CLASSIFICATION OF RIGID ANALYTIC SURFACES

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1. Introduction

The main purpose of the present paper is to give classification of proper regular rigid analytic surfaces. The classification is similar to Kodaira's classification of complex analytic surfaces. Some analogous theorems hold and certain positive characteristic phenomena occur.

We treat proper regular rigid analytic surfaces of algebraic dimension one or two, especially elliptic surfaces. Bombieri and Mumford studied Enriques' classification of algebraic surfaces in arbitrary characteristic and gave a list of possible invariants and combinations of multiple fibers for certain elliptic surfaces in [1]. We give a similar list for rigid analytic surfaces. In the rigid analytic case, non-algebraic elliptic surfaces appear with

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the invariants and combinations of multiple fibers that can never appear in the algebraic case, nor in the complex analytic case.

Ueno first studied rigid analytic surfaces in [29]. We gave the detailed proofs of the study in [23] and [21] partially. Using these results, we study proper regular rigid analytic surfaces. In the following sketch of the results, for simplicity, we assume that the base field is algebraically closed.

First, we review and develop the fundamental theories of rigid analytic surfaces. We treat the intersection theory for non-proper normal rigid analytic surfaces, flat base change theorems, resolutions of singularities, and fields of meromorphic functions.

Second, we study rigid analytic elliptic fibrations. We give a relative duality theorem, a canonical bundle formula, and a relative Noether's formula. In the algebraic case, these formulas correspond to special cases of the Grothendieck duality theorem, the canonical bundle formula of Bombieri and Mumford [1], and the relative Riemann-Roch theorem of Deligne [12] respectively. As for the relative Noether's formula, we have to modify the relative Riemann-Roch theorem by the study of reductions of curves of genus one [19] since the fibrations are not always smoothly fibered.

Third, we study proper smooth rigid analytic surfaces of algebraic dimension one or two. We prove that proper smooth rigid analytic surfaces of algebraic dimension two are projective rigid analytic surfaces and those of algebraic dimension one are rigid analytic elliptic surfaces.

Finally, we study proper rigid analytic elliptic fibrations. The canonical bundle formula and Noether's formula enable us to classify minimal proper rigid analytic elliptic fibrations that are not properly elliptic. We give a list of possible invariants and combinations of multiple fibers for these surfaces.

Let us be more precise on the first two parts of the story. In the first part, we algebraize the normalizations and blowing-ups of rigid analytic surfaces locally in order to apply Lipman's desingularizations [17]. Then we obtain regular models of quasi-compact rigid analytic surfaces. Using this result, we construct algebraic reduction morphisms.

In the second part, we reduce the discussion on rigid analytic elliptic fibrations to the corresponding algebraic arguments by using Conrad's étale coverings [9] and the Zariski's main theorem of de Jong and van der Put [11]. Since the base changes of rigid analytic elliptic fibrations via the étale coverings are projective, by Köpf's GAGA theorems [15], we may apply algebraic results to the covering elements. Finally, to conclude the proof of the desired theorems, we use the faithfully flat descent theory of Bosch and Görtz [3].

Notations and Conventions. We fix a complete non-Archimedean valuation field K with a non-trivial valuation and assume that rigid analytic spaces are defined over K. We mainly use the terminologies and notations of [4], [23], and [21].

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2. Preliminaries

2.1. **Intersection Theory.** In this subsection, we assume that X is a normal surface. We explain the intersection theory on X. For a divisor D on X, let us denote the line bundle $\mathcal{O}_X(D)$ by $[D]$. For an effective divisor D on X, let us denote a coherent sheaf $\mathcal{O}_X/\mathcal{O}_X(-D)$ on X by $\mathcal{O}_D$.

First, we summarize the results on the intersection theory on proper normal surfaces in [21, 5]. Assume that X is proper over K. For two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X, we define the intersection number of $\mathcal{L}_1$ and $\mathcal{L}_2$ by

$$\chi(\mathcal{O}_X) - \chi(\mathcal{L}_1^\vee) - \chi(\mathcal{L}_2^\vee) + \chi(\mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee).$$

Let us denote this integer by $\mathcal{L}_1 \cdot \mathcal{L}_2$. For a line bundle $\mathcal{L}$ and for two divisors D and E on X, we abbreviate $\mathcal{L} \cdot [D]$, $[D] \cdot \mathcal{L}$, and $[D] \cdot [E]$ to $\mathcal{L} \cdot D$, $D \cdot \mathcal{L}$, and $D \cdot E$ respectively.

For any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, the following equalities hold:

(1) $\mathcal{L}_1 \cdot \mathcal{O}_X = \mathcal{O}_X \cdot \mathcal{L}_1 = 0$;
(2) $\mathcal{L}_1 \cdot \mathcal{L}_2 = \mathcal{L}_2 \cdot \mathcal{L}_1$.

Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ be line bundles on X. Assume that one of these line bundles is isomorphic to $[D]$ where D is a divisor. Then the following equality holds:

$$\mathcal{L}_1 \cdot \mathcal{L}_2 \otimes \mathcal{L}_3 = \mathcal{L}_1 \cdot \mathcal{L}_2 + \mathcal{L}_1 \cdot \mathcal{L}_3.$$

In particular, for any two divisors $\sum_{i\in I} a_i D_i$ and $\sum_{j\in J} b_j D_j$, the following equality holds:

$$\sum_{i\in I} a_i D_i \cdot \sum_{j\in J} b_j D_j = \sum_{i\in I, j\in J} a_i b_j D_i \cdot D_j.$$

Let D be a divisor $\sum_{i\in I} a_i D_i$ on X. Let $\pi_i \colon D_i^n \to D_i$ be the normalization of a prime divisor D_i. Then, for any line bundle $\mathcal{L}$ on X, the following equality holds:

$$(*) \qquad \mathcal{O}_X(D) \cdot \mathcal{L} = \sum_{i\in I} a_i \deg_{D_i^n} \pi_i^* \mathcal{L}.$$

The *local intersection number* of two distinct prime divisors D_1 and D_2 at a point p on $D_1 \cap D_2$ is the integer $\dim_K(\mathcal{O}_X/\mathcal{O}_X(-D_1) + \mathcal{O}_X(-D_2))_p$. Let us denote this number by $I(p, D_1, D_2)$. For any two distinct prime divisors D_1 and D_2 on X, the following equality holds:

$$D_1 \cdot D_2 = \sum_{p \in D_1 \cap D_2} I(p, D_1, D_2).$$

We have the Riemann-Roch theorem for proper smooth surfaces. For any line bundle $\mathcal{L}$ on a proper smooth surface X, the following equality holds:

$$\chi(\mathcal{L}) = \frac{\mathcal{L} \cdot \mathcal{K}_X^\vee - \mathcal{L} \cdot \mathcal{L}^\vee}{2} + \chi(O_X).$$

For a general normal surface X, we define the intersection number of a proper divisor D and a line bundle $\mathcal{L}$ by the equality $(*)$. Then the other results hold.

We define a virtual genus $\pi(D)$ of a divisor D on a normal surface by the equality:

$$\pi(D) := \frac{D \cdot D + D \cdot \mathcal{K}_X}{2} + 1.$$

When the surface X is smooth and the divisor D is proper and smooth, the virtual genus $\pi(D)$ is equal to the arithmetic genus of D. Using a local tubular neighborhood covering of D (Theorem 2.12 in [21]), we can show this fact.

2.2. **Coherent Modules and Flat Base Changes.** In this subsection, we introduce sheaves of morphisms between coherent modules. Then we state the flat base change theorems of coherent sheaves.

Let $\mathcal{F}$ and $\mathcal{G}$ be coherent O_X-modules on a rigid analytic space X. For an admissible affinoid open subset U, put

$$\mathcal{H}\mathrm{om}_{O_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{O_X(U)}(\mathcal{F}(U), \mathcal{G}(U)).$$

For two admissible affinoid open subsets U and V of X, we assume that U contains V. Since any affinoid algebra is Noetherian, by (3.E) in [20], there exists a canonical isomorphism:

$$\mathrm{Hom}_{O_X(V)}\, B(\mathcal{F}(V), \mathcal{G}(V)) \cong \mathrm{Hom}_{O_X(U)}(\mathcal{F}(U), \mathcal{G}(U)) \otimes_{O_X(U)} O_X(V).$$

Thus, by [4, 9.4.2], the above definition gives a sheaf on each admissible affinoid open subset. Then we obtain an O_X-module. Since the right-hand side of the above definition is a finite A-module, the O_X-module is coherent. Let us denote this coherent O_X-module by $\mathcal{H}\mathrm{om}_{O_X}(\mathcal{F}, \mathcal{G})$. For simplicity, we put

$$\mathcal{F}^\vee := \mathcal{H}\mathrm{om}_{O_X}(\mathcal{F}, O_X).$$

This notation is compatible with the same one for line bundles.

A morphism $f\colon X \to Y$ of rigid analytic spaces is said to be *flat* if the morphism f satisfies the following condition, Let $\operatorname{Sp} A$ and $\operatorname{Sp} B$ be admissible affinoid open subsets of X and Y respectively. Assume that the preimage $f^{-1}(\operatorname{Sp} B)$ contains $\operatorname{Sp} A$. Then the corresponding K-algebra homomorphism $B \to A$ is flat. This condition is equivalent to the following local one. For any point x in X, the associated local ring homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We show this fact in the following way. First, note that any stalk of the structure sheaf of any rigid analytic space is Noetherian (Proposition 7 in [4, 7.3.2]). Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of A and B corresponding to the points x and $f(x)$ respectively. Then the diagram

$$\begin{tikzcd} B_{\mathfrak{n}} \arrow[r] \arrow[d] & \mathcal{O}_{Y,f(x)} \arrow[r] \arrow[d] & \widehat{\mathcal{O}}_{Y,f(x)} \arrow[d] \\ A_{\mathfrak{m}} \arrow[r] & \mathcal{O}_{X,x} \arrow[r] & \widehat{\mathcal{O}}_{X,x} \end{tikzcd}$$

is commutative where $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{Y,f(x)}$ are the completions of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$ with respect to the maximal ideals respectively. By Proposition 3 in [4, 7.3.2] and (24.A) Theorem 56 in [20], the horizontal homomorphisms in the above diagram are faithfully flat. Then Proposition 3 in [4, 7.3.2] and (20.G) Application 3 in [20] shows that the morphism $B_{\mathfrak{n}} \to A_{\mathfrak{m}}$ is flat if and only if the morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. This proves the fact.

By (3.E) in [20], we obtain the following proposition.

Proposition 2.1. *Let $f\colon X \to Y$ be a flat morphism of rigid analytic spaces. Then, for two coherent $\mathcal{O}_Y$-modules $\mathcal{F}$ and $\mathcal{G}$, there exists a canonical isomorphism:*

$$f^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G}).$$

In particular, there exists a canonical isomorphism:

$$f^*(\mathcal{F}^\vee) \cong (f^*\mathcal{F})^\vee.$$

The following proposition shows that base change preserves flatness.

Proposition 2.2. *Any base change of any flat morphism of rigid analytic spaces is flat.*

Proof. Let $f\colon X \to Y$ and $u\colon Z \to Y$ be morphisms of rigid analytic spaces. Assume that the morphism f is flat. We show that the base change of f via u is flat. We may assume that the rigid analytic spaces X, Y, and Z are affinoid spaces. Thus, it suffices to show that, for two homomorphisms $B \to A$ and $B \to C$ between affinoid algebras, if the homomorphism $B \to A$ is flat, then the base change $C \to A \widehat{\otimes}_B C$ is flat. Further, we may assume that C is finite over B or isomorphic to the free affinoid algebra $B\langle X\rangle$ in one indeterminate over B.

The first case follows from the isomorphism $A \mathbin{\widehat{\otimes}}_B C \cong A \otimes_B C$ (Proposition 6 in [4, 3.7.3]). Then we consider the second case. We take a maximal ideal $\mathfrak{m}'$ of the free affinoid algebra $A\langle X\rangle$ over A. Let $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{n}'$ be the prime ideals of $A[X]$, $B[X]$, and $B\langle X\rangle$ that are the pull-backs of $\mathfrak{m}$ under the natural homomorphisms respectively. Then the prime ideals are maximal since Corollary 3 in [4, 6.1.2] implies that the quotient rings $A[X]/\mathfrak{m}$, $B[X]/\mathfrak{n}$, and $B\langle X\rangle/\mathfrak{n}'$ are finite extension fields of the base field K.

The same corollary gives an isomorphism $A[X]/\mathfrak{m}^n \cong A\langle X\rangle/\mathfrak{m}'^n$ for any positive integer n. Thus, the completion $\widehat{A[X]}_{\mathfrak{m}}$ of $A[X]_{\mathfrak{m}}$ with respect to $\mathfrak{m}$ is isomorphic to the completion $\widehat{A\langle X\rangle}_{\mathfrak{m}'}$ of $A\langle X\rangle_{\mathfrak{m}'}$ with respect to $\mathfrak{m}'$. By (24.A) Theorem 56 in [20], the natural homomorphisms $A[X]_{\mathfrak{m}} \to \widehat{A[X]}_{\mathfrak{m}}$ and $A\langle X\rangle_{\mathfrak{m}'} \to \widehat{A\langle X\rangle}_{\mathfrak{m}'}$ are faithfully flat. Therefore, the natural homomorphism $A[X]_{\mathfrak{m}} \to A\langle X\rangle_{\mathfrak{m}'}$ is flat. In the same way, the natural homomorphism $B[X]_{\mathfrak{n}} \to B\langle X\rangle_{\mathfrak{n}'}$ is flat. Since the ring A is flat over B, the localization $A[X]_{\mathfrak{m}}$ is flat over $B[X]_{\mathfrak{n}}$. Therefore, by (20.G) Application 3 in [20], the natural homomorphism $B\langle X\rangle_{\mathfrak{n}'} \to A\langle X\rangle_{\mathfrak{m}'}$ is flat. This proves the proposition. □

The following proposition shows that cohomology commutes with flat base change under certain assumption.

Proposition 2.3. *For a proper morphism $f\colon X \to Y$ of rigid analytic spaces and a flat morphism $u\colon Z \to Y$ of rigid analytic spaces, let g and v be the morphism of rigid analytic spaces in the following Cartesian diagram:*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r, "u"] & Y. \end{tikzcd}$$

Assume that the preimage of any point on Y is of dimension zero. Then for a coherent $\mathcal{O}_X$-module $\mathcal{F}$, there exists a canonical isomorphism

$$u^* R^q f_* \mathcal{F} \cong R^q g_* v^* \mathcal{F}$$

for all $q \geq 0$.

Proof. First, note that Lemma 1 in [4, 9.6.2] shows that the base change g is proper. We may assume that the rigid analytic spaces Y and Z are affinoid spaces. We write $Y = \operatorname{Sp} A$ and $Z = \operatorname{Sp} B$. Since the sheaves appearing in the above isomorphism are all coherent modules, it suffices to show that there exists a canonical isomorphism:

$$H^q(X, \mathcal{F}) \otimes_A B \cong H^q(X \times_{\operatorname{Sp} A} \operatorname{Sp} B, v^*\mathcal{F}).$$

By the same method as in the proof of Satz 3.5, we may assume that the affinoid algebra A is an Artin ring. Since the affinoid algebra B is also an Artin ring by assumption, Corollary 3 in [4, 6.1.2] shows that B is a finite A-algebra. Thus, Proposition 6 [4, 3.7.3] shows that, for any affinoid algebra C over A, the natural homomorphism $C \otimes_A B \to C \widehat{\otimes}_A B$ is an isomorphism. Therefore, since the affinoid algebra C is a Noetherian ring, for any finite C-module M, the natural homomorphism $M \otimes_A B \to M \widehat{\otimes}_A B$ is an isomorphism.

Since the morphism f is proper, there exists a finite admissible affinoid covering $\mathcal{U}$ of X. Since the morphism f is separated, the covering $\mathcal{U}$ is acyclic for $\mathcal{F}$. Then the pull-back of the covering $\mathcal{U}$ under v is a finite admissible affinoid covering of $X \times_{\mathrm{Sp}\, A} \mathrm{Sp}\, B$ that is acyclic for $v^*\mathcal{F}$. Therefore, the proposition follows from the flatness of u. □

A rigid analytic space is said to be *geometrically connected* (resp. *geometrically reduced*) if any base field extension of the rigid analytic space is connected (resp. reduced). If the base field is perfect, any rigid analytic space is geometrically reduced (Lemma 3.3.1 1 in [8])

Proposition 2.4. *If a proper reduced connected rigid analytic space X is geometrically connected (resp. geometrically connected and geometrically reduced), then the finite K-algebra $\mathcal{O}_X(X)$ is a purely inseparable extension field of K (resp. isomorphic to K).*

Proof. First note that cohomology commutes with base change of the base field (Theorem A.1.2 in [10]). Using the completion of the separable closure, and algebraic closure of K and Corollary 2 in [4, 9.3.3], we can prove this proposition by the same method as in the proof of Corollary 3.3.21 in [18]. □

2.3. **Resolutions of Singularities.** Schoutens gave the resolutions of singularities of rigid analytic spaces over an algebraically closed base field of characteristic zero in [28]. Here, we give the resolutions of singularities of rigid analytic spaces of dimension two over an arbitrary base field.

Theorem 2.5. *For any quasi-compact reduced rigid analytic space X of dimension two, there exists a quasi-compact reduced regular rigid analytic space Y and a bimeromorphic morphism $Y \to X$.*

Proof. First, note that any affinoid algebra is excellent ([8, 1]). Therefore, the regular locus of any reduced affinoid space is a non-empty Zariski open subset and any normal affinoid space of pure dimension two has at most finitely many singular points. We normalize X (see [8, 2.1]) and blow up all the singular points of the resulting surface (see Definition 4.1.1 in [10]). We have only to show that we can obtain a regular surface in a finite succession of this procedure. To show this, we may assume that X is an affinoid space. Then the analytic blowing-up of X coincides with the analytification of the algebraic one. Thus, the termination follows from the remark B in the introduction of [17]. □

2.4. **Fields of Meromorphic Functions.** In this subsection, we assume that X is a proper irreducible surface. We review some facts on the field of meromorphic functions $\mathcal{M}(X)$ on X. Then we construct the algebraic reduction morphism.

The *algebraic dimension* of X is the transcendental degree of $\mathcal{M}(X)$ over K. By $a(X)$ we denote this integer. The following theorem on $\mathcal{M}(X)$ is a special case of Theorem 5.1 (ii) in [2].

Theorem 2.6. *The field of meromorphic functions $\mathcal{M}(X)$ is an algebraic function field over K. Moreover, the algebraic dimension $a(X)$ is at most two.*

Lemma 2.7. *Assume that X is normal. Let Y be a projective regular model of the algebraic function field $\mathcal{M}(X)$. Then there exist a proper regular surface Z, a bimeromorphic morphism $\pi\colon Z \to X$, and a proper morphism $\rho\colon Z \to Y$ such that the associated morphism $\mathcal{O}_Y \to f_*\mathcal{O}_Z$ is an isomorphism. Moreover, the morphism ρ is surjective and has connected fibers.*

Proof. Let f_0 be the constant function 1_X on X. We choose a generating family $\{f_1, \ldots, f_n\}$ of $\mathcal{M}(X)$ over K such that the map $(f_0 : \cdots : f_n)$ gives the embedding of Y to a projective space. We define the analytic subset Γ of $X \times Y$ in the following way. For each $i = 0, \ldots, n$, we write $(f_i) = \sum_D a_{iD} D$. Put $b_D := \min_{i=0,\ldots,n} a_{iD}$. Let $\{\mathrm{Sp}\, A_j\}_{j\in J}$ be a finite locally principal covering for a divisor $\sum_D b_D D$. For each $j \in J$, let g_j be a local defining function of this divisor on $\mathrm{Sp}\, A_j$. Put

$$U_j := \mathrm{Proj}^{\mathrm{an}}\, A_j[X_0, \ldots, X_n]/(X_i - (f_i/g_i)X_0)_{i=1,\ldots,n}$$

where the functor $\mathrm{Proj}^{\mathrm{an}}$ is defined in [10]. Let V_j be the intersection of two analytic subspaces U_j and $\mathrm{Sp}\, A_j \times Y$ of $\mathbb{P}^n_{\mathrm{Sp}\, A_j}$. Then each V_j is proper over $\mathrm{Sp}\, A_j$. Gluing all V_j, we obtain the proper rigid analytic space Γ, the proper modification $\Gamma \to X$, and the proper morphism $\Gamma \to Y$. We resolve the singularities of the reduction of Γ (Theorem 2.5). Let Z be the resulting proper regular surface. We also obtain the bimeromorphic morphism $\pi\colon Z \to X$ and the proper morphism $\rho\colon Z \to Y$.

We have the Stein factorization $\beta \circ \alpha\colon Z \to S \to Y$ (Proposition 5 in [4, 9.6.3]). Since the morphism β is finite, the rigid analytic space S is projective (Theorem 5.4 in [23]). Since the rigid analytic space Z is reduced and irreducible, the rigid analytic space S has the same property. Therefore, the rigid analytic space S is the analytification of a projective integral scheme over K (Theorem 5.3 in [23]). On the other hand, by Theorem 4.3 in [2], we have the K-algebra isomorphism $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}(Z)$. Thus, the K-algebra homomorphism $\mathcal{M}(Y) \to \mathcal{M}(Z)$ is an isomorphism. Since the isomorphism factors $\mathcal{M}(S)$, the K-algebra homomorphism $\mathcal{M}(Y) \to \mathcal{M}(S)$ is an isomorphism. Therefore, the morphism β is an isomorphism. This proves that the triple Z, π, and ρ is the desired one.

The last statement follows from Lemma 4 in [4, 9.6.3]. □

3. Elliptic Fibrations and Jacobian Fibrations

3.1. **Definitions and Fundamental Results.** The following lemma reduces the study of rigid analytic fibered surfaces to the study of algebraic fibered surfaces.

Lemma 3.1. *For any proper flat surjective morphism $\rho\colon Y \to T$ to a regular curve with fibers of dimension at most one, there exist two morphisms $\alpha_1\colon T_1 \to T$ and $\alpha_{12}\colon T_2 \to T_1$ satisfying the following conditions.*

(1) *The morphism α_1 is an admissible affinoid covering.*
(2) *The morphism α_{12} is a finite étale Galois covering.*
(3) *The base change $\rho \times_T T_2$ is locally projective.*

Proof. By Theorem 2.1.4 in [9], there exist two morphisms $\alpha_3\colon T_3 \to T$ and $\alpha_{34}\colon T_4 \to T_3$ satisfying the following conditions.

(1) The morphism α_3 is an admissible affinoid covering.
(2) The morphism α_{34} is étale, quasi-compact, and surjective.
(3) The base change $\rho \times_T T_4$ is locally projective.

By Proposition 3.1.4 in [11] and Lemma 4.4 in [23], there exist three morphisms $\alpha_{35}\colon T_5 \to T_3$, $\alpha_{56}\colon T_6 \to T_5$ and $\alpha_{64}\colon T_4 \to T_6$ satisfying the following conditions.

(1) The composite $\alpha_{35} \circ \alpha_{56} \circ \alpha_{64}$ is equal to the morphism α_{34}.
(2) The morphism α_{35} is an admissible affinoid covering.
(3) The morphism α_{56} is étale, finite, and surjective.
(4) The morphism α_{64} is an open immersion.

We define the morphism $\alpha_1\colon T_1 \to T$ by the collection of all the connected components of T_5. We choose a connected component $\operatorname{Sp} A$ of T_5 and a connected component $\operatorname{Sp} B$ of $\alpha_{56}^{-1}(\operatorname{Sp} A)$. Since the morphism $\operatorname{Sp} B \to \operatorname{Sp} A$ is finite, it is the analytification of $\operatorname{Spec} B \to \operatorname{Spec} A$. The corresponding K-algebra homomorphism $A \to B$ induces a K-algebra homomorphism $\mathcal{M}(A) \to \mathcal{M}(B)$. Since the field extension $\mathcal{M}(B)/\mathcal{M}(A)$ is finite and separated, we take a normal closure of $\mathcal{M}(B)$ over $\mathcal{M}(A)$. Let C be the integral closure of A in $\mathcal{M}(B)$. Since any affinoid algebra is excellent ([8, 1]), the algebra C is a finite A-affinoid algebra.

Applying Lemma 2.4 in [23] to each image of ramification points of $\operatorname{Sp} C \to \operatorname{Sp} A$, we may assume that the morphism $\operatorname{Sp} C \to \operatorname{Sp} A$ is étale.

We define the morphism $\alpha_{12}\colon T_2 \to T_1$ by the collection of the morphisms $\operatorname{Sp} C \to \operatorname{Sp} A$. Then the desired conditions are fulfilled. $\square$

An *elliptic fibration* is a fibered surface with fibers of arithmetic genus one satisfying the following condition. There exists a nowhere dense analytic subset S_0 of S such that, for any $p \in S - S_0$, the fiber $\pi^{-1}(p)$ is smooth and geometrically connected over $K(p)$. A proper regular surface X is called an *elliptic surface* if X admits a structure of an elliptic fibration. The analytification of an algebraic elliptic fibration over an affinoid algebra A is an elliptic fibration over $\operatorname{Sp} A$.

We fix a regular curve S. Let C be the category of regular curves with étale quasi-compact morphisms to S. We sometimes abbreviate an object $f\colon T \to S$ of the category C to T. A morphism from $f_1\colon T_1 \to S$ to $f_2\colon T_2 \to S$ in C is a morphism $\alpha\colon T_1 \to T_2$ of rigid analytic spaces satisfying the following conditions.

(1) The morphism α is étale and quasi-compact.
(2) The composite $f_2 \circ \alpha$ is equal to f_1.

Remark. The étaleness of α follows from that of f_1 and f_2. Lemma 4.4 in [23] shows that the image of any morphism $\alpha\colon T_1 \to T_2$ in the category C is always an admissible open subset of T_2.

The category C admits fiber products.

Let $\mathcal{F}$ be the category of elliptic fibrations over C. An object of the category $\mathcal{F}$ is a quadruple (Y, T, ρ, f) satisfying the following conditions.

(1) The element $f\colon T \to S$ is an object of the category C.
(2) The triple (Y, T, ρ) is an elliptic fibration.

We sometimes abbreviate this quadruple to Y. A morphism from (Y_1, T_1, ρ_1, f_1) to (Y_2, T_2, ρ_2, f_2) in $\mathcal{F}$ is a pair (α, β) satisfying the following conditions.

(1) The morphism $\alpha\colon T_1 \to T_2$ is a morphism in the category C
(2) The morphism $\beta\colon Y_1 \to Y_2$ is a morphism of rigid analytic spaces such that the diagram

$$\begin{tikzcd} Y_1 \arrow[r, "\beta"] \arrow[d, "\rho_1"] & Y_2 \arrow[d, "\rho_2"] \\ T_1 \arrow[r, "\alpha"] & T_2 \end{tikzcd}$$

is Cartesian.

The category $\mathcal{F}$ admits fiber products.

Let $\mathcal{F}_s$ be the category of elliptic fibrations over C with a section, which is a subcategory of $\mathcal{F}$. An object of $\mathcal{F}_s$ is a quintuple (Y, T, ρ, f, s) satisfying the following conditions.

(1) The element $s\colon T \to Y$ is a section of the fibration ρ.
(2) The quadruple (Y, T, ρ, f) is an object of the category $\mathcal{F}$.

We sometimes abbreviate this quintuple to Y. A morphism from $(Y_1, T_1, \rho_1, f_1, s_1)$ to $(Y_2, T_2, \rho_2, f_2, s_2)$ in $\mathcal{F}$ is a morphism (α, β) of the category $\mathcal{F}$ such that $\beta \circ s_1 = s_2 \circ \alpha$. We sometimes abbreviate this morphism to β. The category $\mathcal{F}_s$ admits fiber products.

We fix an elliptic fibration $\pi\colon X \to S$. Let $\mathcal{F}_X$ be a full subcategory of $\mathcal{F}$ whose object (Y, T, ρ, f) satisfies that the fibration $\rho\colon Y \to T$ is given by the base change $X \times_S T \to T$ of the fibration $\pi\colon X \to S$ via a morphism $f\colon T \to S$ of C. We also define a full subcategory $\mathcal{F}_{X,s}$ of $\mathcal{F}_s$ in the same way.

To study the category $\mathcal{F}_X$, we introduce a more manageable subcategory $\mathcal{F}_1$. Let C_1 be the full subcategory of C whose object $f\colon T \to S$ satisfies that the quasi-compact regular curve T is an affinoid space. The category C admits fiber products and contains fiber products over S. Let $\mathcal{F}_1$ be the full subcategory of $\mathcal{F}_X$ whose object (Y, T, ρ, f) satisfies that the morphism f is an object of C_1 and the morphism ρ is projective. In this case, we can algebraize ρ (see [23, 5.1]), i.e., the elliptic fibration $\rho\colon Y \to \operatorname{Sp} A$ is the analytification of an algebraic elliptic fibration $p\colon \mathcal{Y} \to \operatorname{Spec} A$.

A *coherent module* on C_1 is the following data.

(1) Coherent $\mathcal{O}_T$-module $\mathcal{G}_T$ for each object T of C_1.
(2) Isomorphism $\gamma_{12}\colon \alpha_{21}^{*}\mathcal{G}_{T_1} \to \mathcal{G}_{T_2}$ for each two objects T_1 and T_2 and each morphism $\alpha_{21}\colon T_1 \to T_2$ of C_1. The isomorphisms satisfy the equality $\gamma_{12} \circ \alpha_{21}^{*}\gamma_{23} = \gamma_{13}$ when the equality $\alpha_{32} \circ \alpha_{21} = \alpha_{31}$ holds.

Using coherent modules on total spaces, we define coherent modules on $\mathcal{F}_1$ in the same way.

A *coherent module on the base spaces* of $\mathcal{F}_1$ is the following data.

(1) Coherent $\mathcal{O}_T$-module $\mathcal{G}_T$ for each object (Y, T, ρ, f) of $\mathcal{F}_1$.
(2) Isomorphism $\gamma_{12}\colon \alpha_{21}^{*}\mathcal{G}_{T_1} \to \mathcal{G}_{T_2}$ for each two objects (Y_1, T_1, ρ_1, f_1) and (Y_2, T_2, ρ_2, f_2) and each morphism $\beta_{21}\colon Y_1 \to Y_2$ of $\mathcal{F}_1$. The isomorphisms satisfy the equality $\gamma_{12} \circ \alpha_{21}^{*}\gamma_{23} = \gamma_{13}$ when the equality $\beta_{32} \circ \beta_{21} = \beta_{31}$ holds.

We define morphisms between coherent modules in the natural way.

We define coherent modules on C and $\mathcal{F}_X$, and coherent modules on the base spaces of $\mathcal{F}_X$ in the same way. Then these are completely determined by the coherent modules on S and X respectively. Thus, we have the following proposition.

Proposition 3.2. *The inclusion functors $C_1 \to C$ and $\mathcal{F}_1 \to \mathcal{F}_X$ induces the equivalences of the categories of coherent modules respectively. In particular, the categories of coherent modules on C_1 and $\mathcal{F}_1$ are equivalent to the categories of coherent modules on S and X respectively.*

The inclusion functor $\mathcal{F}_1 \to \mathcal{F}_X$ also induces the equivalence of categories of coherent modules on the base spaces. In particular, the category of coherent modules on the base spaces on $\mathcal{F}_1$ is equivalent to the category of coherent modules on S.

Proof. This proposition follows from Lemma 3.1 and the faithfully flat descent theory (Theorem 3.1 in [3], Theorem 4.2.8 in [10]). □

Lemma 3.3. *If a fibered surface (X, S, π) admits a section and the base space S is quasi-compact, the projection π is projective. In particular, any fibered surface over a proper regular curve with a section is projective over K.*

Proof. Since the section of π defines an S-ample line bundle (see Definition 3.2.2 in [10]), gluing a finite number of the projective morphisms that is given in Theorem 3.2.7 in [10], we obtain the desired result. □

3.2. **Canonical Bundle Formula.** We prove the canonical bundle formula for minimal elliptic fibrations.

Let $p\colon \mathcal{Y} \to \mathcal{T}$ be an algebraic fibered surface over an algebraic regular curve. Assume that the base space $\mathcal{T}$ is given by the spectrum $\operatorname{Spec} A$ of an affinoid algebra A. In this case, the morphism p is automatically projective (Theorem 2.6 in [16]). Thus, we have a relative dualizing sheaf $\omega_{\mathcal{Y}/\mathcal{T}}$ for the fibration p (see Definition 6.4.7 in [18] and Theorem 6.4.32 in [18]).

Lemma 3.4. *We keep the above notations. Then, for any quasi-coherent $O_{\mathcal{Y}}$-module $\mathcal{F}$, there exists a canonical isomorphism:*

$$p_* \mathcal{H}\mathrm{om}_{O_{\mathcal{Y}}}(\mathcal{F}, \omega_{\mathcal{Y}/\mathcal{T}}) \cong \left(R^1 p_* \mathcal{F}\right)^{\vee}.$$

The isomorphism commutes with étale base change of affinoid algebras.

Proof. The isomorphism follows from the definition of the relative dualizing sheaf. The base change property follows from Theorem 6.4.9 (b) in [18]. □

Lemma 3.5. *We keep the above notations. Assume that the arithmetic genus of the fibers is equal to one. Suppose that the fibration $\mathcal{Y}$ is minimal. Let C be the fiber at a closed point of $\mathcal{T}$. We write $C = \sum_{i \in I} b_i C_i$. Then each intersection number $\omega_{\mathcal{Y}/\mathcal{T}} \cdot C_i$ is equal to zero.*

Proof. The intersection number $\omega_{\mathcal{Y}/\mathcal{T}} \cdot C$ is equal to zero by the adjunction formula (Theorem 9.1.37 in [18]). Since the fibration is minimal, Proposition 9.3.10 (b) in [18] shows that each intersection number $\omega_{\mathcal{Y}/\mathcal{T}} \cdot C_i$ is non-negative. Therefore, each integer $\omega_{\mathcal{Y}/\mathcal{T}} \cdot C_i$ is equal to zero. □

For a positive integer n, let us denote the relative dualizing sheaf on nD_t by ω_{nD_t}.

Lemma 3.6. *We keep the above notations. Assume that the fibration p is a minimal elliptic fibration. Let $\mathcal{T}_0$ be the finite set of closed points where the fiber is not smooth. For each $t \in \mathcal{T}_0$, let m_t be a multiplicity of the fiber $p^{-1}(t)$. We write the fiber m_tD_t. Then there exists a canonical isomorphism*

$$\omega_{\mathcal{Y}/\mathcal{T}} \cong p^* p_* \omega_{\mathcal{Y}/\mathcal{T}} \otimes_{O_{\mathcal{Y}}} O_{\mathcal{Y}}(D)$$

where the divisor D is given by the following equality:

$$D = \sum_{t \in \mathcal{T}_0} a_t D_t.$$

Here, each integer a_t satisfies the inequalities $0 \leq a_t < m_t$. The isomorphism commutes with étale base change of affinoid algebras.

In particular, if the total space $\mathcal{Y}$ and the base space $\mathcal{T}$ are smooth, then there exists a canonical isomorphism

$$\mathcal{K}_{\mathcal{Y}} \cong p^* \left(p_* \omega_{\mathcal{Y}/\mathcal{T}} \otimes_{O_{\mathcal{T}}} \mathcal{K}_{\mathcal{T}} \right) \otimes_{O_{\mathcal{Y}}} O_{\mathcal{Y}}(D)$$

where $\mathcal{K}_{\mathcal{Y}}$ and $\mathcal{K}_{\mathcal{T}}$ are the canonical bundles on $\mathcal{Y}$ and $\mathcal{T}$ respectively. In this case, the relative dualizing sheaf $\omega_{\mathcal{Y}/\mathcal{T}}$ is isomorphic to $\mathcal{K}_{\mathcal{Y}} \otimes_{O_{\mathcal{Y}}} p^* \mathcal{K}_{\mathcal{T}}^{\vee}$.

Proof. Since the generic fibers are genus one curves, the direct image sheaf $p_* \omega_{\mathcal{Y}/\mathcal{T}}$ is free of rank one. Thus, the natural morphism $p^* p_* \omega_{\mathcal{Y}/\mathcal{T}} \to \omega_{\mathcal{Y}/\mathcal{T}}$ is injective. Since the inclusion morphism is an isomorphism on the generic fibers, we obtain the isomorphism $\omega_{\mathcal{Y}/\mathcal{T}} \cong p^* p_* \omega_{\mathcal{Y}/\mathcal{T}} \otimes_{O_{\mathcal{Y}}} O_Y(D)$ where D is the zero divisor of the inclusion morphism, which is vertical with respect to p.

We determine this effective divisor D. Let C be the fiber at a closed point of $\mathcal{T}$. We write $C = \sum_i b_i C_i$. Then, by Lemma 3.5, the integer $D \cdot C_i$ is equal to zero. Therefore, by Theorem 9.1.23 in [18], we may write $D = \sum_{t \in \mathcal{T}} a_t D_t$. Since the two line bundles $p^* p_* \omega_{\mathcal{Y}/\mathcal{T}}$ and $\omega_{\mathcal{Y}/\mathcal{T}}$ admit the same global sections over any admissible open subset of $\mathcal{T}$, the inequalities follow. This proves the first assertion.

We prove the second assertion. Since the dualizing sheaf $\omega_{\mathcal{Y}/\mathcal{T}}$ is given by $\mathcal{K}_{\mathcal{Y}} \otimes_{O_{\mathcal{Y}}} p^* \mathcal{K}_{\mathcal{T}}^{\vee}$ in this case, the second assertion follows from the first assertion. □

We fix a fibered surface $\pi\colon X \to S$. The analytifications of the isomorphisms in the above lemmas give an isomorphism in the category of coherent modules on the base spaces of $\mathcal{F}_1$ and the category of coherent modules on $\mathcal{F}_i$. Proposition 3.2 gives the corresponding isomorphism in that of $\mathcal{F}_X$. In particular, we obtain the following theorems. Since the canonical bundle commutes with étale base change (Theorem 6.4.9 in [18]), the relative dualizing sheaf $\omega_{X/S}$ corresponds to the coherent module on $\mathcal{F}_1$ given by the relative dualizing sheaves on the total spaces of the fibrations.

Theorem 3.7 (relative duality theorem)**.** *We keep the above notations. Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, there exists a canonical isomorphism:*

$$\pi_* \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \omega_{X/S}) \cong \left(R^1\pi_*\mathcal{F}\right)^\vee .$$

In particular, there exists a canonical isomorphism:

$$\pi_*\omega_{X/S} \cong \left(R^1\pi_*\mathcal{O}_X\right)^\vee .$$

Theorem 3.8 (canonical bundle formula)**.** *We keep the above notations. Assume that the fibration π is a minimal elliptic fibration. Let S_0 be the finite set of closed points where the fiber is not smooth. For each $s \in S_0$, let m_s be a multiplicity of the fiber $p^{-1}(s)$. We write the fiber m_sD_s. Then there exists a canonical isomorphism*

$$\omega_{X/S} \cong p^*p_*\omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$$

where the divisor D is given by the following equality:

$$D = \sum_{s\in S_0} a_sD_s.$$

Here, each integer a_s satisfies the inequalities $0 \le a_s < m_s$.

In particular, if the total space $\mathcal{Y}$ and the base space $\mathcal{T}$ are smooth, then there exists a canonical isomorphism

$$\mathcal{K}_X \cong \pi^* \left(\pi_*\omega_{X/S} \otimes_{\mathcal{O}_S} \mathcal{K}_S\right) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$$

where $\mathcal{K}_X$ and $\mathcal{K}_S$ are the canonical bundles on X and S respectively. In this case, the relative dualizing sheaf $\omega_{X/S}$ is isomorphic to $\mathcal{K}_X \otimes_{\mathcal{O}_X} p^*\mathcal{K}_S^\vee$.

From now on, we assume that the fibration π is a minimal elliptic fibration. To study multiple fibers, we refer to the arguments in [1]. Let $\mathcal{T}_X$ be the torsion submodule of $R^1\pi_*\mathcal{O}_X$. A multiple fiber m_sD_s is said to be *tamely ramified* if $\mathcal{T}_{X,s}$ is equal to zero. Otherwise, it is said to be *wildly ramified*. By ν_s we denote the order of the normal bundle $\mathcal{N}_{D_s/X}$ of D_s.

Lemma 3.9. *We keep the above notations. Then, for any positive integer n, the equality $\chi(\mathcal{O}_{nD_s}) = 0$ holds. In particular, the inequality $h^1(\mathcal{O}_{nD_s}) > 0$ hold.*

Proof. By Lemma 2.4 in [23], it suffices to show the corresponding lemma in the algebraic case. The adjunction formula (Theorem 9.1.37 in [18]) gives the relative dualizing sheaf ω_{nD_s} of nD_s:

$$\omega_{nD_s} \cong (O_X(nD_s) \otimes \omega_{X/S})|_{nD_s}.$$

By Corollary 7.3.31 in [18], the equality $2\chi(O_{nD_s}) = \deg \omega_{nD_s}$ holds. Thus, the adjunction formula on the generic fibers yields the equality $\chi(O_{nD_s}) = 0$. Therefore, since the dimension $h^0(O_{nD_s})$ is not equal to zero, the dimension $h^1(O_{nD_s})$ is also not equal to zero. □

Proposition 3.10. *We keep the above notations. Assume that the fibration X is minimal. Then the order ν_s divides both m_s and $a_s + 1$. If the base field K is separably closed, then the quotient m_s/ν_s is a power of the characteristic of K.*

Proof. Since the normal bundle $\mathcal{N}_{D_s/X}$ is given by $O_X(D)|_D$, the order ν_s divides m.

We show that the order ν_s divides a_s+1. Lemma 3.9 and the Grothendieck duality theorem show that the relative dualizing sheaf ω_{D_s} admits a non-zero section h. Let D_1 be the maximal effective divisor such that $D_1 \leq m_s D_s$ and $h|_{D_1} = 0$. Put $D_2 := D_s - D_1$. Then we define the coherent O_{D_2}-module $\mathcal{S}$ by the following sheaf exact sequence:

$$0 \longrightarrow O_{D_2} \xrightarrow{h} O_{D_2}(-D_1) \otimes \omega_{D_s} \longrightarrow \mathcal{S} \longrightarrow 0.$$

The intersection number of the divisor D_s and any prime divisor appearing in D_s is equal to zero. Thus, by Lemma 3.5, the degree of the restriction of the line bundle ω_{D_s} to any prime divisor appearing in D_s is equal to zero. Therefore, by the Riemann-Roch theorem (Theorem 7.3.26 in [18]) for D_2, the equality

$$\chi(O_{D_2}(-D_1) \otimes \omega_{D_s}) = \chi(O_{D_2}(-D_1))$$

holds. Finally, we obtain the equalities:

$$-D_1 \cdot D_2 = \chi(O_{D_2}(-D_1) \otimes \omega_{D_s}) - \chi(O_{D_2}) = \chi(\mathcal{S}).$$

Since the support of $\mathcal{S}$ is a finite number of points, the right-hand side of the equalities is at least zero. Therefore, by Theorem 9.1.23 in [18], we conclude that the divisor D_1 is equal to zero. In particular, the line bundle ω_{D_s} is trivial. Thus, the order ν_s divides $a_s + 1$.

The last statement follows from Proposition 6.3.5 in [25]. □

Lemma 3.11. *We keep the above notations. Then the function $h^0(O_{nD_s})$ of n is non-decreasing. Moreover, the inequality $h^0(O_{(\nu_s+1)D_s}) > 1$ holds.*

Proof. The function $h^1(\mathcal{O}_{nD_s})$ of n is non-decreasing since the morphism $\mathcal{O}_{n_1D_s} \to \mathcal{O}_{n_2D_s}$ is surjective for any positive integers n_1 and n_2 such that $n_1 > n_2$. Thus, the first assertion follows from Lemma 3.9. Since the line bundle $\mathcal{N}_{D_s/X}^{\otimes \nu_s}$ is trivial, we have the sheaf exact sequence:

$$0 \longrightarrow \mathcal{O}_{D_s} \longrightarrow \mathcal{O}_{(\nu_s+1)D_s} \xrightarrow{\phi} \mathcal{O}_{\nu_s D_s} \longrightarrow 0.$$

Then the second assertion follows from the induced long exact sequence since the homomorphism $H^0(\phi)$ maps constants on $(\nu_s+1)D_s$ into constants on $\nu_s D_s$. □

Proposition 3.12. *We keep the above notations. Assume that the fibration X is minimal and proper, and the total space X is smooth and geometrically connected. If the dimension $h^1(\mathcal{K}_X)$ is at most one, then the multiplicity m_s is equal to either $a_s + 1$ or $\nu_s + a_s + 1$.*

Proof. The above lemma and the Grothendieck duality theorem give the inequality $h^0(\omega_{(\nu_s+1)D_s}) > 1$. The long exact sequence induced by the sheaf exact sequence

$$0 \longrightarrow \mathcal{K}_X \longrightarrow \mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X((\nu_s + 1)D_s) \longrightarrow \omega_{(\nu_s+1)D_s} \longrightarrow 0$$

gives the inequality:

$$h^0(\mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X((\nu_s + 1)D_s)) > h^0(\mathcal{K}_X).$$

Therefore, the inequality $\nu_s + a_s + 1 \geq m_s$ holds. The previous proposition gives the inequality $1 + (a_s + 1)/\nu_s \geq m_s/\nu_s$ of integers. This proves the proposition. □

3.3. **Jacobian Fibrations.** To obtain Noether's formula for elliptic surfaces, we construct the Jacobian fibrations of elliptic fibrations. The Jacobian fibration is defined for all complex analytic elliptic surfaces with reduced fibers. Here, we define the Jacobian fibration for all rigid analytic elliptic surfaces. In the algebraic case, we may use minimal regular models of the Jacobian varieties of the generic fibers to analyze elliptic fibrations ([19], [26], [24]). We use this idea to construct the Jacobian fibrations over (étale) affinoid open subsets. Then we glue these fibrations to obtain the desired Jacobian fibrations.

We define the *algebraic Jacobian fibration* $(\mathcal{J}, \mathcal{T}, q)$ of an algebraic elliptic fibration $(\mathcal{Y}, \mathcal{T}, p)$ over an excellent regular ring of pure dimension one by a minimal regular model of the Jacobian varieties of the generic fibers of p.

Remark. The existence of the minimal regular model of the Jacobian variety follows from the desingularization ([17]) and the existence theorem of minimal regular model (Theorem 4.4 in [16], Theorem 1.2 in [7], Theorem 9.3.21 in [18]). The construction of algebraic Jacobian fibration commutes with étale base change and localization (Proposition 9.3.28 in [18] and Corollary 9.3.30 in [18]). When the projection p is smooth, the algebraic Jacobian fibration is canonically isomorphic to the dual of the identity component of the Picard scheme (Théorème 3.1 in [13]) of the fibration p (see the proof of Proposition 4 in [5, 9.4]).

We fix an elliptic fibration $\pi\colon X \to S$. The *Jacobian functor* over C is a functor J from $\mathcal{F}_X$ to $\mathcal{F}_{X,s}$ such that the restriction J_1 of J to $\mathcal{F}_1$ satisfies the following conditions.

(1) For an object $(Y, \mathrm{Sp}\, A, \rho, f)$ of $\mathcal{F}_1$, let $p\colon \mathcal{Y} \to \mathrm{Spec}\, A$ be an algebraization of ρ (see [23, 5.1]). Then the object $J_1(Y)$ is given by a quintuple $(J_1(Y), \mathrm{Sp}\, A, \sigma, f, s)$ where the fibration $\sigma\colon J_1(Y) \to \mathrm{Sp}\, A$ with the section s is the analytification of the algebraic Jacobian fibration p. Note that any affinoid algebra is excellent ([8, 1]).

(2) Let (α, β) be a morphism from $(Y_1, T_1, \rho_1, f_1, s_1)$ to $(Y_2, T_2, \rho_2, f_2, s_2)$ in $\mathcal{F}_{X,s}$. Then the morphism $J_1(\beta)$ is equal to the composite of the unique canonical isomorphism $l_{Y_2T_2T_1}\colon J_1(Y_1) \xrightarrow{\sim} J_1(Y_2) \times_{T_2} T_1$ (see Lemma 4.3 in [23] and Proposition 9.3.18 in [18]) and the projection $J_1(Y_2) \times_{T_2} T_1 \to J_1(Y_2)$. Note that the composite is equal to the composite of the unique canonical isomorphism $r_{T_1T_2Y_1}\colon J_1(Y_1) \xrightarrow{\sim} T_1 \times_{T_2} J_1(Y_2)$ and the projection $T_1 \times_{T_2} J_1(Y_2) \to J_1(Y_2)$.

We show that there exists the Jacobian functor J up to canonical isomorphism. Then we define the *Jacobian fibration* of the elliptic fibration $\pi\colon X \to S$ by $J(X) \to S$.

Assume that the Jacobian functor J exists. Let (Y, T, ρ, f) be an object of $\mathcal{F}_X$. For each $i = 1, 2$, let $\alpha_i\colon T_i \to T$ be an object of C. Put $Y_i := Y \times_T T_i$. The Cartesian diagram

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y \arrow[d] \\ T_1 \arrow[r] & T \end{tikzcd}$$

gives the Cartesian diagram:

$$\begin{tikzcd} J(Y_1) \arrow[r] \arrow[d] & J(Y) \arrow[d] \\ T_1 \arrow[r] & T. \end{tikzcd}$$

Then the above diagram and the base change of $J(Y_1) \to T_1$ via $T_1 \times_T T_2 \to T_1$ give the commutative diagram consisting of Cartesian squares:

$$\begin{tikzcd} J(Y_1) \times_T T_2 \arrow[r] \arrow[d] & J(Y_1) \arrow[r] \arrow[d] & J(Y) \arrow[d] \\ T_1 \times_T T_2 \arrow[r] & T_1 \arrow[r] & T. \end{tikzcd}$$

We may exchange 1 by 2 in the above diagram. Therefore, we obtain the canonical isomorphism $\psi_{Y_1 T Y_2} \colon J(Y_1) \times_T T_2 \xrightarrow{\sim} T_1 \times_T J(Y_2)$.

We prove the following two lemmas, which enable us to construct the Jacobian functor J by gluing the functor J_1 on the subcategory $\mathcal{F}_1$ of $\mathcal{F}_X$.

Lemma 3.13. *Assume that the Jacobian functor J exists. We use the above notations. Let $\phi \colon Y_1 \times_T T_2 \to T_1 \times_T Y_2$ be the canonical isomorphism. Then the diagram*

$$\begin{tikzcd} J(Y_1 \times_T T_2) \arrow[r, "J(\phi)"] \arrow[d, "l_{Y_1 T T_2}"] & J(T_1 \times_T J_2) \arrow[d, "r_{T_1 T Y_2}"] \\ J(Y_1) \times_T T_2 \arrow[r, "\psi_{Y_1 T Y_2}"] & T_1 \times_T J(Y_2) \end{tikzcd}$$

is commutative.

Remark. It is important that by the above commutative diagram, we can construct the morphism $\psi_{Y_1 T Y_2}$ from the morphisms $l_{Y_1 T T_2}$, $r_{T_1 T Y_2}$, and ϕ, which derive from the base changes of the fibrations $Y \to T$, $J(Y_1) \to T_1$, and $J(Y_2) \to T_2$. In other words, we may construct the morphism $\psi_{Y_1 T Y_2}$ without the fibration $J(Y) \to T$.

Proof. The commutative diagram consisting of Cartesian squares

$$\begin{tikzcd} Y_1 \times_T T_2 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Y \arrow[d] \\ T_1 \times_T T_2 \arrow[r] & T_1 \arrow[r] & T \end{tikzcd}$$

gives the commutative diagram consisting of Cartesian squares:

$$\begin{tikzcd} J(Y_1 \times_T T_2) \arrow[r] \arrow[d] & J(Y_1) \arrow[r] \arrow[d] & J(Y) \arrow[d] \\ T_1 \times_T T_2 \arrow[r] & T_1 \arrow[r] & T. \end{tikzcd}$$

We may exchange 1 and 2 in the above diagram. The two resulting diagrams prove the lemma. □

Lemma 3.14. *Assume that the Jacobian functor J exists. For each $i = 1, 2, 3$, let Y_i and Y'_i be objects in $\mathcal{F}_X$. Assume that the two squares in the commutative diagram*

$$\begin{tikzcd} Y_2 \arrow[d, "\beta_2"] & Y_1 \arrow[l] \arrow[r] \arrow[d] & Y_3 \arrow[d, "\beta_3"] \\ Y'_2 & Y'_1 \arrow[l, "p_2"'] \arrow[r, "p_3"] & Y'_3 \end{tikzcd}$$

are Cartesian. Then the equality $J(p_2)^* J(\beta_2) = J(p_3)^* J(\beta_3)$ holds.

Proof. The above diagram gives the commutative diagram consisting of Cartesian squares:

$$\begin{tikzcd} J(Y_2) \arrow[d, "J(\beta_2)"] & J(Y_1) \arrow[l] \arrow[r] \arrow[d] & J(Y_3) \arrow[d, "J(\beta_3)"] \\ J(Y'_2) & J(Y'_1) \arrow[l, "J(p_2)"'] \arrow[r, "J(p_3)"] & J(Y'_3). \end{tikzcd}$$

The lemma follows from this diagram. □

Using the Galois descent of projective varieties over the generic points of Dedekind rings, by Lemma 3.1, Lemma 3.3, Lemma 3.13, and Lemma 3.14, we obtain the Jacobian functor.

Theorem 3.15. *The Jacobian functor exists up to canonical isomorphism.*

3.4. **Noether's Formula.** In this subsection, we show Noether's formula for minimal elliptic fibrations over the perfect base field K. Note that, when the base field K is perfect, regularity is equivalent to smoothness over K in both the rigid analytic case and the algebraic case.

First, for the Jacobian fibration (J, S, σ, s) for a minimal elliptic fibration (X, S, π), we construct the canonical O_S-module homomorphism:

$$\tau_X : R^1\pi_* O_X \to R^1\sigma_* O_J.$$

Let $\mathcal{T}$ be the spectrum $\operatorname{Spec} A$ of an excellent regular ring A of dimension at most one. For the algebraic Jacobian fibration $(\mathcal{J}, \mathcal{S}, q)$ of a smoothly fibered algebraic elliptic fibration $(\mathcal{Y}, \mathcal{T}, p)$, we construct the canonical A-isomorphism

$$\tau_{\mathcal{Y}} : H^1(\mathcal{Y}, O_{\mathcal{Y}}) \to H^1(\mathcal{J}, O_{\mathcal{J}})$$

in the following way. By Theorem 1 in [5, 8.4], we have the isomorphisms $\mathrm{Lie}(\mathrm{Pic}_{\mathcal{Y}/\mathcal{T}}) \xrightarrow{\sim} H^1(\mathcal{Y}, O_{\mathcal{Y}})$ and $\mathrm{Lie}(\mathrm{Pic}_{\mathcal{J}/\mathcal{T}}) \xrightarrow{\sim} H^1(\mathcal{J}, O_{\mathcal{J}})$. By the proof of Proposition 4 in [5, 9.4], we have the isomorphism from $\mathcal{J}$ to the identity component of $\mathrm{Pic}_{\mathcal{J}/\mathcal{T}}$. These isomorphisms give the desired isomorphism. The A-module isomorphism $\tau_{\mathcal{Y}}$ commutes with flat base change.

Let $\mathcal{T}$ be the spectrum $\operatorname{Spec} A$ of a finite direct sum A of excellent discrete valuation rings. For the algebraic Jacobian fibration $(\mathcal{J}, \mathcal{S}, q)$ of an algebraic elliptic fibration $(\mathcal{Y}, \mathcal{T}, p)$, by Theorem 3.8 in [19], there exists a canonical A-module homomorphism

$$\tau_{\mathcal{Y}} \colon H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to H^1(\mathcal{J}, \mathcal{O}_{\mathcal{J}})$$

such that the following diagram

$$\begin{tikzcd}
H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \arrow[r, "\tau_{\mathcal{Y}}"] \arrow[d] & H^1(\mathcal{J}, \mathcal{O}_{\mathcal{J}}) \arrow[d] \\
H^1(\mathcal{Y}_\eta, \mathcal{O}_{\mathcal{Y}_\eta}) \arrow[r, "\tau_{\mathcal{Y}_\eta}"] & H^1(\mathcal{J}_\eta, \mathcal{O}_{\mathcal{J}_\eta})
\end{tikzcd}$$

is commutative where $\mathcal{Y}_\eta$ is the generic fibers over the generic points $\mathcal{J}_\eta$ and the vertical arrows are the localizations induced by the open immersions. The A-module homomorphism $\tau_{\mathcal{Y}}$ commutes with étale base change.

Let $\mathcal{T}$ be the spectrum $\operatorname{Spec} A$ of an excellent rings A of pure dimension one. For the algebraic Jacobian fibration $(\mathcal{J}, \mathcal{S}, q)$ of an algebraic elliptic fibration $(\mathcal{Y}, \mathcal{T}, p)$, we construct the canonical A-module homomorphism

$$\tau_{\mathcal{Y}} \colon H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to H^1(\mathcal{J}, \mathcal{O}_{\mathcal{J}})$$

in the following way. Let $\mathcal{T}_0$ be the finite number of closed points where the fiber is not smooth. We write the generic fibers $\mathcal{Y}_\eta$ over the generic points $\mathcal{T}_\eta$. Let $\mathcal{T}_1$ be the open subscheme $\mathcal{T} - \mathcal{T}_0$ of $\mathcal{T}$. Let $\mathcal{T}_2$ be the union of all localizations at points in $\mathcal{T}_0$. For $i = 1, 2$, put $\mathcal{Y}_i := \mathcal{Y} \times_{\mathcal{T}} \mathcal{T}_i$ and $\mathcal{J}_i := \mathcal{J} \times_{\mathcal{T}} \mathcal{T}_i$. Since the A-module $H^1(\mathcal{J}, \mathcal{O}_{\mathcal{J}})$ is free of rank one, we may paste $\tau_{\mathcal{Y}_1}$ and $\tau_{\mathcal{Y}_2}$ to construct $\tau_{\mathcal{Y}}$ such that the diagram

$$\begin{tikzcd}
H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \arrow[rr, "\tau_{\mathcal{Y}}"] \arrow[dr] \arrow[dd] & & H^1(\mathcal{J}, \mathcal{O}_{\mathcal{J}}) \arrow[dr] \arrow[dd] & \\
& H^1(\mathcal{Y}_2, \mathcal{O}_{\mathcal{Y}_2}) \arrow[rr, "\tau_{\mathcal{Y}_2}"] \arrow[dd] & & H^1(\mathcal{J}_2, \mathcal{O}_{\mathcal{J}_2}) \arrow[dd] \\
H^1(\mathcal{Y}_1, \mathcal{O}_{\mathcal{Y}_1}) \arrow[rr, "\tau_{\mathcal{Y}_1}"] \arrow[dr] & & H^1(\mathcal{J}_1, \mathcal{O}_{\mathcal{J}_1}) \arrow[dr] & \\
& H^1(\mathcal{Y}_\eta, \mathcal{O}_{\mathcal{Y}_\eta}) \arrow[rr, "\tau_{\mathcal{Y}_\eta}"] & & H^1(\mathcal{J}_\eta, \mathcal{O}_{\mathcal{J}_\eta})
\end{tikzcd}$$

is commutative where the non-horizontal arrows are the localizations. The A-module homomorphism $\tau_{\mathcal{Y}}$ commutes with étale base change.

Lemma 3.16. *Using the above notations, the kernel of $\tau_{\mathcal{Y}}$ is equal to the torsion submodule of the A-module $H^1(\mathcal{Y}, O_{\mathcal{Y}})$. Moreover, the kernel and cokernel of $\tau_{\mathcal{Y}}$ have the same length.*

Proof. This lemma follows from the result on $\tau_{\mathcal{Y}_2}$ in Theorem 3.8 in [19] and Theorem 3.1 in [19]. □

By Proposition 3.2, we obtain the desired O_S-module homomorphism

$$\tau_X \colon R^1\pi_* O_X \to R^1\sigma_* O_J$$

for a minimal elliptic fibration (X, S, π) with its Jacobian fibration (J, S, σ). Let $\mathcal{T}_X$ be the torsion submodule of $R^1 f_* O_X$. Then the quotient $R^1 f_* O_X/\mathcal{T}_X$ is a line bundle $\mathcal{L}_X$ on S. The above lemma gives the following corresponding theorem.

Theorem 3.17. *Using the above notations, the kernel of τ_X is equal to $\mathcal{T}_X$. Moreover, the kernel and cokernel of τ_X have the same length.*

Next, for an algebraic elliptic fibration $(\mathcal{Y}, \mathcal{T}, p)$ without multiple fibers, we construct the canonical injective $O_{\mathcal{Y}}$-module homomorphism

$$\Delta_{\mathcal{Y}/\mathcal{T}} \colon O_{\mathcal{Y}} \to (p_* \omega_{\mathcal{Y}/\mathcal{T}})^{\otimes 12}$$

where $\omega_{\mathcal{Y}/\mathcal{T}}$ is a relative dualizing sheaf for the fibration p. Note that the direct image sheaf $p_*\omega_{\mathcal{Y}/\mathcal{T}}$ is a line bundle on $\mathcal{T}$. Since the fibration p has no multiple fibers, the higher direct image sheaf $R^1 p_* O_{\mathcal{Y}}$ is also a line bundle on $\mathcal{T}$.

First, we assume that the base space is an affine scheme $\operatorname{Spec} A$. We define the canonical injective A-module homomorphism

$$\Delta_{\mathcal{Y}/\mathcal{T}} \colon A \to H^0(\mathcal{Y}, \omega_{\mathcal{Y}/\mathcal{T}})^{\otimes 12},$$

which commute with étale base change, in the following way. For each $i = \eta, 1, 2$, we define $\mathcal{T}_i$ and $\mathcal{Y}_i$ in the same way as in the above discussion. For each $i = 1, 2$, put $A_i := H^0(\mathcal{T}_i, O_{\mathcal{T}_i})$. Note that the A-module $H^0(\mathcal{Y}, \omega_{\mathcal{Y}/\mathcal{T}})$ and the A_i-module $H^0(\mathcal{Y}_i, \omega_{\mathcal{Y}_i/\mathcal{T}_i})$ for each $i = \eta, 1, 2$ are free of rank one. We take the injective A_1-module homomorphism $\Delta_{\mathcal{Y}_1/\mathcal{T}_1}$ and the injective $Q(A)$-module homomorphism $\Delta_{\mathcal{Y}_\eta/\mathcal{T}_\eta}$ in [19, 5] such that the diagram

$$\begin{tikzcd} A_1 \arrow[r, "\Delta_{\mathcal{Y}_1/\mathcal{T}_1}"] \arrow[d] & H^0(\mathcal{Y}_1, \omega_{\mathcal{Y}_1/\mathcal{T}_1})^{\otimes 12} \arrow[d] \\ Q(A) \arrow[r, "\Delta_{\mathcal{Y}_\eta/\mathcal{T}_\eta}"] & H^0(\mathcal{Y}_\eta, \omega_{\mathcal{Y}_\eta/\mathcal{T}_\eta})^{\otimes 12} \end{tikzcd}$$

is commutative where the vertical A_1-module homomorphisms are localizations (see Théorèm 9.9 in [12], Construction 7.5 in [12], and Proposition

5.5 in [19]). We also define the injective A_2-module homomorphism $\Delta_{\mathcal{Y}_2/\mathcal{T}_2}$ such that the diagram

$$\begin{tikzcd}
A_2 \arrow[r, "\Delta_{\mathcal{Y}_2/\mathcal{T}_2}"] \arrow[d] & H^0(\mathcal{Y}_2, \omega_{\mathcal{Y}_2/\mathcal{T}_2})^{\otimes 12} \arrow[d] \\
Q(A) \arrow[r, "\Delta_{\mathcal{Y}_\eta/\mathcal{T}_\eta}"] & H^0(\mathcal{Y}_\eta, \omega_{\mathcal{Y}_\eta/\mathcal{T}_\eta})^{\otimes 12}
\end{tikzcd}$$

is commutative where the vertical arrows are localizations.

Finally, we define the injective A-module homomorphism $\Delta_{\mathcal{Y}/\mathcal{T}}$ such that the diagram

$$\begin{tikzcd}
A \arrow[rr, "\Delta_{\mathcal{Y}/\mathcal{T}}"] \arrow[dr] \arrow[dd] & & H^0(\mathcal{Y}, \omega_{\mathcal{Y}/\mathcal{T}})^{\otimes 12} \arrow[dr] \arrow[dd] & \\
& A_2 \arrow[rr, "\Delta_{\mathcal{Y}_2/\mathcal{T}_2}"] \arrow[dd] & & H^0(\mathcal{Y}_2, \omega_{\mathcal{Y}_2/\mathcal{T}_2})^{\otimes 12} \arrow[dd] \\
A_1 \arrow[rr, "\Delta_{\mathcal{Y}_1/\mathcal{T}_1}"] \arrow[dr] & & H^0(\mathcal{Y}_1, \omega_{\mathcal{Y}_1/\mathcal{T}_1})^{\otimes 12} \arrow[dr] & \\
& Q(A) \arrow[rr, "\Delta_{\mathcal{Y}_\eta/\mathcal{Y}_\eta}"] & & H^0(\mathcal{Y}_\eta, \omega_{\mathcal{Y}_\eta/\mathcal{T}_\eta})^{\otimes 12}
\end{tikzcd}$$

is commutative.

The discriminant of the fibration $p\colon \mathcal{Y} \to \mathcal{T}$ at a point t on $\mathcal{T}$ is given by the length of the cokernel of $\Delta_{\mathcal{Y}/\mathcal{T}}$ at t (see [19, 5]). By $\mathrm{disc}_t(\mathcal{Y}/\mathcal{T})$ we denote this number.

From now on, we consider elliptic surfaces (X, S, π). We define $\deg R^1\pi_*\mathcal{O}_X$ by the integer $\deg \mathcal{L}_X + \operatorname{length} \mathcal{T}_X$.

Proposition 3.18. *We keep the above notations. Then the equality*

$$\chi(\mathcal{O}_X) = -\deg R^1\pi_*\mathcal{O}_X$$

holds.

Proof. By Leray's spectral sequence for π and $\mathcal{O}_X$, we obtain the two equalities

$$h^0(\mathcal{O}_X) = h^0(\pi_*\mathcal{O}_X)$$

and

$$h^2(\mathcal{O}_X) = h^1(R^1\pi_*\mathcal{O}_X)$$

and the exact sequence:

$$0 \longrightarrow H^1(\pi_*\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^0(R^1\pi_*\mathcal{O}_X) \longrightarrow 0.$$

These give the equalities:

$$\chi(\mathcal{O}_X) = \chi(\pi_*\mathcal{O}_X) - \chi(R^1\pi_*\mathcal{O}_X) = \chi(\mathcal{O}_S) - \chi(\mathcal{L}_X) - \operatorname{length}\mathcal{T}_X.$$

Then the proposition follows from the Riemann-Roch theorem for S. □

Theorem 3.17 and the above proposition give the following theorem.

Theorem 3.19. *For any minimal proper elliptic fibration X with its Jacobian fibration J, the equality*

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_J)$$

holds.

By Lemma 3.3, the Jacobian fibration over a proper base space is projective over K. By Chow's theorem (Theorem 5.3 in [23]), we may algebraize this fibration.

Theorem 3.20 (Noether's formula)**.** *For any minimal proper elliptic fibration X over S with its Jacobian fibration J, the equality*

$$12\chi(\mathcal{O}_X) = \sum_{s\in S} \operatorname{disc}_s(J/S)$$

holds.

Remark. Using Lemma 2.4 in [23] and Theorem 5.9 in [19], we can calculate the discriminants without constructing the Jacobian fibration.

Proof. The sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\Delta_{J/S}} (\sigma_*\omega_{J/S})^{\otimes 12} \longrightarrow \operatorname{Coker}(\Delta_{J/S}) \longrightarrow 0$$

gives the equality:

$$\sum_{s\in S} \operatorname{disc}_s(J/S) = \chi\left((\sigma_*\omega_{J/S})^{\otimes 12}\right) - \chi(\mathcal{O}_S).$$

Since the Grothendieck duality theorem gives the isomorphism

$$\sigma_*\omega_{J/S} \cong (R^1\sigma_*\mathcal{O}_J)^\vee,$$

by the above proposition and the Riemann-Roch theorem for S, the equality

$$\sum_{s\in S} \operatorname{disc}_s(J/S) = 12\chi(\mathcal{O}_J)$$

holds. Therefore, by the above theorem, we obtain the desired equality. □

4. Classifications

4.1. **Proper Regular Surfaces of Algebraic Dimension Two.** In this subsection, we prove that any proper regular surface of algebraic dimension two is a projective surface. We also give a criterion for algebraicity.

We have shown that we can blow down exceptional curves of the first kind on proper smooth surfaces (Theorem 4.1) in [21]. Here, we give more precise result in the following lemma when the surfaces are projective.

Lemma 4.1. *If we blow down an exceptional curve of the first kind on a regular projective surface, then the resulting surface is also regular and projective.*

Proof. By the same method as in the proof of Theorem 5.7 in [14], we can show that the blowing-downs of a projective regular surface is a projective regular surface. □

Theorem 4.2. *Any proper regular surface of algebraic dimension two is a projective surface.*

Proof. Assume that the algebraic dimension $a(X)$ of X is equal to two. We take Y, Z, π, and ρ in Lemma 2.7. By Hopf's theorem (Theorem 2.2 in [23]), the morphism ρ is a finite succession of blowing-ups. Thus, the rigid analytic space Z is projective. The theorem follows from the previous lemma. □

We give a criterion for algebraicity.

Theorem 4.3. *If a proper regular surface admits a divisor whose self-intersection number is positive, then the surface is projective.*

Proof. Assume that a divisor D on a proper regular surface X has a positive self-intersection number. We may assume that the divisor D is effective. We take Y, Z, π, and ρ in Lemma 2.7. We have only to show that the algebraic dimension $a(Z)$ is equal to two. Let E be the pull-back of the divisor D under π. Then the effective divisor E have the same self-intersection number as D's by Proposition 3.1 in [23]. The long exact sequence induced by the sheaf exact sequence

$$0 \longrightarrow [(n-1)E] \longrightarrow [nE] \longrightarrow O_E \otimes_{O_X} [nE] \longrightarrow 0$$

shows that, for all sufficiently large integers n, the dimension of the kernel

$$\mathrm{Ker}(H^1(X, [(n-1)E]) \to H^1(X, [nE]))$$

is equal to zero. Therefore, the same long exact sequence gives the inequality

$$(*) \qquad h^0([nE]) > \frac{E \cdot E}{2} n^2 + an + b$$

where a and b are suitable constant integers. In particular, the algebraic dimension $a(Z)$ is positive.

Suppose that the algebraic dimension $a(Z)$ was equal to one. In this case, the rigid analytic space Y is a proper regular curve. We write $E = E_1 + E_2$ where the image of any prime divisor appearing in E_1 under ρ is a point on Y and that of any prime divisor appearing in E_2 is equal to Y. Since the K-algebra homomorphism $\mathcal{M}(Y) \to \mathcal{M}(Z)$ is an isomorphism, the equality

$$h^0([nE]) = h^0([nE_1])$$

holds. We choose an effective divisor F on Y such that $E_1 \leq \rho^* F$. Since the inequality

$$h^0([nE_1]) \leq h^0([nF])$$

holds, by the Riemann-Roch theorem and the above equality, the inequality

$$h^0([nE]) \leq (\deg F)\, n + c$$

holds where c is a suitable constant integer. The above inequality contradicts the inequality $(*)$. Thus, the algebraic dimension $a(Z)$ is equal to two. Therefore, the theorem follows from the previous theorem. □

4.2. **Proper Regular Surfaces of Algebraic Dimension One.** In this subsection, we prove that any proper regular surface of algebraic dimension one is a fibered surface with fibers of arithmetic genus one. In particular, when the base field K is perfect, we prove that any proper smooth surface is an elliptic surface.

Assume that the algebraic dimension of a proper regular surface X is equal to one. We take Y, Z, π, and ρ in Lemma 2.7.

Lemma 4.4. *We keep the above notations. Then the image of any divisor on Z under π is a finite number of points.*

Proof. Suppose the image of a prime divisor D on Z under π was Y. Choose be a point p on Y. Then the preimage E of p under π is a divisor on Z. Since the self-intersection $E \cdot E$ is equal to zero and the intersection number $D \cdot E$ is positive, there exists an integer n such that the self-intersection of the divisor $D + nE$ is positive. However, this contradicts Theorem 4.3. This proves the claim. □

Lemma 4.5. *We keep the above notations. There exists a proper morphism $\sigma \colon X \to Y$ such that the composite $\pi \circ \sigma$ is equal to ρ.*

Proof. By Hopf's theorem (Theorem 2.2 in [23]), we have a finite sequence of blowing-ups at a single point

$$Z_n \longleftarrow Z_{n-1} \longleftarrow \cdots \longleftarrow Z_1 \longleftarrow Z_0.$$

where Z_0 and Z_n are equal to Z and X respectively. The extension theorem (Theorem 2.1 in [23]) shows that if the morphism ρ factors Z_i, then the morphism also factors Z_{i+1}. Thus, we obtain the morphism $\sigma\colon X \to Y$ such that the composite $\pi \circ \sigma$ is equal to ρ. By Proposition 4 in [4, 9.6.2], the morphism σ is proper. □

Lemma 4.6. *Any proper regular surface of algebraic dimension one is a fibered surface with fibers of arithmetic genus one.*

Proof. We show that the morphism σ in the above lemma gives a structure of a desired fibration. Choose a point p on Y. Let D be the pull-back of the divisor p under σ. By Lemma 2.4 in [23] and the adjunction formula (Theorem 9.1.37 in [18]), we have only to show that the intersection number $D \cdot \omega_{X/Y}$ is equal to zero. Suppose that the integer $D \cdot \omega_{X/Y}$ was not equal to zero. The Grothendieck duality theorem gives the equality:

$$h^1(O_D \otimes_{O_X} [mD] \otimes_{O_X} \omega^n_{X/Y}) = h^0(O_D \otimes_{O_X} [-mD] \otimes_{O_X} \omega^{(1-n)}_{X/Y}).$$

We may take the integer n such that the above dimensions vanishes and the absolute value $|n|$ is greater than one. Then the long exact sequence that is induced by the sheaf exact sequence

$$0 \longrightarrow [(m-1)D] \otimes_{O_X} \omega^{\otimes n}_{X/Y} \longrightarrow [mD] \otimes_{O_X} \omega^{\otimes n}_{X/Y} \longrightarrow O_D \otimes_{O_X} [mD] \otimes_{O_X} \omega^{\otimes n}_{X/Y} \longrightarrow 0$$

shows that, for all sufficiently large integers $|n|$, the dimension of the kernel

$$\mathrm{Ker}(H^1(X, [(m-1)E] \otimes_{O_X} \omega^{\otimes n}_{X/Y}) \to H^1(X, [mE] \otimes_{O_X} \omega^{\otimes n}_{X/Y}))$$

is equal to zero. Therefore, the long exact sequence gives the inequality

$$(*) \qquad h^0([mE] \otimes_{O_X} \omega^{\otimes n}_{X/Y}) > |mnD \cdot \omega_{X/Y}| + c$$

for all positive integers m where c is a suitable constant integer. In particular, for a sufficiently big integer m_0, the line bundle $[m_0 D] \otimes \mathcal{K}_X^{\otimes n}$ admits a non-zero section s. Suppose that the line bundle $[(m_0+1)D] \otimes \mathcal{K}_X^{\otimes n}$ admits a section t that is not a section of the line bundle $[m_0 D] \otimes \mathcal{K}_X^{\otimes n}$. Then the quotient t/s is a section of $[D]$. Thus, the K-algebra isomorphism $\mathcal{M}(Y) \xrightarrow{\sim} \mathcal{M}(X)$ gives the inequality:

$$h^0\left([(m_0+1)D] \otimes \mathcal{K}_X^{\otimes n}\right) \leq h^0\left([m_0 D] \otimes \mathcal{K}_X^{\otimes n}\right) + 1.$$

However, this contradicts the inequality $(*)$. Therefore, the intersection number $D \cdot \mathcal{K}_X$ is equal to zero. □

Theorem 4.7. *Assume that the base field K is perfect. Then any proper smooth surface of algebraic dimension one is an elliptic surface.*

Proof. We show that the morphism σ in the above lemma gives a structure of an elliptic fibration. We have only to show that all fibers of σ except for finitely many points are prover smooth curves. By the Jacobi criterion

(Proposition 2.5 in [6]) and Theorem 2.6 in [27], the singular locus of σ is an analytic subset that is not equal to X. The Lemma 4.4 shows that the image of the singular locus is a finite number of points. Then the theorem follows from Proposition 2.9 in [6]. □

4.3. **Proper Elliptic Fibrations.** In this subsection, we classify minimal proper elliptic fibrations that are not properly elliptic when the base field is algebraically closed.

For a proper irreducible surface X, put

$$N(X) := \{m \in \mathbb{Z} \mid H^0(X, \mathcal{K}_X^{\otimes m}) \neq 0, m \geq 0\}.$$

For any integer $m \in N(X)$ and any two sections f and g of $\mathcal{K}_X^m$ with $g \neq 0$, the quotient f/g defines a meromorphic function on X. Then, for each $m \in N(X)$, the quotients $\{f/g \mid f, g \in H^0(X, \mathcal{K}_X^{\otimes m}), g \neq 0\}$ form a subfield of $\mathcal{M}(X)$. Let us denote this field by $K_m(X)$. We define the *Kodaira dimension* $\kappa(X)$ of X by the equality:

$$\kappa(X) := \begin{cases} \max\limits_{m \in N(X)} \operatorname{tr.deg}_K K_m(X), & \text{if } N(X) \neq \emptyset, \\ -\infty, & \text{otherwise.} \end{cases}$$

By definition, the Kodaira dimension of X is at most the algebraic dimension of X.

The relative duality theorem (Theorem 3.7), the canonical bundle formula (Theorem 3.8), and Proposition 3.18 show the following theorem.

Theorem 4.8. *Assume that the base field is perfect. Let $\pi \colon X \to S$ be a minimal proper elliptic fibration. Then there exists a canonical isomorphism*

$$\mathcal{K}_X \cong \pi^* \mathcal{L} \otimes_{O_X} O_X(D)$$

where the line bundle $\mathcal{L}$ on S and the divisor D on X satisfy the following conditions:

(1) *Let $\mathcal{T}_X$ be the torsion submodule of $R^1\pi_* O_X$. Then the degree of $\mathcal{L}$ is equal to $\chi(O_X) + \operatorname{length} \mathcal{T}_X - 2\chi(O_S)$.*
(2) *Let S_0 be the subset of points on S where the fiber is not smooth. We write the multiplicity m_s and the fiber $m_s D_s$ at $s \in S_0$. Then the divisor D is given by the equality:*

$$D = \sum_{s \in S_0} a_s D_s.$$

Here, each integer a_s satisfies the inequalities $0 \leq a_s < m_s$.

An elliptic surface X is said to be *properly elliptic* if the Kodaira dimension of X is equal to one. By the Serre duality theorem (Theorem in [30, 5.1]), the equality $h^1(\mathcal{K}_X) = h^0(O_X)$ holds for a proper smooth surface X. Proposition 3.10 and Proposition 3.12 give the following classification.

Type	$\kappa(X)$	$\chi(\mathcal{O}_X)$	$g(S)$	length($\mathcal{T}_X$)	multiple fibers	char K
(i)	$-\infty$	0	0	0	none $(m - 1/m)$ $(m_1 - 1/m_1, m_2 - 1/m_2)$ $(1/2, 1/2, m - 1/m)$	
(ii)	$-\infty$	0	0	1	(a/m^*) $(a/m_1^*, m_2 - 1/m_2)$	$\neq 0$ $\neq 0$
(iii)	$-\infty$	1	0	0	none $(m - 1/m)$	
(iv)	0	0	0	0	$(1/2, 2/3, 5/6)$ $(1/2, 3/4, 3/4)$ $(2/3, 2/3, 2/3)$ $(1/2, 1/2, 1/2, 1/2)$	
(v)	0	0	0	1	$(0/2^*, 1/2, 1/2)$ $(1/2^*, 1/2)$ $(1/3^*, 2/3)$ $(1/4^*, 3/4)$ $(2/4^*, 1/2)$ $(2/6^*, 2/3)$ $(3/6^*, 1/2)$	2 2 3 2 2 2 3
(vi)	0	0	0	2	$(0/p^{a*})$ $(0/p^{a*}, 0/p^{b*})$	$p \neq 0$ $p \neq 0$
(vii)	0	0	1	0	none	
(viii)	0	1	0	0	$(1/2, 1/2)$	
(ix)	0	1	0	1	$(0/2^*)$	2
(x)	0	2	0	0	none	

Table 1. The possible invariants of minimal proper elliptic fibrations $\pi\colon X \to S$ that are not properly elliptic. The integer $\chi(\mathcal{O}_X)$ is the Euler characteristic of the structure sheaf $\mathcal{O}_X$. The integer $g(S)$ is the genus of the base curve S. The $\mathcal{O}_X$-module $\mathcal{T}_X$ is the torsion submodule of the higher direct image sheaf $R^1\pi_*\mathcal{O}_X$. The notation $(a_1/m_1, \ldots, a_n/m_n)$ describes the multiple fibers of the fibration π. The integer m_i is the multiplicity of the i-th multiple fiber. The integer a_i is the coefficient of the divisor in the canonical bundle formula. The symbol $*$ means that the multiple fiber is wildly ramified.

Theorem 4.9. *Assume that the base field K is algebraically closed. Then Table 1 gives the possible invariants of minimal proper elliptic fibrations that are not properly elliptic.*

Remark. For example, in positive characteristic p, there exists an elliptic fibration of Type (ii) with the combination $(pn - n - 1/pn^{*})$ for any positive integer n. If the integer n is greater than one, then the surface is not algebraic. We discuss details in the forthcoming paper [22].

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