CRITERION FOR MINIMALITY OF RIGID ANALYTIC SURFACES

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1. Introduction

The main purpose of the present paper is to give a criterion for relative minimality of proper regular rigid analytic surfaces. We have proved the existence of relatively minimal regular models of proper regular surfaces in [8]. First, we study coverings of analytic subsets. Second, we study deformations of divisors. Finally, we apply these studies to the study of proper regular rigid analytic surfaces. Ueno studied these objects in [10] in the case of smooth rigid analytic spaces over algebraically closed base fields of characteristic zero. We give detailed proofs of the theorems concerned with them in the general case.

First, to analyze analytic subsets on rigid analytic spaces, we study coverings of analytic subsets. Further, we study small tubular neighborhoods of smooth divisors on smooth rigid analytic spaces. In the affinoid case, these tubular neighborhoods were studied by Kiehl in [4].

Second, we study deformations of divisors on rigid analytic spaces. We prove that we can deform effective Cartier divisors on quasi-compact separated rigid analytic spaces whenever certain obstructions vanish. The study of coverings enables us to generalize the result to the case when the rigid analytic spaces are not quasi-compact. The point of the proof is to show the convergence of certain series. We show this convergence by using the non-Archimedean open mapping theorem.

This work was supported by Grant-in-Aid for JSPS Fellows (21-1111).

Finally, we give a criterion for relative minimality of proper regular surfaces which is similar to Castelnuovo's criterion in the case of algebraic surfaces and complex analytic surfaces. The study of deformations enables us to prove that we can blow down exceptional curves of the first kind on regular rigid analytic surfaces. Then, by the study of bimeromorphic morphisms in [8], we obtain the desired criterion. When we study deformations and blowing-downs, we refer to the arguments of complex analytic cases ([7], [6]).

In the appendix, we give methods of calculating intersection numbers and prove the Riemann-Roch theorem for proper smooth rigid analytic surfaces.

Notations and Conventions. We fix a complete non-Archimedean valuation field K with a non-trivial valuation and assume that rigid analytic spaces are defined over K. We mainly use the terminologies and notations of [1] and [8].

Acknowledgment. The author thanks Kenji Ueno for support and suggestions, and Fumiharu Kato for support and helpful comments.

2. Tubular Neighborhoods

In this section, we show the existence of small tubular neighborhoods for proper smooth divisors on separated smooth rigid analytic spaces of pure dimension. More precisely, we show that, when a proper smooth divisor on such a rigid analytic space satisfies a certain condition, there exists a small admissible open subset that contains the divisor and admits a finite admissible affinoid covering that consists of tubular neighborhoods of the divisor. If the rigid analytic space is proper, then the condition is fulfilled for any smooth divisors.

We start with preparing terminologies and notations. For a special real number r (see [8]), let us denote a one-dimensional closed disk with the radius r by $B(r)$. Let V be an affinoid subdomain of an affinoid space U. When V is relatively compact in U (over K), we write $V \Subset U$. Let $\{U_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ be two finite families of admissible affinoid open subsets of a rigid analytic space X. When V_i is relatively compact in U_i for all i, we write $\{V_i\}_{i\in I} \Subset \{U_i\}_{i\in I}$. If the family $\{V_i\}_{i\in I}$ covers a subset S of X, then we say that the family $\{U_i\}_{i\in I}$ is a *relatively big covering* of S and that the family $\{V_i\}_{i\in I}$ is a *relatively small covering* of S associated to the relatively big covering $\{U_i\}_{i\in I}$. When S equals X, we say that X admits a relatively big covering.

Let $\{V_i\}_{i\in I}$ be a *relatively small covering* of a subset S of a rigid analytic space X associated to the relatively big covering $\{U_i\}_{i\in I}$. When X is quasi-separated, for any subset J of I, the family $\{U_j\}_{j\in J}$ is an admissible covering of an admissible open subset $\bigcup_{j\in J} U_j$ of X by the following lemma. When

X is separated, the family $\{\bigcup_{i\in I} U_i\} \cup \{X - \bigcup_{i\in I} V_i\}$ is an admissible covering of X by Lemma 1.1 in [11].

For two coverings $\mathcal{U}$ and $\mathcal{V}$ of a rigid analytic space X and a subset W of X, let us denote $\{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$ and $\{U \cap W \mid U \in \mathcal{U}\}$ by $\mathcal{U} \cap \mathcal{V}$ and $\mathcal{U} \cap W$ respectively.

Lemma 2.1. *Let $\mathcal{U}$ be a family of admissible affinoid open subsets of a quasi-separated rigid analytic space X. Assume that X admits an admissible affinoid covering $\mathcal{V}$ all of whose covering elements intersect at most finite number of elements of $\mathcal{U}$. Then the family $\mathcal{U} \cap \mathcal{V}$ is an admissible covering of an admissible open subset $\bigcup \mathcal{U}$ of X. Therefore, the family $\mathcal{U}$ is an admissible covering of $\bigcup \mathcal{U}$.*

Proof. Put $U := \bigcup \mathcal{U}$. Let V be an element of $\mathcal{V}$. Since $U \cap V$ is a finite union of an admissible affinoid open subsets of V, Corollary 4 in [1, 9.1.4] shows that $U \cap V$ is an admissible open subset of V. Thus, the union U is an admissible open subset of X. The same corollary shows that $\mathcal{U} \cap V$ has a refinement which is an admissible covering of $U \cap V$. Therefore, since $\mathcal{V} \cap U$ is an admissible covering of U, the family $\mathcal{U} \cap \mathcal{V}$ is an admissible covering of U. This proves the first assertion. Since the admissible covering $\mathcal{U} \cap \mathcal{V}$ is finer than the covering $\mathcal{U}$, the last assertion follows. □

The next lemma enables us to enlarge the elements of relatively small coverings.

Lemma 2.2. *Assume that two finite families $\mathcal{U}$ and $\mathcal{V}$ of admissible affinoid open subsets satisfy the relation $\mathcal{V} \Subset \mathcal{U}$. Then there exists a finite family $\mathcal{W}$ of admissible affinoid open subsets such that the relations $\mathcal{V} \Subset \mathcal{W} \Subset \mathcal{U}$ hold.*

Proof. It suffices to show that for an affinoid subdomain V which is relatively compact in an affinoid space U, there exists an affinoid subdomain W of U such that $V \Subset W \Subset U$. Proposition 4 [1, 6.2.1] implies that we may write $U = \operatorname{Sp} K\langle f_1, \dots, f_n\rangle$ such that for a suitable special real number ϵ, the affinoid subdomain V is contained in the set $U(f_1/\epsilon, \dots, f_n/\epsilon)$. Take a special real number ϵ_0 such that the inequalities $\epsilon < \epsilon_0 < 1$ holds. Then we have only to put $W := U(f_1/\epsilon_0, \dots, f_n/\epsilon_0)$. □

The next lemma shows that the relatively compactness is stable under the pull-backs by closed immersions.

Lemma 2.3. *Let U and V be two admissible affinoid open subsets of a rigid analytic space X such that V is relatively compact in U. Let Y be an analytic subset of X. Then $V \cap Y$ is relatively compact in $U \cap Y$. Therefore, if an analytic subset admits a relatively big covering, then the analytic subset is proper.*

Proof. We may write $U = \operatorname{Sp} K\langle f_1, \ldots, f_n\rangle$ such that V is contained in the subset $\{x \in U \mid |f_1(x)| < 1, \ldots, |f_n(x)| < 1\}$. For each i, let g_i be the image of f_i by the homomorphism corresponding to the closed immersion $U \cap Y \to U$. Then $U \cap Y = \operatorname{Sp} K\langle g_1, \ldots, g_n\rangle$, and $V \cap Y$ is contained in the subset $\{x \in U \cap Y \mid |f_1(x)| < 1, \ldots, |f_n(x)| < 1\}$, which is equal to the subset $\{x \in U \cap Y \mid |g_1(x)| < 1, \ldots, |g_n(x)| < 1\}$. This proves the lemma. □

The following lemma shows that the relatively compactness is stable under product and intersection.

Lemma 2.4. *Let V_i be an affinoid subdomain of an affinoid space U_i for $i = 1, 2$. Assume that each V_i is relatively compact in U_i. Then the product $V_1 \times V_2$ is relatively compact in the product $U_1 \times U_2$. If V_1 and V_2 are admissible affinoid open subsets of a separated rigid analytic space X, then the intersection $V_1 \cap V_2$ is relatively compact in the intersection $U_1 \cap U_2$.*

Proof. The first statement follows from Lemma 1 in [1, 9.6.2]. Since X is separated, the second statement follows from Lemma 2.3. □

Lemma 2.5. *Let V be an affinoid subdomain of an affinoid space Y. Assume that affinoid subdomains U and W are relatively compact in the affinoid spaces X and V respectively. Then for any morphism $\phi\colon X \to Y$ of rigid analytic spaces, the affinoid subdomain $U \cap \phi^{-1}(W)$ is relatively compact in the affinoid subdomain $\phi^{-1}(V)$. In particular, when an affinoid subdomain T is relatively compact in an affinoid subdomain S, for a family $\{g_i\}_{1\le i\le m+n}$ of analytic functions on X and a family $\{\epsilon_i, \epsilon'_i\}_{1\le i\le m+n}$ of special real numbers such that $\epsilon_i < \epsilon'_i$ for all i, the affinoid subdomain*

$$T(g_1/\epsilon_1, \ldots, g_m/\epsilon_m, \epsilon'_{m+1}/g_{m+1}, \ldots, \epsilon'_{m+n}/g_{m+n})$$

is relatively compact in the affinoid subdomain:

$$S(g_1/\epsilon'_1, \ldots, g_m/\epsilon'_m, \epsilon_{m+1}/g_{m+1}, \ldots, \epsilon_{m+n}/g_{m+n}).$$

Proof. The graph of ϕ is an analytic subset of the product $X \times Y$. By Lemma 2.4, the product $U \times W$ is relatively compact in the product $X \times V$. Therefore, the first statement follows from Lemma 2.3. The last statement follows from the first statement if we put

$$X := S(g_1/\epsilon'_1, \ldots, g_m/\epsilon'_m, \epsilon_{m+1}/g_{m+1}, \ldots, \epsilon_{m+n}/g_{m+n}),$$

$$Y := B(1)^{m+n},$$

and

$$\phi := (g_1/\epsilon'_1, \ldots, g_m/\epsilon'_m, \epsilon_{m+1}/g_{m+1}, \ldots, \epsilon_{m+n}/g_{m+n}).$$

□

Lemma 2.6. *Assume that an analytic subset Y of a rigid analytic space X is defined by analytic functions $f_1, \dots, f_n$ on X. Let U be an admissible open subset of X that contains Y. Then there exists a special real number ϵ such that the affinoid subdomain $X(f_1/\epsilon, \dots, f_n/\epsilon)$ is contained in U.*

Proof. This lemma is shown in Lemma 2.3 in [5]. See also Lemma 1.1.4 in [3]. □

We can replace relatively big coverings with enough smaller coverings.

Lemma 2.7. *Let Y be an analytic subset of a rigid analytic space X. Assume that there exists a relatively big covering of Y. Let U be an admissible open subset of X that contains Y. Then there exists a relatively big covering of Y that is contained in U.*

Proof. Let $\{V_i\}_{i\in I}$ be a relatively small covering associated to a relatively big covering $\{U_i\}_{i\in I}$ of Y. We fix an arbitrary element i of I. We may write $U_i = \operatorname{Sp} K\langle f_1, \dots, f_m\rangle$. Put $Y_i := Y \cap U_i$. We have a finite number of defining functions $g_1, \dots, g_n$ of Y_i on U_i. By Lemma 2.6, there exists a special real number ϵ such that $U_i(g_1/\epsilon, \dots, g_n/\epsilon) \subset U$. It suffices to show that for special real numbers ϵ_0, ϵ_1 such that $\epsilon_0 < \epsilon_1 < \epsilon$, the affinoid subdomain $V_i(g_1/\epsilon_0, \dots, g_n/\epsilon_0)$ is relatively compact in $U_i(g_1/\epsilon_1, \dots, g_n/\epsilon_1)$. This follows from Lemma 2.5. □

We can refine relatively big coverings by Zariski coverings.

Lemma 2.8. *Let $\mathcal{U}$ be a Zariski covering of an affinoid space X. Let V be an affinoid subdomain that is relatively compact in X. Then there exists a relatively big covering of V that refines $\mathcal{U}$.*

Proof. Refining $\mathcal{U}$, we may assume that X is covered by a finite number of admissible open subsets of the type $\{x \in X \mid |f(x)| > 0\}$. Further, by Lemma 6 in [1, 9.1.4], we may assume that the covering elements are rational subdomains of the type $X(\epsilon/f)$. Therefore, it suffices to show that for special real numbers ϵ_0 and ϵ_1 such that $\epsilon_0 < \epsilon_1 < \epsilon$, the affinoid subdomain $V(\epsilon_1/f)$ is relatively compact in $X(\epsilon_0/f)$. This follows from Lemma 2.5. □

Proposition 2.9. *Assume that the family $\{U_i\}_{i\in I}$ is a relatively big covering of an analytic subset D of a separated rigid analytic space X. We take a family $\{f_{ij}\}_{j\in J_i}$ of defining functions of D on each U_i. Put $U_{i\epsilon} := \{x \in U \mid \forall j \in J_i, |f_{ij}(x)| \le \epsilon\}$. Then there exist an admissible open subset U_0 of X and a special real number ϵ satisfying the following conditions.*

(1) *The family $\{U_i\}_{i\in I} \cup \{U_0\}$ is an admissible covering of X.*
(2) *For all $i \in I$, the restriction $U_{i\epsilon}$ is disjoint from U_0.*

Proof. Let V be the union of the relatively small covering of $\{U_i\}_{i\in I}$. Put $U_0 := X - V$. The first condition is fulfilled. Since Lemma 2.1 implies that V is an admissible open subset of X, Lemma 2.6 shows that there exists a special real number ϵ such that V contains $U_{i\epsilon}$ for all $i \in I$. □

Proposition 2.10. *Let $\pi\colon X \to Y$ be a morphism of rigid analytic spaces. Assume that the rigid analytic space X is quasi-separated. Suppose that the preimage of a point p on Y admits a relatively big covering. Then there exists an admissible open subset V of Y that contains p such that the restriction π to the preimage of V under π is proper.*

Proof. Let D be the preimage of the point p under π. We choose an admissible affinoid open subset W of Y that contains p and a family $\{g_i\}_{i\in I}$ of defining functions of p on W. For a special real number ϵ, we put $W_\epsilon := \{y \in Y \mid \forall i \in I, |g_i(x)| \leq \epsilon\}$.

We take a relatively small covering $\{V_i\}_{i\in I}$ of D associated to the relatively big covering $\{U_i\}_{i\in I}$. Since the family $\{\pi^* g_i\}_{i\in I}$ is a family of defining functions of D on the preimage $\pi^{-1}(W)$, by Lemma 2.6, we obtain a special real number ϵ such that the preimage of the admissible affinoid open subset W_ϵ under π is contained in the union of the relatively small covering $\{V_i\}_{i\in I}$. Put $Z := \pi^{-1}(W_\epsilon)$. Since the relation $V_i \cap Z \Subset_W U_i \cap Z$ holds for all $i \in I$, the restriction π to Z is proper. Thus, by Lemma 2.1, the admissible open subset W_ϵ is a desired one. □

From now on, we assume that a rigid analytic space X is separated and normal. Let D be a smooth prime divisor on X. Let U be an affinoid open subset $\mathrm{Sp}\,A$ of X. Put $R := D \cap U$. We say that U is a *local tubular neighborhood* of D if there exist a defining function f of D on U and an isomorphism $U \cong R \times B(1)$ such that the corresponding K-algebra homomorphism

$$\phi\colon A/fA\langle X\rangle \longrightarrow A$$

satisfies the following conditions.

(1) The homomorphism ϕ sends X to f.
(2) The restriction of ϕ to A/fA is a section of the natural homomorphism $A \to A/fA$.

In this case, we call R an *extendable open subset* of D. An admissible affinoid open subset of an extendable open subset is also extendable. For a special real number $\epsilon \leq 1$, let us denote the admissible open subset $\mathrm{Sp}\,A/fA\langle X/\epsilon\rangle$ of X by U_ϵ or $R(\epsilon)$. When a smooth prime divisor D admits a relatively big covering that consists of extendable open subsets, we say that the divisor D admits an *extendable covering*. A *local tubular neighborhood covering* of a smooth prime divisor D is a relatively big covering of D such that each covering element is a local tubular neighborhood of

D. A *tubular neighborhood* of a smooth prime divisor D is an admissible open subset P of X such that there exist an analytic function F on P and an admissible covering $\{U_i\}_{i\in I}$ of P satisfying the following conditions.

(1) The family $\{U_i\}_{i\in I}$ is a local tubular neighborhood covering of D.
(2) For each $i \in I$, the restriction $F|_{U_i}$ is the composite of the isomorphism $U_i \to (U_i \cap D) \times B(1)$ and the projection $(U_i \cap D) \times B(1) \to B(1)$.

For a special real number $\epsilon \leq 1$, let us denote the preimage $F^{-1}(B(\epsilon))$ by P_ϵ.

Theorem 2.11. *Let $f_1, \dots, f_n$ be a generating family of an ideal I of an affinoid algebra A. Assume that the affinoid space $\operatorname{Sp} A$ is smooth on the closed subspace $\operatorname{Sp} A/I$. Further, suppose that the closed subspace $\operatorname{Sp} A/I$ is smooth. Then there exist three special real numbers ϵ, ϵ_1, and ϵ_2, and a finite family $\{g_j\}_{j\in J}$ of elements of A satisfying the following conditions. For each $j \in J$, put $B_j := A\langle f_1/\epsilon, \dots, f_n/\epsilon, \epsilon_0/g_j\rangle$.*

(1) The inequality $\epsilon_0 < \epsilon_1$ holds.
(2) The family $\{\operatorname{Sp} A\langle \epsilon_1/g_j\rangle\}_{j\in J}$ is an admissible covering of $\operatorname{Sp} A$.
(3) For all $j \in J$, there exists an integer n_j and an isomorphism

$$\phi_j\colon (B_j/IB_j)\langle X_{j1}, \dots, X_{jn_j}\rangle \xrightarrow{\sim} B_j$$

where the left-hand side is a free affinoid algebra over B_j/IB_j.
(4) For all $j \in J$, the family $\{\phi_j(X_{jk})\}_{1\leq k\leq n_j}$ generates the ideal IB_j.
(5) For all $j \in J$, the restriction ϕ_j on B_j/IB_j is a section of the natural homomorphism $B_j \to B_j/IB_j$.

If the affinoid space $\operatorname{Sp} A$ is of pure dimension m and the affinoid space $\operatorname{Sp} A/I$ is of pure dimension $m - n$, then we can require the following additional conditions.

(6) For all $j \in J$, the integer n_j is equal to n.
(7) For all $j \in J$ and for all integer k such that $1 \leq k \leq n$, there exists a non-zero element ϵ_j of K such that the isomorphism ϕ_j maps X_{jk} to f_k/ϵ_j.

Proof. This theorem follows from Theorem 1.18 in [4] and the following remark of the theorem. We have only to note that, when we construct a family $\{\operatorname{Sp} B_j\}$ of affinoid subdomains of $\operatorname{Sp} A$, we restrict a refinement of a Zariski covering of $\operatorname{Sp} A$ to $\operatorname{Sp} A\langle f_1/\epsilon, \dots, f_n/\epsilon\rangle$. □

We give some equivalent conditions of the existence of local tubular neighborhood coverings.

Theorem 2.12. *An extendable covering of a smooth prime divisor D on X can be extend to a local tubular neighborhood covering of D. Conversely,* *if the rigid analytic space X is smooth on D and the divisor D admits an extendable covering, then there exists a local tubular neighborhood covering of D. Therefore, if the rigid analytic space X is smooth on D, then the following conditions are equivalent.*

(1) *There exists an extendable covering of D.*
(2) *There exists a local tubular neighborhood covering of D.*
(3) *There exists a relatively big covering of D.*

Remark. If the rigid analytic space X is proper, the third condition is fulfilled for any divisors.

Proof. First, we show the first statement. Let $\{S_i\}_{i\in I}$ be a relatively small covering associated to an extendable covering $\{R_i\}_{i\in I}$ of D. Take a special real number δ that is less than one. Then, by Lemma 2.4, $S_i(\delta) \Subset R_i(1)$ for all i. Therefore, the family $\{R_i(1)\}_{i\in I}$ is a local tubular neighborhood covering of D.

Next, we show the second statement. Let $\{Y_i\}_{i\in I}$ be a relatively small covering associated to a relatively big covering $\{X_i\}_{i\in I}$ of D. By Lemma 5.8 in [8] and Lemma 2.8, we may assume that the family $\{X_i\}_{i\in I}$ is a locally principal covering. Further, by Theorem 2.11, we may assume that there exists a local tubular neighborhood of D on each X_i. Put $R_i := X_i \cap D$ and $S_i := Y_i \cap D$ for each i. Lemma 2.3 shows that $S_i \Subset R_i$ for all i. Therefore, the divisor D admits an extendable covering $\{R_i\}_{i\in I}$. □

Proposition 2.13. *Let $\{U_i\}_{i\in I}$ be a local tubular neighborhood covering of a smooth prime divisor D on the rigid analytic space X. Then the following statements hold.*

(1) *There exists a local tubular neighborhood covering $\{V_i\}_{i\in I}$ such that $V_i \Subset U_i$ for each i.*
(2) *For all family $\{\delta_i\}_{i\in I}$ of special real numbers such that $\delta_i \leq 1$ for all i, the family $\{U_{i\delta_i}\}_{i\in I}$ is a local tubular neighborhood covering of D.*
(3) *Let U be an admissible open subset of X that contains D. Then there exists a special real number ϵ such that $U_{i\epsilon}$ is contained in U for each i.*
(4) *Let U be an admissible affinoid open subset of X. Let V be an admissible open subset of U that contains $U \cap D$. Then there exists a special real number ϵ such that V contains $U_{i\epsilon} \cap U$ for each i.*

Proof. Put $R_i := D \cap U_i$. By Lemma 2.3 and Lemma 2.2, we have two coverings $\{S_i\}_{i\in I}, \{T_i\}_{i\in I}$ of D such that $T_i \Subset S_i \Subset R_i$ for all i. Take two special real numbers ϵ_0, ϵ_1 such that $\epsilon_0 < \epsilon_1 < 1$. By Lemma 2.4, $T_i(\epsilon_0) \Subset S_i(\epsilon_1) \Subset U_i$ for all i. Therefore, the first statement holds. Take a special real number ϵ_i such that $\epsilon_i < \delta_i$. Then the relation $S_i(\epsilon_i) \Subset U_{i\delta_i}$ holds for all i.

Therefore, the second statement holds. By Lemma 2.6, the third and fourth statements hold if we replace the above special real numbers ϵ_i and δ_i. □

From the above proposition, we deduce the following proposition.

Proposition 2.14. *Let P be a tubular neighborhood of a smooth prime divisor D on the rigid analytic space X. Then the following statements hold.*

(1) *For any special real number ϵ such that $\epsilon \leq 1$, the subset P_ϵ is a tubular neighborhood of D.*
(2) *Let U be an admissible open subset of X that contains D. Then there exists a special real number ϵ such that P_ϵ is contained in U.*
(3) *Let U be an admissible affinoid open subset of X. Let V be an admissible open subset of U that contains $U \cap D$. Then there exists a special real number ϵ such that V contains $P_\epsilon \cap U$.*

3. Deformations of Divisors

In this section, we prove that we can deform a quasi-compact effective Cartier divisor whenever a certain cohomology group vanishes.

Let D be a quasi-compact reduced Cartier divisor on a quasi-compact separated rigid analytic space X. Let $\{U_i\}_{i\in I}$ be an admissible affinoid covering of X. Put $\mathcal{U} := \{U_i\}_{i\in I}$. For each $i \in I$, put $D_i := D \cap U_i$. We may write $D_i = \operatorname{Sp} B_i$ and $U_i = \operatorname{Sp} A_i$. Assume that the kernel of the natural surjective homomorphism $\theta_i \colon A_i \to B_i$ is generated by the single non-zero-divisor h_i of A_i. For each $i, j, k \in I$, put $A_{ij} := A_i \widehat{\otimes} A_j$, $U_{ij} := \operatorname{Sp} A_i \cap \operatorname{Sp} A_j$, $U_{ijk} := \operatorname{Sp} A_i \cap \operatorname{Sp} A_j \cap \operatorname{Sp} A_k$, $B_{ij} := B_i \widehat{\otimes} B_j$, and $D_{ij} := D \cap U_{ij} = \operatorname{Sp} B_{ij}$. Let h_{ij} be the invertible element h_i/h_j of A_{ij}.

Let δ be a special real number. For each $i \in I$, put $V_{i\delta} := U_i \times \operatorname{Sp} K\langle T/\delta\rangle$. Put $Y_\delta := X \times \operatorname{Sp} K\langle T/\delta\rangle$, and $\mathcal{V}_\delta := \{V_{i\delta}\}_{i\in I}$.

All complete K-algebra norms on any affinoid algebra is equivalent to each other (Proposition 2 in [1, 6.1.3] and Corollary 3 in [1, 2.1.8]). For each $i, j \in I$, we set complete K-algebra norms on affinoid algebras A_i, A_{ij}, B_i, and B_{ij}.

Using a coherent $\mathcal{O}_X$-ideal $\mathcal{O}_X(-D)$, we put $\mathcal{O}_D := \mathcal{O}_X/\mathcal{O}_X(-D)$. The *normal bundle* of D is the line bundle $\mathcal{O}_X(D)|_D$ on D. Let us denote this line bundle by $\mathcal{N}_{D/X}$. For a presheaf $\mathcal{F}$ of abelian groups on X, let us denote the q-cochain group (resp. q-cocycle group) of $\mathcal{U}$ with coefficient in $\mathcal{F}$ by $C^q(\mathcal{U}, \mathcal{F})$ (resp. $Z^q(\mathcal{U}, \mathcal{F})$).

Theorem 3.1. *We use the above notations. Let $\{t_i\}_{i\in I}$ be a section of the normal bundle $\mathcal{N}_{D/X}$ of D. Assume that the cohomology group $H^1(D, \mathcal{N}_{D/X})$ vanishes. Then there exist a special real number δ, $s = \{s_i\}_{i\in I} \in C^0(\mathcal{V}_\delta, \mathcal{O}_{Y_\delta})$, and $f = \{f_{ij}\}_{i,j\in I} \in C^1(\mathcal{V}_\delta, \mathcal{O}_{Y_\delta})$ satisfying the following conditions:*

(1) *For all $i \in I$, the first two terms in the expansion of s_i with respect to T is equal to $h_i + s_{i1}T$ where the image of s_{i1} under θ_i is equal to t_i.*
(2) *For all $i, j, k \in I$, the two equalities $s_i = f_{ij}s_j$ and $f_{ik} = f_{ij}f_{jk}$ hold.*
(3) *For all $i, j \in I$, the analytic function f_{ij} is invertible.*

We prove the above theorem. For each $i, j \in I$, we write

$$s_i = h_i + \sum_{m=1}^{\infty} s_{im}T^m$$

and

$$f_{ij} = h_{ij} + \sum_{m=1}^{\infty} f_{ijm}T^m.$$

First, note that it suffices to construct $s \in C^0(\mathcal{V}_1, \mathcal{O}_{Y_1})$ and $f \in C^1(\mathcal{V}_1, \mathcal{O}_{Y_1})$ that satisfy the conditions (1)–(2). Indeed, if we choose a special real number δ such that the inequality

$$\sup_{m\geq 1} \left| \frac{f_{ijm}\delta^m}{h_{ij}} \right|_{A_{ij}} < 1$$

holds for all $i, j \in I$, then the condition (3) is fulfilled.

We construct a formal solution. For formal power series $F, G \in A[[T]]$ over an affinoid algebra A, if $F \equiv G \pmod{T^{\mu+1}}$, then we write $F \equiv_\mu G$. Put $s_{i0} := h_i$, $s_{i1} := t_i$, and $f_{ij0} := h_{ij}$. For $\mu \geq 0$, put

$$s_i^\mu := \sum_{m=0}^{\mu} s_{im}T^m$$

and

$$f_{ij}^\mu := \sum_{m=0}^{\mu} f_{ijm}T^m.$$

Claim. There exist $s_m = \{s_{im}\}_{i\in I} \in C^0(\mathcal{U}, \mathcal{O}_X)$ and $f_m = \{f_{ijm}\}_{i,j\in I} \in C^1(\mathcal{U}, \mathcal{O}_X)$ for all $m \geq 0$ such that the equation

$$(\mu) \qquad s_i^\mu \equiv_\mu f_{ij}^\mu s_j^\mu \text{ on } U_{ij}.$$

holds for all $\mu \geq 0$. If the above equation (μ) is satisfied, then the congruence

$$f_{ik}^\mu \equiv_\mu f_{ij}^\mu f_{jk}^\mu \text{ on } U_{ijk}$$

holds.

Proof. Clearly, the equation (0) holds.

For a positive integer μ, assume that the families $\{s_m\}_{0\le m\le \mu-1}$ and $\{f_m\}_{0\le m\le \mu-1}$ satisfy the equation $(\mu-1)$. We define s_μ and f_μ such that the equation (μ) is fulfilled. The equation (μ) is equivalent to the equation

$$f_{ij\mu}h_j = s_{i\mu} - h_{ij}s_{j\mu} + g_{ij\mu} \text{ on } U_{ij}$$

where $g_{ij\mu} \in A_{ij}$ is defined by the following congruence:

$$g_{ij\mu}T^\mu \equiv_\mu s_i^{\mu-1} - f_{ij}^{\mu-1}s_j^{\mu-1} \text{ on } U_{ij}.$$

The cochain $\{g_{ij\mu}|_{D_{ij}}\}_{i,j\in I}$ satisfies the one-cocycle condition:

$$g_{ik\mu} = g_{ij\mu} + h_{ij}g_{jk\mu} \text{ on } D_{ijk}.$$

First, we show that there exists s_μ such that the equality

$$(*) \qquad s_{i\mu} - h_{ij}s_{j\mu} + g_{ij\mu} = 0 \text{ on } D_{ij}$$

holds.

If μ is equal to one, then the equality $g_{ij\mu} = 0$ holds. For each $i \in I$, put $s_{i\mu} := \iota_i(t_i)$ where ι_i is a set-theoretical section of the surjective homomorphism $\theta_i\colon A_i \to B_i$. Then, since t_1 is a section of the normal bundle $\mathcal{N}_{D/X}$, the equation $(*)$ is fulfilled.

If μ is greater than one, by the assumption $H^1(D, \mathcal{N}_{D/X}) = 0$, there exists $t_{i\mu} \in B_i$ such that the equality

$$t_{i\mu} - h_{ij}t_{j\mu} + g_{ij\mu} = 0 \text{ on } D_{ij}$$

holds. For each $i \in I$, put $s_{i\mu} := \iota_i(t_{i\mu})$. Then s_μ satisfies the equation $(*)$. Since the analytic function $s_{i\mu} - h_{ij}s_{j\mu} + g_{ij\mu}$ vanishes on D_{ij} and the restriction $h_j|_{U_{ij}}$ is a defining function of D_{ij}, the analytic function $s_{i\mu} - h_{ij}s_{j\mu} + g_{ij\mu}$ is divisible by h_j. Set

$$f_{ij\mu} := (s_{i\mu} - h_{ij\mu}s_{j\mu} + g_{ij\mu})/h_j \text{ on } U_{ij}.$$

Then the pair s_μ and f_μ is a solution of the equation (μ). □

Lemma 3.2. *Let h be a defining function of an effective Cartier divisor D on an affinoid space* $\operatorname{Sp} A$*. Then there exists a positive real number α such that if an element f of A vanishes on D, then the equality $|f/h|_A \le \alpha|f|_A$ holds.*

Proof. Since, for all K-linear maps between K-normed spaces, continuity is equivalent to boundedness (Corollary 3 [1, 2.1.8]), Banach's theorem (Theorem 1 in [2, I, §3]) shows that the isomorphism $A \to fA$ is homeomorphism. This proves the lemma. □

For each $i, j \in I$, applying the above lemma to the affinoid space $\mathrm{Sp}\, A_{ij}$ and the defining function $h_j|_{U_{ij}}$ of $D|_{U_{ij}}$, we obtain the positive real number α_{ij}. Put $\alpha := \max_{i,j\in I} \alpha_{ij}$.

We introduce K-Banach space norms on cochain groups. Put $A^q := C^q(\mathcal{U}|_D, \mathcal{O}_X)$ and $C^q := C^q(\mathcal{U}|_D, \mathcal{N}_{D/X})$. Since the family $\mathcal{U}|_D$ is a finite admissible affinoid covering of D, for each q, we may regard the q-cochain group A^q as an affinoid algebra that is the finite direct sum of the affinoid algebras. Then the q-cochain group A^q is a K-Banach algebra and the q-cochain group C^q is an A^q-Banach space. For each $q = 0, 1$, we define q-coboundary operator $\delta^q \colon C^q \to C^{q+1}$ as follows. For $u = \{u_i\}_{i\in I} \in C^0$, put

$$(\delta^0 u)_{ij} := h_{ij}u_j - u_i \text{ on } U_{ij}.$$

For $v = \{v_{ij}\}_{i,j\in I} \in C^1$, put

$$(\delta^1 v)_{ijk} := h_{ij}v_{jk} - v_{ik} + v_{ij} \text{ on } U_{ijk}.$$

By Z^1 and B^1 we denote the K-normed subspaces $\mathrm{Ker}\, \delta^1$ and $\mathrm{Im}\, \delta^0$ of C^1 respectively.

Claim. There exists a positive real number β satisfying the following condition. For any $v \in C^1$ such that $\delta^1 v = 0$, there exists $u \in C^0$ such that the equation $v = \delta^0 u$ and the inequality $|u|_{C^0} < \beta |v|_{C^1}$ hold.

Proof. Since addition, subtraction, multiplication, and restriction are continuous, the coboundary operators are continuous K-linear maps. Therefore, the kernel Z^1 of δ^1 is a K-Banach space. Put $\eta := \delta^0 \colon C^0 \to Z^1$. Since the cohomology group $H^1(D, \mathcal{N}_{D/X})$ vanishes, the equality $Z^1 = B^1$ holds. Then the K-linear map η between K-Banach spaces is surjective and continuous. Thus, Banach's theorem (Theorem 1 in [2, I, §3]) shows that the K-linear map η is open. Therefore, we obtain a desired positive real number β. □

We take β in the above claim. Put $t_\mu := \{t_{i\mu}\}_{i\in I} \in C^0$ and $g_\mu := \{g_{ij\mu}\}_{i,j\in I} \in C^1$. If μ is greater than one, then we may assume that $|t_\mu|_{\mathrm{Sp}\, C^0} < \beta |g_\mu|_{\mathrm{Sp}\, C^1}$. By Banach's theorem (Theorem 1 in [2, I, §3]), we may take a special real number γ such that the inequality $|\iota(t)|_{A_i} < \gamma |t|_{B_i}$ holds for all $i \in I$ and all $t \in B_i$.

By R we denote a real number field or an affinoid algebra with a norm. For two formal power series $F = \sum_{m=0}^{\infty} a_m T^m \in R[[T]]$ and $G = \sum_{m=0}^{\infty} b_m T^m \in R[[t]]$, we write $F \ll G$ if $|a_m| \leq |b_m|$ for all $m \geq 0$. We set a formal power series over $\mathbb{R}$

$$P(T) := \frac{b}{5c} \sum_{m=1}^{\infty} \frac{c^m T^m}{m^2}$$

where b and c is a positive real number. Then the relation

$$P(T)^2 \ll \frac{b}{c} P(T).$$

holds. We choose four positive real numbers a, b, c, and d satisfying the following inequalities:

(1) $\max_{i \in I} |s_{i1}|_{A_i} < b/5$;
(2) $\max_{i,j \in I} |h_{ij}|_{A_{ij}} < d$;
(3) $\alpha(1 + (ab/c) + d) < a$;
(4) $\beta\gamma(1 + (ab/c) + d) < c$.

Claim. For all $i, j \in I$ and all $\mu \geq 0$, the relations

$$(i, j, \mu) \qquad f_{ij}^{\mu} - h_{ij} \ll aP(T)$$

and

$$(i, \mu) \qquad s_i^{\mu} - h_i \ll P(T)$$

hold.

Proof. Clearly, the relations $(i, j, 0)$ and $(i, 1)$ hold.

Suppose that the integer μ is positive. Assume that for all $i, j \in I$, the relations $(i, j, \mu - 1)$ and (i, μ) hold. We show that for all $i, j \in I$, the relations (i, j, μ) and $(i, \mu + 1)$ hold. The equation (μ) implies that the congruence

$$f_{ij\mu} T^{\mu} h_j \equiv_{\mu} s_i^{\mu} - f_{ij}^{\mu-1} s_j^{\mu}$$

holds. The right side of the above congruence is equal to

$$s_i^{\mu} - h_i - (f_{ij}^{\mu-1} - h_{ij})(s_j^{\mu} - h_j) - h_{ij}(s_j^{\mu} - h_j) + h_i - (f_{ij}^{\mu-1} - h_{ij})h_j.$$

Since the last two terms do not contribute to the term containing T^{ν} for each $\nu \geq \mu$, the relation

$$s_i^{\mu} - f_{ij}^{\mu-1} s_j^{\mu} \ll (1 + (ab/c) + d)P(T)$$

holds. Thus, the relation

$$f_{ij\mu} T^{\mu} h_j \ll (1 + (ab/c) + d)P(T)$$

holds. By the choice of α, the relation

$$f_{ij\mu} T^{\mu} \ll \alpha(1 + (ab/c) + d)P(T)$$

holds. Therefore, the relation (i, j, μ) holds. By the definition of $g_{ij\mu+1}$, the congruence

$$g_{ij\mu+1} T^{\mu+1} \equiv_{\mu+1} s_i^{\mu} - f_{ij}^{\mu} s_j^{\mu}$$

holds. The right side of the above congruence is equal to

$$s_i^{\mu} - h_i - (f_{ij}^{\mu} - h_{ij})(s_j^{\mu} - h_j) - h_{ij}(s_j^{\mu} - h_j) + h_i - (f_{ij}^{\mu} - h_{ij})h_j.$$

Since the last two terms do not contribute to the term containing T^ν for each $\nu \geq \mu$, the relation

$$s_i^\mu - f_{ij}^\mu s_j^\mu \ll (1 + (ab/c) + d)P(T)$$

holds. Thus, the relation

$$g_{ij\mu+1}T^{\mu+1} \ll (1 + (ab/c) + d)P(T)$$

holds. Since the inequality $|s_{i\mu+1}|_{A_i} < \gamma |t_{i\mu+1}|_{B_i}$ holds, by the choice of β, the relation

$$s_{i\mu+1}T^{\mu+1} \ll \beta\gamma(1 + (ab/c) + d)P(T)$$

holds. Therefore, the relation $(i, \mu + 1)$ holds. □

Since $P(T)$ converges on a neighborhood of 0, the formal solutions converge for some special real number δ, i.e., $s \in C^0(\mathcal{V}_\delta, \mathcal{O}_{Y_\delta})$ and $f \in C^1(\mathcal{V}_\delta, \mathcal{O}_{Y_\delta})$. This proves Theorem 3.1.

We say that a Cartier divisor D admits a *global defining function* if there exist an admissible open subset P of X that contains D and a defining function of D on P.

Theorem 3.3. *Let D be an effective Cartier divisor on a quasi-compact separated rigid analytic space X. Assume that the following conditions are satisfied.*

(i) The cohomology group $H^1(D, \mathcal{N}_{D/X})$ vanishes.
(ii) The normal bundle $\mathcal{N}_{D/X}$ admits a nowhere vanishing section.

Then the divisor D admits a global defining function.

Proof. There exist δ, s, and f satisfying the conditions in Theorem 3.1. By the condition (2)–(3) in the same theorem, we obtain the divisor E_δ on Y_δ. By lemma 2.6, for all sufficiently small special real number η, the section s_{i1} is nowhere vanishing on $\mathrm{Sp}\, A'$ for all $i \in I$ where $A_i' = A_i\langle h_i/\eta\rangle$. We take a special real number δ such that the inequality

$$\sup_{m\geq 2}\left|\frac{s_{im}\delta^{m-1}}{s_{i1}}\right|_{A_i'} < 1$$

holds for all $i \in I$. Take a special real number μ that is less than one. For each $i \in I$, put $U_i' := \mathrm{Sp}\, A_i'\langle s_{i0}/s_{i1}\delta\mu\rangle$. Since the condition (1) in Theorem 3.1 is satisfied, the restriction of the projection $\pi\colon E_\delta \to X$ to the preimage $\pi^{-1}(U_i')$ is an isomorphism. Put $P := \bigcup_{i\in I} U_i'$. Then, by Lemma 2.1, the union P is an admissible open subset of X. Thus, the composite $P \to B(\delta)$ of the restriction of the inverse of the morphism π to $\pi^{-1}(P)$ and the second projection $Y_\delta \to B(\delta)$ is a global defining function of D. □

The *supremum semi-norm* on an affinoid algebra A is the K-algebra semi-norm $|\cdot|_{\mathrm{Sp}\,A}\colon A \to \mathbb{R}_{\geq 0}$ defined by the equality:

$$|f|_{\mathrm{Sp}\,A} := \sup_{x \in \mathrm{Sp}\,A} |f(x)|, \quad f \in A.$$

Corollary 2 in [1, 3.8.2] shows the following lemma.

Lemma 3.4. *For an affinoid algebra A, the inequality $|f|_{\mathrm{Sp}\,A} \leq |f|_A$ holds for all $f \in A$.*

Let D be an effective Cartier divisor on a separated rigid analytic space X that admits a relatively big covering. By Lemma 2.8 and Proposition 2.9, we obtain the admissible covering of $\mathcal{U} \cup \{U_0\}$ of X and the special real number ϵ such that the family $\mathcal{U}$ is an admissible covering in Theorem 3.1. Put $J := I \cup \{0\}$, $V_{0\delta} := U_0 \times \mathrm{Sp}\,K\langle T/\delta\rangle$ and $\mathcal{V}_\delta := \{V_{j\delta}\}_{j \in J}$.

Theorem 3.5. *We use the above notations. Assume that the cohomology group $H^1(D, \mathcal{N}_{D/X})$ vanishes. Then we may take a special real number δ in Theorem 3.1 satisfying the following additional condition.*

(4) For all $i \in I$, the analytic function s_i dose not vanish on $V_{i\delta} \cap V_{0\delta}$.

Proof. There exist δ, s, and f satisfying the conditions in Theorem 3.1. By the above lemma, we may take a special real number δ such that the inequality

$$\sup_{m \geq 1} |s_{im}\delta^m|_{\mathrm{Sp}\,A_i} < \epsilon$$

holds for all $i \in I$. This proves the theorem. □

Theorem 3.6. *Let D be an effective Cartier divisor on a separated rigid analytic space X that admits a relatively big covering. Assume that the following conditions are satisfied.*

(i) The cohomology group $H^1(D, \mathcal{N}_{D/X})$ vanishes.
(ii) The normal bundle $\mathcal{N}_{D/X}$ admits a nowhere vanishing section.

Then the divisor D admits a global defining function.

Further, if the rigid analytic space X and the divisor D are smooth, then there exists a tubular neighborhood P of D.

Proof. The first assertion follows from the above two theorems.

We show the second assertion. By Lemma 2.12, we may assume that the family $\mathcal{U}$ is a local tubular neighborhood covering. We may write $D_i = \mathrm{Sp}\,B_i$ and $U_i = \mathrm{Sp}\,A_i$ where $A_i = B_i\langle X_i\rangle$. Further, we may assume that each h_i is the corresponding element to X_i in A_i.

We define a real valued function $|\cdot|_{C\langle X\rangle}$ on a free affinoid algebra $C\langle X\rangle$ over an affinoid algebra C by the following equality:

$$\left|\sum_{e=0}^{\infty} a_e X^e\right|_{C\langle X\rangle} := \max_{e\geq 0} |a_e|_C.$$

Then the function $|\cdot|_{C\langle X\rangle}$ is a complete K-algebra norm on $C\langle X\rangle$. For each $i \in I$, we set such a complete K-algebra norm on free affinoid algebras A_i and $A_i\langle T\rangle$ over B_i and $B_i\langle T\rangle$ respectively.

Let $\{t_i\}_{i\in I}$ be a nowhere vanishing section of the normal bundle $\mathcal{N}_{D/X}$ of D. Then there exist δ, s, and f satisfying the conditions in Theorem 3.1 and Theorem 3.5. We may assume that the inequality

$$\sup_{m\geq 1} |s_{im}\delta^m|_{A_i} < 1$$

holds for all $i \in I$. Then the K-algebra homomorphism

$$\phi_i \colon A_i\langle T/\delta\rangle/(s_i) \xrightarrow{\sim} B_i\langle T/\delta\rangle$$

is an isomorphism for all $i \in I$.

Further, we may take special real numbers δ and η in the following way such that the K-algebra homomorphism

$$\psi_i \colon B_i\langle X_i/\eta\rangle \xrightarrow{\sim} B_i\langle X_i/\eta, T/\delta\rangle/(s_i)$$

is an isomorphism for all $i \in I$. By Lemma 2.6, for all sufficiently small η, the section s_{i1} is nowhere vanishing on $D_i(\eta)$ for all i. Since the condition (1) in Theorem 3.1 is satisfied, we have only to take special real numbers δ and η such that the inequalities

$$\sup_{m\geq 2} \left|\frac{s_{im}\delta^{m-1}}{s_{i1}}\right|_{A_i} < 1$$

and

$$\left|\frac{\eta}{s_{i1}\delta}\right|_{A_i} < 1$$

hold for all $i \in I$.

By the conditions (2)–(4) in Theorem 3.1 and Theorem 3.5, the cochain s defines a divisor E on Y_δ. Let Ψ be the projection $E \to X$. For each $i \in I$, put $E_i := \Psi^{-1}(U_{i\eta})$. Since the restriction $\Psi|\,E_i \to U_{i\eta}$ corresponds to the isomorphism ψ_i, the restriction is an isomorphism.

The isomorphism ϕ_i gives an open immersion $\Phi_i \colon D_i \times B(\delta) \to E$. By Lemma 2.6, there exists a special real number δ' such that $D_i \times B(\delta')$ is contained in $\Phi_i^{-1}(E_i)$ for all $i \in I$. For each $i \in I$, put $W_i := \Psi \circ \Phi_i(D_i \times B(\delta'))$. Then the family $\{W_i\}_{i\in I}$ is a local tubular neighborhood covering of D. Let P be the union $\bigcup_{i\in I} W_i$. Gluing the composites of the isomorphisms $(\Psi \circ \Phi_i)^{-1}|\,W_i \to D_i \times B(\delta')$ and the second projections $D_i \times B(\delta') \to B(\delta')$

(Proposition 1 in [1, 9.3.3]), we obtain the analytic function $F\colon P \to B(\delta')$. Therefore, the admissible open subset P is a tubular neighborhood of D. □

4. Blowing-downs of Exceptional Curves of the First Kind

In this section, we show that we can blow down exceptional curves of the first kind on regular surfaces.

An *exceptional curve of the first kind* is a prime divisor D on a regular surface X satisfying the following conditions. By $C(D)$, we denote the field of constant functions $H^0(D, O_D)$ on D, which is a finite extension of K.

(1) The divisor D is isomorphic to a projective line over $C(D)$.
(2) The degree of the normal bundle of D is equal to $-\dim_K C(D)$.

When X is proper, by Proposition 5.3, the last condition is equivalent to that the self-intersection number of the curve is $-\dim_K C(D)$ (see Appendix). A local calculation shows that the center of the blowing-up is an exceptional curve of the first kind. In this section, we show the converse.

Theorem 4.1 (Castelnuove's criterion)**.** *Let D be an exceptional curve of the first kind on a regular surface X that admits a relatively big covering. Then we can blow down X along D to a regular surface.*

Hopf's theorem (Theorem 2.2 in [8]) and the above theorem give a necessary and sufficient condition for relative minimality.

Corollary 4.2 (criterion for relative minimality)**.** *A proper regular surface is relatively minimal if and only if the surface does not contain any exceptional curves of the first kind.*

Proof of Castelnuovo's criterion. To prove the above theorem, we have only to show that the divisor D contracts to a regular point. We choose two distinct $C(D)$-rational points 0 and ∞ on D. We take an admissible affinoid open subset U and two analytic functions f and g on U satisfying the following conditions.

(1) The subset U contains 0.
(2) The analytic function f defines the divisor D on U.
(3) The two analytic functions f and g generate the maximal ideal of the local ring $O_{X,0}$.

Since the rigid space X is regular and the divisor D is quasi-compact, we have a finite covering of D on whose element the restriction of the divisor D is defined by a single analytic function. Since the rigid space X is separated, by Lemma 2.1, the union Y of the covering elements is an admissible open subset of X. Applying Lemma 2.6 to the intersection W of the covering element U and another covering element V, and the restriction of defining functions of 0 on V to W, we may assume that the single element U of the covering contains 0. By the same method, we may assume that the divisor (g) dose not intersect the other covering elements. Adding two divisors D and (g) on Y, we obtain the divisor E on Y. Since the normal bundle of the divisor E is trivial, by Theorem 3.3, we obtain a global defining function ϕ of E on a quasi-compact admissible open subset of X. Repeating the same procedure for the point ∞, we obtain two analytic functions ϕ and ψ on a quasi-compact admissible open subset W of X satisfying the following conditions.

(1) A divisor $(\phi) - D$ intersects D at the single point 0.
(2) A divisor $(\psi) - D$ intersects D at the single point ∞.

Then the two analytic functions ϕ and ψ define the divisor D. We may define a morphism $\pi\colon W \to B(r) \times B(r)$ for a sufficiently large special real number r. By Proposition 2.10 and Lemma 2.4 in [8], for a sufficiently small special real number ϵ, the restriction of π to $B(\epsilon) \times B(\epsilon)$ is projective. Chow's theorem (Theorem 5.3) shows that the preimage of $B(\epsilon)\times B(\epsilon)$ under π is the analytification of a regular Noethering scheme of pure dimension two. Therefore, the theorem follows from Castelnuovo's criterion for such schemes [9, 6, p102]. □

5. Appendix: Intersection Theory

In this section, we assume that X is a proper normal surface. We give some methods for calculating intersection numbers of divisors and line bundles. Then we prove the Riemann-Roch theorem for proper smooth surfaces.

For two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X, we define the *intersection number* of $\mathcal{L}_1$ and $\mathcal{L}_2$ by the integer:

$$\chi(\mathcal{O}_X) - \chi(\mathcal{L}_1^\vee) - \chi(\mathcal{L}_2^\vee) + \chi(\mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee).$$

Let us denote this integer by $\mathcal{L}_1 \cdot \mathcal{L}_2$. For divisors D_1 and D_2 on X, define $D_1 \cdot D_2$ as $\mathcal{O}_X(D_1) \cdot \mathcal{O}_X(D_2)$. The following proposition is immediate consequence of the definition.

Proposition 5.1. *For any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, the following equalities hold:*

(1) $\mathcal{L}_1 \cdot \mathcal{O}_X = \mathcal{O}_X \cdot \mathcal{L}_1 = 0$;
(2) $\mathcal{L}_1 \cdot \mathcal{L}_2 = \mathcal{L}_2 \cdot \mathcal{L}_1$.

Lemma 5.2. *For any effective divisor D on X and any line bundle $\mathcal{L}$ on X, the following equality holds:*

$$\mathcal{O}_X(D) \cdot \mathcal{L} = \chi(\mathcal{O}_D) - \chi(\mathcal{O}_D \otimes \mathcal{L}^\vee).$$

Proof. Since we have the sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

tensoring $\mathcal{L}^\vee$, we obtain the sheaf exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-D) \otimes \mathcal{L}^\vee \longrightarrow \mathcal{L}^\vee \longrightarrow \mathcal{O}_D \otimes \mathcal{L}^\vee \longrightarrow 0.$$

These sequences give the equalities

$$\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D))$$

and

$$\chi(\mathcal{O}_D \otimes \mathcal{L}^\vee) = \chi(\mathcal{L}^\vee) - \chi(\mathcal{O}_X(-D) \otimes \mathcal{L}^\vee).$$

Subtracting the second equality from the first equality, we obtain the desired equality. □

The fundamental calculation methods are given by the following proposition.

Proposition 5.3. *Let $\pi\colon D^n \to D$ be a normalization of a prime divisor D on X. Then, for any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X, the following equalities hold:*

(1) $\mathcal{O}_X(D) \cdot \mathcal{L} = \deg_{D^n} \pi^* \mathcal{L}$;
(2) $\mathcal{O}_X(D) \cdot \mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{O}_X(D) \cdot \mathcal{L}_1 + \mathcal{O}_X(D) \cdot \mathcal{L}_2$.

Proof. The first equality follows from the previous lemma and Theorem 5.13 in [8]. Therefore, the second equality holds since the equality

$$\deg_{D^n} \pi^*(\mathcal{L}_1 \otimes \mathcal{L}_2) = \deg_{D^n} \pi^* \mathcal{L}_1 + \deg_{D^n} \pi^* \mathcal{L}_2$$

holds. □

Lemma 5.4. *For any effective divisor $\sum_{i \in I} a_i D_i$ on X and any line bundle $\mathcal{L}$ on X, the following equality holds:*

$$\mathcal{O}_X\left(\sum_{i \in I} a_i D_i\right) \cdot \mathcal{L} = \sum_{i \in I} a_i \mathcal{O}_X(D_i) \cdot \mathcal{L}.$$

Proof. By induction on $\sum_{i \in I} a_i$, it suffices to show that for any effective divisor D_1, any prime divisor D_2, and any line bundle $\mathcal{L}$ on X, the following equality holds:

$$\mathcal{O}_X(D_1 + D_2) \cdot \mathcal{L} = \mathcal{O}_X(D_1) \cdot \mathcal{L} + \mathcal{O}_X(D_2) \cdot \mathcal{L}.$$

Since we have the isomorphism

$$\mathcal{O}_{D_2} \otimes \mathcal{O}_X(-D_1) \cong \mathcal{O}_X(-D_1)/\mathcal{O}_X(-D_1 - D_2),$$

we obtain the sheaf exact sequence:

$$0 \longrightarrow \mathcal{O}_{D_2} \otimes \mathcal{O}_X(-D_1) \longrightarrow \mathcal{O}_{D_1 + D_2} \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0.$$

Tensoring $\mathcal{L}^\vee$, we obtain the exact sequence:

$$0 \longrightarrow \mathcal{O}_{D_2} \otimes \mathcal{O}_X(-D_1) \otimes \mathcal{L}^\vee \longrightarrow \mathcal{O}_{D_1+D_2} \otimes \mathcal{L}^\vee \longrightarrow \mathcal{O}_{D_1} \otimes \mathcal{L}^\vee \longrightarrow 0.$$

These sequences give the equalities

$$\chi(\mathcal{O}_{D_1+D_2}) = \chi(\mathcal{O}_{D_1}) + \chi(\mathcal{O}_{D_2} \otimes \mathcal{O}_X(-D_1))$$

and

$$\chi(\mathcal{O}_{D_1+D_2} \otimes \mathcal{L}^\vee) = \chi(\mathcal{O}_{D_1} \otimes \mathcal{L}^\vee) + \chi(\mathcal{O}_{D_2} \otimes \mathcal{O}_X(-D_1) \otimes \mathcal{L}^\vee).$$

Subtracting the second equality from the first equality, we obtain the desired equality by Lemma 5.2 and Proposition 5.3. □

By a direct calculation of intersection numbers, we obtain the following lemma.

Lemma 5.5. *For any three line bundles $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ on X, the following equations are equivalent:*

(1) $\mathcal{L}_1 \cdot \mathcal{L}_2 \otimes \mathcal{L}_3 = \mathcal{L}_1 \cdot \mathcal{L}_2 + \mathcal{L}_1 \cdot \mathcal{L}_3$*;*
(2) $\chi(\mathcal{O}_X) - \chi(\mathcal{L}_1^\vee) - \chi(\mathcal{L}_2^\vee) - \chi(\mathcal{L}_3^\vee) + \chi(\mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee) + \chi(\mathcal{L}_2^\vee \otimes \mathcal{L}_3^\vee) + \chi(\mathcal{L}_3^\vee \otimes \mathcal{L}_1^\vee) - \chi(\mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee \otimes \mathcal{L}_3^\vee) = 0.$

Lemma 5.6. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ be line bundles on X. Assume that one of these line bundles is isomorphic to $\mathcal{O}_X(D)$ where D is an effective divisor. Then the following equality holds:*

$$\mathcal{L}_1 \cdot \mathcal{L}_2 \otimes \mathcal{L}_3 = \mathcal{L}_1 \cdot \mathcal{L}_2 + \mathcal{L}_1 \cdot \mathcal{L}_3.$$

Proof. By Lemma 5.5, it suffices to show that the following equality holds:

$$\mathcal{O}_X(D) \cdot \mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{O}_X(D) \cdot \mathcal{L}_1 + \mathcal{O}_X(D) \cdot \mathcal{L}_2.$$

This equality follows from Lemma 5.4 and Proposition 5.3. □

Proposition 5.7. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ be three line bundles on X. Assume that one of these line bundles is isomorphic to $\mathcal{O}_X(D)$ where D is a divisor. Then the following equality holds:*

$$\mathcal{L}_1 \cdot \mathcal{L}_2 \otimes \mathcal{L}_3 = \mathcal{L}_1 \cdot \mathcal{L}_2 + \mathcal{L}_1 \cdot \mathcal{L}_3.$$

Proof. By Lemma 5.5, it suffices to show that the equality

$$\mathcal{O}_X(D) \cdot \mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{O}_X(D) \cdot \mathcal{L}_1 + \mathcal{O}_X(D) \cdot \mathcal{L}_2.$$

holds. We may write $D = D_1 - D_2$ where D_1 and D_2 are effective divisors. Lemma 5.6 shows that the equalities

$$\mathcal{O}_X(D) \cdot \mathcal{L} = \mathcal{O}_X(D_1) \cdot \mathcal{L} + \mathcal{O}_X(-D_2) \cdot \mathcal{L}$$

and

$$0 = \mathcal{O}_X(D_2) \cdot \mathcal{L} + \mathcal{O}_X(-D_2) \cdot \mathcal{L}$$

hold. Subtracting the second equality from the first equation, we obtain the equality:

$$\mathcal{O}_X(D) \cdot \mathcal{L} = \mathcal{O}_X(D_1) \cdot \mathcal{L} - \mathcal{O}_X(D_2) \cdot \mathcal{L}.$$

Thus, the desired equality follows from Lemma 5.6. □

Corollary 5.8. *Let D be a divisor $\sum_{i \in I} a_i D_i$ on X. Let $\pi_i \colon D_i^n \to D_i$ be a normalization of a prime divisor D_i. Then, for any line bundle $\mathcal{L}$ on X, the following equality holds:*

$$\mathcal{O}_X(D) \cdot \mathcal{L} = \sum_{i \in I} a_i \deg_{D_i^n} \pi_i^* \mathcal{L}.$$

Corollary 5.9. *For any two divisors $\sum_{i \in I} a_i D_i$ and $\sum_{j \in J} b_j D_j$, the following equality holds:*

$$\sum_{i \in I} a_i D_i \cdot \sum_{j \in J} b_j D_j = \sum_{i \in I, j \in J} a_i b_j D_i \cdot D_j.$$

We show that local calculations yield the intersection number of two distinct prime divisors. The *local intersection number* of two distinct prime divisors D_1 and D_2 at a point p on $D_1 \cap D_2$ is the integer $\dim_K(\mathcal{O}_X/\mathcal{O}_X(-D_1) + \mathcal{O}_X(-D_2))_p$. Let us denote this number by $I(p, D_1, D_2)$. Since the intersection $D_1 \cap D_2$ is a finite number of points, the sum $\sum_{p \in D_1 \cap D_2} I(p, D_1, D_2)$ is finite.

Proposition 5.10. *For any two distinct prime divisors D_1 and D_2 on X, the following equality holds:*

$$D_1 \cdot D_2 = \sum_{p \in D_1 \cap D_2} I(p, D_1, D_2).$$

Proof. Put

$$\mathcal{F} := \mathcal{O}_X/\mathcal{O}_X(-D_1) + \mathcal{O}_X(-D_2).$$

By definition, the equality

$$\chi(\mathcal{F}) = \sum_{p \in D_1 \cap D_2} I(p, D_1, D_2)$$

holds. Since we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D_1 - D_2) \longrightarrow \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0,$$

we obtain the equality:

$$\chi(\mathcal{F}) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D_1)) - \chi(\mathcal{O}_X(-D_2)) + \chi(\mathcal{O}_X(-D_1 - D_2)).$$

Since the right side of the above equality is equal to $D_1 \cdot D_2$, we obtain the desired equality. □

The line bundle Ω_Y^2 of regular two-forms on a smooth surface Y is called the *canonical line bundle* on Y. Let us denote this line bundle by $\mathcal{K}_Y$. We prove the Riemann-Roch theorem for proper smooth surfaces.

Theorem 5.11 (Riemann-Roch theorem for proper smooth surfaces)**.** *Assume that X is smooth. For any line bundle $\mathcal{L}$ on X, the following equality holds:*

$$\chi(\mathcal{L}) = \frac{\mathcal{L} \cdot \mathcal{K}_X^\vee - \mathcal{L} \cdot \mathcal{L}^\vee}{2} + \chi(\mathcal{O}_X).$$

Proof. The Serre duality theorem (Theorem in [11, 5.1]) gives the equality:

$$\mathcal{L} \cdot \mathcal{K}_X^\vee = \chi(\mathcal{L}) - \chi(\mathcal{L}^\vee).$$

Adding the above equality to the equality

$$-\mathcal{L} \cdot \mathcal{L}^\vee = \chi(\mathcal{L}) + \chi(\mathcal{L}^\vee) - 2\chi(\mathcal{O}_X),$$

we obtain the desired equality. □

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