs$|m_1 L|$. By applying Theorem 2.3 to

$$(3cm_1 + a)L - (K_X + B + cD) \sim_{\mathbb{Q}} aL - (K_X + B)$$

on $(X, B + cD)$, we see that Bs$|m_2 m_1 L|$ does not contain C for $m_2 \gg 0$. Therefore, Bs$|m_2 m_1 L| \subsetneq$ Bs$|m_1 L|$ holds. By the noetherian induction, we obtain that L is semi-ample. With a little care, we can check that $|mL|$ is base point free for $m \gg 0$. We omit some details here. See the proof of Theorem 13.1. $\qquad\square$

One of the main purposes of this paper is to generalize Theorem 2.3 for pairs (X, B), where X is an n-dimensional normal variety and B is an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier (see Theorems 12.1 and 13.1).

3. Kawamata–Viehweg, Nadel, Kollár, $\cdots$

In this section, we explain our philosophy on vanishing theorems. The reader can skip this section.

In the traditional X-method, the following type of the Kawamata–Viehweg vanishing theorem plays crucial roles (cf. [KM, Theorem 3.1], [L, Theorem 9.1.18]).

3.1 (The Kawamata–Viehweg vanishing theorem)**.** Let X be a smooth projective variety and B an effective $\mathbb{Q}$-divisor such that $\mathrm{Supp}B$ is simple normal crossing and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X such that $L - (K_X + B)$ is nef and big. Then

$$H^i(X, \mathcal{O}_X(L)) = 0$$

for every $i > 0$.

Recently, the (algebraic version of) Nadel vanishing theorem, which is a generalization of the above Kawamata–Viehweg vanishing theorem, is very often used for the study of linear systems (cf. [L, Theorem 9.4.17]).

3.2 (The Nadel vanishing theorem)**.** Let X be a normal projective variety and B an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let L be a Cartier divisor on X such that $L - (K_X + B)$ is nef and big. Then

$$H^i(X, \mathcal{O}_X(L) \otimes \mathcal{J}(X, B)) = 0$$

for every $i > 0$, where $\mathcal{J}(X, B)$ is the multiplier ideal sheaf of the pair (X, B) (see Remark 7.3 below).

The following relative version of the Kawamata–Viehweg vanishing theorem sometimes plays very important roles implicitly (cf. [L, Theorem 9.4.17], [KM, Corollary 2.68]).

3.3 (The relative Kawamata–Viehweg vanishing theorem)**.** Let X be a normal projective variety and B an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a projective resolution such that $K_Y + B_Y = f^*(K_X + B)$ and that $\mathrm{Supp}B_Y$ is simple normal crossing. Then

$$R^i f_* \mathcal{O}_Y(-\llcorner B_Y \lrcorner) = 0$$

for every $i > 0$.

It is obvious that 3.1 is a special case of 3.2. It is a routine work to prove 3.3 by 3.1. We note that 3.2 can be obtained as a consequence of 3.1 and 3.3 by Hironaka's resolution and Leray's spectral sequence. In this paper, we see the Nadel vanishing theorem 3.2 (resp. the relative Kawamata–Viehweg vanishing theorem 3.3) as a special case of Kollár's vanishing theorem 3.4 (ii) (resp. Kollár's torsion-free theorem 3.4 (i)).

Let us recall Kollár's theorems (cf. [Ko2, 10.15 Corollary]).

3.4 (Kollár's torsion-free and vanishing theorems)**.** Let Y be a smooth projective variety and Δ an effective $\mathbb{Q}$-divisor on Y such that $\mathrm{Supp}\Delta$ is simple normal crossing and $\llcorner\Delta\lrcorner = 0$. Let $f : Y \to X$ be a surjective morphism onto a projective variety X and D a Cartier divisor on Y.

(i) If $D-(K_Y+\Delta) \sim_{\mathbb{Q}} f^*M$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor M on X, then $R^i f_* \mathcal{O}_Y(D)$ is torsion-free for every $i \geq 0$. In particular, $R^i f_* \mathcal{O}_Y(D) = 0$ for every $i > 0$ if f is birational.

(ii) If $D - (K_Y + \Delta) \sim_{\mathbb{Q}} f^*M$, where M is an ample $\mathbb{Q}$-divisor on X, then
$$H^i(X, R^j f_* \mathcal{O}_Y(D)) = 0$$
for every $i > 0$ and $j \geq 0$.

We will completely generalize it in Theorem 6.3. As we stated above, in this paper, 3.2 is not seen as a combination of 3.1 and 3.3. It should be recognized as a special case of Kollár's vanishing theorem 3.4 (ii). We do not see the vanishing theorem 3.3 as a relative vanishing theorem but as a special case of Kollár's torsion-free theorem 3.4 (i). This change of viewpoint opens the door to the study of log canonical pairs.

3.5 (Philosophy)**.** We note that 3.4 follows from the pure Hodge theory. In our philosophy, we have the following correspondences.

$$\boxed{\text{Kawamata log terminal pairs}} \Longleftrightarrow \boxed{\text{Pure Hodge structures}}$$

and

$$\boxed{\text{Log canonical pairs}} \Longleftrightarrow \boxed{\text{Mixed Hodge structures}}$$

Therefore, it is very natural to prove a "mixed" version of 3.4 for the study of log canonical pairs. We will carry it out in Sections 5 and 6. There is a big difference between our framework discussed in this paper (cf. Sections 12, 13, and 15) and the traditional X-method from the Hodge theoretic viewpoint. We believe that all the results for klt pairs can be proved without using the mixed Hodge theory (cf. [F12]).

3.6 (Further discussions). When we consider various extension theorems, which play crucial roles in the proof of the existence of pl flips (cf. [HM]), we think the following correspondence is natural.

$$\boxed{\text{Kawamata log terminal pairs}} \Longleftrightarrow \boxed{L^2\text{-method}}$$

The extension theorem in [HM] can be proved as a consequence of the usual vanishing theorems. However, we note that the origin of the extension theorem is the Ohsawa–Takegoshi L^2-extension theorem. The Nadel vanishing theorem also has its origin in the L^2-method. It is very natural to try to generalize the above correspondence for log canonical pairs. However, we do not know what should be in the right box in the correspondence below.

$$\boxed{\text{Log canonical pairs}} \Longleftrightarrow \boxed{?}$$

It is very desirable to fill the right box correctly. Here, we do not discuss this topic any more.

4. PRELIMINARIES

We will work over the complex number field $\mathbb{C}$ throughout this paper. But we note that by using the Lefschetz principle, we can extend almost everything to the case where the base field is an algebraically closed field of characteristic zero. In this paper, an *algebraic scheme* denotes a scheme which is separated and of finite type over $\mathbb{C}$. We collect the basic notation and definitions.

4.1 ($m \gg 0$). The expression '... for $m \gg 0$' means that 'there exists a positive number m_0 such that ... for every $m \geq m_0$.'

4.2 (Operations on $\mathbb{R}$-divisors). For an $\mathbb{R}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$ (resp. the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$), where for every real number x, $\ulcorner x \urcorner$ (resp. $\llcorner x \lrcorner$) is the integer defined by $x \leq \ulcorner x \urcorner < x + 1$ (resp. $x - 1 < \llcorner x \lrcorner \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$. We define

$$D^{=1} = \sum_{d_j = 1} D_j, \quad D^{\leq 1} = \sum_{d_j \leq 1} d_j D_j,$$

$$D^{<1} = \sum_{d_j < 1} d_j D_j, \quad \text{and} \quad D^{>1} = \sum_{d_j > 1} d_j D_j.$$

We call D a *boundary* $\mathbb{R}$-divisor if $0 \leq d_j \leq 1$ for every j. We note that $\sim_{\mathbb{Q}}$ (resp. $\sim_{\mathbb{R}}$) denotes the $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) equivalence of $\mathbb{Q}$-divisors (resp. $\mathbb{R}$-divisors). Let D_1 and D_2 be $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X and $f : X \to Y$ a morphism. We say that D_1 and D_2 are $\mathbb{R}$-*linearly f-equivalent*, denoted by $D_1 \sim_{\mathbb{R},f} D_2$, if and only if there is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on Y such that $D_1 \sim_{\mathbb{R}} D_2 + f^*B$. We can define $D_1 \sim_{\mathbb{Q},f} D_2$ for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors D_1 and D_2 similarly.

Definition 4.3 (Exceptional locus). For a proper birational morphism $f : X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

4.4 (Discrepancy, singularities of pairs, and non-lc locus). Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $f_*^{-1}B$ is the strict transform of B on Y. We write

$$K_Y = f^*(K_X + B) + \sum_i a_i E_i$$

and $a(E_i, X, B) = a_i$. We say that (X, B) is *lc* (resp. *klt*) if and only if $a_i \geq -1$ (resp. $a_i > -1$) for every i. Note that the *discrepancy* $a(E, X, B) \in \mathbb{R}$ can be defined for every prime divisor E *over* X. If $a(E, X, B) > -1$ for every exceptional divisor E over X, then the pair (X, B) is called *plt*. Here, lc (resp. klt, plt) is an abbreviation of *log canonical* (resp. *Kawamata log terminal*, *purely log terminal*). By the definition, there exists the largest Zariski open set U of X such that (X, B) is lc on U. We put $\mathrm{Nlc}(X, B) = X \setminus U$ and call it the *non-lc locus* of the pair (X, B). We sometimes simply denote $\mathrm{Nlc}(X, B)$ by X_{NLC}.

Let (X, B) be a log canonical pair and M an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. The *log canonical threshold* of (X, B) with respect to M is defined by

$$c = \sup\{t \in \mathbb{R} \,|\, (X, B + tM) \text{ is log canonical}\}.$$

Definition 4.5 (Center). Let E be a prime divisor over X. The closure of the image of E on X is denoted by $c_X(E)$ and called the *center* of E on X.

Definition 4.6 (Lc center). Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. If $a(E, X, B) = -1$ and $c_X(E)$ is not contained in $\mathrm{Nlc}(X, B)$, then $c_X(E)$ is called an *lc center* of (X, B).

We note that our definition of lc centers is slightly different from the usual one.

Definition 4.7 (Stratum). Let (X, B) be a log canonical pair. A *stratum* of (X, B) denotes X itself or an lc center of (X, B).

Let T be a simple normal crossing divisor on a smooth variety Y. A *stratum* of T denotes a stratum of the pair (Y, T) contained in T.

4.8 (Kleiman–Mori cone). Let X be an algebraic scheme over $\mathbb{C}$ and $\pi : X \to S$ a proper morphism to an algebraic scheme S. Let $\mathrm{Pic}(X)$ be the group of line bundles on X. Take a complete curve on X which is mapped to a point by π. For $\mathcal{L} \in \mathrm{Pic}(X)$, we define the intersection number $\mathcal{L} \cdot C = \deg_{\overline{C}} f^* \mathcal{L}$, where $f : \overline{C} \to C$ is the normalization of C. Via this intersection pairing, we introduce a bilinear form

$$\cdot : \mathrm{Pic}(X) \times Z_1(X/S) \to \mathbb{Z},$$

where $Z_1(X/S)$ is the free abelian group generated by integral curves which are mapped to points on S by π.

Now we have the notion of numerical equivalence both in $Z_1(X/S)$ and in $\mathrm{Pic}(X)$, which is denoted by $\equiv$, and we obtain a perfect pairing

$$N^1(X/S) \times N_1(X/S) \to \mathbb{R},$$

where

$$N^1(X/S) = \{\mathrm{Pic}(X)/ \equiv\} \otimes \mathbb{R} \quad \text{and} \quad N_1(X/S) = \{Z_1(X/S)/ \equiv\} \otimes \mathbb{R},$$

namely $N^1(X/S)$ and $N_1(X/S)$ are dual to each other through this intersection pairing. It is well known that

$$\dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S) < \infty.$$

We write $\rho(X/S) = \dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S)$. We define the Kleiman–Mori cone $\overline{NE}(X/S)$ as the closed convex cone in $N_1(X/S)$ generated by integral curves on X which are mapped to points on S by π. When $S = \mathrm{Spec}\,\mathbb{C}$, we drop $/\mathrm{Spec}\,\mathbb{C}$ from the notation, e.g., we simply write $N_1(X)$ in stead of $N_1(X/\mathrm{Spec}\,\mathbb{C})$.

Definition 4.9. An element $D \in N^1(X/S)$ is called π-*nef* (or *relatively nef for π*), if $D \geq 0$ on $\overline{NE}(X/S)$. When $S = \mathrm{Spec}\,\mathbb{C}$, we simply say that D is *nef*.

Definition 4.10 (Semi-ample $\mathbb{R}$-divisors). An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X is π-*semi-ample* if $D \sim_{\mathbb{R}} \sum_i a_i D_i$, where D_i is a π-semi-ample Cartier divisor on X and a_i is a positive real number for every i.

Remark 4.11. In Definition 4.10, we can replace $D \sim_{\mathbb{R}} \sum_i a_i D_i$ with $D = \sum_i a_i D_i$ since every principal Cartier divisor on X is π-semi-ample.

The following two lemmas seem to be missing in the literature.

Lemma 4.12. *Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then the following conditions are equivalent.*

(1) *D is π-semi-ample.*
(2) *There exists a morphism $f : X \to Y$ over S such that $D \sim_{\mathbb{R}} f^*A$, where A is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y which is ample over S.*

Proof. It is obvious that (1) follows from (2). If D is π-semi-ample, then we can write $D \sim_{\mathbb{R}} \sum_i a_i D_i$ as in Definition 4.10. By replacing D_i with its multiple, we can assume that $\pi^* \pi_* \mathcal{O}_X(D_i) \to \mathcal{O}_X(D_i)$ is surjective for every i. Let $f : X \to Y$ be a morphism over S obtained by the surjection $\pi^* \pi_* \mathcal{O}_X(\sum_i D_i) \to \mathcal{O}_X(\sum_i D_i)$. Then it is easy to see that $f : Y \to X$ has the desired property. $\square$

Lemma 4.13. *Let D be a Cartier divisor on X. If D is π-semi-ample in the sense of* Definition 4.10, *then D is π-semi-ample in the usual sense, that is, $\pi^* \pi_* \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective for some positive integer m. In particular,* Definition 4.10 *is well-defined.*

Proof. We write $D \sim_{\mathbb{R}} \sum_i a_i D_i$ as in Definition 4.10. Let $f : X \to Y$ be a morphism in Lemma 4.12 (2). By taking the Stein factorization, we can assume that f has connected fibers. By the construction, $D_i \sim_{\mathbb{Q},f} 0$ for every i. By replacing D_i with its multiple, we can assume that $D_i \sim f^* D_i'$ for some Cartier divisor D_i' on Y for every i. Let U be any Zariski open set of Y on which $D_i' \sim 0$ for every i. On $f^{-1}(U)$, we have $D \sim_{\mathbb{R}} 0$. This implies $D \sim_{\mathbb{Q}} 0$ on $f^{-1}(U)$ since D is Cartier. Therefore, there exists a positive integer m such that $f^* f_* \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective. By this surjection, we have $mD \sim f^*A$ for a Cartier divisor A on Y which is ample over S. This means that D is π-semi-ample in the usual sense. $\square$

Theorem 4.14 (Kleiman's criterion for ampleness)**.** *Let $\pi : X \to S$ be a* projective *morphism between algebraic schemes. Then $\mathcal{L} \in \mathrm{Pic}(X)$ is π-ample if and only if the numerical class of $\mathcal{L}$ in $N^1(X/S)$ gives a positive function on $\overline{NE}(X/S) \setminus \{0\}$.*

In Theorem 4.14, we note that the projectivity of π is indispensable (cf. [F2]).

We will repeatedly use the following easy lemma. We give a detailed proof for the reader's convenience.

Lemma 4.15. *Let X be a normal variety and B an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\lfloor B \rfloor = 0$. Let A be a Cartier divisor on X.*

Assume that $A \sim_{\mathbb{R}} B$. Then there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X such that $A \sim_{\mathbb{Q}} C$, $\llcorner C \lrcorner = 0$, and $\mathrm{Supp} C = \mathrm{Supp} B$.

Proof. We can write $B = A + \sum_{i=1}^{k} r_i(f_i)$, where $r_i \in \mathbb{R}$ and f_i is a rational function on X for every i. We put

$$E = \mathrm{Supp} A \cup \mathrm{Supp} B \cup \bigcup_{i=1}^{k} \mathrm{Supp}(f_i).$$

Let $E = \sum_{j=1}^{n} E_j$ be the irreducible decomposition of E. We can write

$$A = \sum_{j} a_j E_j, \quad B = \sum_{j} b_j E_j,$$

and

$$(f_i) = \sum_{j} m_{ij} E_j \quad \text{for every} \quad i.$$

We can assume that $b_j \in \mathbb{Q}$ for $1 \le j \le l$ and $b_j \notin \mathbb{Q}$ for $l+1 \le j \le n$. We note that $a_j \in \mathbb{Z}$ for every j and that $m_{ij} \in \mathbb{Z}$ for every i, j. We define

$$\mathcal{S} = \left\{ (v_1, \cdots, v_k) \in \mathbb{R}^k \; ; \; b_j = a_j + \sum_{i=1}^{k} v_i m_{ij} \text{ for } 1 \le j \le l \right\}.$$

Then $\mathcal{S}$ is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. We note that $\mathcal{S}$ is not empty since $(r_1, \cdots, r_k) \in \mathcal{S}$. If we take $(r'_1, \cdots, r'_k) \in \mathcal{S} \cap \mathbb{Q}^k$ which is very close to $(r_1, \cdots, r_k)$ and put $C = A + \sum_i r'_i(f_i)$, then it is obvious that C satisfies the desired properties. $\square$

The next lemma is well known as the negativity lemma.

Lemma 4.16 (Negativity lemma). *Let $h : Z \to Y$ be a proper birational morphism between normal varieties. Let $-B$ be an h-nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Z. Then we have the following statements.*

    (1) *B is effective if and only if $h_* B$ is.*
    (2) *Assume that B is effective. Then for every $y \in Y$, either $h^{-1}(y) \subset \mathrm{Supp} B$ or $h^{-1}(y) \cap \mathrm{Supp} B = \emptyset$.*

Sketch of the proof. By Chow's lemma, we can assume that h is projective. We can also assume that Y is affine. By taking general hypersurfaces, we can reduce the problem to the case when $\dim Y = 2$. Then we use the Hodge index theorem on Z. For the details, see [KM, Lemma 3.39]. $\square$

We close this section with the following useful lemma. It is a consequence of Szabó's resolution lemma.

Lemma 4.17. *Let Z be a smooth variety and B an $\mathbb{R}$-divisor on Z such that $\operatorname{Supp}B$ is simple normal crossing. Let $f : Z \to X$ be a projective morphism and $\overline{X}$ a projective variety such that $\overline{X}$ contains X as a Zariski open set. Then there exist a smooth projective variety $\overline{Z}$ and an $\mathbb{R}$-divisor $\overline{B}$ on $\overline{Z}$ such that*

    (i) *$f : Z \to X$ is extended to $\overline{f} : \overline{Z} \to \overline{X}$.*
    (ii) *$\operatorname{Supp}\overline{B}$ is simple normal crossing.*
    (iii) *$\operatorname{Supp}\overline{B} \cup \operatorname{Supp}(\overline{Z} \setminus Z)$ is simple normal crossing.*
    (iv) *$\overline{B}|_Z = B$.*

Proof. Let Z' be an arbitrary compactification of Z. By taking the graph of $f : Z' \dashrightarrow \overline{X}$ and using Hironaka's resolution, we can assume that Z' is smooth projective, $\operatorname{Supp}(Z' \setminus Z)$ is simple normal crossing, and $f : Z \to X$ is extended to $f' : Z' \to \overline{X}$. Let B' be the closure of B on Z'. We apply Szabó's resolution lemma (see, for example, [F4]) to $\operatorname{Supp}B' \cup \operatorname{Supp}(Z' \setminus Z)$. Then we obtain the desired variety $\overline{Z}$ and $\overline{B}$. By the construction, f can be extended to $\overline{f} : \overline{Z} \to \overline{X}$. $\square$

5. Hodge theoretic injectivity theorem

In this section, we will prove the following injectivity theorem, which is a generalization of [EV, 5.1. b)] for $\mathbb{R}$-divisors. We use the classical topology throughout this section.

Proposition 5.1 (Fundamental injectivity theorem)**.** *Let X be a smooth projective variety and $S + B$ a boundary $\mathbb{R}$-divisor on X such that the support of $S + B$ is simple normal crossing and $\llcorner S + B \lrcorner = S$. Let L be a Cartier divisor on X and D an effective Cartier divisor whose support is contained in $\operatorname{Supp}B$. Assume that $L \sim_{\mathbb{R}} K_X + S + B$. Then the natural homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Let us recall some results on the mixed Hodge theory.

5.2. Let V be a smooth projective variety and Σ a simple normal crossing divisor on V. Let $\iota : V \setminus \Sigma \to V$ be the natural open immersion. Then $\iota_! \mathbb{C}_{V \setminus \Sigma}$ is quasi-isomorphic to the complex $\Omega_V^\bullet(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)$. By this quasi-isomorphism, we can construct the following spectral sequence

$$E_1^{pq} = H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)) \Rightarrow H_c^{p+q}(V \setminus \Sigma, \mathbb{C}).$$

By the Serre duality, the right hand side

$$H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma))$$

is dual to

$$H^{n-q}(V, \Omega_V^{n-p}(\log \Sigma)),$$

where $n = \dim V$. By the Poincaré duality, $H_c^{p+q}(V \setminus \Sigma, \mathbb{C})$ is dual to $H^{2n-(p+q)}(V \setminus \Sigma, \mathbb{C})$. Therefore,

$$\dim H_c^k(V \setminus \Sigma, \mathbb{C}) = \sum_{p+q=k} \dim H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma))$$

by Deligne (cf. [D, Corollaire (3.2.13) (ii)]). Thus, the above spectral sequence degenerates at E_1. We will use this E_1-degeneration in the proof of Proposition 5.1. By the above E_1-degeneration, we obtain

$$H_c^k(V \setminus \Sigma, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)).$$

In particular, the natural inclusion $\iota_! \mathbb{C}_{V \setminus \Sigma} \subset \mathcal{O}_V(-\Sigma)$ induces surjections

$$H_c^p(V \setminus \Sigma, \mathbb{C}) \simeq H^p(V, \iota_! \mathbb{C}_{V \setminus \Sigma}) \to H^p(V, \mathcal{O}_V(-\Sigma))$$

for all p.

Proof of Proposition 5.1. By Lemma 4.15, we can assume that B is a $\mathbb{Q}$-divisor and that $L \sim_{\mathbb{Q}} K_X + S + B$. We put $\mathcal{L} = \mathcal{O}_X(L - K_X - S)$. Let ν be the smallest positive integer such that $\nu L \sim \nu(K_X + S + B)$. In particular, νB is an integral Weil divisor. We take the ν-fold cyclic cover $\pi' : Y' = \operatorname{Spec}_X \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i} \to X$ associated to the section $\nu B \in |\mathcal{L}^\nu|$. More precisely, let $s \in H^0(X, \mathcal{L}^\nu)$ be a section whose zero divisor is νB. Then the dual of $s : \mathcal{O}_X \to \mathcal{L}^\nu$ defines an $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i}$. Let $Y \to Y'$ be the normalization and $\pi : Y \to X$ the composition morphism. For the details, see [EV, 3.5. Cyclic covers]. We can take a finite cover $\varphi : V \to Y$ such that V is smooth and that T is a simple normal crossing divisor on V, where $\psi = \pi \circ \varphi$ and $T = \psi^* S$, by Kawamata's covering trick (cf. [EV, 3.17. Lemma]). Let $\iota' : Y \setminus \pi^* S \to Y$ be the natural open immersion and U the smooth locus of Y. We denote the natural open immersion $U \to Y$ by j. We put $\widetilde{\Omega}_Y^p(\log(\pi^* S)) = j_* \Omega_U^p(\log(\pi^* S))$ for every p. Then it can be checked easily that

$$\iota'_! \mathbb{C}_{Y \setminus \pi^* S} \xrightarrow{qis} \widetilde{\Omega}_Y^\bullet(\log(\pi^* S)) \otimes \mathcal{O}_Y(-\pi^* S)$$

is a direct summand of

$$\varphi_*(\iota_! \mathbb{C}_{V \setminus T}) \xrightarrow{qis} \varphi_*(\Omega_V^\bullet(\log T) \otimes \mathcal{O}_V(-T)),$$

where qis means a quasi-isomorphism. On the other hand, we can decompose $\pi_*(\widetilde{\Omega}_Y^\bullet(\log(\pi^*S)) \otimes \mathcal{O}_Y(-\pi^*S))$ and $\pi_*(\iota_!'\mathbb{C}_{Y\backslash\pi^*S})$ into eigen components of the Galois action of $\pi : Y \to X$. We write these decompositions as follows,

$$\pi_*(\iota_!'\mathbb{C}_{Y\backslash\pi^*S}) = \bigoplus_{i=0}^{\nu-1}\mathcal{C}_i \subset \bigoplus_{i=0}^{\nu-1}\mathcal{L}^{-i}(\llcorner iB \lrcorner - S) = \pi_*\mathcal{O}_Y(-\pi^*S),$$

where $\mathcal{C}_i \subset \mathcal{L}^{-i}(\llcorner iB \lrcorner - S)$ for every i. We put $\mathcal{C} = \mathcal{C}_1$. We have that

$$\mathcal{C} \xrightarrow{qis} \Omega_X^\bullet(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)$$

is a direct summand of

$$\psi_*(\iota_!\mathbb{C}_{V\backslash T}) \xrightarrow{qis} \psi_*(\Omega_V^\bullet(\log T) \otimes \mathcal{O}_V(-T)).$$

The E_1-degeneration of the spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(V, \Omega_V^p(\log T) \otimes \mathcal{O}_V(-T)) \\ &\Rightarrow \mathbb{H}^{p+q}(V, \Omega_V^\bullet(\log T) \otimes \mathcal{O}_V(-T)) \simeq H^{p+q}(V, \iota_!\mathbb{C}_{V\backslash T}) \end{aligned}$$

implies the E_1-degeneration of

$$\begin{aligned} E_1^{pq} &= H^q(X, \Omega_X^p(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \simeq H^{p+q}(X, \mathcal{C}) \end{aligned}$$

Therefore, the inclusion $\mathcal{C} \subset \mathcal{L}^{-1}(-S)$ induces surjections

$$H^p(X, \mathcal{C}) \to H^p(X, \mathcal{L}^{-1}(-S))$$

for all p. We can check the following simple property by seeing the monodromy action of the Galois group of $\pi : Y \to X$ on $\mathcal{C}$ around $\mathrm{Supp}B$.

Corollary 5.3 (cf. [KM, Corollary 2.54]). *Let $U \subset X$ be a connected open set such that $U \cap \mathrm{Supp}B \neq \emptyset$. Then $H^0(U, \mathcal{C}|_U) = 0$.*

This property is utilized via the following fact. The proof is obvious.

Lemma 5.4 (cf. [KM, Lemma 2.55]). *Let F be a sheaf of Abelian groups on a topological space X and $F_1, F_2 \subset F$ subsheaves. Let $Z \subset X$ be a closed subset. Assume that*

(1) $F_2|_{X\backslash Z} = F|_{X\backslash Z}$, and
(2) if U is connected, open and $U \cap Z \neq \emptyset$, then $H^0(U, F_1|_U) = 0$.

Then F_1 is a subsheaf of F_2.

As a corollary, we obtain:

Corollary 5.5 (cf. [KM, Corollary 2.56]). *Let $M \subset \mathcal{L}^{-1}(-S)$ be a subsheaf such that $M|_{X \setminus \operatorname{Supp} B} = \mathcal{L}^{-1}(-S)|_{X \setminus \operatorname{Supp} B}$. Then the injection*

$$\mathcal{C} \to \mathcal{L}^{-1}(-S)$$

factors as

$$\mathcal{C} \to M \to \mathcal{L}^{-1}(-S).$$

Therefore,

$$H^i(X, M) \to H^i(X, \mathcal{L}^{-1}(-S))$$

is surjective for every i.

Proof. The first part is clear from Corollary 5.3 and Lemma 5.4. This implies that we have maps

$$H^i(X, \mathcal{C}) \to H^i(X, M) \to H^i(X, \mathcal{L}^{-1}(-S)).$$

As we saw above, the composition is surjective. Hence so is the map on the right. $\square$

Therefore, we obtain that

$$H^q(X, \mathcal{L}^{-1}(-S - D)) \to H^q(X, \mathcal{L}^{-1}(-S))$$

is surjective for every q. By the Serre duality, we obtain

$$H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S)) \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S + D))$$

is injective for every q. This means that

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

is injective for every q. $\square$

6. Injectivity, torsion-free, and vanishing theorems

In this section, we prove generalizations of Kollár's torsion-freeness and vanishing theorem (cf. Theorem 6.3). First, we prove a generalization of Kollár's injectivity theorem (cf. [A1, Theorem 3.1]). It is a straightforward consequence of Proposition 5.1 and will produce the desired torsion-free and vanishing theorems.

Theorem 6.1 (Injectivity theorem). *Let X be a smooth projective variety and Δ a boundary $\mathbb{R}$-divisor such that $\operatorname{Supp} \Delta$ is simple normal crossing. Let L be a Cartier divisor on X and D an effective Cartier divisor that contains no lc centers of (X, Δ). Assume the following conditions.*

(i) *$L \sim_{\mathbb{R}} K_X + \Delta + H$,*
(ii) *H is a semi-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and*

(iii) $tH \sim_{\mathbb{R}} D + D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that contains no lc centers of (X, Δ).

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Proof. We put $S = \llcorner \Delta \lrcorner$ and $B = \{\Delta\}$. We can take a resolution $f : Y \to X$ such that f is an isomorphism outside $\mathrm{Supp}(D + D' + B)$, and that the union of the support of $f^*(S + B + D + D')$ and the exceptional locus of f has a simple normal crossing support on Y. Let B' be the strict transform of B on Y. We write

$$K_Y + S' + B' = f^*(K_X + S + B) + E,$$

where S' is the strict transform of S and E is f-exceptional. It is easy to see that $E_+ = \ulcorner E \urcorner \geq 0$. We put $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor with $\llcorner E_- \lrcorner = 0$. Since f^*H is semi-ample, we can write $f^*H \sim_{\mathbb{R}} \sum_i a_i H'_i$, where $0 < a_i < 1$ and H'_i is a general Cartier divisor on Y for every i. We put

$$B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D + D') + (1 - \varepsilon) \sum_i a_i H'_i$$

for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{R}} K_Y + S' + B''$. By the construction, $\llcorner B'' \lrcorner = 0$, the support of $S' + B''$ is simple normal crossing on Y, and $\mathrm{Supp} B'' \supset \mathrm{Supp} f^*D$. So, Proposition 5.1 implies that the homomorphisms

$$H^q(Y, \mathcal{O}_Y(L')) \to H^q(Y, \mathcal{O}_Y(L' + f^*D))$$

are injective for all q. It is easy to see that $f_* \mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By Lemma 4.15, we can write $L' \sim_{\mathbb{Q}} K_Y + S' + B'''$, where B''' is a $\mathbb{Q}$-divisor on Y such that $\llcorner B''' \lrcorner = 0$ and $\mathrm{Supp} B''' = \mathrm{Supp} B''$. Thus, by Lemma 6.2 below, $R^q f_* \mathcal{O}_Y(L') = 0$ for all $q > 0$. By the Leray spectral sequence, the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

are injective for all q. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let us recall the following well-known easy lemma.

Lemma 6.2 (Reid–Fukuda type)**.** *Let V be a smooth projective variety and B a boundary $\mathbb{Q}$-divisor on V such that $\mathrm{Supp} B$ is simple normal crossing. Let $f : V \to W$ be a projective birational morphism onto a*

variety W. Assume that f is an isomorphism at the generic point of every lc center of (V, B) and that D is a Cartier divisor on V such that $D - (K_V + B)$ is nef. Then $R^i f_* \mathcal{O}_V(D) = 0$ for every $i > 0$.

Proof. We use the induction on the number of irreducible components of $\llcorner B \lrcorner$ and on the dimension of V. If $\llcorner B \lrcorner = 0$, then the lemma follows from the Kawamata–Viehweg vanishing theorem (cf. [KM, Corollary 2.68]). Therefore, we can assume that there is an irreducible divisor $S \subset \llcorner B \lrcorner$. We consider the following short exact sequence

$$0 \to \mathcal{O}_V(D - S) \to \mathcal{O}_V(D) \to \mathcal{O}_S(D) \to 0.$$

By induction, we see that $R^i f_* \mathcal{O}_V(D - S) = 0$ and $R^i f_* \mathcal{O}_S(D) = 0$ for every $i > 0$. Thus, we have $R^i f_* \mathcal{O}_V(D) = 0$ for $i > 0$. $\qquad\square$

Theorem 6.3 (Torsion-freeness and vanishing theorem). *Let Y be a smooth variety and B a boundary $\mathbb{R}$-divisor such that $\mathrm{Supp} B$ is simple normal crossing. Let $f : Y \to X$ be a projective morphism and L a Cartier divisor on Y such that $H \sim_\mathbb{R} L - (K_Y + B)$ is f-semi-ample.*

(i) *Let q be an arbitrary non-negative integer. Every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y, B).*

(ii) *Let $\pi : X \to S$ be a projective morphism. Assume that $H \sim_\mathbb{R} f^* H'$ for some π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H' on X. Then $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.*

Let us start the proof of Theorem 6.3.

Proof of Theorem 6.3. (i) We divide the proof into two steps.

Step 1. First, we assume that X is projective. We can assume that H is semi-ample by replacing L (resp. H) with $L + f^* A'$ (resp. $H + f^* A'$), where A' is a very ample Cartier divisor. Assume that $R^q f_* \mathcal{O}_Y(L)$ has a local section whose support does not contain the image of any (Y, B)-stratum. Then we can find a very ample Cartier divisor A with the following properties.

(a) $f^* A$ contains no lc centers of (Y, B), and

(b) $R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

We can assume that $H - f^* A$ is semi-ample by replacing L (resp. H) with $L + f^* A$ (resp. $H + f^* A$). If necessary, we replace L (resp. H) with $L + f^* A''$ (resp. $H + f^* A''$), where A'' is a very ample Cartier divisor. Then, we have

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \simeq H^q(Y, \mathcal{O}_Y(L))$$

and

$$H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \simeq H^q(Y, \mathcal{O}_Y(L + f^* A)).$$

We see that

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \to H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A))$$

is not injective by (b) if A'' is sufficiently ample. So,

$$H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + f^* A))$$

is not injective. It contradicts Theorem 6.1. We finish the proof when X is projective.

Step 2. Next, we assume that X is not projective. Note that the problem is local. So, we can shrink X and assume that X is affine. By the argument similar to the one in Step 1 in the proof of (ii) below, we can assume that H is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We compactify X and apply Lemma 4.17. Then we obtain a compactification $\overline{f} : \overline{Y} \to \overline{X}$ of $f : Y \to X$. Let $\overline{H}$ be the closure of H on $\overline{Y}$. If $\overline{H}$ is not a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we take blowing-ups of $\overline{Y}$ inside $\overline{Y} \setminus Y$ and obtain a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\overline{H}$ on $\overline{Y}$ such that $\overline{H}|_Y = H$. Let $\overline{L}$ (resp. $\overline{B}$) be the closure of L (resp. B) on $\overline{Y}$. We note that $\overline{H} \sim_{\mathbb{R}} \overline{L} - (K_{\overline{Y}} + \overline{B})$ does not necessarily hold. We can write $H + \sum_i a_i(f_i) = L - (K_Y + B)$, where a_i is a real number and f_i is a rational function on Y for every i. We put

$$E = \overline{H} + \sum_i a_i(f_i) - (\overline{L} - (K_{\overline{Y}} + \overline{B})).$$

We replace $\overline{L}$ (resp. $\overline{B}$) with $\overline{L} + \lceil E \rceil$ (resp. $\overline{B} + \{-E\}$). Then we obtain the desired property of $R^q \overline{f}_* \mathcal{O}_{\overline{Y}}(\overline{L})$ since $\overline{X}$ is projective. We note that $\mathrm{Supp} E$ is in $\overline{Y} \setminus Y$. So, this completes the whole proof.

(ii) We divide the proof into three steps.

Step 1. We assume that $\dim S = 0$. The following arguments are well known and standard. We describe them for the reader's convenience. In this case, we can write $H' \sim_{\mathbb{R}} H_1' + H_2'$, where H_1' (resp. H_2') is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) on X. So, we can write $H_2' \sim_{\mathbb{R}} \sum_i a_i H_i$, where $0 < a_i < 1$ and H_i is a general very ample Cartier divisor on X for every i. Replacing B (resp. H') with $B + \sum_i a_i f^* H_i$ (resp. H_1'), we can assume that H' is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We take a general member $A \in |mH'|$, where m is a sufficiently divisible positive integer, such that $A' = f^* A$ and $R^q f_* \mathcal{O}_Y(L + A')$ is π_*-acyclic for all q. By (i), we have the following short exact sequences,

$$0 \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L + A') \to R^q f_* \mathcal{O}_{A'}(L + A') \to 0.$$

for all q. Note that $R^q f_* \mathcal{O}_{A'}(L + A')$ is π_*-acyclic by induction on dim X and $R^q f_* \mathcal{O}_Y(L + A')$ is also π_*-acyclic by the above assumption. Thus, $E_2^{pq} = 0$ for $p \geq 2$ in the following commutative diagram of spectral sequences.

$$
\begin{array}{ccc}
E_2^{pq} = R^p \pi_* R^q f_* \mathcal{O}_Y(L) & \Longrightarrow & R^{p+q}(\pi \circ f)_* \mathcal{O}_Y(L) \\
\varphi^{pq} \downarrow & & \varphi^{p+q} \downarrow \\
\overline{E}_2^{pq} = R^p \pi_* R^q f_* \mathcal{O}_Y(L + A') & \Longrightarrow & R^{p+q}(\pi \circ f)_* \mathcal{O}_Y(L + A')
\end{array}
$$

We note that φ^{1+q} is injective by Theorem 6.1. We have $E_2^{1q} \to R^{1+q}(\pi \circ f)_* \mathcal{O}_Y(L)$ is injective by the fact that $E_2^{pq} = 0$ for $p \geq 2$. We also have that $\overline{E}_2^{1q} = 0$ by the above assumption. Therefore, we obtain $E_2^{1q} = 0$ since the injection $E_2^{1q} \to R^{1+q}(\pi \circ f)_* \mathcal{O}_Y(L + A')$ factors through $\overline{E}_2^{1q} = 0$. This implies that $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$.

Step 2. We assume that S is projective. By replacing H' (resp. L) with $H' + \pi^* G$ (resp. $L + (\pi \circ f)^* G$), where G is a very ample Cartier divisor on S, we can assume that H' is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. By the same argument as in Step 1, we can assume that H' is ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and $H \sim_{\mathbb{Q}} f^* H'$. If G is a sufficiently ample Cartier divisor on S, $H^k(S, R^p \pi_* R^q f_* \mathcal{O}_Y(L) \otimes G) = 0$ for every $k \geq 1$, $H^0(S, R^p \pi_* R^q f_* \mathcal{O}_Y(L) \otimes G) \simeq H^p(X, R^q f_* \mathcal{O}_Y(L) \otimes \pi^* G)$, and $R^p \pi_* R^q f_* \mathcal{O}_Y(L) \otimes G$ is generated by its global sections. Since $H + f^* \pi^* G \sim_{\mathbb{R}} L + f^* \pi^* G - (K_Y + B)$, $H + f^* \pi^* G \sim_{\mathbb{Q}} f^*(H' + \pi^* G)$, and $H' + \pi^* G$ is ample, we can apply Step 1 and obtain $H^p(X, R^q f_* \mathcal{O}_Y(L + f^* \pi^* G)) = 0$ for every $p > 0$. Thus, $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ by the above arguments.

Step 3. When S is not projective, we shrink S and assume that S is affine. By the same argument as in Step 1 above, we can assume that H' is $\mathbb{Q}$-Cartier. We compactify S and X, and can assume that S and X are projective. By Lemma 4.17, we can reduce it to the case when S is projective. This step is essentially the same as Step 2 in the proof of (i). So, we omit the details here.

We obtained the statement (ii). $\square$

7. Non-lc ideal sheaves

We introduce the notion of *non-lc ideal sheaves*. It is an analogue of the usual *multiplier ideal sheaves* (see, for example, [L, Chapter 9]). For the details, see [F8] and [FT].

Definition 7.1 (Non-lc ideal sheaf)**.** Let X be a normal variety and B an $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution with $K_Y + B_Y = f^*(K_X + B)$ such that $\operatorname{Supp} B_Y$ is simple normal crossing. Then we put

$$\mathcal{J}_{NLC}(X, B) = f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1} \lrcorner)$$
$$= f_*\mathcal{O}_Y(-\llcorner B_Y \lrcorner + B_Y^{=1})$$

and call it the *non-lc ideal sheaf associated to* (X, B). If B is effective, then $\mathcal{J}_{NLC}(X, B) \subset \mathcal{O}_X$.

The ideal sheaf $\mathcal{J}_{NLC}(X, B)$ is well-defined by the following easy lemma.

Lemma 7.2. *Let* $g : Z \to Y$ *be a proper birational morphism between smooth varieties and* B_Y *an* $\mathbb{R}$-*divisor on* Y *such that* $\operatorname{Supp} B_Y$ *is simple normal crossing. Assume that* $K_Z + B_Z = g^*(K_Y + B_Y)$ *and that* $\operatorname{Supp} B_Z$ *is simple normal crossing. Then we have*

$$g_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1} \lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1} \lrcorner).$$

Proof. By $K_Z + B_Z = g^*(K_Y + B_Y)$, we obtain

$$K_Z = g^*(K_Y + B_Y^{=1} + \{B_Y\})$$
$$+ g^*(\llcorner B_Y^{<1} \lrcorner + \llcorner B_Y^{>1} \lrcorner) - (\llcorner B_Z^{<1} \lrcorner + \llcorner B_Z^{>1} \lrcorner) - B_Z^{=1} - \{B_Z\}.$$

If $a(\nu, Y, B_Y^{=1} + \{B_Y\}) = -1$ for a prime divisor ν over Y, then we can check that $a(\nu, Y, B_Y) = -1$ by using [KM, Lemma 2.45]. Since $g^*(\llcorner B_Y^{<1} \lrcorner + \llcorner B_Y^{>1} \lrcorner) - (\llcorner B_Z^{<1} \lrcorner + \llcorner B_Z^{>1} \lrcorner)$ is Cartier, we can easily see that

$$g^*(\llcorner B_Y^{<1} \lrcorner + \llcorner B_Y^{>1} \lrcorner) = \llcorner B_Z^{<1} \lrcorner + \llcorner B_Z^{>1} \lrcorner + E,$$

where E is an effective f-exceptional Cartier divisor. Thus, we obtain

$$g_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1} \lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1} \lrcorner).$$

This completes the proof. □

Remark 7.3. We use the same notation as in Definition 7.1. We put

$$\mathcal{J}(X, B) = f_*\mathcal{O}_Y(-\llcorner B_Y \lrcorner).$$

This sheaf $\mathcal{J}(X, B)$ is well known as the (algebraic version of) *multiplier ideal sheaf* of the pair (X, B). See, for example, [L, Chapter 9].

By the definition, the following proposition is obvious.

Proposition 7.4. *Let* X *be a normal variety and* B *an effective* $\mathbb{R}$-*divisor on* X *such that* $K_X + B$ *is* $\mathbb{R}$-*Cartier. Then* (X, B) *is log canonical if and only if* $\mathcal{J}_{NLC}(X, B) = \mathcal{O}_X$.

The next proposition is a kind of Bertini's theorem.

Proposition 7.5. *Let X be a smooth variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let Λ be a linear system on X and $D \in \Lambda$ a general member of Λ. Then*

$$\mathcal{J}_{NLC}(X, B + tD) = \mathcal{J}_{NLC}(X, B)$$

outside the base locus $\mathrm{Bs}\Lambda$ of Λ for all $0 \le t \le 1$.

Proof. By replacing X with $X \setminus \mathrm{Bs}\Lambda$, we can assume that $\mathrm{Bs}\Lambda = \emptyset$. Let $f : Y \to X$ be a resolution as in Definition 7.1. Since D is a general member of Λ, $f^*D = f_*^{-1}D$ is a smooth divisor on Y such that $\mathrm{Supp} f^*D \cup \mathrm{Supp} B_Y$ is simple normal crossing. Therefore, we can check that $\mathcal{J}_{NLC}(X, B + tD) = \mathcal{J}_{NLC}(X, B)$ for all $0 \le t \le 1$. $\square$

We close this section with an important remark.

Remark 7.6. In the following sections (Sections 8, 12, 13, and 15), we consider the scheme structure of $\mathrm{Nlc}(X, B)$ defined by $\mathcal{J}_{NLC}(X, B)$. However, we can use $\mathcal{J}'(X, B)$ or $\mathcal{J}'_l(X, B)$ for any $l = 0, -1, \cdots, -\infty$ in place of $\mathcal{J}_{NLC}(X, B)$. For the definitions and basic properties of $\mathcal{J}'(X, B)$ and $\mathcal{J}'_l(X, B)$, see [FT]. We adopt $\mathcal{J}_{NLC}(X, B)$ since we think $\mathcal{J}_{NLC}(X, B)$ is the most natural defining ideal sheaf of $\mathrm{Nlc}(X, B)$.

8. Vanishing theorem

The following vanishing theorem is a key result in this paper. It is a special case of [A1, Theorem 4.4]. For the details, see [F9, Theorem 3.39].

Theorem 8.1. *Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let D be a Cartier divisor on X. Assume that $D - (K_X + B)$ is π-ample, where $\pi : X \to S$ is a projective morphism onto a variety S. Let $\{C_i\}$ be any set of lc centers of the pair (X, B). We put $W = \bigcup C_i$ with a reduced scheme structure. Assume that W is disjoint from $\mathrm{Nlc}(X, B)$. Then we have*

$$R^i \pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$, where $\mathcal{J} = \mathcal{I}_W \cdot \mathcal{J}_{NLC}(X, B) \subset \mathcal{O}_X$ and $\mathcal{I}_W$ is the defining ideal sheaf of W on X. Therefore, the restriction map

$$\pi_* \mathcal{O}_X(D) \to \pi_* \mathcal{O}_W(D) \oplus \pi_* \mathcal{O}_{\mathrm{Nlc}(X,B)}(D)$$

is surjective and

$$R^i \pi_* \mathcal{O}_W(D) = 0$$

for every $i > 0$. In particular, the restriction maps

$$\pi_* \mathcal{O}_X(D) \to \pi_* \mathcal{O}_W(D)$$

and

$$\pi_*\mathcal{O}_X(D) \to \pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(D)$$

are surjective.

Proof. Let $f : Y \to X$ be a resolution such that $\mathrm{Supp} f_*^{-1}B \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor. We can further assume that $f^{-1}(W)$ is a simple normal crossing divisor on Y. We can write

$$K_Y + B_Y = f^*(K_X + B).$$

Let T be the union of the irreducible components of $B_Y^{=1}$ that are mapped into W by f. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A - N - T) \to \mathcal{O}_Y(A - N) \to \mathcal{O}_T(A - N) \to 0,$$

where $A = \ulcorner -(B_Y^{<1}) \urcorner$ and $N = \llcorner B_Y^{>1} \lrcorner$. Note that A is an effective f-exceptional divisor. We obtain the following long exact sequence

$$0 \to f_*\mathcal{O}_Y(A - N - T) \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_T(A - N)$$

$$\xrightarrow{\delta} R^1 f_*\mathcal{O}_Y(A - N - T) \to \cdots .$$

Since

$$A - N - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) = -(K_Y + B_Y)$$

$$\sim_{\mathbb{R}} -f^*(K_X + B),$$

every non-zero local section of $R^1 f_*\mathcal{O}_Y(A - N - T)$ contains in its support the f-image of some strata of $(Y, \{B_Y\} + B_Y^{=1} - T)$ by Theorem 6.3 (i). On the other hand, $W = f(T)$. Therefore, the connecting homomorphism δ is a zero map. Thus, we have a short exact sequence

$$(\diamondsuit) \quad 0 \to f_*\mathcal{O}_Y(A - N - T) \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_T(A - N) \to 0.$$

We put $\mathcal{J} = f_*\mathcal{O}_Y(A - N - T) \subset \mathcal{O}_X$. Since W is disjoint from $\mathrm{Nlc}(X, B)$, the ideal sheaf $\mathcal{J}$ coincides with $\mathcal{I}_W$ (resp. $\mathcal{J}_{NLC}(X, B)$) around W (resp. $\mathrm{Nlc}(X, B)$). Therefore, $\mathcal{J} = \mathcal{I}_W \cdot \mathcal{J}_{NLC}(X, B)$. We put $U = X \setminus \mathrm{Nlc}(X, B)$ and $V = f^{-1}(U)$. By restricting $(\diamondsuit)$ to U, we obtain

$$0 \to f_*\mathcal{O}_V(A - T) \to f_*\mathcal{O}_V(A) \to f_*\mathcal{O}_T(A) \to 0.$$

Since $f_*\mathcal{O}_V(A) \simeq \mathcal{O}_U$, we have $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$. The isomorphism $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ plays crucial roles in this paper. Thus we write it as a proposition.

Proposition 8.2. *We have $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$. It obviously implies that $f_*\mathcal{O}_T \simeq \mathcal{O}_W$ since A is effective.*

Remark 8.3. We did not use D nor $\pi : X \to S$ to obtain Proposition 8.2.

Since

$$f^*D + A - N - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) \sim_{\mathbb{R}} f^*(D - (K_X + B)),$$

we have

$$R^i\pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) \simeq R^i\pi_*(f_*\mathcal{O}_Y(A - N - T) \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$ by Theorem 6.3 (ii). By considering the short exact sequence

$$0 \to \mathcal{J} \to \mathcal{J}_{NLC}(X, B) \to \mathcal{O}_W \to 0,$$

we obtain

$$\cdots \to R^i\pi_*(\mathcal{J}_{NLC}(X, B) \otimes \mathcal{O}_X(D))$$
$$\to R^i\pi_*\mathcal{O}_W(D) \to R^{i+1}\pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) \to \cdots.$$

Since we have already checked

$$R^i\pi_*(\mathcal{J}_{NLC}(X, B) \otimes \mathcal{O}_X(D)) = R^i\pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$, we have $R^i\pi_*\mathcal{O}_W(D) = 0$ for all $i > 0$. Finally, we consider the following short exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_W \oplus \mathcal{O}_{\mathrm{Nlc}(X,B)} \to 0.$$

By taking $\otimes \mathcal{O}_X(D)$ and $R^i\pi_*$, we obtain

$$0 \to \pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) \to \pi_*\mathcal{O}_X(D) \to \pi_*\mathcal{O}_W(D) \oplus \pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(D) \to 0.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

9. Lc centers

We prove the basic properties of lc centers as an application of the result in the preceding section (cf. Proposition 8.2). Theorem 9.1 is very useful in the study of linear systems on log canonical pairs.

Theorem 9.1 (cf. [A1, Propositions 4.7 and 4.8])**.** *Let X be a normal variety and B an effective $\mathbb{R}$-divisor such that (X, B) is log canonical. Then we have the following properties.*

    (1) *(X, B) has at most finitely many lc centers.*
    (2) *An intersection of two lc centers is a union of lc centers.*
    (3) *Any union of lc centers of (X, B) is semi-normal.*
    (4) *Let $x \in X$ be a closed point such that (X, B) is lc but not klt at x. Then there is a unique minimal lc center W_x passing through x, and W_x is normal at x.*

Proof. We use the notation in the proof of Theorem 8.1. (1) is obvious. (3) is also obvious by Proposition 8.2 since T is a simple normal crossing divisor. Let C_1 and C_2 be two lc centers of (X, B). We fix a closed point $P \in C_1 \cap C_2$. For the proof of (2), it is enough to find an lc center C such that $P \in C \subset C_1 \cap C_2$. We put $W = C_1 \cup C_2$. By Proposition 8.2, we obtain $f_* \mathcal{O}_T \simeq \mathcal{O}_W$. This means that $f : T \to W$ has connected fibers. We note that T is a simple normal crossing divisor on Y. Thus, there exist irreducible components T_1 and T_2 of T such that $T_1 \cap T_2 \cap f^{-1}(P) \neq \emptyset$ and that $f(T_i) \subset C_i$ for $i = 1, 2$. Therefore, we can find an lc center C with $P \in C \subset C_1 \cap C_2$. We finish the proof of (2). Finally, we will prove (4). The existence and the uniqueness of the minimal lc center follow from (2). We take the unique minimal lc center $W = W_x$ passing through x. By Proposition 8.2, we have $f_* \mathcal{O}_T \simeq \mathcal{O}_W$. By shrinking W around x, we can assume that every stratum of T dominates W. Thus, $f : T \to W$ factors through the normalization W^ν of W. Since $f_* \mathcal{O}_T \simeq \mathcal{O}_W$, we obtain that $W^\nu \to W$ is an isomorphism. So, we obtain (4). □

10. DLT MODIFICATIONS

In this section, we discuss *dlt modifications* by Hacon. In the following sections, we will not use the arguments in this section. We will only use the statements of Lemma 10.2, which is well known to the experts, and Theorem 10.4 in Sections 11 and 18.

Let us recall the definition of *dlt pairs*. For another definition and the basic properties of dlt pairs, see [KM, Section 2.3] and [F4].

Definition 10.1 (Dlt pair)**.** Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. If there exists a resolution such that

    (i) both $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1} B$ are simple normal crossing divisors on Y, and

    (ii) $a(E, X, B) > -1$ for every exceptional divisor $E \subset Y$,

then (X, B) is called *divisorial log terminal* (*dlt*, for short).

We will use the following lemma in Section 11. For the details, see [F4, 3.9 Adjunction for dlt pairs].

Lemma 10.2. *Let (X, B) be a dlt pair and V an lc center of (X, B). Then $K_V + B_V = (K_X + B)|_V$ is dlt by adjunction.*

We borrow the next theorem from [BCHM].

Theorem 10.3 (cf. [BCHM, Theorem 1.2])**.** *Let (X, B) be a klt pair, where $K_X + B$ is $\mathbb{R}$-Cartier. Let $\pi : X \to S$ be a projective birational*

*morphism of quasi-projective varieties. Then (X, B) has a log termi-
nal model over S. This means that there exists a projective birational
morphism $f : X' \to S$ such that*

    (i) *X' is $\mathbb{Q}$-factorial,*
    (ii) *ϕ^{-1} has no exceptional divisors, where $\phi = f^{-1} \circ \pi : X \dashrightarrow X'$,*
    (iii) *$K_{X'} + B'$ is f-nef, where $B' = \phi_* B$, and*
    (iv) *$a(E, X, B) < a(E, X', B')$ for every ϕ-exceptional divisor $E \subset X$.*

The following theorem is very useful. It is a consequence of Theorem
10.3.

Theorem 10.4 (Hacon)**.** *Let X be a normal quasi-projective variety
and B a boundary $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. In
this case, we can construct a projective birational morphism $f : Y \to X$
from a normal quasi-projective variety with the following properties.*

    (i) *Y is $\mathbb{Q}$-factorial.*
    (ii) *$a(E, X, B) \le -1$ for every f-exceptional divisor E on Y.*
    (iii) *We put*

$$B_Y = f_*^{-1}B + \sum_{E : f\text{-}exceptional} E.$$

Then (Y, B_Y) is dlt and

$$K_Y + B_Y = f^*(K_X + B) + \sum_{a(E,X,B)<-1} (a(E, X, B) + 1)E.$$

*In particular, if (X, B) is lc, then $K_Y + B_Y = f^*(K_X + B)$.
Moreover, if (X, B) is dlt, then we can make f small, that is,
f is an isomorphism in codimension one.*

Proof. Let $\pi : V \to X$ be a resolution such that $\pi_*^{-1}B \cup \mathrm{Exc}(\pi)$ has a
simple normal crossing support. We can assume that π is a composite
of blow-ups of centers of codimension at least two. Then there exists an
effective π-exceptional Cartier divisor C on V such that $-C$ is π-ample.
We put

$$F = \sum_{\substack{a(E,X,B)>-1, \\ E:\pi\text{-exceptional}}} E$$

and

$$E^+ = -\sum_{a(E,X,B)\le-1} a(E, X, B)E.$$

We note that E^+ is not necessarily π-exceptional. We put $E = \mathrm{Supp}E^+$.
We note that $E^+ - E$ is π-exceptional.

Let H be a sufficiently ample Cartier divisor on X. We choose $0 < \epsilon, \nu, \mu \ll 1$ and note that

$$(\spadesuit) \qquad E + (1 - \nu)F + \mu(-C + \pi^* H)$$
$$= (1 - \epsilon\mu)E + (1 - \nu)F + \mu(\epsilon E - C + \pi^* H).$$

Since $-C + \pi^* H$ and $\epsilon E - C + \pi^* H$ are ample, we can take effective $\mathbb{Q}$-divisors H_1 and H_2 with small coefficients such that $E + F + \pi^* B + H_1 + H_2$ has a simple normal crossing support and that $-C + \pi^* H \sim_{\mathbb{Q}} H_1$, $\epsilon E - C + \pi^* H \sim_{\mathbb{Q}} H_2$. Then $(V, (1 - \epsilon\mu)E + (1 - \nu)F + \pi_*^{-1}B^{<1} + \mu H_2)$ is klt. By Theorem 10.3, it has a log terminal model $f : Y \to X$. By the above equation $(\spadesuit)$, this is also a relative minimal model of the pair $(V, E + (1 - \nu)F + \pi_*^{-1}B^{<1} + \mu H_1)$, which is therefore dlt.

For any divisor G on V appearing above, let G' denote its transform on Y. By the above construction,

$$N = K_Y + (1 - \epsilon\mu)E' + (1 - \nu)F' + f_*^{-1}B^{<1} + \mu H_2'$$

is f-nef and $K_Y + \overline{B} = f^*(K_X + B)$ is $\mathbb{R}$-linearly f-trivial. We put

$$D = \overline{B} - E' - (1 - \nu)F' - f_*^{-1}B^{<1} + \mu C'.$$

Then

$$-D \sim_{\mathbb{R}, f} N - (K_Y + \overline{B})$$
$$= -\overline{B} + (1 - \epsilon\mu)E' + (1 - \nu)F' + f_*^{-1}B^{<1} + \mu H_2',$$

hence it is f-nef. Since $f_* D = 0$, we see that D is effective by the negativity lemma (cf. Lemma 4.16).

Every divisor in F has a negative coefficient in

$$\widetilde{B} - E - (1 - \nu)F - \pi_*^{-1}B^{<1} + \mu C,$$

where $K_V + \widetilde{B} = \pi^*(K_X + B)$. Therefore, F is contracted on Y. So, every f-exceptional divisor has discrepancy ≤ -1. By the above construction, $(Y, E' + f_*^{-1}B^{<1} + \mu H_1')$ is dlt since $F' = 0$. Therefore, $(Y, E' + f_*^{-1}B^{<1})$ is also dlt. This means that (Y, B_Y) is dlt because $B_Y = E' + \sum f_*^{-1}B^{<1}$.

When (X, B) is dlt, we can assume that $E^+ = \pi_*^{-1}B^{=1}$ by the definition of dlt pairs. Therefore, we can make f small. $\qquad\square$

11. Vanishing theorem for minimal lc centers

In this section, we prove a vanishing theorem on minimal lc centers. It is very powerful and will play crucial roles in the proof of Theorem 12.1.

Theorem 11.1 (Vanishing theorem for minimal lc centers)**.** *Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let W be a minimal lc center of (X, B) such that W is disjoint from $\mathrm{Nlc}(X, B)$. Let $\pi : X \to S$ be a projective morphism onto a variety S. Let D be a Cartier divisor on W such that $D - (K_X + B)|_W$ is π-ample. Then*

$$R^i \pi_* \mathcal{O}_W(D) = 0$$

for every $i > 0$.

Proof. We shrink X around W and assume that (X, B) is log canonical. By Theorem 10.4, we can make a projective birational morphism $f : Y \to X$ such that $K_Y + B_Y = f^*(K_X + B)$ and (Y, B_Y) is dlt. We take an lc center V of (Y, B_Y) such that $f(V) = W$ and put $K_V + B_V = (K_Y + B_Y)|_V$. Then (V, B_V) is dlt by Lemma 10.2 and $K_V + B_V \sim_{\mathbb{R}} f^*((K_X + B)|_W)$. Let $g : Z \to V$ be a resolution such that $K_Z + B_Z = g^*(K_V + B_V)$ and $\mathrm{Supp} B_Z$ is simple normal crossing. Then we have $K_Z + B_Z \sim_{\mathbb{R}} h^*((K_X + B)|_W)$, where $h = f \circ g$. Since

$$h^*(D - (K_X + B)|_W) \sim_{\mathbb{R}} h^*D + \ulcorner -(B_Z^{\leq 1}) \urcorner - (K_Z + B_Z^{=1} + \{B_Z\}),$$

we obtain

$$R^i \pi_* h_* \mathcal{O}_Z(h^*D + \ulcorner -(B_Z^{\leq 1}) \urcorner) = 0$$

for every $i > 0$ by Theorem 6.3 (ii). We note that

$$h_* \mathcal{O}_Z(h^*D + \ulcorner -(B_Z^{\leq 1}) \urcorner) \simeq f_* \mathcal{O}_V(f^*D)$$

by the projection formula since $\ulcorner -(B_Z^{\leq 1}) \urcorner$ is effective and g-exceptional. We note that $\mathcal{O}_W(D)$ is a direct summand of $f_* \mathcal{O}_V(f^*D) \simeq \mathcal{O}_W(D) \otimes f_* \mathcal{O}_V$ since W is normal (cf. Theorem 9.1 (4)). Therefore, we have $R^i \pi_* \mathcal{O}_W(D) = 0$ for every $i > 0$. $\square$

We close this section with a very important remark.

Remark 11.2. We can prove Theorem 11.1 without using Theorem 10.4. In particular, Theorem 11.1 is independent of the results in [BCHM]. For the original argument, see [A1, Theorem 4.4] and [F9, Theorem 3.39]. It depends on the mixed Hodge theory on reducible varieties (cf. [F9, Chapter 2]).

12. Non-vanishing theorem

In this section, we prove the non-vanishing theorem. It is a generalization of the main theorem of [F14].

Theorem 12.1 (Non-vanishing theorem)**.** *Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $\pi : X \to S$ be a projective morphism onto a variety S and L a π-nef Cartier divisor on X. Assume that*

(i) $aL - (K_X + B)$ is π-ample for some real number $a > 0$, and
(ii) $\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is $\pi|_{\mathrm{Nlc}(X,B)}$-generated for $m \gg 0$.

Then the relative base locus $\mathrm{Bs}_\pi|mL|$ contains no lc centers of (X, B) and is disjoint from $\mathrm{Nlc}(X, B)$ for $m \gg 0$.

Proof. Without loss of generality, we can assume that S is affine.

Step 1. In this step, we will prove that $\mathcal{O}_X(mL)$ is π-generated around $\mathrm{Nlc}(X, B)$ for $m \gg 0$.

By the assumption, $\pi^*\pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL) \to \mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is surjective for $m \gg 0$. On the other hand, $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is surjective for $m \geq a$ since

$$R^1\pi_*(\mathcal{J}_{NLC}(X, B) \otimes \mathcal{O}_X(mL)) = 0$$

for $m \geq a$ by Theorem 8.1. Therefore, $\pi^*\pi_*\mathcal{O}_X(mL) \to \mathcal{O}_X(mL)$ is surjective around $\mathrm{Nlc}(X, B)$ for $m \gg 0$. See the following commutative diagram.

$$
\begin{array}{ccccc}
\pi^*\pi_*\mathcal{O}_X(mL) & \longrightarrow & \pi^*\pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{\mathrm{Nlc}(X,B)}(mL) & \longrightarrow & 0,
\end{array}
$$

Let W be a minimal lc center of (X, B). Then it is sufficient to see that W is not contained in $\mathrm{Bs}|mL|$ for $m \gg 0$.

Step 2. If $W \cap \mathrm{Nlc}(X, B) \neq \emptyset$, then $\mathrm{Bs}|mL|$ does not contain W by Step 1. So, from now on, we can assume that $W \cap \mathrm{Nlc}(X, B) = \emptyset$.

Step 3. We assume that $L|_{W_\eta}$ is numerically trivial, where W_η is the generic fiber of $W \to \pi(W)$. In this case,

$$
\begin{aligned}
h^0(W_\eta, \mathcal{O}_{W_\eta}(L)) &= \chi(W_\eta, \mathcal{O}_{W_\eta}(L)) \\
&= \chi(W_\eta, \mathcal{O}_{W_\eta}) = h^0(W_\eta, \mathcal{O}_{W_\eta}) > 0
\end{aligned}
$$

by [Kl, Chapter II §2 Theorem 1] and the vanishing theorem: Theorem 11.1. On the other hand,

$$\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_W(mL) \oplus \pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$$

is surjective for any $m \geq a$ by Theorem 8.1. In particular, $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_W(mL)$ is surjective for any $m \geq a$. Thus, $\mathrm{Bs}|mL|$ does not contain W for any $m \geq a$.

Step 4. We assume that $L|_{W_\eta}$ is not numerically trivial. We take a general subvariety V of W such that $V \to \pi(W)$ is generically finite. If l is a positive large integer, then we can write

$$lL - (K_X + B) = N_1 + a_2N_2 + \cdots + a_kN_k$$

with the following properties.

(a) N_1 is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that

$$((N_1|_W)|_F)^{\dim F} > d(\operatorname{codim}_W V)^{\dim F},$$

where d is the mapping degree of $V \to \pi(W)$ and F is a general fiber of $W \to \pi(W)$.

(b) a_i is a positive real number and N_i is a π-very ample Cartier divisor on X for every $i \geq 2$.

By Lemma 12.2, we can find an effective $\mathbb{Q}$-divisor D_1 on W such that $D_1 \sim_{\mathbb{Q}} N_1|_W$ with $\operatorname{mult}_V D_1 > \operatorname{codim}_W V$. If b is sufficiently large and divisible, then $bD_1 \sim bN_1|_W$, $\mathcal{I}_W \otimes \mathcal{O}_X(bN_1)$ is π-generated, and $R^1\pi_*(\mathcal{I}_W \otimes \mathcal{O}_X(bN_1)) = 0$ since N_1 is π-ample, where $\mathcal{I}_W$ is the defining ideal sheaf of W. By using the following short exact sequence

$$0 \to f_*(\mathcal{I}_W \otimes \mathcal{O}_X(bN_1)) \to f_*\mathcal{O}_X(bN_1) \to f_*\mathcal{O}_W(bN_1) \to 0,$$

we can find an effective $\mathbb{Q}$-divisor M_1 on X with the following properties.

(1) $M_1|_W = D_1$.
(2) $M_1 \sim_{\mathbb{Q}} N_1$.
(3) $(X, B + M_1)$ is lc outside $W \cup \operatorname{Nlc}(X, B)$.
(4) $\mathcal{J}_{NLC}(X, B + M_1) = \mathcal{J}_{NLC}(X, B)$ outside W.

Let M_i be a general member of $|N_i|$ for every $i \geq 2$. We put $M = M_1 + a_2M_2 + \cdots + a_kM_k$. Then we have

(i) $M|_W \geq D_1$.
(ii) $M \sim_{\mathbb{R}} lL - (K_X + B)$.
(iii) $(X, B + M)$ is lc outside $W \cup \operatorname{Nlc}(X, B)$.
(iv) $\mathcal{J}_{NLC}(X, B + M) = \mathcal{J}_{NLC}(X, B)$ outside W.

We take the log canonical threshold c of (X, B) with respect to M outside $\operatorname{Nlc}(X, B)$. By the above construction, we have $0 < c < 1$. We replace (X, B) with $(X, B + cM)$, a with $a - ac + cl$. Then we have that

$$(a - ac + cl)L - (K_X + B + cM) \sim_{\mathbb{R}} (1 - c)(L - (K_X + B))$$

is π-ample. Moreover, we can find a smaller lc center W' of $(X, B + cM)$ contained in W. By repeating this process, we reach the situation where $L|_{W_\eta}$ is numerically trivial.

Anyway, we proved that $\mathrm{Bs}|mL|$ contains no lc centers of (X, B) for $m \gg 0$. $\qquad\square$

The following lemma is a relative version of Shokurov's concentration method. We used it in the proof of Theorem 12.1.

Lemma 12.2. *Let $f : Y \to Z$ be a projective morphism from a normal variety Y onto an affine variety Z. Let V be a general closed subvariety of Y such that $f : V \to Z$ is generically finite. Let M be an f-ample $\mathbb{R}$-divisor on Y. Assume that*

$$(M|_F)^d > km^d,$$

where F is a general fiber of $f : Y \to Z$, $d = \dim F$, and k is the mapping degree of $f : V \to Z$. Then we can find an effective $\mathbb{R}$-divisor D on Y such that

$$D \sim_{\mathbb{R}} M$$

and that $\mathrm{mult}_V D > m$. If M is a $\mathbb{Q}$-divisor, then we can make D a $\mathbb{Q}$-divisor with $D \sim_{\mathbb{Q}} M$.

Proof. We can write

$$M = M_1 + a_2 M_2 + \cdots + a_l M_l,$$

where M_1 is an f-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $(M_1|_F)^d > km^d$, a_i is a positive real number, and M_i is an f-ample Cartier divisor for every i. Let $\mathcal{I}_V$ be the defining ideal sheaf of V on Y. We consider the following exact sequence

$$0 \to f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm}) \to f_*\mathcal{O}_Y(pM_1)$$
$$\to f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{O}_Y/\mathcal{I}_V^{pm}) \to \cdots$$

for a sufficiently large and divisible integer p. By restricting the above sequence to a general fiber F of f, we can check that the rank of $f_*\mathcal{O}_Y(pM_1)$ is greater than that of $f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{O}_Y/\mathcal{I}_V^{pm})$ by the usual estimates (see Lemma 12.3 below). Therefore, $f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm}) \neq 0$. Let D_1 be a member of

$$H^0(Z, f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm})) = H^0(Y, \mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm})$$

and D_i an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $D_i \sim_{\mathbb{Q}} M_i$ for $i \geq 2$. Then $D = (1/p)D_1 + a_2 D_2 + \cdots + a_l D_l$ satisfies the desired properties. $\qquad\square$

We close this section with the following well-known lemma. The proof is obvious.

Lemma 12.3. *Let X be a normal projective variety with $\dim X = d$ and A an ample $\mathbb{Q}$-divisor on X such that rA is Cartier for some positive integer r. Then*

$$h^0(X, \mathcal{O}_X(trA)) = \chi(X, \mathcal{O}_X(trA))$$
$$= \frac{(trA)^d}{d!} + (\text{lower terms in } t)$$

by the Riemann-Roch formula and the Serre vanishing theorem for $t \gg 0$.

Let $P \in X$ be a smooth point. Then

$$\dim_{\mathbb{C}} \mathcal{O}_X / m_P^\alpha = \binom{\alpha - 1 + d}{d}$$
$$= \frac{\alpha^d}{d!} + (\text{lower terms in } \alpha)$$

for all $\alpha \geq 1$, where m_P is the maximal ideal associated to P.

13. BASE POINT FREE THEOREM

The base point free theorem is one of the most important theorems in the log minimal model program. Theorem 13.1 is a special case of the base point free theorem for quasi-log varieties obtained by Ambro (cf. [A1, Theorem 5.1]). See also [F9, Theorem 3.66].

Theorem 13.1 (Base point free theorem)**.** *Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $\pi : X \to S$ be a projective morphism onto a variety S and L a π-nef Cartier divisor on X. Assume that*

(i) *$aL - (K_X + B)$ is π-ample for some real number $a > 0$, and*
(ii) *$\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is $\pi|_{\mathrm{Nlc}(X,B)}$-generated for $m \gg 0$.*

Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.

We will prove the base point free theorem for $\mathbb{R}$-divisors in Section 17 as an application of the cone theorem: Theorem 16.5.

Proof. We can assume that S is affine.

Step 1. We assume that (X, B) is klt and that L_η is numerically trivial, where $L_\eta = L|_{X_\eta}$ and X_η is the generic fiber of $\pi : X \to S$. Then we have

$$h^0(X_\eta, \mathcal{O}_{X_\eta}(L_\eta)) = \chi(X_\eta, \mathcal{O}_{X_\eta}(L_\eta))$$
$$= \chi(X_\eta, \mathcal{O}_{X_\eta}) = h^0(X_\eta, \mathcal{O}_{X_\eta}) > 0$$

by [Kl, Chapter II §2 Theorem 1] and the vanishing theorem. Here, the Kawamata–Viehweg vanishing theorem is sufficient. Therefore, $|L| \neq \emptyset$. Let D be a member of $|L|$. If $D = 0$, then it is obvious that $|mL|$ is free for every m. Thus, we can assume that $D \neq 0$. Let c be the log canonical threshold of (X, B) with respect to D. We replace (X, B) with $(X, B + cD)$, a with $a + c$. Then we can assume that (X, B) is lc but not klt. This case will be treated in Step 3.

Step 2. We assume that (X, B) is klt and that L_η is not numerically trivial. We take a general subvariety V on X such that $\pi : V \to S$ is generically finite. By Lemma 12.2, we can find an effective $\mathbb{R}$-divisor D on X such that

$$D \sim_{\mathbb{R}} lL - (K_X + B)$$

for some large l and that $\mathrm{mult}_V D > \mathrm{codim}_X V$. Let c be the log canonical threshold of (X, B) with respect to D. By the above construction, we obtain $0 < c < 1$. We replace (X, B) with $(X, B + cD)$, a with $a - ac + cl$ and can assume that (X, B) is lc but not klt. We note that

$$(a - ac + cl)L - (K_X + B + cD) \sim_{\mathbb{R}} (1 - c)(aL - (K_X + B)).$$

So, the problem is reduced to the case when (X, B) is lc but not klt. It will be treated in Step 3.

Step 3. We assume that (X, B) is not klt. Let p be a prime integer. We will prove that $\mathrm{Bs}|p^m L| = \emptyset$ for some positive integer m.

By Theorem 12.1, $|p^{m_1} L| \neq \emptyset$ for some positive integer m_1. If $\mathrm{Bs}|p^{m_1} L| = \emptyset$, then there are nothing to prove. So, we can assume that $\mathrm{Bs}|p^{m_1} L| \neq \emptyset$. We take general members $D_1, \cdots, D_{n+1} \in |p^{m_1} L|$, where $n = \dim X$. Since $D_1, \cdots, D_{n+1}$ are general, $(X, B + D_1 + \cdots + D_{n+1})$ is lc outside $\mathrm{Bs}|p^{m_1} L| \cup \mathrm{Nlc}(X, B)$. It is easy to see that $(X, B + D)$, where $D = D_1 + \cdots + D_{n+1}$, is not lc at the generic point of every irreducible component of $\mathrm{Bs}|p^{m_1} L|$ (see Lemma 13.2 below). Let c be the log canonical threshold of (X, B) with respect to D outside $\mathrm{Nlc}(X, B)$. Then $(X, B + cD)$ is lc but not klt outside $\mathrm{Nlc}(X, B)$, $0 < c < 1$, and $\mathcal{J}_{NLC}(X, B + cD) = \mathcal{J}_{NLC}(X, B)$ (see Proposition 7.5). We note that

$$(c(n + 1)p^{m_1} + a)L - (K_X + B + cD) \sim_{\mathbb{R}} aL - (K_X + B)$$

is f-ample. By the construction, there exists an lc center of $(X, B + cD)$ contained in $\mathrm{Bs}|p^{m_1} L|$. By Theorem 12.1, we can find $m_2 > m_1$ such that $\mathrm{Bs}|p^{m_2} L| \subsetneq \mathrm{Bs}|p^{m_1} L|$. By the noetherian induction, there exists m such that $\mathrm{Bs}|p^m L| = \emptyset$.

Step 4. Let p' be a prime integer such that $p' \neq p$. Then, by Step 3 again, we can find a positive integer m' such that $\mathrm{Bs}|p'^{m'} L| = \emptyset$. So,

there exists a positive integer m_0 such that $|kL|$ is free for every $k \geq m_0$ by $\mathrm{Bs}|p^m L| = \emptyset$ and $\mathrm{Bs}|p'^{m'} L| = \emptyset$.

This completes the proof. $\square$

We close this section with the following lemma. We used it in the proof of Theorem 13.1.

Lemma 13.2. *Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let P be a closed point of X and $P \in D_i$ a Cartier divisor for every i. If $(X, B + \sum_{i=1}^{k} D_i)$ is log canonical at P, then $k \leq \dim X$.*

Proof. The proof is by the induction on $\dim X$. The assertion is clear if $\dim X = 1$. We put $S = D_1$. Let $\nu : S^\nu \to S$ be the normalization and B_{S^ν} the *different* of $(X, S + B)$ on S^ν (see Section 14 below). So, we have $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$. Since $(X, B + S + \sum_{i=2}^{k} D_i)$ is log canonical at P, $(S^\nu, B_{S^\nu} + \sum_{i=2}^{k} \nu^* D_i)$ is log canonical at $Q \in \nu^{-1}(P)$. Thus, $k - 1 \leq \dim S^\nu$ by induction. This means that $k \leq \dim X$. $\square$

14. Shokurov's differents

Let us recall the definition and basic properties of Shokurov's *differents* following [S2, §3] and [A2, 9.2.1].

14.1. Let X be a normal variety and $S + B$ an $\mathbb{R}$-divisor on X such that $K_X + S + B$ is $\mathbb{R}$-Cartier. Assume that S is reduced and that S and B have no common irreducible components. Let $f : Y \to X$ be a resolution such that

$$K_Y + S_Y + B_Y = f^*(K_X + S + B)$$

and $\mathrm{Supp}(S_Y + B_Y)$ is simple normal crossing and S_Y is smooth, where S_Y is the strict transform of S on Y. Let $\nu : S^\nu \to S$ be the normalization. Then $f : S_Y \to S$ can be decomposed as

$$f : S_Y \xrightarrow{\pi} S^\nu \xrightarrow{\nu} S.$$

We define $B_{S_Y} = B_Y|_{S_Y}$. Then we obtain

$$(K_Y + S_Y + B_Y)|_{S_Y} = K_{S_Y} + B_{S_Y}$$

by adjunction. We put $B_{S^\nu} = \pi_* B_{S_Y}$. Then we have

$$K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B).$$

The $\mathbb{R}$-divisor B_{S^ν} on S^ν is called the *different* of $(X, S + B)$ on S^ν. We can easily check that B_{S^ν} is independent of the resolution $f : Y \to X$. So, B_{S^ν} is a well-defined $\mathbb{R}$-divisor on S^ν. We can check the following properties.

(i) $K_{S^\nu} + B_{S^\nu}$ is $\mathbb{R}$-Cartier and $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$.

(ii) If B is a $\mathbb{Q}$-divisor, then B_{S^ν} is a $\mathbb{Q}$-divisor.

(iii) B_{S^ν} is effective if B is effective in a neighborhood of S.

(iv) (S^ν, B_{S^ν}) is log canonical if $(X, S + B)$ is log canonical near S.

(v) Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that S and D have no common irreducible components. Then we have

$$(B + D)_{S^\nu} = B_{S^\nu} + \nu^*D.$$

We sometimes write $D|_{S^\nu} = \nu^*D$ for simplicity.

The properties except (iii) are obvious by the definition. We give a proof of (iii) for the reader's convenience.

Proof of (iii). By shrinking X, we can assume that X is quasi-projective and B is effective. By taking hyperplane cuts, we can also assume that X is a surface. Run the log minimal model program over X with respect to $K_Y + S_Y$. Let C be a curve on Y such that $(K_Y + S_Y) \cdot C < 0$ and $f(C)$ is a point. Then $K_Y \cdot C < 0$ because S_Y is the strict transform of S. Therefore, each step of the log minimal model program over X with respect to $K_Y + S_Y$ is a contraction of a (-1)-curve E with $(K_Y + S_Y) \cdot E < 0$. So, by replacing (Y, S_Y) with the output of the above log minimal model program, we can assume that Y is smooth, (Y, S_Y) is plt, and $K_Y + S_Y$ is f-nef. We note that S_Y is a smooth curve since (Y, S_Y) is plt (cf. [KM, Proposition 5.51]). By the negativity lemma (see Lemma 4.16) and the assumption that B is effective, B_Y is effective. We note the following equality

$$-B_Y = K_Y + S_Y - f^*(K_X + S + B).$$

By adjunction, we obtain

$$(K_Y + S_Y + B_Y)|_{S_Y} = K_{S_Y} + B_Y|_{S_Y}.$$

It is obvious that $B_Y|_{S_Y}$ is effective. This implies that $B_{S^\nu} = B_Y|_{S_Y}$ is effective. $\square$

When X is singular, B_{S^ν} is not necessarily zero even if $B = 0$.

15. Rationality theorem

In this section, we prove the following rationality theorem. It is a special case of [A1, Theorem 5.9].

Theorem 15.1 (Rationality theorem). *Let X be a normal variety and B an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $\pi : X \to S$ be a projective morphism and H a π-ample Cartier divisor on X. Assume that $K_X + B$ is not π-nef and that r is a positive number such that*

(1) $H + r(K_X + B)$ is π-nef but not π-ample, and
(2) $(H + r(K_X + B))|_{\text{Nlc}(X,B)}$ is $\pi|_{\text{Nlc}(X,B)}$-ample.

Then r is a rational number, and in reduced form, r has denominator at most $a(\dim X + 1)$, where $a(K_X + B)$ is a Cartier divisor on X.

Before the proof of Theorem 15.1, we recall the following lemmas.

Lemma 15.2 (cf. [KM, Lemma 3.19]). *Let $P(x, y)$ be a non-trivial polynomial of degree $\leq n$ and assume that P vanishes for all sufficiently large integral solutions of $0 < ay - rx < \varepsilon$ for some fixed positive integer a and positive ε for some $r \in \mathbb{R}$. Then r is rational, and in reduced form, r has denominator $\leq a(n+1)/\varepsilon$.*

Proof. We assume that r is irrational. Then an infinite number of integral points in the (x, y)-plane on each side of the line $ay - rx = 0$ are closer than $\varepsilon/(n+2)$ to that line. So there is a large integral solution (x', y') with $0 < ay' - rx' < \varepsilon/(n + 2)$. In this case, we see that

$$(2x', 2y'), \cdots, ((n + 1)x', (n + 1)y')$$

are also solutions by hypothesis. So $(y'x - x'y)$ divides P, since P and $(y'x - x'y)$ have $(n + 1)$ common zeroes. We choose a smaller ε and repeat the argument. We do this $n + 1$ times to get a contradiction.

Now we assume that $r = u/v$ in lowest terms. For given j, let (x', y') be a solution of $ay - rx = aj/v$. Note that an integral solution exists for every j. Then we have $a(y' + ku) - r(x' + akv) = aj/v$ for all k. So, as above, if $aj/v < \varepsilon$, $(ay - rx) - (aj/v)$ must divide P. So we can have at most n such values of j. Thus $a(n + 1)/v \geq \varepsilon$. $\square$

Lemma 15.3. *Let C be a projective variety and D_1 and D_2 Cartier divisors on X. Consider the Hilbert polynomial*

$$P(u_1, u_2) = \chi(C, \mathcal{O}_C(u_1 D_1 + u_2 D_2)).$$

If D_1 is ample, then $P(u_1, u_2)$ is a non-trivial polynomial by Serre's theorems A and B, and its degree is $\leq \dim C$.

Proof of Theorem 15.1. Let m be a positive integer such that $H' = mH$ is π-very ample. If $H' + r'(K_X + B)$ is π-nef but not π-ample, and $(H' + r'(K_X + B))|_{\text{Nlc}(X,B)}$ is $\pi|_{\text{Nlc}(X,B)}$-ample, then we have

$$H + r(K_X + B) = \frac{1}{m}(H' + r'(K_X + B)).$$

This gives $r = \frac{1}{m}r'$. Thus, r is rational if and only if r' is rational. Assume furthermore that r' has denominator v. Then r has denominator dividing mv. Since m can be arbitrary sufficiently large integer, this

implies that r has denominator dividing v. Therefore, by replacing H with mH, we can assume that H is very ample over S.

For each $(p,q) \in \mathbb{Z}^2$, let $L(p,q)$ denote the relative base locus of the linear system $M(p,q)$ on X (with reduced scheme structure), that is,

$$L(p,q) = \mathrm{Supp}(\mathrm{Coker}(\pi^*\pi_*\mathcal{O}_X(M(p,q)) \to \mathcal{O}_X(M(p,q)))),$$

where $M(p,q) = pH + qa(K_X + B)$. By the definition, $L(p,q) = X$ if and only if $\pi_*\mathcal{O}_X(M(p,q)) = 0$.

Claim 1 (cf. [KM, Claim 3.20]). *Let ε be a positive number. For (p,q) sufficiently large and $0 < aq - rp < \varepsilon$, $L(p,q)$ is the same subset of X. We call this subset L_0. Let $I \subset \mathbb{Z}^2$ be the set of (p,q) for which $0 < aq - rp < 1$ and $L(p,q) = L_0$. We note that I contains all sufficiently large (p,q) with $0 < aq - rp < 1$.*

Proof. We fix $(p_0, q_0) \in \mathbb{Z}^2$ such that $0 < aq_0 - rp_0 < 1$. Since H is π-very ample, there exists a positive integer m_0 such that $\mathcal{O}_X(mH + ja(K_X + B))$ is π-generated for every $m > m_0$ and every $0 \le j \le q_0 - 1$. Let M be the round-up of

$$\left(m_0 + \frac{1}{r}\right) \Big/ \left(\frac{a}{r} - \frac{p_0}{q_0}\right).$$

If $(p', q') \in \mathbb{Z}^2$ such that $0 < aq' - rp' < 1$ and $q' \ge M + q_0 - 1$, then we can write

$$p'H + q'a(K_X + B) = k(p_0 H + q_0 a(K_X + B)) + (lH + ja(K_X + B))$$

for some $k \ge 0$, $0 \le j \le q_0 - 1$ with $l > m_0$. It is because we can uniquely write $q' = kq_0 + j$ with $0 \le j \le q_0 - 1$. Thus, we have $kq_0 \ge M$. So, we obtain

$$l = p' - kp_0 > \frac{a}{r}q' - \frac{1}{r} - (kq_0)\frac{p_0}{q_0} \ge \left(\frac{a}{r} - \frac{p_0}{q_0}\right)M - \frac{1}{r} \ge m_0.$$

Therefore, $L(p', q') \subset L(p_0, q_0)$. By the noetherian induction, we obtain the desired closed subset $L_0 \subset X$ and $I \subset \mathbb{Z}^2$. $\square$

Claim 2. *We have $L_0 \cap \mathrm{Nlc}(X,B) = \emptyset$.*

Proof of Claim 2. We take $(\alpha, \beta) \in \mathbb{Q}^2$ such that $\alpha > 0$, $\beta > 0$, and $\beta a/\alpha > r$ is sufficiently close to r. Then $(\alpha H + \beta a(K_X + B))|_{\mathrm{Nlc}(X,B)}$ is $\pi|_{\mathrm{Nlc}(X,B)}$-ample because $(H + r(K_X + B))|_{\mathrm{Nlc}(X,B)}$ is $\pi|_{\mathrm{Nlc}(X,B)}$-ample. If $0 < aq - rp < 1$ and $(p,q) \in \mathbb{Z}^2$ is sufficiently large, then

$$M(p,q) = mM(\alpha, \beta) + (M(p,q) - mM(\alpha, \beta))$$

such that $M(p,q) - mM(\alpha, \beta)$ is π-very ample and that

$$m(\alpha H + \beta a(K_X + B))|_{\mathrm{Nlc}(X,B)}$$

is also $\pi|_{\mathrm{Nlc}(X,B)}$-very ample. It can be checked by the same argument as in the proof of Claim 1. Therefore, $\mathcal{O}_{\mathrm{Nlc}(X,B)}(M(p,q))$ is π-very ample. Since

$$\pi_*\mathcal{O}_X(M(p,q)) \to \pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(M(p,q))$$

is surjective by the vanishing theorem: Theorem 8.1, we obtain $L(p,q) \cap \mathrm{Nlc}(X,B) = \emptyset$. We note that $M(p,q) - (K_X + B)$ is π-ample because (p,q) is sufficiently large and $aq - rp < 1$. By Claim 1, we have $L_0 \cap \mathrm{Nlc}(X,B) = \emptyset$. $\qquad\square$

Claim 3. *We assume that r is not rational or that r is rational and has denominator $> a(n+1)$ in reduced form, where $n = \dim X$. Then, for (p,q) sufficiently large and $0 < aq - rp < 1$, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of every lc center of (X, B).*

Proof of Claim 3. We note that

$$M(p,q) - (K_X + B) = pH + (qa - 1)(K_X + B).$$

If $aq - rp < 1$ and (p,q) is sufficiently large, then $M(p,q) - (K_X + B)$ is π-ample. Let C be an lc center of (X, B). We note that we can assume $C \cap \mathrm{Nlc}(X,B) = \emptyset$ by Claim 2. Then $P_{C_\eta}(p,q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$ is a non-zero polynomial of degree at most $\dim C_\eta \le \dim X$ by Lemma 15.3. Note that C_η is the generic fiber of $C \to \pi(C)$. By Lemma 15.2, there exists (p,q) such that $P_{C_\eta}(p,q) \ne 0$, (p,q) sufficiently large, and $0 < aq - rp < 1$. By the π-ampleness of $M(p,q) - (K_X + B)$,

$$P_{C_\eta}(p,q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) = h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$$

and

$$\pi_*\mathcal{O}_X(M(p,q)) \to \pi_*\mathcal{O}_C(M(p,q))$$

is surjective by Theorem 8.1. We note that $C \cap \mathrm{Nlc}(X,B) = \emptyset$. Therefore, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of C. By combining this with Claim 1, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of every lc center of (X, B) if (p,q) is sufficiently large with $0 < aq - rp < 1$. So, we obtain Claim 3. $\qquad\square$

Note that $\mathcal{O}_X(M(p,q))$ is not π-generated for $(p,q) \in I$ because $M(p,q)$ is not π-nef. Therefore, $L_0 \ne \emptyset$. We shrink S to an affine open subset intersecting $\pi(L_0)$. Let $D_1, \cdots, D_{n+1}$ be general members of $\pi_*\mathcal{O}_X(M(p_0, q_0)) = H^0(X, \mathcal{O}_X(M(p_0, q_0)))$ with $(p_0, q_0) \in I$. We can check that $K_X + B + \sum_{i=1}^{n+1} D_i$ is not lc at the generic point of every irreducible component of L_0 by Lemma 13.2. On the other hand, $K_X + B + \sum_{i=1}^{n+1} D_i$ is lc outside $L_0 \cup \mathrm{Nlc}(X,B)$ since D_i is a general member of $|M(p_0, q_0)|$ for every i. Let $0 < c < 1$ be the log canonical threshold of (X, B) with respect to $D = \sum_{i=1}^{n+1} D_i$ outside $\mathrm{Nlc}(X,B)$.

Note that $c > 0$ by Claim 3. Thus, the pair $(X, B + cD)$ has some lc centers contained in L_0. Let C be an lc center contained in L_0. We note that $\mathcal{J}_{NLC}(X, B+cD) = \mathcal{J}_{NLC}(X, B)$ by Proposition 7.5 and that $C \cap \mathrm{Nlc}(X, B + cD) = C \cap \mathrm{Nlc}(X, B) = \emptyset$. We consider

$$K_X + B + cD = c(n + 1)p_0H + (1 + c(n + 1)q_0a)(K_X + B).$$

Thus we have

$$
\begin{aligned}
&pH + qa(K_X + B) - (K_X + B + cD) \\
&= (p - c(n + 1)p_0)H + (qa - (1 + c(n + 1)q_0a))(K_X + B).
\end{aligned}
$$

If p and q are large enough and $0 < aq - rp \le aq_0 - rp_0$, then

$$pH + qa(K_X + B) - (K_X + B + cD)$$

is π-ample. It is because

$$
\begin{aligned}
&(p - c(n + 1)p_0)H + (qa - (1 + c(n + 1)q_0a))(K_X + B) \\
&= (p - (1 + c(n + 1))p_0)H + (qa - (1 + c(n + 1))q_0a)(K_X + B) \\
&\quad + p_0H + (q_0a - 1)(K_X + B).
\end{aligned}
$$

Suppose that r is not rational. There must be arbitrarily large (p, q) such that $0 < aq - rp < \varepsilon = aq_0 - rp_0$ and $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) \ne 0$ by Lemma 15.2 because $P_{C_\eta}(p, q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q)))$ is a non-trivial polynomial of degree at most $\dim C_\eta$ by Lemma 15.3. Since $M(p, q) - (K_X + B + cD)$ is π-ample by $0 < aq - rp < aq_0 - rp_0$, we have $h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) \ne 0$ by the vanishing theorem: Theorem 8.1. By the vanishing theorem: Theorem 8.1,

$$\pi_*\mathcal{O}_X(M(p, q)) \to \pi_*\mathcal{O}_C(M(p, q))$$

is surjective because $M(p, q) - (K_X + B + cD)$ is π-ample. We note that $C \cap \mathrm{Nlc}(X, B + cD) = \emptyset$. Thus C is not contained in $L(p, q)$. Therefore, $L(p, q)$ is a proper subset of $L(p_0, q_0) = L_0$, giving the desired contradiction. So now we know that r is rational.

We next suppose that the assertion of the theorem concerning the denominator of r is false. We choose $(p_0, q_0) \in I$ such that $aq_0 - rp_0$ is the maximum, say it is equal to d/v. If $0 < aq - rp \le d/v$ and (p, q) is sufficiently large, then $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) = h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p, q)))$ since $M(p, q) - (K_X + B + cD)$ is π-ample. There exists sufficiently large (p, q) in the strip $0 < aq - rp < 1$ with $\varepsilon = 1$ for which $h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) \ne 0$ by Lemma 15.2 since $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q)))$ is a non-trivial polynomial of degree at most $\dim C_\eta$ by Lemma 15.3. Note that $aq - rp \le d/v = aq_0 - rp_0$ holds automatically for $(p, q) \in I$. Since

$$\pi_*\mathcal{O}_X(M(p, q)) \to \pi_*\mathcal{O}_C(M(p, q))$$

is surjective by the π-ampleness of $M(p,q) - (K_X + B + cD)$, we obtain the desired contradiction by the same reason as above. So, we finish the proof of the rationality theorem. $\qquad\square$

Remark 15.4. In Theorem 15.1, it is sufficient to assume that B is an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-linearly equivalent to a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor ω on X with the condition that $a\omega$ is Cartier. All we have to do is to replace $a(K_X + B)$ with $a\omega$ in the proof of Theorem 15.1. We need this generalization in the proof of the cone theorem: Theorem 16.5.

16. Cone theorem

The main theorem of this section is the cone theorem. Before we state the main theorem, let us fix the notation.

Definition 16.1. Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $\pi : X \to S$ be a projective morphism. We put

$$\overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} = \mathrm{Im}(\overline{NE}(\mathrm{Nlc}(X,B)/S) \to \overline{NE}(X/S)).$$

For an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D, we define

$$D_{\geq 0} = \{z \in N_1(X/S) \mid D \cdot z \geq 0\}.$$

Similarly, we can define $D_{>0}$, $D_{\leq 0}$, and $D_{<0}$. We also define

$$D^\perp = \{z \in N_1(X/S) \mid D \cdot z = 0\}.$$

We use the following notation

$$\overline{NE}(X/S)_{D \geq 0} = \overline{NE}(X/S) \cap D_{\geq 0},$$

and similarly for > 0, ≤ 0, and < 0.

Definition 16.2. An *extremal face* of $\overline{NE}(X/S)$ is a non-zero subcone $F \subset \overline{NE}(X/S)$ such that $z, z' \in F$ and $z + z' \in F$ imply that $z, z' \in F$. Equivalently, $F = \overline{NE}(X/S) \cap H^\perp$ for some π-nef $\mathbb{R}$-divisor H, which is called a *supporting function* of F. An *extremal ray* is a one-dimensional extremal face.

  (1) An extremal face F is called $(K_X + B)$-*negative* if

$$F \cap \overline{NE}(X/S)_{K_X + B \geq 0} = \{0\}.$$

  (2) An extremal face F is called *rational* if we can choose a π-nef $\mathbb{Q}$-divisor H as a support function of F.

(3) An extremal face F is called *relatively ample at* $\mathrm{Nlc}(X, B)$ if

$$F \cap \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} = \{0\}.$$

Equivalently, $H|_{\mathrm{Nlc}(X,B)}$ is $\pi|_{\mathrm{Nlc}(X,B)}$-ample for every supporting function H of F.

(4) An extremal face F is called *contractible at* $\mathrm{Nlc}(X, B)$ if it has a rational supporting function H such that $H|_{\mathrm{Nlc}(X,B)}$ is $\pi|_{\mathrm{Nlc}(X,B)}$-semi-ample.

The following theorem is a direct consequence of Theorem 13.1.

Theorem 16.3 (Contraction theorem)**.** *Let X be a normal variety, B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier, and $\pi : X \to S$ a projective morphism. Let H be a π-nef Cartier divisor such that $F = H^{\perp} \cap \overline{NE}(X/S)$ is $(K_X + B)$-negative and contractible at $\mathrm{Nlc}(X, B)$. Then there exists a projective morphism $\varphi_F : X \to Y$ over S with the following properties.*

(1) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.*

(2) *$\mathcal{O}_Y \simeq (\varphi_F)_* \mathcal{O}_X$.*

(3) *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Assume that $L^{\otimes m}|_{\mathrm{Nlc}(X,B)}$ is $\varphi_F|_{\mathrm{Nlc}(X,B)}$-generated for $m \gg 0$. Then there is a line bundle L_Y on Y such that $L \simeq \varphi_F^* L_Y$.*

Proof. By the assumption, $qH - (K_X + B)$ is π-ample for some positive integer q and we can assume that $H|_{\mathrm{Nlc}(X,B)}$ is $\pi|_{\mathrm{Nlc}(X,B)}$-semi-ample. By Theorem 13.1, $\mathcal{O}_X(mH)$ is π-generated for some $m > 0$. We take the Stein factorization of the associated morphism. Then, we have the contraction morphism $\varphi_F : X \to Y$ with the properties (1) and (2).

We consider $\varphi_F : X \to Y$ and $\overline{NE}(X/Y)$. Then $\overline{NE}(X/Y) = F$, L is numerically trivial over Y, and $-(K_X + B)$ is φ_F-ample. Applying the base point free theorem (cf. Theorem 13.1) over Y, both $L^{\otimes m}$ and $L^{\otimes(m+1)}$ are pull-backs of line bundles on Y. Their difference gives a line bundle L_Y such that $L \simeq \varphi_F^* L_Y$. $\square$

Example 16.4. Let S be a cone over a smooth cubic curve and $\pi : X \to S$ the blow-up at the vertex of S. Then $K_X + E = \pi^* K_S$, where E is the π-exceptional curve. We put $B = 2E$ and consider the pair (X, B). In this case, $\varphi_F = \pi : X \to Y = S$ with $F = \overline{NE}(X/S) = \mathbb{R}_{\geq 0}[E]$ is an example of contraction morphisms in Theorem 16.3.

Theorem 16.5 (Cone theorem)**.** *Let X be a normal variety, B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier, and $\pi : X \to S$ a projective morphism. Then we have the following properties.*

(1) $\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+B\geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} + \sum R_j$,
where R_j's are the (K_X+B)-negative extremal rays of $\overline{NE}(X/S)$ that are rational and relatively ample at $\mathrm{Nlc}(X,B)$. In particular, each R_j is spanned by an integral curve C_j on X such that $\pi(C_j)$ is a point.

(2) Let H be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many R_j's included in $(K_X + B + H)_{<0}$. In particular, the R_j's are discrete in the half-space $(K_X + B)_{<0}$.

(3) Let F be a $(K_X + B)$-negative extremal face of $\overline{NE}(X/S)$ that is relatively ample at $\mathrm{Nlc}(X,B)$. Then F is a rational face. In particular, F is contractible at $\mathrm{Nlc}(X,B)$.

Proof. First, we assume that $K_X + B$ is $\mathbb{R}$-linearly equivalent to a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X (see Remark 15.4). We can assume that $\dim_{\mathbb{R}} N_1(X/S) \geq 2$ and $K_X + B$ is not π-nef. Otherwise, the theorem is obvious.

Step 1. We have

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+B\geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} + \sum_F F,$$

where F's vary among all rational proper $(K_X + B)$-negative faces that are relatively ample at $\mathrm{Nlc}(X,B)$ and $\overline{\quad}$ denotes the closure with respect to the real topology.

Proof. We put

$$\mathfrak{B} = \overline{NE}(X/S)_{K_X+B\geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} + \sum_F F.$$

It is clear that $\overline{NE}(X/S) \supset \mathfrak{B}$. We note that each F is spanned by curves on X mapped to points on S by Theorem 16.3 (1). Supposing $\overline{NE}(X/S) \neq \mathfrak{B}$, we shall derive a contradiction. There is a separating function M which is Cartier and is not a multiple of K_X+B in $N^1(X/S)$ such that $M > 0$ on $\mathfrak{B} \setminus \{0\}$ and $M \cdot z_0 < 0$ for some $z_0 \in \overline{NE}(X/S)$. Let C be the dual cone of $\overline{NE}(X/S)_{K_X+B\geq 0}$, that is,

$$C = \{D \in N^1(X/S) \mid D \cdot z \geq 0 \text{ for } z \in \overline{NE}(X/S)_{K_X+B\geq 0}\}.$$

Then C is generated by π-nef divisors and $K_X + B$. Since $M > 0$ on $\overline{NE}(X/S)_{K_X+B\geq 0}\setminus\{0\}$, M is in the interior of C, and hence there exists a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A such that $M - A = L' + p(K_X + B)$ in $N^1(X/S)$, where L' is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and p is a non-negative rational number. Therefore, M is expressed in the form $M = H + p(K_X + B)$ in $N^1(X/S)$, where $H = A + L'$ is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The rationality theorem (see Theorem 15.1) implies

that there exists a positive rational number $r < p$ such that $L = H + r(K_X + B)$ is π-nef but not π-ample, and $L|_{\mathrm{Nlc}(X,B)}$ is $\pi|_{\mathrm{Nlc}(X,B)}$-ample. Note that $L \neq 0$ in $N^1(X/S)$, since M is not a multiple of $K_X + B$. Thus the extremal face F_L associated to the supporting function L is contained in $\mathfrak{B}$, which implies $M > 0$ on F_L. Therefore, $p < r$. It is a contradiction. This completes the proof of our first claim. $\qquad\square$

Step 2. In the equality of Step 1, we may take such L that has the extremal face F_L of dimension one.

Proof. Let F be a rational proper $(K_X + B)$-negative extremal face that is relatively ample at $\mathrm{Nlc}(X, B)$, and assume that $\dim F \geq 2$. Let $\varphi_F : X \to W$ be the associated contraction. Note that $-(K_X + B)$ is φ_F-ample. By Step 1, we obtain

$$F = \overline{NE}(X/W) = \overline{\sum_G G},$$

where the G's are the rational proper $(K_X + B)$-negative extremal faces of $\overline{NE}(X/W)$. We note that $\overline{NE}(X/W)_{\mathrm{Nlc}(X,B)} = 0$ because φ_F embeds $\mathrm{Nlc}(X, B)$ into W. The G's are also $(K_X + B)$-negative extremal faces of $\overline{NE}(X/S)$ that are ample at $\mathrm{Nlc}(X, B)$, and $\dim G < \dim F$. By induction, we obtain

$$(\clubsuit) \quad \overline{NE}(X/S) = \overline{\overline{NE}(X/S)_{K_X + B \geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} + \sum R_j},$$

where the R_j's are $(K_X + B)$-negative rational extremal rays. Note that each R_j does not intersect $\overline{NE}(X/S)_{\mathrm{Nlc}(X,B)}$. $\qquad\square$

Step 3. The contraction theorem (cf. Theorem 16.3) guarantees that for each extremal ray R_j there exists a reduced irreducible curve C_j on X such that $[C_j] \in R_j$. Let $\psi_j : X \to W_j$ be the contraction morphism of R_j, and let A be a π-ample Cartier divisor. We set

$$r_j = -\frac{A \cdot C_j}{(K_X + B) \cdot C_j}.$$

Then $A + r_j(K_X + B)$ is ψ_j-nef but not ψ_j-ample, and $(A + r_j(K_X + B))|_{\mathrm{Nlc}(X,B)}$ is $\psi_j|_{\mathrm{Nlc}(X,B)}$-ample. By the rationality theorem (see Theorem 15.1), expressing $r_j = u_j/v_j$ with $u_j, v_j \in \mathbb{Z}_{>0}$ and $(u_j, v_j) = 1$, we have the inequality $v_j \leq a(\dim X + 1)$.

Step 4. Now take π-ample Cartier divisors $H_1, H_2, \cdots, H_{\rho-1}$ such that $K_X + B$ and the H_i's form a basis of $N^1(X/S)$, where $\rho =$

$\dim_{\mathbb{R}} N^1(X/S)$. By Step 3, the intersection of the extremal rays R_j with the hyperplane

$$\{z \in N_1(X/S) \mid a(K_X + B) \cdot z = -1\}$$

in $N_1(X/S)$ lie on the lattice

$\Lambda = \{z \in N_1(X/S) \mid a(K_X+B)\cdot z = -1, H_i \cdot z \in (a(a(\dim X+1))!)^{-1}\mathbb{Z}\}.$

This implies that the extremal rays are discrete in the half space

$$\{z \in N_1(X/S) \mid (K_X + B) \cdot z < 0\}.$$

Thus we can omit the closure sign $\overline{}$ from the formula (♣) and this completes the proof of (1) when $K_X + B$ is $\mathbb{R}$-linearly equivalent to a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

Step 5. Let H be a π-ample $\mathbb{R}$-divisor on X. We choose $0 < \varepsilon_i \ll 1$ for $1 \le i \le \rho - 1$ such that $H - \sum_{i=1}^{\rho-1} \varepsilon_i H_i$ is π-ample. Then the R_j's included in $(K_X + B + H)_{<0}$ correspond to some elements of the above lattice Λ for which $\sum_{i=1}^{\rho-1} \varepsilon_i H_i \cdot z < 1/a$. Therefore, we obtain (2).

Step 6. Let F be a $(K_X + B)$-negative extremal face as in (3). The vector space $V = F^\perp \subset N^1(X/S)$ is defined over $\mathbb{Q}$ because F is generated by some of the R_j's (cf. the proof of Theorem 17.1 below). There exists a π-ample $\mathbb{R}$-divisor H such that F is contained in $(K_X + B + H)_{<0}$. Let $\langle F \rangle$ be the vector space spanned by F. We put

$$W_F = \overline{NE}(X/S)_{K_X+B+H\geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} + \sum_{R_j \not\subset F} R_j.$$

Then W_F is a closed cone, $\overline{NE}(X/S) = W_F + F$, and $W_F \cap \langle F \rangle = \{0\}$. The supporting functions of F are the elements of V that are positive on $W_F \setminus \{0\}$. This is a non-empty open set and thus it contains a rational element that, after scaling, gives a π-nef Cartier divisor L such that $F = L^\perp \cap \overline{NE}(X/S)$. Therefore, F is rational. So, we have (3).

From now on, $K_X + B$ is $\mathbb{R}$-Cartier.

Step 7. Let H be a π-ample $\mathbb{R}$-divisor on X. We shall prove (2). We assume that there are infinitely many R_j's in $(K_X + B + H)_{<0}$ and get a contradiction. There exists an affine open subset U of S such that $\overline{NE}(\pi^{-1}(U)/U)$ has infinitely many $(K_X + B + H)$-negative extremal rays. So, we shrink S and can assume that S is affine. We can write $H = E + H'$ such that H' is π-ample, $\mathcal{J}_{NLC}(X, B + E) = \mathcal{J}_{NLC}(X, B)$, and $K_X + B + E$ is $\mathbb{R}$-linearly equivalent to a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Since $K_X + B + H = K_X + B + E + H'$, we have

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+B+H\geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,B)} + \sum_{\mathrm{finite}} R_j.$$

It is a contradiction. Thus, we obtain (2). The statement (1) is a direct consequence of (2). Of course, (3) holds by Step 6 once we obtain (1).

So, we complete the proof of the cone theorem. $\qquad\square$

We close this section with the following elementary example.

Example 16.6. We consider $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi_i : Y \to \mathbb{P}^1$ be the i-th projection for $i = 1, 2$. Let F_i be a fiber of π_i for $i = 1, 2$. We put $P = F_1 \cap F_2$ and consider the blow-up $f : X \to Y$ at P. Let E be the exceptional curve of f and $C_i = f_*^{-1}F_i$ for $i = 1, 2$. In this situation, we can check that $-K_X$ is ample, $\rho(X) = 3$, and

$$\overline{NE}(X) = \mathbb{R}_{\geq 0}[C_1] + \mathbb{R}_{\geq 0}[C_2] + \mathbb{R}_{\geq 0}[E].$$

We put

$$B = \frac{3}{2}E + \frac{1}{2}C_1 + C_2.$$

Then we have

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \overline{NE}(X)_{\mathrm{Nlc}(X,B)} + \mathbb{R}_{\geq 0}[C_2],$$

where

$$\overline{NE}(X)_{\mathrm{Nlc}(X,B)} = \mathbb{R}_{\geq 0}[E], \quad \overline{NE}(X)_{K_X+B\geq 0} = \mathbb{R}_{\geq 0}[C_1],$$

and

$$C_2 \cdot (K_X + B) < 0.$$

17. BASE POINT FREE THEOREM REVISITED

This section is a supplement to the base point free theorem: Theorem 13.1.

Theorem 17.1 (Base point free theorem for $\mathbb{R}$-divisors)**.** *Let (X, B) be a log canonical pair and $\pi : X \to S$ a projective morphism onto a variety S. Let D be a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $aD - (K_X + B)$ is ample for some real number $a > 0$. Then D is π-semi-ample.*

Proof. We can assume that $a = 1$ by replacing D with aD. We put

$$F = \{z \in \overline{NE}(X/S) \,|\, D \cdot z = 0\}.$$

Then F is a face of $\overline{NE}(X/S)$ and $(K_X + B) \cdot z < 0$ for $z \in F$. We claim that F contains only finitely many $(K_X + B)$-negative extremal rays $R_1, \cdots, R_k$ of $\overline{NE}(X/S)$. If F contains infinitely many $(K_X + B)$-negative extremal rays of $\overline{NE}(X/S)$, then it also holds after shrinking S suitably. Therefore, we can assume that S is affine. In this situation, X is quasi-projective. We take a general small ample $\mathbb{Q}$-divisor A on

X such that $D - (K_X + B + A)$ is ample and that $(X, B + A)$ is log canonical. Let R be a $(K_X + B)$-negative extremal ray such that $R \subset F$. Then R is a $(K_X + B + A)$-negative extremal ray since $D \cdot R = 0$ and $D - (K_X + B + A)$ is ample. On the other hand, there are only finitely many $(K_X + B + A)$-negative extremal rays in $\overline{NE}(X/S)$ by Theorem 16.5 (2). It is a contradiction. Therefore, F is spanned by the extremal rays $R_1, \cdots, R_k$. We consider the finite dimensional real vector space $V = \bigoplus_j \mathbb{R} D_j$, where $\sum_j D_j = \mathrm{Supp} D$ is the irreducible decomposition. Then

$$\mathcal{R} = \{E \in V \mid E \text{ is } \mathbb{R}\text{-Cartier and } E \cdot z = 0 \text{ for any } z \in F\}$$

is a rational affine subspace of V and $D \in \mathcal{R}$. Thus, we can find positive real numbers $r_1, r_2, \cdots, r_m$ and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $E_1, E_2, \cdots, E_m$ such that $D = \sum_{i=1}^m r_i E_i$ and that $E_i - (K_X + B)$ is ample for every i. By Theorem 13.1, E_i is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for every i. Therefore, D is semi-ample. $\square$

18. Lengths of extremal rays

In this section, we discuss the estimate of lengths of extremal rays. It is indispensable for the log minimal model program with scaling (see, for example, [BCHM]). The results in this section were obtained in [S3], [S4], and [B1] with some extra assumptions. We include them for the reader's convenience.

Let us recall the following easy lemma.

Lemma 18.1 (cf. [S4, Lemma 1]). *Let (X, B) be a log canonical pair, where B is an $\mathbb{R}$-divisor. Then there are positive real numbers r_i and effective $\mathbb{Q}$-divisors B_i for $1 \leq i \leq l$ and a positive integer m such that $\sum_{i=1}^l r_i = 1$, $K_X + B = \sum_{i=1}^l r_i(K_X + B_i)$, (X, B_i) is lc for every i, and $m(K_X + B_i)$ is Cartier for every i.*

Proof. Let $\sum_k D_k$ be the irreducible decomposition of $\mathrm{Supp} B$. We consider the finite dimensional real vector space $V = \bigoplus_k \mathbb{R} D_k$. We put

$$\mathcal{Q} = \{D \in V \mid K_X + D \text{ is } \mathbb{R}\text{-Cartier}\}.$$

Then, it is easy to see that $\mathcal{Q}$ is an affine subspace of V defined over $\mathbb{Q}$. We put

$$\mathcal{P} = \{D \in \mathcal{Q} \mid K_X + D \text{ is log canonical}\}.$$

Thus, by the definition of log canonicity, it is also easy to check that $\mathcal{P}$ is a closed convex rational polyhedron in V. We note that $\mathcal{P}$ is compact

in the classical topology of V. By the assumption, $B \in \mathcal{P}$. Therefore, we can find the desired $\mathbb{Q}$-divisors $B_i \in \mathcal{P}$. $\square$

The next result is essentially due to [Ka2] and [S4, Proposition 1].

Theorem 18.2. *Let X be a normal variety, B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier, and $\pi : X \to S$ a projective morphism onto a variety S. Let R be a $(K_X + B)$-negative extremal ray such that $\operatorname{Exc}(\varphi_R) \cap \operatorname{Nlc}(X, B) = \emptyset$, where $\varphi_R : X \to Y$ is the extremal contraction over S associated to R. Then we can find a rational curve C on X such that $[C] \in R$ and $-(K_X + B) \cdot C \leq 2 \dim X$.*

Proof. By replacing $\pi : X \to S$ with the extremal contraction $\varphi_R : X \to Y$ over S, we can assume that the relative Picard number $\rho(X/S) = 1$. In particular, $-(K_X + B)$ is π-ample. Since $\operatorname{Exc}(\varphi_R) \cap \operatorname{Nlc}(X, B) = \emptyset$, we can also assume that (X, B) is log canonical by shrinking S. Let $K_X + B = \sum_{i=1}^{l} r_i (K_X + B_i)$ be as in Lemma 18.1. We assume that $-(K_X + B_1)$ is π-ample and $-(K_X + B_i) = -s_i(K_X + B_1)$ in $N^1(X/S)$ with $s_i \leq 1$ for every $i \geq 2$. Thus, it is sufficient to find a rational curve C such that $\pi(C)$ is a point and that $-(K_X + B_1) \cdot C \leq 2 \dim X$. So, we can assume that $K_X + B$ is $\mathbb{Q}$-Cartier and lc. By Theorem 10.4, there is a birational morphism $f : (V, B_V) \to (X, B)$ such that $K_V + B_V = f^*(K_X + B)$, V is $\mathbb{Q}$-factorial, and (V, B_V) is dlt. By [Ka2, Theorem 1] and [Ma, Theorem 10-2-1], we can find a rational curve C' on V such that $-(K_V + B_V) \cdot C' \leq 2 \dim V = 2 \dim X$ and that C' spans a $(K_V + B_V)$-negative extremal ray. By the projection formula, the f-image of C' is a desired rational curve. So, we finish the proof. $\square$

Remark 18.3. It is conjectured that the estimate $\leq 2 \dim X$ in Theorem 18.2 should be replaced by $\leq \dim X + 1$. When X is smooth projective, it is true by Mori's famous result (cf. [Mo]). See, for example, [KM, Theorem 1.13]. When X is a toric variety, it is also true by [F1] and [F3].

Remark 18.4. In the proof of Theorem 18.2, we need Kawamata's estimate on the length of an extremal rational curve (cf. [Ka2, Theorem 1] and [Ma, Theorem 10-2-1]). It depends on Mori's bend and break technique to create rational curves. So, we need the mod p reduction technique there.

By the proof of Theorem 18.2, we have the following corollary.

Corollary 18.5. *Let (X, B) be a log canonical pair and let $K_X + B = \sum_{i=1}^{l} r_i(K_X + B_i)$ and m as in Lemma 18.1. Let $\varphi : X \to Y$ be*

a projective surjective morphism with connected fibers such that the relative Picard number $\rho(X/Y) = 1$. Then we can find a curve C on X such that C spans $N_1(X/Y)$ and

$$-(K_X + B_i) \cdot C = \frac{n_i}{m}$$

with $n_i \leq 2m \dim X$ for every i. Of course, we have

$$-(K_X + B) \cdot C = \sum_i \frac{r_i n_i}{m} \leq 2 \dim X.$$

If $-(K_X + B_i)$ is φ-ample for some i, then we can find a rational curve C in the above statement. We note that φ is not necessarily assumed to be a $(K_X + B)$-negative extremal contraction.

The following important lemma is a very special case of [S3, 6.2. First Main Theorem].

Lemma 18.6. *Let (X, B) be a log canonical pair and $\pi : X \to S$ a projective morphism onto a variety S. We take $\sum_k D_k$ such that $\mathrm{Supp} B \subset \sum_k D_k$, where D_i is an irreducible Weil divisor for every i and $D_i \neq D_j$ for every $i \neq j$. We put*

$$\mathcal{P} = \left\{ \sum_k d_k D_k \; ; \; 0 \leq d_k \leq 1 \text{ for all } k \text{ and } K_X + \sum_k d_k D_k \text{ is lc} \right\}.$$

Then $\mathcal{P}$ is a closed convex rational polyhedron.

Let $\{R_j\}$ be any set of $(K_X + B)$-negative extremal rays of the lc pair (X, B) over S. We put

$$\mathcal{N} = \bigcap_j \left\{ \sum_k d_k D_k \in \mathcal{P}; (K_X + \sum_k d_k D_k) \cdot R_j \geq 0 \right\}.$$

Then $\mathcal{N}$ is a closed convex subset of $\mathcal{P}$.

We take $B' \in \mathcal{P}$. Let $\mathcal{F}$ be the minimal face of $\mathcal{P}$ containing B'. Assume that $(K_X + B') \cdot R_j > 0$ for every j. Then there is an open subset U of $\mathcal{F}$ in the classical topology such that $B' \in U \subset \mathcal{N} \cap \mathcal{F}$. In particular, we can write

$$K_X + B' = \sum_{i=1}^{d+1} r'_i(K_X + B'_i)$$

with the following properties.

(a) $d = \dim \mathcal{F}$.
(b) $B'_i \in \mathcal{F}$ for every i.
(c) $m'(K_X + B'_i)$ is Cartier for some positive integer m' for every i.

(d) $\sum_{i=1}^{d+1} r_i' = 1$ *and* $0 \leq r_i' \leq 1$ *for every* i.
(e) $(K_X + B_i') \cdot R_j > 0$ *for every* i *and* j.

Proof. It is obvious that $\mathcal{P}$ is a closed convex rational polyhedron (see the proof of Lemma 18.1). By the definition, $\mathcal{N}$ is a closed convex subset of $\mathcal{P}$. Since $\mathcal{F}$ is a face of $\mathcal{P}$ and contains B', we can take a d-dimensional rational simplex spanned by Δ_i for $1 \leq i \leq d+1$ in $\mathcal{F}$ containing B' inside it. Thus, we can write

$$K_X + B' = \sum_{i=1}^{d+1} r_i(K_X + \Delta_i)$$

such that $\sum_{i=1}^{d+1} r_i = 1$ and $0 < r_i < 1$ for every i, and $m(K_X + \Delta_i)$ is Cartier for every i, where m is a positive integer.

We take an extremal ray R_j. By Corollary 18.5, we can find a curve C_j on X such that C_j spans R_j and that $m(K_X + \Delta_i) \cdot C_j = n_{ij}$ with $n_{ij} \geq -2m \dim X$ for every i. By the assumption, we have

$$(K_X + B') \cdot C_j = \sum_i \frac{r_i n_{ij}}{m} > 0.$$

We define

$$\alpha = \inf \left\{ \sum_i \frac{r_i n_i}{m} > 0 \; ; n_i \geq -2m \dim X \text{ and } n_i \in \mathbb{Z} \text{ for every } i \right\}.$$

Then we obtain $\alpha > 0$. We put

$$c = \frac{\alpha}{2 \dim X + \alpha + 1} > 0.$$

It is obvious that

$$B' + c(\Delta_i - B') \in \mathcal{F}$$

for every i since $0 < c < 1$ and that

$$(K_X + B' + c(\Delta_i - B')) \cdot C_j > 0$$

for every i and j by the definition of c. Thus, the d-dimensional simplex spanned by $B' + c(\Delta_i - B')$ for $1 \leq i \leq d+1$ is contained in $\mathcal{N} \cap \mathcal{F}$ and contains B' in its interior. So, the interior of the above simplex is a desired open set contained in $\mathcal{N} \cap \mathcal{F}$. Thus, we can write

$$K_X + B' = \sum_{i=1}^{d+1} r_i'(K_X + B_i')$$

with the required properties. $\square$

Remark 18.7. In [S3, 6.2. First Main Theorem], it is proved that $\mathcal{N}$ is a closed convex rational polyhedron. For the details, see [S3]. See also [B2, Section 3].

By Corollary 18.5 and Lemma 18.6, Lemma 2.6 in [B1] holds for lc pairs. It may be useful for the log minimal model program with scaling. We copy Birkar's proof in [BP].

Theorem 18.8 (cf. [B1, Lemma 2.6]). *Let (X, B) be an lc pair, B an $\mathbb{R}$-divisor, and $\pi : X \to S$ a projective morphism between algebraic varieties. Let H be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $K_X + B + H$ is π-nef and $(X, B + H)$ is lc. Then, either $K_X + B$ is also π-nef or there is a $(K_X + B)$-negative extremal ray R such that $(K_X + B + \lambda H) \cdot R = 0$, where*

$$\lambda := \inf\{t \geq 0 \,|\, K_X + B + tH \ \text{is } \pi\text{-nef}\}.$$

Of course, $K_X + B + \lambda H$ is π-nef.

Proof. Assume that $K_X + B$ is not π-nef. Let $\{R_j\}$ be the set of $(K_X + B)$-negative extremal rays over S. Let C_j be the rational curve spanning R_j with the estimate as in Corollary 18.5 for every j. We put $\mu = \sup_j\{\mu_j\}$, where

$$\mu_j = \frac{-(K_X + B) \cdot C_j}{H \cdot C_j}.$$

Obviously, $\lambda = \mu$ and $0 < \mu \leq 1$. So, it is sufficient to prove that $\mu = \mu_l$ for some l. By Corollary 18.5, there are positive real numbers $r_1, \cdots, r_l$ and a positive integer m, which are independent of j, such that

$$-(K_X + B) \cdot C_j = \sum_{i=1}^{l} \frac{r_i n_{ij}}{m} > 0,$$

where n_{ij} is an integer with $n_{ij} \leq 2m \dim X$ for every i and j. If $(K_X + B + H) \cdot R_l = 0$ for some l, then there are nothing to prove since $\lambda = 1$ and $(K_X + B + H) \cdot R = 0$ with $R = R_l$. Thus, we assume that $(K_X + B + H) \cdot R_j > 0$ for every j. Therefore, we can apply Lemma 18.6 and obtain

$$K_X + B + H = \sum_{p=1}^{q} r'_p(K_X + \Delta_p),$$

where $r'_1, \cdots, r'_q$ are positive real numbers, (X, Δ_p) is lc for every p, $m'(K_X + \Delta_p)$ is Cartier for some positive integer m' and every p, and

$(K_X + \Delta_p) \cdot C_j > 0$ for every p and j. So, we obtain

$$(K_X + B + H) \cdot C_j = \sum_{p=1}^{q} \frac{r'_p n'_{pj}}{m'}$$

with $0 < n'_{pj} = m'(K_X + \Delta_p) \cdot C_j \in \mathbb{Z}$. Note that m' and r'_p are independent of j for every p. We also note that

$$\frac{1}{\mu_j} = \frac{H \cdot C_j}{-(K_X + B) \cdot C_j} = \frac{(K_X + B + H) \cdot C_j}{-(K_X + B) \cdot C_j} + 1$$

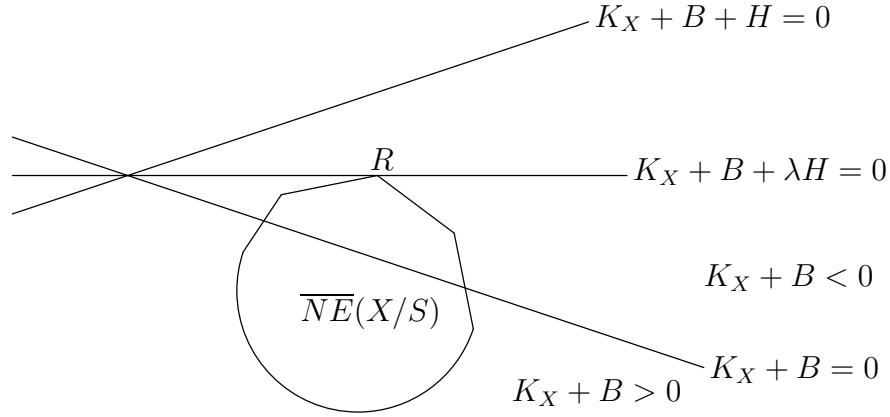$$= \frac{m \sum_{p=1}^{q} r'_p n'_{pj}}{m' \sum_{i=1}^{l} r_j n_{ij}} + 1.$$

Since

$$\sum_{i=1}^{l} \frac{r_i n_{ij}}{m} > 0$$

for every j and $n_{ij} \leq 2m \dim X$ with $n_{ij} \in \mathbb{Z}$ for every i and j, the number of the set $\{n_{ij}\}_{i,j}$ is finite. Thus,

$$\inf_{j} \left\{ \frac{1}{\mu_j} \right\} = \frac{1}{\mu_l}$$

for some l. Therefore, we obtain $\mu = \mu_l$. We finish the proof. $\square$

The following picture helps the reader to understand Theorem 18.8.

19. Ambro's theory of quasi-log varieties

In this section, we make some comments on Ambro's theory of quasi-log varieties. We strongly recommend the reader to see [F13] for an introduction to the theory of quasi-log varieties.

In the acknowledgements in [A1], Ambro wrote "The motivation behind this work is his (Professor Shokurov's) idea that log varieties and their LCS loci should be treated on an equal footing." So, in the theory of quasi-log varieties, we have to treat highly reducible non-equidimensional varieties (see Example 19.2 below). Therefore, our approach explained in this paper is completely different from the theory of quasi-log varieties. We recommend the reader to compare our proof of the base point free theorem for projective lc surfaces in Section 2 with Ambro's proof (see, for example, [F13, Section 4]).

Let us explain some results of the theory of quasi-log varieties which can not be covered by our approach.

19.1. Let (X, B) be a projective log canonical pair and $\{C_i\}$ any set of lc centers of the pair (X, B). We put $W = \bigcup C_i$ with a reduced scheme structure. Then $[W, \omega]$ is a *qlc pair*, where $\omega = (K_X + B)|_W$. For the definition of *qlc pairs*, see [F9, Definition 3.29] or [F13, Definition 3.1].

Example 19.2. Let V be a projective toric variety and D the complement of the big torus. Then (V, D) is log canonical and $K_V + D \sim 0$. In this case, any torus invariant closed subvariety W of V with $\omega = 0$ is a qlc pair. In particular, W is not necessarily pure-dimensional (cf. [F5, §5]).

We can prove the cone theorem for $[W, \omega]$.

Theorem 19.3 (Cone theorem)**.** *We have*

$$\overline{NE}(W) = \overline{NE}(W)_{\omega \geq 0} + \sum_j R_j.$$

For the details, see [F9, 3.3.3 Cone Theorem]. We can also prove the base point free theorem.

Theorem 19.4 (Base point free theorem)**.** *Let L be a nef Cartier divisor on W such that $aL - \omega$ is ample for some $a > 0$. Then $|mL|$ is base point free for $m \gg 0$.*

See, for example, [F9, 3.3.1 Base point free theorem]. By these theorems, we have the following statement.

Theorem 19.5 (Contraction theorem)**.** *Let F be an ω-negative extremal face of $\overline{NE}(W)$. Then there is a contraction morphism $\varphi_F : W \to V$ with the following properties.*

   (i) *Let C be an integral curve on W. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.*

   (ii) *$\mathcal{O}_V \simeq (\varphi_F)_* \mathcal{O}_W$.*

(iii) *Let L be a line bundle on W such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_V on V such that $L \simeq \varphi_F^* L_V$.*

For the details of the theory of quasi-log varieties, see [F9]. The book [F9] treats some various other topics which can not be covered by this paper.

20. Related topics

In this final section, we briefly explain some related topics obtained by the author.

In this paper, we did not describe the notion of singularities of pairs. However, it is very important when we read some papers on the log minimal model program. We think that [F4] helps the reader to understand the subtlety of the notion of dlt pairs.

The reader can find that all the injectivity, vanishing, and torsion-free theorems in this paper are discussed in full generality in [F9, Sections 2 and 3]. They heavily depends on the mixed Hodge theory on reducible varieties.

We omitted the explanation of the log minimal model program for log canonical pairs. It is because the framework is the same as for klt pairs. The reader can find it in [F9, Section 3]. We note that the existence problem of log canonical flips is still open in dimension ≥ 5 and the termination of log canonical flips follows from the termination of klt flips. For the details, see [F9, Section 3].

In [F6], we prove an effective version of the base point free theorem for log canonical pairs. It is a log canonical version of Kollár's effective freeness. In [F7], the Angehrn–Siu type effective base point free theorems are proved for log canonical pairs. The reader can find that the proof of our non-vanishing theorem (cf. Theorem 12.1 and [F14, Theorem 1.1]) grew out from the arguments in [F6] and [F7].

In [F8], we systematically treat the basic properties of non-lc ideal sheaves, especially, the restriction theorem of non-lc ideal sheaves for normal divisors. It is a generalization of Kawakita's inversion of adjunction on log canonicity.

In [F11], we prove the finite generation of the log canonical ring in dimension four and discuss related topics. It induces the existence of four-fold log canonical flips.

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