Splitting of gauge groups

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1 Introduction

We will always assume each space has the homotopy type of a CW-complex.

Let G be a topological group and let P be a principal G-bundle over a space B. The gauge group of P, denoted $\mathcal{G}(P)$, is the group of automorphisms of P covering the identity of B.

Fix a basepoint b_0 of B. Then the basepoint inclusion $b_0 \hookrightarrow B$ induces a homomorphism of topological groups

$$\mathcal{G}(P) \to \mathcal{G}(P|_{b_0}) \cong G.$$

Since we work with CW-complexes which are normal, this homomorphism is easily seen to be a surjection. We call the kernel of this homomorphism the based gauge group of P and denote by $\mathcal{G}_0(P)$. Namely, $\mathcal{G}_0(P)$ consists of automorphisms of P covering 1_B which restrict to the identity on the fibre at the basepoint b_0. Now we have an extension of topological groups:

$$1 \to \mathcal{G}_0(P) \xrightarrow{\iota} \mathcal{G}(P) \xrightarrow{\pi} G \to 1 \tag{1.1}$$

The second author [14] classified the homotopy types of $\mathcal{G}(P)$ as spaces, not as topological groups, when P runs all over principal SU(2)-bundles over S^4. Later, Crabb and Sutherland [4] studied the homotopy type of $\mathcal{G}(P)$ as H-spaces for a general P. Moreover, when B is a simply connected 4-manifold and $G = \mathrm{SU}(2)$, Tsukuda and the second author [15], [22] classified the homotopy types of the classifying spaces $B\mathcal{G}(P)$, equivalently, the homotopy types of $\mathcal{G}(P)$ as loop spaces. These results suggest us to study the homotopy theory of gauge groups as spaces with intermediate higher homotopy associativity in the sense of Stasheff [18], that is, as A_n-spaces. In particular, we may study the group extension (1.1) in the category of A_n-spaces and A_n-maps. The aim of this article is to study a splitting of (1.1) in the category of A_n-spaces and A_n-maps which we call an A_n-splitting. More precisely, we will formulate the A_n-splitting and consider:

Question 1.1. *1. What does a geometric meaning of an A_n-splitting of* (1.1)*?*

2. *Give a criterion for an A_n-splitting of* (1.1).

Regarding the first question, we consider a relation between an A_n-splitting of (1.1) and the bundle P. Let $\mathrm{ad} : G \to \mathrm{Aut}G$ be the adjoint action of G on itself and let $\mathrm{ad}P = P \times_{\mathrm{ad}} G$, the adjoint bundle of P. Introducing fibrewise analogue of A_n-maps between topological monoids, we obtain:

Theorem 1.1. *There is an A_n-splitting of* (1.1) *if and only if* $\mathrm{ad}P$ *is fibrewise A_n-equivalent to the trivial bundle $B \times G$.*

Let $\mathrm{map}(X, Y; f)$ be the path component of the space of maps from X to Y containing f, where we will always take f to be basepoint preserving. Denote the universal G-bundle by $EG \to BG$. Regarding the second question, we will be concerned with the classical result of Atiyah and Bott [2]:

$$B\mathcal{G}(P) \simeq \mathrm{map}(B, BG; \alpha), \tag{1.2}$$

where α is the classifying map of P. Naturality of this homotopy equivalence allows us to identify the map $B\pi : B\mathcal{G}(P) \to BG$ with the evaluation fibration $\mathrm{map}(B, BG; \alpha) \to BG$. This leads us to the definition of $H(k, l)$-spaces having the following property.

Theorem 1.2. *There is an A_l-splitting of* (1.1) *if BG is an $H(k, l)$-space and* $\mathrm{cat}B \leq k$.

As above, $H(k, l)$-space is motivated by the evaluation fibration $\mathrm{map}(B, BG; \alpha) \to BG$ and, in particular, $H(1, n)$-space can be described by the connecting map $\delta : G \to \mathrm{map}_0(B, BG; \alpha)$ in the fibre sequence $G \xrightarrow{\delta} \mathrm{map}_0(B, BG; \alpha) \to \mathrm{map}(B, BG; \alpha) \to BG$, where $\mathrm{map}_0(X, Y; f)$ is the subspace of $\mathrm{map}(X, Y; f)$ consisting of based maps. Note that the adjoint action $\mathrm{ad} : G \to \mathrm{aut}(G)$ induces a map $B\mathrm{ad} : G \to \mathrm{map}_0(BG, BG; 1)$ which assigns each $g \in G$ to the map $B\mathrm{ad}(g) : BG \to BG$. Here we must notice that $B\mathrm{ad}$ does not mean the map $BG \to B\mathrm{aut}(G)$ induced from the adjoint action $\mathrm{ad} : G \to \mathrm{aut}(G)$. Then we obtain:

Theorem 1.3. *The connecting map $\delta : G \to \mathrm{map}_0(B, BG; \alpha)$ is given by $\delta(g) = B\mathrm{ad}(g) \circ \alpha$ for $g \in G$.*

Let $E_nG \to B_nG$ be the n-th stage of Milnor's construction of the universal bundle $EG \to BG$ [16]. By definition, BG is an $H(1, n)$-space if and only if the connecting map δ in Theorem 1.3 is trivial for the inclusion $i_n : B_nG \to BG$. Then we have:

Corollary 1.1. *BG is an $H(1, n)$-space if and only if $B\mathrm{ad} \circ i_n : G \to \mathrm{map}_0(B_nG, BG; i_n)$ is null-homotopic.*

We will investigate an $H(k,l)$-space further in view of higher homotopy commutativity as follows. By definition, the loop space of an $H(1,1)$-space is homotopy commutative and an $H(\infty,\infty)$-space is an H-space. On the other hand, Sugawara [21] constructed a class of spaces between homotopy commutative topological monoids and the loop spaces of H-spaces, called higher homotopy commutativity. Then we expect that the loop spaces of $H(k,l)$-spaces form a new class of higher homotopy commutativity. Kawamoto and Hemmi [12] introduced $H_k(n)$-spaces in order to unify Aguadé's T_k-spaces [1] and Félix and Tanré's $H(n)$-spaces [6]. They also introduced higher homotopy commutativity called $C_k(n)$-spaces in order to describe $H_k(n)$-spaces by higher homotopy. An $H_k(n)$-space is, in fact, given by patchinig together $H(i,j)$-spaces for $i+j=n$ and $i=1,\ldots,k$. Moreover, in describing an $H_k(n)$-space by a $C_k(n)$-space, they worked at the level of $H(i,j)$-spaces. This leads us to define a new class of higher homotopy commutativity, $C(k,l)$-spaces, by cutting $C_k(n)$-spaces into pieces and obtain:

Theorem 1.4. *A connected topological monoid is a $C(k,l)$-space if and only if its classifying space is an $H(k,l)$-space.*

Combining Theorem 1.1, Theorem 1.2 and Theorem 1.4, we can conclude:

Corollary 1.2. *Let G be a connected topological group and let P be a principal G-bundle over B. If G is a $C(k,l)$-space, then there is an A_n-splitting of* (1.1)*, equivalently,* ad$P$ *is fibrewise A_n-homotopy equivalent to the trivial bundle $B\times G$.*

The authors would like to thank Y. Kawamoto for pointing out similarity of $H_k(n)$-spaces and $H(k,l)$-spaces, and for letting the second author know his work with Hemmi [12].

2 A_n-splitting

In this section, we formulate a splitting of an extension of topological groups in the category of A_n-spaces and A_n-maps which we call an A_n-splitting. An A_n-space was introduced by Stasheff [18] to be a space with a multiplication which enjoys a certain higher homotopy associativity. Then an A_n-map should be a map between A_n-spaces preserving their A_n-space structures. Stasheff [19] defined an A_n-map between A_∞-spaces. Later, he [20] defined an A_n-map from an A_n-space to an A_∞-space and implied an A_n-map between A_n-spaces. Finally, Iwase and Mimura [13] described an A_n-map between A_n-spaces completely. Of course, these definitions of A_n-maps are consistent and then we will use convenient one case by case.

An A_n-splitting of an extension of topological groups should be analogous to a splitting in the category of topological groups and their homomorphisms. Then we define an A_n-splitting of an extension of topological groups as follows.

Definition 2.1. An A_n-splitting of an extension of topological groups $1 \to K \to \tilde{H} \to H \to 1$ consists of the following:

1. There is an A_n-structure on $H \times K$, the direct product as spaces, not as topological groups, which restricts to the canonical group structures on $H \times \{1\}$ and $\{1\} \times K$.

2. There is an A_n-map $\theta : H \times K \to \tilde{H}$ with respect to the above A_n-structure on $H \times K$ satisfying the homotopy commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d, equal] & \tilde{H} \arrow[r] & H \arrow[r] \arrow[d, equal] & 1 \arrow[d, equal] \\ 1 \arrow[r] & K \arrow[r] & H \times K \arrow[r, "\pi"] \arrow[u, "\theta"] & H \arrow[r] & 1, \end{tikzcd}$$

where π is the second projection.

Let $1 \to K \to \tilde{H} \xrightarrow{\pi} H \to 1$ be an extension of topological groups. A splitting of this extension as groups can be completely described by a section of π which is a group homomorphism. We shall show that there is an analogy for an A_n-splitting. Namely, a homotopy section of π which is an A_n-map, called an A_n-section, implies an A_n-splitting of the extension, where homotopy section of π is a map $s : H \to \tilde{H}$ such that $\pi \circ s \simeq 1_H$.

Let us first recall Stasheff's polytope, the associahedron, which was used to define an A_n-space and an A_n-map from A_n-space to an A_∞-space (See [18] and [20]). The i-th associahedron K_i is an $(i-2)$-dimensional convex polytope having the face maps

$$\partial_k(r,s) : K_r \times K_s \to K_i$$

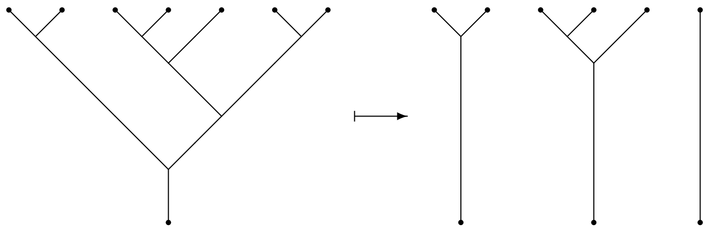
for $r + s = i + 1$ and $1 \leq k \leq i - s + 1$ and the degeneracy maps

$$s_j : K_i \to K_{i-1}$$

for $1 \leq j \leq i$. In particular, there are relations:

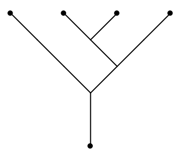
$$s_j \circ \partial_k(r,s) = \begin{cases} \partial_k(r, s-1) \circ (1 \times s_{j-k+1}) & k \leq j < k+s \\ \partial_k(r-1, s) \circ (s_{j-s+1} \times 1) & j \geq k+s \end{cases} \tag{2.1}$$

There is a one to one correspondence between vertices of K_i and connected binary trees with n-leaves. In order to define an A_n-space structure from an A_n-section, we consider the following operations of binary trees. Let T_n be the set of connected binary trees with n leaves and let $\widehat{T}_n$ be the set of ordered binary trees, not necessarily connected, with n leaves. Then we can label each leaf of an element of $\widehat{T}_n$ from 1 to n in the obvious way. Define a map $\delta : T_{n+1} \to \widehat{T}_n$ by deleting the branches from the root to the n-th leaf. For example, $\delta : T_7 \to \widehat{T}_6$ is:



Then δ is a bijection. Analogously we define a map $\hat{\delta} : \widehat{T}_n \to \widehat{T}_{n-1}$ by applying the above map δ to the connected binary tree having the leaf labelled by n. Then $\delta : T_n \to \widehat{T}_{n-1}$ is the restriction of $\hat{\delta} : \widehat{T}_n \to \widehat{T}_{n-1}$.

Let X be an H-space. For $x_1, \dots, x_n \in X$ and $t \in T_n$, we define $t(x_1, \dots, x_n)$ as in [20], which is consistent with the definition of A_n-spaces. For example, if $t \in T_4$ is



then $t(x_1, x_2, x_3, x_4) = x_1((x_2x_3)x_4)$. Let G be a topological group. Using the above map t, for a map $f : X \to G$, we define a map $\hat{f} : \widehat{T}_n \times X^n \to G$ by

$$\hat{f}(\hat{t}, x_1, \dots, x_n) = f(t_1(x_1, \dots, x_{n_1}))f(t_2(x_{n_1+1}, \dots, t_{n_1+n_2})) \cdots f(t_k(x_{n_1+\cdots+n_{k-1}+1}, \dots, x_n)),$$

where $\hat{t} = t_1 \sqcup \cdots \sqcup t_k \in \widehat{T}_n$ such that $t_1 < \cdots < t_k$ and $t_i \in T_{n_i}$.

Now we consider an extension of topological groups $1 \to K \to \tilde{H} \xrightarrow{\pi} H \to 1$. Suppose that π admits an A_n-section s whose A_n-form is $\{m_i : K_{i+1} \times H^i \to \tilde{H}\}_{1 \le i \le n}$. As noted above, for a vertex $v \in K_{i+1}$ corresponding to $\hat{t} \in \widehat{T}_i$, we have

$$h_i(v, x_1, \dots, x_i) = s(\hat{t}, x_1, \dots, x_i).$$

We write $\gamma_j(\tau, x) = h_j(s_{j+1}s_{j+2} \cdots s_i(\tau), \pi_{j+1}\pi_{j+2} \cdots \pi_i(x))$ for $\tau \in K_i, x \in K^i$ and the projection $\pi_j : K^i \to K^{i-1}$ omitting the i-th entry. Then, for a vertex $v \in K_i$ corresponding to $t \in T_i$ and $x = (x_1, \dots, x_i) \in K^i$, it follows from (2.1) that

$$\gamma_j(v, x) = s(\hat{d}^{i-j}t, x_1, \dots, x_i). \tag{2.2}$$

Define $M_i : K_i \times (H \times K)^i \to H \times K$ by

$$M_i(\tau, (h_1, k_1), \dots, (h_i, k_i)) = (h_1 h_2^{\gamma_1(\tau,k)} h_3^{\gamma_2(\tau,k)} \cdots h_i^{\gamma_{i-1}(\tau,k)}, k_1 \cdots k_i)$$

for $\tau \in K_i$ and $k = (k_1, \ldots, k_i) \in K^i$, where $g^h = hgh^{-1}$ for $g, h \in H$. Then, by (2.2), it is straightforward to check that $\{M_i : K_i \times (H \times K)^i \to H \times K\}_{2 \le i \le n+1}$ is an A_{n+1}-form on $H \times K$ such that, for a vertex $v \in K_i$ corresponding to $t \in T_i$, $M_i(v, (h_1, k_1), \ldots (h_i, k_i)) = t((h_1, k_1), \ldots, (h_i, k_i))$. In particular, the multiplication of $H \times K$ is defined by

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1(h_2^{\sigma(k_1)}), k_1 k_2)$$

which is analogous to semidirect products of groups.

By a quite analogous observation, we can see that the map $\theta : H \times K \to \tilde{H}$ defined by

$$\theta(h, k) = h \cdot s(k)$$

for $h \in H$ and $k \in K$ admits an A_n-form. Summarizing, we have established:

Lemma 2.1. *An extension of topological groups $1 \to K \to \tilde{H} \xrightarrow{\pi} H \to 1$ has an A_n-splitting if and only if π admits an A_n-section.*

3 Fibrewise A_n-map

In this section, we introduce fibrewise analogue of A_n-maps between topological monoids and characterize them by using fibrewise analogue of projective spaces. Let us first recall from [3] some notations and terminologies of fibrewise homotopy theory. Fix a space B. A fibrewise space over B is an arrow $X \xrightarrow{\pi_X} B$. π_X is called the projection and $\pi_X^{-1}(b)$ for $b \in B$ is called a fibre at b. Then the direct product $A \times B$ is a fibrewise space over B and, in particular, so is B itself. A fibrewise map from a fibrewise space $X \xrightarrow{\pi_X} B$ to $Y \xrightarrow{\pi_Y} B$ is a commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Then fibrewise spaces over B and fibrewise maps between them form a category which is nothing but the comma category $\underline{\underline{\mathrm{Top}}} \downarrow B$, where $\underline{\underline{\mathrm{Top}}}$ is the category of topological spaces and continuous maps. Fibrewise homotopy theory is not homotopy theory by the canonical model category structure on $\underline{\underline{\mathrm{Top}}} \downarrow B$ induced from $\underline{\underline{\mathrm{Top}}}$, but it respects fibre homotopy equivalence in the classical sense. With this in mind, we recall basic constructions in fibrewise homotopy theory. The fibrewise product $X \times_B Y$ of $X \xrightarrow{\pi_X} B$ and $Y \xrightarrow{\pi_Y} B$ is the pullback of the triad $X \xrightarrow{\pi_X} B \xleftarrow{\pi_Y} Y$, that is,

$$X \times_B Y = \{(x, y) \in X \times Y | \pi_X(x) = \pi_Y(y)\}.$$

Then the diagonal map restricts to the fibrewise diagonal map $X \to X \times_B X$, denoted Δ_B. We often abbreviate the fibrewise product of n-copies of a fibrewise space $X \to B$ by X^n by abuse of notation. We denote the fibrewise space $[0,1] \times B \to B$ by I_B and call it the fibrewise interval, here the projection is the second projection. A fibrewise homotopy is a fibrewise map $X \times_B I_B \to Y$ and we have a fibrewise homotopy equivalence in the obvious sense, which are the classical fibre homotopy and fibre homotopy equivalence respectively. With this notion of fibrewise homotopies, we have a fibrewise fibration and a fibrewise cofibration which are characterized by a fibrewise homotopy lifting property and a fibrewise homotopy extension property respectively (See [3]).

The fibrewise unreduced cone of $X \xrightarrow{\pi_X} B$, denoted $C_B X$, is a pushout of the cotriad $I_B \times_B X \hookleftarrow \{0\} \times X \xrightarrow{\pi_X} B$. Similarly, the fibrewise unreduced suspension of $X \xrightarrow{\pi_X} B$, denoted $\Sigma_B X$, is a pushout of the cotriad $I_B \times_B X \hookleftarrow \{0,1\} \times X \xrightarrow{1 \times \pi_X} \{0,1\} \times B$.

A fibrewise pointed space is a fibrewise space $X \to B$ with a distinguished section and then we assume $B \subset X$. We have a fibrewise pointed map in the obvious sense. The fibrewise reduced cone $C^B_B X$ and the fibrewise reduced suspension $\Sigma^B_B X$ are the fibrewise collapses $C_B X/_B C_B B$ and $\Sigma_B X/_B \Sigma_B B$ respectively (See [3, p.55]). A fibrewise pointed space is said to be well-pointed if the section is a fibrewise cofibration. Then, as in the usual case, if a fibrewise pointed space X over B is well-pointed, then $C_B X$ is fibrewise homotopy equivalent to $C^B_B X$ relative to X. In particular, $\Sigma_B X$ is fibrewise homotopy equivalent to $\Sigma^B_B X$.

In order to introduce a fibrewise analogue of A_n-maps between topological monoids, we need to have a fibrewise analogue of topological monoids which is given by replacing spaces and structure maps with fibrewise spaces and fibrewise maps of topological monoids as follows. A fibrewise topological monoid over B is a fibrewise space $X \xrightarrow{\pi_X} B$ with fibrewise maps $\epsilon : B \to X$ and $\mu : X \times_B X \to X$ satisfying two conditions:

$$\mu \circ (\mu \times 1) = \mu \circ (1 \times \mu), \quad \mu \circ (1 \times \epsilon\pi_X) \circ \Delta_B = 1 = \mu \circ (\epsilon \circ \pi_X \times 1).$$

In particular, a fibrewise topological monoid is a fibrewise pointed space and each of its fibre is a topological monoid. We usually abbreviate $\mu(x,y)$ by xy. A fibrewise topological monoid $X \xrightarrow{\pi_X} B$ is a fibrewise topological group, if it has a fibrewise map $\iota : X \to X$ satisfying

$$\mu \circ (1 \times \iota) \circ \Delta_B = \epsilon \circ \pi_X = \mu \circ (\iota \times 1) \circ \Delta_B.$$

Let us look at examples of fibrewise topological monoids.

Example 3.1. Let $X \xrightarrow{\pi} B$ be a fibrewise pointed space with a distinguished section s. The fibrewise Moore path space of X is

$$\Omega'_B X = \coprod_{b \in B} \Omega'(\pi^{-1}(b))$$

equipped with an appropriate topology (See [3]), where $\Omega' Y$ is the Moore path space of a space Y. Then the loop multiplication of $\Omega'(\pi^{-1}(b))$ makes $\Omega'_B X$ be a fibrewise topological monoid.

Example 3.2. Let G be a topological group and let $\pi : P \to B$ be a principal G-bundle. Then the adjoint bundle $\mathrm{ad}P$ is a fibrewise topological group with the structure maps:

$$\epsilon : B \to \mathrm{ad}P, \ \epsilon(b) = [\pi^{-1}(b), 1],$$

$$\mu : \mathrm{ad}P \times_B \mathrm{ad}P \to \mathrm{ad}P, \ \mu([x,g],[x,h]) = [x, gh],$$

$$\iota : \mathrm{ad}P \to \mathrm{ad}P, \ \iota([x,g]) = [x, g^{-1}],$$

where $[x,g]$ is a equivalence class of $(x,g) \in P \times G$ in $\mathrm{ad}P$.

Now we define a fibrewise A_n-map between fibrewise topological monoids just by replacing objects and arrows with fibrewise ones and the interval $[0,1]$ with the fibrewise interval I_B (See [19] for the definition of the usual A_n-maps between topological monoids).

Definition 3.1. Let X and Y be fibrewise topological monoids over B. A fibrewise map $f : X \to Y$ is called a fibrewise A_n-map if there exists a sequence of fibrewise maps $\{h_i : I_B^{i-1} \times_B X^i \to Y\}_{1 \le i \le n}$ such that $h_1 = f$ and

$$h_i(t_1, \ldots, t_{i-1}, x_1, \ldots, x_i) \\ = \begin{cases} h_{i-1}(t_1, \ldots, \widehat{t_j}, \ldots, t_{i-1}, x_1, \ldots, x_j x_{j+1}, \ldots, x_i) & t_j = 0 \\ h_j(t_1, \ldots, t_{j-1}, x_1, \ldots, x_j) h_{i-j}(t_{j+1}, \ldots, t_{i-1}, x_{j+1}, \ldots, x_i) & t_j = 1. \end{cases}$$

By a quite analogous proof to [20] and [7], we can see the following properties of fibrewise A_n-maps.

Proposition 3.1. *1. A fibrewise map f is fibrewise homotopic to a fibrewise A_n-map, then so is f.*

2. The composition of fibrewise A_n-maps is a fibrewise A_n-map.

3. A homotopy inverse of a fibrewise homotopy equivalence which is a fibrewise A_n-map is a fibrewise A_n-map.

It follows from the above proposition that fibrewise homotopy equivalences which are fibrewise A_n-maps give an equivalence relation among fibrewise topological monoids. We call this equivalence by a fibrewise A_n-equivalence.

Let us characterize fibrewise A_n-maps using fibrewise analogue of projective spaces as in [19]. Note that we do not have appropriate quasi-fibrations in our fibrewise category. That is, we do not have weak equivalences nor quasi-fibrations, which can be replaced with fibrewise fibrations by weak equivalences, in our fibrewise category. Then it seems impossible to mimic the proof of [19, Theorem 4.5] directly. However, we only need to deal with fibrewise topological groups and we can overcome the above difficulty by restricting to fibrewise topological groups.

Let G be a fibrewise topological group over B. Then, by [3, p.37], we have a fibrewise analogue of the Milnor construction for classifying spaces. Denote the n-th stage of the fibrewise Milnor construction for G by $\mathbf{E}^n_B G \to \mathbf{B}^n_B G$ which is a finite numerable fibrewise fibre bundle. Thus, by a quite analogous observation of [17, Corollary 14], we have:

Lemma 3.1. *The fibrewise map $\mathbf{E}^n_B G \to \mathbf{B}^n_B G$ is a fibrewise fibration.*

It will be convenient for later use to state a characterization of fibrewise A_n-maps by using fibrewise analogue of the Dold-Lashof construction which coincides with the Milnor construction in the usual case (See, for example, [8]). Then we define the fibrewise Dold-Lashof construction only by replacing everything in the Dold-Lashof construction with a fibrewise one as follows. Let H be a fibrewise topological monoid having a fibrewise action on E, denoted $m : H \times_B E \to E$ (See [3, p.15]). Start with a fibrewise map $q : E \to X$ enjoying $q(m(h, x)) = p(x)$ for $(h, x) \in H \times_B X$. Let $\mathrm{DL}_B(E)$ be the fibrewise quotient of $(H \times_B C_B E) \sqcup E$ by the relation $(h, (1, x)) \sim \mu(h, x)$ for $(h, (1, x)) \in H \times_B C_B E$ and let $\mathrm{DL}_B(X)$ be the fibrewise quotient of $C_B E \sqcup X$ by $(1, x) \sim q(x)$ for $(1, x) \in C_B E$. Then the Dold-Lashof construction for q is the fibrewise map

$$\mathrm{DL}_B(q) : \mathrm{DL}_B(E) \to \mathrm{DL}_B(X),\ (h, (t, x)) \mapsto (t, x).$$

Note that we do not have to take much care for topologies of $\mathrm{DL}_B(E)$ and $\mathrm{DL}_B(X)$ since we work in the category of spaces having the homotopy types of CW-complexes. Since H is fibrewise associative, we can apply the Dold-Lashof construction iteratively. We denote the iterated Dold-Lashof construction $\mathrm{DL}^n_B(\pi_H) : \mathrm{DL}^n_B(H) \to \mathrm{DL}^n_B(B)$ for the projection $\pi_H : H \to B$ by $\pi^n_B : E^n_B H \to P^n_B H$. As in the usual case, we can easily verify that if H is a fibrewise topological group, $\pi^n_B : E^n_B H \to P^n_B H$ coincides with the n-th stage of the Milnor construction $\mathbf{E}^n_B H \to \mathbf{B}^n_B H$.

We follow [13] to characterize fibrewise A_n-maps then we first define a fibrewise A_n-structure of a fibrewise A_n-map. Let $D^n_B X = C_B E^n_B X$ for a fibrewise topological monoid X.

Definition 3.2. Let X and Y be fibrewise topological monoids over B. A fibrewise A_n-structure of a fibrewise map $f : X \to Y$ consists of:

1. f respects fibrewise units of X and Y.

2. There are sequences of commutative squares of fibrewise maps

$$\begin{array}{ccc} (D_B^{i+1}X, E_B^iX) & \xrightarrow{f_E^i} & (D_B^{i+1}Y, E_B^iY) \\ {\scriptstyle \pi_{i+1}}\downarrow & & \downarrow{\scriptstyle \pi_{i+1}} \\ (P_B^{i+1}X, P_B^iX) & \xrightarrow{f_P^i} & (P_B^{i+1}Y, P_B^iY) \end{array}$$

for $1 \leq i \leq n-1$ such that $f_E^1|X = f$, $f_E^i|_{D_B^iX} = f_E^{i-1}$, $f_P^i|_{P_B^iX} = f_P^{i-1}$.

Now we give a characterization of fibrewise A_n-map.

Theorem 3.1. *Let X be a fibrewise topological monoid over B and let Y be a fibrewise well-pointed topological group over B. A fibrewise map $f : X \to Y$ is a fibrewise A_n-map if and only if f possesses a fibrewise A_n-structure.*

Proof. The if part is done by Sugawara's construction [21]. In order to prove the only if part, we can mimic the proof of [19, Theorem 4.5] instead of replacing quasi-fibrations with fibrations. Then if we can replace $\pi_B^n : E_B^nY \to P_B^nY$ with a fibrewise fibration, the proof is completed.

Consider the Dold-Lashof construction for the projection $\pi_Y : Y \to B$ in which the unreduced cone is replaced by the reduced cone. Then, as in [8], we obtain the Milnor construction $\mathbf{E}_B^nY \to \mathbf{B}_B^nY$and hence a commutative diagram:

$$\begin{array}{ccc} E_B^nY & \longrightarrow & \mathbf{E}_B^nY \\ {\scriptstyle \pi_B^n}\downarrow & & \downarrow \\ P_B^nY & \longrightarrow & \mathbf{B}_B^nY \end{array}$$

Moreover, it follows from induction with the hypothesis that Y is fibrewise well-pointed, $\mathbf{E}_B^nY$ and $\mathbf{B}_B^nY$ are fibrewise well-pointed. This implies that the horizontal arrows in the above diagram are fibrewise homotopy equivalences and thus, by Lemma 3.1, we assume that $\pi_B^n : E_B^nY \to P_B^nY$ is a fibrewise fibration. □

4 Set of sections

In this section, we consider the set of sections of a fibrewise space and prove Theorem 1.1. Let X be a fibrewise space over B. We denote the set of sections of X by $\Gamma(X)$. Then it is obvious that Γ is a functor from $\underline{\underline{\mathrm{Top}}} \downarrow B$ to $\underline{\underline{\mathrm{Top}}}$. Note that, by the pointwise multiplication, $\Gamma(X)$ is a topological monoid and a topological group according as X is a fibrewise topological monoid and a fibrewise topological group. In particular, for a principal bundle P, $\Gamma(\mathrm{ad}P)$ is a topological group by which we have an isomorphism of topological groups

$$\mathcal{G}(P) \cong \Gamma(\mathrm{ad}P) \tag{4.1}$$

(See [2]).

Let $C : \underline{\underline{\mathrm{Top}}} \to \underline{\underline{\mathrm{Top}}}$ be the unreduced cone functor. We define a natural transformation $\lambda : C\Gamma \to \Gamma C_B$ by

$$\lambda : C\Gamma(X) \to \Gamma(C_B X),\ \lambda(t,s)(b) = (t, s(b)) \tag{4.2}$$

for $b \in B$. Let H be a fibrewise topological monoid with a fibrewise action $\mu : H \times_B E \to E$ and let $q : E \to X$ be a fibrewise map such that $q(\mu(h,x)) = x$ for $(h,x) \in H \times_B E$. Then, by definition, the natural transformation λ induces a commutative diagram

$$\begin{tikzcd} \mathrm{DL}(\Gamma(E)) \arrow[r, "\bar{\lambda}"] \arrow[d, "\mathrm{DL}(\Gamma(q))"'] & \Gamma(\mathrm{DL}_B(E)) \arrow[d, "\Gamma(\mathrm{DL}_B(q))"] \\ \mathrm{DL}(\Gamma(X)) \arrow[r, "\lambda"] & \Gamma(\mathrm{DL}_B(X)) \end{tikzcd}$$

in which all maps respects the action of $\Gamma(H)$, where $\mathrm{DL}(-)$ means the usual Dold-Lashof construction. Then it follows that we have a commutative square

$$\begin{tikzcd} (D^{n+1}\Gamma(H), E^n\Gamma(H)) \arrow[r, "\bar{\lambda}_n"] \arrow[d, "\pi^{n+1}"'] & (\Gamma(D_B^{n+1}H), \Gamma(E_B^n H)) \arrow[d, "\pi_B^{n+1}"] \\ (P^{n+1}\Gamma(H), P^n\Gamma(H)) \arrow[r, "\lambda_n"] & (\Gamma(P_B^{n+1}H), \Gamma(P_B^n H)) \end{tikzcd}$$

for all n such that

$$\bar{\lambda}_n|_{D^n\Gamma(H)} = \bar{\lambda}_{n-1},\ \ \lambda_n|_{P^n\Gamma(H)} = \lambda_{n-1},$$

where, for a topological monoid Y, $\pi^n : E^nY \to P^nY$ is $\mathrm{DL}^n(*) : \mathrm{DL}(Y) \to \mathrm{DL}(*)$ and $D^{n+1}Y = CE^nY$.

Proof of Theorem 1.1. Suppose that we have an A_n-splitting of (1.1). Then, by Lemma 2.1, we have an A_n-section σ of $\pi : \mathcal{G}(P) \to G$ which is identified with the evaluation at the basepoint $\Gamma(\mathrm{ad}P) \to G$ through the isomorphism (4.1). Define a fibrewise map

$$\theta : B \times G \to \mathrm{ad}P,\ \theta(b,g) = \sigma(g)(b).$$

Then we have $\theta|_{\{b_0\}\times G} \simeq 1_G$ since σ is a section of the evaluation at the basepoint $\Gamma(\mathrm{ad}P) \to G$, where b_0 is the basepoint of B. Thus, by Dold's theorem [5], θ is a fibrewise homotopy equivalence.

Since σ is an A_n-map, it possesses an A_n-structure in the sense of [13], that is, there is a sequence of homotopy commutative square

$$\begin{tikzcd} (D^{i+1}G, E^iG) \arrow[r, "\sigma_E^i"] \arrow[d] & (D^{i+1}\Gamma(\mathrm{ad}P), E^i\Gamma(\mathrm{ad}P)) \arrow[d] \\ (P^{i+1}G, P^iG) \arrow[r, "\sigma_P^i"] & (P^{i+1}\Gamma(\mathrm{ad}P), P^i\Gamma(\mathrm{ad}P)) \end{tikzcd}$$

for $i = 1, \dots, n-1$ in which $\sigma_1^E|_G = \sigma, \sigma_E^i|_{D^iG} = f_E^{i-1}, f_P^i|_{P^iG} = f_P^{i-1}$. Note that

$$D_B^i(B \times G) = B \times G,\ E_B^i(B \times G) = B \times E^iG,\ P^i(B \times G) = B \times P^iG$$

and then we shall make these identifications. Define fibrewise maps

$$\theta_E^i : (D_B^{i+1}(B \times G), E_B^i(B \times G)) \to (D_B^{i+1}\mathrm{ad}P, E_B^i\mathrm{ad}P)$$

and

$$\theta_P^i : (P_B^{i+1}(B \times G), P_B^i(B \times G)) \to (P_B^{i+1}\mathrm{ad}P, P_B^i\mathrm{ad}P)$$

by

$$\theta_E^i(b, x) = \bar{\lambda}_i(\sigma_E^i(x))(b),\ \theta_P^i(b, y) = \lambda_i(\sigma_P^i(y))(b)$$

for $b \in B, x \in D^{i+1}G, y \in P^{i+1}G$. Then these fibrewise maps gives a fibrewise A_n-structure of θ and therefore, by Theorem 3.1, θ is a fibrewise A_n-equivalence.

Let X be a fibrewise space over B. As in (4.2), we have a map

$$\rho : [0,1] \times \Gamma(V) \to \Gamma(I_B \times_B X),\ \rho(t, s)(b) = (t, s(b))$$

for $(t, s) \in [0,1] \times \Gamma(V)$ and $b \in B$. Then a fibrewise A_n-map $f : X \to Y$ for fibrewise topological monoids X, Y induces an A_n-map $\Gamma(f) : \Gamma(X) \to \Gamma(Y)$ in the sense of [19].

Suppose that we have a fibrewise A_n-equivalence $\theta : B \times G \to \mathrm{ad}P$. Then it follows that we have an A_n-equivalence $\Gamma(\theta) : \Gamma(B \times G) \to \Gamma(\mathrm{ad}P)$. Now we have an isomorphism of topological groups $\Gamma(B \times G) \cong \mathrm{map}(B, G)$ which is natural with respect to B. Then the evaluation at the basepoint $\Gamma(B \times G) \to G$ is nothing but the evaluation at the basepoint $\mathrm{map}(B, G) \to G$ which admits a section as topological groups. Then we obtain an A_n-section of $\pi : \Gamma(\mathrm{ad}P) \to G$ and thus, by Lemma 2.1, we have established an A_n-splitting of (1.1). □

5 $H(k, l)$-space

In this section, we consider the second question, that is, a criterion for an A_n-splitting of (1.1). Our major tool is the homotopy equivalence (1.2). Then let us first recall the construction of the construction of the homotopy equivalence (1.2). Let G be a topological group. We denote by $\mathrm{map}^G(X, Y)$ the space of all G-equivariant maps from X to Y for G-spaces X, Y. Let P and Q be principal G-bundles. Then $\mathcal{G}(P)$ acts on $\mathrm{map}^G(X, Y)$ by composition. Now we consider the case $Q = EG$. Then we have:

Lemma 5.1 ([11, Theorem 5.2], [2, Proposition 2.4]). *1.* $\mathrm{map}^G(P, EG)$ *is contractible.*

2. The action of $\mathcal{G}(P)$ *on* $\mathrm{map}^G(P, EG)$ *is free.*

Then we have the universal $\mathcal{G}(P)$-bundle:

$$\mathcal{G}(P) \to \mathrm{map}^G(P, EG) \to \mathrm{map}^G(P, EG)/\mathcal{G}(P) \tag{5.1}$$

Let us denote by θ the map $\mathrm{map}^G(P, EG) \to \mathrm{map}(B, BG; \alpha)$ induced from the projections $P \to B$ and $EG \to BG$, where B is the base space of P and α is the classifying map of P. Then one can easily see that the map θ induces a homeomorphism

$$\bar{\theta} : \mathrm{map}^G(P, EG)/\mathcal{G}(P) \xrightarrow{\cong} \mathrm{map}(B, BG; \alpha) \tag{5.2}$$

which is natural with respect to P. Thus we obtain a homotopy equivalence

$$\hat{\theta} : B\mathcal{G}(P) \xrightarrow{\simeq} \mathrm{map}(B, BG; \alpha)$$

which is natural with respect to P.

Consider the topological group G as the principal G-bundle over a point and identify $\mathcal{G}(G)$ with G. Then the basepoint inclusion $i : b_0 \to B$ induces a homotopy commutative diagram:

$$\begin{tikzcd} B\mathcal{G}(P) \arrow[r, "B\pi"] \arrow[d, "\simeq"', "\hat{\theta}"] & BG \arrow[d, equal, "\hat{\theta}"] \\ \mathrm{map}(B, BG; \alpha) \arrow[r, "i^*"] & \mathrm{map}(b_0, BG; 0) \end{tikzcd} , \tag{5.3}$$

where 0 stands for the constant map. Then the evaluation at the basepoint $e : \mathrm{map}(B, BG; \alpha) \to BG$ is a model for $B\pi : B\mathcal{G}(P) \to BG$ and this leads us to the following definition of $H(k,l)$-spaces. Let $i_k : P^k\Omega X \to P^\infty\Omega X \simeq X$ denote the canonical inclusion.

Definition 5.1. A space X is called an $H(k,l)$-space if there is a map $m : P^k\Omega X \times P^l\Omega X \to X$ satisfying a homotopy commutative diagram:

$$\begin{tikzcd} P^k\Omega X \vee P^l\Omega X \arrow[r, "i_k \vee i_l"] \arrow[d, "j"] & X \arrow[d, equal] \\ P^k\Omega X \times P^l\Omega X \arrow[r, "m"] & X, \end{tikzcd}$$

where j is the inclusion.

It is obvious that an $H(k,l)$-space is an $H(k',l')$-space if $k \geq k'$ or $l \geq l'$. The loop space of an $H(1,1)$-space is homotopy commutative and an $H(\infty,\infty)$-space is an H-space. The loop spaces of $H(k,l)$-spaces give intermediate states between H-spaces and the loop spaces of H-spaces which will be discussed in section 7. On the other hand, an $H(\infty,k)$-space is Aguadé's T_k-space [1]. In particular, an $H(1,\infty)$-space is Aguadé's T-space and this can be seen also by the fibrewise homotopy equivalence $\mathrm{ad}EG \simeq LBG$ over BG, where LX is the free loop space of X.

An $H(k,l)$-space is defined to satisfy the following lemma:

Lemma 5.2. *If a classifying space of a topological group G is an $H(k,l)$-space, then there is an A_n-splitting of the exact sequence $1 \to \mathcal{G}_0(E^kG) \to \mathcal{G}(E^kG) \xrightarrow{\pi} G \to 1$.*

Proof. Recall first from [20] that, for A_∞-spaces X, Y, a map $f : X \to Y$ is an A_n-map if and only if its adjoint $\bar{f} : \Sigma X \to P^\infty Y$ extends to $P^nX \to P^\infty Y$ up to homotopy.

Suppose that X is an $H(k,l)$-space by $m : P^k\Omega X \times P^l\Omega X \to X$. Then, by the exponential law, the adjoint of m restricts to a map $\hat{m} : \Sigma\Omega X \to \mathrm{map}(P^k\Omega X, X; i_k)$ such that $e \circ \hat{m} \simeq i_1$, where $e : \mathrm{map}(P^k\Omega X, X; i_k) \to X$ is the evaluation at the basepoint. Then the adjoint of $\hat{m}$, say $\bar{m} : \Omega X \to \Omega\mathrm{map}(P^k\Omega X, X; i_k)$, is a homotopy section of Ωe and thus $\bar{m}$ is an A_n-map. Therefore, by Lemma 2.1 and (5.3), Lemma 5.2 is established. □

Proof of Theorem 1.2. It is well-known that $\mathrm{cat}B \le k$ if and only if $i_k : P^k\Omega B \to B$ admits a homotopy section. Then, by naturality of i_k, if $\mathrm{cat}B \le k$, each map $f : B \to BG$ admits a map $\bar{f} : B \to P^kG$ such that $i_k \circ \bar{f} \simeq f$. This implies that $\bar{f}^{-1}E^kG \cong P$ and then an A_n-section for $\pi : \mathcal{G}(E^kG) \to G$ induces that of $\pi : \mathcal{G}(P) \to G$. Thus, by Lemma 5.2, the proof is completed. □

6 Investigating $H(1,n)$-spaces

In the previous section, we have obtained the universal $\mathcal{G}(P)$-bundle (5.1). Then it follows from (5.2) that there is a homotopy equivalence $\varphi : \mathrm{map}^G(P, EG; \alpha)/\mathcal{G}_0(P) \to \mathrm{map}_0(B, BG; \alpha)$ and $\bar{\varphi} : B\mathcal{G}_0(P) \to \mathrm{map}^G(P, EG; \alpha)/\mathcal{G}_0(P)$ such that the following diagram of fibre sequences is homotopy commutative.

$$\begin{tikzcd}
G \arrow[r] \arrow[d, equal] & B\mathcal{G}_0(P) \arrow[r, "B\iota"] \arrow[d, "\bar{\varphi}"', "\simeq"] & B\mathcal{G}(P) \arrow[r, "B\pi"] \arrow[d, "\simeq"', "\hat{\theta}"] & BG \arrow[d, equal] \\
\mathcal{G}(P)/\mathcal{G}_0(P) \arrow[r] \arrow[d, equal] & \mathrm{map}^G(P, EG; f)/\mathcal{G}_0(P) \arrow[r] \arrow[d, "\varphi"', "\simeq"] & \mathrm{map}^G(P, EG; \alpha)/\mathcal{G}(P) \arrow[r] \arrow[d, "\bar{\theta}"', "\simeq"] & BG \arrow[d, equal] \\
G \arrow[r, "\delta_\alpha"] & \mathrm{map}_0(B, BG; \alpha) \arrow[r] & \mathrm{map}(B, BG; \alpha) \arrow[r, "e"] & BG
\end{tikzcd}$$

The aim of this section is to study the connecting map δ_α and characterize $H(1,n)$-spaces by it. Consider the following commutative diagram.

$$\begin{tikzcd}
G \arrow[r, "\delta"] \arrow[d, equal] & \mathrm{map}_0(BG, BG; 1) \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{map}(BG, BG; 1) \arrow[r, "e"] \arrow[d, "\alpha^*"] & BG \arrow[d, equal] \\
G \arrow[r, "\delta_\alpha"] & \mathrm{map}_0(B, BG; \alpha) \arrow[r] & \mathrm{map}(B, BG; \alpha) \arrow[r, "e"] & BG
\end{tikzcd} \tag{6.1}$$

Then it is sufficient to consider the universal connecting map $\delta : G \to \mathrm{map}_0(BG, BG; 1)$.

Put $\mathcal{E} = \mathrm{map}^G(EG, EG)$, $\mathcal{G} = \mathcal{G}(EG)$ and $\mathcal{G}_0 = \mathcal{G}_0(EG)$. Let $\mathcal{E}_0$ be the subspace of $\mathcal{E}$ consisting of G-equivariant maps $EG \to EG$ restricting to the identity on the fibre at the basepoint. Then we have a fibre sequence $\mathcal{E}_0 \to \mathcal{E} \to \mathrm{map}^G(G, EG)$ induced from the basepoint inclusion of BG. Then it follows from Lemma 5.1 that $\mathcal{E}_0$ is contractible and $\mathcal{G}_0$ acts freely on $\mathcal{E}_0$ by composition. Then we have the universal $\mathcal{G}_0$-bundle

$$\mathcal{G}_0 \to \mathcal{E}_0 \to \mathcal{E}_0/\mathcal{G}_0.$$

On the other hand, the projection $\theta_0 : \mathcal{E}_0 \to \mathrm{map}_0(BG, BG; 1)$ induces a homeomorphism

$$\bar{\theta}_0 : \mathcal{E}_0/\mathcal{G}_0 \xrightarrow{\cong} \mathrm{map}_0(BG, BG; 1).$$

Note that the inclusion $\kappa : \mathcal{E}_0 \to \mathcal{E}$ induces a map $\bar{\kappa} : \mathcal{E}_0/\mathcal{G}_0 \to \mathcal{E}/\mathcal{G}_0$ by which the diagram

$$\begin{tikzcd}
\mathcal{E}_0/\mathcal{G}_0 \arrow[r, "\bar{\kappa}"] \arrow[d, "\bar{\theta}_0"] & \mathcal{E}/\mathcal{G}_0 \arrow[r] \arrow[d, "\varphi"] & \mathcal{E}/\mathcal{G} \arrow[d, "\bar{\theta}"] \\
\mathrm{map}_0(BG, BG; 1) \arrow[r, equal] & \mathrm{map}_0(BG, BG; 1) \arrow[r] & \mathrm{map}(BG, BG; 1)
\end{tikzcd} \tag{6.2}$$

commutes up to homotopy.

Let us construct an alternative universal $\mathcal{G}$-bundle to describe the connecting map δ. Following Milnor [16], we denote an element of EG by $t_0g_0 \oplus t_1g_1 \oplus \cdots$ for $\sum_i t_i = 1$, $t_i \geq 0$ and $g_i \in G$ such that finite t_i's are positive. The basepoint of EG is $1e \oplus 0 \oplus 0 \oplus \cdots$, where e is unity of G. For $g \in G$, we denote by ξ_g the principal bundle map

$$EG \to EG,\ t_0g_0 \oplus t_1g_1 \oplus \cdots \mapsto t_0g^{-1}g_0 \oplus t_1g^{-1}g_1 \oplus \cdots.$$

Then we have a commutative diagram:

$$\begin{tikzcd}
EG \arrow[r, "\xi_g"] \arrow[d] & EG \arrow[d] \\
BG \arrow[r, "B\mathrm{ad}(g)"] & BG
\end{tikzcd} \tag{6.3}$$

Now we let $\mathcal{G}$ act on $\mathcal{E}_0 \times EG$ from right by

$$(f, x) \cdot g = (\xi_{\pi(g)^{-1}} \circ f \circ g, x \cdot \pi(g))$$

for $g \in \mathcal{G}$ and $(f, x) \in \mathcal{E}_0 \times EG$. One can easily check that this action is free and then we have established the universal $\mathcal{G}$-bundle

$$\mathcal{G} \to \mathcal{E}_0 \times EG \to (\mathcal{E}_0 \times EG)/\mathcal{G}.$$

Thus there exist a homotopy equivalence $\mathcal{E}/\mathcal{G} \to (\mathcal{E}_0 \times EG)/\mathcal{G}$ and a $\mathcal{G}$-equivariant homotopy equivalence $\nu : \mathcal{E} \to \mathcal{E}_0 \times EG$ by which the diagram

$$
\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[rr] \arrow[d, "\nu"', "\simeq"] & & \mathcal{E}/\mathcal{G} \arrow[r, equal] \arrow[d, "\simeq"] & B\mathcal{G} \arrow[d, equal] \\
\mathcal{G} \arrow[r] & \mathcal{E}_0 \times EG \arrow[rr] & & (\mathcal{E}_0 \times EG)/\mathcal{G} \arrow[r, equal] & B\mathcal{G}
\end{tikzcd}
$$

commutes up to homotopy. Since the above diagram is that of $\mathcal{G}_0$-spaces and $\mathcal{G}_0$-equivariant maps, we obtain a homotopy commutative diagram:

$$
\begin{tikzcd}
G \arrow[r, equal] & \mathcal{G}/\mathcal{G}_0 \arrow[r] \arrow[d, equal] & \mathcal{E}/\mathcal{G}_0 \arrow[r] \arrow[d, "\bar{\nu}"', "\simeq"] & \mathcal{E}/\mathcal{G} \arrow[r, equal] \arrow[d, "\simeq"] & B\mathcal{G} \arrow[d, equal] \\
& \mathcal{G}/\mathcal{G}_0 \arrow[r] & (\mathcal{E}_0 \times EG)/\mathcal{G}_0 \arrow[r] & (\mathcal{E}_0 \times EG)/\mathcal{G}_0 \arrow[r, equal] & B\mathcal{G}
\end{tikzcd}
$$

Note that the above action of $\mathcal{G}$ on $\mathcal{E}_0 \times EG$ restricts to the product of the usual action of $\mathcal{G}_0$ on $\mathcal{E}_0$ and the trivial action of $\mathcal{G}_0$ on EG. Then we have $(\mathcal{E}_0 \times EG)/\mathcal{G}_0 = \mathcal{E}_0/\mathcal{G}_0 \times EG$ and thus the first projection $\pi_1 : \mathcal{E}_0 \times EG \to \mathcal{E}_0$ induces a homotopy equivalence $\bar{\pi}_0 : (\mathcal{E}_0 \times EG)/\mathcal{G}_0 \xrightarrow{\simeq} \mathcal{E}_0/\mathcal{G}_0$. Since $\mathrm{map}^{\mathcal{G}_0}(\mathcal{E}_0, \mathcal{E}_0)$ is contractible, in particular, path connected, the $\mathcal{G}_0$-equivariant map $\pi_1 \circ \nu \circ \kappa$ is homotopic to the identity of $\mathcal{E}_0$ as $\mathcal{G}_0$-equivariant maps. Then, by (6.2), we have established a homotopy commutative diagram:

$$
\begin{tikzcd}
& \mathcal{E}_0/\mathcal{G}_0 \arrow[ddd, equal, bend right=60] & \\
G \arrow[r] \arrow[d, equal] & (\mathcal{E}_0 \times EG)/\mathcal{G}_0 \arrow[r] \arrow[u, "\bar{\pi}_1"] & (\mathcal{E}_0 \times EG)/\mathcal{G} \\
G \arrow[r] \arrow[dd, equal] & \mathcal{E}/\mathcal{G}_0 \arrow[r] \arrow[u, "\bar{\nu}"] \arrow[dd, bend left=40, "\varphi"] & \mathcal{E}/\mathcal{G} \arrow[u, "\simeq"'] \arrow[dd, "\bar{\theta}"', "\simeq"] \\
& \mathcal{E}_0/\mathcal{G}_0 \arrow[u, "\bar{\kappa}"] \arrow[d, "\bar{\theta}_0"] & \\
G \arrow[r] & \mathrm{map}_0(BG, BG; 1) \arrow[r] & \mathrm{map}(BG, BG; 1)
\end{tikzcd}
$$

Therefore we have obtained:

Lemma 6.1. *There is a homotopy commutative diagram:*

$$
\begin{tikzcd}
G \arrow[r] \arrow[d, equal] & \mathcal{E}_0/\mathcal{G}_0 \arrow[r] \arrow[d, "\simeq"', "\bar{\theta}_0"] & (\mathcal{E}_0 \times EG)/\mathcal{G} \arrow[d, "\simeq"'] \\
G \arrow[r, "\delta"] & \mathrm{map}_0(BG, BG; 1) \arrow[r] & \mathrm{map}(BG, BG; 1)
\end{tikzcd}
$$

*In particular, the connecting map δ is B*ad.

Theorem 1.3 follows from (6.1).

7 $C(k,l)$-space

In this section, we discuss a relation between $H(k,l)$-spaces and higher homotopy commutativity as promised in section 5. Higher homotopy commutativity was first introduced by Sugawara [21] as intermediate states between loop spaces and loop spaces of H-spaces. Later, Williams [23] introduced another kind of higher homotopy commutativity using associahedra in section 2. Recently, Hemmi and Kawamoto [12] studied a relation between those higher homotopy commutativity, Aguadé's T_k-spaces [1] and Félix and Tanré's $H(n)$-spaces [6]. In order to relate them, They introduced $H_k(n)$-spaces and $C_k(n)$-spaces. $H_k(n)$-spaces collect Aguadé's T_k-spaces and Félix and Tanré's $H(n)$-spaces whose definition is given by a sequence of $H(k,l)$-spaces for $k+l=n$ (See [12]). On the other hand, $C_k(n)$-spaces are defined as follows by using Gel'fand, Kapranov and Zelevinsky's polytopes called resultohedra (See [9], [10] for definition of resultohedra).

Let $\mathbf{R}_+ = \{x \in \mathbf{R} | x \geq 0\}$. The resultohedron $N_{m,n}$ is an $(m+n-1)$-dimensional polytope in $\mathbf{R}_+^{m+n+2}$ which consists of all points $(p_0, \ldots, p_m, q_0, \ldots, q_n) \in \mathbf{R}_+^{m+n+2}$ satisfying:

$$\sum_{i=0}^{m} p_i = n,\ \sum_{i=0}^{n} q_i = m,\ h_{i,j} \geq 0,\ h_{m,n} = 0,$$

where

$$h_{i,j} = \sum_{k=0}^{i} (i-k)p_k + \sum_{l=0}^{j} (j-l)q_l - ij \tag{7.1}$$

for $0 \leq i \leq m$ and $0 \leq j \leq n$. Then, in particular, $N_{0,0}$ is the one point set and $N_{k,1}$ and $N_{1,k}$ are affinely homeomorphic to the k-simplex Δ^k. Vertices of $N_{m,n}$ is labelled by integer lattice paths from $(0,0)$ to (m,n).

For $x = p_i, q_j$ and $h_{i,j}$ in (7.1), we put

$$N(x) = \{(p_0, \ldots, p_m, q_0, \ldots, q_n) \in N_{m,n} | x = 0\}.$$

Gel'fand, Kapranov and Zelevinsky [10] described the face maps

$$\epsilon^{(p_i)} : N_{m-1,n} \to N(p_i),\ \epsilon^{(q_j)} : N_{m,n-1} \to N(q_j),\ \epsilon^{(h_{i,j})} : N_{i,j} \times N_{m-i,n-j} \to N(h_{i,j}).$$

On the other hand, Hemmi and Kawamoto [12] described the degeneracy maps

$$\delta_i : N_{m,n} \to N_{m-1,m},\ \delta'_j : N_{m,n} \to N_{m,n-1}.$$

Now a $C_k(n)$-space is defined by a coherent sequence of maps $Q_{r,s} : N_{r,s} \times X^{r+s} \to X$ for a topological monoid X, $r+s \leq n$ and $s \leq k$ (See [12] for precise definition). The main result of [12] is:

Theorem 7.1 ([12, Theorem A]). *A connected topological monoid is a $C_k(n)$-space if and only if its classifying space is an $H_k(n)$-space.*

As noted above, definition of an $H_k(n)$-space is a collection of that of $H(k,l)$-spaces for $k+l \le n$ and, actually, the proof of Theorem 7.1 is done by collecting constructions on $H(k,l)$-spaces.Then, by defining $C(k,l)$-spaces as follows which is a modification of that of $C_k(n)$-spaces, we obtain Theorem 1.4.

Definition 7.1. A topological monoid X is a $C(k,l)$-space if there exists a sequence of maps $Q_{r,s} : N_{r,s} \times X^{r+s} \to X$ for $0 \le r \le k$ and $0 \le s \le l$ satisfying:

$$
\begin{aligned}
&Q_{r,0}(*, x_1, \dots, x_r) = x_1 \cdots x_r,\ Q_{0,s}(*, y_1, \dots, y_s) = y_1 \cdots y_s\\
&Q_{r,s}(\epsilon^{(p_i)}(\sigma), x_1, \dots, x_r, y_1, \dots, y_s) = \begin{cases} x_1 \cdot Q_{r-1,s}(\sigma, x_2, \dots, y_s) & i = 0\\ Q_{r-1,s}(\sigma, x_1, \dots, x_i x_{i+1}, \dots, y_s) & 0 < i < r\\ Q_{r-1,s}(\sigma, x_1, \dots, x_{r-1}, y_1, \dots, y_s) & i = r \end{cases}\\
&Q_{r,s}(\epsilon^{(q_j)}(\sigma), x_1, \dots, x_r, y_1, \dots, y_s) = \begin{cases} y_1 \cdot Q_{r,s-1}(\sigma, x_1, \dots, x_r, y_2, \dots, y_s) & j = 0\\ Q_{r,s-1}(\sigma, x_1, \dots, y_j y_{j+1}, \dots, y_s) & 0 < j < s\\ Q_{r,s-1}(\sigma, x_1, \dots, y_{s-1}) & j = s \end{cases}\\
&Q_{r,s}(\epsilon^{(h_{i,j})}(\sigma_1, \sigma_2), x_1, \dots, x_r, y_1, \dots, y_s)\\
&\qquad\quad = Q_{i,j}(\sigma_1, x_1, \dots, x_i, y_1, \dots, y_j) \cdot Q_{r-i,s-j}(\sigma_2, x_{i+1}, \dots, x_y, y_{j+1}, \dots, y_s)\\
&Q_{r,s}(\sigma, x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, y_s) = Q_{r-1,s}(\delta_i(\sigma), x_1, \dots, x_{i-1}, x_{i+1}, \dots, y_s)\\
&Q_{r,s}(\sigma, x_1, \dots, y_{j-1}, *, y_{j+1}, \dots, y_s) = Q_{r,s-1}(\delta'_j(\sigma), x_1, \dots, y_{j-1}, y_{j+1}, \dots, y_s)
\end{aligned}
$$

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