ON THE COMMUTATIVITY OF THE LOCALIZED SELF HOMOTOPY GROUPS OF $SU(n)$

HIROAKI HAMANAKA AND AKIRA KONO

1. Introduction

For a connected Lie group G and a based space X, the set $[X, G]$ of homotopy classes of based maps from X to G inherits a group structure from G. If X is a finite dimensional CW complex, the group $[X, G]$ is nilpotent by [13]. Let p be a prime. We consider the localization $[X, G]_{(p)}$ of $[X, G]$ (see [7]) and $[X, G]_{(p)}$ is isomorphic to $[X, G_{(p)}]$ (see [2]).

An important special case occurs when $X = G$. For then $\mathcal{H}(G) = [G, G]$ studied extensively ([1], [6], [8], [10], [11]).

For an integer $l \geq 2$, G_l denotes $SU(l)$ or $U(l)$ and $\bar{G}_l = SU(l)/H$, where H is a subgroup of the centre of $SU(l) = \mathbf{Z}/l$. G_∞ denotes $SU(\infty)$ or $U(\infty)$.

The purpose of this paper is to show the following:

Theorem 1.1. *Assume $n \geq 3$.*

(i). *$\mathcal{H}(SU(n))_{(p)}$ is commutative if and only if $p = 2n-1$ for $n = 3, 4, 6, 7$ or $p > 2n-1$.*

(ii). *$\mathcal{H}(U(n))_{(p)}$ is commutative if and only if $p = 2n-1$ for $n = 3, 4$, or $p > 2n-1$.*

(iii). *$\mathcal{H}(\bar{G}_n)_{(p)}$ is commutative if and only if $\mathcal{H}(SU(n))_{(p)}$ is commutative.*

Remark 1.2. (i). If $n = 2$, $SU(2) \cong S^3$ and $SU(2)/(\mathbf{Z}/2) \cong SO(3)$. $\mathcal{H}(SU(2))$ and $\mathcal{H}(SO(3))$ are commutative (see [8]). $\mathcal{H}(U(2))$ is known by [10]. $\mathcal{H}(U(2))_{(2)}$ is not commutative and $\mathcal{H}(U(2))_{(p)}$ is commutative for $p \geq 3$.

(ii). If $p > 2n-1$, $G_{n(p)}$ is homotopy commutative by McGibbon [9]. Therefore $\mathcal{H}(G_n)_{(p)}$ is commutative if $p > 2n-1$.

(iii). If $n \geq 3$ and $p \geq 2n-1$, then $p > n$. Therefore if $p \geq 2n-1$ the natural projection $q\colon SU(n) \to \bar{G}_n$ induces a Hopf equivalence $q\colon SU(n)_{(p)} \to \bar{G}_{n(p)}$ denoted by the same symbol. Therefore $q^*\colon \mathcal{H}(\bar{G}_{n(p)}) \to [SU(n), \bar{G}_{n(p)}]$ and $q_*\colon [SU(n), \bar{G}_{n(p)}] \to \mathcal{H}(SU(n))_{(p)}$ are isomorphism of groups and

$$q_*^{-1} \circ q^*\colon \mathcal{H}(\bar{G}_n)_{(p)} \to \mathcal{H}(SU(n))_{(p)}$$

is an isomorphism of groups.

Denote the commutator of G_n and $\bar{G}_n$ by γ_n and $\bar{\gamma}_n$ respectively. Define a map $\alpha\colon G_n \to G_n$ by $\alpha(A) = \bar{A}$ for $A \in G_n$. α induces a map $\bar{\alpha}\colon \bar{G}_n \to \bar{G}_n$ satisfying $\bar{\alpha} \circ q = q \circ \alpha$. To prove Theorem 1.1, we show the following:

Proposition 1.3. (i). *If $p < 2n-1$, then $[\alpha, 1] = \gamma_n \circ (\alpha \wedge 1) \circ \Delta \neq 0$ in $\mathcal{H}(G_n)_{(p)}$.*

(ii). *If $p < 2n-1$, then $[\bar{\alpha}, 1] \neq 0$ in $\mathcal{H}(\bar{G}_n)_{(p)}$.*

Proposition 1.4. (i). *If $n \geq 8$ and $p = 2n-1$ is a prime, then $\mathcal{H}(SU(n))_{(p)}$ is not commutative. If $n \geq 5$ and $p = 2n-1$ is a prime, then $\mathcal{H}(U(n))_{(p)}$ is not commutative.*

(ii). *If $n = 3, 4, 6$ or 7 and $p = 2n-1$, $\mathcal{H}(SU(n))_{(p)}$ is commutative. If $n = 3$ or 4 and $p = 2n-1$, $\mathcal{H}(U(n))_{(p)}$ is commutative.*

Consider the fibre sequence

$$\Omega G_\infty \xrightarrow{\Omega\pi} \Omega W_l \xrightarrow{\delta} G_l \xrightarrow{j} G_\infty \xrightarrow{\pi} W_l = G_\infty/G_l$$

where π is the projection and $j\colon G_l \to G_\infty$ is the inclusion. A lift $\tilde{\gamma}_l\colon G_l \wedge G_l \to \Omega W_l$ of γ_l ($\gamma_l \simeq \delta \circ \tilde{\gamma}_l$) constructed in [5] plays an important role in this paper. We review results on unstable K-theory, $[\ \ , G_l]$ in section 2. Using the results in section 2, Proposition 1.3 and Proposition 1.4 are proved in section 3 and section 4 respectively.

2. Unstable K-theory

In this section $l \geq 2$. Let $W_l = G_\infty/G_l$ and $\pi\colon G_\infty \to W_l$ be the projection. As an algebra

$$H^*(G_l) \cong \bigwedge(x_1, x_3, \dots, x_{2l-1})$$

where $\deg x_{2j-1} = 2j-1$, $x_{2j-1} = \sigma(c_j)$ and $x_1 = 0$ if $G = SU$ and

$$H^*(W_l) \cong \bigwedge(\bar{x}_{2l+1}, \bar{x}_{2l+3}, \dots)$$

where $\deg \bar{x}_{2j+1} = 2j+1$, $\pi^*(\bar{x}_{2j+1} = 2j+1)$. Moreover $\mathcal{P}^1\rho(\bar{x}_{2j+1}) = j\rho(\bar{x}_{2j+2p-1})$ where ρ is the mod p reduction. Put $a_{2j} = \sigma(\bar{x}_{2j+1})$. Consider the fibre sequence

$$\Omega G_\infty \xrightarrow{\Omega\pi} \Omega W_l \xrightarrow{\delta} G_l \xrightarrow{j} G_\infty \xrightarrow{\pi} W_l.$$

In [5] a lift $\tilde{\gamma}_l\colon G_l \wedge G_l \to \Omega W_l$ of γ_l satisfying $\delta \circ \tilde{\gamma}_l \simeq \gamma_l$ and

$$\tilde{\gamma}_l^*(a_{2l}) = \sum_{i+j=l-1} x_{2i+1} \otimes x_{2j+1}$$

is constructed. Moreover by [4]

$$\tilde{\gamma}_l^*(a_{2k}) = \sum_{i+j=k-1} x_{2i+1} \otimes x_{2j+1}$$

for $k \geq l$. Define a map

$$\bar{x} = \prod_{j\geq 0} x_{2j+1}\colon W_l \to K = \prod_{j \geq l} K_j$$

where $K_j = K(\mathbf{Z}_{(p)}, 2j+1)$. Then $\bar{x}_{(0)}\colon W_{l(0)} \to K_{(0)}$ is a homotopy equivalence. For a finite complex X, consider the map

$$\lambda = (\Omega\bar{x})_*\colon [X, \Omega W_l] \to \bigoplus H^{2j}(X; \mathbf{Z}_{(p)}).$$

We have the following:

Lemma 2.1. (i). *λ is a group homomorphism and*

$$\lambda([f]) = (f^*(a_{2l}), f^*(a_{2l+2}), \dots),$$

where $f\colon X \to \Omega W_l$.

(ii). *$\lambda \otimes \mathbf{Q}$ is an isomorphism.*

(iii). *$\lambda \circ (\Omega\pi)_*(\alpha) = (l!ch_l(\alpha), (l+1)!ch_{l+1}(\alpha), \dots)$ where $\alpha \in [X, \Omega G_\infty] = \widetilde{K}(X)$.*

(iv). *If* $\dim X = 2l$, *then* $\Theta = (\Omega x_{2l+1})_*\colon [X, \Omega W_l] \to H^{2l}(X)$ *is an isomorphism of groups (see* [5]*).*

Let E_j be the homotopy fibre of a generator of $H^{2j+2p}(K_j; \mathbf{Z}_{(p)}) \cong \mathbf{Z}_p$ where $j \geq 2$. As an algebra

$$H^*(E_j; \mathbf{Z}_{(p)}) \cong \bigwedge(v_{2j+1}, v_{2j+2p-1})$$

for $* \leq 2j + 4p - 3$, where $\deg v_k = k$ and $\mathcal{P}^1\rho(v_{2j+1}) = \rho(v_{2j+2p-1})$. Consider the fibering:

$$K_{j+p-1} \to E_j \xrightarrow{\theta_j} K_j$$

if $j \geq l$, there is a lift $\hat{x}_{2j+1}$ of $\bar{x}_{2j+1}$ satisfying $\theta_j \circ \hat{x}_{2j+1} \simeq \bar{x}_{2j+1}$, $(\hat{x}_{2j+1})^*(v_{2j+1}) = \bar{x}_{2j+1}$ and

$$(\hat{x}_{2j+1})^*(v_{2j+2p-1}) \equiv j\bar{x}_{2j+2p-1}$$

modulo $pH^{2j+2p-1}(W_l; \mathbf{Z}_{(p)})$, since $\mathcal{P}^1\rho(\bar{x}_{2j+1}) = j\rho(x_{2j+2p-1})$ and $\beta\mathcal{P}^1\rho(\bar{x}_{2j+1}) = 0$. Define

$$\hat{x} = \hat{x}_l = \prod_{j=l}^{l+p-2} \hat{x}_{2j+1}\colon W_{l(p)} \to E = \prod_{j=l}^{l+p-2} E_j.$$

We have the following:

Lemma 2.2. *If* $l < p$, *then* $\hat{x}$ *is a* $(4p-2)$*-equivalence.*

Proof. $(\hat{x})^*\colon H^*(E; \mathbf{Z}_{(p)}) \to H^*(W_{l(p)}; \mathbf{Z}_{(p)})$ is an isomorphism for $* \leq 4p - 2$ and injective for $* = 4p-1$. Therefore $H^*(\hat{x}; \mathbf{Z}_{(p)}) = 0$ for $* \leq 4p-1$ and $H_*(\hat{x}; \mathbf{Z}_{(p)}) = 0$ for $* \leq 4p - 2$. $\square$

Remark 2.3. Since $E_j \simeq \Omega^2 E_{j+1}$, ΩE_j is a homotopy commutative Hopf space. Therefore for a finite complex X, the group $[X, \Omega E_j]$ is commutative. Note that $E_{j(0)} \simeq (K_j \times K_{j+p-1})_{(0)}$.

Consider the following exact commutative diagram

$$\begin{array}{ccccccccc} & & H^{2j+2p-2}(X; \mathbf{Z}_{(p)}) & \longrightarrow & [X, \Omega E_j] & \longrightarrow & H^{2j}(X; \mathbf{Z}_{(p)}) & & \\ & & \downarrow{\scriptstyle \otimes\mathbf{Q}} & & \downarrow{\scriptstyle \otimes\mathbf{Q}} & & \downarrow & & \\ 0 & \longrightarrow & H^{2j+2p-2}(X; \mathbf{Q}) & \longrightarrow & [X, \Omega E_j]_{(0)} & \longrightarrow & H^{2j}(X, \mathbf{Q}) & \longrightarrow & 0 \end{array}$$

Therefore if $H^*(X; \mathbf{Z}_{(p)})$ is free $\mathbf{Z}_{(p)}$-module,

$$0 \to H^{2j+2p-2}(X; \mathbf{Z}_{(p)}) \to [X, \Omega E_j] \to H^{2j}(X; \mathbf{Z}_{(p)}) \to 0$$

is exact and $[X, \Omega E_j]$ is a free $\mathbf{Z}_{(p)}$-module and we have

Lemma 2.4. *If* $H^*(X; \mathbf{Z}_{(p)})$ *is a free* $\mathbf{Z}_{(p)}$*-module, then*

$$[X, \Omega E]$$

is a free $\mathbf{Z}_{(p)}$*-module.*

3. Proof of Proposition 1.3

Assume $n \geq 3$. Denote a generator of $H^j(S^j)$ by u_j. Let m be an integer satisfying $m < n \leq 2m$. Define a map

$$\theta\colon X = S^{2m-1} \times S^{2m+1} \to G_n$$

by $\theta = \mu \circ (\epsilon_m \times \epsilon_{m+1})$, where ϵ_j denotes a generator of $\pi_{2j-1}(G_n) \cong \mathbf{Z}$ for $2 \leq j \leq n$ and μ the multiplication of G_n. Let $i\colon G_n \to G_{2m}$ be the inclusion. Note that $\epsilon_j^*(x_{2j-1}) = (j-1)!u_{2j-1}$. Then we have

Lemma 3.1. *If $j \neq m$ or $m+1$, $\theta^*(x_{2j-1}) = 0$ and*

$$\theta^*(x_{2j-1}) = \begin{cases} (m-1)!u_{2m-1} & \text{if } j = m \\ m!u_{2m+1} & \text{if } j = m+1. \end{cases}$$

Note that $\alpha^*(x_{2j-1}) = (-1)^j x_{2j-1}$. Put $\xi = i \circ [\alpha, 1] \circ \theta$ and

$$\tilde{\xi} = \tilde{\gamma}_{2m} \circ (i \wedge i) \circ (\alpha \wedge 1) \circ \Delta \circ \theta.$$

Then $\delta \circ \tilde{\xi} \simeq \gamma_{2m} \circ (i \wedge i) \circ (\alpha \wedge 1) \circ \Delta \circ \theta = i \circ \gamma_n \circ (\alpha \wedge 1) \circ \Delta \circ \theta = i \circ [\alpha, 1] \circ \theta = \xi$.

Lemma 3.2. $\tilde{\xi}^*(a_{4m}) = 2(-1)^m((m-1)!m!)u_{2m-1}u_{2m+1}$.

Proof.

$$\begin{aligned} a_{4m} &\overset{\tilde{\gamma}_{2m}^*}{\longmapsto} \sum x_{2i+1} \otimes x_{2j+1} \overset{(i\wedge i)^*}{\longmapsto} \sum x_{2i+1} \otimes x_{2j+1} \overset{(\alpha\wedge 1)^*}{\longmapsto} \sum (-1)^{i+1} x_{2i+1} \otimes x_{2j+1} \\ &\longmapsto (-1)^m((m-1)!u_{2m-1})(m!u_{2m+1}) + (-1)^{m+1}(m!u_{2m+1})((m-1)!u_{2m-1}) \\ &= 2(-1)^m((m-1)!m!)u_{2m-1}u_{2m+1}. \end{aligned}$$

∎

Consider the following exact commutative diagram

$$\begin{tikzcd} \widetilde{K}(X) \arrow[r, "\Theta"] & H^{4m}(X) \arrow[r] & {[X, G_{2m}]} \\ \widetilde{K}(S^{4m}) \arrow[r, "\Theta"'] \arrow[u, "\cong"', "q^*"] & H^{4m}(S^{4m}) \arrow[r] \arrow[u, "\cong"', "q^*"] & {[S^{4m}, G_{2m}]} \arrow[u] \end{tikzcd}$$

where $q\colon X \to S^{4m}$ is the natural projection. Note that $\mathrm{Im}\{\Theta\colon \widetilde{K}(S^{4m}) \to \tilde{H}(S^{4m})\} = (2m)!\mathbf{Z}$ (see [5]) and $\Theta(\tilde{\xi}) = 2((m-1)!m!)u_{2m-1}u_{2m+1}$. Since the localization is an exact functor, if there exists m satisfying $m < n \leq 2m$ and

$$\frac{(2m)!}{2(m-1)!m!} = m\binom{2m-1}{m} \equiv 0 \mod p,$$

then $\xi \notin \mathrm{Im}\{(\Omega\pi)_*\colon \widetilde{K}(X)_{(p)} \to [X, \Omega W_{2m}]_{(p)}\}$ and $\xi \simeq \delta \circ \tilde{\xi} \neq 0$ in $[X, G_{2m}]_{(p)}$. Therefore $[\alpha, 1] \neq 0$ in $[G_n, G_n]_{(p)}$.

Lemma 3.3. *If $p < 2n-1$, then there exists m satisfying $m < n \leq 2m$ and $m\binom{2m-1}{m} \equiv 0 \mod p$.*

Proof. If $n > p$, put $m = [(n-1)/p]p$. Then $m < n \leq 2m$ and $m\binom{2m-1}{m} \equiv 0 \mod p$. If $n \leq p < 2n-1$, then p is odd and $p \leq 2n-3$. Put $m = \frac{p+1}{2}$ then $2m = p+1 \geq n$ and $\binom{2m-1}{m} \equiv 0 \mod p$ since $1 \leq m \leq p-1$. ∎

Proof of (ii). Since $\gamma_n(ga, g'a') = \gamma_n(g, g')$ for $g, g' \in SU(n)$ and $a, a' \in H$, γ_n induces a map $\gamma_n' \colon \bar{G}_n \wedge \bar{G}_n \to SU(n)$ satisfying $\gamma_n' \circ (q \wedge q) = \gamma_n$ and $q \circ \gamma_n' = \bar{\gamma}_n$. Since q is a finite covering,

$$q_* \colon [\bar{G}_n, SU(n)] \to [\bar{G}_n, \bar{G}_n] = \mathcal{H}(\bar{G}_n)$$

is monic. If $p < 2n-1$, $[\alpha, 1] \neq 0$ in $\mathcal{H}(SU(n))_{(p)}$. Since

$$\begin{aligned}\gamma_n' \circ (\bar{\alpha} \wedge 1) \circ \Delta \circ q &= \gamma_n' \circ (\bar{\alpha} \wedge 1) \circ (q \wedge q) \circ \Delta \\ &= \gamma_n' \circ (q \wedge q) \circ (\alpha \wedge 1) \circ \Delta \\ &= \gamma_n \circ (\alpha \wedge 1) \circ \Delta = [\alpha, 1] \neq 0\end{aligned}$$

in $\mathcal{H}(SU(n))_{(p)}$, $\gamma_n' \circ (\bar{\gamma} \wedge 1) \circ \Delta \neq 0$ in $[\bar{G}_n, SU(n)]_{(p)}$. Therefore

$$[\bar{\alpha}, 1] = \bar{\gamma}_n \circ (\bar{\alpha} \wedge 1) \circ \Delta = q \circ \gamma_n' \circ (\bar{\alpha} \wedge 1) \circ \Delta \neq 0$$

in $\mathcal{H}(\bar{G}_n)_{(p)}$. □

4. Proof of Proposition 1.4

Proof of (i). Note that G_n is p-regular if $p = 2n-1$ (see [12]). If $n \geq 8$, then $2n - 9 \geq 7$. Consider the following maps:

$$\beta_1 \colon SU(n)_{(p)} \simeq \prod_{j=2}^{n} S^{2j-1}_{(p)} \xrightarrow{\pi''} S^3_{(p)} \times S^5_{(p)} \times S^{2n-9}_{(p)} \xrightarrow{q} S^{2n-1}_{(p)} \xrightarrow{\epsilon_n} SU(n)_{(p)}$$

$$\beta_2 \colon SU(n)_{(p)} \xrightarrow{\pi'''} S^{2n-1}_{(p)} \xrightarrow{\epsilon_n} SU(n)_{(p)}$$

where π'', π''' and q are projections. Using the fact that the Samelson product $\langle \epsilon_n, \epsilon_n \rangle \neq 0$ in $\pi_{4n-2}(SU(2n-1))_{(p)}$ (see [3]), $\langle \epsilon_n, \epsilon_n \rangle \neq 0$ in $\pi_{4n-2}(SU(n))_{(p)}$. We can prove (i) by a quite similar method to that in the proof of Proposition 4.1 of [6]. If $n \geq 5$, then $2n-5 \geq 5$ and (i) for $G = U$ is shown similarly. □

Proof of (ii). Assume $n \leq 7$ and $p = 2n-1$ is a prime.

Lemma 4.1. *If $X_{(p)} \simeq \prod S^{n_\alpha}_{(p)}$, then*

$$\mathrm{Im}\{ch \colon \widetilde{K}(X)_{(p)} \to \bigoplus_{k>0} H^{2k}(X; \mathbf{Q})\}$$

is equal to $\bigoplus_{k>0} H^{2k}(X; \mathbf{Z})$.

Proof. As is well known $\mathrm{Im}\{ch_l \colon \widetilde{K}(S^{2l}) \to H^{2l}(S^{2l}; \mathbf{Q})\} = H^{2l}(S^{2l})$. Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{K}(X)_{(p)} \cong \widetilde{K}(X_{(p)})_{(p)} & \xrightarrow[\cong]{\beta} & \widetilde{K}(\Sigma^2 \wedge X_{(p)})_{(p)} \\ \downarrow & & \downarrow{\scriptstyle ch} \\ \bigoplus_{k>0} H^{2k}(X_{(p)}; \mathbf{Q}) & \longrightarrow & \bigoplus_{k>0} H^{2k}(\Sigma^2 \wedge X_{(p)}; \mathbf{Q}) \end{array}$$

where β is the Bott map. Since $\Sigma^2 \wedge X_{(p)}$ is homotopy equivalent to a wedge of localized spheres, we get the lemma. □

Since $SU(n)$ is p-regular we can apply Lemma 4.1.

Lemma 4.2. *For any $b_j \in H^{2j}(SU(n);\mathbf{Z}_{(p)})$ for $n \leq j < p$, there is $\alpha \in \widetilde{K}(SU(n))_{(p)}$ such that*

$$ch_j(\alpha) = \begin{cases} (j!)^{-1}b_j & n \leq j < p \\ 0 & j \geq p. \end{cases}$$

Proof. Since $SU(n)$ is p-regular and $(j!)^{-1} \in \mathbf{Z}_{(p)}^{\times}$ for $n \leq j < p$, Lemma 4.2 follows from Lemma 4.1. □

Lemma 4.3. *For any $f, f' \colon SU(n)_{(p)} \to SU(n)_{(p)}$,*

$$(\tilde{\gamma}_n \circ (f \wedge f') \circ \Delta)^*(a_{2k}) = 0$$

for $k \geq p$.

Proof. If $k > p$, then $\tilde{\gamma}^*(a_{2k}) = 0$ by the dimensional reasons. $\tilde{\gamma}_n(a_{2p}) = x_{2n-1} \otimes x_{2n-1}$. Since $3+5+7 = 15 > 2n-1$, $f^*(x_{2n-1}) = \eta x_{2n-1}$ and $f'^*(x_{2n-1}) = \eta' x_{2n-1}$ for $\eta, \eta' \in \mathbf{Z}_{(p)}$. Therefore

$$\Delta^* \circ (f \wedge f')^* \circ \tilde{\gamma}_n^*(a_{2p}) = \eta\eta' x_{2n-1}^2 = 0$$

□

Now we can prove (ii). Note that $\dim SU(n) = n^2 - 1$. Since $4p - 3 = 8n - 7 > n^2 - 1$ if $n \leq 7$, the group homomorphism $(\Omega\hat{x})_* \colon [SU(n), \Omega W_n]_{(p)} \to [SU(n), \Omega E]$ is an isomorphism by Lemma 2.2. Since $H^*(SU(n);\mathbf{Z}_{(p)})$ is a free $\mathbf{Z}_{(p)}$-module, $[SU(n), \Omega E]$ is a free $\mathbf{Z}_{(p)}$-module by Lemma 2.4. Since by 2.1, $\lambda \otimes \mathbf{Q}$ is an isomorphism, λ is monic. For any maps $f, f' \colon SU(n)_{(p)} \to SU(n)_{(p)}$,

$$\lambda(\tilde{\gamma}_n \circ (f \wedge f') \circ \Delta) \in \mathrm{Im}(\lambda \circ (\Omega\pi)_*)$$

by Lemma 4.2, Lemma 4.3 and (iii) of Lemma 2.1. Since λ is monic,

$$\tilde{\gamma}_n \circ (f \wedge f') \circ \Delta \in \mathrm{Im}(\Omega\pi)_*$$

and

$$[f, f'] = \gamma_n \circ (f \wedge f') \circ \Delta \simeq \delta \circ \tilde{\gamma}_n \circ (f \wedge f') \circ \Delta = 0$$

in $[SU(n), SU(n)]_{(p)} = \mathcal{H}(SU(n))_{(p)}$.

For the case $U(n)$, if $2 \leq n \leq 4$, then $\dim U(n) = n^2 < 8n - 6 = 4p - 2$. $[U(n), \Omega W_n]_{(p)}$ is a free $\mathbf{Z}_{(p)}$-module. Since $1 + 3 + 5 = 9 > 2n - 1$ for $n \leq 4$, we can prove for any $f, f' \colon U(n)_{(p)} \to U(n)_{(p)}$, $(\tilde{\gamma}_n \circ (f \wedge f') \circ \Delta)^*(a_{2k}) = 0$ for $k \geq p$. Therefore we can prove (ii) for $G = U(n)$ similarly. □

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Department of Natural Sciences, Hyogo University of Teacher Education, Kato, Hyogo 673-1494, JAPAN

Department of Mathematics Kyoto University, Kitashirakawa, Kyoto 606-8502, JAPAN