Splitting of gauge groups

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1 Introduction

We will always assume each space has the homotopy type of a CW-complex.

Let $G$ be a topological group and let $P$ be a principal $G$-bundle over a space $B$. The gauge group of $P$, denoted $\mathcal{G}(P)$, is the group of automorphisms of $P$ covering the identity of $B$.

Fix a basepoint $b_0$ of $B$. Then the basepoint inclusion $b_0 \hookrightarrow B$ induces a homomorphism of topological groups

$$\mathcal{G}(P) \rightarrow \mathcal{G}(P)\vert_{b_0} \cong G.$$ Since we work with CW-complexes which are normal, this homomorphism is easily seen to be a surjection. We call the kernel of this homomorphism the based gauge group of $P$ and denote by $\mathcal{G}_0(P)$. Namely, $\mathcal{G}_0(P)$ consists of automorphisms of $P$ covering $1_B$ which restrict to the identity on the fibre at the basepoint $b_0$. Now we have an extension of topological groups:

$$1 \rightarrow \mathcal{G}_0(P) \rightarrow \mathcal{G}(P) \rightarrow G \rightarrow 1 \quad (1.1)$$

The second author [14] classified the homotopy types of $\mathcal{G}(P)$ as spaces, not as topological groups, when $P$ runs all over principal SU(2)-bundles over $S^4$. Later, Crabb and Sutherland [4] studied the homotopy type of $\mathcal{G}(P)$ as $H$-spaces for a general $P$. Moreover, when $B$ is a simply connected 4-manifold and $G = \text{SU}(2)$, Tsukuda and the second author [15], [22] classified the homotopy types of the classifying spaces $BG(P)$, equivalently, the homotopy types of $\mathcal{G}(P)$ as loop spaces. These results suggest us to study the homotopy theory of gauge groups as spaces with intermediate higher homotopy associativity in the sense of Stasheff [18], that is, as $A_n$-spaces. In particular, we may study the group extension (1.1) in the category of $A_n$-spaces and $A_n$-maps. The aim of this article is to study a splitting of (1.1) in the category of $A_n$-spaces and $A_n$-maps which we call an $A_n$-splitting. More precisely, we will formulate the $A_n$-splitting and consider:

**Question 1.1.** 1. What does a geometric meaning of an $A_n$-splitting of (1.1)?
2. Give a criterion for an $A_n$-splitting of (1.1).

Regarding the first question, we consider a relation between an $A_n$-splitting of (1.1) and the bundle $P$. Let $\text{ad} : G \to \text{Aut} G$ be the adjoint action of $G$ on itself and let $\text{ad} P = P \times_{\text{ad}} G$, the adjoint bundle of $P$. Introducing fibrewise analogue of $A_n$-maps between topological monoids, we obtain:

**Theorem 1.1.** There is an $A_n$-splitting of (1.1) if and only if $\text{ad} P$ is fibrewise $A_n$-equivalent to the trivial bundle $B \times G$.

Let $\text{map}(X,Y;f)$ be the path component of the space of maps from $X$ to $Y$ containing $f$, where we will always take $f$ to be basepoint preserving. Denote the universal $G$-bundle by $EG \to BG$. Regarding the second question, we will be concerned with the classical result of Atiyah and Bott [2]:

$$BG(P) \simeq \text{map}(B,BG;\alpha),$$

where $\alpha$ is the classifying map of $P$. Naturality of this homotopy equivalence allows us to identify the map $B\pi : BG(P) \to BG$ with the evaluation fibration $\text{map}(B,BG;\alpha) \to BG$. This leads us to the definition of $H(k,l)$-spaces having the following property.

**Theorem 1.2.** There is an $A_l$-splitting of (1.1) if $BG$ is an $H(1,n)$-space and $\text{cat} B \leq k$.

As above, $H(k,l)$-space is motivated by the evaluation fibration $\text{map}(B,BG;\alpha) \to BG$ and, in particular, $H(1,n)$-space can be described by the connecting map $\delta : G \to \text{map}_0(B,BG;\alpha)$ in the fibre sequence $G \to \delta \to \text{map}_0(B,BG;\alpha) \to \text{map}(B,BG;\alpha) \to BG$, where $\text{map}_0(X,Y;f)$ is the subspace of $\text{map}(X,Y;f)$ consisting of based maps. Note that the adjoint action $\text{ad} : G \to \text{aut}(G)$ induces a map $\text{Bad} : G \to \text{map}_0(BG,BG;1)$ which assigns each $g \in G$ to the map $\text{Bad}(g) : BG \to BG$. Here we must notice that $\text{Bad}$ does not mean the map $BG \to \text{Baut}(G)$ induced from the adjoint action $\text{ad} : G \to \text{aut}(G)$. Then we obtain:

**Theorem 1.3.** The connecting map $\delta : G \to \text{map}_0(B,BG;\alpha)$ is given by $\delta(g) = \text{Bad}(g) \circ \alpha$ for $g \in G$.

Let $E_n G \to B_n G$ be the $n$-th stage of Milnor’s construction of the universal bundle $EG \to BG$ [16]. By definition, $BG$ is an $H(1,n)$-space if and only if the connecting map $\delta$ in Theorem 1.3 is trivial for the inclusion $i_n : B_n G \to BG$. Then we have:

**Corollary 1.1.** $BG$ is an $H(1,n)$-space if and only if $\text{Bad} \circ i_n : G \to \text{map}_0(B_n G,BG;i_n)$ is null-homotopic.
We will investigate an $H(k,l)$-space further in view of higher homotopy commutativity as follows. By definition, the loop space of an $H(1,1)$-space is homotopy commutative and an $H(\infty,\infty)$-space is an H-space. On the other hand, Sugawara [21] constructed a class of spaces between homotopy commutative topological monoids and the loop spaces of H-spaces, called higher homotopy commutativity. Then we expect that the loop spaces of $H(k,l)$-spaces form a new class of higher homotopy commutativity. Kawamoto and Hemmi [12] introduced $H_k(n)$-spaces in order to unify Aguadé’s $T_k$-spaces [1] and Félix and Tanrê’s $H(n)$-spaces [6]. They also introduced higher homotopy commutativity called $C_k(n)$-spaces in order to describe $H_k(n)$-spaces by higher homotopy. An $H_k(n)$-space is, in fact, given by patching together $H(i,j)$-spaces for $i + j = n$ and $i = 1, \ldots, k$. Moreover, in describing an $H_k(n)$-space by a $C_k(n)$-space, they worked at the level of $H(i,j)$-spaces. This leads us to define a new class of higher homotopy commutativity, $C(k,l)$-spaces, by cutting $C_k(n)$-spaces into pieces and obtain:

**Theorem 1.4.** A connected topological monoid is a $C(k,l)$-space if and only if its classifying space is an $H(k,l)$-space.

Combining Theorem 1.1, Theorem 1.2 and Theorem 1.4, we can conclude:

**Corollary 1.2.** Let $G$ be a connected topological group and let $P$ be a principal $G$-bundle over $B$. If $G$ is a $C(k,l)$-space, then there is an $A_n$-splitting of (1.1), equivalently, $\text{ad}P$ is fibrewise $A_n$-homotopy equivalent to the trivial bundle $B \times G$.

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## 2 $A_n$-splitting

In this section, we formulate a splitting of an extension of topological groups in the category of $A_n$-spaces and $A_n$-maps which we call an $A_n$-splitting. An $A_n$-space was introduced by Stasheff [18] to be a space with a multiplication which enjoys a certain higher homotopy associativity. Then an $A_n$-map should be a map between $A_n$-spaces preserving their $A_n$-space structures. Stasheff [19] defined an $A_n$-map between $A_\infty$-spaces. Later, he [20] defined an $A_n$-map from an $A_n$-space to an $A_\infty$-space and implied an $A_n$-map between $A_n$-spaces. Finally, Iwase and Mimura [13] described an $A_n$-map between $A_n$-spaces completely. Of course, these definitions of $A_n$-maps are consistent and then we will use convenient one case by case.

An $A_n$-splitting of an extension of topological groups should be analogous to a splitting in the category of topological groups and their homomorphisms. Then we define an $A_n$-splitting of an extension of topological groups as follows.
**Definition 2.1.** An $A_n$-splitting of an extension of topological groups $1 \to K \to \tilde{H} \to H \to 1$ consists of the following:

1. There is an $A_n$-structure on $H \times K$, the direct product as spaces, not as topological groups, which restricts to the canonical group structures on $H \times \{1\}$ and $\{1\} \times K$.

2. There is an $A_n$-map $\theta : H \times K \to \tilde{H}$ with respect to the above $A_n$-structure on $H \times K$ satisfying the homotopy commutative diagram:

$$
\begin{array}{cccccc}
1 & \to & K & \to & \tilde{H} & \to & H & \to & 1 \\
\| & & \| & & \uparrow \theta & & \| & & \| \\
1 & \to & K & \to & H \times K & \xrightarrow{\pi} & H & \to & 1,
\end{array}
$$

where $\pi$ is the second projection.

Let $1 \to K \to \tilde{H} \xrightarrow{\pi} H \to 1$ be an extension of topological groups. A splitting of this extension as groups can be completely described by a section of $\pi$ which is a group homomorphism. We shall show that there is an analogy for an $A_n$-splitting. Namely, a homotopy section of $\pi$ which is an $A_n$-map, called an $A_n$-section, implies an $A_n$-splitting of the extension, where homotopy section of $\pi$ is a map $s : H \to \tilde{H}$ such that $\pi \circ s \simeq 1_H$.

Let us first recall Stasheff’s polytope, the associahedron, which was used to define an $A_n$-space and an $A_n$-map from $A_n$-space to an $A_\infty$-space (See [18] and [20]). The $i$-th associahedron $K_i$ is an $(i - 2)$-dimensional convex polytope having the face maps

$$
\partial_k(r, s) : K_r \times K_s \to K_i
$$

for $r + s = i + 1$ and $1 \leq k \leq i - s + 1$ and the degeneracy maps

$$
s_j : K_i \to K_{i-1}
$$

for $1 \leq j \leq i$. In particular, there are relations:

$$
s_j \circ \partial_k(r, s) = \begin{cases} 
\partial_k(r, s - 1) \circ (1 \times s_{j-k+1}) & k \leq j < k + s \\
\partial_k(r - 1, s) \circ (s_{j-s+1} \times 1) & j \geq k + s
\end{cases}
$$

(2.1)

There is a one to one correspondence between vertices of $K_i$ and connected binary trees with $n$-leaves. In order to define an $A_n$-space structure from an $A_n$-section, we consider the following operations of binary trees. Let $T_n$ be the set of connected binary trees with $n$ leaves and let $\widehat{T}_n$ be the set of ordered binary trees, not necessarily connected, with $n$ leaves. Then we can label each leaf of an element of $\widehat{T}_n$ from 1 to $n$ in the obvious way. Define a map $\delta : T_{n+1} \to \widehat{T}_n$ by deleting the branches from the root to the $n$-th leaf. For example, $\delta : T_7 \to \widehat{T}_6$ is:
Then $\delta$ is a bijection. Analogously we define a map $\hat{\delta} : \hat{T}_n \to \hat{T}_{n-1}$ by applying the above map $\delta$ to the connected binary tree having the leaf labelled by $n$. Then $\delta : T_n \to \hat{T}_{n-1}$ is the restriction of $\hat{\delta} : \hat{T}_n \to \hat{T}_{n-1}$.

Let $X$ be an $H$-space. For $x_1, \ldots, x_n \in X$ and $t \in T_n$, we define $t(x_1, \ldots, x_n)$ as in [20], which is consistent with the definition of $A_n$-spaces. For example, if $t \in T_4$ is

![Diagram]

then $t(x_1, x_2, x_3, x_4) = x_1((x_2 x_3) x_4)$. Let $G$ be a topological group. Using the above map $t$, for a map $f : X \to G$, we define a map $\hat{f} : \hat{T}_n \times X^n \to G$ by

$$\hat{f}(\hat{t}, x_1, \ldots, x_n) = f(t_1(x_1, \ldots, x_n)) f(t_2(x_{n+1}, \ldots, t_{n+2})) \cdots f(t_k(x_1, \ldots, x_n)),$$

where $\hat{t} = t_1 \sqcup \cdots \sqcup t_k \in \hat{T}_n$ such that $t_k < \cdots < t_1$ and $t_i \in T_n$.

Now we consider an extension of topological groups $1 \to K \to \hat{H} \xrightarrow{\pi} H \to 1$. Suppose that $\pi$ admits an $A_n$-section $s$ whose $A_n$-form is $\{m_i : K_{i+1} \times H^i \to \hat{H}, 1 \leq i \leq n\}$. As noted above, for a vertex $v \in K_{i+1}$ corresponding to $\hat{t} \in \hat{T}_i$, we have

$$h_i(v, x_1, \ldots, x_i) = s(\hat{t}, x_1, \ldots, x_i).$$

We write $\gamma_j(\tau, x) = h_j(s_{j+1} s_{j+2} \cdots s_1(\tau), \pi_{j+1} \pi_{j+2} \cdots \pi_1(x))$ for $\tau \in K_i, x \in K^i$ and the projection $\pi_j : K^i \to K^{i-1}$ omitting the $i$-th entry. Then, for a vertex $v \in K_i$ corresponding to $t \in T_i$ and $x = (x_1, \ldots, x_i) \in K^i$, it follows from (2.1) that

$$\gamma_j(v, x) = s(\hat{d}^{i-j} t, x_1, \ldots, x_i).$$

(2.2)

Define $M_i : K_i \times (H \times K)^{i} \to H \times K$ by

$$M_i(\tau, (h_1, k_1), \ldots, (h_i, k_i)) = (h_1 h_{\gamma_1}^{1(\tau, k)} h_{\gamma_2}^{1(\tau, k)} \cdots h_{\gamma_{i-1}}^{1(\tau, k)} k_1 \cdots k_i).$$
for $\tau \in K_i$ and $k = (k_1, \ldots, k_i) \in K^i$, where $g^h = hgh^{-1}$ for $g, h \in H$. Then, by (2.2), it is straightforward to check that $\{M_i : K_i \times (H \times K)^i \to H \times K\}_{2 \leq i \leq n+1}$ is an $A_{n+1}$-form on $H \times K$ such that, for a vertex $v \in K_i$ corresponding to $t \in T_i$, $M_i(v, (h_1, k_1), \ldots, (h_i, k_i)) = t((h_1, k_1), \ldots, (h_i, k_i))$. In particular, the multiplication of $H \times K$ is defined by

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1(h_2^{\sigma(k_1)}), k_1k_2)$$

which is analogous to semidirect products of groups.

By a quite analogous observation, we can see that the map $\theta : H \times K \to \tilde{H}$ defined by

$$\theta(h, k) = h \cdot s(k)$$

for $h \in H$ and $k \in K$ admits an $A_n$-form. Summarizing, we have established:

**Lemma 2.1.** An extension of topological groups $1 \to K \to \tilde{H} \overset{\pi}{\to} H \to 1$ has an $A_n$-splitting if and only if $\pi$ admits an $A_n$-section.

### 3 Fibrewise $A_n$-map

In this section, we introduce fibrewise analogue of $A_n$-maps between topological monoids and characterize them by using fibrewise analogue of projective spaces. Let us first recall from [3] some notations and terminologies of fibrewise homotopy theory. Fix a space $B$. A fibrewise space over $B$ is an arrow $X \overset{\pi}{\to} B$. $\pi_X$ is called the projection and $\pi_X^{-1}(b)$ for $b \in B$ is called a fibre at $b$. Then the direct product $A \times B$ is a fibrewise space over $B$ and, in particular, so is $B$ itself. A fibrewise map from a fibrewise space $X \overset{\pi_X}{\to} B$ to $Y \overset{\pi_Y}{\to} B$ is a commutative diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\pi_X & & \pi_Y \\
\downarrow & & \downarrow \\
B & \longrightarrow & B
\end{array}
$$

Then fibrewise spaces over $B$ and fibrewise maps between them form a category which is nothing but the comma category $\underbrace{\text{Top} \downarrow B}$, where Top is the category of topological spaces and continuous maps. Fibrewise homotopy theory is not homotopy theory by the canonical model category structure on $\underbrace{\text{Top} \downarrow B}$ induced from Top, but it respects fibre homotopy equivalence in the classical sense. With this in mind, we recall basic constructions in fibrewise homotopy theory. The fibrewise product $X \times_B Y$ of $X \overset{\pi_X}{\to} B$ and $Y \overset{\pi_Y}{\to} B$ is the pullback of the triad $X \overset{\pi_X}{\to} B \overset{\pi_Y}{\to} Y$, that is,

$$X \times_B Y = \{(x, y) \in X \times Y | \pi_X(x) = \pi_Y(y)\}.$$
Then the diagonal map restricts to the fibrewise diagonal map \( X \to X \times_B X \), denoted \( \Delta_B \). We often abbreviate the fibrewise product of \( n \)-copies of a fibrewise space \( X \to B \) by \( X^n \) by abuse of notation. We denote the fibrewise space \([0, 1] \times B \to B\) by \( I_B \) and call it the fibrewise interval, here the projection is the second projection. A fibrewise homotopy is a fibrewise map \( X \times_B I_B \to Y \) and we have a fibrewise homotopy equivalence in the obvious sense, which are the classical fibre homotopy and fibre homotopy equivalence respectively. With this notion of fibrewise homotopies, we have a fibrewise fibration and a fibrewise cofibration which are characterized by a fibrewise homotopy lifting property and a fibrewise homotopy extension property respectively (See [3]).

The fibrewise unreduced cone of \( X \to B \), denoted \( C_B X \), is a pushout of the cotriad \( I_B \times_B X \leftarrow \{0\} \times X \to B \). Similarly, the fibrewise unreduced suspension of \( X \to B \), denoted \( \Sigma_B X \), is a pushout of the cotriad \( I_B \times_B X \leftarrow \{0, 1\} \times X \to B \).

A fibrewise pointed space is a fibrewise space \( X \to B \) with a distinguished section and then we assume \( B \subseteq X \). We have a fibrewise pointed map in the obvious sense. The fibrewise reduced cone \( C^B_B X \) and the fibrewise reduced suspension \( \Sigma^B_B X \) are the fibrewise collapses \( C_B X/B C_B B \) and \( \Sigma_B X/B \Sigma_B B \) respectively (See [3, p.55]). A fibrewise pointed space is said to be well-pointed if the section is a fibrewise cofibration. Then, as in the usual case, if a fibrewise pointed space \( X \) over \( B \) is well-pointed, then \( C_B X \) is fibrewise homotopy equivalent to \( C^B_B X \) relative to \( X \). In particular, \( \Sigma_B X \) is fibrewise homotopy equivalent to \( \Sigma^B_B X \).

In order to introduce a fibrewise analogue of \( A_n \)-maps between topological monoids, we need to have a fibrewise analogue of topological monoids which is given by replacing spaces and structure maps with fibrewise spaces and fibrewise maps of topological monoids as follows. A fibrewise topological monoid over \( B \) is a fibrewise space \( X \to B \) with fibrewise maps \( \mu: X \times_B X \to X \) and \( \mu: X \times_B X \to X \) satisfying two conditions:

\[
\mu \circ (\mu \times 1) = \mu \circ (1 \times \mu), \quad \mu \circ (1 \times \pi_X) \circ \Delta_B = 1 = \mu \circ (\epsilon \circ \pi_X \times 1).
\]

In particular, a fibrewise topological monoid is a fibrewise pointed space and each of its fibre is a topological monoid. We usually abbreviate \( \mu(x, y) \) by \( xy \). A fibrewise topological monoid \( X \to B \) is a fibrewise topological group, if it has a fibrewise map \( \iota: X \to X \) satisfying

\[
\mu \circ (1 \times \iota) \circ \Delta_B = \epsilon \circ \pi_X = \mu \circ (\iota \times 1) \circ \Delta_B.
\]

Let us look at examples of fibrewise topological monoids.

**Example 3.1.** Let \( X \to B \) be a fibrewise pointed space with a distinguished section \( s \). The fibrewise Moore path space of \( X \) is

\[
\Omega'_B X = \coprod_{b \in B} \Omega'(\pi^{-1}(b))
\]
equipped with an appropriate topology (See [3]), where \( \Omega Y \) is the Moore path space of a space \( Y \). Then the loop multiplication of \( \Omega (\pi^{-1}(b)) \) makes \( \Omega'_B X \) be a fibrewise topological monoid.

**Example 3.2.** Let \( G \) be a topological group and let \( \pi : P \to B \) be a principal \( G \)-bundle. Then the adjoint bundle \( \text{ad}P \) is a fibrewise topological group with the structure maps:

\[
\begin{align*}
\epsilon : B & \to \text{ad}P, \quad \epsilon(b) = [\pi^{-1}(b), 1], \\
\mu : \text{ad}P \times_B \text{ad}P & \to \text{ad}P, \quad \mu([x, g], [x, h]) = [x, gh], \\
\iota : \text{ad}P & \to \text{ad}P, \quad \iota([x, g]) = [x, g^{-1}],
\end{align*}
\]

where \([x, g]\) is a equivalence class of \((x, g) \in P \times G\) in \( \text{ad}P \).

Now we define a fibrewise \( A_n \)-map between fibrewise topological monoids just by replacing objects and arrows with fibrewise ones and the interval \([0, 1]\) with the fibrewise interval \( I_B \) (See [19] for the definition of the usual \( A_n \)-maps between topological monoids).

**Definition 3.1.** Let \( X \) and \( Y \) be fibrewise topological monoids over \( B \). A fibrewise map \( f : X \to Y \) is called a fibrewise \( A_n \)-map if there exists a sequence of fibrewise maps \( \{ h_i : I^{-1}_B \times_B X^i \to Y \}_{1 \leq i \leq n} \) such that \( h_1 = f \) and

\[
\begin{align*}
h_i(t_1, \ldots, t_{i-1}, x_1, \ldots, x_i) \\
= \begin{cases} 
    h_{i-1}(t_1, \ldots, \hat{t}_j, \ldots, t_{i-1}, x_1, \ldots, x_jx_{j+1}, \ldots, x_i) & t_j = 0 \\
    h_j(t_1, \ldots, t_{j-1}, x_1, \ldots, x_j)h_{i-j}(t_{j+1}, \ldots, t_{i-1}, x_{j+1}, \ldots, x_i) & t_j = 1.
\end{cases}
\end{align*}
\]

By a quite analogous proof to [20] and [7], we can see the following properties of fibrewise \( A_n \)-maps.

**Proposition 3.1.**

1. A fibrewise map \( f \) is fibrewise homotopic to a fibrewise \( A_n \)-map, then so is \( f \).

2. The composition of fibrewise \( A_n \)-maps is a fibrewise \( A_n \)-map.

3. A homotopy inverse of a fibrewise homotopy equivalence which is a fibrewise \( A_n \)-map is a fibrewise \( A_n \)-map.

It follows from the above proposition that fibrewise homotopy equivalences which are fibrewise \( A_n \)-maps give an equivalence relation among fibrewise topological monoids. We call this equivalence by a fibrewise \( A_n \)-equivalence.

Let us characterize fibrewise \( A_n \)-maps using fibrewise analogue of projective spaces as in [19]. Note that we do not have appropriate quasi-fibrations in our fibrewise category. That is,
we do not have weak equivalences nor quasi-fibrations, which can be replaced with fibrewise fibrations by weak equivalences, in our fibrewise category. Then it seems impossible to mimic the proof of [19, Theorem 4.5] directly. However, we only need to deal with fibrewise topological groups and we can overcome the above difficulty by restricting to fibrewise topological groups.

Let $G$ be a fibrewise topological group over $B$. Then, by [3, p.37], we have a fibrewise analogue of the Milnor construction for classifying spaces. Denote the $n$-th stage of the fibrewise Milnor construction for $G$ by $E^n_B G \to B^n_B G$ which is a finite numerable fibrewise fibre bundle. Thus, by a quite analogous observation of [17, Corollary 14], we have:

**Lemma 3.1.** The fibrewise map $E^n_B G \to B^n_B G$ is a fibrewise fibration.

It will be convenient for later use to state a characterization of fibrewise $A_n$-maps by using fibrewise analogue of the Dold-Lashof construction which coincides with the Milnor construction in the usual case (See, for example, [8]). Then we define the fibrewise Dold-Lashof construction only by replacing everything in the Dold-Lashof construction with a fibrewise one as follows. Let $H$ be a fibrewise topological monoid having a fibrewise action on $E$, denoted $m : H \times_B E \to E$ (See [3, p.15]). Start with a fibrewise map $q : E \to X$ enjoying $q(m(h, x)) = p(x)$ for $(h, x) \in H \times_B X$. Let $DL_B(E)$ be the fibrewise quotient of $(H \times_B C_B E) \sqcup E$ by the relation $(h, (1, x)) \sim \mu(h, x)$ for $(h, (1, x)) \in H \times_B C_B E$ and let $DL_B(X)$ be the fibrewise quotient of $C_B E \sqcup X$ by $(1, x) \sim q(x)$ for $(1, x) \in C_B E$. Then the Dold-Lashof construction for $q$ is the fibrewise map

$$DL_B(q) : DL_B(E) \to DL_B(X), \quad (h, (t, x)) \mapsto (t, x).$$

Note that we do not have to take much care for topologies of $DL_B(E)$ and $DL_B(X)$ since we work in the category of spaces having the homotopy types of CW-complexes. Since $H$ is fibrewise associative, we can apply the Dold-Lashof construction iteratively. We denote the iterated Dold-Lashof construction

$$DL^n_B(\pi_H) : DL^n_B(H) \to DL^n_B(B)$$

by $\pi^n_B : E^n_B H \to P^n_B H$. As in the usual case, we can easily verify that if $H$ is a fibrewise topological group, $\pi^n_B : E^n_B H \to P^n_B H$ coincides with the $n$-th stage of the Milnor construction $E^n_B H \to B^n_B H$.

We follow [13] to characterize fibrewise $A_n$-maps then we first define a fibrewise $A_n$-structure of a fibrewise $A_n$-map. Let $D^n_B X = C_B E^n_B X$ for a fibrewise topological monoid $X$.

**Definition 3.2.** Let $X$ and $Y$ be fibrewise topological monoids over $B$. A fibrewise $A_n$-structure of a fibrewise map $f : X \to Y$ consists of:

1. $f$ respects fibrewise units of $X$ and $Y$. 


2. There are sequences of commutative squares of fibrewise maps

\[
\begin{array}{ccc}
(D_{i+1}^X, E_i^X) & \xrightarrow{f_i} & (D_{i+1}^Y, E_i^Y) \\
\downarrow{\pi_{i+1}} & & \downarrow{\pi_{i+1}} \\
(P_{i+1}^X, P_i^X) & \xrightarrow{f_i} & (P_{i+1}^Y, P_i^Y)
\end{array}
\]

for \(1 \leq i \leq n-1\) such that \(f_i|_X = f, f_i|_{D_{i}^X} = f_{i-1}^E, f_i|_{P_{i}^X} = f_{i-1}^P\).

Now we give a characterization of fibrewise \(A_n\)-map.

**Theorem 3.1.** Let \(X\) be a fibrewise topological monoid over \(B\) and let \(Y\) be a fibrewise well-pointed topological group over \(B\). A fibrewise map \(f : X \to Y\) is a fibrewise \(A_n\)-map if and only if \(f\) possesses a fibrewise \(A_n\)-structure.

**Proof.** The if part is done by Sugawara’s construction [21]. In order to prove the only if part, we can mimic the proof of [19, Theorem 4.5] instead of replacing quasi-fibrations with fibrations. Then if we can replace \(\pi^n_B : E^n_B Y \to P^n_B Y\) with a fibrewise fibration, the proof is completed.

Consider the Dold-Lashof construction for the projection \(\pi_Y : Y \to B\) in which the unreduced cone is replaced by the reduced cone. Then, as in [8], we obtain the Milnor construction \(E^n_B Y \to B^n_B Y\) and hence a commutative diagram:

\[
\begin{array}{ccc}
E^n_B Y & \xrightarrow{\pi^n_B} & E^n_B Y \\
\downarrow{\pi^n_B} & & \downarrow{\pi^n_B} \\
P^n_B Y & \xrightarrow{} & B^n_B Y
\end{array}
\]

Moreover, it follows from induction with the hypothesis that \(Y\) is fibrewise well-pointed, \(E^n_B Y\) and \(B^n_B Y\) are fibrewise well-pointed. This implies that the horizontal arrows in the above diagram are fibrewise homotopy equivalences and thus, by Lemma 3.1, we assume that \(\pi^n_B : E^n_B Y \to P^n_B Y\) is a fibrewise fibration. \(\square\)

### 4 Set of sections

In this section, we consider the set of sections of a fibrewise space and prove Theorem 1.1. Let \(X\) be a fibrewise space over \(B\). We denote the set of sections of \(X\) by \(\Gamma(X)\). Then it is obvious that \(\Gamma\) is a functor from \(\text{Top} \downarrow B\) to \(\text{Top}\). Note that, by the pointwise multiplication, \(\Gamma(X)\) is a topological monoid and a topological group according as \(X\) is a fibrewise topological monoid and a fibrewise topological group. In particular, for a principal bundle \(P\), \(\Gamma(\text{ad}P)\) is a topological group by which we have an isomorphism of topological groups

\[
\mathcal{G}(P) \cong \Gamma(\text{ad}P) \tag{4.1}
\]
Let \( C : \text{Top} \to \text{Top} \) be the unreduced cone functor. We define a natural transformation \( \lambda : C \Gamma \to \Gamma C_B \) by
\[
\lambda : C \Gamma(X) \to \Gamma(C_B X), \quad \lambda(t, s)(b) = (t, s(b))
\] for \( b \in B \). Let \( H \) be a fibrewise topological monoid with a fibrewise action \( \mu : H \times B E \to E \) and let \( q : E \to X \) be a fibrewise map such that \( q(\mu(h, x)) = x \) for \( (h, x) \in H \times B E \). Then, by definition, the natural transformation \( \lambda \) induces a commutative diagram
\[
\begin{array}{ccc}
DL(\Gamma(E)) & \xrightarrow{\lambda} & DL_B(E) \\
\downarrow \text{DL(}\Gamma(q)) & & \downarrow \text{DL(}\Gamma_B(q)) \\
DL(\Gamma(X)) & \xrightarrow{\lambda} & DL_B(X)
\end{array}
\]
in which all maps respects the action of \( \Gamma(H) \), where \( DL(\cdot) \) means the usual Dold-Lashof construction. Then it follows that we have a commutative square
\[
\begin{array}{ccc}
(D^{n+1} \Gamma(H), E^n \Gamma(H)) & \xrightarrow{\lambda_n} & (\Gamma(D^{n+1} H), \Gamma(E^n_B H)) \\
\downarrow \pi^{n+1} & & \downarrow \pi^{n+1}_B \\
(P^{n+1} \Gamma(H), P^n \Gamma(H)) & \xrightarrow{\lambda_n} & (\Gamma(P^{n+1} B H), \Gamma(P^n_B H))
\end{array}
\]
for all \( n \) such that
\[
\tilde{\lambda}_n|_{D^n \Gamma(H)} = \lambda_{n-1}, \quad \lambda_n|_{P^n \Gamma(H)} = \lambda_{n-1},
\]
where, for a topological monoid \( Y \), \( \pi^n : E^n Y \to P^n Y \) is \( DL^n(\cdot) : DL(Y) \to DL(\cdot) \) and \( D^{n+1} Y = CE^n Y \).

**Proof of Theorem 1.1.** Suppose that we have an \( A_n \)-splitting of (1.1). Then, by Lemma 2.1, we have an \( A_n \)-section \( \sigma : G(P) \to G \) which is identified with the evaluation at the basepoint \( \Gamma(\text{ad} P) \to G \) through the isomorphism (4.1). Define a fibrewise map
\[
\theta : B \times G \to \text{ad} P, \quad \theta(b, g) = \sigma(g)(b).
\]
Then we have \( \theta|_{(b_0) \times G} \simeq 1_G \) since \( \sigma \) is a section of the evaluation at the basepoint \( \Gamma(\text{ad} P) \to G \), where \( b_0 \) is the basepoint of \( B \). Thus, by Dold’s theorem [5], \( \theta \) is a fibrewise homotopy equivalence.

Since \( \sigma \) is an \( A_n \)-map, it possesses an \( A_n \)-structure in the sense of [13], that is, there is a sequence of homotopy commutative square
\[
\begin{array}{ccc}
(D^{i+1} G, E^i G) & \xrightarrow{\sigma^i} & (D^{i+1} \Gamma(\text{ad} P), E^i \Gamma(\text{ad} P)) \\
\downarrow & & \downarrow \\
(P^{i+1} G, P^i G) & \xrightarrow{\sigma^i_p} & (P^{i+1} \Gamma(\text{ad} P), P^i \Gamma(\text{ad} P))
\end{array}
\]

for $i = 1, \ldots, n - 1$ in which $\sigma^E_i|_G = \sigma, \sigma^i|_{D^i G} = f^E_i, f^j|_{P^i G} = f^j_1$. Note that
\[ D^i_B(B \times G) = B \times G, \quad E^i_B(B \times G) = B \times E^i G, \quad P^i(B \times G) = B \times P^i G \]
and then we shall make these identifications. Define fibrewise maps
\[ \theta^i_E : (D^i_B + 1 B \times G), E^i_B(B \times G)) \to (D^i_B + 1 \text{ad} P, E^i_B \text{ad} P) \]
and
\[ \theta^i_P : (P^{i+1}_B(B \times G), P^i_B(B \times G)) \to (P^{i+1}_B \text{ad} P, P^i_B \text{ad} P) \]
by
\[ \theta^i_E(b, x) = \bar{\lambda}_i(\sigma^i_E(x))(b), \quad \theta^i_P(b, y) = \lambda_i(\sigma^i_P(y))(b) \]
for $b \in B, x \in D^i + 1 G, y \in P^{i+1} G$. Then these fibrewise maps gives a fibrewise $A_n$-structure of $\theta$ and therefore, by Theorem 3.1, $\theta$ is a fibrewise $A_n$-equivalence.

Let $X$ be a fibrewise space over $B$. As in (4.2), we have a map
\[ \rho : [0, 1] \times \Gamma(V) \to \Gamma(I_B \times_B X), \quad \rho(t, s)(b) = (t, s(b)) \]
for $(t, s) \in [0, 1] \times \Gamma(V)$ and $b \in B$. Then a fibrewise $A_n$-map $f : X \to Y$ for fibrewise topological monoids $X, Y$ induces an $A_n$-map $\Gamma(f) : \Gamma(X) \to \Gamma(Y)$ in the sense of [19].

Suppose that we have a fibrewise $A_n$-equivalence $\theta : B \times G \to \text{ad} P$. Then it follows that we have an $A_n$-equivalence $\Gamma(\theta) : \Gamma(B \times G) \to \Gamma(\text{ad} P)$. Now we have an isomorphism of topological groups $\Gamma(B \times G) \cong \text{map}(B, G)$ which is natural with respect to $B$. Then the evaluation at the basepoint $\Gamma(B \times G) \to G$ is nothing but the evaluation at the basepoint map$(B, G) \to G$ which admits a section as topological groups. Then we obtain an $A_n$-section of $\pi : \Gamma(\text{ad} P) \to G$ and thus, by Lemma 2.1, we have established an $A_n$-splitting of (1.1).

\section{5 $H(k, l)$-space}

In this section, we consider the second question, that is, a criterion for an $A_n$-splitting of (1.1). Our major tool is the homotopy equivalence (1.2). Then let us first recall the construction of the construction of the homotopy equivalence (1.2). Let $G$ be a topological group. We denote by $\text{map}^G(X, Y)$ the space of all $G$-equivariant maps from $X$ to $Y$ for $G$-spaces $X, Y$. Let $P$ and $Q$ be principal $G$-bundles. Then $G(P)$ acts on $\text{map}^G(X, Y)$ by composition. Now we consider the case $Q = EG$. Then we have:

\textbf{Lemma 5.1} ([11, Theorem 5.2], [2, Proposition 2.4]).

1. $\text{map}^G(P, EG)$ is contractible.

2. The action of $G(P)$ on $\text{map}^G(P, EG)$ is free.
Then we have the universal $\mathcal{G}(P)$-bundle:

$$\mathcal{G}(P) \to \text{map}^G(P, EG) \to \text{map}^G(P, EG)/\mathcal{G}(P) \quad (5.1)$$

Let us denote by $\theta$ the map $\text{map}^G(P, EG) \to \text{map}(B, BG; \alpha)$ induced from the projections $P \to B$ and $EG \to BG$, where $B$ is the base space of $P$ and $\alpha$ is the classifying map of $P$. Then one can easily see that the map $\theta$ induces a homeomorphism

$$\tilde{\theta} : \text{map}^G(P)/\mathcal{G}(P) \cong \text{map}(B, BG; \alpha) \quad (5.2)$$

which is natural with respect to $P$. Thus we obtain a homotopy equivalence

$$\hat{\theta} : BG(P) \cong \text{map}(B, BG; \alpha)$$

which is natural with respect to $P$.

Consider the topological group $G$ as the principal $G$-bundle over a point and identify $G(G)$ with $G$. Then the basepoint inclusion $i : b_0 \to B$ induces a homotopy commutative diagram:

$$
\begin{array}{ccc}
BG(P) & \xrightarrow{B\pi} & BG \\
\searrow{\approx} & & \searrow{\approx}
\end{array}
$$

where $0$ stands for the constant map. Then the evaluation at the basepoint $e : \text{map}(B, BG; \alpha) \to BG$ is a model for $B\pi : BG(P) \to BG$ and this leads us to the following definition of $H(k, l)$-spaces. Let $i_k : P^k\Omega X \to P^{\infty}\Omega X \simeq X$ denote the canonical inclusion.

**Definition 5.1.** A space $X$ is called an $H(k, l)$-space if there is a map $m : P^k\Omega X \times P^l\Omega X \to X$ satisfying a homotopy commutative diagram:

$$
\begin{array}{ccc}
P^k\Omega X \vee P^l\Omega X & \xrightarrow{i_k \vee i_l} & X \\
\downarrow{j} & & \downarrow{m} \\
P^k\Omega X \times P^l\Omega X & \xrightarrow{m} & X,
\end{array}
$$

where $j$ is the inclusion.

It is obvious that an $H(k, l)$-space is an $H(k', l')$-space if $k \geq k'$ or $l \geq l'$. The loop space of an $H(1, 1)$-space is homotopy commutative and an $H(\infty, \infty)$-space is an H-space. The loop spaces of $H(k, l)$-spaces give intermediate states between H-spaces and the loop spaces of H-spaces which will be discussed in section 7. On the other hand, an $H(\infty, k)$-space is Aguadé’s $T_k$-space [1]. In particular, an $H(1, \infty)$-space is Aguadé’s $T$-space and this can be seen also by the fibrewise homotopy equivalence $adEG \simeq LBG$ over $BG$, where $LX$ is the free loop space of $X$.

An $H(k, l)$-space is defined to satisfy the following lemma:
Lemma 5.2. If a classifying space of a topological group $G$ is an $H(k, l)$-space, then there is an $A_n$-splitting of the exact sequence $1 \to G_0(E^kG) \to G(E^kG) \xrightarrow{p} G \to 1$.

Proof. Recall first from [20] that, for $A_\infty$-spaces $X, Y$, a map $f : X \to Y$ is an $A_n$-map if and only if its adjoint $\bar{f} : \Sigma X \to P^\infty Y$ extends to $P^n X \to P^\infty Y$ up to homotopy.

Suppose that $X$ is an $H(k, l)$-space by $m : P^k \Omega X \times P^l \Omega X \to X$. Then, by the exponential law, the adjoint of $m$ restricts to a map $\hat{m} : \Sigma \Omega X \to \map(P^k \Omega X, X; i_k)$ such that $e \circ \hat{m} \simeq i_1$, where $e : \map(P^k \Omega X, X; i_k) \to X$ is the evaluation at the basepoint. Then the adjoint of $\hat{m}$, say $\tilde{m} : \Omega X \to \Omega \map(P^k \Omega X, X; i_k)$, is a homotopy section of $\Omega e$ and thus $\tilde{m}$ is an $A_n$-map.

Therefore, by Lemma 2.1 and (5.3), Lemma 5.2 is established.

Proof of Theorem 1.2. It is well-known that $\text{cat} B \leq k$ if and only if $i_k : P^k \Omega B \to B$ admits a homotopy section. Then, by naturality of $i_k$, if $\text{cat} B \leq k$, each map $f : B \to BG$ admits a map $\bar{f} : B \to P^k G$ such that $i_k \circ \bar{f} \simeq f$. This implies that $\bar{f}^{-1} P^k G \simeq P$ and then an $A_n$-section for $\pi : G(E^k G) \to G$ induces that of $\pi : G(P) \to G$. Thus, by Lemma 5.2, the proof is completed.

6 Investigating $H(1, n)$-spaces

In the previous section, we have obtained the universal $G(P)$-bundle (5.1). Then it follows from (5.2) that there is a homotopy equivalence $\varphi : \map^G(P, EG; \alpha) / G_0(P) \to \map_0(B, BG; \alpha)$ and $\bar{\varphi} : B G_0(P) \to \map^G(P, EG; \alpha) / G_0(P)$ such that the following diagram of fibre sequences is homotopy commutative.

$$
\begin{array}{cccccc}
G & \xrightarrow{\delta} & BG_0(P) & \xrightarrow{B\alpha} & BG(P) & \xrightarrow{B\pi} & BG \\
\downarrow \delta & \cong & \downarrow \varphi & \cong & \downarrow \delta & \cong & \\
G / G_0(P) & \to & \map^G(P, EG; f) / G_0(P) & \to & \map^G(P, EG; \alpha) / G(P) & \to & BG \\
\downarrow \delta & \cong & \downarrow \varphi & \cong & \downarrow \delta & \cong & \\
G & \xrightarrow{\delta} & \map_0(B, BG; \alpha) & \xrightarrow{e} & BG \\
\end{array}
$$

The aim of this section is to study the connecting map $\delta$ and characterize $H(1, n)$-spaces by it. Consider the following commutative diagram.

$$
\begin{array}{cccccc}
G & \xrightarrow{\delta} & \map_0(BG, BG; 1) & \xrightarrow{\alpha^*} & \map(BG, BG; 1) & \xrightarrow{e} & BG \\
\downarrow \delta & \cong & \downarrow \alpha^* & & \downarrow \alpha^* & & \\
G & \xrightarrow{\delta} & \map_0(B, BG; \alpha) & \xrightarrow{e} & \map(B, BG; \alpha) & \xrightarrow{e} & BG \\
\end{array}
$$

Then it is sufficient to consider the universal connecting map $\delta : G \to \map_0(BG, BG; 1)$.
Let us construct an alternative universal $G$-bundle to describe the connecting map $\delta$. Following Milnor [16], we denote an element of $EG$ by $t_0g_0 \oplus t_1g_1 \oplus \cdots$ for $\sum_i t_i = 1$, $t_i \geq 0$ and $g_i \in G$ such that finite $t_i$'s are positive. The basepoint of $EG$ is $1e \oplus 0 \oplus 0 \oplus \cdots$, where $e$ is unity of $G$. For $g \in G$, we denote by $\xi_g$ the principal bundle map

$$EG \to EG, \ t_0g_0 \oplus t_1g_1 \oplus \cdots \mapsto t_0g^{-1}g_0 \oplus t_1g^{-1}g_1 \oplus \cdots.$$ 

Then we have a commutative diagram:

$$EG \xrightarrow{\xi_g} EG \xrightarrow{\delta} BG \implies map_0(BG, BG; 1).$$

Now we let $G$ act on $E_0 \times EG$ from right by

$$(f, x) \cdot g = (\xi_{\pi(g)^{-1}} \circ f \circ g, x \cdot \pi(g))$$

for $g \in G$ and $(f, x) \in E_0 \times EG$. One can easily check that this action is free and then we have established the universal $G$-bundle

$$G \to E_0 \times EG \to (E_0 \times EG)/G.$$
Thus there exist a homotopy equivalence $E/G \to (E_0 \times EG)/G$ and a $G$-equivariant homotopy equivalence $\nu : E \to E_0 \times EG$ by which the diagram

$$
\begin{array}{cccccc}
G & \longrightarrow & E & \longrightarrow & E/G & \longrightarrow & BG \\
\| & & \| & & \| & & \| \\
G & \longrightarrow & E_0 \times EG & \longrightarrow & (E_0 \times EG)/G & \longrightarrow & BG
\end{array}
$$

commutes up to homotopy. Since the above diagram is that of $G_0$-spaces and $G_0$-equivariant maps, we obtain a homotopy commutative diagram:

$$
\begin{array}{cccccc}
G & \longrightarrow & G/G_0 & \longrightarrow & E/G_0 & \longrightarrow & E/G & \longrightarrow & BG \\
\| & & \| & & \| & & \| & & \| \\
G & \longrightarrow & (E_0 \times EG)/G_0 & \longrightarrow & (E_0 \times EG)/G & \longrightarrow & BG
\end{array}
$$

Note that the above action of $G$ on $E_0 \times EG$ restricts to the product of the usual action of $G_0$ on $E_0$ and the trivial action of $G_0$ on $EG$. Then we have $(E_0 \times EG)/G_0 = E_0/G_0 \times EG$ and thus the first projection $\pi_1 : E_0 \times EG \to E_0$ induces a homotopy equivalence $\tilde{\pi}_0 : (E_0 \times EG)/G_0 \cong E_0/G_0$. Since $\text{map}^{G_0}(E_0, E_0)$ is contractible, in particular, path connected, the $G_0$-equivariant map $\pi_1 \circ \nu \circ \kappa$ is homotopic to the identity of $E_0$ as $G_0$-equivariant maps. Then, by (6.2), we have established a homotopy commutative diagram:

Therefore we have obtained:

**Lemma 6.1.** There is a homotopy commutative diagram:

$$
\begin{array}{cccccc}
G & \longrightarrow & E_0/G_0 & \longrightarrow & (E_0 \times EG)/G_0 & \longrightarrow & (E_0 \times EG)/G \\
\| & & \| & & \| & & \| \\
G & \longrightarrow & E/G_0 & \longrightarrow & E/G & \longrightarrow & BG \\
\| & & \| & & \| & & \| \\
G & \longrightarrow & \text{map}_0(BG, BG; 1) & \longrightarrow & \text{map}(BG, BG; 1)
\end{array}
$$

In particular, the connecting map $\delta$ is $\text{Bad}$.

Theorem 1.3 follows from (6.1).
\section{\textbf{\textit{C}}(k, l)-space}

In this section, we discuss a relation between $H(k, l)$-spaces and higher homotopy commutativity as promised in section 5. Higher homotopy commutativity was first introduced by Sugawara \cite{21} as intermediate states between loop spaces and loop spaces of H-spaces. Later, Williams \cite{23} introduced another kind of higher homotopy commutativity using associahedra in section 2. Recently, Hemmi and Kawamoto \cite{12} studied a relation between those higher homotopy commutativity, Aguadé’s $T_k$-spaces \cite{1} and Félix and Tanrè’s $H(n)$-spaces \cite{6}. In order to relate them, They introduced $H_k(n)$-spaces and $C_k(n)$-spaces. $H_k(n)$-spaces collect Aguadé’s $T_k$-spaces and Félix and Tanrè’s $H(n)$-spaces whose definition is given by a sequence of $H(k, l)$-spaces for $k + l = n$ (See \cite{12}). On the other hand, $C_k(n)$-spaces are defined as follows by using Gel’fand, Kapranov and Zelevinsky’s polytopes called resultohedra (See \cite{9}, \cite{10} for definition of resultohedra).

Let $R_+ = \{x \in R | x \geq 0\}$. The resultohedron $N_{m,n}$ is an $(m + n - 1)$-dimensional polytope in $R_+^{m+n+2}$ which consists of all points $(p_0, \ldots, p_m, q_0, \ldots, q_n) \in R_+^{m+n+2}$ satisfying:

$$\sum_{i=0}^{m} p_i = n, \quad \sum_{i=0}^{n} q_i = m, \quad h_{i,j} \geq 0, \quad h_{m,n} = 0,$$

where

$$h_{i,j} = \sum_{k=0}^{i} (i - k)p_k + \sum_{l=0}^{j} (j - l)q_l - ij$$

(7.1)

for $0 \leq i \leq m$ and $0 \leq j \leq n$. Then, in particular, $N_{0,0}$ is the one point set and $N_{k,1}$ and $N_{1,k}$ are affinely homeomorphic to the $k$-simplex $\Delta^k$. Vertices of $N_{m,n}$ is labelled by integer lattice paths from $(0, 0)$ to $(m, n)$.

For $x = p_i, q_j$ and $h_{i,j}$ in (7.1), we put

$$N(x) = \{(p_0, \ldots, p_m, q_0, \ldots, q_n) \in N_{m,n} | x = 0\}.$$

Gel’fand, Kapranov and Zelevinsky \cite{10} described the face maps

$$\epsilon^{(p_i)} : N_{m-1,n} \rightarrow N(p_i), \quad \epsilon^{(q_j)} : N_{m,n-1} \rightarrow N(q_j), \quad \epsilon^{(h_{i,j})} : N_{i,j} \times N_{m-i,n-j} \rightarrow N(h_{i,j}).$$

On the other hand, Hemmi and Kawamoto \cite{12} described the degeneracy maps

$$\delta_i : N_{m,n} \rightarrow N_{m-1,m}, \quad \delta'_j : N_{m,n} \rightarrow N_{m,n-1}.$$

Now a $C_k(n)$-space is defined by a coherent sequence of maps $Q_{r,s} : N_{r,s} \times X^{r+s} \rightarrow X$ for a topological monoid $X$, $r + s \leq n$ and $s \leq k$ (See \cite{12} for precise definition). The main result of \cite{12} is:
**Theorem 7.1** ([12, Theorem A]). A connected topological monoid is a $C_k(n)$-space if and only if its classifying space is an $H_k(n)$-space.

As noted above, definition of an $H_k(n)$-space is a collection of that of $H(k,l)$-spaces for $k + l \leq n$ and, actually, the proof of Theorem 7.1 is done by collecting constructions on $H(k,l)$-spaces. Then, by defining $C(k,l)$-spaces as follows which is a modification of that of $C_k(n)$-spaces, we obtain Theorem 1.4.

**Definition 7.1.** A topological monoid $X$ is a $C(k,l)$-space if there exists a sequence of maps $Q_{r,s} : N_{r,s} \times X^{r+s} \to X$ for $0 \leq r \leq k$ and $0 \leq s \leq l$ satisfying:

$$
Q_{r,0}(*, x_1, \ldots, x_r) = x_1 \cdots x_r, \quad Q_{0,s}(*, y_1, \ldots, y_s) = y_1 \cdots y_s
$$

$$
Q_{r,s}(\epsilon^{(p_i)}(\sigma), x_1, \ldots, x_r, y_1, \ldots, y_s) =
\begin{cases} 
  x_1 \cdot Q_{r-1,s}(\sigma, x_2, \ldots, y_s) & i = 0 \\
  Q_{r-1,s}(\sigma, x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_s) & 0 < i < r \\
  Q_{r-1,s}(\sigma, x_1, \ldots, x_{r-1}, y_1, \ldots, y_s) & i = r \\
  y_1 \cdot Q_{r,s-1}(\sigma, x_1, \ldots, x_r, y_2, \ldots, y_s) & j = 0 \\
  Q_{r,s-1}(\sigma, x_1, \ldots, y_j y_{j+1}, \ldots, y_s) & 0 < j < s \\
  Q_{r,s-1}(\sigma, x_1, \ldots, y_{s-1}) & j = s
\end{cases}
$$

$$
Q_{r,s}(\epsilon^{(h_{i,j})}(\sigma_1, \sigma_2), x_1, \ldots, x_r, y_1, \ldots, y_s) = Q_{i,j}(\sigma_1, x_1, \ldots, x_i, y_1, \ldots, y_j) \cdot Q_{r-i,s-j}(\sigma_2, x_{i+1}, \ldots, x_y, y_{j+1}, \ldots, y_s)
$$

$$
Q_{r,s}(\sigma, x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, y_s) = Q_{r-1,s}(\delta_i(\sigma), x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, y_s)
$$

$$
Q_{r,s}(\sigma, x_1, \ldots, y_{j-1}, *, y_{j+1}, \ldots, y_s) = Q_{r,s-1}(\delta_j(\sigma), x_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_s)
$$

**References**


