INSTANTON APPROXIMATION, PERIODIC ASD CONNECTIONS, AND MEAN DIMENSION

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ABSTRACT. We study a moduli space of framed ASD connections over $S^3 \times \mathbb{R}$. We consider not only finite energy ASD connections but also infinite energy ones. So the moduli space is infinite dimensional. We study the (local) mean dimension of this infinite dimensional moduli space. We show the upper bound on the mean dimension by using a "Runge-approximation" for ASD connections, and we prove its lower bound by constructing an infinite dimensional deformation theory of periodic ASD connections.

1. INTRODUCTION

Since Donaldson [4] discovered his revolutionary theory, many mathematicians have intensively studied the Yang-Mills gauge theory. There are several astonishing results on the structures of the ASD moduli spaces and their applications. But most of them study only finite energy ASD connections and their finite dimensional moduli spaces. Almost nothing is known about infinite energy ASD connections and their infinite dimensional moduli spaces. (One of the authors struggled to open the way to this direction in [18, 19].) This paper studies an infinite dimensional moduli space coming from the Yang-Mills theory over $S^3 \times \mathbb{R}$. Our main purposes are to prove estimates on its "mean dimension" (Gromov [12]) and to show that there certainly exists a non-trivial structure in this infinite dimensional moduli space. (Mean dimension is a "dimension of an infinite dimensional space averaged by a group action".)

The reason why we consider $S^3 \times \mathbb{R}$ is that it is one of the simplest non-compact anti-self-dual 4-manifolds of (uniformly) positive scalar curvature. (Indeed it is conformally flat.) These metrical conditions are used via the Weitzenböck formula (see Section 4.1). Recall that one of the important results of the pioneering work of Atiyah-Hitchin-Singer [1, Theorem 6.1] is the calculation of the dimension of the moduli space of (irreducible) self-dual connections over a compact self-dual 4-manifold of positive scalar curvature. So our work is an attempt to develop an infinite dimensional analogue of [1, Theorem 6.1].

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Of course, the study of the mean dimension is just a one-step toward the full understanding of the structures of the infinite dimensional moduli space. (But the authors believe that "dimension" is one of the most fundamental invariants of spaces and that the study of mean dimension is a crucial step toward the full understanding.) So we need much more studies, and the authors hope that this paper becomes a stimulus to a further study of infinite dimensional moduli spaces in the Yang-Mills gauge theory.

Set $X := S^3 \times \mathbb{R}$. Throughout the paper, the variable t means the variable of the $\mathbb{R}$-factor of $X = S^3 \times \mathbb{R}$. (That is, $t : X \to \mathbb{R}$ is the natural projection.) $S^3 \times \mathbb{R}$ is endowed with the product metric of a positive constant curvature metric on S^3 and the standard metric on $\mathbb{R}$. (Therefore X is $S^3(r) \times \mathbb{R}$ for some $r > 0$ as a Riemannian manifold, where $S^3(r) = \{x \in \mathbb{R}^4 | \, |x| = r\}$.) Let $\boldsymbol{E} := X \times SU(2)$ be the product principal $SU(2)$-bundle over X. The additive group $\mathbb{Z}$ acts on X by $X \times \mathbb{Z} \ni ((\theta, t), s) \mapsto (\theta, t+s) \in X$. This action trivially lifts to the action on $\boldsymbol{E}$ by $\boldsymbol{E} \times \mathbb{Z} \ni ((\theta, t, u), s) \mapsto (\theta, t+s, u) \in \boldsymbol{E}$.

Fix a point $\theta_0 \in S^3$. Let $d \geq 0$. We define the "periodically framed moduli space" $\mathcal{M}_d$ as the set of all gauge equivalence classes $[\boldsymbol{A}, \boldsymbol{p}]$ satisfying the following conditions. $\boldsymbol{A}$ is an ASD connection on $\boldsymbol{E}$ satisfying

$$\|F(\boldsymbol{A})\|_{L^\infty} \leq d, \tag{1}$$

and $\boldsymbol{p}$ is a map from $\mathbb{Z}$ to $\boldsymbol{E}$ satisfying $\boldsymbol{p}(n) \in \boldsymbol{E}_{(\theta_0, n)}$ for each $n \in \mathbb{Z}$. Here $\boldsymbol{E}_{(\theta_0,n)}$ is the fiber of $\boldsymbol{E}$ over $(\theta_0, n) \in X$. We have $[\boldsymbol{A}, \boldsymbol{p}] = [\boldsymbol{B}, \boldsymbol{q}]$ if there exists a gauge transformation $g : \boldsymbol{E} \to \boldsymbol{E}$ satisfying $g(\boldsymbol{A}) = \boldsymbol{B}$ and $g(\boldsymbol{p}(n)) = \boldsymbol{q}(n)$ for all $n \in \mathbb{Z}$.

$\mathcal{M}_d$ is equipped with the topology of $\mathcal{C}^\infty$-convergence on compact subsets: a sequence $[\boldsymbol{A}_n, \boldsymbol{p}_n]$ $(n \geq 1)$ converges to $[\boldsymbol{A}, \boldsymbol{p}]$ in $\mathcal{M}_d$ if there exists a sequence of gauge transformations g_n of $\boldsymbol{E}$ such that $g_n(\boldsymbol{A}_n)$ converges to $\boldsymbol{A}$ as $n \to \infty$ in the $\mathcal{C}^\infty$-topology over every compact subset in X and that $g_n(\boldsymbol{p}_n(k)) \to \boldsymbol{p}(k)$ as $n \to \infty$ for every $k \in \mathbb{Z}$. $\mathcal{M}_d$ becomes a compact metrizable space by the Uhlenbeck compactness ([21, 22]). The additive group $\mathbb{Z}$ continuously acts on $\mathcal{M}_d$ by

$$\mathcal{M}_d \times \mathbb{Z} \to \mathcal{M}_d, \quad ([\boldsymbol{A}, \boldsymbol{p}], \gamma) \mapsto [\gamma^* \boldsymbol{A}, \gamma^* \boldsymbol{p}],$$

where γ^* is the pull-back by $\gamma : \boldsymbol{E} \to \boldsymbol{E}$. Then we can consider the mean dimension $\dim(\mathcal{M}_d : \mathbb{Z})$. Intuitively,

$$\dim(\mathcal{M}_d : \mathbb{Z}) = \frac{\dim \mathcal{M}_d}{|\mathbb{Z}|}.$$

(This is ∞/∞. The precise definition will be given in Section 2.) Our first main result is the following estimate on the mean dimension.

Theorem 1.1.

$$3 \leq \dim(\mathcal{M}_d : \mathbb{Z}) < \infty.$$

Moreover, $\dim(\mathcal{M}_d : \mathbb{Z}) \to +\infty$ *as* $d \to +\infty$.

For an ASD connection $\boldsymbol{A}$ on $\boldsymbol{E}$ we define $\rho(\boldsymbol{A})$ by setting

$$\rho(\boldsymbol{A}) := \lim_{T\to+\infty} \frac{1}{8\pi^2 T} \sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T]} |F(\boldsymbol{A})|^2 d\mathrm{vol}. \tag{2}$$

This limit always exists because we have the following subadditivity.

$$\sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T_1+T_2]} |F(\boldsymbol{A})|^2 d\mathrm{vol}$$
$$\leq \sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T_1]} |F(\boldsymbol{A})|^2 d\mathrm{vol} + \sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T_2]} |F(\boldsymbol{A})|^2 d\mathrm{vol}.$$

$\rho(\boldsymbol{A})$ is translation invariant; For $s \in \mathbb{R}$, we have $\rho(s^*\boldsymbol{A}) = \rho(\boldsymbol{A})$, where $s^*\boldsymbol{A}$ is the pull-back of $\boldsymbol{A}$ by the map $s : \boldsymbol{E} = S^3\times\mathbb{R}\times SU(2) \to \boldsymbol{E}$, $(\theta, t, u) \mapsto (\theta, t+s, u)$. We define $\rho(d)$ as the supremum of $\rho(\boldsymbol{A})$ over all ASD connections $\boldsymbol{A}$ on $\boldsymbol{E}$ satisfying $\|F(\boldsymbol{A})\|_{L^\infty} \leq d$.

Let $\boldsymbol{A}$ be an ASD connection on $\boldsymbol{E}$. We call $\boldsymbol{A}$ a periodic ASD connection if there exist $T > 0$, a principal $SU(2)$-bundle $\underline{\mathrm{E}}$ over $S^3 \times (\mathbb{R}/T\mathbb{Z})$, and an ASD connection $\underline{\mathrm{A}}$ on $\underline{\mathrm{E}}$ such that $(\boldsymbol{E}, \boldsymbol{A})$ is gauge equivalent to $(\pi^*(\underline{\mathrm{E}}), \pi^*(\underline{\mathrm{A}}))$ where $\pi : S^3 \times \mathbb{R} \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ is the natural projection. (Here $S^3 \times (\mathbb{R}/T\mathbb{Z})$ is equipped with the metric induced by the covering map π.) Then we have

$$\rho(\boldsymbol{A}) = \frac{1}{8\pi^2 T} \int_{S^3\times[0,T]} |F(\boldsymbol{A})|^2 d\mathrm{vol} = \frac{c_2(\underline{E})}{T}. \tag{3}$$

We define $\rho_{peri}(d)$ as the supremum of $\rho(\boldsymbol{A})$ over all periodic ASD connections $\boldsymbol{A}$ on $\boldsymbol{E}$ satisfying $\|F(\boldsymbol{A})\|_{L^\infty} < d$. (Note that we impose the strict inequality condition here.) If $d = 0$, then such an $\boldsymbol{A}$ does not exist. Hence we set $\rho_{peri}(0) := 0$. (If $d > 0$, then the product connection $\boldsymbol{A}$ is a periodic ASD connection satisfying $\|F(\boldsymbol{A})\|_{L^\infty} = 0 < d$.) Obviously we have $\rho_{peri}(d) \leq \rho(d)$. Our second main result is the following estimates on the "local mean dimensions".

Theorem 1.2. *For any* $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d$,

$$\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) \leq 8\rho(\boldsymbol{A}) + 3.$$

Moreover, if $\boldsymbol{A}$ is a periodic ASD connection satisfying $\|F(\boldsymbol{A})\|_{L^\infty} < d$, then

$$\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) = 8\rho(\boldsymbol{A}) + 3.$$

Therefore,

$$8\rho_{peri}(d) + 3 \leq \dim_{loc}(\mathcal{M}_d : \mathbb{Z}) \leq 8\rho(d) + 3.$$

Here $\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z})$ is the "local mean dimension" of $\mathcal{M}_d$ at $[\boldsymbol{A}, \boldsymbol{p}]$, and $\dim_{loc}(\mathcal{M}_d : \mathbb{Z}) := \sup_{[\boldsymbol{A},\boldsymbol{p}]\in\mathcal{M}_d} \dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z})$ is the "local mean dimension" of $\mathcal{M}_d$. These notions will be defined in Section 2.2.

Note that

$$\lim_{d\to\infty} \rho_{peri}(d) = +\infty.$$

This obviously follows from the fact that for any integer $n \geq 0$ there exists an ASD connection on $S^3 \times (\mathbb{R}/\mathbb{Z})$ whose second Chern number is equal to n. This is a special case of the famous theorem of Taubes [16]. (Note that the intersection form of $S^3 \times S^1$ is zero.) We have $\dim(\mathcal{M}_d : \mathbb{Z}) \geq \dim_{loc}(\mathcal{M}_d : \mathbb{Z})$ (see (5) in Section 2.2). Hence the statements that $\dim(\mathcal{M}_d : \mathbb{Z}) \geq 3$ and $\dim(\mathcal{M}_d : \mathbb{Z}) \to \infty$ $(d \to \infty)$ in Theorem 1.1 follow from the inequality $\dim_{loc}(\mathcal{M}_d : \mathbb{Z}) \geq 8\rho_{peri}(d) + 3$ in Theorem 1.2.

Remark 1.3. All principal $SU(2)$-bundle over $S^3 \times \mathbb{R}$ is gauge equivalent to the product bundle $\boldsymbol{E}$. Hence the moduli space $\mathcal{M}_d$ is equal to the space of gauge equivalence classes $[E, A, p]$ satisfying the following conditions. E is a principal $SU(2)$-bundle over X, and A is an ASD connection on E satisfying $|F(A)| \leq d$. $p : \mathbb{Z} \to E$ is a map satisfying $p(n) \in E_{(\theta_0, n)}$. We have $[E_1, A_1, p_1] = [E_2, A_2, p_2]$ if and only if there exists a bundle map $g : E_1 \to E_2$ satisfying $g(A_1) = A_2$ and $g(p_1(n)) = p_2(n)$ for all $n \in \mathbb{Z}$. In this description, the topology of $\mathcal{M}_d$ is described as follows. A sequence $[E_n, A_n, p_n]$ $(n \geq 1)$ in $\mathcal{M}_d$ converges to $[E, A, p]$ if and only if there exist gauge transformations $g_n : E_n \to E$ $(n \gg 1)$ such that $g_n(A_n)$ converges to A as $n \to \infty$ in $\mathcal{C}^\infty$ over every compact subset in X and that $g_n(p_n(k)) \to p(k)$ as $n \to \infty$ for every $k \in \mathbb{Z}$.

Remark 1.4. An ASD connection satisfying the condition (1) is a Yang-Mills analogue of a "Brody curve" (cf. Brody [3]) in the entire holomorphic curve theory (Nevanlinna theory). It is widely known that there exist several similarities between the Yang-Mills gauge theory and the theory of (pseudo-)holomorphic curves (e.g. Donaldson invariant vs. Gromov-Witten invariant). On the holomorphic curve side, several researchers in the Nevanlinna theory have systematically studied the value distributions of holomorphic curves (of infinite energy) from the complex plane $\mathbb{C}$. They have found several deep structures of such infinite energy holomorphic curves. Therefore the authors hope that infinite energy ASD connections also have deep structures.

The rough ideas of the proofs of the main theorems are as follows. (For more about the outline of the proofs, see Section 3.) The upper bounds on the (local) mean dimension are proved by using the Runge-type approximation of ASD connections (originally due to Donaldson [5]). This "instanton approximation" technique gives a method to approximate infinite energy ASD connections by finite energy ones (instantons). Then we can construct "finite dimensional approximations" of $\mathcal{M}_d$ by moduli spaces of (framed) instantons. This gives a upper bound on $\dim(\mathcal{M}_d : \mathbb{Z})$. The lower bound on the local mean dimension is proved by constructing an infinite dimensional deformation theory of periodic ASD connections. This method is a Yang-Mills analogue of the deformation theory of "elliptic Brody curves" developed in Tsukamoto [20].

A big technical difficulty in the study of $\mathcal{M}_d$ comes from the point that ASD equation is not elliptic. When we study the Yang-Mills theory over compact manifolds, this point can be easily overcome by using the Coulomb gauge. But in our situation (perhaps) there is no such good way to recover the ellipticity. So we will consider some "partial gauge fixings" in this paper. In the proof of the upper bound, we will consider the Coulomb gauge over S^3 instead of $S^3 \times \mathbb{R}$ (see Proposition 6.1). In the proof of the lower bound, we will consider the Coulomb gauge over $S^3 \times \mathbb{R}$, but it is less powerful and more technical than the usual Coulomb gauges over compact manifolds (see Proposition 8.6).

Organization of the paper: In Section 2 we review the definition of mean dimension and define local mean dimension. In Section 3 we explain the outline of the proofs of Theorem 1.1 and 1.2. Sections 4, 5 and 6 are preparations for the proof of the upper bounds on the (local) mean dimension. In Section 7 we prove the upper bounds. Section 8 is a preparation for the proof of the lower bound. In Section 9 we develop the deformation theory of periodic ASD connections and prove the lower bound on the local mean dimension. In Appendix A we prepare some basic results on the Green kernel of $\Delta + 1$.

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2. Mean dimension and local mean dimension

2.1. **Review of mean dimension.** We review the definitions and basic properties of mean dimension in this subsection. For the detail, see Gromov [12] and Lindenstrauss-Weiss [14]. We need only the mean dimension for $\mathbb{Z}$-actions. So we consider only $\mathbb{Z}$-action cases, but we try to formulate the notions so that it can be easily generalized to the case of actions of general amenable groups.

Let (X, d) be a compact metric space, Y be a topological space, and $f : X \to Y$ be a continuous map. For $\varepsilon > 0$, f is called an ε-embedding if we have $\mathrm{Diam} f^{-1}(y) \leq \varepsilon$ for all $y \in Y$. We define $\mathrm{Widim}_\varepsilon(X, d)$ as the minimum integer $n \geq 0$ such that there exist a polyhedron P of dimension n and an ε-embedding $f : X \to P$. We have

$$\lim_{\varepsilon \to 0} \mathrm{Widim}_\varepsilon(X, d) = \dim X,$$

where $\dim X$ denotes the topological covering dimension of X. For example, consider $[0,1] \times [0, \varepsilon]$ with the Euclidean distance. Then the natural projection $\pi : [0,1] \times [0,\varepsilon] \to [0,1]$ is an ε-embedding. Hence $\mathrm{Widim}_\varepsilon([0,1] \times [0,\varepsilon], \mathrm{Euclidean}) \leq 1$. The following is given in Gromov [12, p. 333]. (For the detailed proof, see also Gournay [10, Lemma 2.5] and Tsukamoto [20, Appendix].)

Lemma 2.1. *Let $(V, \|\cdot\|)$ be a finite dimensional normed linear space over $\mathbb{R}$. Let $B_r(V)$ be the closed ball of radius $r > 0$ in V. Then*

$$\mathrm{Widim}_\varepsilon(B_r(V), \|\cdot\|) = \dim V \quad (\varepsilon < r).$$

$\mathrm{Widim}_\varepsilon(X, d)$ satisfies the following subadditivity. (The proof is obvious.)

Lemma 2.2. *For compact metric spaces* (X, d_X), (Y, d_Y), *we set* $(X, d_X) \times (Y, d_Y) := (X \times Y, d_{X\times Y})$ *with* $d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \max(d_X(x_1, x_2), d_Y(y_1, y_2))$. *Then we have*

$$\mathrm{Widim}_\varepsilon((X, d_X) \times (Y, d_Y)) \leq \mathrm{Widim}_\varepsilon(X, d_X) + \mathrm{Widim}_\varepsilon(Y, d_Y).$$

The following will be used in Section 7.1

Lemma 2.3. *Let* (X, d) *be a compact metric space and suppose* $X = X_1 \cup X_2$ *with closed sets* X_1 *and* X_2. *Then*

$$\mathrm{Widim}_\varepsilon(X, d) \leq \mathrm{Widim}_\varepsilon(X_1, d) + \mathrm{Widim}_\varepsilon(X_2, d) + 1.$$

In general, if $X = X_1 \cup X_2 \cup \cdots \cup X_n$ *(*X_i*: closed), then*

$$\mathrm{Widim}_\varepsilon(X, d) \leq \sum_{i=1}^{n} \mathrm{Widim}_\varepsilon(X_i, d) + n - 1.$$

Proof. There exist a finite polyhedron P_i $(i = 1, 2)$ with $\dim P_i = \mathrm{Widim}_\varepsilon(X_i, d)$ and an ε-embedding $f_i : (X_i, d) \to P_i$. Let $P_1 * P_2 = \{tx \oplus (1-t)y | \, x \in X_1, y \in X_2, 0 \leq t \leq 1\}$ be the join of P_1 and P_2. ($P_1 * P_2 = [0, 1] \times P_1 \times P_2/\sim$, where $(0, x, y) \sim (0, x', y)$ for any $x, x' \in X$ and $(1, x, y) \sim (1, x, y')$ for any $y, y' \in Y$. $tx \oplus (1-t)y$ is the equivalence class of (t, x, y).) $P_1 * P_2$ is a finite polyhedron of dimension $\mathrm{Widim}_\varepsilon(X_1, d) + \mathrm{Widim}_\varepsilon(X_2, d) + 1$. Since a finite polyhedron is ANR, there exists a open set $U_i \supset X_i$ over which the map f_i continuously extends. Let ρ be a cut-off function such that $0 \leq \rho \leq 1$, $\mathrm{supp}\rho \subset U_1$ and $\rho(x) = 1$ if and only if $x \in X_1$. Then $\mathrm{supp}(1 - \rho) = \overline{X \setminus X_1} \subset X_2 \subset U_2$. We define a continuous map $F : X \to P_1 * P_2$ by setting $F(x) := \rho(x) f_1(x) \oplus (1 - \rho(x)) f_2(x)$. F becomes an ε-embedding; Suppose $F(x) = F(y)$. If $\rho(x) = \rho(y) = 1$, then $x, y \in X_1$ and $f_1(x) = f_1(y)$. Then $d(x, y) \leq \varepsilon$. If $\rho(x) = \rho(y) < 1$, then $x, y \in X_2$ and $f_2(x) = f_2(y)$. Then $d(x, y) \leq \varepsilon$. Thus $\mathrm{Widim}_\varepsilon(X, d) \leq \dim P_1 * P_2 = \mathrm{Widim}_\varepsilon(X_1, d) + \mathrm{Widim}_\varepsilon(X_2, d) + 1$. □

Suppose that the additive group $\mathbb{Z}$ continuously acts on X. For a subset $\Omega \subset \mathbb{Z}$, we define a distance $d_\Omega(\cdot, \cdot)$ on X by

$$d_\Omega(x, y) := \sup_{n\in\Omega} d(n.x, n.y) \quad (x, y \in X).$$

A sequence $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$ of finite subsets in $\mathbb{Z}$ is called an amenable sequence if for each $r > 0$

$$|\partial_r \Omega_n| / |\Omega_n| \to 0 \quad (n \to \infty),$$

where $\partial_r \Omega_n$ is the r-boundary of Ω_n given by

$$\partial_r \Omega_n := \{x \in \mathbb{Z} | \, \exists y \in \Omega_n : |x - y| \leq r \text{ and } \exists z \in \mathbb{Z} \setminus \Omega_n : |x - z| \leq r\}.$$

For example, $\Omega_n := \{0, 1, 2, \cdots, n-1\}$ $(n \geq 1)$ is an amenable sequence. $\Omega_n := \{-n, -n+1, \cdots, -1, 0, 1, \cdots, n-1, n\}$ $(n \geq 1)$ is also an amenable sequence. We need the following "Ornstein-Weiss lemma" ([12, pp. 336-338] and [14, Appendix]).

Lemma 2.4. *Let $h : \{\text{finite subsets in } \mathbb{Z}\} \to \mathbb{R}_{\geq 0}$ be a map satisfying the following.*
(i) If $\Omega_1 \subset \Omega_2$, then $h(\Omega_1) \leq h(\Omega_2)$.
(ii) $h(\Omega_1 \cup \Omega_2) \leq h(\Omega_1) + h(\Omega_2)$.
(iii) For any $\gamma \in \mathbb{Z}$ and a finite subset $\Omega \subset \mathbb{Z}$, $h(\gamma + \Omega) = h(\Omega)$, where $\gamma + \Omega := \{\gamma + x \in \mathbb{Z} | \, x \in \Omega\}$.

Then for any amenable sequence $\{\Omega_n\}_{n\geq 1}$, the limit $\lim_{n\to\infty} h(\Omega_n)/|\Omega_n|$ always exists and is independent of the choice of $\{\Omega_n\}$.

Lemma 2.5. *The map $\Omega \mapsto \mathrm{Widim}_\varepsilon(X, d_\Omega)$ satisfies the conditions in Lemma 2.4.*

Proof. If $\Omega_1 \subset \Omega_2$, then the identity map $(X, d_{\Omega_1}) \to (X, d_{\Omega_2})$ is distance non-decreasing. Hence $\mathrm{Widim}_\varepsilon(X, d_{\Omega_1}) \leq \mathrm{Widim}_\varepsilon(X, d_{\Omega_2})$. The map $(X, d_{\Omega_1 \cup \Omega_2}) \to (X, d_{\Omega_1}) \times (X, d_{\Omega_2})$, $x \to (x, x)$, is distance preserving. Hence, by using Lemma 2.2, $\mathrm{Widim}_\varepsilon(X, d_{\Omega_1 \cup \Omega_2}) \leq \mathrm{Widim}_\varepsilon(X, d_{\Omega_1}) + \mathrm{Widim}_\varepsilon(X, d_{\Omega_2})$. The map $(X, d_{\gamma+\Omega}) \to (X, d_\Omega)$, $x \mapsto \gamma x$, is an isometry. Hence $\mathrm{Widim}_\varepsilon(X, d_{\gamma+\Omega}) = \mathrm{Widim}_\varepsilon(X, d_\Omega)$. □

Suppose that an amenable sequence $\{\Omega_n\}_{n\geq 1}$ is given. For $\varepsilon > 0$, we set

$$\mathrm{Widim}_\varepsilon(X : \mathbb{Z}) := \lim_{n\to\infty} \frac{1}{|\Omega_n|} \mathrm{Widim}_\varepsilon(X, d_{\Omega_n}).$$

This limit exists and is independent of the choice of an amenable sequence $\{\Omega_n\}_{n\geq 1}$. The value of $\mathrm{Widim}_\varepsilon(X : \mathbb{Z})$ depends on the distance d. Hence, strictly speaking, we should use the notation $\mathrm{Widim}_\varepsilon((X, d) : \mathbb{Z})$. But we use the above notation for simplicity. We define $\dim(X : \mathbb{Z})$ (the mean dimension of $(X, \mathbb{Z})$) by

$$\dim(X : \mathbb{Z}) := \lim_{\varepsilon \to 0} \mathrm{Widim}_\varepsilon(X : \mathbb{Z}).$$

This becomes a topological invariant, i.e., the value of $\dim(X : \mathbb{Z})$ does not depend on the choice of a distance compatible with the given topology of X.

Example 2.6. Let $B \subset \mathbb{R}^N$ be the closed ball. $\mathbb{Z}$ acts on $B^{\mathbb{Z}}$ by the shift. Then

$$\dim(B^{\mathbb{Z}} : \mathbb{Z}) = N.$$

For the proof of this equation, see Lindenstrauss-Weiss [14, Proposition 3.1, 3.3] or Tsukamoto [19, Example B.2].

Let $Y \subset X$ be a closed subset. Then the map $\Omega \mapsto \sup_{k\in\mathbb{Z}} \mathrm{Widim}_\varepsilon(Y, d_{k+\Omega})$ satisfies the conditions in Lemma 2.4. Hence we can set

$$\mathrm{Widim}_\varepsilon(Y \subset X : \mathbb{Z}) := \lim_{n\to\infty} \left(\frac{1}{|\Omega_n|} \sup_{k\in\mathbb{Z}} \mathrm{Widim}_\varepsilon(Y, d_{k+\Omega_n}) \right),$$

where $\{\Omega_n\}_{n\geq 1}$ is an amenable sequence. We define

$$\dim(Y \subset X : \mathbb{Z}) := \lim_{\varepsilon \to 0} \mathrm{Widim}_\varepsilon(Y \subset X : \mathbb{Z}).$$

This does not depend on the choice of a distance compatible with the given topology of X. Note that we have $\mathrm{Widim}_\varepsilon(Y, d_{k+\Omega}) = \mathrm{Widim}_\varepsilon(k.Y, d_\Omega) \leq \mathrm{Widim}_\varepsilon(X, d_\Omega)$ because $(Y, d_{k+\Omega}) \to (k.Y, d_\Omega)$, $x \mapsto kx$, is an isometry. Therefore

$$\dim(Y \subset X : \mathbb{Z}) \leq \dim(X : \mathbb{Z}) = \dim(X \subset X : \mathbb{Z}).$$

If $Y_1 \subset Y_2$, then

$$\dim(Y_1 \subset X : \mathbb{Z}) \leq \dim(Y_2 \subset X : \mathbb{Z}).$$

If $Y \subset X$ is a $\mathbb{Z}$-invariant closed subset, then

$$\dim(Y \subset X : \mathbb{Z}) = \dim(Y : \mathbb{Z}),$$

where the right-hand-side is the ordinary mean dimension of $(Y, \mathbb{Z})$. Let X_1 and X_2 be compact metric spaces with continuous $\mathbb{Z}$-actions. Let $Y_1 \subset X_1$ and $Y_2 \subset X_2$ be closed subsets. If there exists a $\mathbb{Z}$-equivariant topological embedding $f : X_1 \to X_2$ satisfying $f(Y_1) \subset Y_2$, then

$$\dim(Y_1 \subset X_1 : \mathbb{Z}) \leq \dim(Y_2 \subset X_2 : \mathbb{Z}). \tag{4}$$

2.2. **Local mean dimension.** Let (X, d) be a compact metric space. The usual topological dimension $\dim X$ is a "local notion" as follows: For each point $p \in X$, we define the "local dimension" $\dim_p X$ at p by $\dim_p X := \lim_{r\to 0} \dim B_r(x)$. (Here $B_r(p)$ is the closed r-ball centered at p.) Then we have $\dim X = \sup_{p\in X} \dim_p X$. The authors don't know whether a similar description of the mean dimension is possible or not. Instead, in this subsection we will introduce a new notion "local mean dimension".

Suppose that the additive group $\mathbb{Z}$ continuously acts on X. For each point $p \in X$ and $r > 0$, we denote $B_r(p)_\mathbb{Z}$ as the closed r-ball centered at p with respect to the distance $d_\mathbb{Z}(\cdot, \cdot)$:

$$B_r(p)_\mathbb{Z} := \{x \in X \,|\, d_\mathbb{Z}(x, p) \leq r\}.$$

Note that $d_\mathbb{Z}(x, y) \leq r \Leftrightarrow d(n.x, n.y) \leq r$ for all $n \in \mathbb{Z}$. $B_r(p)_\mathbb{Z}$ is a closed set in X. We define the local mean dimension of X at p by

$$\dim_p(X : \mathbb{Z}) := \lim_{r\to 0} \dim(B_r(p)_\mathbb{Z} \subset X : \mathbb{Z}).$$

This is independent of the choice of a distance compatible with the topology of X. We define the local mean dimension of X by

$$\dim_{loc}(X : \mathbb{Z}) := \sup_{p\in X} \dim_p(X : \mathbb{Z}).$$

Obviously we have

$$\dim_{loc}(X : \mathbb{Z}) \leq \dim(X : \mathbb{Z}). \tag{5}$$

Let X, Y be compact metric spaces with continuous $\mathbb{Z}$-actions. If there exists a $\mathbb{Z}$-equivariant topological embedding $f : X \to Y$, then, from (4), for all $p \in X$

$$\dim_p(X : \mathbb{Z}) \leq \dim_{f(p)}(Y : \mathbb{Z}).$$

Example 2.7. Let $B \subset \mathbb{R}^N$ be the closed ball centered at the origin. Then we have

$$\dim_{\mathbf{0}}(B^{\mathbb{Z}} : \mathbb{Z}) = \dim_{loc}(B^{\mathbb{Z}} : \mathbb{Z}) = \dim(B^{\mathbb{Z}} : \mathbb{Z}) = N,$$

where $\mathbf{0} = (\cdots, 0, 0, 0, \cdots) \in B^{\mathbb{Z}}$.

Proof. Fix a distance on $B^{\mathbb{Z}}$. Then it is easy to see that for any $r > 0$ there exists $s > 0$ such that $B_s^{\mathbb{Z}} \subset B_r(\mathbf{0})_{\mathbb{Z}}$, where B_s is the s-ball in $\mathbb{R}^N$. Then

$$N = \dim(B_s^{\mathbb{Z}} : \mathbb{Z}) \leq \dim(B_r(\mathbf{0})_{\mathbb{Z}} \subset B^{\mathbb{Z}} : \mathbb{Z}) \leq \dim(B^{\mathbb{Z}} : \mathbb{Z}) = N.$$

Hence $\dim_{\mathbf{0}}(B^{\mathbb{Z}} : \mathbb{Z}) = N$. □

We will use the following formula in Section 7.2. Since $k.B_r(p)_{\mathbb{Z}} = B_r(kp)_{\mathbb{Z}}$, we have

$$\mathrm{Widim}_\varepsilon(B_r(p)_{\mathbb{Z}}, d_{k+\Omega}) = \mathrm{Widim}_\varepsilon(k.B_r(p)_{\mathbb{Z}}, d_\Omega) = \mathrm{Widim}_\varepsilon(B_r(kp)_{\mathbb{Z}}, d_\Omega),$$

and hence

$$\mathrm{Widim}_\varepsilon(B_r(p)_{\mathbb{Z}} \subset X : \mathbb{Z}) = \lim_{n\to\infty}\left(\frac{1}{|\Omega_n|}\sup_{k\in\mathbb{Z}} \mathrm{Widim}_\varepsilon(B_r(kp)_{\mathbb{Z}}, d_{\Omega_n})\right). \tag{6}$$

The following will be used in Section 9.2.

Example 2.8. Let G be a compact Lie group, and $G^{\mathbb{Z}}$ be the infinite product of G indexed by integers. G acts on $G^{\mathbb{Z}}$ by $g(u_n)_{n\in\mathbb{Z}} := (gu_n)_{n\in\mathbb{Z}}$. Let $G^{\mathbb{Z}}/G$ be the quotient by this action. Then for any point $[p] \in G^{\mathbb{Z}}/G$,

$$\dim_{[p]}(G^{\mathbb{Z}}/G : \mathbb{Z}) = \dim(G^{\mathbb{Z}}/G : \mathbb{Z}) = \dim G.$$

Proof. We define a distance $d(\cdot,\cdot)$ on $G^{\mathbb{Z}}/G$ by setting

$$d([(x_n)_{n\in\mathbb{Z}}], [(y_n)_{n\in\mathbb{Z}}]) := \inf_{g\in G}\sum_{n\in\mathbb{Z}} 2^{-|n|}\underline{d}(gx_n, y_n),$$

where $\underline{d}(\cdot,\cdot)$ is a two-sided-invariant distance on G. Set $\Omega_N := \{0, 1, 2, \cdots, N-1\}$ $(N \geq 1)$. For any $\varepsilon > 0$, let $L = L(\varepsilon)$ be a positive integer satisfying

$$\sum_{|n|>L} 2^{-|n|} \leq \varepsilon/\mathrm{Diam}(G).$$

Then it is easy to see that the map

$$G^{\mathbb{Z}}/G \to G^{N+2L+1}/G, \quad [(x_n)_{n\in\mathbb{Z}}] \mapsto [(x_{-L}, x_{-L+1}, \cdots, x_{N+L})].$$

is an ε-embedding with respect to the distance d_{Ω_N}. G^{N+2L+1}/G is a manifold of dimension $(N+2L)\dim G$. Hence $\mathrm{Widim}_\varepsilon(G^{\mathbb{Z}}/G, d_{\Omega_N}) \leq (N+2L)\dim G$, and

$$\lim_{N\to\infty}\frac{1}{|\Omega_N|}\mathrm{Widim}_\varepsilon(G^{\mathbb{Z}}/G, d_{\Omega_N}) \leq \dim G.$$

Then for any point $[p] \in G^{\mathbb{Z}}/G$, $\dim_{[p]}(G^{\mathbb{Z}}/G : \mathbb{Z}) \leq \dim(G^{\mathbb{Z}}/G : \mathbb{Z}) \leq \dim G$.

Let $p = (p_n)_{n\in\mathbb{Z}}$ be a point in $G^{\mathbb{Z}}$, and $r > 0$. Let $\mathrm{Lie}(G)$ be the Lie algebra of G with a norm $|\cdot|$. We take $r_0 > 0$ so that the map $\{x \in \mathrm{Lie}(G) | \, |x| \leq r_0\} \ni x \mapsto e^x \in G$

becomes a topological embedding. We choose $r' = r'(r) \leq r_0$ so that every $x \in \mathrm{Lie}(G)$ with $|x| \leq r'$ satisfies $\underline{d}(e^x, 1) \leq r/3$ ($\Leftrightarrow \underline{d}(e^x p_n, p_n) \leq r/3$). Fix an integer $m > 0$, and define $A_m \subset (\mathrm{Lie}(G))^{\mathbb{Z}}$ by

$$A_m := \{(x_n)_{n\in\mathbb{Z}} |\, |x_n| \leq r' \text{ for all } n, \text{ and } x_n = 0 \text{ for } n \in m\mathbb{Z}.\}.$$

Define $f : A_m \to G^{\mathbb{Z}}/G$ by $f((x_n)_{n\in\mathbb{Z}}) := [(e^{x_n} p_n)_{n\in\mathbb{Z}}]$. Then $f(A_m) \subset B_r([p])_{\mathbb{Z}}$. For $g \in G$ and $(x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}$ in A_m, since $x_0 = y_0 = 0$ and $\underline{d}(g, 1) = \underline{d}(ge^{x_n}, e^{x_n})$,

$$\begin{aligned}\sum_{n=0}^{m-1} 2^{-n} \underline{d}(ge^{x_n}, e^{y_n}) &\geq \sum_{n=1}^{m-1} 2^{-n}(\underline{d}(ge^{x_n}, e^{y_n}) + \underline{d}(g, 1)),\\ &\geq \sum_{n=1}^{m-1} 2^{-n}\underline{d}(e^{x_n}, e^{y_n}) \geq 2^{-m} \max_{0\leq n\leq m-1} \underline{d}(e^{x_n}, e^{y_n}).\end{aligned}$$

Hence $d_{\Omega_{mN}}([(e^{x_n}p_n)_{n\in\mathbb{Z}}], [(e^{y_n}p_n)_{n\in\mathbb{Z}}]) \geq 2^{-m} \max_{0\leq n\leq mN-1} \underline{d}(e^{x_n}, e^{y_n})$. We choose $\varepsilon = \varepsilon(r, m) > 0$ so that if $x, y \in \mathrm{Lie}(G)$ with $|x|, |y| \leq r'$ satisfy $\underline{d}(e^x, e^y) \leq 2^m\varepsilon$ then $|x - y| \leq r'/2$. Then, if $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ in A_m satisfy $d_{\Omega_{mN}}([(e^{x_n}p_n)_{n\in\mathbb{Z}}], [(e^{y_n}p_n)_{\in\mathbb{Z}}]) \leq \varepsilon$, we have $\max_{0\leq n\leq mN-1} |x_n - y_n| \leq r'/2$. Then, by using Lemma 2.1,

$$\mathrm{Widim}_\varepsilon(B_r([p])_{\mathbb{Z}}, d_{\Omega_{mN}}) \geq \mathrm{Widim}_{r'/2}(B_{r'}(\mathrm{Lie}(G))^{(m-1)N}, \|\cdot\|_{\ell^\infty}) = (m-1)N \dim G.$$

From this estimate, we get

$$\dim_{[p]}(G^{\mathbb{Z}}/G : \mathbb{Z}) \geq (1 - 1/m) \dim G.$$

Let $m \to \infty$. Then we get the conclusion: $\dim_{[p]}(G^{\mathbb{Z}}/G : \mathbb{Z}) \geq \dim G$. □

3. Outline of the proofs of the main theorems

The ideas of the proofs of Theorem 1.1 and 1.2 are simple. But the completion of the proofs needs lengthy technical arguments. So we want to describe the outline of the proofs in this section. Here we don't pursue the accuracy of the arguments for simplicity of the explanation. Some of the arguments will be replaced with different ones in the later sections.

First we explain how to get the upper bound on the mean dimension of $\mathcal{M}_d$. We define a distance on $\mathcal{M}_d$ by setting

$$\begin{aligned}&\mathrm{dist}([\boldsymbol{A}, \boldsymbol{p}], [\boldsymbol{B}, \boldsymbol{q}])\\ &\quad:= \inf_{g:\boldsymbol{E}\to\boldsymbol{E}} \left\{\sum_{n\geq 1} 2^{-n} \frac{\|g(\boldsymbol{A}) - \boldsymbol{B}\|_{L^\infty(|t|\leq n)}}{1 + \|g(\boldsymbol{A}) - \boldsymbol{B}\|_{L^\infty(|t|\leq n)}} + \sum_{n\in\mathbb{Z}} 2^{-|n|}|g(\boldsymbol{p}(n)) - \boldsymbol{q}(n)|\right\},\end{aligned}$$

where g runs over gauge transformations of $\boldsymbol{E}$, and $|t| \leq n$ means the region $\{(\theta, t) \in S^3 \times \mathbb{R} |\, |t| \leq n\}$. For $R = 1, 2, 3, \cdots$, we define an amenable sequence $\Omega_R \subset \mathbb{Z}$ by $\Omega_R := \{n \in \mathbb{Z} | -R \leq n \leq R\}$.

Let $\varepsilon > 0$ be a positive number and define a positive integer $L = L(\varepsilon)$ so that

$$\sum_{n>L} 2^{-n} < \frac{\varepsilon}{2(1+2\mathrm{Diam}(SU(2)))}. \tag{7}$$

Let $D = D(\varepsilon)$ be a large positive integer which depends on ε but is independent of R, and set $T := R + L + D$. (D is chosen so that the condition (8) below is satisfied. Here we don't explain how to define D precisely.) For $c \geq 0$ we define $M_T(c)$ as the space of the gauge equivalence classes $[A, p]$ where A is an ASD connection on $\boldsymbol{E}$ satisfying

$$\frac{1}{8\pi^2}\int_X |F_A|^2 d\mathrm{vol} \leq c,$$

and p is a map from $\{n \in \mathbb{Z} | -T \leq n \leq T\}$ to $\boldsymbol{E}$ with $p(n) \in \boldsymbol{E}_{(\theta_0,n)}$ $(-T \leq n \leq T)$. The index theorem gives the estimate:

$$\dim M_T(c) \leq 8c + 6T.$$

We want to construct an ε-embedding from $(\mathcal{M}_d, \mathrm{dist}_{\Omega_R})$ to $M_T(c)$ for an appropriate $c \geq 0$.

Let $(\boldsymbol{A}, \boldsymbol{p})$ be a framed connection on $\boldsymbol{E}$ with $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d$. We "cut-off" $(\boldsymbol{A}, \boldsymbol{p})$ over the region $T < |t| < T + 1$ and produce a new framed connection $(\boldsymbol{A}', \boldsymbol{p}')$ satisfying the following conditions. $\boldsymbol{A}'$ is a (not necessarily ASD) connection on $\boldsymbol{E}$ satisfying $\boldsymbol{A}'|_{|t|\leq T} = \boldsymbol{A}|_{|t|\leq T}$, $F(\boldsymbol{A}') = 0$ over $|t| \geq T + 1$, and

$$\frac{1}{8\pi^2}\int_X tr(F(\boldsymbol{A}')^2) \leq \frac{1}{8\pi^2}\int_{|t|\leq T} |F(\boldsymbol{A})|^2 d\mathrm{vol} + \mathrm{const} \leq \frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + \mathrm{const},$$

where const is a positive constant independent of ε and R. $\boldsymbol{p}'$ is a map from $\{n \in \mathbb{Z} | -T \leq n \leq T\}$ to $\boldsymbol{E}$ with $\boldsymbol{p}'(n) = \boldsymbol{p}(n) \in \boldsymbol{E}_{(\theta_0,n)}$. Next we "perturb" $\boldsymbol{A}'$ and produce an ASD connection $\boldsymbol{A}''$ on $\boldsymbol{E}$ satisfying

$$|\boldsymbol{A} - \boldsymbol{A}''| = |\boldsymbol{A}' - \boldsymbol{A}''| \leq \varepsilon/4 \quad (|t| \leq T - D = R + L), \tag{8}$$

$$\frac{1}{8\pi^2}\int_X |F(\boldsymbol{A}'')|^2 d\mathrm{vol} = \frac{1}{8\pi^2}\int_X tr(F(\boldsymbol{A}')^2) \leq \frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + \mathrm{const}.$$

Then we can define the map

$$\mathcal{M}_d \to M_T\left(\frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + \mathrm{const}\right), \quad [\boldsymbol{A}, \boldsymbol{p}] \mapsto [\boldsymbol{A}'', \boldsymbol{p}'].$$

The conditions (7), (8) and $\boldsymbol{p}'(n) = \boldsymbol{p}(n)$ $(|n| \leq T)$ imply that this map is an ε-embedding with respect to the distance dist_{Ω_R}. Hence

$$\mathrm{Widim}_\varepsilon(\mathcal{M}_d, \mathrm{dist}_{\Omega_R}) \leq \dim M_T\left(\frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + \mathrm{const}\right) \leq \frac{2Td^2\mathrm{vol}(S^3)}{\pi^2} + 8\cdot\mathrm{const} + 6T.$$

(Caution! This estimate will *not* be proved in this paper. The above argument contains a gap.) Recall $T = R + L + D$. Since L, D and const are independent of R, we get

$$\lim_{R\to\infty} \frac{\mathrm{Widim}_\varepsilon(\mathcal{M}_d, \mathrm{dist}_{\Omega_R})}{|\Omega_R|} \leq \frac{d^2\mathrm{vol}(S^3)}{\pi^2} + 3.$$

Hence we get

$$\dim(\mathcal{M}_d : \mathbb{Z}) \leq \frac{d^2\mathrm{vol}(S^3)}{\pi^2} + 3 < +\infty. \tag{9}$$

This is the outline of the proof of the upper bound on the mean dimension. (The upper bound on the local mean dimension can be proved by investigating the above procedure more precisely.) Strictly speaking, the above argument contains a gap. Actually we have not so far succeeded to prove the estimate $\dim(\mathcal{M}_d : \mathbb{Z}) \leq d^2\mathrm{vol}(S^3)/\pi^2 + 3$. In this paper we prove only $\dim(\mathcal{M}_d : \mathbb{Z}) < +\infty$. A problem occurs in the cut-off construction. Indeed (we think that) there exists no canonical way to cut-off connections compatible with the gauge symmetry. Therefore we cannot define a suitable cut-off construction all over $\mathcal{M}_d$. Instead we will decompose $\mathcal{M}_d$ as $\mathcal{M}_d = \bigcup_{1\leq i,j\leq N} \mathcal{M}_{d,T}(i,j)$ (N is independent of ε and R) and define a cut-off construction for each piece $\mathcal{M}_{d,T}(i,j)$ independently. Then we will get an upper bound worse than (9) (cf. Lemma 2.3). We study the cut-off construction (the procedure $[\boldsymbol{A}, \boldsymbol{p}] \mapsto [\boldsymbol{A}', \boldsymbol{p}']$) in Section 6. This construction uses the framing $\boldsymbol{p}$ as an essential data. In Section 4 and 5 we study the perturbation procedure ($\boldsymbol{A}' \mapsto \boldsymbol{A}''$). The perturbation does not use the framing. The upper bounds on the (local) mean dimension are proved in Section 7.

Next we explain how to prove the lower bound on the local mean dimension. Let $T > 0$, $\underline{\mathrm{E}}$ be a principal $SU(2)$-bundle over $S^3 \times (\mathbb{R}/T\mathbb{Z})$, and $\underline{\mathrm{A}}$ be a non-flat ASD connection on $\underline{\mathrm{E}}$ satisfying $|F(\underline{\mathrm{A}})| < d$. For simplicity of the explanation, we assume $T = 1$. When $T \neq 1$ (in particular, when T is an irrational number), we need some modifications of the arguments below.

Let $\pi : S^3 \times \mathbb{R} \to S^3 \times (\mathbb{R}/\mathbb{Z})$ be the natural projection, and set $E := \pi^*(\underline{\mathrm{E}})$ and $A := \pi^*(\underline{\mathrm{A}})$. We define the infinite dimensional Banach space H^1_A by

$$H^1_A := \{a \in \Omega^1(\mathrm{ad}E) | \, (d_A^* + d_A^+)a = 0, \, \|a\|_{L^\infty} < \infty\}.$$

There exists a natural $\mathbb{Z}$-action on H^1_A. Let $r > 0$ be a sufficiently small number. For each $a \in H^1_A$ with $\|a\|_{L^\infty} \leq r$ we can construct $\tilde{a} \in \Omega^1(\mathrm{ad}E)$ (a small perturbation of a) satisfying $F^+(A + \tilde{a}) = 0$ and $|F(A + \tilde{a})| \leq d$. If $a = 0$, then $\tilde{a} = 0$.

For $n \geq 1$, let $\pi_n : S^3 \times (\mathbb{R}/n\mathbb{Z}) \to S^3 \times (\mathbb{R}/\mathbb{Z})$ be the natural projection, and set $E_n := \pi_n^*(\underline{\mathrm{E}})$ and $A_n := \pi_n^*(\underline{\mathrm{A}})$. We define $H^1_{A_n}$ as the space of $a \in \Omega(\mathrm{ad}E_n)$ satisfying $(d^*_{A_n} + d^+_{A_n})a = 0$. We can identify $H^1_{A_n}$ with the subspace of H^1_A consisting of $n\mathbb{Z}$-invariant elements. The index theorem gives

$$\dim H^1_{A_n} = 8nc_2(\underline{\mathrm{E}}).$$

Let $\ell^\infty(\mathbb{Z}, su(2))$ be the Banach space of $(u_n)_{n\in\mathbb{Z}}$ in $su(2)^{\mathbb{Z}}$ satisfying $\|(u_n)_{n\in\mathbb{Z}}\|_{\ell^\infty} := \sup_{n\in\mathbb{Z}} |u_n| < \infty$. $\ell^\infty(\mathbb{Z}, su(2))$ also admits a natural $\mathbb{Z}$-action. Set $V := H^1_A \times \ell^\infty(\mathbb{Z}, su(2))$ with $\|(a,(u_n)_{n\in\mathbb{Z}})\|_V := \max(\|a\|_{L^\infty}, \|(u_n)_{n\in\mathbb{Z}}\|_{\ell^\infty})$. We define $V_n \subset V$ $(n \geq 1)$ as the subspace of V consisting of $n\mathbb{Z}$-invariant elements. $V_n \cong H^1_{A_n} \times \ell^\infty(\mathbb{Z}/n\mathbb{Z}, su(2))$ and hence

$$\dim V_n = 8nc_2(\underline{\mathrm{E}}) + 3n.$$

Take $p \in \underline{\mathrm{E}}_{(\theta_0,[0])}$ and let $p_n \in E_{(\theta_0,n)}$ $(n \in \mathbb{Z})$ be its lifts. We define the map from $B_r(V)$ (the r-ball of V centered at the origin) to $\mathcal{M}_d$ by

$$B_r(V) \to \mathcal{M}_d, \quad (a, (u_n)_{n\in\mathbb{Z}}) \mapsto [E, A + \tilde{a}, (p_n e^{u_n})_{n\in\mathbb{Z}}].$$

(cf. the description of $\mathcal{M}_d$ in Remark 1.3.) This map becomes a $\mathbb{Z}$-equivariant topological embedding for $r \ll 1$. (Here $B_r(V)$ is endowed with the following topology. A sequence $\{(a_n, (u^{(n)}_k)_{k\in\mathbb{Z}})\}_{n\geq 1}$ in $B_r(V)$ converges to $(a, (u_k)_{k\in\mathbb{Z}})$ in $B_r(V)$ if and only if a_n uniformly converges to a over every compact subset and $u^{(n)}_k$ converges to u_k for every $k \in \mathbb{Z}$.) Then we have

$$\dim_{[E,A,(p_n)_{n\in\mathbb{Z}}]}(\mathcal{M}_d : \mathbb{Z}) \geq \dim_0(B_r(V) : \mathbb{Z}).$$

The right-hand-side is the local mean dimension of $B_r(V)$ at the origin. We can prove that $\dim_0(B_r(V) : \mathbb{Z})$ can be estimated from below by "the growth of periodic points":

$$\dim_0(B_r(V) : \mathbb{Z}) \geq \lim_{n\to\infty} \dim V_n/n = 8c_2(\underline{\mathrm{E}}) + 3 = 8\rho(A) + 3.$$

(This is not difficult to prove. This is just an application of Lemma 2.1.) Therefore

$$\dim_{[E,A,(p_n)_{n\in\mathbb{Z}}]}(\mathcal{M}_d : \mathbb{Z}) \geq 8\rho(A) + 3.$$

This is the outline of the proof of the lower bound. Here we consider only the "1-periodic" ASD connection A and "the periodic framing" $(p_n)_{n\in\mathbb{Z}}$. Hence the real proof needs some modifications.

4. Perturbation

In this section we construct the method of constructing ASD connections from "approximately ASD" connections over $X = S^3 \times \mathbb{R}$. We basically follow the argument of Donaldson [5]. As we promised in the introduction, the variable t means the variable of the $\mathbb{R}$-factor of $S^3 \times \mathbb{R}$.

4.1. **Construction of the perturbation.** Let T be a positive integer, and d, d' be two non-negative real numbers. Set $\varepsilon_0 = 1/(1000)$. (The value $1/(1000)$ itself has no meaning. The point is that it is an explicit number which satisfies (13) below.) Let E be a principal $SU(2)$-bundle over X, and A be a connection on E satisfies
(i) $F_A = 0$ over $|t| > T + 1$,
(ii) F^+_A is supported in $\{(\theta, t) \in S^3 \times \mathbb{R} | \, T < |t| < T + 1\}$, and $\|F^+_A\|_T \leq \varepsilon_0$. Here $\|\cdot\|_T$ is the "Taubes norm" defined below ((16) and (17)).

(iii) $|F_A| \leq d$ on $|t| \leq T$ and $\left\| F_A^+ \right\|_{L^\infty(X)} \leq d'$. (The condition (iii) is not used in Section 4.1, 4.2, 4.3. It will be used in Section 4.4.)

Let $\Omega^+(\mathrm{ad}E)$ be the set of smooth self-dual 2-forms valued in $\mathrm{ad}E$ (not necessarily compact supported). The first main purpose of this section is to solve the equation $F^+(A + d_A^*\phi) = 0$ for $\phi \in \Omega^+(\mathrm{ad}E)$. We have $F^+(A+d_A^*\phi) = F_A^+ + d_A^+ d_A^* \phi + (d_A^*\phi \wedge d_A^*\phi)^+$. The Weitzenböck formula gives ([8, Chapter 6])

$$d_A^+ d_A^* \phi = \frac{1}{2}\nabla_A^*\nabla_A\phi + \left(\frac{S}{6} - W^+\right)\phi + F_A^+ \cdot \phi, \tag{10}$$

where S is the scalar curvature of X and W^+ is the self-dual part of the Weyl curvature. Since X is conformally flat, we have $W^+ = 0$. The scalar curvature S is a positive constant. Then the equation $F^+(A + d_A^*\phi) = 0$ becomes

$$(\nabla_A^*\nabla_A + S/3)\phi + 2F_A^+ \cdot \phi + 2(d_A^*\phi \wedge d_A^*\phi)^+ = -2F_A^+. \tag{11}$$

Set $c_0 = 10$. Then

$$|F_A^+ \cdot \phi| \leq c_0 |F_A^+| \cdot |\phi|, \quad |(d_A^*\phi_1 \wedge d_A^*\phi_2)^+| \leq c_0 |\nabla_A\phi_1| \cdot |\nabla_A\phi_2|. \tag{12}$$

(These are not best possible.[1]) The positive constant $\varepsilon_0 = 1/1000$ in the above satisfies

$$50 c_0 \varepsilon_0 < 1. \tag{13}$$

Let $\Delta = \nabla^*\nabla$ be the Laplacian on functions over X, and $g(x,y)$ be the Green kernel of $\Delta + S/3$. We prepare basic facts on $g(x,y)$ in Appendix A. Here we state some of them without the proofs. For the proofs, see Appendix A. $g(x,y)$ satisfies

$$(\Delta_y + S/3) g(x,y) = \delta_x(y).$$

This equation means that, for any compact supported smooth function φ,

$$\varphi(x) = \int_X g(x,y)(\Delta_y + S/3)\varphi(y) d\mathrm{vol}(y),$$

where $d\mathrm{vol}(y)$ denotes the volume form of X. $g(x,y)$ is smooth outside the diagonal and it has a singularity of order $1/d(x,y)^2$ along the diagonal:

$$\mathrm{const}_1/d(x,y)^2 \leq g(x,y) \leq \mathrm{const}_2/d(x,y)^2, \quad (d(x,y) \leq \mathrm{const}_3), \tag{14}$$

where $d(x,y)$ is the distance on X, and const_1, const_2, const_3 are positive constants. $g(x,y) > 0$ for $x \neq y$ (Lemma A.1), and it has an exponential decay (Lemma A.2):

$$0 < g(x,y) < \mathrm{const}_4 \cdot e^{-\sqrt{S/3}\, d(x,y)} \quad (d(x,y) \geq 1). \tag{15}$$

Since $S^3 \times \mathbb{R} = SU(2) \times \mathbb{R}$ is a Lie group and its Riemannian metric is two-sided invariant, we have $g(zx, zy) = g(xz, yz) = g(x,y)$. In particular, for $x = (\theta_1, t_1)$ and $y = (\theta_2, t_2)$, we

[1]Strictly speaking, the choice of c_0 depends on the convention of the metric (inner product) on $su(2)$. Our convention is: $\langle A, B\rangle = -tr(AB)$ for $A, B \in su(2)$.

have $g((\theta_1, t_1 - t_0), (\theta_2, t_2 - t_0)) = g((\theta_1, t_1), (\theta_2, t_2))$ ($t_0 \in \mathbb{R}$). That is, $g(x, y)$ is invariant under the translation $t \mapsto t - t_0$.

For $\phi \in \Omega^+(\mathrm{ad}E)$, we define the pointwise Taubes norm $|\phi|_T(x)$ by setting

$$|\phi|_T(x) := \int_X g(x, y)|\phi(y)|d\mathrm{vol}(y) \quad (x \in X), \tag{16}$$

(Note that $g(x, y) > 0$ for $x \neq y$.) This may be infinity. We define the Taubes norm $\|\phi\|_T$ by

$$\|\phi\|_T := \sup_{x \in X} |\phi|_T(x). \tag{17}$$

Set

$$K := \int_X g(x, y)d\mathrm{vol}(y) \quad \text{(this is independent of } x \in X).$$

(This is finite by (14) and (15).) We have

$$\|\phi\|_T \leq K \|\phi\|_{L^\infty}.$$

We define $\Omega^+(\mathrm{ad}E)_0$ as the set of $\phi \in \Omega^+(\mathrm{ad}E)$ which vanish at infinity: $\lim_{x\to\infty} |\phi(x)| = 0$. (Here $x = (\theta, t) \to \infty$ means $|t| \to +\infty$.) If $\phi \in \Omega^+(\mathrm{ad}E)_0$, then $\|\phi\|_T < \infty$ and $\lim_{x\to\infty} |\phi|_T(x) = 0$. (See the proof of Proposition A.7.) For $\eta \in \Omega^+(\mathrm{ad}E)_0$, there uniquely exists $\phi \in \Omega^+(\mathrm{ad}E)_0$ satisfying $(\nabla_A^*\nabla_A + S/3)\phi = \eta$. (See Proposition A.7.) We set $(\nabla_A^*\nabla_A + S/3)^{-1}\eta := \phi$. This satisfies

$$|\phi(x)| \leq |\eta|_T(x), \text{ and hence } \|\phi\|_{L^\infty} \leq \|\eta\|_T. \tag{18}$$

Lemma 4.1. $\lim_{x\to\infty} |\nabla_A\phi(x)| = 0$.

Proof. From the condition (i) in the beginning of this section, A is flat over $|t| > T + 1$. Therefore there exists a bundle map $g : E|_{|t|>T+1} \to X_{|t|>T+1} \times SU(2)$ such that $g(A)$ is the product connection. Here $X_{|t|>T+1} = \{(\theta, t) \in S^3 \times \mathbb{R} |\, |t| > T + 1\}$ and $E|_{|t|>T+1}$ is the restriction of E to $X_{|t|>T+1}$. We sometimes use similar notations in this paper. Set $\phi' := g(\phi)$ and $\eta' := g(\eta)$. They satisfy $(\nabla^*\nabla + S/3)\phi' = \eta'$. (Here ∇ is defined by the product connection on $X|_{|t|>T+1} \times SU(2)$ and the Levi-Civita connection.)

For $|t| > T + 2$, we set $B_t := S^3 \times (t - 1, t + 1)$. From the elliptic estimates, for any $\theta \in S^3$,

$$|\nabla\phi'(\theta, t)| \leq C(\|\phi'\|_{L^\infty(B_t)} + \|\eta'\|_{L^\infty(B_t)}),$$

where C is a constant independent of t. This means

$$|\nabla_A\phi(\theta, t)| \leq C(\|\phi\|_{L^\infty(B_t)} + \|\eta\|_{L^\infty(B_t)}).$$

The right-hand-side goes to 0 as $|t|$ goes to infinity. □

The following Lemma shows a power of the Taubes norm.

Lemma 4.2.

$$\left||\nabla_A\phi|^2\right|_T(x) := \int_X g(x,y)|\nabla_A\phi(y)|^2 d\mathrm{vol}(y) \le \|\eta\|_T\,|\eta|_T(x).$$

In particular, $\||\nabla_A\phi|^2\|_T := \sup_{x\in X}\left||\nabla_A\phi|^2\right|_T(x) \le \|\eta\|_T^2$ *and* $\|(d_A^*\phi\wedge d_A^*\phi)^+\|_T \le c_0\,\|\eta\|_T^2$.

Proof. $\nabla|\phi|^2 = 2(\nabla_A\phi,\phi)$ vanishes at infinity (Lemma 4.1).

$$(\Delta+2S/3)|\phi|^2 = 2(\nabla_A^*\nabla_A\phi + (S/3)\phi,\phi) - 2|\nabla_A\phi|^2 = 2(\eta,\phi) - 2|\nabla_A\phi|^2.$$

In particular, $(\Delta+S/3)|\phi|^2$ vanishes at infinity (Lemma 4.1). Hence $|\phi|^2, \nabla|\phi|^2, (\Delta+S/3)|\phi|^2$ vanish at infinity (in particular, they are contained in L^∞). Then we can apply Lemma A.3 in Appendix A to $|\phi|^2$ and get

$$\int_X g(x,y)(\Delta_y + S/3)|\phi(y)|^2 d\mathrm{vol}(y) = |\phi(x)|^2.$$

We have

$$\begin{aligned}|\nabla_A\phi|^2 &= (\eta,\phi) - \frac{1}{2}(\Delta+S/3)|\phi|^2 - \frac{S}{6}|\phi|^2,\\ &\le (\eta,\phi) - \frac{1}{2}(\Delta+S/3)|\phi|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\int_X g(x,y)|\nabla_A\phi(y)|^2 d\mathrm{vol}(y) &\le \int_X g(x,y)(\eta(y),\phi(y))d\mathrm{vol}(y) - \frac{1}{2}|\phi(y)|^2,\\ &\le \int_X g(x,y)(\eta(y),\phi(y))d\mathrm{vol}(y),\\ &\le \|\phi\|_{L^\infty}\int_X g(x,y)|\eta(y)|d\mathrm{vol}(y) \le \|\eta\|_T\,|\eta|_T(x).\end{aligned}$$

In the last line we have used (18). $\square$

For $\eta_1,\eta_2\in\Omega^+(\mathrm{ad}E)_0$, set $\phi_i := (\nabla_A^*\nabla_A + S/3)^{-1}\eta_i \in \Omega^+(\mathrm{ad}E)_0$ and

$$\beta(\eta_1,\eta_2) := (d_A^*\phi_1\wedge d_A^*\phi_2)^+ + (d_A^*\phi_2\wedge d_A^*\phi_1)^+. \tag{19}$$

β is symmetric and $|\beta(\eta_1,\eta_2)| \le 2c_0|\nabla_A\phi_1|\cdot|\nabla_A\phi_2|$. In particular, $\beta(\eta_1,\eta_2)\in\Omega^+(\mathrm{ad}E)_0$ (Lemma 4.1).

Lemma 4.3. $\|\beta(\eta_1,\eta_2)\|_T \le 4c_0\,\|\eta_1\|_T\,\|\eta_2\|_T$.

Proof. From Lemma 4.2, $\|\beta(\eta,\eta)\|_T \le 2c_0\,\|\eta\|_T^2$. Suppose $\|\eta_1\|_T = \|\eta_2\|_T = 1$. Since $4\beta(\eta_1,\eta_2) = \beta(\eta_1+\eta_2,\eta_1+\eta_2) - \beta(\eta_1-\eta_2,\eta_1-\eta_2)$,

$$4\,\|\beta(\eta_1,\eta_2)\|_T \le 2c_0\,\|\eta_1+\eta_2\|_T^2 + 2c_0\,\|\eta_1-\eta_2\|_T^2 \le 16c_0.$$

Hence $\|\beta(\eta_1,\eta_2)\|_T \le 4c_0$. The general case follows from this. $\square$

For $\eta \in \Omega^+(\mathrm{ad}E)_0$, we set $\phi := (\nabla_A^*\nabla_A + S/3)^{-1}\eta \in \Omega^+(\mathrm{ad}E)_0$ and define

$$\Phi(\eta) := -2F_A^+ \cdot \phi - \beta(\eta,\eta) - 2F_A^+ \in \Omega^+(\mathrm{ad}E)_0.$$

If η satisfies $\eta = \Phi(\eta)$, then ϕ satisfies the ASD equation (11).

Lemma 4.4. *For $\eta_1, \eta_2 \in \Omega^+(\mathrm{ad}E)_0$,*

$$\|\Phi(\eta_1) - \Phi(\eta_2)\|_T \leq 2c_0(\|F_A^+\|_T + 2\|\eta_1+\eta_2\|_T)\|\eta_1 - \eta_2\|_T.$$

Proof.

$$\Phi(\eta_1) - \Phi(\eta_2) = -2F_A^+ \cdot (\phi_1 - \phi_2) + \beta(\eta_1+\eta_2, \eta_2 - \eta_1).$$

From Lemma 4.3 and $\|\phi_1 - \phi_2\|_{L^\infty} \leq \|\eta_1 - \eta_2\|_T$ (see (18)),

$$\begin{aligned}\|\Phi(\eta_1) - \Phi(\eta_2)\|_T &\leq 2c_0\|F_A\|_T\|\phi_1-\phi_2\|_{L^\infty} + 4c_0\|\eta_1+\eta_2\|_T\|\eta_1-\eta_2\|_T, \\ &\leq 2c_0(\|F_A^+\|_T + 2\|\eta_1+\eta_2\|_T)\|\eta_1-\eta_2\|_T.\end{aligned}$$

□

Proposition 4.5. *The sequence $\{\eta_n\}_{n\geq 0}$ in $\Omega^+(\mathrm{ad}E)_0$ defined by*

$$\eta_0 = 0, \quad \eta_{n+1} = \Phi(\eta_n),$$

becomes a Cauchy sequence with respect to the Taubes norm $\|\cdot\|_T$ and satisfies

$$\|\eta_n\|_T \leq 3\varepsilon_0,$$

for all $n \geq 1$.

Proof. Set $B := \{\eta \in \Omega^+(\mathrm{ad}E)_0 | \, \|\eta\|_T \leq 3\varepsilon_0\}$. For $\eta \in B$ (recall: $\|F_A^+\|_T \leq \varepsilon_0$),

$$\begin{aligned}\|\Phi(\eta)\|_T &\leq 2c_0\|F_A^+\|_T\|\phi\|_{L^\infty} + 2c_0\|\eta\|_T^2 + 2\|F_A^+\|_T, \\ &\leq 2c_0\varepsilon_0\|\eta\|_T + 2c_0\|\eta\|_T^2 + 2\varepsilon_0, \\ &\leq (24c_0\varepsilon_0 + 2)\varepsilon_0 \leq 3\varepsilon_0\end{aligned}$$

Here we have used (13). Hence $\Phi(\eta) \in B$. Lemma 4.4 implies (for $\eta_1, \eta_2 \in B$)

$$\|\Phi(\eta_1) - \Phi(\eta_2)\|_T \leq 2c_0(\|F_A^+\|_T + 2\|\eta_1+\eta_2\|_T)\|\eta_1-\eta_2\|_T \leq 26c_0\varepsilon_0\|\eta_1-\eta_2\|_T.$$

$26c_0\varepsilon_0 < 1$ by (13). Hence $\Phi : B \to B$ becomes a contraction map with respect to the norm $\|\cdot\|_T$. Thus $\eta_{n+1} = \Phi(\eta_n)$ $(\eta_0 = 0)$ becomes a Cauchy sequence. □

The sequence $\phi_n \in \Omega^+(\mathrm{ad}E)_0$ $(n \geq 0)$ defined by $\phi_n := (\nabla_A^*\nabla_A + S/3)^{-1}\eta_n$ satisfies $\|\phi_n - \phi_m\|_{L^\infty} \leq \|\eta_n - \eta_m\|_T$. Hence it becomes a Cauchy sequence in $L^\infty(\Lambda^+(\mathrm{ad}E))$. Therefore ϕ_n converges to some ϕ_A in $L^\infty(\Lambda^+(\mathrm{ad}E))$. ϕ_A is continuous since every ϕ_n is continuous. Indeed we will see later that ϕ_A is smooth and satisfies the ASD equation $F^+(A + d_A^*\phi_A) = 0$.

We have $\eta_{n+1} = \Phi(\eta_n) = -2F_A^+ \cdot \phi_n - 2(d_A^*\phi_n \wedge d_A^*\phi_n)^+ - 2F_A^+$.

$$|2F_A^+ \cdot \phi_n|_T(x) \leq 2c_0 \int g(x,y)|F_A^+(y)||\phi_n(y)|d\mathrm{vol}(y),$$
$$\leq 2c_0|F_A^+|_T(x)\,\|\phi_n\|_{L^\infty} \leq 2c_0|F_A^+|_T(x)\,\|\eta_n\|_T\,.$$

$$|2(d_A^*\phi_n \wedge d_A^*\phi_n)^+|_T(x) \leq 2c_0\,\|\eta_n\|_T\,|\eta_n|_T(x) \quad \text{(Lemma 4.2)}.$$

Hence

$$|\eta_{n+1}|_T(x) \leq 2c_0\,\|\eta_n\|_T\,|F_A^+|_T(x) + 2c_0\,\|\eta_n\|_T\,|\eta_n|_T(x) + 2|F_A^+|_T(x).$$

Since $\|\eta_n\|_T \leq 3\varepsilon_0$,

$$|\eta_{n+1}|_T(x) \leq 6c_0\varepsilon_0|\eta_n|_T(x) + (6c_0\varepsilon_0 + 2)|F_A^+|_T(x).$$

By (13),

$$|\eta_n|_T(x) \leq \frac{(6c_0\varepsilon_0 + 2)|F_A^+|_T(x)}{1 - 6c_0\varepsilon_0} \leq 3|F_A^+|_T(x).$$

Recall that F_A^+ is supported in $\{T < |t| < T+1\}$. Set

$$\delta(x) := \int_{T<|t|<T+1} g(x,y)d\mathrm{vol}(y) \quad (x \in X).$$

Then $|F_A^+|_T(x) \leq \delta(x)\left\|F_A^+\right\|_{L^\infty}$. Note that $\delta(x)$ vanishes at infinity because $g(x,y) \leq \mathrm{const}\cdot e^{-\sqrt{S/3}d(x,y)}$ for $d(x,y) \geq 1$. (See (15).) We get the following decay estimate.

Proposition 4.6. $|\phi_n(x)| \leq |\eta_n|_T(x) \leq 3\delta(x)\left\|F_A^+\right\|_{L^\infty}$. *Hence* $|\phi_A(x)| \leq 3\delta(x)\left\|F_A^+\right\|_{L^\infty}$. *In particular,* ϕ_A *vanishes at infinity.*

4.2. **Regularity and the behavior at the end.** From the definition of ϕ_n, we have

$$(\nabla_A^*\nabla_A + S/3)\phi_{n+1} = \eta_{n+1} = -2F_A^+ \cdot \phi_n - 2(d_A^*\phi_n \wedge d_A^*\phi_n)^+ - 2F_A^+. \tag{20}$$

Lemma 4.7. $\sup_{n\geq 1}\|\nabla_A\phi_n\|_{L^\infty} < +\infty$.

Proof. We use the rescaling argument of Donaldson [5, Section 2.4]. Recall that ϕ_n are uniformly bounded and uniformly go to zero at infinity (Proposition 4.6). Moreover $\|\nabla_A\phi_n\|_{L^\infty} < \infty$ for each $n \geq 1$ by Lemma 4.1. Suppose $\sup_{n\geq 1}\|\nabla_A\phi_n\|_{L^\infty} = +\infty$. Then there exists a sequence $n_1 < n_2 < n_3 < \cdots$ such that $R_k := \|\nabla_A\phi_{n_k}\|_{L^\infty}$ go to infinity and $R_k \geq \max_{1\leq n\leq n_k}\|\nabla_A\phi_n\|_{L^\infty}$. Since $|\nabla_A\phi_n|$ vanishes at infinity (see Lemma 4.1), we can take $x_k \in X$ satisfying $R_k = |\nabla_A\phi_{n_k}(x_k)|$. From the equation (20), $|\nabla_A^*\nabla_A\phi_{n_k}| \leq \mathrm{const}_A \cdot R_k^2$. Here "$\mathrm{const}_A$" means a positive constant depending on A (but independent of $k \geq 1$). Let $r_0 > 0$ be a positive number less than the injectivity radius of X. We consider the geodesic coordinate centered at x_k for each $k \geq 1$, and we take a bundle trivialization of E over each geodesic ball $B(x_k, r_0)$ by the exponential gauge centered at

x_k. Then we can consider ϕ_{n_k} as a vector-valued function in the ball $B(x_k, r_0)$. Under this setting, ϕ_{n_k} satisfies

$$\left|\sum_{i,j} g^{ij}_{(k)} \partial_i \partial_j \phi_{n_k}\right| \leq \mathrm{const}_A \cdot R_k^2 \quad \text{on } B(x_k, r_0), \tag{21}$$

where $(g^{ij}_{(k)}) = (g_{(k),ij})^{-1}$ and $g_{(k),ij}$ is the Riemannian metric tensor in the geodesic coordinate centered at x_k. (Indeed $S^3 \times \mathbb{R} = SU(2) \times \mathbb{R}$ is a Lie group. Hence we can take the geodesic coordinates so that $g_{(k),ij}$ are independent of k.) Set $\tilde{\phi}_k(x) := \phi_{n_k}(x/R_k)$. $\tilde{\phi}_k(x)$ is a vector-valued function defined over the $r_0 R_k$-ball in $\mathbb{R}^4$. $\tilde{\phi}_k$ ($k \geq 1$) satisfy $|\nabla \tilde{\phi}_k(0)| = 1$, and they are uniformly bounded. From (21), they satisfy

$$\left|\sum_{i,j} \tilde{g}^{ij}_{(k)} \partial_i \partial_j \tilde{\phi}_k\right| \leq \mathrm{const}_A,$$

where $\tilde{g}^{ij}_{(k)}(x) = g^{ij}_{(k)}(x/R_k)$. $\{\tilde{g}^{ij}_{(k)}\}_{k\geq 1}$ converges to δ^{ij} (the Kronecker delta) as $k \to +\infty$ in the $\mathcal{C}^\infty$-topology over compact subsets in $\mathbb{R}^4$. Hence there exists a subsequence $\{\tilde{\phi}_{k_l}\}_{l\geq 1}$ which converges to some $\tilde{\phi}$ in the $\mathcal{C}^1$-topology over compact subsets in $\mathbb{R}^4$. Since $|\nabla \tilde{\phi}_k(0)| = 1$, we have $|\nabla \tilde{\phi}(0)| = 1$.

If $\{x_{k_l}\}_{l\geq 1}$ is a bounded sequence, then $\{\tilde{\phi}_{k_l}\}$ has a subsequence which converges to a constant function uniformly over every compact subset because ϕ_n converges to ϕ_A (a continuous section) in the $\mathcal{C}^0$-topology ($= L^\infty$-topology) and $R_k \to \infty$. But this contradicts the above $|\nabla \tilde{\phi}(0)| = 1$. Hence $\{x_{k_l}\}$ is an unbounded sequence. Since ϕ_n uniformly go to zero at infinity, $\{\tilde{\phi}_{k_l}\}$ has a subsequence which converges to 0 uniformly over every compact subset. Then this also contradicts $|\nabla \tilde{\phi}(0)| = 1$. □

From Lemma 4.7 and the equation (20), the elliptic estimates show that ϕ_n converges to ϕ_A in the $\mathcal{C}^\infty$-topology over every compact subset in X. In particular, ϕ_A is smooth. (Indeed $\phi_A \in \Omega^+(\mathrm{ad}E)_0$ from Proposition 4.6.) From the equation (20),

$$(\nabla_A^* \nabla_A + S/3)\phi_A = -2F_A^+ \cdot \phi_A - 2(d_A^*\phi_A \wedge d_A^*\phi_A)^+ - 2F_A^+. \tag{22}$$

This implies that $A + d_A^*\phi_A$ is an ASD connection.

Lemma 4.1 shows $\lim_{x\to\infty} |\nabla_A \phi_n(x)| = 0$ for each n. Indeed we can prove a stronger result:

Lemma 4.8. *For each $\varepsilon > 0$, there exists a compact set $K \subset X$ such that for all n*

$$|\nabla_A \phi_n(x)| \leq \varepsilon \quad (x \in X \setminus K).$$

Therefore, $\lim_{x\to\infty} |\nabla_A \phi_A(x)| = 0$.

Proof. Suppose the statement is false. Then there are $\delta > 0$, a sequence $n_1 < n_2 < n_3 < \cdots$, and a sequence of points $x_1, x_2, x_3, \cdots$ in X which goes to infinity such that

$$|\nabla_A \phi_{n_k}(x_k)| \geq \delta \quad (k = 1, 2, 3, \cdots).$$

Let $x_k = (\theta_k, t_k) \in S^3 \times \mathbb{R} = X$. $|t_k|$ goes to infinity. We can suppose $|t_k| > T+2$.

Since A is flat in $|t| > T+1$, there exists a bundle trivialization $g : E|_{|t|>T+1} \to X_{|t|>T+1} \times SU(2)$ such that $g(A)$ is equal to the product connection. (Here $X_{|t|>T+1} = \{(\theta, t) \in S^3 \times \mathbb{R} |\, |t| > T+1\}$.) Set $\phi'_n := g(\phi_n)$. We have

$$(\nabla^*\nabla + S/3)\phi'_n = -2(d^*\phi'_{n-1} \wedge d^*\phi'_{n-1})^+ \quad (|t| > T+1),$$

where ∇ is defined by using the product connection on $X_{|t|>T+1} \times SU(2)$. From this equation and Lemma 4.7,

$$|(\nabla^*\nabla + S/3)\phi'_n| \le \mathrm{const} \quad (|t| > T+1),$$

where const is independent of n. We define $\varphi_k \in \Gamma(S^3 \times (-1,1), \Lambda^+ \otimes su(2))$ by $\varphi_k(\theta, t) := \phi'_{n_k}(\theta, t_k + t)$. We have $|(\nabla^*\nabla + S/3)\varphi_k| \le \mathrm{const}$. Since $|\phi'_n(x)| \le 3\delta(x) \left\|F_A^+\right\|_{L^\infty}$ and $|t_k| \to +\infty$, the sequence φ_k converges to 0 in $L^\infty(S^3 \times (-1,1))$. Using the elliptic estimate, we get $\varphi_k \to 0$ in $\mathcal{C}^1(S^3 \times [-1/2, 1/2])$. On the other hand, $|\nabla\varphi_k(\theta_k, 0)| = |\nabla_A \phi_{n_k}(\theta_k, t_k)| \ge \delta > 0$. This is a contradiction. □

Set

$$\eta_A := (\nabla_A^*\nabla_A + S/3)\phi_A = -2F_A^+ \cdot \phi_A - 2(d_A^*\phi_A \wedge d_A^*\phi_A)^+ - 2F_A^+. \tag{23}$$

This is contained in $\Omega^+(\mathrm{ad}E)_0$ (Lemma 4.8). The sequence η_n defined in Proposition 4.5 satisfies

$$\eta_{n+1} = -2F_A^+ \cdot \phi_n - 2(d_A^*\phi_n \wedge d_A^*\phi_n)^+ - 2F_A^+.$$

Corollary 4.9. *The sequence η_n converges to η_A in L^∞. In particular, $\|\eta_n - \eta_A\|_T \to 0$ as $n \to \infty$. Hence $\|\eta_A\|_T \le 3\varepsilon_0$. (Proposition 4.5.)*

Proof.

$$\eta_{n+1} - \eta_A = -2F_A^+ \cdot (\phi_n - \phi_A) + 2\{d_A^*(\phi_A - \phi_n) \wedge d_A^*\phi_A + d_A^*\phi_n \wedge d_A^*(\phi_A - \phi_n)\}^+.$$

Hence

$$|\eta_{n+1} - \eta_A| \le 2c_0 \left\|F_A^+\right\|_{L^\infty} \|\phi_n - \phi_A\|_{L^\infty} + 2c_0(|\nabla_A\phi_n| + |\nabla_A\phi_A|)|\nabla_A\phi_A - \nabla_A\phi_n|.$$

$\phi_n \to \phi_A$ in $L^\infty(X)$ and in $\mathcal{C}^\infty$ over every compact subset. Moreover $|\nabla_A\phi_n|$ are uniformly bounded and uniformly go to zero at infinity (Lemma 4.7 and Lemma 4.8). Then the above inequality implies that $\|\eta_{n+1} - \eta_A\|_{L^\infty}$ goes to 0. □

Lemma 4.10. $\|d_A d_A^*\phi_A\|_{L^\infty} < \infty$.

Proof. It is enough to prove $|d_A d_A^*\phi_A(\theta, t)| \le \mathrm{const}$ for $|t| > T+2$. Take a trivialization g of E over $|t| > T+1$ such that $g(A)$ is the product connection, and set $\phi' := g(\phi_A)$. This satisfies

$$(\nabla^*\nabla + S/3)\phi' = -2(d^*\phi' \wedge d^*\phi')^+ \quad (|t| > T+1).$$

Since $|\phi'|$ and $|\nabla\phi'|$ go to zero at infinity (Proposition 4.6 and Lemma 4.8), this shows (by using the elliptic estimates) that $|dd^*\phi'|$ is bounded. $\square$

Lemma 4.11.
$$\frac{1}{8\pi^2}\int_X |F(A+d_A^*\phi_A)|^2 d\mathrm{vol} = \frac{1}{8\pi^2}\int_X tr(F_A^2).$$
Recall that A is flat over $|t|>T+1$. Hence the right hand side is finite. (Indeed it is a non-negative integer by the Chern-Weil theory.)

Proof. Set $a := d_A^*\phi_A$ and $cs_A(a) := \frac{1}{8\pi^2}tr(2a\wedge F_A + a\wedge d_A a + \frac{2}{3}a^3)$. We have $\frac{1}{8\pi^2}tr(F(A+a)^2) - \frac{1}{8\pi^2}tr(F(A)^2) = dcs_A(a)$. Since $A+a$ is ASD, we have $|F(A+a)|^2 = tr(F(A+a)^2)$ and
$$\frac{1}{8\pi^2}\int_{|t|\le R} tr(F(A+a)^2) - \frac{1}{8\pi^2}\int_{|t|\le R} tr(F(A)^2) = \int_{t=R} cs_A(a) - \int_{t=-R} cs_A(a).$$
From Lemma 4.8, $|a| = |d_A^*\phi_A|$ goes to zero at infinity. From Lemma 4.10, $|d_A a| = |d_A d_A^*\phi_A|$ is bounded. F_A vanishes over $|t|>T+1$. Hence $|cs_A(a)|$ goes to zero at infinity. Thus the right-hand-side of the above equation goes to zero as $R\to\infty$. $\square$

4.3. **Conclusion of the construction.** The following is the conclusion of Section 4.1 and 4.2. This will be used in Section 5 and 7. (Notice that we have not so far used the condition (iii) in the beginning of Section 4.1.)

Proposition 4.12. *Let E be a principal $SU(2)$-bundle over X, and A be a connection on E satisfying $F_A = 0$ ($|t|>T+1$), $\mathrm{supp}F_A^+ \subset \{T<|t|<T+1\}$ and $\|F_A^+\|_T \le \varepsilon_0 = 1/1000$. Then we can construct $\phi_A \in \Omega^+(\mathrm{ad}E)_0$ satisfying the following conditions.*
(a) $A+d_A^*\phi_A$ is an ASD connection: $F^+(A+d_A^*\phi_A)=0$.
(b)
$$\frac{1}{8\pi^2}\int_X |F(A+d_A^*\phi_A)|^2 d\mathrm{vol} = \frac{1}{8\pi^2}\int_X tr(F_A^2).$$
(c) $|\phi_A(x)| \le 3\delta(x)\|F_A^+\|_{L^\infty}$, where $\delta(x) = \int_{T<|t|<T+1} g(x,y)d\mathrm{vol}(y)$.
(d) $\eta_A := (\nabla_A^*\nabla_A + S/3)\phi_A$ is contained in $\Omega^+(\mathrm{ad}E)_0$ and $\|\eta_A\|_T \le 3\varepsilon_0$.
Moreover this construction $(E,A)\mapsto \phi_A$ is gauge equivariant, i.e., if F is another principal $SU(2)$-bundle over X admitting a bundle map $g: E\to F$, then $\phi_{g(A)} = g(\phi_A)$.

Proof. The conditions (a), (b), (c), (d) have been already proved. The gauge equivariance is obvious by the construction of ϕ_A in Section 4.1. $\square$

4.4. **Interior estimate.** In the proof of the upper bound on the mean dimension, we need to use an "interior estimate" of ϕ_A (Lemma 4.14 below), which we investigate in this subsection. We use the argument of Donaldson [5, pp. 189-190]. Recall that $|F_A|\le d$ on $|t|\le T$ and $\|F_A^+\|_{L^\infty(X)} \le d'$ by the condition (iii) in the beginning of Section 4.1. We fix $r_0>0$ so that r_0 is less than the injectivity radius of $S^3\times\mathbb{R}$ (cf. the proof of Lemma 4.7).

Lemma 4.13. *For any $\varepsilon > 0$, there exists a constant $\delta_0 = \delta_0(d, \varepsilon) > 0$ depending only on d and ε such that the following statement holds. For any $\phi \in \Omega^+(\mathrm{ad}E)$ and any closed r_0-ball B contained in $S^3 \times [-T+1, T-1]$, if ϕ satisfies*

$$(\nabla_A^* \nabla_A + S/3)\phi = -2(d_A^*\phi \wedge d_A^*\phi)^+ \text{ over } B \text{ and } \|\phi\|_{L^\infty(B)} \leq \delta_0, \tag{24}$$

then we have

$$\sup_{x \in B} |\nabla_A \phi(x)| d(x, \partial B) \leq \varepsilon.$$

Here $d(x, \partial B)$ is the distance between x and ∂B.

Proof. Suppose ϕ satisfies

$$\sup_{x \in B} |\nabla_A \phi(x)| d(x, \partial B) > \varepsilon,$$

and the supremum is attained at $x_0 \in B$ (x_0 is an inner point of B). Set $R := |\nabla_A \phi(x_0)|$ and $r_0' := d(x_0, \partial B)/2$. Let B' be the closed r_0'-ball centered at x_0. We have $|\nabla_A \phi| \leq 2R$ on B'. We consider the geodesic coordinate over B' centered at x_0, and we trivialize the bundle E over B' by the exponential gauge centered at x_0. Since A is ASD and $|F_A| \leq d$ over $-T \leq t \leq T$, the $\mathcal{C}^1$-norm of the connection matrix of A in the exponential gauge over B' is bounded by a constant depending only on d. From the equation (24) and $|\nabla_A \phi| \leq 2R$ on B',

$$\left| \sum g^{ij} \partial_i \partial_j \phi \right| \leq \mathrm{const}_{d,\varepsilon} \cdot R^2 \quad \text{over } B',$$

where $(g^{ij}) = (g_{ij})^{-1}$ and g_{ij} is the Riemannian metric tensor in the geodesic coordinate over B'. Here we consider ϕ as a vector valued function over B'. Set $\tilde{\phi}(x) := \phi(x/R)$. Since $2r_0'R > \varepsilon$, $\tilde{\phi}$ is defined over the $\varepsilon/2$-ball $B(\varepsilon/2)$ centered at the origin in $\mathbb{R}^4$, and it satisfies

$$\left| \sum \tilde{g}^{ij} \partial_i \partial_i \tilde{\phi} \right| \leq \mathrm{const}_{d,\varepsilon} \quad \text{over } B(\varepsilon/2).$$

Here $\tilde{g}^{ij}(x) := g^{ij}(x/R)$. The eigenvalues of the matrix $(\tilde{g}^{ij})$ are bounded from below by a positive constant depending only on the geometry of X, and the $\mathcal{C}^1$-norm of $\tilde{g}^{ij}$ is bounded from above by a constant depending only on ε and the geometry of X. (Note that $R > \varepsilon/(2r_0') \geq \varepsilon/(2r_0)$.) Then by using the elliptic estimate [9, Theorem 9.11] and the Sobolev embedding $L_2^8(B(\varepsilon/4)) \hookrightarrow \mathcal{C}^{1,1/2}(B(\varepsilon/4))$ (the Hölder space), we get

$$\|\tilde{\phi}\|_{\mathcal{C}^{1,1/2}(B(\varepsilon/4))} \leq \mathrm{const}_\varepsilon \cdot \|\tilde{\phi}\|_{L_2^8(B(\varepsilon/4))} \leq C = C(d, \varepsilon).$$

Hence $|\nabla \tilde{\phi}(x) - \nabla \tilde{\phi}(0)| \leq C|x|^{1/2}$ on $B(\varepsilon/4)$. Set $u := \nabla \tilde{\phi}(0)$. From the definition, we have $|u| = 1$.

$$\tilde{\phi}(tu) - \tilde{\phi}(0) = t \int_0^1 \nabla \tilde{\phi}(tsu) \cdot u ds = t + t \int_0^1 (\nabla \tilde{\phi}(tsu) - u) \cdot u ds.$$

Hence

$$|\tilde{\phi}(tu) - \tilde{\phi}(0)| \geq t - t \int_0^1 C|tsu|^{1/2} ds = t - 2Ct^{3/2}/3.$$

We can suppose $C \geq 2/\sqrt{\varepsilon}$. Then $u/C^2 \in B(\varepsilon/4)$ and

$$|\tilde{\phi}(u/C^2) - \tilde{\phi}(0)| \geq 1/(3C^2).$$

If $|\phi| \leq \delta_0 < 1/(6C^2)$, then this inequality becomes a contradiction. $\square$

The following will be used in Section 7.

Lemma 4.14. *For any $\varepsilon > 0$ there exists a positive integer $D = D(d, d', \varepsilon)$ such that*

$$\|d_A^* \phi_A\|_{L^\infty(S^3 \times [-T+D, T-D])} \leq \varepsilon.$$

(If $D > T$, then $S^3 \times [-T+D, T-D]$ is the empty set.) Here the important point is that D is independent of T.

Proof. Note that $|d_A^* \phi_A| \leq \sqrt{3/2}|\nabla_A \phi_A|$. We have $|\phi_A(x)| \leq 3d'\delta(x)$ by Proposition 4.12 (c) (or Proposition 4.6) and

$$\delta(x) = \int_{T<|t|<T+1} g(x,y) d\mathrm{vol}(y).$$

Set $D' := D - r_0$. (We choose D so that $D' \geq 1$.) Since $g(x,y) \leq \mathrm{const} \cdot e^{-\sqrt{S/3}d(x,y)}$ for $d(x,y) \geq 1$, we have

$$\delta(x) \leq C \cdot e^{-\sqrt{S/3}D'} \quad \text{for } x \in S^3 \times [-T+D', T-D'].$$

We choose $D = D(d, d', \varepsilon) \geq r_0 + 1$ so that

$$3d'Ce^{-\sqrt{S/3}D'} \leq \delta_0(d, r_0\varepsilon\sqrt{2/3}).$$

Here $\delta_0(d, r_0\varepsilon\sqrt{2/3})$ is the positive constant introduced in Lemma 4.13. Note that this condition is independent of T. Then ϕ_A satisfies, for $x \in S^3 \times [-T+D', T-D']$,

$$|\phi_A(x)| \leq \delta_0(d, r_0\varepsilon\sqrt{2/3}).$$

ϕ_A satisfies $(\nabla_A^* \nabla_A + S/3)\phi_A = -2(d_A^*\phi_A \wedge d_A^*\phi_A)^+$ over $|t| \leq T$. Then Lemma 4.13 implies

$$|\nabla_A \phi_A(x)| \leq \varepsilon\sqrt{2/3} \quad \text{for } x \in S^3 \times [-T+D, T-D].$$

(Note that, for $x \in S^3 \times [-T+D, T-D]$, we have $B(x, r_0) \subset S^3 \times [-T+D', T-D']$ and hence $|\phi_A| \leq \delta_0(d, r_0\varepsilon\sqrt{2/3})$ over $B(x, r_0)$.) Then, for $x \in S^3 \times [-T+D, T-D]$,

$$|d_A^* \phi_A(x)| \leq \sqrt{3/2}|\nabla_A \phi_A(x)| \leq \varepsilon.$$

$\square$

5. Continuity of the perturbation

The purpose of this section is to show the continuity of the perturbation construction in Section 4. The conclusion of Section 5 is Proposition 5.6. As in Section 4, $X = S^3\times\mathbb{R}$, $T > 0$ is a positive integer, and $E \to X$ is a principal $SU(2)$-bundle. Let ρ be a flat connection on $E|_{|t|>T+1}$. ($E|_{|t|>T+1}$ is the restriction of E to $X_{|t|>T+1} = \{(\theta, t) \in S^3\times\mathbb{R}|\, |t| > T+1\}$.) We define $\mathcal{A}'$ as the set of connections A on E satisfying the following.
(i) $A|_{|t|>T+1} = \rho$, i.e., A coincides with ρ over $|t| > T + 1$.
(ii) F_A^+ is supported in $\{(\theta, t) \in S^3 \times \mathbb{R}| \,T < |t| < T + 1\}$.
(iii) $\|F_A^+\|_T \le \varepsilon_0 = 1/1000$.

By Proposition 4.12, for each $A \in \mathcal{A}'$, we have $\phi_A \in \Omega^+(\mathrm{ad}E)_0$ and $\eta_A := (\nabla_A^*\nabla_A + S/3)\phi_A \in \Omega^+(\mathrm{ad}E)_0$ satisfying

$$\eta_A = -2F_A^+\cdot\phi_A - 2(d_A^*\phi_A\wedge d_A^*\phi_A)^+ - 2F_A^+,\quad \|\eta_A\|_T \le 3\varepsilon_0. \tag{25}$$

The first equation in the above is equivalent to the ASD equation $F^+(A + d_A^*\phi_A) = 0$. Since $\phi_A = (\nabla_A^*\nabla_A + S/3)^{-1}\eta_A$, we have ((18) and Lemma 4.2)

$$\|\phi_A\|_{L^\infty} \le \|\eta_A\|_T \le 3\varepsilon_0,\quad \left\||\nabla_A\phi_A|^2\right\|_T \le \|\eta_A\|_T^2 \le 9\varepsilon_0^2.$$

Then (by the Cauchy-Schwartz inequality)

$$\|\nabla_A\phi_A\|_T := \sup_{x\in X}\int_X g(x,y)|\nabla_A\phi_A(y)|d\mathrm{vol}(y) \le 3\varepsilon_0\sqrt{K},$$

where $K = \int_X g(x,y)d\mathrm{vol}(y)$.

Let $A, B \in \mathcal{A}'$. We want to estimate $\|\phi_A - \phi_B\|_{L^\infty}$. Set $a := B - A$. Since both A and B coincide with ρ (the fixed flat connection) over $|t| > T + 1$, a is compact-supported. We set

$$\|a\|_{\mathcal{C}^1_A} := \|a\|_{L^\infty} + \|\nabla_A a\|_{L^\infty}.$$

We suppose

$$\|a\|_{\mathcal{C}^1_A} \le 1.$$

Lemma 5.1. $\|\phi_A - \phi_B\|_{L^\infty} \le \|\eta_A - \eta_B\|_T + \mathrm{const}\,\|a\|_{\mathcal{C}^1_A}$, *where* const *is an universal constant independent of A, B.*

Proof. We have $\eta_A = (\nabla_A^*\nabla_A + S/3)\phi_A$ and

$$\eta_B = (\nabla_B^*\nabla_B + S/3)\phi_B = (\nabla_A^*\nabla_A + S/3)\phi_B + (\nabla_A^* a)*\phi_B + a*\nabla_B\phi_B + a*a*\phi_B,$$

where $*$ are algebraic multiplications. Then

$$\begin{aligned}\|\phi_A - \phi_B\|_{L^\infty} &\le \|(\nabla_A^*\nabla_A + S/3)(\phi_A - \phi_B)\|_T\,,\\ &\le \|\eta_A - \eta_B\|_T + \mathrm{const}\left(\|\nabla_A a\|_{L^\infty}\|\phi_B\|_T + \|a\|_{L^\infty}\|\nabla_B\phi_B\|_T + \|a\|_{L^\infty}^2\|\phi_B\|_T\right),\\ &\le \|\eta_A - \eta_B\|_T + \mathrm{const}\,\|a\|_{\mathcal{C}^1_A}\,.\end{aligned}$$

□

Lemma 5.2.

$$\left\|(d_A^*\phi_A\wedge d_A^*\phi_A)^+ - (d_B^*\phi_B\wedge d_B^*\phi_B)^+\right\|_T \leq \left(\frac{1}{4}+\text{const}\,\|a\|_{\mathcal{C}^1_A}\right)\|\eta_A-\eta_B\|_T + \text{const}\,\|a\|_{\mathcal{C}^1_A}.$$

Proof.

$$\begin{aligned}(d_A^*\phi_A\wedge & d_A^*\phi_A)^+ - (d_B^*\phi_B\wedge d_B^*\phi_B)^+ = \\ & \underbrace{(d_A^*\phi_A\wedge d_A^*\phi_A)^+ - (d_A^*\phi_B\wedge d_A^*\phi_B)^+}_{(I)} + \underbrace{(d_A^*\phi_B\wedge d_A^*\phi_B)^+ - (d_B^*\phi_B\wedge d_B^*\phi_B)^+}_{(II)}.\end{aligned}$$

We first estimate the term (II). Since $B = A+a$,

$$\begin{aligned}(d_B^*\phi_B\wedge d_B^*\phi_B)^+ &- (d_A^*\phi_B\wedge d_A^*\phi_B)^+ = \\ &(d_A^*\phi_B\wedge(a*\phi_B))^+ + ((a*\phi_B)\wedge d_A^*\phi_B)^+ + ((a*\phi_B)\wedge(a*\phi_B))^+.\end{aligned}$$

$$\begin{aligned}\|(II)\|_T &\leq \text{const}\,\|\nabla_A\phi_B\|_T\,\|a\|_{L^\infty}\,\|\phi_B\|_{L^\infty} + \text{const}\,\|a\|_{L^\infty}^2\,\|\phi_B\|_{L^\infty}^2, \\ &\leq \text{const}\cdot\|\nabla_A\phi_B\|_T\,\|a\|_{L^\infty} + \text{const}\cdot\|a\|_{L^\infty}.\end{aligned}$$

We have

$$\|\nabla_A\phi_B\|_T = \|\nabla_B\phi_B + a*\phi_B\|_T \leq \|\nabla_B\phi_B\|_T + \text{const}\,\|a\|_{L^\infty}\,\|\phi_B\|_{L^\infty} \leq \text{const}.$$

Hence $\|(II)\|_T \leq \text{const}\,\|a\|_{L^\infty}$.

Next we estimate the term (I). For $\eta_1,\eta_2\in\Omega^+(\mathrm{ad}E)_0$, set $\phi_i := (\nabla_A^*\nabla_A + S/3)^{-1}\eta_i \in \Omega^+(\mathrm{ad}E)_0$, and define (see (19))

$$\beta_A(\eta_1,\eta_2) := (d_A^*\phi_1\wedge d_A^*\phi_2)^+ + (d_A^*\phi_2\wedge d_A^*\phi_1)^+.$$

Set $\eta'_B := (\nabla_A^*\nabla_A + S/3)\phi_B = \eta_B + (\nabla_A^* a)*\phi_B + a*\nabla_B\phi_B + a*a*\phi_B$. Then $(d_A^*\phi_B\wedge d_A^*\phi_B)^+ = \beta_A(\eta'_B,\eta'_B)/2$ and $(I) = (\beta_A(\eta_A,\eta_A)-\beta_A(\eta'_B,\eta'_B))/2 = \beta_A(\eta_A+\eta'_B,\eta_A-\eta'_B)/2$. From Lemma 4.3,

$$\|(I)\|_T \leq 2c_0\,\|\eta_A+\eta'_B\|_T\,\|\eta_A-\eta'_B\|_T.$$

$\|\eta_A+\eta'_B\|_T \leq \|\eta_A+\eta_B\|_T + \|\eta'_B-\eta_B\|_T \leq 6\varepsilon_0 + \text{const}\,\|a\|_{\mathcal{C}^1_A}$, and $\|\eta_A-\eta'_B\|_T \leq \|\eta_A-\eta_B\|_T + \text{const}\,\|a\|_{\mathcal{C}^1_A}$. From (13), we have $12c_0\varepsilon_0 \leq 1/4$. Then

$$\|(I)\|_T \leq \left(\frac{1}{4}+\text{const}\,\|a\|_{\mathcal{C}^1_A}\right)\|\eta_A-\eta_B\|_T + \text{const}\,\|a\|_{\mathcal{C}^1_A}.$$

□

We have $F_B^+ = F_A^+ + d_A^+ a + (a\wedge a)^+$. Recall that we have supposed $\|a\|_{\mathcal{C}^1_A}\leq 1$. Hence

$$|F_B^+ - F_A^+| \leq \text{const}\,\|a\|_{\mathcal{C}^1_A}.$$

Proposition 5.3. *There exists $\delta>0$ such that if $\|a\|_{\mathcal{C}^1_A}\leq\delta$ then*

$$\|\eta_A-\eta_B\|_T \leq \text{const}\,\|a\|_{\mathcal{C}^1_A}.$$

Proof. From (25),

$$\begin{aligned}\eta_A - \eta_B = &2(F_B^+ - F_A^+)\cdot\phi_B + 2F_A^+\cdot(\phi_B - \phi_A)\\ &+ 2((d_B^*\phi_B\wedge d_B^*\phi_B)^+ - (d_A^*\phi_A\wedge d_A^*\phi_A)^+) + 2(F_B^+ - F_A^+)\end{aligned}$$

Using $\|\phi_B\|_{L^\infty} \le 3\varepsilon_0$, $\left\|F_A^+\right\|_T \le \varepsilon_0$ and Lemma 5.2,

$$\|\eta_A - \eta_B\|_T \le \text{const}\,\|a\|_{\mathcal{C}^1_A} + 2c_0\varepsilon_0\,\|\phi_A - \phi_B\|_{L^\infty} + \left(\frac{1}{2} + \text{const}\,\|a\|_{\mathcal{C}^1_A}\right)\|\eta_A - \eta_B\|_T\,.$$

Using Lemma 5.1,

$$\|\eta_A - \eta_B\|_T \le \text{const}\,\|a\|_{\mathcal{C}^1_A} + \left(\frac{1}{2} + \text{const}\,\|a\|_{\mathcal{C}^1_A} + 2c_0\varepsilon_0\right)\|\eta_A - \eta_B\|_T\,.$$

From (13), we can choose $\delta > 0$ so that if $\|a\|_{\mathcal{C}^1_A} \le \delta$ then

$$\left(\frac{1}{2} + \text{const}\,\|a\|_{\mathcal{C}^1_A} + 2c_0\varepsilon_0\right) \le 3/4.$$

Then we get

$$\|\eta_A - \eta_B\|_T \le \text{const}\,\|a\|_{\mathcal{C}^1_A} + (3/4)\,\|\eta_A - \eta_B\|_T\,.$$

Then $\|\eta_A - \eta_B\|_T \le \text{const}\,\|a\|_{\mathcal{C}^1_A}$. □

From Lemma 5.1, we get (under the condition $\|a\|_{\mathcal{C}^1_A} \le \delta$)

$$\|\phi_A - \phi_B\|_{L^\infty} \le \|\eta_A - \eta_B\|_T + \text{const}\,\|a\|_{\mathcal{C}^1_A} \le \text{const}\,\|a\|_{\mathcal{C}^1_A}\,.$$

Therefore we get the following.

Corollary 5.4. *The map*

$$(\mathcal{A}', \mathcal{C}^1\textit{-topology}) \to (\Omega^+(\mathrm{ad}E), \|\cdot\|_{L^\infty}), \quad A \mapsto \phi_A,$$

is continuous.

Let A_n ($n \ge 1$) be a sequence in $\mathcal{A}'$ which converges to $A \in \mathcal{A}'$ in the $\mathcal{C}^1$-topology: $\|A_n - A\|_{\mathcal{C}^1_A} \to 0$ ($n \to \infty$). By Corollary 5.4, we get $\|\phi_{A_n} - \phi_A\|_{L^\infty} \to 0$. Set $a_n := A_n - A$.

Lemma 5.5. $\sup_{n\ge1}\|\nabla_{A_n}\phi_{A_n}\|_{L^\infty} < \infty$. *(Equivalently,* $\sup_{n\ge1}\|\nabla_A\phi_{A_n}\|_{L^\infty} < \infty$.*)*

Proof. Note that $|\nabla_{A_n}\phi_{A_n}|$ vanishes at infinity (see Lemma 4.8). Hence we can take a point $x_n \in S^3\times\mathbb{R}$ satisfying $|\nabla_{A_n}\phi_{A_n}(x_n)| = \|\nabla_{A_n}\phi_{A_n}\|_{L^\infty}$. ϕ_{A_n} uniformly converge to ϕ_A and uniformly go to zero at infinity (see Proposition 4.12 (c) or Proposition 4.6). Then the rescaling argument as in the proof of Lemma 4.7 shows the above statement. □

Since $(\nabla_{A_n}^*\nabla_{A_n} + S/3)\phi_{A_n} = -2F_{A_n}^+\cdot\phi_{A_n} - 2(d_{A_n}^*\phi_{A_n}\wedge d_{A_n}^*\phi_{A_n})^+ - 2F_{A_n}^+$,

$$\sup_{n\ge1}\left\|\nabla_{A_n}^*\nabla_{A_n}\phi_{A_n}\right\|_{L^\infty} < \infty.$$

We have $\nabla^*_{A_n}\nabla_{A_n}\phi_{A_n} = \nabla^*_A\nabla_A\phi_{A_n} + (\nabla^*_A a_n) * \phi_{A_n} + a_n * \nabla_{A_n}\phi_{A_n} + a_n * a_n * \phi_{A_n}$. Hence

$$\sup_{n\geq 1} \|\nabla^*_A\nabla_A\phi_{A_n}\|_{L^\infty} < \infty.$$

By the elliptic estimate, we conclude that ϕ_{A_n} converges to ϕ_A in $\mathcal{C}^1$ over every compact subset. Then we get the following conclusion. This will be used in Section 7.

Proposition 5.6. *Let $\{A_n\}_{n\geq 1}$ be a sequence in $\mathcal{A}'$ which converges to $A \in \mathcal{A}'$ in the $\mathcal{C}^1$-topology. Then ϕ_{A_n} converges to ϕ_A in the $\mathcal{C}^1$-topology over every compact subset in X. Therefore $d^*_{A_n}\phi_{A_n}$ converges to $d^*_A\phi_A$ in the $\mathcal{C}^0$-topology over every compact subset in X. Moreover, for any $n \geq 1$,*

$$\int_X |F(A_n + d^*_{A_n}\phi_{A_n})|^2 d\mathrm{vol} = \int_X |F(A + d^*_A\phi_A)|^2 d\mathrm{vol}.$$

(This means that no energy is lost at the end.)

Proof. The last statement follows from Proposition 4.12 (b) (or Lemma 4.11) and the fact that for any A and B in $\mathcal{A}'$ we have

$$\int_X tr(F_A^2) = \int_X tr(F_B^2).$$

This is because $trF_B^2 - trF_A^2 = dtr(2a \wedge F_A + a \wedge d_A a + \frac{2}{3}a^3)$ $(a = B - A)$, and both A and B coincide with the fixed flat connection ρ over $|t| > T + 1$. $\square$

6. Cut-off constructions

As we explained in Section 3, we need to define a 'cut-off' for $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d$. Section 6.1 is a preparation to define a cut-off construction, and we define it in Section 6.2.

Let $\delta_1 > 0$. We define $\delta_1' = \delta_1'(\delta_1)$ by

$$\delta_1' := \sup_{x\in S^3\times\mathbb{R}} \left(\int_{S^3\times(-\delta_1,\delta_1)} g(x,y) d\mathrm{vol}(y)\right).$$

Since we have $g(x,y) \leq \mathrm{const}/d(x,y)^2$ (see (14) and (15)),

$$\int_{d(x,y)\leq(\delta_1)^{1/4}} g(x,y)d\mathrm{vol}(y) \leq \mathrm{const}\int_0^{(\delta_1)^{1/4}} rdr = \mathrm{const}\sqrt{\delta_1},$$

$$\int_{\{d(x,y)\geq(\delta_1)^{1/4}\}\cap S^3\times(-\delta_1,\delta_1)} g(x,y)d\mathrm{vol}(y) \leq \mathrm{const}\cdot\delta_1\frac{1}{\sqrt{\delta_1}} = \mathrm{const}\sqrt{\delta_1}.$$

Hence $\delta_1' \leq \mathrm{const}\sqrt{\delta_1}$ (this calculation is due to [5, pp. 190-191]). In particular, we have $\delta_1' \to 0$ as $\delta_1 \to 0$. For $d \geq 0$, we choose $\delta_1 = \delta_1(d)$ so that $0 < \delta_1 < 1$ and $\delta_1' = \delta_1'(\delta_1(d))$ satisfies

$$(5 + 7d + d^2)\delta_1' \leq \varepsilon_0/4 = 1/(4000). \tag{26}$$

The reason of this choice will be revealed in Proposition 6.4.

6.1. **Gauge fixing on S^3 and gluing instantons.** Fix a point $\theta_0 \in S^3$ as in Section 1. Let $F := S^3 \times SU(2)$ be the product principal $SU(2)$-bundle over S^3. Let $\mathcal{A}_{S^3}$ be the space of framed connections (A, p) where A is a connection on F and $p \in F_{\theta_0}$. (F_{θ_0} is the fiber of F over $\theta_0 \in S^3$.) Let $\mathcal{G}$ be the gauge transformation group of F. $\mathcal{A}_{S^3}$ and $\mathcal{G}$ are equipped with the $\mathcal{C}^\infty$-topology. Set $\mathcal{B}_{S^3} := \mathcal{A}_{S^3}/\mathcal{G}$ (with the quotient topology), and let $\pi : \mathcal{A}_{S^3} \to \mathcal{B}_{S^3}$ be the natural projection. Note that $\mathcal{G}$ freely acts on $\mathcal{A}_{S^3}$.

Proposition 6.1. *Let $d \geq 0$. For any $(A, p) \in \mathcal{A}_{S^3}$ there exist a closed neighborhood $U_{(A,p)}$ of $[A, p]$ in $\mathcal{B}_{S^3}$ and a continuous map $\Phi_{(A,p)} : \pi^{-1}(U_{(A,p)}) \to \mathcal{G}$ such that, for any $(B, q) \in \pi^{-1}(U_{(A,p)})$, $g := \Phi_{(A,p)}(B, q)$ satisfies the following.*
(i) $g(B) = A + a$ with $\|a\|_{L^\infty} \leq \delta_1 = \delta_1(d)$. ($\delta_1$ is the positive constant chosen in the above (26).)
(ii) For any gauge transformation h of F, we have $\Phi_{(A,p)}(h(B, q)) = gh^{-1}$.

Proof. Let $H_A^0 := \{u \in \Omega^0(\mathrm{ad}F) | \, d_A u = 0\}$. The restriction map $H_A^0 \to (\mathrm{ad}F)_{\theta_0}$ is an injection. ($(\mathrm{ad}F)_{\theta_0}$ is the fiber of $\mathrm{ad}F$ over the point $\theta_0 \in S^3$.) Hence we can consider H_A^0 as a subspace of $(\mathrm{ad}F)_{\theta_0}$. Let $(H_A^0)^\perp \subset (\mathrm{ad}F)_{\theta_0}$ be a complement of H_A^0 in $(\mathrm{ad}F)_{\theta_0}$. ($(\mathrm{ad}F)_{\theta_0} = H_A^0 \oplus (H_A^0)^\perp$.) Let $\nu > 0$ be a small number, and we define $V \subset F_{\theta_0}$ by $V = \{e^u p | \, u \in (H_A^0)^\perp, |u| \leq \nu\}$. We take $\nu > 0$ so small that the map $\{u \in (H_A^0)^\perp | \, |u| \leq \nu\} \ni u \mapsto e^u p \in V$ becomes an embedding. V is a slice for the action of Γ_A (the isotropy group of A) on F_{θ_0} at p.

Let $\varepsilon > 0$ be sufficiently small, and we take a closed neighborhood $U_{(A,p)}$ of $[A, p]$ in $\mathcal{B}_{S^3}$ such that

$$U_{(A,p)} \subset \{[B, q] | \, \exists g\text{: gauge transformation of } F \text{ s.t. } \|g(B) - A\|_{L_1^4} + |gq - p| < \varepsilon\}.$$

The usual Coulomb gauge construction shows that, for each $(B, q) \in \pi^{-1}(U_{(A,p)})$, there uniquely exists a gauge transformation g such that $g(B) = A + a$ with $d_A^* a = 0$, $gq \in V$, and $\|a\|_{L_1^4} + |gq - p| \leq \mathrm{const} \cdot \varepsilon$. Since $L_1^4 \hookrightarrow L^\infty$, we have $\|a\|_{L^\infty} \leq \mathrm{const} \cdot \varepsilon \leq \delta_1$ for sufficiently small ε. We define $\Phi_{(A,p)}(B, q) := g$. Then the condition (i) is obviously satisfied, and the condition (ii) follows from the uniqueness of g. $\square$

Recall the settings in Section 1. Let $d \geq 0$. The moduli space $\mathcal{M}_d$ is the space of all the gauge equivalence classes $[\boldsymbol{A}, \boldsymbol{p}]$ where $\boldsymbol{A}$ is an ASD connection on $\boldsymbol{E} := X \times SU(2)$ satisfying $|F(\boldsymbol{A})| \leq d$ and $\boldsymbol{p}$ is a map from $\mathbb{Z}$ to $\boldsymbol{E}$ with $\boldsymbol{p}(n) \in \boldsymbol{E}_{(\theta_0, n)}$ for every $n \in \mathbb{Z}$.

We define $K_d \subset \mathcal{B}_{S^3}$ by

$$K_d := \{[\boldsymbol{A}|_{S^3 \times \{0\}}, \boldsymbol{p}(0)] \in \mathcal{B}_{S^3} | \, [\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d\},$$

where we identify $\boldsymbol{E}|_{S^3 \times \{0\}}$ with F. From the Uhlenbeck compactness [21, 22], $\mathcal{M}_d$ is compact, and hence K_d is also compact. Hence there exist $(A_1, p_1), (A_2, p_2), \cdots, (A_N, p_N) \in \mathcal{A}_{S^3}$ ($N = N(d)$) such that $K_d \subset \mathrm{Int}(U_{(A_1,p_1)}) \cup \cdots \cup \mathrm{Int}(U_{(A_N,p_N)})$ and $[A_i, p_i] \in K_d$ ($1 \leq i \leq N$). Here $\mathrm{Int}(U_{(A_i,p_i)})$ is the interior of the closed set $U_{(A_i,p_i)}$ introduced in

Proposition 6.1. Note that we can naturally identify K_d with the space $\{[\boldsymbol{A}|_{S^3\times\{n\}}, \boldsymbol{p}(n)] \in \mathcal{B}_{S^3} | \, [\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d\}$ for each integer n because $\mathcal{M}_d$ admits the natural $\mathbb{Z}$-action.

For the statement of the next proposition, we introduce a new notation. We denote $F\times\mathbb{R}$ as the pull-back of F by the natural projection $X = S^3\times\mathbb{R}\to S^3$. So $F\times\mathbb{R}$ is a principal $SU(2)$-bundle over X. Of course, we can naturally identify $F\times\mathbb{R}$ with $\boldsymbol{E}$, but here we use this notation for the later convenience.

Proposition 6.2. *For each $i = 1, 2, \cdots, N$ there exists a connection $\hat{A}_i$ on $F\times\mathbb{R}$ satisfying the following. (Recall $0<\delta_1<1$.)*
(i) $\hat{A}_i = A_i$ over $S^3\times[-\delta_1, \delta_1]$. Here A_i (a connection on $F\times\mathbb{R}$) means the pull-back of A_i (a connection on F) by the natural projection $X\to S^3$.
(ii) $F(\hat{A}_i)$ is supported in $S^3\times(-1,1)$.
(iii) $\left\| F^+(\hat{A}_i)|_{\delta_1<|t|<1}\right\|_T \le \varepsilon_0/4 = 1/(4000)$, where $F^+(\hat{A}_i)|_{\delta_1<|t|<1} = F^+(\hat{A}_i)\times 1_{\delta_1<|t|<1}$ and $1_{\delta_1<|t|<1}$ is the characteristic function of the set $\{(\theta,t)\in S^3\times\mathbb{R}|\, \delta_1<|t|<1\}$.

Proof. By using a cut-off function, we can construct a connection A_i' on $F\times\mathbb{R}$ such that $A_i' = A_i$ over $S^3\times[-\delta_1,\delta_1]$ and $\mathrm{supp}F(A_i')\subset S^3\times(-1,1)$. We can reduce the self-dual part of $F(A_i')$ by "gluing instantons" to A_i' over $\delta_1<|t|<1$. This technique is essentially well-known for the specialists in the gauge theory. For the detail, see Donaldson [5, pp. 190-199].

By the argument of [5, pp. 196-198], we get the following situation. For any $\varepsilon>0$, there exists a connection $\hat{A}_i$ satisfying the following. $\hat{A}_i = A_i' = A_i$ over $|t|\le\delta_1$, and $\mathrm{supp}F(\hat{A}_i)\subset S^3\times(-1,1)$. Moreover $F^+(\hat{A}_i) = F_1^+ + F_2^+$ over $\delta_1<|t|<1$ such that $|F_1^+|\le\varepsilon$ and

$$|F_2^+|\le \mathrm{const}, \quad \mathrm{vol}(\mathrm{supp}(F_2^+))\le\varepsilon,$$

where const is a positive constant depending only on A_i' and independent of ε. If we take ε sufficiently small, then

$$\left\| F^+(\hat{A}_i)|_{\delta_1<|t|<1}\right\|_T \le \varepsilon_0/4.$$

□

6.2. **Cut-off construction.** Let T be a positive integer. We define a closed subset $\mathcal{M}_{d,T}(i,j)\subset\mathcal{M}_d$ $(1\le i,j\le N = N(d))$ as the set of $[\boldsymbol{A},\boldsymbol{p}]\in\mathcal{M}_d$ satisfying $[\boldsymbol{A}|_{S^3\times\{T\}}, \boldsymbol{p}(T)]\in U_{(A_i,p_i)}$ and $[\boldsymbol{A}|_{S^3\times\{-T\}}, \boldsymbol{p}(-T)]\in U_{(A_j,p_j)}$. Here we naturally identify $E_T := \boldsymbol{E}|_{S^3\times\{T\}}$ and $E_{-T} := \boldsymbol{E}|_{S^3\times\{-T\}}$ with F, and (A_i, p_i) $(1\le i\le N)$ are the framed connections on F introduced in the previous subsection. We have

$$\mathcal{M}_d = \bigcup_{1\le i,j\le N}\mathcal{M}_{d,T}(i,j). \tag{27}$$

Of course, this decomposition depends on the parameter $T>0$. But N is independent of T. This is an important point. We will define a cut-off construction for each piece $\mathcal{M}_{d,T}(i,j)$.

Let $(\boldsymbol{A}, \boldsymbol{p})$ be a framed connection on $\boldsymbol{E}$ satisfying $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_{d,T}(i,j)$. Let $u_+ : \boldsymbol{E}|_{t\geq T} \to E_T \times [T, +\infty)$ be the temporal gauge of $\boldsymbol{A}$ with $u_+ = \mathrm{id}$ on $\boldsymbol{E}|_{S^3\times\{T\}} = E_T$. (See Donaldson [6, Chapter 2].) Here $\boldsymbol{E}|_{t\geq T}$ is the restriction of $\boldsymbol{E}$ to $S^3 \times [T, +\infty)$, and $E_T \times [T, +\infty)$ is the pull-back of E_T by the projection $S^3 \times [T, \infty) \to S^3 \times \{T\}$. We will repeatedly use these kinds of notations. In the same way, let $u_- : \boldsymbol{E}|_{t\leq -T} \to E_{-T} \times (-\infty, -T]$ be the temporal gauge of $\boldsymbol{A}$ with $u_- = \mathrm{id}$ on $\boldsymbol{E}|_{S^3\times\{-T\}} = E_{-T}$. We define $A(t)$ ($|t| \geq T$) by setting $A(t) := u_+(\boldsymbol{A})$ for $t \geq T$ and $A(t) := u_-(\boldsymbol{A})$ for $t \leq -T$. $A(t)$ becomes dt-part free. Since $\boldsymbol{A}$ is ASD, we have

$$\frac{\partial A(t)}{\partial t} = *_3 F(A(t))_3, \tag{28}$$

where $*_3$ is the Hodge star on $S^3\times\{t\}$ and $F(A(t))_3$ is the curvature of $A(t)$ as a connection on the 3-manifold $S^3 \times \{t\}$.

Since $[A(T), \boldsymbol{p}(T)] \in U_{(A_i,p_i)}$ and $[A(-T), \boldsymbol{p}(-T)] \in U_{(A_j,p_j)}$, we have the gauge transformations $g_+ := \Phi_{(A_i,p_i)}(A(T), \boldsymbol{p}(T))$ and $g_- := \Phi_{(A_j,p_j)}(A(-T), \boldsymbol{p}(-T))$ by Proposition 6.1. We consider g_+ (resp. g_-) as the gauge transformation of E_T (resp. E_{-T}). They satisfy

$$\|g_+(A(T)) - A_i\|_{L^\infty} \leq \delta_1, \quad \|g_-(A(-T)) - A_j\|_{L^\infty} \leq \delta_1. \tag{29}$$

We define a principal $SU(2)$-bundle $\boldsymbol{E}'$ over X by

$$\boldsymbol{E}' := \boldsymbol{E}|_{|t|<T+\delta_1/4} \sqcup E_T \times (T, +\infty) \sqcup E_{-T} \times (-\infty, -T)/\sim,$$

where the identification $\sim$ is given as follows. $\boldsymbol{E}|_{|t|<T+\delta_1/4}$ is identified with $E_T \times (T, +\infty)$ over the region $T < t < T + \delta_1/4$ by the map $g_+ \circ u_+ : \boldsymbol{E}|_{T<t<T+\delta_1/4} \to E_T \times (T, T + \delta_1/4)$. Here we consider g_+ as a gauge transformation of $E_T \times (T, T + \delta_1/4)$ by $g_+ : E_T \times (T, T + \delta_1/4) \to E_T \times (T, T + \delta_1/4)$, $(p, t) \mapsto (g_+(p), t)$. Similarly, we identify $\boldsymbol{E}|_{|t|<T+\delta_1/4}$ with $E_{-T} \times (-\infty, -T)$ over the region $-T - \delta/4 < t < -T$ by the map $g_- \circ u_- : \boldsymbol{E}|_{-T-\delta_1/4<t<-T} \to E_{-T} \times (-T - \delta_1/4, -T)$.

Let $\rho(t)$ be a smooth function on $\mathbb{R}$ such that $0 \leq \rho \leq 1$, $\rho = 0$ ($|t| \leq \delta_1/4$), $\rho = 1$ ($|t| \geq 3\delta_1/4$) and

$$|\rho'| \leq 4/\delta_1.$$

We define a (not necessarily ASD) connection $\boldsymbol{A}'$ on $\boldsymbol{E}'$ as follows. Over the region $|t| < T + \delta_1/4$ where $\boldsymbol{E}'$ is equal to $\boldsymbol{E}$, we set

$$\boldsymbol{A}' := \boldsymbol{A} \quad \text{on } \boldsymbol{E}|_{|t|<T+\delta_1/4}. \tag{30}$$

Over the region $t > T$, we set

$$\boldsymbol{A}' := (1 - \rho(t - T))g_+(A(t)) + \rho(t - T)\hat{A}_{i,T} \quad \text{on } E_T \times (T, +\infty), \tag{31}$$

where $\hat{A}_{i,T}$ is the pull-back of the connection $\hat{A}_i$ in Proposition 6.2 by the map $t \mapsto t - T$. So, in particular, $\hat{A}_{i,T} = A_i$ over $T - \delta_1 < t < T + \delta_1$ and $F(\hat{A}_{i,T}) = 0$ over $t > T + 1$.

(31) is compatible with (30) over $T < t < T + \delta_1/4$ where $\rho(t - T) = 0$. In the same way, over the region $t < -T$, we set

$$\boldsymbol{A}' := (1 - \rho(t + T))g_-(A(t)) + \rho(t + T)\hat{A}_{j,-T} \quad \text{on } E_{-T} \times (-\infty, -T).$$

We have $F(\boldsymbol{A}') = 0$ ($|t| \geq T + 1$).

Finally, we define $\boldsymbol{p}' : \{n \in \mathbb{Z} |\, |n| \leq T\} \to \boldsymbol{E}'$ by setting $\boldsymbol{p}'(n) := \boldsymbol{p}(n) \in \boldsymbol{E}_{(\theta_0,n)} = \boldsymbol{E}'_{(\theta_0,n)}$. Then we have constructed $(\boldsymbol{E}', \boldsymbol{A}', \boldsymbol{p}')$ from $(\boldsymbol{A}, \boldsymbol{p})$ with $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_{d,T}(i, j)$. It is routine to check that the gauge equivalence class of $(\boldsymbol{E}', \boldsymbol{A}', \boldsymbol{p}')$ depends only on the gauge equivalence class of $(\boldsymbol{A}, \boldsymbol{p})$. (We need Proposition 6.1 (ii) for the proof of this fact.) Hence the map $\mathcal{M}_{d,T}(i, j) \ni [\boldsymbol{A}, \boldsymbol{p}] \mapsto [\boldsymbol{E}', \boldsymbol{A}', \boldsymbol{p}']$ is well-defined.

Lemma 6.3.

$$|F^+(\boldsymbol{A}')| \leq 5 + 7d + d^2 \quad \text{on } T \leq |t| \leq T + \delta_1.$$

Proof. We consider the case $T < t \leq T + \delta_1$ where $\hat{A}_{i,T} = A_i$. We have $\boldsymbol{A}' = (1 - \rho)g_+(A(t)) + \rho A_i$, $\rho = \rho(t - T)$. Set $a := A_i - g_+(A(t))$. Then $\boldsymbol{A}' = g_+(A(t)) + \rho a$. We have

$$F^+(\boldsymbol{A}') = (\rho' dt \wedge a)^+ + \frac{\rho}{2}(F(A_i) + *_3 F(A_i) \wedge dt) + (\rho^2 - \rho)(a \wedge a)^+.$$

We have $|F(A_i)| \leq d$ and $|\rho'| \leq 4/\delta_1$. From (29), $|A_i - g_+(A(T))| \leq \delta_1$. From the ASD equation (28) and $|F(\boldsymbol{A})| \leq d$, $|A(t) - A(T)| \leq d|t - T| \leq d\delta_1$. Hence

$$|a| \leq |A_i - g_+(A(T))| + |g_+(A(T)) - g_+(A(t))| \leq (1 + d)\delta_1 \quad (T < t \leq T + \delta_1). \tag{32}$$

Therefore, for $T < t \leq T + \delta_1$,

$$|F^+(\boldsymbol{A}')| \leq 4(1 + d) + d + (1 + d)^2 = 5 + 7d + d^2.$$

□

Proposition 6.4. *$F(\boldsymbol{A}') = 0$ over $|t| \geq T + 1$, and $F^+(\boldsymbol{A}')$ is supported in $\{T < |t| < T + 1\}$. We have $|F(\boldsymbol{A}')| \leq d$ over $|t| \leq T$, and*

$$\left\|F^+(\boldsymbol{A}')\right\|_{L^\infty} \leq d', \quad \left\|F^+(\boldsymbol{A}')\right\|_T \leq \varepsilon_0 = 1/(1000), \tag{33}$$

where $d' = d'(d)$ is a positive constant depending only on d. Moreover

$$\frac{1}{8\pi^2}\int_X tr(F(\boldsymbol{A}')^2) \leq \frac{1}{8\pi^2}\int_{|t|\leq T} |F(\boldsymbol{A})|^2 d\mathrm{vol} + C_1(d) \leq \frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d).$$

Here $C_1(d)$ depends only on d.

Proof. The statements about the supports of $F(\boldsymbol{A}')$ and $F^+(\boldsymbol{A}')$ are obvious by the construction. Since $\boldsymbol{A}' = \boldsymbol{A}$ over $|t| \leq T$, $|F(\boldsymbol{A}')| \leq d$ over $|t| \leq T$. We have $\boldsymbol{A}' = \hat{A}_{i,T}$ for $t \geq T + \delta_1$ and $\boldsymbol{A}' = \hat{A}_{j,-T}$ for $t \leq -T - \delta_1$. Hence (from Lemma 6.3)

$$\left\|F^+(\boldsymbol{A}')\right\|_{L^\infty} \leq d' := \max\left(5 + 7d + d^2, \left\|F^+(\hat{A}_1)\right\|_{L^\infty}, \left\|F^+(\hat{A}_2)\right\|_{L^\infty} \cdots, \left\|F^+(\hat{A}_N)\right\|_{L^\infty}\right).$$

By using Lemma 6.3, (26) and Proposition 6.2 (iii) (note that $g(x,y)$ is invariant under the translations $t \mapsto t-T$ and $t \mapsto t+T$),

$$\left\| F^+(\boldsymbol{A}') \right\|_T \leq 2(5+7d+d^2)\delta_1' + \varepsilon_0/2 \leq \varepsilon_0.$$

We have $\boldsymbol{A}' = \boldsymbol{A}$ over $|t| \leq T$ and

$$F(\boldsymbol{A}') = (1-\rho) g_+ \circ u_+(F(\boldsymbol{A})) + \rho F(A_i) + \rho' dt \wedge a + (\rho^2 - \rho) a^2,$$

over $T < t < T + \delta_1$. Hence $|F(\boldsymbol{A}')| \leq \mathrm{const}_d$ over $T < |t| < T + \delta_1$ by using (32). Then the last statement can be easily proved. □

6.3. **Continuity of the cut-off.** Fix $1 \leq i, j \leq N$. Let $[\boldsymbol{A}_n, \boldsymbol{p}_n]$ $(n \geq 1)$ be a sequence in $\mathcal{M}_{d,T}(i,j)$ converging to $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_{d,T}(i,j)$ in the $\mathcal{C}^\infty$-topology over every compact subset in X. Let $[\boldsymbol{E}_n', \boldsymbol{A}_n', \boldsymbol{p}_n']$ (respectively $[\boldsymbol{E}', \boldsymbol{A}', \boldsymbol{p}']$) be the framed connections constructed by cutting off $[\boldsymbol{A}_n, \boldsymbol{p}_n]$ (respectively $[\boldsymbol{A}, \boldsymbol{p}]$) as in Section 6.2.

Lemma 6.5. *There are bundle maps $h_n : \boldsymbol{E}_n' \to \boldsymbol{E}'$ $(n \gg 1)$ such that $h_n(\boldsymbol{A}_n') = \boldsymbol{A}'$ for $|t| \geq T+1$ and $h_n(\boldsymbol{A}_n')$ converges to $\boldsymbol{A}'$ in the $\mathcal{C}^\infty$-topology over X (indeed, we will need only $\mathcal{C}^1$-convergence in the later argument), and that $h_n(\boldsymbol{p}_n'(k))$ converges to $\boldsymbol{p}'(k)$ for $|k| \leq T$.*

Proof. We can suppose that $\boldsymbol{A}_n$ converges to $\boldsymbol{A}$ in the $\mathcal{C}^\infty$-topology over $|t| \leq T+2$ and that $\boldsymbol{p}_n(k) \to \boldsymbol{p}(k)$ for $|k| \leq T$. Let $u_{+,n} : \boldsymbol{E}|_{t \geq T} \to E_T \times [T, +\infty)$ (resp. u_+) be the temporal gauge of $\boldsymbol{A}_n$ (resp. $\boldsymbol{A}$), and set $A_n(t) := u_{+,n}(\boldsymbol{A}_n)$ and $A(t) := u_+(\boldsymbol{A})$ for $t \geq T$. We set $g_{+,n} := \Phi_{(A_i,p_i)}(A_n(T), \boldsymbol{p}_n(T))$ and $g_+ := \Phi_{(A_i,p_i)}(A(T), \boldsymbol{p}(T))$.

$u_{+,n}$ converges to u_+ in the $\mathcal{C}^\infty$-topology over $T \leq t \leq T+1$, and $g_{+,n}$ converges to g_+ in the $\mathcal{C}^\infty$-topology. Hence there are $\chi_n \in \Gamma(S^3 \times [T, T+1], \mathrm{ad}E_T \times [T, T+1])$ $(n \gg 1)$ satisfying $g_+ \circ u_+ = e^{\chi_n} g_{+,n} \circ u_{+,n}$. $\chi_n \to 0$ in the $\mathcal{C}^\infty$-topology over $T \leq t \leq T+1$. Let φ be a smooth function on X such that $0 \leq \varphi \leq 1$, $\varphi = 1$ over $t \leq T + \delta_1$ and $\varphi = 0$ over $t \geq T+1$. We define $h_n : \boldsymbol{E}_n' \to \boldsymbol{E}'$ $(n \gg 1)$ as follows.
(i) In the case of $|t| < T + \delta_1/4$, we set $h_n := \mathrm{id} : \boldsymbol{E} \to \boldsymbol{E}$.
(ii) In the case of $t > T$, we set $h_n := e^{\varphi \chi_n} : E_T \times (T, +\infty) \to E_T \times (T, +\infty)$. This is compatible with the case (i).
(iii) In the case of $t < -T$, we define $h_n : E_{-T} \times (-\infty, -T) \to E_{-T} \times (-\infty, -T)$ in the same way as in the above (ii).

Then we can easily check that these h_n satisfy the required properties. □

7. Proofs of the upper bounds

7.1. **Proof of** $\dim(\mathcal{M}_d : \mathbb{Z}) < \infty$**.** As in Section 1, $\boldsymbol{E} = X \times SU(2)$ and $\mathcal{M}_d$ $(d \geq 0)$ is the space of the gauge equivalence classes $[\boldsymbol{A}, \boldsymbol{p}]$ where $\boldsymbol{A}$ is an ASD connection on $\boldsymbol{E}$

satisfying $\|F(\boldsymbol{A})\|_{L^\infty} \leq d$ and $\boldsymbol{p} : \mathbb{Z} \to \boldsymbol{E}$ is a map satisfying $\boldsymbol{p}(n) \in \boldsymbol{E}_{(\theta_0,n)}$ for every $n \in \mathbb{Z}$. We define a distance on $\mathcal{M}_d$ as follows. For $[\boldsymbol{A},\boldsymbol{p}], [\boldsymbol{B},\boldsymbol{q}] \in \mathcal{M}_d$, we set

$$\begin{aligned}&\mathrm{dist}([\boldsymbol{A},\boldsymbol{p}],[\boldsymbol{B},\boldsymbol{q}])\\ &:= \inf_{g:\boldsymbol{E}\to\boldsymbol{E}} \left\{ \sum_{n\geq 1} 2^{-n} \frac{\|g(\boldsymbol{A})-\boldsymbol{B}\|_{L^\infty(|t|\leq n)}}{1+\|g(\boldsymbol{A})-\boldsymbol{B}\|_{L^\infty(|t|\leq n)}} + \sum_{n\in\mathbb{Z}} 2^{-|n|}|g(\boldsymbol{p}(n))-\boldsymbol{q}(n)| \right\},\end{aligned}$$

where g runs over gauge transformations of $\boldsymbol{E}$, and $|t| \leq n$ means the region $\{(\theta,t) \in S^3 \times \mathbb{R}|\, |t| \leq n\}$. This distance is compatible with the topology of $\mathcal{M}_d$ introduced in Section 1. For $R = 1,2,3,\cdots$, we define an amenable sequence $\Omega_R \subset \mathbb{Z}$ by $\Omega_R = \{n \in \mathbb{Z}| -R \leq n \leq R\}$. We define $\mathrm{dist}_{\Omega_R}([\boldsymbol{A},\boldsymbol{p}],[\boldsymbol{B},\boldsymbol{q}])$ as in Section 2.1, i.e.,

$$\mathrm{dist}_{\Omega_R}([\boldsymbol{A},\boldsymbol{p}],[\boldsymbol{B},\boldsymbol{q}]) := \max_{k\in\Omega_R} \mathrm{dist}(k^*[\boldsymbol{A},\boldsymbol{p}], k^*[\boldsymbol{B},\boldsymbol{q}]),$$

where $k^*[\boldsymbol{A},\boldsymbol{p}] = [k^*\boldsymbol{A}, k^*\boldsymbol{p}]$ is the pull-back by $k : \boldsymbol{E} \to \boldsymbol{E}$.

Let $\varepsilon > 0$. We take a positive integer $L = L(\varepsilon)$ so that

$$\sum_{n>L} 2^{-n} < \frac{\varepsilon}{2(1+2\mathrm{Diam}(SU(2)))}. \tag{34}$$

We define $D = D(d,d',\varepsilon/4)$ as the positive integer introduced in Lemma 4.14, where $d' = d'(d)$ is the positive constant introduced in Proposition 6.4. We set $T = T(R,d,\varepsilon) = R + L + D$. T is a positive integer.

We have the decomposition $\mathcal{M}_d = \bigcup_{1\leq i,j\leq N} \mathcal{M}_{d,T}(i,j)$ $(N = N(d))$ as in Section 6.2. $\mathcal{M}_{d,T}(i,j)$ is the space of $[\boldsymbol{A},\boldsymbol{p}] \in \mathcal{M}_d$ satisfying $[\boldsymbol{A}|_{S^3\times\{T\}}, \boldsymbol{p}(T)] \in U_{(A_i,p_i)}$ and $[\boldsymbol{A}|_{S^3\times\{-T\}}, \boldsymbol{p}(-T)] \in U_{(A_j,p_j)}$. Fix $1 \leq i,j \leq N$. Let $(\boldsymbol{A},\boldsymbol{p})$ be a framed connection on $\boldsymbol{E}$ satisfying $[\boldsymbol{A},\boldsymbol{p}] \in \mathcal{M}_{d,T}(i,j)$. By the cut-off construction in Section 6.2, we have constructed $(\boldsymbol{E}',\boldsymbol{A}',\boldsymbol{p}')$ satisfying the following conditions (see Proposition 6.4). $\boldsymbol{E}'$ is a principal $SU(2)$-bundle over X, and $\boldsymbol{A}'$ is a connection on $\boldsymbol{E}'$ such that $F(\boldsymbol{A}') = 0$ for $|t| \geq T+1$, $F^+(\boldsymbol{A}')$ is supported in $\{T < |t| < T+1\}$, and that

$$\left\|F^+(\boldsymbol{A}')\right\|_T \leq \varepsilon_0, \quad \left\|F^+(\boldsymbol{A}')\right\|_{L^\infty} \leq d', \quad \|F(\boldsymbol{A}')\|_{L^\infty(|t|\leq T)} \leq d.$$

$\boldsymbol{p}'$ is a map from $\{n \in \mathbb{Z}| -T \leq n \leq T\}$ to $\boldsymbol{E}'$ satisfying $\boldsymbol{p}'(n) \in \boldsymbol{E}'_{(\theta_0,n)}$. We can identify $\boldsymbol{E}'$ with $\boldsymbol{E}$ over $|t| < T + \delta_1/4$ by the definition, and

$$\boldsymbol{A}'|_{|t|<T+\delta_1/4} = \boldsymbol{A}|_{|t|<T+\delta_1/4}, \quad \boldsymbol{p}'(n) = \boldsymbol{p}(n)\ (|n| \leq T). \tag{35}$$

$(\boldsymbol{E}',\boldsymbol{A}')$ satisfies the conditions (i), (ii), (iii) in the beginning of Section 4.1. Therefore, by using the perturbation argument in Section 4 (see Proposition 4.12), we can construct the ASD connection $\boldsymbol{A}'' := \boldsymbol{A}' + d^*_{\boldsymbol{A}'}\phi_{\boldsymbol{A}'}$ on $\boldsymbol{E}'$. By Lemma 4.14

$$|\boldsymbol{A} - \boldsymbol{A}''| = |\boldsymbol{A}' - \boldsymbol{A}''| \leq \varepsilon/4 \quad (|t| \leq T - D = R + L). \tag{36}$$

From Proposition 6.4 and Proposition 4.12 (b),

$$\begin{aligned}\frac{1}{8\pi^2}\int_X |F(\boldsymbol{A}'')|^2 d\mathrm{vol} &= \frac{1}{8\pi^2}\int_X tr(F(\boldsymbol{A}')^2) \\ &\leq \frac{1}{8\pi^2}\int_{|t|\leq T} |F(\boldsymbol{A})|^2 d\mathrm{vol} + C_1(d) \leq \frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d),\end{aligned} \tag{37}$$

where $C_1(d)$ is a positive constant depending only on d. Since the cut-off and perturbation constructions are gauge equivariant (see Proposition 4.12), the gauge equivalence class $[\boldsymbol{E}', \boldsymbol{A}'', \boldsymbol{p}']$ depends only on the gauge equivalence class $[\boldsymbol{A}, \boldsymbol{p}]$. We set $F_{i,j}([\boldsymbol{A}, \boldsymbol{p}]) := [\boldsymbol{E}', \boldsymbol{A}'', \boldsymbol{p}']$.

For $c \geq 0$ we define $M_T(c)$ as the space of the gauge equivalence classes $[E, A, p]$ satisfying the following. E is a principal $SU(2)$-bundle over X, A is an ASD connection on E satisfying

$$\frac{1}{8\pi^2}\int_X |F_A|^2 d\mathrm{vol} \leq c,$$

and p is a map from $\{n \in \mathbb{Z} \,|\, |n| \leq T\}$ to E satisfying $p(n) \in E_{(\theta_0, n)}$ ($|n| \leq T$). The topology of $M_T(c)$ is defined as follows. A sequence $[E_n, A_n, p_n] \in M_T(c)$ ($n \geq 1$) converges to $[E, A, p] \in M_T(c)$ if the following two conditions are satisfied:
(i) $\int_X |F(A_n)|^2 d\mathrm{vol} = \int_X |F(A)|^2 d\mathrm{vol}$ for $n \gg 1$.
(ii) There are gauge transformations $g_n : E_n \to E$ ($n \gg 1$) such that for any compact set $K \subset X$ and any integer k with $|k| \leq T$ we have $\|g_n(A_n) - A\|_{\mathcal{C}^0(K)} \to 0$ and $g_n(p_n(k)) \to p(k)$ as $n \to \infty$.

Using the index theorem, we have

$$\dim M_T(c) \leq 8c - 3 + 3(2T+1) = 8c + 6T. \tag{38}$$

Here $\dim M_T(c)$ denotes the topological covering dimension of $M_T(c)$. By (37), we get the map

$$F_{i,j} : \mathcal{M}_{d,T}(i,j) \to M_T\left(\frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d)\right), \quad [\boldsymbol{A}, \boldsymbol{p}] \mapsto [\boldsymbol{E}', \boldsymbol{A}'', \boldsymbol{p}'].$$

Lemma 7.1. *For* $[\boldsymbol{A}_1, \boldsymbol{p}_1]$ *and* $[\boldsymbol{A}_2, \boldsymbol{p}_2]$ *in* $\mathcal{M}_{d,T}(i,j)$, *if* $F_{i,j}([\boldsymbol{A}_1, \boldsymbol{p}_1]) = F_{i,j}([\boldsymbol{A}_2, \boldsymbol{p}_2])$, *then*

$$\mathrm{dist}_{\Omega_R}([\boldsymbol{A}_1, \boldsymbol{p}_1], [\boldsymbol{A}_2, \boldsymbol{p}_2]) < \varepsilon.$$

Proof. From (35) and (36), there exists a gauge transformation g of $\boldsymbol{E}$ defined over $|t| < T + \delta_1/4$ such that $|g(\boldsymbol{A}_1) - \boldsymbol{A}_2| \leq \varepsilon/2$ over $|t| \leq R + L$ and $g(\boldsymbol{p}_1(n)) = \boldsymbol{p}_2(n)$ for $n \in \mathbb{Z}$ with $|n| \leq T$. There exists a gauge transformation $\tilde{g}$ of $\boldsymbol{E}$ defined all over X satisfying $\tilde{g} = g$ on $|t| \leq T$. Then we have $|\tilde{g}(\boldsymbol{A}_1) - \boldsymbol{A}_2| \leq \varepsilon/2$ on $|t| \leq R + L$ and $\tilde{g}(\boldsymbol{p}_1(n)) = \boldsymbol{p}_2(n)$

for $|n| \leq T$. For $k \in \Omega_R$, by using (34),

$$\begin{aligned}
&\mathrm{dist}(k^*[\boldsymbol{A}_1, \boldsymbol{p}_1], k^*[\boldsymbol{A}_2, \boldsymbol{p}_2]) \\
&\quad \leq \sum_{n\geq 1} 2^{-n} \frac{\|\tilde{g}(\boldsymbol{A}_1) - \boldsymbol{A}_2\|_{L^\infty(|t-k|\leq n)}}{1 + \|\tilde{g}(\boldsymbol{A}_1) - \boldsymbol{A}_2\|_{L^\infty(|t-k|\leq n)}} + \sum_{n\in\mathbb{Z}} 2^{-|n|} |\tilde{g}(\boldsymbol{p}_1(n+k)) - \boldsymbol{p}_2(n+k)| \\
&\quad \leq \sum_{n=1}^{L} 2^{-n}(\varepsilon/2) + \sum_{n>L} 2^{-n} + \sum_{|n|>L} 2^{-|n|}\mathrm{Diam}(SU(2)) \\
&\quad < (\varepsilon/2) + (1 + 2\mathrm{Diam}(SU(2))) \sum_{n>L} 2^{-n} < (\varepsilon/2) + (\varepsilon/2) = \varepsilon.
\end{aligned}$$

□

Lemma 7.2. *The map* $F_{i,j} : \mathcal{M}_{d,T}(i,j) \to M_T\left(\frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d)\right)$ *is continuous.*

Proof. Let $[\boldsymbol{A}_n, \boldsymbol{p}_n] \in \mathcal{M}_{d,T}(i,j)$ be a sequence converging to $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_{d,T}(i,j)$. From Lemma 6.5, there are bundle maps $h_n : \boldsymbol{E}'_n \to \boldsymbol{E}'$ ($n \gg 1$) such that $h_n(\boldsymbol{A}'_n) = \boldsymbol{A}'$ over $|t| \geq T+1$, $h_n(\boldsymbol{A}'_n)$ converges to $\boldsymbol{A}'$ in the $\mathcal{C}^\infty$-topology over X and that $h_n(\boldsymbol{p}'_n(k)) \to \boldsymbol{p}'(k)$ for $|k| \leq T$. Since the perturbation construction in Section 4 is gauge equivariant (Proposition 4.12), we have

$$(\boldsymbol{E}', h_n(\boldsymbol{A}'_n) + d^*_{h_n(\boldsymbol{A}'_n)}\phi_{h_n(\boldsymbol{A}'_n)}, h_n(\boldsymbol{p}'_n)) = h_n(\boldsymbol{E}'_n, \boldsymbol{A}''_n, \boldsymbol{p}'_n).$$

From Proposition 5.6, $d^*_{h_n(\boldsymbol{A}'_n)}\phi_{h_n(\boldsymbol{A}'_n)}$ converges to $d^*_{\boldsymbol{A}'}\phi_{\boldsymbol{A}'}$ in the $\mathcal{C}^0$-topology over every compact subset in X and

$$\int_X |F(h_n(\boldsymbol{A}'_n) + d^*_{h_n(\boldsymbol{A}'_n)}\phi_{h_n(\boldsymbol{A}'_n)})|^2 d\mathrm{vol} = \int_X |F(\boldsymbol{A}' + d^*_{\boldsymbol{A}'}\phi_{\boldsymbol{A}'})|^2 d\mathrm{vol} \quad \text{for } n \gg 1.$$

This shows $[\boldsymbol{E}'_n, \boldsymbol{A}''_n, \boldsymbol{p}'_n] = [\boldsymbol{E}', h_n(\boldsymbol{A}'_n) + d^*_{h_n(\boldsymbol{A}'_n)}\phi_{h_n(\boldsymbol{A}'_n)}, h_n(\boldsymbol{p}'_n)]$ converges to $[\boldsymbol{E}', \boldsymbol{A}'', \boldsymbol{p}']$ in $M_T\left(\frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d)\right)$. □

From Lemma 7.1 and 7.2, $F_{i,j}$ becomes an ε-embedding with respect to the distance dist_{Ω_R}. Hence

$$\mathrm{Widim}_\varepsilon(\mathcal{M}_{d,T}(i,j), \mathrm{dist}_{\Omega_R}) \leq \dim M_T\left(\frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d)\right).$$

Since $\mathcal{M}_d = \bigcup_{1\leq i,j\leq N} \mathcal{M}_{d,T}(i,j)$ (each $\mathcal{M}_{d,T}(i,j)$ is a closed set), by using Lemma 2.3, we get

$$\mathrm{Widim}_\varepsilon(\mathcal{M}_d, \mathrm{dist}_{\Omega_R}) \leq N^2 \dim M_T\left(\frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d)\right) + N^2 - 1.$$

From (38) and $T = R + L + D$,

$$\dim M_T\left(\frac{2Td^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d)\right) \leq \frac{2(R+L+D)d^2\mathrm{vol}(S^3)}{\pi^2} + 8C_1(d) + 6(R+L+D).$$

Since $N = N(d)$, $L = L(\varepsilon)$ and $D = D(d, d'(d), \varepsilon/4)$ are independent of R, we get

$$\mathrm{Widim}_\varepsilon(\mathcal{M}_d : \mathbb{Z}) = \lim_{R\to\infty} \frac{\mathrm{Widim}_\varepsilon(\mathcal{M}_d, \mathrm{dist}_{\Omega_R})}{|\Omega_R|} \le \frac{N^2 d^2 \mathrm{vol}(S^3)}{\pi^2} + 3N^2.$$

This holds for any $\varepsilon > 0$. Thus

$$\dim(\mathcal{M}_d : \mathbb{Z}) = \lim_{\varepsilon\to 0} \mathrm{Widim}_\varepsilon(\mathcal{M}_d : \mathbb{Z}) \le \frac{N^2 d^2 \mathrm{vol}(S^3)}{\pi^2} + 3N^2 < \infty.$$

7.2. **Upper bound on the local mean dimension.**

Lemma 7.3. *There exists $r_1 = r_1(d) > 0$ satisfying the following. For any $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d$ and $n \in \mathbb{Z}$, there exists an integer i $(1 \le i \le N)$ such that if $[\boldsymbol{B}, \boldsymbol{q}] \in \mathcal{M}_d$ satisfies $\mathrm{dist}_{\mathbb{Z}}([\boldsymbol{A}, \boldsymbol{p}], [\boldsymbol{B}, \boldsymbol{q}]) \le r_1$ then*

$$[\boldsymbol{B}|_{S^3\times\{n\}}, \boldsymbol{q}(n)] \in U_{(A_i, p_i)}.$$

Here we identifies $\boldsymbol{E}|_{S^3\times\{n\}}$ with F, and $U_{(A_i,p_i)}$ is the closed set introduced in Section 6.1. Recall $\mathrm{dist}_{\mathbb{Z}}([\boldsymbol{A}, \boldsymbol{p}], [\boldsymbol{B}, \boldsymbol{q}]) = \sup_{k\in\mathbb{Z}} \mathrm{dist}(k^*[\boldsymbol{A}, \boldsymbol{p}], k^*[\boldsymbol{B}, \boldsymbol{q}])$.

Proof. There exists $r_1 > 0$ (the Lebesgue number) satisfying the following. For any $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d$, there exists $i = i([\boldsymbol{A}, \boldsymbol{p}])$ such that if $[\boldsymbol{B}, \boldsymbol{q}] \in \mathcal{M}_d$ satisfies $\mathrm{dist}([\boldsymbol{A}, \boldsymbol{p}], [\boldsymbol{B}, \boldsymbol{q}]) \le r_1$ then $[\boldsymbol{B}|_{S^3\times\{0\}}, \boldsymbol{q}(0)] \in U_{(A_i,p_i)}$. If $\mathrm{dist}_{\mathbb{Z}}([\boldsymbol{A}, \boldsymbol{p}], [\boldsymbol{B}, \boldsymbol{q}]) \le r_1$, then for each $n \in \mathbb{Z}$ we have $\mathrm{dist}([n^*\boldsymbol{A}, n^*\boldsymbol{p}], [n^*\boldsymbol{B}, n^*\boldsymbol{q}]) \le r_1$ and hence

$$[\boldsymbol{B}|_{S^3\times\{n\}}, \boldsymbol{q}(n)] = [(n^*\boldsymbol{B})|_{S^3\times\{0\}}, (n^*\boldsymbol{q})(0)] \in U_{(A_i,p_i)},$$

for $i = i([n^*\boldsymbol{A}, n^*\boldsymbol{p}])$. □

Lemma 7.4. *For any $\varepsilon' > 0$, there exists $r_2 = r_2(\varepsilon') > 0$ such that if $[\boldsymbol{A}, \boldsymbol{p}]$ and $[\boldsymbol{B}, \boldsymbol{q}]$ in $\mathcal{M}_d$ satisfy* $\mathrm{dist}_{\mathbb{Z}}([\boldsymbol{A}, \boldsymbol{p}], [\boldsymbol{B}, \boldsymbol{q}]) \le r_2$ *then*

$$\left\| |F(\boldsymbol{A})|^2 - |F(\boldsymbol{B})|^2 \right\|_{L^\infty(X)} \le \varepsilon'.$$

Proof. The map $\mathcal{M}_d \ni [\boldsymbol{A}, \boldsymbol{p}] \mapsto |F(\boldsymbol{A})|^2 \in \mathcal{C}^0(S^3 \times [0,1])$ is continuous. Hence there exists $r_2 > 0$ such that if $\mathrm{dist}([\boldsymbol{A}, \boldsymbol{p}], [\boldsymbol{B}, \boldsymbol{q}]) \le r_2$ then

$$\left\| |F(\boldsymbol{A})|^2 - |F(\boldsymbol{B})|^2 \right\|_{L^\infty(S^3\times[0,1])} \le \varepsilon'.$$

Then for each $k \in \mathbb{Z}$, if $\mathrm{dist}(k^*[\boldsymbol{A}, \boldsymbol{p}], k^*[\boldsymbol{B}, \boldsymbol{q}]) \le r_2$,

$$\left\| |F(\boldsymbol{A})|^2 - |F(\boldsymbol{B})|^2 \right\|_{L^\infty(S^3\times[k,k+1])} \le \varepsilon'.$$

Therefore if $\mathrm{dist}_{\mathbb{Z}}([\boldsymbol{A}, \boldsymbol{p}], [\boldsymbol{B}, \boldsymbol{q}]) \le r_2$, then $\| |F(\boldsymbol{A})|^2 - |F(\boldsymbol{B})|^2 \|_{L^\infty(X)} \le \varepsilon'$. □

Let $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d$, and $\varepsilon, \varepsilon' > 0$ be arbitrary two positive numbers. There exists $T_0 = T_0([\boldsymbol{A}, \boldsymbol{p}], \varepsilon') > 0$ such that for any $T_1 \ge T_0$

$$\frac{1}{8\pi^2 T_1} \sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T_1]} |F(\boldsymbol{A})|^2 d\mathrm{vol} \le \rho(\boldsymbol{A}) + \varepsilon'/2.$$

The important point for the later argument is the following: We can arrange T_0 so that $T_0(k^*[\boldsymbol{A},\boldsymbol{p}],\varepsilon') = T_0([\boldsymbol{A},\boldsymbol{p}],\varepsilon')$ for all $k\in\mathbb{Z}$. We set

$$r = r(d,\varepsilon') = \min\left(r_1(d), r_2\left(\frac{4\pi^2\varepsilon'}{\mathrm{vol}(S^3)}\right)\right),$$

where $r_1(\cdot)$ and $r_2(\cdot)$ are the positive constants in Lemma 7.3 and 7.4. By Lemma 7.4, if $[\boldsymbol{B},\boldsymbol{q}]\in B_r([\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}}$ (the closed ball of radius r in $\mathcal{M}_d$ with respect to the distance $\mathrm{dist}_{\mathbb{Z}}$), then for any $T_1\geq T_0$

$$\frac{1}{8\pi^2 T_1}\sup_{t\in\mathbb{R}}\int_{S^3\times[t,t+T_1]}|F(\boldsymbol{B})|^2 d\mathrm{vol} \leq \rho(\boldsymbol{A})+\varepsilon'/2+\varepsilon'/2 \leq \rho(\boldsymbol{A})+\varepsilon'. \tag{39}$$

We define positive integers $L = L(\varepsilon)$ and $D = D(d,d'(d),\varepsilon/4)$ as in the previous subsection. ($L=L(\varepsilon)$ is a positive integer satisfying (34), and $D = D(d,d'(d),\varepsilon/4)$ is the positive integer introduced in Lemma 4.14.) Let R be an integer with $R\geq T_0$, and set $T := R+L+D$. By Lemma 7.3, there exist i,j $(1\leq i,j\leq N)$ depending on $[\boldsymbol{A},\boldsymbol{p}]$ and T such that all $[\boldsymbol{B},\boldsymbol{q}]\in B_r([\boldsymbol{A},\boldsymbol{q}])_{\mathbb{Z}}$ satisfy $[\boldsymbol{B}|_{S^3\times\{T\}},\boldsymbol{q}(T)]\in U_{(A_i,p_i)}$ and $[\boldsymbol{B}|_{S^3\times\{-T\}},\boldsymbol{q}(-T)]\in U_{(A_j,p_j)}$. (That is, $B_r([\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}}\subset\mathcal{M}_{d,T}(i,j)$.) As in the previous subsection, by using the cut-off construction and perturbation, for each $[\boldsymbol{B},\boldsymbol{q}]\in B_r([\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}}$ we can construct the framed ASD connection $[\boldsymbol{E}',\boldsymbol{B}'',\boldsymbol{q}']$. By (37), (39) and $T\geq T_0$,

$$\frac{1}{8\pi^2}\int_X |F(\boldsymbol{B}'')|^2 d\mathrm{vol} \leq \frac{1}{8\pi^2}\int_{|t|\leq T}|F(\boldsymbol{B})|^2 d\mathrm{vol} + C_1(d) \leq 2T(\rho(\boldsymbol{A})+\varepsilon')+C_1(d),$$

where $C_1(d)$ depends only on d. Therefore we get the map

$$B_r([\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}}\to M_T(2T(\rho(\boldsymbol{A})+\varepsilon')+C_1(d)),\quad [\boldsymbol{B},\boldsymbol{q}]\mapsto[\boldsymbol{E}',\boldsymbol{B}'',\boldsymbol{q}'].$$

This is an ε-embedding with respect to the distance dist_{Ω_R} by Lemma 7.1 and 7.2. Therefore we get (by (38))

$$\mathrm{Widim}_\varepsilon(B_r([\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}},\mathrm{dist}_{\Omega_R}) \leq 16T(\rho(\boldsymbol{A})+\varepsilon')+8C_1(d)+6T,$$

for $R\geq T_0([\boldsymbol{A},\boldsymbol{p}],\varepsilon')$ and $r=r(d,\varepsilon')$. As we pointed out before, we have $T_0(k^*[\boldsymbol{A},\boldsymbol{p}],\varepsilon') = T_0([\boldsymbol{A},\boldsymbol{p}],\varepsilon')$ for $k\in\mathbb{Z}$. Hence for all $k\in\mathbb{Z}$ and $R\geq T_0 = T_0([\boldsymbol{A},\boldsymbol{p}],\varepsilon'])$, we have the same upper bound on $\mathrm{Widim}_\varepsilon(B_r(k^*[\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}},\mathrm{dist}_{\Omega_R})$. Then for $R\geq T_0$,

$$\frac{1}{|\Omega_R|}\sup_{k\in\mathbb{Z}}\mathrm{Widim}_\varepsilon(B_r(k^*[\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}},\mathrm{dist}_{\Omega_R}) \leq \frac{16T(\rho(\boldsymbol{A})+\varepsilon')+8C_1(d)+6T}{2R+1}.$$

$T=R+L+D$. $L=L(\varepsilon)$ and $D=D(d,d'(d),\varepsilon/4)$ are independent of R. Hence

$$\begin{aligned}\mathrm{Widim}_\varepsilon&(B_r([\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}}\subset\mathcal{M}_d:\mathbb{Z})\\ &= \lim_{R\to\infty}\left(\frac{1}{|\Omega_R|}\sup_{k\in\mathbb{Z}}\mathrm{Widim}_\varepsilon(B_r(k^*[\boldsymbol{A},\boldsymbol{p}])_{\mathbb{Z}},\mathrm{dist}_{\Omega_R})\right)\\ &\leq 8(\rho(\boldsymbol{A})+\varepsilon')+3.\end{aligned}$$

Here we have used (6). This holds for any $\varepsilon > 0$. (Note that $r = r(d, \varepsilon')$ is independent of ε.) Hence

$$\begin{aligned}\dim(B_r([\boldsymbol{A}, \boldsymbol{p}])_{\mathbb{Z}} \subset \mathcal{M}_d : \mathbb{Z}) &= \lim_{\varepsilon \to 0} \operatorname{Widim}_\varepsilon(B_r([\boldsymbol{A}, \boldsymbol{p}])_{\mathbb{Z}} \subset \mathcal{M}_d : \mathbb{Z}) \\ &\le 8(\rho(\boldsymbol{A}) + \varepsilon') + 3.\end{aligned}$$

Since $\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) \le \dim(B_r([\boldsymbol{A}, \boldsymbol{p}])_{\mathbb{Z}} \subset \mathcal{M}_d : \mathbb{Z})$,

$$\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) \le 8(\rho(\boldsymbol{A}) + \varepsilon') + 3.$$

This holds for any $\varepsilon' > 0$. Thus

$$\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) \le 8\rho(\boldsymbol{A}) + 3.$$

Therefore we get the conclusion:

Theorem 7.5. *For any* $[\boldsymbol{A}, \boldsymbol{p}] \in \mathcal{M}_d$,

$$\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) \le 8\rho(\boldsymbol{A}) + 3.$$

8. Analytic preliminaries for the lower bound

8.1. **"Non-flat" implies "irreducible".** Note that the following trivial fact: if a smooth function u on $\mathbb{R}$ is bounded and convex ($u'' \ge 0$) then u is a constant function.

Lemma 8.1. *If a smooth function f on $S^3 \times \mathbb{R}$ is bounded, non-negative and sub-harmonic ($\Delta f \le 0$)*[2]*, then f is a constant function.*

Proof. We have $\Delta = -\partial^2/\partial t^2 + \Delta_{S^3}$ where t is the coordinate of the $\mathbb{R}$-factor of $S^3 \times \mathbb{R}$ and Δ_{S^3} is the Laplacian of S^3. We have

$$\frac{\partial^2}{\partial t^2} f^2 = 2\left(\frac{\partial f}{\partial t}\right)^2 + 2f\Delta_{S^3} f - 2f\Delta f.$$

Then we have

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\int_{S^3\times\{t\}} f^2 d\mathrm{vol} = \int_{S^3\times\{t\}}\left(\left|\frac{\partial f}{\partial t}\right|^2 + |\nabla_{S^3} f|^2 + f(-\Delta f)\right) d\mathrm{vol} \ge 0.$$

Here we have used $f \ge 0$ and $\Delta f \le 0$. This shows that $u(t) = \int_{S^3\times\{t\}} f^2$ is a bounded convex function. Hence it is a constant function. In particular $u'' \equiv 0$. Then the above formula implies $\partial f/\partial t \equiv \nabla_{S^3} f \equiv 0$. This means that f is a constant function. □

Lemma 8.2. *If A is a $U(1)$-ASD connection on $S^3 \times \mathbb{R}$ satisfying $\|F_A\|_{L^\infty} < \infty$, then A is flat.*

[2] Our convention of the sign of the Laplacian is geometric; we have $\Delta = -\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \partial^2/\partial x_3^2 - \partial^2/\partial x_4^2$ on $\mathbb{R}^4$

Proof. We have $F_A \in \sqrt{-1}\Omega^-$. The Weitzenböck formula (cf. (10)) gives $(\nabla^*\nabla + S/3)F_A = 2d^-d^*F_A = 0$. We have

$$\Delta|F_A|^2 = -2|\nabla F_A|^2 + 2(F_A, \nabla^*\nabla F_A) = -2|\nabla F_A|^2 - (2S/3)|F_A|^2 \le 0.$$

This shows that $|F_A|^2$ is a non-negative, bounded, subharmonic function. Hence it is a constant function. In particular $\Delta|F_A|^2 \equiv 0$. Then the above formula implies $F_A \equiv 0$. □

Corollary 8.3. *If A is a non-flat $SU(2)$-ASD connection on $S^3 \times \mathbb{R}$ satisfying $\|F_A\|_{L^\infty} < \infty$, then A is irreducible.*

8.2. **Periodic ASD connections.** Let $T > 0$ be a real number, $\underline{\mathrm{E}}$ be a principal $SU(2)$-bundle over $S^3 \times (\mathbb{R}/T\mathbb{Z})$, and $\underline{\mathrm{A}}$ be an ASD connection on $\underline{\mathrm{E}}$. Suppose $\underline{\mathrm{A}}$ is not flat. Let $\pi : S^3 \times \mathbb{R} \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ be the natural projection, and $E := \pi^*\underline{\mathrm{E}}$ and $A := \pi^*\underline{\mathrm{A}}$ be the pull-backs. Obviously A is a non-flat ASD connection satisfying $\|F_A\|_{L^\infty} < \infty$. Hence it is irreducible (Corollary 8.3). Some constants introduced below (e.g. C_2, C_3, ε_1) will depend on $(\underline{\mathrm{E}}, \underline{\mathrm{A}})$. But we consider that $(\underline{\mathrm{E}}, \underline{\mathrm{A}})$ is fixed, and hence the dependence on it will not be explicitly written.

Lemma 8.4. *There exists $C_2 > 0$ such that for any $u \in \Omega^0(\mathrm{ad}E)$*

$$\int_{S^3\times[0,T]} |u|^2 \le C_2 \int_{S^3\times[0,T]} |d_A u|^2.$$

Then, from the natural T-periodicity of A, for every $n \in \mathbb{Z}$

$$\int_{S^3\times[nT,(n+1)T]} |u|^2 \le C_2 \int_{S^3\times[nT,(n+1)T]} |d_A u|^2.$$

Proof. Since A is ASD and irreducible, the restriction of A to $S^3 \times (0,T)$ is also irreducible (by the unique continuation [7, Section 4.3.4]). Suppose the above is false, then there exist u_n $(n \ge 1)$ such that

$$1 = \int_{S^3\times[0,T]} |u_n|^2 > n \int_{S^3\times[0,T]} |d_A u_n|^2.$$

We can suppose that the restrictions of u_n to $S^3 \times (0,T)$ converge to some u weakly in $L^2_1(S^3 \times (0,T))$ and strongly in $L^2(S^3 \times (0,T))$. We have $\|u\|_{L^2} = 1$ (in particular $u \ne 0$) and $d_A u = 0$. This means that A is reducible over $S^3 \times (0,T)$. This is a contradiction. □

Let $N > 0$ be a large positive integer which will be fixed later, and set $R := NT$. Let φ be a smooth function on $\mathbb{R}$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on $[0,R]$, $\varphi = 0$ over $t \ge 2R$ and $t \le -R$, and $|\varphi'|, |\varphi''| \le 2/R$. Then for any $u \in \Omega^0(\mathrm{ad}E)$ (not necessarily compact supported),

$$\int_{S^3\times[0,R]} |d_A u|^2 \le \int_{S^3\times\mathbb{R}} |d_A(\varphi u)|^2 = \int_{S^3\times\mathbb{R}} (\Delta_A(\varphi u), \varphi u).$$

Here $\Delta_A := \nabla_A^*\nabla_A = -*d_A*d_A$ on $\Omega^0(\mathrm{ad}E)$. We have $\Delta_A(\varphi u) = \varphi\Delta_A u + \Delta\varphi\cdot u + *(*d\varphi\wedge d_A u - d\varphi\wedge *d_A u)$. Then $\Delta_A(\varphi u) = \Delta_A u$ over $S^3\times[0,R]$ and

$$|\Delta_A(\varphi u)| \leq (2/R)|u| + (4/R)|d_A u| + |\Delta_A u|.$$

Hence

$$\int_{S^3\times[0,R]}|d_A u|^2 \leq (2/R)\int_{t\in[-R,0]\cup[R,2R]}|u|^2 + (4/R)\int_{t\in[-R,0]\cup[R,2R]}|u||d_A u| + \int_{S^3\times[-R,2R]}|\Delta_A u||u|.$$

From Lemma 8.4,

$$\int_{t\in[-R,0]\cup[R,2R]}|u|^2 \leq C_2\int_{t\in[-R,0]\cup[R,2R]}|d_A u|^2,$$

$$\int_{t\in[-R,0]\cup[R,2R]}|u||d_A u| \leq \sqrt{\int_{t\in[-R,0]\cup[R,2R]}|u|^2}\sqrt{\int_{t\in[-R,0]\cup[R,2R]}|d_A u|^2} \leq \sqrt{C_2}\int_{t\in[-R,0]\cup[R,2R]}|d_A u|^2.$$

Hence

$$\int_{S^3\times[0,R]}|d_A u|^2 \leq \frac{2C_2+4\sqrt{C_2}}{R}\int_{t\in[-R,0]\cup[R,2R]}|d_A u|^2 + \int_{S^3\times[-R,2R]}|\Delta_A u||u|.$$

For a function (or a section of some Riemannian vector bundle) f on $S^3\times\mathbb{R}$ and $p\in[1,\infty]$, we set

$$\|f\|_{\ell^\infty L^p} := \sup_{n\in\mathbb{Z}}\|f\|_{L^p(S^3\times(nR,(n+1)R))}.$$

Then the above implies

$$\int_{S^3\times[0,R]}|d_A u|^2 \leq \frac{4C_2+8\sqrt{C_2}}{R}\|d_A u\|_{\ell^\infty L^2}^2 + 3\,\||\Delta_A u|\cdot|u|\|_{\ell^\infty L^1}.$$

In the same way, for any $n\in\mathbb{Z}$,

$$\int_{S^3\times[nR,(n+1)R]}|d_A u|^2 \leq \frac{4C_2+8\sqrt{C_2}}{R}\|d_A u\|_{\ell^\infty L^2}^2 + 3\,\||\Delta_A u|\cdot|u|\|_{\ell^\infty L^1}.$$

Then we have

$$\|d_A u\|_{\ell^\infty L^2}^2 \leq \frac{4C_2+8\sqrt{C_2}}{R}\|d_A u\|_{\ell^\infty L^2}^2 + 3\,\||\Delta_A u|\cdot|u|\|_{\ell^\infty L^1}.$$

We fix $N>0$ so that $(4C_2+8\sqrt{C_2})/R\leq 1/2$ (recall: $R=NT$). If $\|d_A u\|_{\ell^\infty L^2}<\infty$, then we get

$$\|d_A u\|_{\ell^\infty L^2}^2 \leq 6\,\||\Delta_A u|\cdot|u|\|_{\ell^\infty L^1}.$$

From Hölder's inequality and Lemma 8.4,

$$\||\Delta_A u|\cdot|u|\|_{\ell^\infty L^1} \leq \|\Delta_A u\|_{\ell^\infty L^2}\|u\|_{\ell^\infty L^2} \leq \sqrt{C_2}\,\|\Delta_A u\|_{\ell^\infty L^2}\|d_A u\|_{\ell^\infty L^2}.$$

Hence $\|d_A u\|_{\ell^\infty L^2} \leq 6\sqrt{C_2}\,\|\Delta_A u\|_{\ell^\infty L^2}$, and $\|u\|_{\ell^\infty L^2} \leq \sqrt{C_2}\,\|d_A u\|_{\ell^\infty L^2} \leq 6C_2\,\|\Delta_A u\|_{\ell^\infty L^2}$. Then we get the following conclusion.

Lemma 8.5. *There exists a constant $C_3 > 0$ such that, for any $u \in \Omega^0(\mathrm{ad}E)$ with $\|d_A u\|_{\ell^\infty L^2} < \infty$, we have*

$$\|u\|_{\ell^\infty L^2} + \|d_A u\|_{\ell^\infty L^2} \le C_3 \|\Delta_A u\|_{\ell^\infty L^2}.$$

Recall that we fixed a point $\theta_0 \in S^3$. The following result gives the "partial Coulomb gauge slice" in our situation.

Proposition 8.6. *There exists $\varepsilon_1 > 0$ satisfying the following. For any a and b in $\Omega^1(\mathrm{ad}E)$ satisfying $d_A^* a = d_A^* b = 0$ and $\|a\|_{L^\infty}, \|b\|_{L^\infty} \le \varepsilon_1$, if there is a gauge transformation g of E satisfying*

$$g(A+a) = A+b, \quad |g(\theta_0, n) - 1| \le \varepsilon_1 \ (\forall n \in \mathbb{Z}),$$

then $g = 1$ and $a = b$.

Proof. Since $g(A+a) = A+b$, we have $d_A g = ga - bg$. Then we have $|d_A g| \le 2\varepsilon_1$. From the condition $|g(\theta_0, n) - 1| \le \varepsilon_1$ $(n \in \mathbb{Z})$, we get $\|g-1\|_{L^\infty} \le \mathrm{const} \cdot \varepsilon_1 \ll 1$. Therefore there exists $u \in \Omega^0(\mathrm{ad}E)$ satisfying $g = e^u$ and $\|u\|_{L^\infty} \le \mathrm{const} \cdot \varepsilon_1$. We have

$$d_A e^u = d_A u + (d_A u \cdot u + u d_A u)/2! + (d_A u \cdot u^2 + u d_A u \cdot u + u^2 d_A u)/3! + \cdots.$$

Since $|u| \le \mathrm{const} \cdot \varepsilon_1 \ll 1$,

$$|d_A e^u| \ge |d_A u|(2 - e^{|u|}) \ge |d_A u|/2.$$

Hence $|d_A u| \le 2|d_A g| \le 4\varepsilon_1$. In the same way we get $|d_A g| \le 2|d_A u|$, and hence

$$\|d_A g\|_{\ell^\infty L^2} \le 2 \|d_A u\|_{\ell^\infty L^2} \le 2C_3 \|\Delta_A u\|_{\ell^\infty L^2}.$$

Here we have used Lemma 8.5. Since $d_A^* a = d_A^* b = 0$, we have

$$\Delta_A g = - * d_A * d_A g = - * (d_A g \wedge *a + *b \wedge d_A g).$$

Therefore

$$\|\Delta_A g\|_{\ell^\infty L^2} \le (\|a\|_{L^\infty} + \|b\|_{L^\infty}) \|d_A g\|_{\ell^\infty L^2} \le 4C_3 \varepsilon_1 \|\Delta_A u\|_{\ell^\infty L^2}. \tag{40}$$

A direct calculation shows $|\Delta_A u^n| \le n(n-1)|u|^{n-2}|d_A u|^2 + n|u|^{n-1}|\Delta_A u|$. Hence

$$|\Delta_A(e^u - u)| \le e^{|u|}|d_A u|^2 + (e^{|u|} - 1)|\Delta_A u| \le \mathrm{const} \cdot \varepsilon_1(|d_A u| + |\Delta_A u|).$$

Here we have used $|u|, |d_A u| \le \mathrm{const} \cdot \varepsilon_1 \ll 1$. Therefore

$$\|\Delta_A g - \Delta_A u\|_{\ell^\infty L^2} \le \mathrm{const} \cdot \varepsilon_1(\|d_A u\|_{\ell^\infty L^2} + \|\Delta_A u\|_{\ell^\infty L^2}).$$

Using Lemma 8.5, we get

$$\|\Delta_A g - \Delta_A u\|_{\ell^\infty L^2} \le \mathrm{const} \cdot \varepsilon_1 \|\Delta_A u\|_{\ell^\infty L^2}.$$

Then the inequality (40) gives

$$(1 - \mathrm{const} \cdot \varepsilon_1) \|\Delta_A u\|_{\ell^\infty L^2} \le 4C_3 \varepsilon_1 \|\Delta_A u\|_{\ell^\infty L^2}.$$

If we choose ε_1 so small that $1 - \mathrm{const}\cdot\varepsilon_1 > 4C_3\varepsilon_1$, then this estimate gives $\Delta_A u = 0$. Then we get (from Lemma 8.5) $u = 0$. This shows $g = 1$ and $a = b$. $\square$

The following "L^∞-estimate" will be used in the next section. For its proof, see Proposition A.5 in Appendix A.

Proposition 8.7. *Let ξ be a $\mathcal{C}^2$-section of $\Lambda^+(\mathrm{ad}E)$, and set $\eta := (\nabla_A^*\nabla_A + S/3)\xi$. If $\|\xi\|_{L^\infty}, \|\eta\|_{L^\infty} < \infty$, then*

$$\|\xi\|_{L^\infty} \le (24/S)\,\|\eta\|_{L^\infty}.$$

9. Proof of the lower bound

9.1. **Deformation of periodic ASD connections.** The argument in this subsection is a Yang-Mills analogue of the deformation theory developed in Tsukamoto [20]. Let d be a positive number. As in Section 8.2, let $T > 0$ be a positive real number, $\underline{\mathrm{E}}$ be a principal $SU(2)$-bundle over $S^3 \times (\mathbb{R}/T\mathbb{Z})$, and $\underline{\mathrm{A}}$ be an ASD connection on $\underline{\mathrm{E}}$. Suppose that $\underline{\mathrm{A}}$ is not flat and

$$\|F(\underline{\mathrm{A}})\|_{L^\infty} < d. \tag{41}$$

Set $E := \pi^*\underline{\mathrm{E}}$ and $A := \pi^*\underline{\mathrm{A}}$ where $\pi: S^3\times\mathbb{R} \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ is the natural projection. Some constants introduced below depend on $(\underline{\mathrm{E}}, \underline{\mathrm{A}})$. But we don't explicitly write the dependence on it because we consider that $(\underline{\mathrm{E}}, \underline{\mathrm{A}})$ is fixed.

We define the Banach space H_A^1 by setting

$$H_A^1 := \{a \in \Omega^1(\mathrm{ad}E) \,|\, (d_A^* + d_A^+)a = 0,\ \|a\|_{L^\infty} < \infty\}.$$

$(H_A^1, \|\cdot\|_{L^\infty})$ becomes an infinite dimensional Banach space. The additive group $T\mathbb{Z} = \{nT \in \mathbb{R} \,|\, n \in \mathbb{Z}\}$ acts on H_A^1 as follows. From the definition of E and A, we have $(T^*E, T^*A) = (E, A)$ where $T: S^3\times\mathbb{R} \to S^3\times\mathbb{R}, (\theta, t) \mapsto (\theta, t+T)$. Hence for any $a \in H_A^1$, we have $T^*a \in H_A^1$ and $\|T^*a\|_{L^\infty} = \|a\|_{L^\infty}$.

Fix $0 < \alpha < 1$. We want to define the Hölder space $\mathcal{C}^{k,\alpha}(\Lambda^+(\mathrm{ad}E))$ for $k \ge 0$. Let $\{U_\lambda\}_{\lambda=1}^\Lambda$, $\{U'_\lambda\}_{\lambda=1}^\Lambda$, $\{U''_\lambda\}_{\lambda=1}^\Lambda$ be finite open coverings of $S^3\times(\mathbb{R}/T\mathbb{Z})$ satisfying the following conditions.

(i) $\bar{U}_\lambda \subset U'_\lambda$ and $\bar{U}'_\lambda \subset U''_\lambda$. U_λ, U'_λ and U''_λ are connected, and their boundaries are smooth. Each U''_λ is a coordinate chart, i.e., a diffeomorphism between U''_λ and an open set in $\mathbb{R}^4$ is given for each λ.

(ii) The covering map $\pi: S^3\times\mathbb{R} \to S^3\times(\mathbb{R}/T\mathbb{Z})$ can be trivialized over each U''_λ, i.e., we have a disjoint union $\pi^{-1}(U''_\lambda) = \bigsqcup_{n\in\mathbb{Z}} U''_{n\lambda}$ such that $\pi: U''_{n\lambda} \to U''_\lambda$ is diffeomorphic. We set $U_{n\lambda} := U''_{n\lambda} \cap \pi^{-1}(U_\lambda)$ and $U'_{n\lambda} := U''_{n\lambda}\cap\pi^{-1}(U'_\lambda)$. We have $\pi^{-1}(U_\lambda) = \bigsqcup_{n\in\mathbb{Z}} U_{n\lambda}$ and $\pi^{-1}(U'_\lambda) = \bigsqcup_{n\in\mathbb{Z}} U'_{n\lambda}$.

(iii) A trivialization of the principal $SU(2)$-bundle $\underline{\mathrm{E}}$ over each U''_λ is given.

From the conditions (ii) and (iii), we have a coordinate system and a trivialization of E over each $U''_{n\lambda}$. Let u be a section of $\Lambda^i(\mathrm{ad}E)$ $(0 \le i \le 4)$ over $S^3\times\mathbb{R}$. Then $u|_{U''_{n\lambda}}$

can be seen as a vector-valued function over $U''_{n\lambda}$. Hence we can consider the Hölder norm $\|u\|_{\mathcal{C}^{k,\alpha}(\bar{U}_{n\lambda})}$ of u as a vector-valued function over $\bar{U}_{n\lambda}$ (cf. Gilbarg-Trudinger [9, Chapter 4]). We define the Hölder norm $\|u\|_{\mathcal{C}^{k,\alpha}}$ by setting

$$\|u\|_{\mathcal{C}^{k,\alpha}} := \sup_{n\in\mathbb{Z},1\leq\lambda\leq\Lambda} \|u\|_{\mathcal{C}^{k,\alpha}(\bar{U}_{n\lambda})}.$$

For $a \in H^1_A$, we have $\|a\|_{\mathcal{C}^{k,\alpha}} \leq \mathrm{const}_k \|a\|_{L^\infty} < \infty$ for every $k = 0, 1, 2, \cdots$ by the elliptic regularity. We define the Banach space $\mathcal{C}^{k,\alpha}(\Lambda^+(\mathrm{ad}E))$ as the space of sections u of $\Lambda^+(\mathrm{ad}E)$ satisfying $\|u\|_{\mathcal{C}^{k,\alpha}} < \infty$.

Consider the following map:

$$\Phi : H^1_A \times \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}E)) \to \mathcal{C}^{0,\alpha}(\Lambda^+(\mathrm{ad}E)), \quad (a, \phi) \mapsto F^+(A + a + d^*_A\phi).$$

This is a smooth map between the Banach spaces. Since $F^+(A + a) = (a \wedge a)^+$,

$$F^+(A + a + d^*_A\phi) = (a \wedge a)^+ + d^+_A d^*_A\phi + [a \wedge d^*_A\phi]^+ + (d^*_A\phi \wedge d^*_A\phi)^+. \tag{42}$$

The derivative of Φ with respect to the second variable ϕ at the origin $(0, 0)$ is given by

$$\partial_2\Phi_{(0,0)} = d^+_A d^*_A = \frac{1}{2}(\nabla^*_A\nabla_A + S/3) : \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}E)) \to \mathcal{C}^{0,\alpha}(\Lambda^+(\mathrm{ad}E)). \tag{43}$$

Here we have used the Weitzenböck formula (see (10)).

Proposition 9.1. *The map* $(\nabla^*_A\nabla_A + S/3) : \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}E)) \to \mathcal{C}^{0,\alpha}(\Lambda^+(\mathrm{ad}E))$ *is isomorphic.*

Proof. The injectivity follows from the L^∞-estimate of Proposition 8.7. So the problem is the surjectivity. First we prove the following lemma.

Lemma 9.2. *Suppose that* $\eta \in \mathcal{C}^{0,\alpha}(\Lambda^+(\mathrm{ad}E))$ *is compact-supported. Then there exists* $\phi \in \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}E))$ *satisfying* $(\nabla^*_A\nabla_A + S/3)\phi = \eta$ *and* $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leq \mathrm{const}\cdot\|\eta\|_{\mathcal{C}^{0,\alpha}}$.

Proof. Let $L^2_1 := \{\xi \in L^2(\Lambda^+(\mathrm{ad}E)) | \nabla_A\xi \in L^2\}$. For $\xi_1, \xi_2 \in L^2_1$, set $(\xi_1, \xi_2)_{S/3} := (S/3)(\xi_1, \xi_2)_{L^2} + (\nabla_A\xi_1, \nabla_A\xi_2)_{L^2}$. Since S is a positive constant, this inner product defines a norm equivalent to the standard L^2_1-norm. η defines a bounded linear functional $(\cdot, \eta)_{L^2} : L^2_1 \to \mathbb{R}$, $\xi \mapsto (\xi, \eta)_{L^2}$. From the Riesz representation theorem, there uniquely exists $\phi \in L^2_1$ satisfying $(\xi, \phi)_{S/3} = (\xi, \eta)_{L^2}$ for any $\xi \in L^2_1$. This implies that $(\nabla^*_A\nabla_A + S/3)\phi = \eta$ in the sense of distributions. Moreover we have $\|\phi\|_{L^2_1} \leq \mathrm{const}\,\|\eta\|_{L^2}$. From the elliptic regularity (see Gilbarg-Trudinger [9, Chapter 9]) and the Sobolev embedding $L^2_1 \hookrightarrow L^4$,

$$\begin{aligned}\|\phi\|_{L^4_2(U_{n\lambda})} &\leq \mathrm{const}_\lambda(\|\phi\|_{L^4(U'_{n\lambda})} + \|\eta\|_{L^4(U'_{n\lambda})}),\\ &\leq \mathrm{const}_\lambda(\|\phi\|_{L^2_1(U'_{n\lambda})} + \|\eta\|_{L^4(U'_{n\lambda})}),\\ &\leq \mathrm{const}_\lambda(\|\eta\|_{L^2} + \|\eta\|_{L^4}).\end{aligned}$$

Here const_λ are constants depending on $\lambda = 1, 2, \cdots, \Lambda$. The important point is that they are independent of $n \in \mathbb{Z}$. This is because we have the $T\mathbb{Z}$-symmetry of the equation. From the Sobolev embedding $L^4_2 \hookrightarrow L^\infty$, we have

$$\|\phi\|_{L^\infty} \leq \mathrm{const} \cdot \sup_{n,\lambda} \|\phi\|_{L^4_2(U_{n\lambda})} \leq \mathrm{const}(\|\eta\|_{L^2} + \|\eta\|_{L^4}) < \infty.$$

Using the Schauder interior estimate (see Gilbarg-Trudinger [9, Chapter 6]), we get

$$\|\phi\|_{\mathcal{C}^{2,\alpha}(\bar{U}_{n\lambda})} \leq \mathrm{const}_\lambda(\|\phi\|_{L^\infty} + \|\eta\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})}).$$

It is easy to see that

$$\sup_{n,\lambda} \|\eta\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})} \leq \mathrm{const}\, \|\eta\|_{\mathcal{C}^{0,\alpha}}\,. \tag{44}$$

(Recall $\|\eta\|_{\mathcal{C}^{0,\alpha}} = \sup_{n,\lambda} \|\eta\|_{\mathcal{C}^{0,\alpha}(\bar{U}_{n\lambda})}$.) Hence $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leq \mathrm{const}(\|\phi\|_{L^\infty} + \|\eta\|_{\mathcal{C}^{0,\alpha}}) < \infty$. □

Let $\eta \in \mathcal{C}^{0,\alpha}(\Lambda^+(\mathrm{ad}E))$ (not necessarily compact-supported). Let φ_k $(k = 1, 2, \cdots)$ be cut-off functions such that $0 \leq \varphi_k \leq 1$, $\varphi_k = 1$ over $|t| \leq k$ and $\varphi_k = 0$ over $|t| \geq k+1$. Set $\eta_k := \varphi_k \eta$. From the above Lemma 9.2, there exists $\phi_k \in \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}E))$ satisfying $(\nabla_A^*\nabla_A + S/3)\phi_k = \eta_k$. From the L^∞-estimate (Lemma 8.7), we get

$$\|\phi_k\|_{L^\infty} \leq (24/S)\,\|\eta_k\|_{L^\infty} \leq (24/S)\,\|\eta\|_{L^\infty}\,.$$

From the Schauder interior estimate, we get

$$\|\phi_k\|_{\mathcal{C}^{2,\alpha}(\bar{U}_{n\lambda})} \leq \mathrm{const}_\lambda \cdot (\|\phi_k\|_{L^\infty(U'_{n\lambda})} + \|\eta_k\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})}) \leq \mathrm{const}(\|\eta\|_{L^\infty} + \|\eta_k\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})}).$$

We have $\eta_k = \eta$ over each $U'_{n\lambda}$ for $k \gg 1$. Hence $\|\phi_k\|_{\mathcal{C}^{2,\alpha}(\bar{U}_{n\lambda})}$ $(k \geq 1)$ is bounded for each (n, λ). Therefore, if we take a subsequence, ϕ_k converges to a $\mathcal{C}^2$-section ϕ of $\Lambda^+(\mathrm{ad}E)$ in the $\mathcal{C}^2$-topology over every compact subset. ϕ satisfies $(\nabla_A^*\nabla_A + S/3)\phi = \eta$ and $\|\phi\|_{L^\infty} \leq (24/S)\,\|\eta\|_{L^\infty}$. The Schauder interior estimate gives

$$\|\phi\|_{\mathcal{C}^{2,\alpha}(\bar{U}_{n\lambda})} \leq \mathrm{const}_\lambda(\|\phi\|_{L^\infty} + \|\eta\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})}).$$

By (44), we get $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leq \mathrm{const}\,\|\eta\|_{\mathcal{C}^{0,\alpha}} < \infty$. □

Since the map (43) is isomorphic, the implicit function theorem implies that there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that for any $a \in H^1_A$ with $\|a\|_{L^\infty} \leq \delta_2$ there uniquely exists $\phi_a \in \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}E))$ with $\|\phi_a\|_{\mathcal{C}^{2,\alpha}} \leq \delta_3$ satisfying $F^+(A + a + d_A^*\phi_a) = 0$, i.e.,

$$d_A^+ d_A^* \phi_a + [a \wedge d_A^*\phi_a]^+ + (d_A^*\phi_a \wedge d_A^*\phi_a)^+ = -(a \wedge a)^+. \tag{45}$$

Here the "uniqueness" means that if $\phi \in \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}E))$ with $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leq \delta_3$ satisfies $F^+(A + a + d_A^*\phi) = 0$ then $\phi = \phi_a$. From the elliptic regularity, ϕ_a is smooth. We have $\phi_0 = 0$ and

$$\|\phi_a\|_{\mathcal{C}^{2,\alpha}} \leq \mathrm{const}\,\|a\|_{L^\infty}\,, \quad \|\phi_a - \phi_b\|_{\mathcal{C}^{2,\alpha}} \leq \mathrm{const}\,\|a - b\|_{L^\infty}\,, \tag{46}$$

for any $a, b \in H^1_A$ with $\|a\|_{L^\infty}, \|b\|_{L^\infty} \leq \delta_2$. The map $a \mapsto \phi_a$ is T-equivariant, i.e., $\phi_{T^*a} = T^*\phi_a$ where $T : S^3 \times \mathbb{R} \to S^3 \times \mathbb{R}, (\theta, t) \mapsto (\theta, t + T)$.

We have $F(A+a+d_A^*\phi_a) = F(A+a)+d_A d_A^*\phi_a+[a\wedge d_A^*\phi_a]+d_A^*\phi_a\wedge d_A^*\phi_a$. From (41), if we choose $\delta_2>0$ sufficiently small,

$$\|F(A+a+d_A^*\phi_a)\|_{L^\infty} \leq \|F(A)\|_{L^\infty} + \mathrm{const}\cdot\delta_2 \leq d. \tag{47}$$

Moreover we can choose $\delta_2>0$ so that, for any $a\in H_A^1$ with $\|a\|_{L^\infty}\leq\delta_2$,

$$\|a+d_A^*\phi_a\|_{L^\infty} \leq \mathrm{const}\cdot\delta_2 \leq \varepsilon_1, \tag{48}$$

where ε_1 is the positive constant introduced in Proposition 8.6.

Lemma 9.3. *We can take the above constant $\delta_2>0$ sufficiently small so that, if $a,b\in H_A^1$ with $\|a\|_{L^\infty}, \|b\|_{L^\infty}\leq\delta_2$ satisfy $a+d_A^*\phi_a=b+d_A^*\phi_b$, then $a=b$.*

Proof. We have

$$\begin{aligned}
&\frac{1}{2}(\nabla_A^*\nabla_A+S/3)(\phi_a-\phi_b) = d_A^+d_A^*(\phi_a-\phi_b)\\
&= (b\wedge(b-a))^+ + ((b-a)\wedge a)^+ + [b\wedge(d_A^*\phi_b-d_A^*\phi_a)]^+ + [(b-a)\wedge d_A^*\phi_a]^+\\
&+ (d_A^*\phi_b\wedge(d_A^*\phi_b-d_A^*\phi_a))^+ + ((d_A^*\phi_b-d_A^*\phi_a)\wedge d_A^*\phi_a)^+
\end{aligned} \tag{49}$$

Its $\mathcal{C}^{0,\alpha}$-norm is bounded by

$$\begin{aligned}
&\mathrm{const}(\|a\|_{\mathcal{C}^{0,\alpha}}+\|b\|_{\mathcal{C}^{0,\alpha}}+\|d_A^*\phi_a\|_{\mathcal{C}^{0,\alpha}})\,\|a-b\|_{\mathcal{C}^{0,\alpha}}\\
&\qquad + \mathrm{const}(\|b\|_{\mathcal{C}^{0,\alpha}}+\|d_A^*\phi_a\|_{\mathcal{C}^{0,\alpha}}+\|d_A^*\phi_b\|_{\mathcal{C}^{0,\alpha}})\,\|d_A^*\phi_a-d_A^*\phi_b\|_{\mathcal{C}^{0,\alpha}}.
\end{aligned}$$

From (46), this is bounded by $\mathrm{const}\cdot\delta_2\,\|a-b\|_{L^\infty}$. Then Proposition 9.1 implies

$$\|\phi_a-\phi_b\|_{\mathcal{C}^{2,\alpha}} \leq \mathrm{const}\cdot\delta_2\,\|a-b\|_{L^\infty}.$$

Hence, if $a+d_A^*\phi_a=b+d_A^*\phi_b$ then

$$\|a-b\|_{L^\infty} = \|d_A^*\phi_a-d_A^*\phi_b\|_{L^\infty} \leq \mathrm{const}\cdot\delta_2\,\|a-b\|_{L^\infty}.$$

If δ_2 is sufficiently small, then this implies $a=b$. □

For $r>0$, we set $B_r(H_A^1) := \{a\in H_A^1|\ \|a\|_{L^\infty}\leq r\}$.

Lemma 9.4. *Let $\{a_n\}_{n\geq1}\subset B_{\delta_2}(H_A^1)$ and suppose that this sequence converges to $a\in B_{\delta_2}(H_A^1)$ in the topology of compact-uniform convergence, i.e., for any compact set $K\subset S^3\times\mathbb{R}$, $\|a_n-a\|_{L^\infty(K)}\to0$ as $n\to\infty$. Then $d_A^*\phi_{a_n}$ converges to $d_A^*\phi_a$ in the $\mathcal{C}^\infty$-topology over every compact subset in $S^3\times\mathbb{R}$.*

Proof. It is enough to prove that there exists a subsequence (also denoted by $\{a_n\}$) such that $d_A^*\phi_{a_n}$ converges to $d_A^*\phi_a$ in the topology of $\mathcal{C}^\infty$-convergence over compact subsets in $S^3\times\mathbb{R}$. From the elliptic regularity, a_n converges to a in the $\mathcal{C}^\infty$-topology over every compact subset. Hence, for each $k\geq0$ and each compact subset K in X, the $\mathcal{C}^k$-norms of ϕ_{a_n} over K ($n\geq1$) are bounded by the equation (45) and $\|\phi_{a_n}\|_{\mathcal{C}^{2,\alpha}}\leq\delta_3$. Then a subsequence of ϕ_{a_n} converges to some ϕ in the $\mathcal{C}^\infty$-topology over every compact subset.

We have $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leq \delta_3$ and $F^+(A+a+d_A^*\phi) = 0$. Then the uniqueness of ϕ_a implies $\phi = \phi_a$. □

Lemma 9.5. *For any $K>0$ and $\varepsilon>0$ there exist $L>0$ and $\delta>0$ such that, for any $a, b \in B_{\delta_2}(H_A^1)$, if there is a gauge transformation g of E satisfying*

$$\|g(A+a+d_A^*\phi_a) - (A+b+d_A^*\phi_b)\|_{L^\infty(|t|\leq L)} \leq \delta,$$
$$|g(\theta_0, n) - 1| \leq \varepsilon_1 \quad (\forall n \in \mathbb{Z}\cap[-L, L]),$$

then we have

$$\|g-1\|_{L^\infty(|t|\leq K)} \leq \varepsilon, \quad \|a-b\|_{L^\infty(|t|\leq K)} \leq \varepsilon.$$

Recall that the positive constant ε_1 was introduced in Proposition 8.6.

Proof. We prove the statement $\|g-1\|_{L^\infty(|t|\leq K)} \leq \varepsilon$. The statement $\|a-b\|_{L^\infty(|t|\leq K)} \leq \varepsilon$ can be proved in the same way. (In the proof of $\|a-b\|_{L^\infty(|t|\leq K)} \leq \varepsilon$, we need Lemma 9.3.) If the statement is false, then there exist $K>0$, $\varepsilon>0$, $a_n, b_n \in B_{\delta_2}(H_A^1)$ $(n\geq 1)$, gauge transformations g_n $(n \geq 1)$ such that

$$\|g_n(A+a_n+d_A^*\phi_{a_n}) - (A+b_n+d_A^*\phi_{b_n})\|_{L^\infty(|t|\leq n)} \leq 1/n,$$
$$|g_n(\theta_0, k) - 1| \leq \varepsilon_1 \quad (\forall k \in \mathbb{Z}\cap[-n, n]), \quad \|g_n - 1\|_{L^\infty(|t|\leq K)} > \varepsilon.$$

If we take subsequences, a_n and b_n converge to some a and b in $B_{\delta_2}(H_A^1)$ respectively in the $\mathcal{C}^\infty$-topology over compact subsets. Then $d_A^*\phi_{a_n}$ and $d_A^*\phi_{b_n}$ converge to $d_A^*\phi_a$ and $d_A^*\phi_b$ in the $\mathcal{C}^\infty$-topology over compact subsets (Lemma 9.4). Set $c_n := g_n(A+a_n+d_A^*\phi_{a_n}) - (A+b_n+d_A^*\phi_{b_n}) = -(d_A g_n)g_n^{-1} + g_n(a_n+d_A^*\phi_{a_n})g_n^{-1} - (b_n+d_A^*\phi_{b_n})$. We have $|d_A g_n| \leq 1/n + 2\varepsilon_1$ over $|t|\leq n$ (recall (48)). Then, if we take a subsequence, g_n converges to some g in the topology of uniform convergence over compact subsets. Moreover for any $1\leq p<\infty$ and any pre-compact open set $U \subset S^3\times\mathbb{R}$, $g_n|_U$ weakly converges to $g|_U$ in $L_1^p(U)$. We have

$$|g(\theta_0, k) - 1| \leq \varepsilon_1\, (\forall k \in \mathbb{Z}), \quad \|g-1\|_{L^\infty(|t|\leq K)} \geq \varepsilon.$$

We have $d_A g_n = -c_n g_n + g_n(a_n + d_A^*\phi_{a_n}) - (b_n + d_A^*\phi_{b_n})g_n$. Since c_n converges to 0 in the topology of uniform convergence over compact subsets, we have

$$d_A g = g(a+d_A^*\phi_a) - (b+d_A^*\phi_b)g.$$

This means $g(A+a+d_A^*\phi_a) = A+b+d_A^*\phi_b$. Moreover we have $|g(\theta_0,k)-1| \leq \varepsilon_1\,(k\in\mathbb{Z})$. Then Proposition 8.6 implies $g=1$. (Note that we have $d_A^*(a+d_A^*\phi_a) = d_A^*(b+d_A^*\phi_b) = 0$ and $\|a+d_A^*\phi_a\|_{L^\infty}, \|b+d_A^*\phi_b\|_{L^\infty} \leq \varepsilon_1$ by (48).) This contradicts $\|g-1\|_{L^\infty(|t|\leq K)} \geq \varepsilon$. □

Corollary 9.6. *We can take $\delta_2>0$ so small that the following statement holds. For any $K>0$ and $\varepsilon>0$ there exist $L = L(K,\varepsilon)>0$ and $\delta = \delta(K,\varepsilon)>0$ satisfying the following. For any $a, b \in B_{\delta_2}(H_A^1)$, if there exist $t_0\in\mathbb{R}$ and a gauge transformation g of E such that*

$$\|g(A+a+d_A^*\phi_a) - (A+b+d_A^*\phi_b)\|_{L^\infty(|t-t_0|\leq L)} \leq \delta,$$
$$|g(\theta_0, n) - 1| \leq \varepsilon_1/2 \quad (n \in \mathbb{Z}\cap[t_0-L, t_0+L]),$$

then we have

$$\|g-1\|_{L^\infty(|t-t_0|\le K)} \le \varepsilon, \quad \|a-b\|_{L^\infty(|t-t_0|\le K)} \le \varepsilon. \tag{50}$$

Proof. We have $|d_A g| \le \delta + \mathrm{const}\cdot\delta_2$ over $|t-t_0|\le L$. Since $|g(\theta_0,n)-1|\le \varepsilon_1/2$ for $n\in\mathbb{Z}\cap[t_0-L,t_0+L]$, we have

$$\|g-1\|_{L^\infty(|t-t_0|\le L)} \le \varepsilon_1/2 + \mathrm{const}\cdot(\delta+\delta_2) \le \varepsilon_1,$$

if we take δ and δ_2 sufficiently small. Recall that we have the natural $T\mathbb{Z}$-actions on all data. Let n_0 be an integer satisfying $|t_0-n_0T|\le T$, and set $t_0' := t_0-n_0T$. Let $a' := (n_0T)^*a$, $b' := (n_0T)^*b$ and $g' := (n_0T)^*g$ be the pull-backs. (We have $\phi_{a'} = (n_0T)^*\phi_a$.) These satisfy

$$\|g'(A+a'+d_A^*\phi_{a'}) - (A+b'+d_A^*\phi_{b'})\|_{L^\infty(|t-t_0'|\le L)} \le \delta, \quad \|g'-1\|_{L^\infty(|t-t_0'|\le L)} \le \varepsilon_1.$$

Note that $\{|t|\le L-T\}\subset\{|t-t_0'|\le L\}$ and $\{|t-t_0'|\le K\}\subset\{|t|\le K+T\}$. Hence if we take $\delta>0$ sufficiently small and $L>0$ sufficiently large, then Lemma 9.5 implies

$$\|g'-1\|_{L^\infty(|t-t_0'|\le K)} \le \varepsilon, \quad \|a'-b'\|_{L^\infty(|t-t_0'|\le K)} \le \varepsilon.$$

This is equivalent to the above (50). □

9.2. **Proof of the lower bound.** We continue the argument of the previous subsection. For each $n\in\mathbb{Z}$, we take a point $p_n^0\in E_{(\theta_0,n)}$. For $r>0$, we denote $B_r(E_{(\theta_0,n)})$ as the closed r-ball centered at p_n^0 in $E_{(\theta_0,n)}$. Consider the following map:

$$B_{\delta_2}(H_A^1)\times\prod_{n\in\mathbb{Z}} B_{\delta_2}(E_{(\theta_0,n)}) \to \mathcal{M}_d, \quad (a,(p_n)_{n\in\mathbb{Z}}) \mapsto [E, A+a+d_A^*\phi_a, (p_n)_{n\in\mathbb{Z}}]. \tag{51}$$

Note that we have $|F(A+a+d_A^*\phi_a)|\le d$ (see (47)), and hence this map is well-defined. $B_{\delta_2}(H_A^1)$ is equipped with the topology of uniform convergence over compact subsets, and we consider the product topology on $B_{\delta_2}(H_A^1)\times\prod_n B_{\delta_2}(E_{(\theta_0,n)})$. Then the map (51) is continuous (see Lemma 9.4).

Lemma 9.7. *The map (51) is injective for sufficiently small $\delta_2>0$.*

Proof. Let $(a,(p_n)_{n\in\mathbb{Z}}),(b,(q_n)_{n\in\mathbb{Z}})\in B_{\delta_2}(H_A^1)\times\prod_{n\in\mathbb{Z}}B_{\delta_2}(E_{(\theta_0,n)})$, and suppose that there exists a gauge transformation g satisfying

$$g(A+a+d_A^*\phi_a) = A+b+d_A^*\phi_b, \quad g(p_n)=q_n\ (\forall n\in\mathbb{Z}).$$

From $g(p_n)=q_n$, we have $|g(\theta_0,n)-1|\le\varepsilon_1$ ($\delta_2\ll 1$). We have $d_A^*(a+d_A^*\phi_a) = d_A^*(b+d_A^*\phi_b)=0$ and $\|a+d_A^*\phi_a\|_{L^\infty}, \|b+d_A^*\phi_b\|_{L^\infty}\le\varepsilon_1$ (see (48)). Then Proposition 8.6 implies $g=1$ and $a+d_A^*\phi_a = b+d_A^*\phi_b$. Then we have $p_n=q_n$ and $a=b$ (see Lemma 9.3). □

We define a distance on $\mathcal{M}_d$ as follows. For $[E_1, A_1, (p_n)_{n\in\mathbb{Z}}], [E_2, A_2, (q_n)_{n\in\mathbb{Z}}] \in \mathcal{M}_d$ (see Remark 1.3), we set

$$\begin{aligned}\mathrm{dist}([E_1,&A_1,(p_n)_{n\in\mathbb{Z}}],[E_2,A_2,(q_n)_{n\in\mathbb{Z}}])\\ &:= \inf_{g:E_1\to E_2}\left\{\sum_{n\geq 1}2^{-n}\frac{\|g(A_1)-A_2\|_{L^\infty(|t|\leq n)}}{1+\|g(A_1)-A_2\|_{L^\infty(|t|\leq n)}}+\sum_{n\in\mathbb{Z}}2^{-|n|}|g(p_n)-q_n|\right\}.\end{aligned}$$

For $N = 0, 1, 2, \cdots$, we set $\Omega_N := \{0, 1, 2, \cdots, N-1\}$.

Lemma 9.8. *We can take $\delta_2 > 0$ so small that the following statement holds. For any $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that, for any $N \geq 1$ and $(a, (p_n)_{n\in\mathbb{Z}})$, $(b, (q_n)_{n\in\mathbb{Z}}) \in B_{\delta_2}(H^1_A) \times \prod_{n\in\mathbb{Z}} B_{\delta_2}(E_{(\theta_0,n)})$, if*

$$\mathrm{dist}_{\Omega_N}([E, A + a + d_A^*\phi_a, (p_n)_{n\in\mathbb{Z}}], [E, A + b + d_A^*\phi_b, (q_n)_{n\in\mathbb{Z}}]) \leq \varepsilon', \tag{52}$$

then we have

$$\|a - b\|_{L^\infty(S^3\times[0,N])} \leq \varepsilon, \quad |p_n - q_n| \leq \varepsilon\, (n = 0, 1, 2, \cdots, N-1).$$

(For the definition of $\mathrm{dist}_{\Omega_N}(\cdot,\cdot)$*, see Section 2.)*

Proof. Let $L = L(1, \varepsilon/2)$ and $\delta = \delta(1, \varepsilon/2)$ be the constants introduced in Corollary 9.6 for $K = 1$ and $\varepsilon/2$. If we take ε' sufficiently small, then (52) implies that, for each $n = 0, 1, 2, \cdots, N-1$, there exists a gauge transformation g_n of E satisfying

$$\begin{gathered}\|g_n(A + a + d_A^*\phi_a) - (A + b + d_A^*\phi_b)\|_{L^\infty(|t-n|\leq L)} \leq \delta,\\ |g_n(\theta_0, k)p_k - q_k| \leq \min(\varepsilon_1/3, \varepsilon/2) \quad (k \in \mathbb{Z}\cap[n-L, n+L]).\end{gathered}$$

In particular,

$$|g_n(\theta_0, k) - 1| \leq |g_n(\theta_0, k)p_k - q_k| + |q_k - p_k| \leq \varepsilon_1/3 + 2\delta_2 \leq \varepsilon_1/2,$$

for $k \in \mathbb{Z}\cap[n-L, n+L]$ $(\delta_2 \leq \varepsilon_1/12)$. Then we can use Corollary 9.6, and get $(n = 0, 1, \cdots, N-1)$

$$\|g_n - 1\|_{L^\infty(|t-n|\leq 1)} \leq \varepsilon/2, \quad \|a - b\|_{L^\infty(|t-n|\leq 1)} \leq \varepsilon/2.$$

The latter inequality implies $\|a - b\|_{L^\infty(S^3\times[0,N])} \leq \varepsilon/2 < \varepsilon$. The former inequality implies

$$|p_n - q_n| \leq |p_n - g_n(\theta_0, n)p_n| + |g_n(\theta_0, n)p_n - q_n| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

for $n = 0, 1, 2, \cdots, N-1$. □

Set $\ell^\infty(\mathbb{Z}, su(2)) := \{(u_n)_{n\in\mathbb{Z}} \in su(2)^{\mathbb{Z}} |\ \|(u_n)_{n\in\mathbb{Z}}\|_{\ell^\infty} := \sup_n |u_n| < \infty\}$ and $V := H^1_A \times \ell^\infty(\mathbb{Z}, su(2))$. For $(a, (u_n)_{n\in\mathbb{Z}}) \in V$ we define

$$\|(a, (u_n)_{n\in\mathbb{Z}})\|_V := \max(\|a\|_{L^\infty}, \|(u_n)_{n\in\mathbb{Z}}\|_{\ell^\infty}).$$

For $r>0$, we denote $B_r(V)$ as the closed r-ball centered at the origin in V. There exists $\delta_2'=\delta_2'(\delta_2)>0$ such that, for any $(a,(u_n)_{n\in\mathbb{Z}})\in B_{\delta_2'}(V)$, we have $(a,(p_n^0e^{u_n})_{n\in\mathbb{Z}})\in B_{\delta_2}(H_A^1)\times\prod_{n\in\mathbb{Z}}B_{\delta_2}(E_{(\theta_0,n)})$. Then, for $r\le\delta_2'$, we can define

$$P_r: B_r(V)\to\mathcal{M}_d,\quad (a,(u_n)_{n\in\mathbb{Z}})\mapsto[E,A+a+d_A^*\phi_a,(p_n^0e^{u_n})_{n\in\mathbb{Z}}].$$

We consider that $B_r(V)$ is equipped with the topology of uniform convergence over compact subsets, i.e., a sequence $\{(a_n,(u_k^{(n)})_{k\in\mathbb{Z}})\}_{n\ge1}$ in $B_r(V)$ converges to $(a,(u_k)_{k\in\mathbb{Z}})$ in $B_r(V)$ if and only if a_n uniformly converges to a over every compact subset and $u_k^{(n)}$ converges to u_k for every k. Then the above map P_r is continuous.

Lemma 9.9. *For any $s>0$ there exists $r>0$ such that*

$$P_r(B_r(V))\subset B_s([E,A,(p_n^0)_{n\in\mathbb{Z}}])_{\mathbb{Z}}.$$

Here $B_s([E,A,(p_n^0)_{n\in\mathbb{Z}}])_{\mathbb{Z}}$ is the closed s-ball of $\mathcal{M}_d$ centered at $[E,A,(p_n^0)_{n\in\mathbb{Z}}]$ with respect to the distance $\mathrm{dist}_{\mathbb{Z}}$ (see Section 2.2).

Proof. Let $(a,(u_n)_{n\in\mathbb{Z}})\in B_r(V)$. We have $\|a+d_A^*\phi_a\|_{L^\infty}\le\mathrm{const}\,\|a\|_{L^\infty}\le\mathrm{const}\cdot r$. Hence, for any $k\in\mathbb{Z}$,

$$\begin{aligned}&\mathrm{dist}(k^*[E,A,(p_n^0)_{n\in\mathbb{Z}}],k^*[E,A+a+d_A^*\phi_a,(p_n^0e^{u_n})_{n\in\mathbb{Z}}])\\&\quad\le\sum_{n\ge1}2^{-n}\frac{\|a+d_A^*\phi_a\|_{L^\infty(|t-k|\le n)}}{1+\|a+d_A^*\phi_a\|_{L^\infty(|t-k|\le n)}}+\sum_{n\in\mathbb{Z}}2^{-|n|}|p_{n+k}^0e^{u_{n+k}}-p_{n+k}^0|\le\mathrm{const}\cdot r.\end{aligned}$$

Therefore $\mathrm{dist}_{\mathbb{Z}}([E,A,(p_n^0)_{n\in\mathbb{Z}}],[E,A+a+d_A^*\phi_a,(p_n^0e^{u_n})_{n\in\mathbb{Z}}])\le\mathrm{const}\cdot r\le s$. □

For each $n\ge1$, let $\pi_n: S^3\times(\mathbb{R}/nT\mathbb{Z})\to S^3\times(\mathbb{R}/T\mathbb{Z})$ be the natural n-hold covering, and set $E_n:=\pi_n^*(\underline{E})$ and $A_n:=\pi_n^*(\underline{A})$. We denote $H_{A_n}^1$ as the space of $a\in\Omega^1(\mathrm{ad}E_n)$ over $S^3\times(\mathbb{R}/nT\mathbb{Z})$ satisfying $(d_{A_n}^++d_{A_n}^*)a=0$. We can identify $H_{A_n}^1$ with the subspace of H_A^1 consisting of nT-invariant elements. (Here we consider the natural action of $T\mathbb{Z}$ on H_A^1.) The index formula gives $\dim H_{A_n}^1=8nc_2(\underline{E})$. (We have $H_{A_n}^0=H_{A_n}^2=0$.) We define the finite dimensional subspace $V_n\subset V=H_A^1\times\ell^\infty(\mathbb{Z},su(2))$ by

$$V_n:=\{(a,(u_k)_{k\in\mathbb{Z}})\in V\,|\,a\in H_{A_n}^1,u_k=0\,(k<0,\,k\ge[nT])\}.$$

Here $[nT]$ means the maximum integer not greater than nT. We have

$$\dim V_n=8nc_2(\underline{E})+3[nT]. \tag{53}$$

Let $s>0$. We choose $0<r\le\delta_2'(\ll1)$ such that $P_r(B_r(V))\subset B_s([E,A,(p_k^0)_{k\in\mathbb{Z}}])_{\mathbb{Z}}$. Let N be a positive integer and set $n:=[N/T]$. By using Lemma 9.8, there exists $\varepsilon=\varepsilon(r)>0$ (independent of N,n) such that, for $(a,(u_k)_{k\in\mathbb{Z}}),(b,(v_k)_{k\in\mathbb{Z}})\in B_r(V)$, if

$$\mathrm{dist}_{\Omega_N}(P_r(a,(u_k)_{k\in\mathbb{Z}}),P_r(b,(v_k)_{k\in\mathbb{Z}}))\le\varepsilon.$$

then

$$\|a-b\|_{L^\infty(S^3\times[0,N])}\le r/2,\quad |u_k-v_k|\le r/2\,(k=0,1,2,\cdots,N-1).$$

In particular, if $(a,(u_k)_{k\in\mathbb{Z}}), (b,(v_k)_{k\in\mathbb{Z}}) \in B_r(V_n)$ satisfies

$$\mathrm{dist}_{\Omega_N}(P_r(a,(u_k)_{k\in\mathbb{Z}}), P_r(b,(v_k)_{k\in\mathbb{Z}})) \le \varepsilon,$$

then

$$\|(a,(u_k)_{k\in\mathbb{Z}}) - (b,(v_k)_{k\in\mathbb{Z}})\|_V \le r/2.$$

This implies

$$\begin{aligned}\mathrm{Widim}_\varepsilon(B_s([E,A,(p_k^0)_{k\in\mathbb{Z}}])_{\mathbb{Z}}, \mathrm{dist}_{\Omega_N}) &\ge \mathrm{Widim}_{r/2}(B_r(V_n), \|\cdot\|_V),\\ &= \dim V_n = 8nc_2(\underline{\mathrm{E}}) + 3[nT].\end{aligned}$$

Here we have used Lemma 2.1 and (53). Therefore

$$\dim(B_s([E,A,(p_k^0)_{k\in\mathbb{Z}}])_{\mathbb{Z}} \subset \mathcal{M}_d : \mathbb{Z}) \ge 8c_2(\underline{\mathrm{E}})/T + 3 = 8\rho(A) + 3.$$

Here we have used (3). This holds for any $s>0$. Thus

$$\dim_{[E,A,(p_k^0)_{k\in\mathbb{Z}}]}(\mathcal{M}_d : \mathbb{Z}) \ge 8\rho(A) + 3.$$

So we get the following conclusion.

Theorem 9.10. *Suppose $d>0$. If $\boldsymbol{A}$ is a periodic ASD connection satisfying $\|F(\boldsymbol{A})\|_{L^\infty} < d$, then for any framing $\boldsymbol{p}:\mathbb{Z}\to\boldsymbol{E}$ ($\boldsymbol{p}(n)\in\boldsymbol{E}_{(\theta_0,n)}$)*

$$\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) = 8\rho(\boldsymbol{A}) + 3.$$

Proof. The upper-bound $\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) \le 8\rho(\boldsymbol{A})+3$ was already proved in Section 7.2. If $\boldsymbol{A}$ is not flat, then the above argument shows that we also have the lower-bound $\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) \ge 8\rho(\boldsymbol{A})+3$. Hence $\dim_{[\boldsymbol{A},\boldsymbol{p}]}(\mathcal{M}_d : \mathbb{Z}) = 8\rho(\boldsymbol{A})+3$. So we suppose that $\boldsymbol{A}$ is flat. Since every flat connection on $\boldsymbol{E} = X\times SU(2)$ is gauge equivalent to the product connection, we can suppose that $\boldsymbol{A}$ is the product connection. Then the following map becomes a $\mathbb{Z}$-equivariant topological embedding.

$$SU(2)^{\mathbb{Z}}/SU(2) \to \mathcal{M}_d, \quad [(p_n)_{n\in\mathbb{Z}}] \mapsto [\boldsymbol{A},(p_n)_{n\in\mathbb{Z}}],$$

where $SU(2)^{\mathbb{Z}}/SU(2)$ is the quotient space defined as in Example 2.8. From the result of Example 2.8, we get

$$\dim_{[\boldsymbol{A},(p_n)_{n\in\mathbb{Z}}]}(\mathcal{M}_d : \mathbb{Z}) \ge \dim_{[(p_n)_{n\in\mathbb{Z}}]}(SU(2)^{\mathbb{Z}}/SU(2) : \mathbb{Z}) = 3 = 8\rho(\boldsymbol{A}) + 3.$$

□

When $d=0$, we can determine the value of the (local) mean dimension.

Proposition 9.11.

$$\dim_{loc}(\mathcal{M}_0 : \mathbb{Z}) = \dim(\mathcal{M}_0 : \mathbb{Z}) = 3.$$

Proof. $\mathcal{M}_0$ is $\mathbb{Z}$-homeomorphic to $SU(2)^{\mathbb{Z}}/SU(2)$. Hence Example 2.8 gives the above result. □

We have completed all the proofs of Theorem 1.1 and 1.2.

Appendix A. Green kernel

In this appendix, we prepare some basic facts on a Green kernel over $S^3 \times \mathbb{R}$. Let $a > 0$ be a positive constant. Some constants in this appendix depend on a, but we don't explicitly write their dependence on a for simplicity of the explanation. In the main body of the paper we have $a = S/3$ (S is the scalar curvature of $S^3 \times \mathbb{R}$), and its value is fixed throughout the argument. Hence we don't need to care about the dependence on $a = S/3$.

A.1. $(\Delta + a)$ **on functions.** Let $\Delta := \nabla^*\nabla$ be the Laplacian on functions over $S^3 \times \mathbb{R}$. (Notice that the sign convention of our Laplacian $\Delta = \nabla^*\nabla$ is "geometric". For example, we have $\Delta = -\sum_{i=1}^4 \partial^2/\partial x_i^2$ on the Euclidean space $\mathbb{R}^4$.) Let $g(x, y)$ be the Green kernel of $\Delta + a$;

$$(\Delta_y + a)g(x, y) = \delta_x(y).$$

This equation means that

$$\phi(x) = \int_{S^3\times\mathbb{R}} g(x, y)(\Delta_y + a)\phi(y)d\mathrm{vol}(y),$$

for compact-supported smooth functions ϕ. The existence of $g(x, y)$ is essentially standard ([2, Chapter 4]). We briefly explain how to construct it. We fix $x \in S^3 \times \mathbb{R}$ and construct a function $g_x(y)$ satisfying $(\Delta+a)g_x = \delta_x$. As in [2, Chapter 4, Section 2], by using a local coordinate around x, we can construct (by hand) a compact-supported function $g_{0,x}(y)$ satisfying

$$(\Delta + a)g_{0,x} = \delta_x - g_{1,x},$$

where $g_{1,x}$ is a compact supported continuous function. Moreover $g_{0,x}$ is smooth outside $\{x\}$ and it satisfies

$$\mathrm{const}_1/d(x, y)^2 \le g_{0,x}(y) \le \mathrm{const}_2/d(x, y)^2,$$

for some positive constants const_1 and const_2 in some small neighborhood of x. Here $d(x, y)$ is the distance between x and y. Since $(\Delta + a) : L^2_2 \to L^2$ is isomorphic, there exists $g_{2,x} \in L^2_2$ satisfying $(\Delta + a)g_{2,x} = g_{1,x}$. ($g_{2,x}$ is of class $\mathcal{C}^1$.) Then $g_x := g_{0,x} + g_{2,x}$ satisfies $(\Delta+a)g_x = \delta_x$, and $g(x, y) := g_x(y)$ becomes the Green kernel. $g(x, y)$ is smooth outside the diagonal. Since $S^3\times\mathbb{R} = SU(2)\times\mathbb{R}$ is a Lie group and its Riemannian metric is two-sided invariant, we have $g(x, y) = g(zx, zy) = g(xz, yz)$ for $x, y, z \in S^3\times\mathbb{R}$. $g(x, y)$ satisfies

$$c_1/d(x, y)^2 \le g(x, y) \le c_2/d(x, y)^2 \quad (d(x, y) \le \delta), \tag{54}$$

for some positive constants c_1, c_2, δ.

Lemma A.1. *$g(x, y) > 0$ for $x \ne y$.*

Proof. Fix $x \in S^3 \times \mathbb{R}$. We have $(\Delta + a)g_x = 0$ outside $\{x\}$, and hence (by elliptic regularity)

$$|g_x(\theta, t)| \leq \text{const}\, \|g_x\|_{L^2(S^3\times[t-1,t+1])} \quad (|t| > 1).$$

Since the right-hand-side goes to zero as $|t| \to \infty$, g_x vanishes at infinity. Let $R > 0$ be a large positive number and set $\Omega := S^3 \times [-R, R] \setminus B_\delta(x)$. ($\delta$ is a positive constant in (54).) Since $g_x(y) \geq c_1/d(x,y)^2 > 0$ on $\partial B_\delta(x)$, we have $g_x \geq -\sup_{t=\pm R} |g_x(\theta, t)|$ on $\partial\Omega$. Since $(\Delta + a)g_x = 0$ on Ω, we can apply the weak maximum (minimum) principle to g_x (Gilbarg-Trudinger [9, Chapter 3, Section 1]) and get

$$g_x(y) \geq -\sup_{t=\pm R} |g_x(\theta, t)| \quad (y \in \Omega).$$

The right-hand-side goes to zero as $R \to \infty$. Hence we have $g_x(y) \geq 0$ for $y \neq x$. Since g_x is not constant, the strong maximum principle ([9, Chapter 3, Section 2]) implies that g_x cannot achieve zero. Therefore $g_x(y) > 0$ for $y \neq x$. □

Lemma A.2. *There exists $c_3 > 0$ such that*

$$0 < g(x, y) \leq c_3 e^{-\sqrt{a}d(x,y)} \quad (d(x, y) \geq 1).$$

In particular,

$$\int_{S^3\times\mathbb{R}} g(x, y)d\mathrm{vol}(y) < \infty.$$

The value of this integral is independent of $x \in S^3 \times \mathbb{R}$ because of the symmetry of $g(x, y)$.

Proof. We fix $x_0 = (\theta_0, 0) \in S^3 \times \mathbb{R}$. Since $S^3 \times \mathbb{R}$ is homogeneous, it is enough to show that $g_{x_0}(y) = g(x_0, y)$ satisfies

$$g_{x_0}(y) \leq \text{const} \cdot e^{-\sqrt{a}|t|} \quad (y = (\theta, t) \in S^3 \times \mathbb{R} \text{ and } |t| \geq 1).$$

Let $C := \sup_{|t|=1} g_{x_0}(\theta, t) > 0$, and set $u := Ce^{\sqrt{a}(1-|t|)} - g_{x_0}(y)$ ($|t| \geq 1$). We have $u \geq 0$ at $t = \pm 1$ and $(\Delta + a)u = 0$ ($|t| \geq 1$). u goes to zero at infinity. (See the proof of Lemma A.1.) Hence we can apply the weak minimum principle (see the proof of Lemma A.1) to u and get $u \geq 0$ for $|t| \geq 1$. Thus $g_{x_0}(y) \leq Ce^{\sqrt{a}(1-|t|)}$ ($|t| \geq 1$). □

The following technical lemma will be used in the next subsection.

Lemma A.3. *Let f be a smooth function over $S^3 \times \mathbb{R}$. Suppose that there exist non-negative functions $f_1, f_2 \in L^2$, $f_3 \in L^1$ and $f_4, f_5, f_6 \in L^\infty$ such that $|f| \leq f_1 + f_4$, $|\nabla f| \leq f_2 + f_5$ and $|\Delta f + af| \leq f_3 + f_6$. Then we have*

$$f(x) = \int_{S^3\times\mathbb{R}} g(x, y)(\Delta_y + a)f(y)d\mathrm{vol}(y).$$

Proof. We fix $x \in S^3\times\mathbb{R}$. Let ρ_n $(n \geq 1)$ be cut-off functions satisfying $0 \leq \rho_n \leq 1$, $\rho_n = 1$ over $|t| \leq n$ and $\rho_n = 0$ over $|t| \geq n+1$. Moreover $|\nabla\rho_n|, |\Delta\rho_n| \leq \text{const}$ (independent of $n \geq 1$). Set $f_n := \rho_n f$. We have

$$f_n(x) = \int g(x,y)(\Delta_y + a)f_n(y)d\mathrm{vol}(y).$$

$$(\Delta + a)f_n = \Delta\rho_n \cdot f - 2\langle\nabla\rho_n, \nabla f\rangle + \rho_n(\Delta + a)f.$$

Note that $g_x(y) = g(x,y)$ is smooth outside $\{x\}$ and exponentially decreases as y goes to infinity. Hence for $n \gg 1$,

$$\int g_x|\Delta\rho_n \cdot f|d\mathrm{vol} \leq C\sqrt{\int_{\mathrm{supp}(d\rho_n)} f_1^2 \, d\mathrm{vol}} + C\int_{\mathrm{supp}(d\rho_n)} g_x f_4 \, d\mathrm{vol}(y).$$

Since $\mathrm{supp}(d\rho_n) \subset \{t \in [-n-1, -n] \cup [n, n+1]\}$ and $f_1 \in L^2$ and $f_4 \in L^\infty$, the right-hand-side goes to zero as $n \to \infty$. In the same way, we get

$$\int g_x|\langle\nabla\rho_n, \nabla f\rangle|d\mathrm{vol} \to 0 \quad (n \to \infty).$$

We have $g_x|\rho_n(\Delta + a)f| \leq g_x|\Delta f + af|$, and

$$\int g_x(y)|\Delta f + af|d\mathrm{vol} \leq \int_{d(x,y)\leq 1} g_x(y)|\Delta f + af|d\mathrm{vol} + \left(\sup_{d(x,y)>1} g_x(y)\right)\int_{d(x,y)>1} f_3 \, d\mathrm{vol}$$
$$+ \int_{d(x,y)>1} g_x f_6 \, d\mathrm{vol} < \infty.$$

Hence Lebesgue's theorem implies

$$\lim_{n\to\infty}\int g_x\rho_n(\Delta + a)f \, d\mathrm{vol} = \int g_x(\Delta + a)f \, d\mathrm{vol}.$$

Therefore we get

$$f(x) = \int g_x(\Delta + a)f \, d\mathrm{vol}.$$

∎

A.2. $(\nabla^*\nabla + a)$ **on sections.** Let E be a real vector bundle over $S^3 \times \mathbb{R}$ with a fiberwise metric and a connection ∇ compatible with the metric.

Lemma A.4. *Let ϕ be a smooth section of E such that $\|\phi\|_{L^2}$, $\|\nabla\phi\|_{L^2}$ and $\|\nabla^*\nabla\phi + a\phi\|_{L^\infty}$ are finite. Then ϕ satisfies*

$$|\phi(x)| \leq \int_{S^3\times\mathbb{R}} g(x,y)|\nabla^*\nabla\phi(y) + a\phi(y)|d\mathrm{vol}(y).$$

Proof. The following argument is essentially due to Donaldson [5, p. 184]. Let $\underline{\mathbb{R}}$ be the product line bundle over $S^3 \times \mathbb{R}$ with the product metric and the product connection. Set $\phi_n := (\phi, 1/n)$ (a section of $E \oplus \underline{\mathbb{R}}$). Then $|\phi_n| \geq 1/n$ and hence $\phi_n \neq 0$ at all points. We want to apply Lemma A.3 to $|\phi_n|$. $|\phi_n| \leq |\phi| + 1/n$ where $|\phi| \in L^2$ and $1/n \in L^\infty$.

$\nabla\phi_n = (\nabla\phi, 0)$ and $\nabla^*\nabla\phi_n = (\nabla^*\nabla\phi, 0)$. We have the Kato inequality $|\nabla|\phi_n|| \leq |\nabla\phi_n|$. Hence $\nabla|\phi_n| \in L^2$. From $\Delta|\phi_n|^2/2 = (\nabla^*\nabla\phi_n, \phi_n) - |\nabla\phi_n|^2$,

$$(\Delta + a)|\phi_n| = (\nabla^*\nabla\phi_n + a\phi_n, \phi_n/|\phi_n|) - \frac{|\nabla\phi_n|^2 - |\nabla|\phi_n||^2}{|\phi_n|}. \tag{55}$$

Hence (by using $|\phi_n| \geq 1/n$ and $|\nabla|\phi_n|| \leq |\nabla\phi_n|$)

$$|(\Delta + a)|\phi_n|| \leq |\nabla^*\nabla\phi_n + a\phi_n| + n|\nabla\phi_n|^2 \leq |\nabla^*\nabla\phi + a\phi| + a/n + n|\nabla\phi|^2.$$

$|\nabla^*\nabla\phi + a\phi| + a/n \in L^\infty$ and $n|\nabla\phi|^2 \in L^1$. Therefore we can apply Lemma A.3 to $|\phi_n|$ and get

$$|\phi_n(x)| = \int g(x, y)(\Delta_y + a)|\phi_n(y)| d\mathrm{vol}(y).$$

From (55) and the Kato inequality $|\nabla|\phi_n|| \leq |\nabla\phi_n|$,

$$(\Delta_y + a)|\phi_n(y)| \leq |\nabla^*\nabla\phi_n + a\phi_n| \leq |\nabla^*\nabla\phi + a\phi| + a/n.$$

Thus

$$|\phi_n(x)| \leq \int g(x, y)|\nabla^*\nabla\phi(y) + a\phi(y)| d\mathrm{vol}(y) + \frac{a}{n}\int g(x, y) d\mathrm{vol}(y).$$

Let $n \to \infty$. Then we get the desired bound. □

Proposition A.5. *Let ϕ be a section of E of class $\mathcal{C}^2$, and suppose that ϕ and $\eta := (\nabla^*\nabla + a)\phi$ are contained in L^∞. Then*

$$\|\phi\|_{L^\infty} \leq (8/a)\, \|\eta\|_{L^\infty}\,.$$

Proof. There exists a point $(\theta_1, t_1) \in S^3 \times \mathbb{R}$ where $|\phi(\theta_1, t_1)| \geq \|\phi\|_{L^\infty}/2$. We have

$$\Delta|\phi|^2 = 2(\nabla^*\nabla\phi, \phi) - 2|\nabla\phi|^2 = 2(\eta, \phi) - 2a|\phi|^2 - 2|\nabla\phi|^2.$$

Set $M := \|\phi\|_{L^\infty} \|\eta\|_{L^\infty}$. Then

$$(\Delta + 2a)|\phi|^2 \leq 2(\eta, \phi) \leq 2M.$$

Define a function f on $S^3 \times \mathbb{R}$ by $f(\theta, t) := (2M/a)\cosh\sqrt{a}(t - t_1) = (M/a)(e^{\sqrt{a}(t-t_1)} + e^{\sqrt{a}(-t+t_1)})$. Then $(\Delta + a)f = 0$, and hence $(\Delta + 2a)f = af \geq 2M$. Therefore

$$(\Delta + 2a)(f - |\phi|^2) \geq 0.$$

Since $|\phi|$ is bounded and f goes to $+\infty$ at infinity, we have $f - |\phi|^2 > 0$ for $|t| \gg 1$. Then the weak minimum principle ([9, Chapter 3, Section 1]) implies $f(\theta_1, t_1) - |\phi(\theta_1, t_1)|^2 \geq 0$. This means that $\|\phi\|_{L^\infty}^2/4 \leq |\phi(\theta_1, t_1)|^2 \leq (2M/a) = (2/a)\|\phi\|_{L^\infty}\|\eta\|_{L^\infty}$. Thus $\|\phi\|_{L^\infty} \leq (8/a)\|\eta\|_{L^\infty}$. □

Lemma A.6. *Let η be a compact-supported smooth section of E. Then there exists a smooth section ϕ of E satisfying $(\nabla^*\nabla + a)\phi = \eta$ and*

$$|\phi(x)| \leq \int_{S^3\times\mathbb{R}} g(x, y)|\eta(y)| d\mathrm{vol}(y).$$

Proof. Set $L^2_1(E) := \{\xi \in L^2(E) | \nabla\xi \in L^2\}$ and $(\xi_1, \xi_2)_a := (\nabla\xi_1, \nabla\xi_2)_{L^2} + a(\xi_1, \xi_2)_{L^2}$ for $\xi_1, \xi_2 \in L^2_1(E)$. (Since $a > 0$, this inner product defines a norm equivalent to the standard L^2_1-norm.) η defines the bounded functional

$$(\cdot, \eta)_{L^2} : L^2_1(E) \to \mathbb{R}, \quad \xi \mapsto (\xi, \eta)_{L^2}.$$

From the Riesz representation theorem, there uniquely exists $\phi \in L^2_1(E)$ satisfying $(\xi, \phi)_a = (\xi, \eta)_{L^2}$ for any $\xi \in L^2_1(E)$. Then we have $(\nabla^*\nabla + a)\phi = \eta$ in the sense of distribution. From the elliptic regularity, ϕ is smooth. ϕ and $\nabla\phi$ are in L^2, and $(\nabla^*\nabla + a)\phi = \eta$ is in L^∞. Hence we can apply Lemma A.4 to ϕ and get

$$|\phi(x)| \leq \int g(x, y)|\nabla^*\nabla\phi(y) + a\phi(y)|d\mathrm{vol}(y) = \int g(x, y)|\eta(y)|d\mathrm{vol}(y).$$

□

Proposition A.7. *Let η be a smooth section of E satisfying $\|\eta\|_{L^\infty} < \infty$. Then there exists a smooth section ϕ of E satisfying $(\nabla^*\nabla + a)\phi = \eta$ and*

$$|\phi(x)| \leq \int_{S^3\times\mathbb{R}} g(x, y)|\eta(y)|d\mathrm{vol}(y). \tag{56}$$

(Hence ϕ is in L^∞.) In particular, if η vanishes at infinity, then ϕ also vanishes at infinity. Moreover, if a smooth section $\phi' \in L^\infty(E)$ satisfies $(\nabla^*\nabla + a)\phi' = \eta$ (η does not necessarily vanishes at infinity), then $\phi' = \phi$.

Proof. Let ρ_n ($n \geq 1$) be the cut-off functions introduced in Lemma A.3, and set $\eta_n := \rho_n\eta$. From Lemma A.6, there exists a smooth section ϕ_n satisfying $(\nabla^*\nabla + a)\phi_n = \eta_n$ and

$$|\phi_n(x)| \leq \int g(x, y)|\eta_n(y)|d\mathrm{vol}(y) \leq \int g(x, y)|\eta(y)|d\mathrm{vol}(y). \tag{57}$$

Hence $\{\phi_n\}_{n\geq 1}$ is uniformly bounded. Then by using the Schauder interior estimate ([9, Chapter 6]), for any compact set $K \subset S^3 \times \mathbb{R}$, the $\mathcal{C}^{2,\alpha}$-norms of ϕ_n over K are bounded ($0 < \alpha < 1$). Hence there exists a subsequence $\{\phi_{n_k}\}_{k\geq 1}$ and a section ϕ of E such that $\phi_{n_k} \to \phi$ in the $\mathcal{C}^2$-topology over every compact subset in $S^3 \times \mathbb{R}$. Then ϕ satisfies $(\nabla^*\nabla + a)\phi = \eta$. ϕ is smooth by the elliptic regularity, and it satisfies (56) from (57).

Suppose η vanishes at infinity. Set $K := \int g(x, y)d\mathrm{vol}(y) < \infty$ (independent of x). For any $\varepsilon > 0$, there exists a compact set $\Omega_1 \subset S^3 \times \mathbb{R}$ such that $|\eta| \leq \varepsilon/(2K)$ on the complement of Ω_1. There exists a compact set $\Omega_2 \supset \Omega_1$ such that for any $x \notin \Omega_2$

$$\|\eta\|_{L^\infty} \int_{\Omega_1} g(x, y)d\mathrm{vol}(y) \leq \varepsilon/2.$$

Then from (56), for $x \notin \Omega_2$,

$$|\phi(x)| \leq \int_{\Omega_1} g(x, y)|\eta(y)|d\mathrm{vol}(y) + \int_{\Omega_1^c} g(x, y)|\eta(y)|d\mathrm{vol}(y) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that ϕ vanishes at infinity.

Suppose that smooth $\phi' \in L^\infty(E)$ satisfies $(\nabla^*\nabla + a)\phi' = \eta$. We have $(\nabla^*\nabla + a)(\phi - \phi') = 0$, and $\phi - \phi'$ is contained in L^∞. Then the L^∞-estimate in Proposition A.5 implies $\phi - \phi' = 0$. □

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