# LOGARITHMIC TRANSFORMATIONS OF RIGID ANALYTIC ELLIPTIC SURFACES

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Contents



## 1. Introduction

The main purpose of the present paper is to give examples of algebraic elliptic surfaces and non-algebraic rigid analytic elliptic surfaces by means of twistings and logarithmic transformations.

First, we construct twistings and logarithmic transformations of families of Tate curves on a disk. These operations come as surgeries of rigid analytic elliptic fibrations. Although Ueno gave a rough sketch of such surgeries in characteristic zero in [12, 6], we give precise definitions and proofs in arbitrary characteristics and unify these two kinds of surgery to algebraize the resulting surfaces.

In particular, we treat logarithmic transformations which produce multiple fibers whose multiplicities are divided by the characteristic of the base field. When such fibers appear, we obtain plenty of regular one-forms on the resulting surface.

In the algebraic case, Mumford discovered the following pathology in [10, II]: In the case of positive characteristic, there exists a proper smooth algebraic surface $X$ such that the dimension $h^0(\Omega^1_X)$ of regular one-forms is equal to one while the dimension $h^1(\mathcal{O}_X)$ of the first cohomology group of the structure sheaf is equal to zero.

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Algebraizing the above rigid analytic elliptic surfaces, we obtain algebraic surfaces $X$ such that the dimension $h^0(\Omega^1_X)$ is arbitrarily greater than the dimension $h^1(O_X)$.

Second, we treat algebraic elliptic surfaces of Kodaira dimension zero without singular fibers. Bombieri and Mumford classified the possible combinations of multiple fibers of such surfaces in [1]. However, it is unknown that all of possible combinations actually occur. We construct some cases of known and unknown elliptic surfaces of this type in a unified method.

Finally, we construct non-algebraic rigid analytic elliptic surfaces. In the case of characteristic zero, the equality $h^0(\Omega^1_X) = h^1(O_X)$ gives a necessary and sufficient condition for algebraicity of certain elliptic surfaces. In the case of positive characteristic, the condition of multiple fibers of algebraic elliptic surfaces (Theorem 3.3 in [5]) gives a criterion for non-algebraicity. Even in the situation where the criterion does not work, we give another criterion for non-algebraicity by Grothendieck's lifting criterion for projective smooth schemes (Théorème 7.3 in [3, III]).

**Notations and Conventions.** We fix an algebraically closed complete non-Archimedean valuation field $K$ with a non-trivial valuation. We assume that rigid spaces and schemes are defined over $K$. Let us denote the characteristic of $K$ by $p \geq 0$. We mainly use the terminologies and notations of [2], [9], [8], and [7].

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## 2. Preliminaries

In this section, we review some facts on rigid analytic elliptic fibrations. We refer to the previous paper [7] in detail.

An *elliptic fibration* (over a smooth curve) is a triple $(X, C, \pi)$ where $X$ is a smooth surface, $C$ is a smooth curve, and $\pi$ is a proper flat morphism from $X$ to $C$ with connected fibers satisfying the following condition. There exists a nowhere dense analytic subset $C_0$ of $C$ such that for any $p \in C - C_0$, the fiber $\pi^{-1}(p)$ is a proper smooth curve of arithmetic genus one over $K(p)$.

We define algebraic elliptic fibrations over algebraic smooth curves in the same way. An (algebraic) proper smooth surface $X$ is called an *elliptic surface* if $X$ admits a structure of (algebraic) elliptic fibration. The analytification of an algebraic elliptic fibration over an affinoid algebra $A$ is an elliptic fibration over $\mathrm{Sp}\, A$.

Let us denote the genus of a proper smooth curve $C$ by $g(C)$. For an elliptic fibration $(X, C, \pi)$, let us denote the length of the torsion submodule of $R^1\pi_*O_X$ by $l(\pi)$. We have the canonical bundle formula and Noether's formula.

**Proposition 2.1.** *Let $\pi\colon X \to C$ be an elliptic surface. Then there exists a canonical isomorphism*

$$\mathcal{K}_X \cong \pi^*\mathcal{L} \otimes_{O_X} O_X(D)$$

*where the line bundle $\mathcal{L}$ on $C$ and the divisor $D$ on $X$ satisfy the following conditions:*

(1) *The degree of $\mathcal{L}$ is equal to $\chi(O_X) + l(\pi) + 2g(C) - 2$.*
(2) *Let $C_0$ be the subset of points on $C$ where the fiber is not smooth. We write the multiplicity $m_q$ and the fiber $m_q D_q$ at $q \in C_0$. Then the divisor $D$ is given by the following equality:*

$$D = \sum_{q \in C_0} c_q D_q.$$

*Here, each integer $c_q$ satisfies the inequalities $0 \leq c_q < m_q$.*

*Remark.* If the fiber is tame at $q \in C_0$, then the integer $c_q$ is equal to $m_q - 1$. If all the fibers are tame, then the equality $l(\pi) = 0$ holds. This is the case when the characteristic $p$ is equal to zero.

**Proposition 2.2.** *Let $\pi\colon X \to C$ be an elliptic surface. Then the Euler characteristic $\chi(O_X)$ of the structure sheaf is equal to zero if and only if the fibration $\pi$ is free from singular fibers.*

## 3. Twistings and Logarithmic Transformations

In this section, we construct twistings and logarithmic transformations. We treat these two kinds of surgery in the same method. Since we use Tate's uniformization of elliptic curves, we only treat families of elliptic curves with j-invariants whose absolute values are greater than one. Note that, comparing to the logarithmic transformations of complex analytic elliptic surfaces ([6, 9]), we treat the restricted cases in the rigid analytic case.

Put $C := \operatorname{Sp} K\langle T\rangle$. Assume that the elliptic fibration $\pi : X \to C$ admits a *relative uniformization* in the following sense: There exists an analytic function $\alpha(T)$ on $C$ satisfying the following conditions. Put $\mathcal{X} := \mathbb{A}_K^{1\times} \times C$.

(1) The inequalities $0 < \alpha(t) < 1$ hold for all $t$.
(2) The quotient of $\mathcal{X}$ by the automorphism $(x, t) \mapsto (\alpha(t)x, t)$ is isomorphic to $X$ over $C$. Note that we have the fundamental domain $\{(x, t) \in \mathcal{X} \mid \alpha(t) \leq |x| \leq 1\}$ for this action.

In this case, we obtain the commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & \circlearrowleft & X \arrow[dl, "\pi"] \\ & C. & \end{tikzcd}$$

We take integers $a$ and $m$ such that the conditions $\gcd(a, m) = 1$, $a \neq 0$, and $m > 0$ are satisfied. Choose integers $b$ and $n$ such that the equality $ab - mn = 1$ holds. For a positive real number $\epsilon$, put

$$\mathcal{Y}_\epsilon := \{(y, u) \in \mathbb{A}_K^{1\times} \times \mathbb{A}_K^1 \mid |y^b u^m| \leq \epsilon\}.$$

Further, put

$$\mathcal{Y} := \mathcal{Y}_1,$$
$$\mathcal{Y}^* := \{(y, u) \in \mathcal{Y} \mid u \neq 0\},$$
$$C^* := C - \{0\},$$

and

$$\mathcal{X}^* := \psi^{-1}(C^*).$$

We may define the isomorphism $\mathcal{X}^* \to \mathcal{Y}^*$ by the equalities:

$$x = y^n u^a,$$
$$t = y^b u^m.$$

Let $Y$ be the quotient of $\mathcal{Y}$ by the automorphism:

$$(y, u) \mapsto \left(\alpha(y^b u^m)^{-m} y, \alpha(y^b u^m)^b u\right).$$

Note that we have the fundamental domain

$$\{(y, u) \in \mathcal{Y} \mid |\alpha(y^b u^m)|^m \leq |y| \leq 1\}$$

for this action. We define the fibration $\tau \colon Y \to C$ by a projection $(y, u) \mapsto y^b u^m$. Put $Y^* := \tau^{-1}(C^*)$. Let us denote the restrictions of $\pi$ and $\tau$ to the preimages of $C^*$ by $\pi^* \colon X^* \to C^*$ and $\tau^* \colon Y^* \to C^*$ respectively.

We construct the isomorphism $\mu \colon Y^* \to X^*$ over $C^*$ in the following way. We define *relative Tate's elliptic functions* on $\mathcal{Y}$ by the equalities (cf., [11]):

$$X_\alpha(x, t) := \sum_{i=-\infty}^{\infty} \frac{\alpha(t)^i x}{(1 - \alpha(t)^i x)^2} - 2 \sum_{i=1}^{\infty} \frac{t^i}{(1 - t^i)^2}$$

and

$$Y_\alpha(x, t) := \sum_{i=-\infty}^{\infty} \frac{(\alpha(t)^i x)^2}{(1 - \alpha(t)^i x)^3} + \sum_{i=1}^{\infty} \frac{t^i}{(1 - t^i)^2}.$$

These analytic functions give the morphism $\mathcal{X}^* \to \mathbb{P}^2_{C^*}$ by the correspondence:

$$(x, t) \mapsto ((X_\alpha(x, t) : Y_\alpha(x, t) : 1), t).$$

Let $Z^*$ be the image of this morphism. Then the morphism induces an isomorphism $X^* \to Z^*$. We may define a morphism $\mathcal{Y}^* \to Z^*$ by the correspondence:

$$(y, u) \mapsto \left((X_\alpha(y^n u^a, y^b u^m) : Y_\alpha(y^n u^a, y^b u^m) : 1), y^b u^m\right).$$

This morphism induces an isomorphism $Y^* \to Z^*$. Compositing the last isomorphism $Y^* \to Z^*$ and the inverse of the first isomorphism $X^* \to Z^*$, we obtain the isomorphism $\mu\colon Y^* \to X^*$ over $C^*$.

As a result, the fibration $\tau\colon Y \to C$ satisfies the following conditions.

(1) The fibration $\tau$ is an elliptic fibration with only one tame multiple fiber over the origin. The multiplicity of the multiple fiber is equal to $m$.
(2) The fibration $Y^*$ is isomorphic to the fibration $X^*$ over $C^*$. The isomorphism is induced by $\mu$.

The construction of the fibration $\tau$ dose not depend on the choice of $b$ and $n$.

For the relative uniformization $\psi : \mathcal{X} \to C$ of the elliptic fibration $\pi : X \to C$, let us denote the fibration $\tau\colon Y \to C$ by $L(0, m, a)X \to C$. When the positive integer $m$ is equal to one, we call the surgery the *twisting* of the elliptic fibration $\pi$. Otherwise, we call the surgery the *logarithmic transformation* of the elliptic fibration $\pi$.

Assume that the elliptic fibration $\pi$ admits another relative uniformization:

$$\begin{tikzcd} \mathcal{X}' \arrow[rr, "\phi'"] \arrow[dr, "\psi'"'] & \arrow[d, phantom, "\circlearrowleft"] & X \arrow[dl, "\pi"] \\ & C. & \end{tikzcd}$$

We take $\mathcal{Y}'$ and construct $Y'$ in the same way. Then the parameter $(x', t')$ of $\mathcal{X}'$ and the parameter $(y', u')$ of $\mathcal{Y}'$ satisfy the following equalities:

$$x' = y'^n u'^a,$$

$$t' = y'^b u'^m.$$

Suppose that the morphism $\lambda\colon \mathcal{X} \to \mathcal{X}', (x, t) \mapsto (\beta(t)x, \rho(t))$ is an isomorphism where $\beta(t)$ is a nowhere-zero analytic function on $C$ and $\rho$ is an automorphism of $C$. Then the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\lambda"] \arrow[dr, "\phi"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{X}' \arrow[dl, "\phi'"] \\ & X & \end{tikzcd}$$

is commutative. For a sufficiently small positive real number $\epsilon$, we may define the nowhere-zero analytic function $\gamma := \rho(t)/t$ for all $t$ such that $|t| \leq \epsilon$. We define the isomorphism $\widetilde{\lambda}\colon \mathcal{Y}_\epsilon \to \mathcal{Y}'_\epsilon$ over $C$ by the equality:

$$\widetilde{\lambda}(y, u) := \left(\beta(y^b u^m)^{-m} \gamma(y^b u^m)^a y, \beta(y^b u^m)^b \gamma(y^b u^m)^{-n} u\right).$$

Then the isomorphism $\widetilde{\lambda}$ induces the isomorphism $Y \to Y'$ over $C$.

## 4. Regular One-Forms and Regular Vector Fields

In this section, we calculate the dimensions of cohomology groups of the regular one-forms and regular vector fields on the modified elliptic surfaces.

Let $\alpha$ is an element of $K$ such that the inequalities $0 < |\alpha| < 1$ hold. Let $C$ be a smooth curve. Put $E := \mathbb{A}_K^{1\times}/\langle\alpha\rangle$, $\mathcal{X} := \mathbb{A}_K^{1\times} \times C$, and $X := E \times C$. We consider the surgery in the previous section and use the notations in the previous section.

First, we assume that the curve $C$ is isomorphic to the disk $\operatorname{Sp} K\langle T\rangle$. Since we have the isomorphism

$$\Omega_X^1 \cong \pi^* O_C \frac{dx}{x} \oplus \pi^*\Omega_C^1,$$

we obtain the isomorphism:

$$\phi^*\Omega_X^1 \cong \psi^* O_C \frac{dx}{x} \oplus \psi^*\Omega_C^1.$$

Thus, we may write a global section of this sheaf on $\mathcal{X}^*$ in the following way:

$$F(t)\frac{dx}{x} + G(t)\,dt.$$

The global section may be written as

$$F(y^b u^m)\left(n\frac{dy}{y} + a\frac{du}{u}\right) + G(y^b u^m)(by^{b-1}u^m\,dy + my^b u^{m-1}\,du)$$

on $\mathcal{Y}^*$. We assume that the global section extends to the whole space $\mathcal{Y}$.

If the characteristic $p$ divides $m$, then the function $F(t)$ has an at least first order zero, and the function $G(t)$ has an at most first order pole at the origin.

If the characteristic $p$ does not divide $n$, then the function $F(t)$ is analytic at the origin. In this case, if the value $F(0)$ is zero, then the function $G(t)$ is analytic at the origin. Otherwise, the function $G(t)$ has a first order pole with the residue $-aF(0)/m$ at the origin.

In the same method, we may describe a global section of $\Omega_X^1 \otimes O_X((m-1)D_0)$ on $\mathcal{Y}$. We write a global section of this sheaf on $\mathcal{X}^*$ in the following way:

$$F(t)\frac{dx}{x} + G(t)\,dt.$$

We assume that the global section extends to the whole space $\mathcal{Y}$. Then the function $F(t)$ is analytic at the origin, and the function $G(t)$ has an at most first order pole at the origin.

Next, we assume that $C$ is a proper smooth curve of genus $g(C)$. Let $S$ be a finite subset of the points of $C$. For each element $q$ of $S$, we take integers $a_q$ and $m_q$ such that the conditions $\gcd(a_q, m_q) = 1$, $a_q \neq 0$, and $m_q > 0$ holds. Put $T := \{q \mid p \text{ divides } m_q\}$. If the characteristic $p$ is equal to zero, then the set $T$ is empty. For all elements $q$ of $S$, we apply $L(q, m_q, a_q)$ to $X$. Let us denote the resulting elliptic fibration by $X(\{(q, a_q/m_q)\}_{q\in S})$. Put $Y := X(\{(q, a_q/m_q)\}_{q\in S})$.

**Theorem 4.1.** *If the set $T$ is non-empty, then the equality $h^0(\Omega^1_Y) = g(C) + \sharp T - 1$ holds.*

*Proof.* The above calculations show that the equality

$$h^0(\Omega^1_Y) = h^0\left(\Omega^1_C \otimes \mathcal{O}_C(D)\right)$$

holds where the divisor $D$ is given by $\sum_{q\in T} q$. This proves the theorem. □

Assume that the set $T$ is empty. We consider the first Cousin problem of the rational differential one-forms on the proper smooth curve $C$. The sheaf exact sequence

$$0 \longrightarrow \mathcal{K}_C \longrightarrow \mathcal{M}_C \otimes \mathcal{K}_C \longrightarrow (\mathcal{M}_C/\mathcal{O}_C) \otimes \mathcal{K}_C \longrightarrow 0.$$

gives the connecting homomorphism $\delta\colon H^0(C, (\mathcal{M}_C/\mathcal{O}_C)\otimes\mathcal{K}_C) \to H^1(C, \mathcal{K}_C)$. We put

$$\theta := \sum_{q\in S} \frac{a_q}{m_q t_q}\, dt_q$$

in $H^0(C, (\mathcal{M}_C/\mathcal{O}_C) \otimes \mathcal{K}_C)$ where $t_q$ is a parameter at the point $q$. Put

$$\eta := \sum_{q\in S} \frac{a_q}{m_q}$$

in $K$. Then the image of $\delta\theta$ under the residue map $H^1(C, \mathcal{K}_C) \to K$ is equal to $\eta$.

**Theorem 4.2** (cf., Lemma 6.1 in [12])**.** *If the set $T$ is empty, then the equality*

$$h^0(\Omega^1_Y) = \begin{cases} g(C), & \eta \neq 0, \\ g(C) + 1, & \eta = 0 \end{cases}$$

*holds.*

**Theorem 4.3.** *If the base space $C$ is isomorphic to the projective line, then the equality*

$$h^2(\Theta_Y) = \begin{cases} 0, & \sharp S \leq 3, \\ \sharp S - 3, & \textit{otherwise} \end{cases}$$

*holds.*

*Proof.* Since the base space is isomorphic to the projective line, there exists a canonical isomorphism

$$\mathcal{K}_{\mathcal{Y}} \cong \tau^* \mathcal{O}_{\mathbb{P}^1_K}(-2) \otimes \mathcal{O}_{\mathcal{Y}}\left(\sum_{q\in S}(m_q - 1)D_q\right)$$

where the divisor $m_q D_q$ is the multiple fiber over $q$. Calculation of $h^0(\Omega^1_Y \otimes \mathcal{K}_Y)$ and the Serre duality theorem show the theorem. □

*Remark.* In this case, in the case of positive characteristic, if the elliptic surface $Y$ is algebraic and if the inequality $\sharp S \leq 3$ holds, then the surface is liftable to characteristic zero.

## 5. Criterion for Algebraicity

We study the elliptic surface $Y$ over the proper smooth curve that is constructed in the previous section. We use the notations in the previous section.

Let $l$ be the least common multiple of the integers $\{m_q\}_{q\in S}$. We define the divisor $D$ on $C$ by the equality:

$$D := \sum_{q\in S} \frac{a_q l}{m_q} \cdot q.$$

**Theorem 5.1.** *If there exists an integer $m$ such that the divisor $mD$ is principal, then the elliptic surface $Y$ is algebraic.*

*Proof.* The above condition shows that there exists a rational function $f$ on $C$ such that the divisor $(f)$ is equal to $mD$. Put $M := lm$. We define the divisor $\mathcal{D}^*$ on the relative uniformization $\mathbb{A}^{1\times}_K \times (C - S)$ by the equality $x^M - f$. Put $D^* := \phi(\mathcal{D}^*)$. Then the image $D^*$ is a divisor on the preimage of $C - S$ under $\tau$. We show that the divisor $D^*$ extends to the whole rigid analytic space $Y$. For $q \in S$, we take parameters $(x, t)$ and $(y, u)$ for the surgery $L(q, m_q, a_q)$. Take integers $b$ and $n$ such that $a_q b_q - m_q n_q = 1$. We may write $f(t) = t^{a_q M/m_q} g(t)$ on a neighborhood of $q$ where $g(t)$ is a nowhere-zero analytic function. Then the divisor $\mathcal{D}^*$ extends to the divisor $\mathcal{D}$ on $\mathcal{Y}$ by the equality:

$$x^M - f(t) = y^{n_q M} u^{a_q M}\left(1 - y^{M/m_q} g(y^{b_q} u^{m_q})\right).$$

Thus, the divisor $D^*$ extends to the divisor $D$ on $Y$. Since the image $\tau(D)$ is equal to $C$, Lemma 4.4 in [7] proves the theorem. □

*Remark.* In this case, if the set $T$ is empty, then the equality $h^0(\Omega^1_Y) = g(C) + 1$ holds.

## 6. Algebraic Elliptic Surfaces with $\kappa = 0$ and $\chi = 0$

We construct algebraic elliptic surfaces $\tau\colon Y \to C$ of Kodaira dimension zero without singular fibers by Theorem 5.1. The notation $(c_1/m_1, \ldots, c_n/m_n)$ describes the multiple fibers of the fibration $\tau$. The integer $m_i$ is the multiplicity of the $i$-th multiple fiber. The integer $c_i$ is the coefficient of the divisor in the canonical bundle formula. The symbol $*$ over $m_i$ means that the $i$-th multiple fiber is wildly ramified.

From now on, we assume that the curve $C$ is isomorphic to the projective line. By Theorem 5.1, we obtain some examples.

**Example 6.1.** We obtain the algebraic elliptic surface

$$X((0, -1/2), (1, 1/3), (\infty, 1/6))$$

with $\kappa = 0$ and $\chi = 0$. The combination of the multiple fibers is of type $(1/2, 2/3, 5/6)$. In the same way, we obtain the algebraic elliptic surfaces with the desired invariants whose multiple fibers are of type $(1/2, 3/4, 3/4)$, $(2/3, 2/3, 2/3)$, and $(1/2, 1/2, 1/2, 1/2)$.

**Example 6.2.** Let $\lambda$ be a point on $C$ which is not equal to $-1$, $0$, $1$, or $\infty$. We obtain the algebraic elliptic surface

$$X((0, 1/2), (1, 1/2), (\lambda, 1/2), (\lambda + 1, 1/2), (\infty, -2/1))$$

with $\kappa = 0$ and $\chi = 0$. The combination of the multiple fibers is of type $(1/2, 1/2, 1/2, 1/2)$. Assume that the characteristic $p$ is equal to two. By the quotient as in Example 8.1 in [5], we obtain the algebraic elliptic surface with the same invariants. The combination of the multiple fibers is of type $(1/2^*, 1/2, 1/2)$.

**Example 6.3.** Let $\lambda$ be a point on $C$ which is not equal to $0$, $1$, or $\infty$. We obtain the algebraic elliptic surface

$$X((0, 1/3), (1, 1/3), (\lambda, 1/3), (\infty, -1/1))$$

with $\kappa = 0$ and $\chi = 0$. The combination of the multiple fibers is of type $(2/3, 2/3, 2/3)$. Assume that the characteristic $p$ is equal to three. Put $\lambda := 2$. By the quotient as in Example 8.1 in [5], we obtain the algebraic elliptic surface with the same invariants. The combination of the multiple fibers is of type $(1/3^*, 2/3)$.

## 7. Criterion for Non-Algebraicity

In this section, we give criterions for non-algebraicity of proper smooth rigid analytic surfaces and examples of non-algebraic elliptic surfaces.

| $g(C)$ | $l(\tau)$ | multiple fibers | $p$ | examples | |
|---|---|---|---|---|---|
| 0 | 0 | $(1/2, 2/3, 5/6)$ | 2 | ○ | |
| | | | 3 | ○ | |
| | | | $\neq 2, 3$ | ○ | [1, 3] |
| | | $(1/2, 3/4, 3/4)$ | 2 | ○ | |
| | | | $\neq 2$ | ○ | [1, 3] |
| | | $(2/3, 2/3, 2/3)$ | 3 | ○ | [4, 5.2] |
| | | | $\neq 3$ | ○ | [1, 3] |
| | | $(1/2, 1/2, 1/2, 1/2)$ | 2 | ○ | |
| | | | $\neq 2$ | ○ | [1, 3] |
| 0 | 1 | $(0/2^*, 1/2, 1/2)$ | 2 | ○ | |
| | | $(1/2^*, 1/2)$ | 2 | | unknown |
| | | $(1/3^*, 2/3)$ | 3 | ○ | [4, 5.2] |
| | | $(1/4^*, 3/4)$ | 2 | | unknown |
| | | $(2/4^*, 1/2)$ | 2 | | unknown |
| | | $(2/6^*, 2/3)$ | 2 | | [5, 8.4] |
| | | $(3/6^*, 1/2)$ | 3 | | [5, 8.4] |
| 0 | 2 | $(0/p^{a*})$ | $\neq 0$ | | [1, 3], [5, 8.1] (some cases) |
| | | $(0/p^{a*}, 0/p^{b*})$ | $\neq 0$ | | [1, 3] (some cases) |
| 1 | 0 | none | | | product of two elliptic curves |

Table 1. The possible invariants of elliptic surfaces $\tau\colon Y \to C$ of Kodaira dimension zero without singular fibers. The symbol ○ means that we treat the type in the present paper.

**Proposition 7.1** (cf., Theorem 27 in [6]). *Assume that the characteristic $p$ of the base field $K$ is equal to zero. Suppose that the element $\eta$ in Section 4 is not equal to zero. Then the surface $Y$ in Section 4 is not algebraic.*

*Remark.* If the genus of the base curve is equal to zero, then Theorem 5.1 and the above proposition give the necessary and sufficient condition for the algebraicity of the surface $Y$ (cf., Appendix 1 in [5]).

*Proof.* Since the characteristic $p$ is equal to zero, for any proper algebraic surface $Z$, the equality $h^1(\mathcal{O}_Z) = h^0(\Omega^1_Z)$ holds. Since we have the equality $h^0(\mathcal{K}_Y) = g(C)$, by the Serre duality theorem and by the equality $\chi(\mathcal{O}_Y) = 0$, we obtain the equality $h^1(\mathcal{O}_Y) = g(C) + 1$. Thus, Theorem 4.2 shows the proposition. $\square$

**Lemma 7.2.** *Assume that the characteristic of the base field $K$ is positive. Then any proper smooth algebraic surface $Z$ with the invariants $\chi(\mathcal{O}_Z) = 0$, $h^0(\mathcal{K}_Z) = 0$, and $h^0(\Omega_Z^1) = 0$ is non-liftable to characteristic zero.*

*Proof.* Assume that the surface $Z$ admitted a lifting. Let $Y$ be the generic fiber of this lifting. The constancy of the Euler characteristic and the upper-semicontinuity of the dimension of the cohomology groups of coherent sheaves on a proper flat family of schemes shows that the equalities $\chi(\mathcal{O}_Y) = 0$, $h^0(\mathcal{K}_Y) = 0$, and $h^0(\Omega_Y^1) = 0$ hold. The Serre duality theorem (Theorem in [13, 5.1]) shows that the equality $h^0(\mathcal{K}_K) = 0$ holds. Therefore, we obtain the inequality $h^0(\Omega_Y^1) < h^1(\mathcal{O}_Y)$. Since the inequality dose not hold in characteristic zero, this is a contradiction. □

The above lemma, the canonical bundle formula, and Noether's formula give the following proposition.

**Proposition 7.3.** *Assume that the characteristic of the base field $K$ is positive. Then any algebraic elliptic surface $Z$ over the projective line without singular fibers with the invariant $h^0(\Omega_Z^1) = 0$ is non-liftable to characteristic zero.*

The above proposition and Grothendieck's lifting criterion (Théorème 7.3 in [3, III]) show the following proposition.

**Proposition 7.4.** *Assume that the characteristic of the base field $K$ is positive. Then any elliptic surface $Z$ over the projective plane without singular fibers with the invariants $h^0(\Omega_Z^1) = 0$ and $h^2(\Theta_Z) = 0$ is not algebraic.*

The condition $U_i$ (Theorem 3.3 in [5]) gives a criterion for non-algebraicity of some elliptic surfaces. Here, we give an example of non-algebraic elliptic surfaces such that the criterion does not work. Let $q$ and $r$ be two distinct points on the projective line $C$. Assume that the integers $m_q$ and $m_r$ are larger than one and not divided by $p$. Put $a_q = 1$ and $a_r = m_r - 1$. Then we can not conclude that the resulting surface $Y$ is not algebraic only by the conditions $U_1$ and $U_2$. The above proposition proves, however, that the surface $Y$ is not algebraic.

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