

FINITE GENERATION OF THE LOG CANONICAL RING IN DIMENSION FOUR

OSAMU FUJINO

Dedicated to the memory of Professor Masayoshi Nagata

ABSTRACT. We treat three different topics on the log minimal model program, especially for four-dimensional log canonical pairs.

- (a) Finite generation of the log canonical ring in dimension four.
- (b) Abundance theorem for irregular fourfolds.
- (c) Invariance of log plurigenera for threefolds.

We obtain (a) as a direct consequence of Birkar's existence of four-dimensional log minimal models by using Fukuda's theorem on the four-dimensional log abundance conjecture. We can prove (b) only by using traditional arguments. More precisely, we prove the abundance conjecture for irregular $(n+1)$ -folds on the assumption that the minimal model conjecture holds in dimension $\leq n$. The invariance of log plurigenera for threefolds (c) is a consequence of the semi-stable log minimal model program for fourfolds and a generalization of Kollár's torsion-free theorem.

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1. INTRODUCTION

In this paper, we treat three different topics on the log minimal model program, especially for four-dimensional log canonical pairs. We

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will freely use the results on the three-dimensional log minimal model program (cf. [FA], [KeMM], etc.). Sorry, we do not always refer to the original papers since the results are scattered in various places.

1 (Finite generation of the log canonical ring in dimension four). The following theorem is the main result of Section 3 (cf. [F8, Section 3.1]).

Theorem 1.1 (Finite generation of the log canonical ring in dimension four). *Let $\pi : X \rightarrow Z$ be a proper surjective morphism from a smooth fourfold X . Let B be a boundary \mathbb{Q} -divisor on X such that $\text{Supp} B$ is a simple normal crossing divisor on X . Then the relative log canonical ring*

$$R(X/Z, K_X + B) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$$

is a finitely generated \mathcal{O}_Z -algebra.

It is easy to see that Theorem 1.1 is equivalent to Theorem 1.2.

Theorem 1.2. *Let $\pi : X \rightarrow Z$ be a proper surjective morphism from a four-dimensional log canonical pair (X, B) such that B is an effective \mathbb{Q} -divisor. Then the relative log canonical ring*

$$R(X/Z, K_X + B) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$$

is a finitely generated \mathcal{O}_Z -algebra.

In Section 3, we give a proof of Theorem 1.1 by using Birkar's existence theorem of four-dimensional log minimal models (cf. [B]) and Fukuda's result on the log abundance conjecture for fourfolds (cf. [Fk]). A key point of Fukuda's result is the abundance theorem for semi log canonical threefolds in [F1].

2 (Abundance theorem for irregular fourfolds). In Section 4, we prove the abundance theorem for irregular $(n + 1)$ -folds on the assumption that the minimal model conjecture holds in dimension $\leq n$ (see Theorem 4.3). By this result, we know that the abundance conjecture for irregular varieties is the problem for lower dimensional varieties. Since the minimal model conjecture holds in dimension ≤ 3 , we obtain the next theorem (see Corollary 4.5).

Theorem 1.3 (Abundance theorem for irregular fourfolds). *Let X be a normal complete fourfold with only canonical singularities. Assume that K_X is nef and the irregularity $q(X)$ is not zero. Then K_X is semi-ample.*

We also prove that there exists a good minimal model for any smooth projective irregular fourfold (see Theorem 4.6).

Theorem 1.4 (Good minimal models of irregular fourfolds). *Let X be a smooth projective irregular fourfold. If X is not uni-ruled, then there exists a normal projective variety X' such that X' has only \mathbb{Q} -factorial terminal singularities, X' is birationally equivalent to X , and $K_{X'}$ is semi-ample.*

3 (Invariance of log plurigenera for threefolds). In the final section: Section 4, we treat the invariance of log plurigenera (see Corollary 5.5). It is a direct generalization of Nakayama's result (cf. [N]).

Theorem 1.5 (Invariance of log plurigenera). *Let $f : X \rightarrow C$ be a smooth projective morphism onto a non-singular curve C with connected fibers. Let D be a simple normal crossing divisor on X such that D is relatively normal crossing over C . If $\dim X \leq 4$, then*

$$\dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + D_s))),$$

where $D_s = D|_{X_s}$, is constant for any $s \in C$ and any non-negative integer m . In particular, the logarithmic Kodaira dimension of $X_s \setminus D_s$, that is, $\kappa(X_s, K_{X_s} + D_s)$, is constant for any $s \in C$.

We obtain this theorem as a consequence of the semi-stable log minimal model program for fourfolds and a generalization of Kollár's torsion-free theorem. We give an important remark. If we establish the log minimal model program and the log abundance conjecture in dimension $\leq n$, then the arguments in Section 4 works without any changes in dimension $\leq n$. For the details, see Theorem 5.3.

We note that each section can be read independently.

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We will work over \mathbb{C} , the complex number field, throughout this paper. We will freely use the notation in [KMM] and [KM]. Note that we do not use \mathbb{R} -divisors.

2. PRELIMINARIES

In this section, we collect basic definitions.

Definition 2.1 (Divisors, \mathbb{Q} -divisors). Let X be a normal variety. For a \mathbb{Q} -Weil divisor $D = \sum_{j=1}^r d_j D_j$ on X such that $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\lceil D \rceil = \sum_{j=1}^r \lceil d_j \rceil D_j$ (resp. the *round-down*

$\lfloor D \rfloor = \sum_{j=1}^r \lfloor d_j \rfloor D_j$, where for any rational number x , $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x+1$ (resp. $x-1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$.

We call D a *boundary* \mathbb{Q} -divisor if $0 \leq d_j \leq 1$ for any j .

We note that $\sim_{\mathbb{Q}}$ denotes the \mathbb{Q} -linear equivalence of \mathbb{Q} -divisors.

We call X *\mathbb{Q} -factorial* if and only if every Weil divisor on X is \mathbb{Q} -Cartier.

Definition 2.2 (Exceptional locus). For a proper birational morphism $f : X \rightarrow Y$, the *exceptional locus* $\text{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

Let us quickly recall the definitions of singularities of pairs.

Definition 2.3 (Singularities of pairs). Let X be a normal variety and B an effective \mathbb{Q} -divisor on X such that $K_X + B$ is \mathbb{Q} -Cartier. Let $f : Y \rightarrow X$ be a resolution such that $\text{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $f_*^{-1}B$ is the strict transform of B on Y . We write

$$K_Y = f^*(K_X + B) + \sum_i a_i E_i$$

and $a(E_i, X, B) = a_i$. We say that (X, B) is *lc* (resp. *klt*) if and only if $a_i \geq -1$ (resp. $a_i > -1$) for any i . We note that lc (resp. klt) is an abbreviation of *log canonical* (resp. *Kawamata log terminal*). We also note that the *discrepancy* $a(E, X, B) \in \mathbb{Q}$ can be defined for any prime divisor E over X .

In the above notation, if $B = 0$ and $a_i > 0$ (resp. $a_i \geq 0$) for any i , then we say that X has only *terminal* (resp. *canonical*) singularities.

Definition 2.4 (Divisorial log terminal pair). Let X be a normal variety and B a boundary \mathbb{Q} -divisor such that $K_X + B$ is \mathbb{Q} -Cartier. If there exists a resolution $f : Y \rightarrow X$ such that

- (i) both $\text{Exc}(f)$ and $\text{Exc}(f) \cup \text{Supp}(f_*^{-1}B)$ are simple normal crossing divisors on Y , and
- (ii) $a(E, X, B) > -1$ for every exceptional divisor $E \subset Y$,

then (X, B) is called *divisorial log terminal* (*dlt*, for short).

For the details of singularities of pairs, see, for example, [KM] and [F5].

Definition 2.5 (Center, lc center). Let E be a prime divisor over X . The closure of the image of E on X is denoted by $c_X(E)$ and called the *center* of E on X .

Let (X, B) be an lc pair. If $a(E, X, B) = -1$, $c_X(E)$ is called an *lc center* of (X, B) .

The following definitions are now classical.

Definition 2.6 (Iitaka's D -dimension and numerical D -dimension). Let X be a normal complete variety and D a \mathbb{Q} -Cartier \mathbb{Q} -divisor. Assume that mD is Cartier for a positive integer m . Let

$$\Phi_{|tmD|} : X \dashrightarrow \mathbb{P}^{\dim |tmD|}$$

be rational mappings given by linear systems $|tmD|$ for positive integers t . We define *Iitaka's D -dimension*

$$\kappa(X, D) = \begin{cases} \max_{t>0} \dim \Phi_{|tmD|}(X), & \text{if } |tmD| \neq \emptyset \text{ for some } t, \\ -\infty, & \text{otherwise.} \end{cases}$$

In case D is nef, we can also define the *numerical D -dimension*

$$\nu(X, D) = \max\{e \mid D^e \neq 0\},$$

where \equiv denotes *numerical equivalence*. We note that $\nu(X, D) \geq \kappa(X, D)$ holds.

Definition 2.7 (Nef and abundant divisors). Let X be a normal complete variety and D a \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Assume that D is nef. The nef \mathbb{Q} -divisor D is said to be *abundant* if the equality $\kappa(X, D) = \nu(X, D)$ holds. Let $\pi : X \rightarrow Z$ be a proper surjective morphism of normal varieties and D a π -nef \mathbb{Q} -divisor on X . Then D is said to be *π -abundant* if D_η is abundant, where $D_\eta = D|_{X_\eta}$ and X_η is the generic fiber of π .

Definition 2.8 (Irregularity). Let X be a normal complete variety with only rational singularities. We put

$$q(X) = h^0(X, \mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) < \infty$$

and call it the *irregularity* of X .

Let X be as above. If $q(X) \neq 0$, then we call X *irregular*.

If X' is a normal complete variety with only rational singularities such that X' is birationally equivalent to X , then it is easy to see that $q(X) = q(X')$.

3. LOG CANONICAL RING

In this section, we prove the following theorem: Theorem 1.1.

Theorem 3.1 (Finite generation of the four-dimensional log canonical ring). *Let $\pi : X \rightarrow Z$ be a proper surjective morphism from a smooth fourfold X . Let B be a boundary \mathbb{Q} -divisor on X such that $\text{Supp} B$ is*

a simple normal crossing divisor on X . Then the relative log canonical ring

$$R(X/Z, K_X + B) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$$

is a finitely generated \mathcal{O}_Z -algebra.

Let us start the proof of Theorem 1.1. The next proposition is well known and a slight generalization of [K4, Theorem 7.3].

Proposition 3.2. *Let (X, B) be a proper log canonical fourfold such that $K_X + B$ is nef and $\kappa(X, K_X + B) > 0$. Then $K_X + B$ is abundant, that is, $\kappa(X, K_X + B) = \nu(X, K_X + B)$.*

Proof. See, for example, [Fk, Proposition 3.1]. We note that we need the three-dimensional log minimal model program and log abundance theorem here (see [FA], [KeMM], and [KeMM2]). \square

Let us recall Fukuda's result in [Fk]. We will generalize this in Theorem 3.8.

Theorem 3.3 (cf. [Fk, Theorem 1.5]). *Let (X, B) be a proper dlt fourfold. Assume that $K_X + B$ is nef and that $\kappa(X, K_X + B) > 0$. Then $K_X + B$ is semi-ample.*

Proof. By Proposition 3.2, $\kappa(X, K_X + B) = \nu(X, K_X + B)$. We put $S = \lfloor B \rfloor$ and $K_S + B_S = (K_X + B)|_S$. Then the pair (S, B_S) is semi-divisorial log terminal and $K_S + B_S$ is semi-ample by [F1, Theorem 0.1]. Finally, by [F7, Corollary 6.7], we obtain that $K_X + B$ is semi-ample. \square

Remark 3.4. The proof of [Fk, Proposition 3.3] depends on [K4, Theorem 5.1]. It requires [K4, Theorem 4.3] whose proof contains a non-trivial gap. See [F5, Remark 3.10.3] and [F9]. So, we adopted [F7, Corollary 6.7] in the proof of Theorem 3.3.

Let us go to the proof of Theorem 1.1. Here, we adopt Birkar's definition of the log minimal model (see [B, Definition 2.4]), which is a slightly different from [KM, Definition 3.50]. See Remark 3.6 and Example 3.7 below.

Definition 3.5 (cf. [B, Definition 2.4]). Let (X, B) be a log canonical pair over Z . A log minimal model $(Y/Z, B_Y + E)$ of $(X/Z, B)$ consists of a birational map $\phi : X \dashrightarrow Y/Z$, $B_Y = \phi_* B$, and $E = \sum_j E_j$, where E_j is a prime divisor on Y and ϕ^{-1} -exceptional for any i , and satisfies the following conditions:

- (1) Y is \mathbb{Q} -factorial and $(Y, B_Y + E)$ is dlt,

- (2) $K_Y + B_Y + E$ is nef over Z , and
(3) for any prime divisor D on X which is exceptional over Y , we have

$$a(D, X, B) < a(D, Y, B_Y + E),$$

where $a(D, X, B)$ (resp. $a(D, Y, B_Y + E)$) denotes the discrepancy of D with respect to (X, B) (resp. $(Y, B_Y + E)$).

Remark 3.6. In [KM, Definition 3.50], it is required that ϕ^{-1} has no exceptional divisors.

Example 3.7. Let $X = \mathbb{P}^2$ and D_X the complement of the big torus. Then $K_X + D_X$ is dlt and $K_X + D_X \sim 0$. Let $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and D_Y the complement of the big torus. Then (Y, D_Y) is a log minimal model of (X, D_X) in the sense of Definition 3.5 (cf. [B, Definition 2.4]). Of course, $K_Y + D_Y$ is dlt and $K_Y + D_Y \sim 0$. On the other hand, (Y, D_Y) is not a log minimal model of (X, D_X) in the sense of [KM, Definition 3.50].

Proof of Theorem 1.1. We can assume that the fiber of π is connected. First, if $\kappa(X_\eta, K_{X_\eta} + B_\eta) = -\infty$, where η is the generic point of Z , then the statement is trivial. We note that the statement is obvious when Z is a point and $\kappa(X, K_X + B) = 0$. So, we can assume that $\kappa(X_\eta, K_{X_\eta} + B_\eta) \geq 0$ and that $\kappa(X, K_X + B) \geq 1$ when Z is a point. Since the problem is local, we can assume that Z is affine. By compactifying Z and X and taking a resolution of X , we can assume that X and Z are projective and that $\text{Supp} B$ is a simple normal crossing divisor. By the assumption, we can find an effective \mathbb{Q} -divisor M on X such that $K_X + B \sim_{\mathbb{Q}, \pi} M$, that is, there exists a \mathbb{Q} -divisor N on Z such that $K_X + B \sim_{\mathbb{Q}} M + \pi^* N$. We take a log minimal model of (X, B) by using the arguments in [B, Section 3]. Then we obtain a projective surjective morphism $\pi_Y : Y \rightarrow Z$ such that $(Y/Z, B_Y + \sum_j E_j)$ is a log minimal model of $(X/Z, B)$, where B_Y is the pushforward of B on Y by $\phi : X \dashrightarrow Y$ and E_j is exceptional over X and is a prime divisor on Y for any i . Let A be a sufficiently ample general Cartier divisor on Z . Then $(Y, B_Y + E + \pi_Y^* A)$, where $E = \sum_j E_j$, is a log minimal model of $(X, B + \pi^* A)$ by Kawamata's bound of the length of extremal rays (see, for example, [F8, Subsection 3.1.3]). Since $\kappa(Y, K_Y + B_Y + E + \pi_Y^* A) \geq 1$, $K_Y + B_Y + E + \pi_Y^* A$ is semi-ample by Theorem 3.3. In particular,

$$K_Y + B_Y + E = K_Y + B_Y + E + \pi_Y^* A - \pi_Y^* A$$

is π_Y -semi-ample. Thus,

$$\bigoplus_{m \geq 0} \pi_{Y*} \mathcal{O}_Y(\lfloor m(K_Y + B_Y + E) \rfloor)$$

is a finitely generated \mathcal{O}_Z -algebra. Therefore,

$$\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$$

is a finitely generated \mathcal{O}_Z -algebra. We finish the proof. \square

The final theorem in this section is a generalization of Fukuda's theorem (see Theorem 3.3).

Theorem 3.8 (A special case of the log abundance theorem). *Let $\pi : X \rightarrow Z$ be a proper surjective morphism from a four-dimensional log canonical pair (X, B) such that B is an effective \mathbb{Q} -divisor and that $K_X + B$ is π -nef. When Z is a point, we further assume that $\kappa(X, K_X + B) > 0$. Then $K_X + B$ is π -semi-ample.*

Proof. Without loss of generality, we can assume that π has connected fibers. By Proposition 3.2 and the log abundance theorem in dimension ≤ 3 , $K_{X_\eta} + B_\eta$ is nef and abundant, where η is the generic point of Z . By Theorem 1.2, $\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$ is a finitely generated \mathcal{O}_Z -algebra. Therefore, $K_X + B$ is π -semi-ample by Lemma 3.10 below. \square

The next lemma is well known. We omit the proof for the reader's exercise.

Lemma 3.9 (cf. [L, Theorem 2.3.15]). *Let $\pi : X \rightarrow Z$ be a projective surjective morphism from a smooth variety X to a normal variety Z and M a π -nef and π -big Cartier divisor on X . Then $\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(mM)$ is a finitely generated \mathcal{O}_Z -algebra if and only if M is π -semi-ample.*

By [KMM, Proposition 6-1-3], we can reduce Lemma 3.10 to Lemma 3.9.

Lemma 3.10. *Let $\pi : X \rightarrow Z$ be a proper surjective morphism between normal varieties and M a π -nef and π -abundant Cartier divisor on X . Then $\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(mM)$ is a finitely generated \mathcal{O}_Z -algebra if and only if M is π -semi-ample.*

3.1. Appendix. In this appendix, we explicitly state the results in dimension ≤ 3 because we can find no good references for the relative statements (cf. [Ft], [KeMM], and [KeMM2]).

Theorem 3.11. *Let $\pi : X \rightarrow Z$ be a proper surjective morphism between normal varieties. Assume that (X, B) is log canonical with $\dim X \leq 3$ and that B is an effective \mathbb{Q} -divisor. Then*

$$\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$$

is a finitely generated \mathcal{O}_Z -algebra.

Proof. When Z is a point, this theorem is well known (cf. [Ft], [KeMM], and [KeMM2]). So, we assume that $\dim Z \geq 1$. By the arguments in the proof of Theorem 1.1, we can prove that $\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$ is a finitely generated \mathcal{O}_Z -algebra. \square

Theorem 3.12. *Let $\pi : X \rightarrow Z$ be a proper surjective morphism such that (X, B) is log canonical with $\dim X \leq 3$. Assume that $K_X + B$ is π -nef and B is an effective \mathbb{Q} -divisor. Then $K_X + B$ is π -semi-ample.*

Proof. When Z is a point, this theorem is well known (cf. [Ft], [KeMM], and [KeMM2]). So, we assume that $\dim Z \geq 1$. Without loss of generality, we can assume that π has connected fibers. It is well known that $K_X + B$ is π -nef and π -abundant by the log abundance theorem in dimension ≤ 2 . By Theorem 3.11 and Lemma 3.10, $K_X + B$ is π -semi-ample. \square

We close this appendix with a remark.

Remark 3.13. Let $\pi : X \rightarrow Z$ be a proper surjective morphism between normal varieties. Assume that (X, B) is klt and that B is an effective \mathbb{Q} -divisor. Then

$$\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$$

is a finitely generated \mathcal{O}_Z -algebra by [BCHM]. Therefore, by Lemma 3.10, $K_X + B$ is π -semi-ample if and only if $K_X + B$ is π -nef and π -abundant by Lemma 3.10.

Of course, we know that $K_X + B$ is π -semi-ample if and only if $K_X + B$ is π -nef and π -abundant without appealing [BCHM]. See, for example, [F9]. It is known as Kawamata's theorem (cf. [K4]).

4. ABUNDANCE THEOREM FOR IRREGULAR VARIETIES

In this section, we treat the abundance conjecture for irregular varieties. Let us recall the following minimal model conjecture.

Conjecture 4.1 (Minimal model conjecture (weak form)). *Let X be a smooth projective variety. Assume that K_X is pseudo-effective. Then there exists a normal projective variety X' which satisfies the following conditions:*

- (i) X' is birationally equivalent to X .
- (ii) X' has at most canonical singularities.
- (iii) $K_{X'}$ is semi-ample.

In particular, the Kodaira dimension $\kappa(X) = \kappa(X, K_X)$ is non-negative. We sometimes call X' a good minimal model of X .

We know that Conjecture 4.1 holds in dimension ≤ 3 (cf. [KMM], [FA], etc.).

Remark 4.2. Let X be a smooth projective variety. Then X is uniruled if and only if K_X is not pseudo-effective by [BDPP].

The next theorem is the main theorem of this section.

Theorem 4.3 (Abundance theorem for irregular $(n + 1)$ -folds). *Assume that Conjecture 4.1 holds in dimension $\leq n$. Let X be a normal complete $(n + 1)$ -fold with only canonical singularities. If K_X is nef and $q(X) \neq 0$, then K_X is semi-ample.*

Proof. Let $\pi : \bar{X} \rightarrow X$ be a resolution and $\alpha : \bar{X} \rightarrow A = \text{Alb}(\bar{X})$ the Albanese mapping. By the assumption, we have $\dim A \geq 1$. Since X has only rational singularities, $\beta = \alpha \circ \pi^{-1} : X \rightarrow A$ is a morphism (cf. [R, Proposition 2.3], [BS, Lemma 2.4.1])

Claim 1. *We have $\kappa(X, K_X) = \kappa(\bar{X}, K_{\bar{X}}) \geq 0$.*

Proof of Claim 1. Let $f : \bar{X} \rightarrow S$ be the Stein factorization of α and F a general fiber of f . Then, by [K5, Corollary 1.2], we have

$$\kappa(\bar{X}, K_{\bar{X}}) \geq \kappa(F, K_F) + \kappa(\bar{S}, K_{\bar{S}}),$$

where \bar{S} is a resolution of S . We note that $\kappa(\bar{S}, K_{\bar{S}}) \geq 0$ because $S \rightarrow \beta(X) \subset A$ is generically finite (see, for example, [U, Theorem 6.10, Lemma 10.1]). We also note that $\kappa(F, K_F) = \kappa(G, K_G) \geq 0$ since $\dim G \leq n$, G has only canonical singularities, and K_G is nef, where $G = \pi(F)$. Here, we used Conjecture 4.1 in dimension $\dim G = \dim F \leq n$. Therefore, we obtain $\kappa(\bar{X}, K_{\bar{X}}) \geq 0$. \square

Claim 2. *If $\kappa(X, K_X) = 0$, then $\nu(X, K_X) = 0$.*

Proof of Claim 2. By Kawamata's theorem (cf. [K3, Theorem 1]), β is surjective and $\beta_*\mathcal{O}_X \simeq \mathcal{O}_A$. Let G be a general fiber of β . Then $\kappa(G, K_G) = 0$ by

$$0 = \kappa(X, K_X) \geq \kappa(G, K_G) + \kappa(A, K_A) = \kappa(G, K_G)$$

as in Claim 1 and $\kappa(G, K_G) \geq 0$ by Conjecture 4.1 in $\dim G \leq n$ since K_G is nef. We note that X and G have only canonical singularities. By Remark 3.13 and the assumption: Conjecture 4.1 in dimension $\leq n$, K_X is β -semi-ample. Therefore, $\beta : X \rightarrow A$ can be written as follows:

$$\beta : X \xrightarrow{f} S \xrightarrow{g} A,$$

where $K_X \sim_{\mathbb{Q}} f^*D$ for some g -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor D on S , $g : S \rightarrow A$ is a birational morphism, and S is a normal variety. Since

$\kappa(X, K_X) = 0$, we obtain $\kappa(S, D) = 0$. So, it is sufficient to prove that $D \sim_{\mathbb{Q}} 0$. By [A, Theorem 0.2], we can write $D \sim_{\mathbb{Q}} K_S + \Delta_S$ such that (S, Δ_S) is klt. In particular, Δ_S is effective. By Lemma 4.4 below, we obtain that g is an isomorphism. Therefore, $D \sim_{\mathbb{Q}} 0$ since $\kappa(S, D) = 0$ and $S = A$ is an Abelian variety. \square

By Claim 1 and Claim 2, $\nu(X, K_X) > 0$ implies $\kappa(X, K_X) > 0$. In this case, we obtain $\kappa(X, K_X) = \nu(X, K_X)$ by Kawamata's argument and the assumption: Conjecture 4.1 in dimension $\leq n$ (see the proof of [K4, Theorem 7.3]). Therefore, K_X is semi-ample by Remark 3.13. \square

We already used the following lemma in the proof of Claim 2.

Lemma 4.4. *Let $g : S \rightarrow A$ be a projective birational morphism from a klt pair (S, Δ_S) to an Abelian variety A . Assume that $K_S + \Delta_S$ is g -nef and $\kappa(S, K_S + \Delta_S) = 0$. Then g is an isomorphism.*

Proof. By replacing S with its small projective \mathbb{Q} -factorialization (cf. [BCHM]), we can assume that S is \mathbb{Q} -factorial. We note that $K_S = E$, where E is effective and $\text{Supp} E = \text{Exc}(g)$ since A is an Abelian variety. If $B = g_* \Delta_S \neq 0$, then $g^* B \leq m(K_S + \Delta_S)$ for some $m > 0$. In this case,

$$1 \leq \kappa(A, B) = \kappa(S, g^* B) \leq \kappa(S, K_S + \Delta_S) = 0.$$

It is a contradiction. Therefore, $B = 0$. This means that Δ_S is g -exceptional. Thus, $K_S + \Delta_S$ is effective, g -exceptional, and $\text{Exc}(g) = \text{Supp}(K_S + \Delta_S)$. By the assumption, $K_S + \Delta_S$ is g -nef. So, g is an isomorphism by the negativity lemma. \square

As a special case of Theorem 4.3, we obtain the abundance theorem for irregular fourfolds.

Corollary 4.5 (Abundance theorem for irregular fourfolds). *Let X be a normal complete fourfold with only canonical singularities. Assume that K_X is nef and the irregularity $q(X)$ is not zero. Then K_X is semi-ample.*

Proof. It is obvious by Theorem 4.3 because Conjecture 4.1 holds for threefolds (cf. [FA], [KM], etc.). \square

We close this section with the following theorem.

Theorem 4.6 (Good minimal models of irregular fourfolds). *Let X be a smooth projective irregular fourfold. If X is not uni-ruled, then X has a good minimal model. More precisely, there exists a normal projective variety X' such that X' has only \mathbb{Q} -factorial terminal singularities, X' is birationally equivalent to X , and $K_{X'}$ is semi-ample.*

Proof. We run the minimal model program. Then we obtain a minimal model X' of X since K_X is pseudo-effective by the assumption. Here, we used the existence and the termination of four-dimensional terminal flips (cf. [KMM, Theorem 5-1-15], [S], and [HM, Corollary 5.1.2]). We note that $q(X') = h^0(X', \mathcal{O}_{X'}) = q(X) \neq 0$. Therefore, by Theorem 4.3, we obtain that $K_{X'}$ is semi-ample. \square

5. INVARIANCE OF LOG PLURIGENERA

Let us recall the definition of dlt morphisms.

Definition 5.1 ([KM, Definition 7.1]). Let X be a normal variety, B an effective \mathbb{Q} -divisor on X and $f : X \rightarrow C$ a non-constant morphism to a non-singular curve C . We say that $f : (X, B) \rightarrow C$ is dlt if $(X, B + f^*P)$ is dlt for every closed point $P \in C$.

The following theorem is an easy consequence of the generalized version of the torsion-free theorem (see, for example, [F8, Theorem 2.39 (i)]).

Theorem 5.2. *Let $f : (X, B) \rightarrow C$ be a proper dlt morphism. Assume that $K_X + B$ is \mathbb{Q} -Cartier and that $K_X + B$ is f -semi-ample. Then $R^i f_* \mathcal{O}_X(m(K_X + B))$ is locally free for any $i \geq 0$, where m is any positive integer such that $m(K_X + B)$ is integral. In particular,*

$$\dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B_s)))$$

is constant for any $s \in C$, where $B_s = B|_{X_s}$.

Proof. Let $g : Y \rightarrow X$ be a resolution such that g is an isomorphism over the generic point of any lc center of (X, B) , $K_Y = g^*(K_X + B) + E$, and that $\text{Supp} E$ is a simple normal crossing divisor on Y . We put $F = \lrcorner f_*^{-1} B \lrcorner$. Then

$$\begin{aligned} & \lrcorner g^*(m(K_X + B)) + E + F \lrcorner \\ & - (K_Y + \{-(g^*(m(K_X + B)) + E + F)\} + F) = g^*((m-1)(K_X + B)). \end{aligned}$$

By the vanishing theorem of Reid–Fukuda type (see, for example, [F8, Lemma 4.10]), we obtain that

$$R^i g_* \mathcal{O}_Y(\lrcorner g^*(m(K_X + B)) + E + F \lrcorner) = 0$$

for any $i > 0$. On the other hand, it is easy to see that

$$g_* \mathcal{O}_Y(\lrcorner g^*(m(K_X + B)) + E + F \lrcorner) \simeq \mathcal{O}_X(m(K_X + B)).$$

By the generalized version of the torsion-free theorem (see, for example, [F8, Theorem 2.39 (i)]), we have that

$$R^i (f \circ g)_* \mathcal{O}_Y(\lrcorner g^*(m(K_X + B)) + E + F \lrcorner)$$

is torsion-free for any i . We note that any lc center of the pair

$$(Y, \{- (g^*(m(K_X + B)) + E + F)\} + F)$$

is dominant onto C by f . Therefore,

$$R^i f_* \mathcal{O}_X(m(K_X + B)) \simeq R^i (f \circ g)_* \mathcal{O}_Y(\Gamma g^*(m(K_X + B)) + E + F^\vee)$$

is locally free for every i . We finish the proof by the base change theorem. \square

The next result is the invariance of log plurigenera in $\dim \leq 3$. It is a direct generalization of [N]. For related topics, see [K1] and [K2].

Theorem 5.3. *Let $f : (X, B) \rightarrow C$ be a projective dlt morphism such that $K_X + B$ is \mathbb{Q} -Cartier and $\dim X \leq 4$. Assume that all the fibers of f are irreducible. Then*

$$\dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B_s)))$$

is constant for any $s \in C$, where $B_s = B|_{X_s}$ and m is any positive integer such that $m(K_X + B)$ is integral.

Proof. First, by the existence of log flips (see [S] and [HM]) and the special termination in dimension four (see [S] and [F6]), there exists a small projective \mathbb{Q} -factorialization $\varphi : (\tilde{X}, \tilde{B}) \rightarrow (X, B)$ such that (\tilde{X}, \tilde{B}) is dlt and $K_{\tilde{X}} + \tilde{B} = \varphi^*(K_X + B)$. By replacing (X, B) with (\tilde{X}, \tilde{B}) , we can assume that X is \mathbb{Q} -factorial. Apply the log minimal model program to (X, B) over C . By the existence of log flips and the termination of four-dimensional semi-stable log flips (see [F4] and Remark 5.4 below), we obtain a sequence of semi-stable log flips and divisorial contractions

$$(X, B) = (X_0, B_0) \dashrightarrow \cdots \dashrightarrow (X_i, B_i) \dashrightarrow \cdots \dashrightarrow (X_k, B_k) = (X', B')$$

such that

$$(1) \quad K_{X'} + B' \text{ is } f'\text{-nef, where } f' : X' \rightarrow C,$$

or

$$(2) \quad \text{there exists a contraction } \psi : X' \rightarrow Z \text{ over } C \text{ such that } \dim Z < \dim X' \text{ and that } -(K_{X'} + B') \text{ is } \psi\text{-ample.}$$

In (1), $K_{X'} + B'$ is f' -semi-ample by the abundance theorem (see Theorem 3.8). Therefore,

$$\dim H^0(X'_s, \mathcal{O}_{X'_s}(m(K_{X'_s} + B'_s))) = \dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B_s)))$$

is constant for any $s \in C$ by Theorem 5.2, where $B'_s = B'|_{X'_s}$. In (2),

$$\begin{aligned} & \dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B_s))) \\ &= \dim H^0(X'_s, \mathcal{O}_{X'_s}(m(K_{X'_s} + B'_s))) = 0 \end{aligned}$$

for any $s \in C$. Thus, we finish the proof. \square

Remark 5.4. In [F2], [F3], and [F4], we assumed that X is projective. However, we do not need this assumption there. Since the exceptional loci of $(2, 1)$ -flips are always projective (see the proof of [AHK, Lemma 3.1]).

The following corollary is a very special case of Theorem 5.3.

Corollary 5.5 (Invariance of log plurigenera). *Let $f : X \rightarrow C$ be a smooth projective morphism onto a non-singular curve C with connected fibers. Let D be a simple normal crossing divisor on X such that D is relatively normal crossing over C . If $\dim X \leq 4$, then*

$$\dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + D_s))),$$

where $D_s = D|_{X_s}$, is constant for any $s \in C$ and any non-negative integer m . In particular, the logarithmic Kodaira dimension of $X_s \setminus D_s$, that is, $\kappa(X_s, K_{X_s} + D_s)$, is constant for any $s \in C$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY,
KYOTO 606-8502 JAPAN

E-mail address: `fujino@math.kyoto-u.ac.jp`