Note on semiclassical analysis for nonlinear Schrödinger equations

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Preface

This article is a de facto doctoral thesis of the author. This consists of the summary of the author's research during the ph.D. course of the Kyoto university. The de jure doctoral thesis is another one which has the same contents as [58].

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Contents

Notations

We list the notations and collect the definitions that we use throughout this article.

(i) $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{N}$ denote the set of complex number, real number, non-negative number, integer, and positive integers.

(ii) We denote by $\mathbb{R}^n$ the Euclidean n-dimensional space with point $x = (a_1, \ldots, x_n)$.

(iii) $A := B$ means that A is defined by B.

(iv) Let u: $\mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}^n \to \mathbb{R}$) is smooth. ∂_{x_i} denotes the partial derivatives of a function u with respect to x_i. We sometimes write ∂_i, for short. When $n = 1$, we denote u' the derivative of a function u. Moreover, ∂^α denotes $(\partial^{\alpha_1}/\partial_{x_1}^{\alpha_1}) \cdots (\partial^{\alpha_n}/\partial_{x_n}^{\alpha_n})$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

(v) We denote by ∇u the gradient of a function u, that is, $\nabla u = (\partial_1 u, \ldots, \partial_n u)$. When $n = 1$, we use d/dx instead of ∇.

(vi) Δ stands for the Laplacian on $\mathbb{R}^n$, that is, $\Delta = \sum_{i=1}^n \partial^2/d_i^2$. When $n = 1$, we use d^2/dx^2 instead of Δ.

(vii) ∇^2 denotes the tensor product of ∇, that is, $\nabla^2 u$ is the $n \times n$ matrix $(\partial_i \partial_j u)_{1 \leqslant i,j \leqslant n}$.

(viii) $f(x) = O(g(x))$ as $x \to x_0$ means that $|f(x)/g(x)|$ is bounded as $x \to x_0$. Moreover, $f(x) = o(g(x))$ as $x \to x_0$ means that $|f(x)/g(x)|$ tends to zero as $x \to x_0$.

(ix) $C^k(\mathbb{R}^n)$ stands for the set of k-time differentiable function on $\mathbb{R}^n$, and $C^\infty(\mathbb{R}^n) := \cap_{k \geqslant 0} C^k(\mathbb{R}^n)$ for the set of infinitely differentiable function on $\mathbb{R}^n$. $C_0^\infty(\mathbb{R})$ is the set of infinitely differentiable function with compact support.

(x) Let I be an interval of $\mathbb{R}$ and let X be a Banach space. $C^k(I, X)$ is the space of k-time continuously differentiable function from I to X.

(xi) $L^p(\mathbb{R}^n)$ denotes the Banach space of measurable functions $u\colon \mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}^n \to \mathbb{R}$) such that $\|u\|_{L^p(\mathbb{R}^n)} < \infty$ with

$$\|u\|_{L^p(\mathbb{R}^n)} = \begin{cases} \left(\displaystyle\int_{\mathbb{R}^n} |u(x)|^p dx\right)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |u(x)| & \text{if } p \in \infty. \end{cases}$$

We write L^p if there is no risk of confusion.

(xii) $\mathcal{S}(\mathbb{R}^n)$ is the set of Schwartz function (rapidly decreasing function) on $\mathbb{R}^n$. $\mathcal{S}'(\mathbb{R}^n)$ is the set of tempered distributions.

(xiii) $\mathcal{F}$ denotes the Fourier transform

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

(xiv) For $s \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we define $(1-\Delta)^{a/2}$ as $((1-\Delta)^{a/2}f)(x) = \mathcal{F}^{-1}[(1+|\xi|^2)^{a/2}\mathcal{F}f](x)$, and $|\nabla|^a$ as $(|\nabla|^a f)(x) = \mathcal{F}^{-1}[|\xi|^a\mathcal{F}f](x)$. We sometimes denote $(1-\Delta)^{1/2}$ by Λ.

(xv) For $s \in \mathbb{R}$ and $p \in [1,\infty]$, $W^{s,p}(\mathbb{R}^n)$ denotes the Sobolev space, that is, Banach space of functions $u\colon \mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}^n \to \mathbb{R}$) equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} := \left\|(1-\Delta)^{s/2}u\right\|_{L^p} < \infty.$$

If there is no risk of confusion, we write $W^{s,p}$.

(xvi) $H^s(\mathbb{R}^n) := W^{s,2}(\mathbb{R}^n)$. If there is no risk of confusion, we write H^s.

Chapter 1

Introduction

1.1 Introduction

In this article, we consider the Cauchy problem of the semiclassical nonlinear Schrödinger equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = N(|u^\varepsilon|)u^\varepsilon; \quad u^\varepsilon(0,x) = u_0(x), \tag{NLS}$$

where $u^\varepsilon = u^\varepsilon(t,x)$ is a complex-valued function on $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. ε is a positive parameter corresponding to the scaled Planck's constant $\varepsilon \sim \hbar$ and N denotes the nonlinearity. We are concerned with the problem of semiclassical limit $\varepsilon \to 0$. The aim of this article is to describe the results about the asymptotic behavior of the solution u^ε in this limit. In particular, we are interested in a phase-amplitude approximation, called WKB approximation, of the solution u^ε:

$$u^\varepsilon(t,x) = e^{i\frac{\phi_0(t,x)}{\varepsilon}}(a_0(t,x) + \varepsilon a_1(t,x) + \varepsilon^2 a_2(t,x) + \cdots), \tag{1.1.1}$$

where ϕ_0 is a real-valued function and a_i $(i = 0,1,2,\dots)$ is a complex-valued function.

The nonlinear Schrödinger equations appears in many physical contexts. For example, (NLS) with the quintic nonlinearity $N(u^\varepsilon)u^\varepsilon = |u^\varepsilon|^4 u^\varepsilon$ is sometimes used as a model for one-dimensional Bose-Einstein condensation in space dimension $n = 1$. When $n = 2$ or $n = 3$, a cubic nonlinearity $N(u^\varepsilon)u^\varepsilon = |u^\varepsilon|^2 u^\varepsilon$ is usually considered. The Schrödinger-Poisson system ((SP) below) is studied as the fundamental equation in semiconductors appication, with $b > 0$ standing for a constant background charge and $\lambda \gg 1$ being the reciprocal of the square of the Debye number.

In Chapter 2, we justify the WKB approximation of the solution to (NLS) with some typical nonlinearities in a time interval which is small (in general) but independent of ε. For this approximation, several approaches are known. We follow the one by the pioneering works of Gérard [30] and

Grenier [34], for the NLS with local nonlinearity. It consists in using the modified Madelung transform $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$. Then, it turns out that the problem boils down to the analysis of the system

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon \cdot \nabla)a^\varepsilon + \dfrac{1}{2}a^\varepsilon \Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t\phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + N(|a^\varepsilon|) = 0, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0). \end{cases} \tag{1.1.2}$$

Note that $u^\varepsilon = a^\varepsilon e^{i\frac{\phi^\varepsilon}{\varepsilon}}$ is an exact solution of (NLS) if $(a^\varepsilon, \phi^\varepsilon)$ solves (1.1.2). The main purpose of this chapter is the following two points: First is to clarify the difficulty of this method which appears in all kinds of nonlinearity. Second is to present how to overcome this difficulty with some typical examples. In this chapter, we treat the (essentially) cubic nonlinear Schrödinger equation, Schrödinger-Poisson system, Hartree equation, and Hartree equation with local nonlinearity. For cubic nonlinear Schrödinger equation, we give a slightly different formulation of the proof of the result in [34], and generalized this result into the above typical equations. Though the basic strategy of the proof is the same. However, the details are quite different and so far there is no general theory to treat them at once.

In Chapter 3, we turn to the analysis of classical trajectories. It is a fundamental principle in quantum mechanics that, when the time and distance scales are large enough relative to the Planck constant $\hbar$, the system will approximately obey the laws of classical Newtonian mechanics. The equations (1.1.2) is a kind of quantum hydrodynamics equations, which is classical hydrodynamics equations with a quantum correction term. In the limit, the Euler equation for an isentropic compressible flow is formally recovered from the nonlinear Schrödinger equation. Indeed, denoting $\rho := \lim_{\varepsilon\to 0} |a^\varepsilon|^2$ and $v := \lim_{\varepsilon\to 0} \nabla\phi^\varepsilon$ for a solution of (1.1.2), one verifies that, at least formally, (ρ, v) solves the Euler equation

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + (v\cdot\nabla)v + \nabla N(\sqrt{\rho}) = 0, \end{cases} \tag{1.1.3}$$

which are the statements of the conservation of mass and Newton's second law, respectively. In this chapter, we analyze (1.1.3) by using the method of characteristic curves. The characteristic curve of v is called *classical trajectory* in the context of Schödinger equations. It is known that when the characteristic curves cross each other, the solution to this equation breaks down by a formation of singularity, a shock. This is also related to the theory of geometrical optics. The classical trajectory is an analogue of the notion of *ray* developed initially to describe the propagation of electro-magnetic waves, such as light. The breakdown of the solution of (1.1.3) is closely related to the occurrence of caustics. We choose the Schrödinger-Poisson system with constant background $b \geqslant 0$:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda V_{\mathrm{P}} u^\varepsilon, \quad -\Delta V_{\mathrm{P}} = |u^\varepsilon|^2 - b, \quad V_{\mathrm{P}} \to 0 \text{ as } |x| \to \infty \quad \text{(SP)}$$

and corresponding Euler-Poisson equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \partial_t v + (v \cdot \nabla)v + \lambda \nabla V_{\mathrm{P}} = 0, \\ -\Delta V_{\mathrm{P}} = \rho - b, \quad V_{\mathrm{P}} \to 0 \text{ as } |x| \to \infty \end{cases} \quad \text{(EP)}$$

as a target equation of this chapter. Under the radial symmetry, we derive the necessary and sufficient conditions that ensure the classical solution to (EP) is global. In the case where $b > 0$ and $n = 1$ and the case where $b = 0$ and $n \geqslant 1$ (and some more case such as the presence of relaxation term) are studied precisely in [25]. We stem the missing parts and give complete descriptions of the necessary and sufficient conditions for all $n \geqslant 1$ and $b \geqslant 0$. As a result, we will see that, under the assumption that $n \geqslant 3$, ρ is integrable, and v decays at spatial infinity (and they are radially symmetric), there is only one possible form of the initial data which admits the global solution (Theorem 3.3.14).

In Chapter 4, we justify the WKB approximation (1.1.1) of the solution to Shcrödinger-Poisson system (SP) for large time. In one dimensional case, Liu and Tadmor [46] show by applying the result [25] that, for a class of initial data admitting a global solution of (EP), (1.1.1) holds for an interval which depends on the parameter ε and becomes arbitrarily large as $\varepsilon \to 0$. In this chapter, we generalize this result to the $n \geqslant 3$ case. This is done by a combination of results in Chapters 2 and 3. An example of global solution to (EP) is given by the results in Chapter 3. Then, using (the modified version of) the analysis in Chapter 2, we justify (1.1.1) for large time.

1.2 Several remarks on function spaces

We give several remarks on the function spaces which we use throughout this article.

1.2.1 Lebesgue space

The first is the Lebesgue space $L^p(\mathbb{R}^n)$. In this article, we often use Lebesgue spaces to investigate the decay property of functions. As usual, it is defined as the Banach space of measurable functions $f \colon \mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}^n \to \mathbb{R}$) such that $\|f\|_{L^p(\mathbb{R}^n)} < \infty$ with

$$\|f\|_{L^p(\mathbb{R}^n)} = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \operatorname*{ess\ sup}_{x \in \mathbb{R}^n} |f(x)| & \text{if } p \in \infty. \end{cases}$$

In general, if a measurable function f is not integrable, then the one of the following holds:

1. There exists a bounded set $\Omega \subset \mathbb{R}^n$ (with arbitrarily small measure) such that $\int_\Omega |f| dx = \infty$.
2. $\int_\Omega |f| dx$ is finite for any bounded set $\Omega \subset \mathbb{R}^n$. However, $\int_\Omega |f| dx \to \infty$ as $|\Omega| \to \infty$.

In the first case, f has a *singularity* at some point. An example of such function is $f(x) = |x|^{-n} \mathbf{1}_{\{|x| \leqslant 1\}}(x)$. In the second case, f is not integrable because the decay of f is not enough. $f(x) \equiv 1$ is an example.

It can be said that the index p of $L^p(\mathbb{R}^n)$ indicates both the strength of the singularity and the rate of the decay. For example, $|x|^{-q} \mathbf{1}_{\{|x| \leqslant 1\}}(x)$ belongs to L^p space if and only if $q < n/p$. Similarly, $|x|^{-r} \mathbf{1}_{\{|x| \leqslant 1\}}(x)$ belongs to L^p space if and only if $r > n/p$. Hence, very roughly speaking, an element of L^p space is a function which has a singularity of order at most $O(|x|^{-\frac{n}{p}+\varepsilon})$ and decays at spatial infinity at least order $O(|x|^{-\frac{n}{p}-\varepsilon})$. L^∞ space is rather special: The functions in L^∞ do not necessarily decay at spatial infinity.

p	small	$\longleftrightarrow$	large	$\cdots$	∞
singularity at a point	strong	$\longleftrightarrow$	weak	$\cdots$	none
decay at spatial infinity	rapid	$\longleftrightarrow$	slow	$\cdots$	none

The Hölder inequality A.1.2 shows that these two properties, singularity and decay, are "monotone" in p. To concentrate on singularities, assume that f is supported on a bounded set $\Omega \subset \mathbb{R}^n$. Then, we have $L^p(\Omega) \supset L^q(\Omega)$ $(1 \leqslant p \leqslant q)$ since

$$\|f\|_{L^p(\Omega)} \leqslant |\Omega|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(\Omega)} \quad \text{for } 1 \leqslant p \leqslant q$$

by Hölder inequality. This suggests that if the singularity of f is so weak that $f \in L^q(\Omega)$ then f automatically belongs to $L^p(\Omega)$. The converse is not true, as the following example shows: For a bounded set $\Omega \ni 0$ and $p < q$, $|x|^{-\frac{2n}{p+q}} \mathbf{1}_\Omega(x) \in L^p(\Omega) \setminus L^q(\Omega)$ since $n/p > 2n/(p+q) > n/q$. To concentrate on the decay property we now assume g is bounded. In this case,

$$\|g\|_{L^p} \leqslant \|g\|_{L^q}^{\frac{q}{p}} \|g\|_{L^\infty}^{1-\frac{q}{p}} \quad \text{for } p \geqslant q \geqslant 1,$$

and so $(L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \subset (L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ for $p \geqslant q \geqslant 1$. This suggests that if the decay of f is so rapid that $f \in L^q(\mathbb{R}^n)$ then this decay is enough for being $f \in L^p(\mathbb{R}^n)$. The converse is false, as the following example shows: For $p > q$, $|x|^{-\frac{2n}{p+q}}\mathbf{1}_{\{|x|\geqslant 1\}}(x) \in L^p(\mathbb{R}^n) \setminus L^q(\mathbb{R}^n)$ since $n/p < 2n/(p+q) < n/q$.

Tail estimate

Let $1 \leqslant p < q \leqslant \infty$. Take a function $f \in L^q \cap L^\infty$ with $f \notin L^p$. If there exists some function g such that $f - g \in L^p \cap L^\infty$ then g is an approximation of the "tail part" of f in such a sense that the decay of $f - g$ is faster than f itself. In this respect, we call such an estimate as a tail estimate.

1.2.2 Sobolev space

The next is the Sobolev space $H^s(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$. For $s \in \mathbb{R}$ and $p \in [1, \infty]$, $W^{s,p}(\mathbb{R}^n)$ denotes the Banach space of functions u: $\mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}^n \to \mathbb{R}$) equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} := \left\|(1-\Delta)^{s/2}u\right\|_{L^p} < \infty.$$

Moreover, $H^s(\mathbb{R}^n) := W^{s,2}(\mathbb{R}^n)$. The Sobolev embedding reveals the connection between the integrability of higher derivative and of lower derivative. Indeed, for $1 \leqslant q \leqslant p < \infty$, we have

$$\|f\|_{L^p} \leqslant C \left\||\nabla|^{\frac{n}{q}-\frac{n}{p}} f\right\|_{L^q}.$$

It suggests the fundamental principle that the differentiation makes the singularity stronger and the decay faster. Indeed, for $|\nabla|^{\frac{n}{q}-\frac{n}{p}} f \in L^q(\mathbb{R}^n)$ ($q < p$) being true, f is required to have so weak singularity that $f \in L^p(\mathbb{R}^n)$ holds, however, the decay is not required no more than $f \in L^p(\mathbb{R}^n)$. The Hardy-Littlewood-Sobolev inequality (Lemma A.1.5) is a counterpart of the Sobolev embedding in a sense. For $\gamma \in (0, n)$, we have

$$\mathcal{F}|x|^{-\gamma} = c_{n,\gamma}|\xi|^{-n+\gamma}$$

(see e.g. [65]). Therefore, the Hardy-Littlewood-Sobolev inequality can be written as

$$\left\||\nabla|^{-(\frac{n}{q}-\frac{n}{p})} f\right\|_{L^p} \leqslant C\,\|f\|_{L^q}$$

for $1 < q < p < \infty$. Hence, it suggests the fundamental principle that the integration makes the singularity weaker and the decay slower.

1.2.3 Zhidkov space

We also use the Zhidkov space $X^s(\mathbb{R}^n)$ and its modified space $Y^s_{p,q}(\mathbb{R}^n)$. The Zhidkov space is defined as follows: For $s > n/2$,

$$X^s(\mathbb{R}^n) := \{u \in L^\infty(\mathbb{R}^n) | \nabla u \in H^{s-1}(\mathbb{R}^n)\}.$$

The norm of X^s is given by

$$\|\cdot\|_{X^s(\mathbb{R}^n)} := \|\cdot\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \cdot\|_{H^{s-1}(\mathbb{R}^n)}.$$

The space was introduced in [74] in the case $n = 1$, and its study was generalized to the multidimensional case in [26] (see also [5, 15]). In general, a function in the Zhidkov space does not have spatial decay at all. For example, constant function $f(x) \equiv 1$ belongs to $X^s(\mathbb{R}^n)$, while it does not belongs to any Lebesgue spaces or Sobolev spaces. However, if $n \geqslant 3$ then we see from Lemma 2.2.1 below that for all $f \in X^s(\mathbb{R}^n)$ there exists a constant C such that $f - C \in L^{2n/(n-2)} \cap L^\infty$. This is a kind of tail estimate. Recall that f itself belongs to L^p only if $p = \infty$. In Chapter 4, we will use the modified one $Y^s_{p,q}(\mathbb{R}^n)$. For $n \geqslant 3$, $s > n/2 + 1$, $p \in [1, \infty]$, and $q \in [1, \infty]$, we define a function space $Y^s_{p,q}(\mathbb{R}^n)$ by

$$Y^s_{p,q}(\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{Y^s_{p,q}(\mathbb{R}^n)}}$$

with norm

$$\|\cdot\|_{Y^s_{p,q}(\mathbb{R}^n)} := \|\cdot\|_{L^p(\mathbb{R}^n)} + \|\nabla \cdot\|_{L^q(\mathbb{R}^n)} + \left\|\nabla^2 \cdot\right\|_{H^{s-2}(\mathbb{R}^n)}.$$

The indices p and q indicate the decay rate of the function and its first derivative, respectively. This is a generalized version of $X^s(\mathbb{R}^n)$ if $n \geqslant 3$. $Y^s_{\infty,2}(\mathbb{R}^n)$ is almost equal to $X^s(\mathbb{R}^n)$. The difference is the fact that all functions in $Y^s_{\infty,2}(\mathbb{R}^n)$ decays at spacial infinity. However, as noted above, for all $f \in X^s(\mathbb{R}^n)$, there exists a constant C such that $f - C \in Y^s_{\infty,2}(\mathbb{R}^n)$. We also deduce form Lemma 2.2.1 that $Y^s_{\infty,2}(\mathbb{R}^n) = Y^s_{\infty,2}(\mathbb{R}^n) \cap Y^s_{2^*,2}(\mathbb{R}^n)$, where $2^* = 2n/(n-2)$. We discuss this space again in Section 4.3.1.

Chapter 2

Small time WKB analysis for nonlinear Schrödinger equations

2.1 Introduction

2.1.1 Equations and main results

In this chapter, we consider the asymptotic behavior of the solution to the Cauchy problem

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = N(|u^\varepsilon|)u^\varepsilon; \quad u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon). \tag{2.1.1}$$

In particular, the purpose of this chapter is to give a WKB-type approximation

$$u^\varepsilon(t,x) \sim e^{i\frac{\phi(t,x)}{\varepsilon}}(b_0(t,x) + \varepsilon b_1(t,x) + \varepsilon^2 b_2(t,x) + \cdots) \tag{2.1.2}$$

in a time interval $[0,T]$ which is small in general but independent of ε. We construct suitable phase ϕ and amplitude b_i, and provide a pointwise description of u^ε as $\varepsilon \to 0$ in the following cases:

- Defocusing nonlinearity which is cubic at the origin: $N(|u^\varepsilon|) = f(|u^\varepsilon|^2)$ with $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f \geqslant 0$, $f' > 0$, and $f(0) = 0$.
- Focusing or defocusing nonlocal nonlinearity: $N(|u^\varepsilon|) = \pm(-\Delta)^{-1}|u^\varepsilon|^2$ or $N(|u^\varepsilon|) = \pm(|x|^{-\gamma} * |u^\varepsilon|^2)$.

We also analyze the case where the nonlinearity is the sum of the above two types. More precisely, the target equations of this chapter are the following:

1. The *defocusing (essentially) cubic nonlinear Schrödinger equation*

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f(|u^\varepsilon|^2)u^\varepsilon; \quad u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon) \tag{CNLS}$$

   with $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f \geqslant 0$, $f'(y) > 0$, and $f(0) = 0$. This is treated in [34] (for generalized or other types of local nonlinearities, see [3, 4, 13, 20, 23, 28, 44]).

2. The *Schrödinger-Poisson system* without background

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda V_{\mathrm{P}}^\varepsilon u^\varepsilon, \\ -\Delta V_{\mathrm{P}}^\varepsilon = |u^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x| \to \infty, \\ u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon), \end{cases} \tag{SP}$$

   where $\lambda = \pm 1$. For this equation, see [5, 43, 46, 72, 73].

3. The *Hartree equation*

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon; \quad u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon), \tag{H}$$

   where $\lambda = \pm 1$. Hartree equation is treated in [15].

4. The nonlinear Schrödinger equation with *local nonlinearity and non-local nonlinearity*

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f(|u^\varepsilon|^2) + \lambda(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon, \\ u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon) \end{cases} \tag{L-NL}$$

   with $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f \geqslant 0$, $f'(y) > 0$, and $f(0) = 0$, and $\lambda = \pm = 1$.

We reformulate and generalize the previous results. Especially, we would like to relax the decay condition on the initial phase Φ_0. This is due to the fact that Φ_0 is not necessarily to be bounded for being $u^\varepsilon(0) = A_0^\varepsilon e^{i\Phi_0/\varepsilon} \in H^s$. From this respect, in the following theorems, we try to make our assumptions on Φ_0 so close to the one "$\nabla\Phi_0$ is bounded (without spatial decay)" as possible.

The following are the main results of this chapter.

Theorem 2.1.1 ([34], WKB analysis for (CNLS))**.** *Let $f \in C^\infty(\mathbb{R}_+ : \mathbb{R}_+)$ with $f(0) = 0$ and $f' > 0$. Let $k \geqslant 1$ be an integer and $s > n/2 + 2k + 4$ be a real number. Assume that $\Phi_0 \in X^{s+1}$, and that A_0^ε writes*

$$A_0^\varepsilon = \sum_{j=0}^{k} \varepsilon^j A_j + o(\varepsilon^k) \quad \text{in } H^s \tag{2.1.3}$$

for $\varepsilon \in [0,1]$. Then, there exist a existence of time $T > 0$ independent of ε and a solution $u^\varepsilon \in C([0,T];H^s)$ of (CNLS). *There also exists $\phi_0 \in C([0,T];X^{s+1})$ and $\beta_j \in H^{s-2j-2}$ such that*

$$u^\varepsilon = e^{i\frac{\phi_0}{\varepsilon}}(\beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^{k-1}\beta_{k-1} + o(\varepsilon^{k-1})) \quad \text{in } C([0,T];H^{s-2k-2}). \tag{2.1.4}$$

Furthermore, ϕ_0 satisfies

$$\phi_0(t,x) - \Phi_0(x) \in W^{s,1}(\mathbb{R}^n).$$

Theorem 2.1.2 (WKB analysis for (SP))**.** *Let $n \geqslant 3$ and $\lambda \in \mathbb{R}$. Let k be a positive integer and let $s > n/2 + 2k + 3$ be a real number. Assume that $\Phi_0 \in C^{2k+5}$ with $\nabla^2\Phi_0 \in H^s$, and that A_0^ε writes* (2.1.3)*. for $\varepsilon \in [0,1]$. Then, there exist a existence of time $T > 0$ independent of ε and a solution $u^\varepsilon \in C([0,T];H^s)$ of* (SP)*. There also exists $\phi_0 \in C([0,T];C^{2k+5})$ and $\beta_j \in H^{s-2j-2}$ such that* (2.1.4) *holds. Furthermore, there uniquely exists a constant c_∞ such that $\nabla\Phi_0 \to c_\infty$ as $|x| \to \infty$ and ϕ_0 satisfies*

$$\begin{aligned} &\nabla\phi_0(t,x) - \nabla\Phi_0(x - c_\infty t) \in (L^{\frac{n}{n-1}+} \cap L^\infty)(\mathbb{R}^n), \\ &\phi_0(t,x) - \Phi_0(x) + \frac{1}{2}\int_0^t |\nabla\Phi_0(x - c_\infty s)|^2 ds \in (L^{\frac{n}{n-2}+} \cap L^\infty)(\mathbb{R}^n). \end{aligned}$$

Theorem 2.1.3 (WKB analysis for (H))**.** *Let $n \geqslant 3$ and $\lambda \in \mathbb{R}$. Let γ be a positive number with $n/2 - 2 < \gamma \leqslant n - 2$. Let k be a positive integer and let $s > n/2 + 2k + 3$ be a real number. Assume that $\Phi_0 \in C^{2k+5}$ with $\nabla^2\Phi_0 \in H^s$, and that A_0^ε writes* (2.1.3)*. or $\varepsilon \in [0,1]$. Then, there exist a existence of time $T > 0$ independent of ε and a solution $u^\varepsilon \in C([0,T];H^s)$ of* (H)*. There also exists $\phi_0 \in C([0,T];C^{2k+5})$ and $\beta_j \in H^{s-2j-2}$ such that* (2.1.4) *holds. Furthermore, there uniquely exists a constant c_∞ such that $\nabla\Phi_0 \to c_\infty$ as $|x| \to \infty$ and ϕ_0 satisfies*

$$\begin{aligned} &\nabla\phi_0(t,x) - \nabla\Phi_0(x - c_\infty t) \in (L^{\frac{n}{\gamma+1}+} \cap L^\infty)(\mathbb{R}^n), \\ &\phi_0(t,x) - \Phi_0(x) + \frac{1}{2}\int_0^t |\nabla\Phi_0(x - c_\infty s)|^2 ds \in (L^{\frac{n}{\gamma}+} \cap L^\infty)(\mathbb{R}^n). \end{aligned}$$

Theorem 2.1.4 (WKB analysis for (L-NL))**.** *Let $n \geqslant 2$ and $\lambda \in \mathbb{R}$. Let $f \in C^\infty(\mathbb{R}_+ : \mathbb{R}_+)$ with $f(0) = 0$ and $f' > 0$. Let $\lambda \in \mathbb{R}$ and let γ be a positive number with $n/2 - 1 < \gamma \leqslant n - 1$. Let k be a positive integer and let $s > n/2 + 2k + 4$ be a real number. Assume that $\Phi_0 \in X^{s+1}$ and A_0^ε writes* (2.1.3)*. for $\varepsilon \in [0,1]$. Then, there exist a existence of time $T > 0$ independent of ε and a solution $u^\varepsilon \in C([0,T];H^s)$ of* (L-NL)*. There also exists $\phi_0 \in C([0,T];X^{s+1})$ and $\beta_j \in H^{s-2j-2}$ such that* (2.1.4) *holds. Furthermore, ϕ_0 satisfies*

$$\phi_0(t,x) - \Phi_0(x) \in (L^{\frac{n}{\gamma}+} \cap L^\infty)(\mathbb{R}^n).$$

Remark 2.1.5. The assumption on the phase function Φ_0 reflects the shape of nonlinearity. Under the assumption in above theorems, Φ_0 does not necessarily decay at spatial infinity and moreover is not necessarily bounded, in general. In Theorems 2.1.1 and 2.1.4, we assume $\Phi_0 \in X^{s+1}$. In this case, Φ_0 is always bounded but does not necessarily decay. For the $n \geqslant 3$ case, Lemma 2.2.1 below shows the existence of the constant $c_0 \in \mathbb{R}$ such that $\Phi_0 - c_0 \in L^{2^*}$. On the other hand, the assumption for Theorems 2.1.2 and 2.1.3 is $\Phi_0 \in C^{2k+5}$ and $\nabla^2\Phi_0 \in H^s$. This class is much larger than $X^{s+1}(\mathbb{R}^n)$. Especially, this Φ_0 can tend to infinity as $|x| \to \infty$. Lemma 2.2.1 below shows that there exists a constant $c_\infty \in \mathbb{R}^n$ such that $\nabla\Phi_0 - c_\infty \in L^{2^*}$ since $n \geqslant 3$, and moreover that there exists a constant c_0 such that $\Phi_0 - c_0 - c_\infty \cdot x \in L^{2^{**}}$ if $n \geqslant 5$, where $2^{**} = (2^*)^* = 2n/(n-4)$. Nevertheless, in all above theorems, we can construct a function $P(t,x)$ explicitly given only by Φ_0 such that $\phi_0(t) - P(t)$ decays at spacial infinity as long as solution exists. The decay rate of this difference also reflects the shape of nonlinearity. One of the most remarkable difference between Theorems 2.1.1 and 2.1.4 is this point.

Remark 2.1.6. We need $f' > 0$ in Theorems 2.1.1 and 2.1.4 because the quantity $1/f'$ appears when we estimate the energy. If we try to treat more general nonlinearity such as quintic nonlinearity $f(y) = y^2$ then what prevents us is the fact that $f'(0) = 0$. We refer to [3, 4, 20] for such generalized local nonlinearities.

Remark 2.1.7. It is remarked in [12, 62] that the above WKB analysis and a geometrical transform can help understand the behavior of a wave function near a focal point, in a supercritical régime (see also [10, 11, 13, 14, 16, 60, 61]).

2.1.2 Two different approaches

We now address an outline for the method to justify the WKB approximation (2.1.2) (see also [13, 28, 66]). One approach to obtain a WKB-type estimate is to use Madelung's transform

$$u^\varepsilon(t,x) = \sqrt{\rho^\varepsilon(t,x)}e^{i\frac{S^\varepsilon(t,x)}{\varepsilon}}.$$

Plugging this to (2.1.1) and separating real and imaginary part, we find the quantum Euler equation

$$\begin{cases} \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon\nabla S^\varepsilon) = 0, \\ \partial_t\nabla S^\varepsilon + (\nabla S^\varepsilon\cdot\nabla)\nabla S^\varepsilon + \nabla N(\sqrt{\rho^\varepsilon}) = \varepsilon^2\nabla\left(\dfrac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right), \\ (\rho^\varepsilon(0,x), \nabla S^\varepsilon(0,x)) = (|A_0^\varepsilon|^2, \nabla(\Phi_0 + \varepsilon\arg A_0^\varepsilon)). \end{cases} \tag{2.1.5}$$

The term $\varepsilon^2\nabla(\Delta\sqrt{\rho^\varepsilon}/\sqrt{\rho^\varepsilon})$ is called quantum pressure. The equations (2.1.5) represent a fluid dynamics formulation of the (2.1.1) and are known as Madelung's fluid equations [48, 49]. Taking $\varepsilon \to 0$, we obtain, at least formally, the compressible Euler equation

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + (v\cdot\nabla)v + \nabla N(\sqrt{\rho}) = 0, \\ (\rho(0,x), v(0,x)) = (|A_0|^2, \nabla\Phi_0), \end{cases} \tag{2.1.6}$$

where $\rho = \lim_{\varepsilon\to 0}\rho^\varepsilon$, $v = \lim_{\varepsilon\to 0}\nabla S^\varepsilon$, and $A_0 = \lim_{\varepsilon\to 0} A_0^\varepsilon$. With this method, the convergence of the quadratic quantities

$$|u^\varepsilon|^2 \to \rho, \quad \varepsilon\operatorname{Im}(\overline{u^\varepsilon}\nabla u^\varepsilon) \to \rho v$$

as $\varepsilon \to 0$ is proved in several situations. The Wigner measure is one of the strong tool for justifying this limit For this limit and the Wigner measure, consult [13, 29, 32, 37, 38, 45, 55, 56, 57, 72, 73] and references therein. Though this convergence suggests that the solution u^ε may have the asymptotics of the form $u^\varepsilon = e^{iS/\varepsilon}(\sqrt{\rho}+o(1))$, it is not satisfactory. In particular, the argument of the solution u^ε is not clear. In fact, the asymptotics $e^{iS/\varepsilon}(\sqrt{\rho}+o(1))$ is not true. The leading order term of the amplitude of the approximate solution cannot be expected to be real-valued, even if so is this at the initial time.

Another way to justify (2.1.2) is to employ a modified Madelung transform

$$u^\varepsilon = a^\varepsilon e^{i\frac{\phi^\varepsilon}{\varepsilon}} \tag{2.1.7}$$

and consider the system

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon\cdot\nabla)a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t\phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + N(|a^\varepsilon|) = 0, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0). \end{cases} \tag{2.1.8}$$

It is essential that a^ε takes complex value. and so, $S^\varepsilon \neq \phi^\varepsilon$, in general. If we know this system has a solution $(a^\varepsilon, \phi^\varepsilon)$ and the solution can be expanded as

$$a^\varepsilon = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots, \quad \phi^\varepsilon = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots,$$

then, by means of (2.1.7), we obtain WKB type estimate (2.1.2) with $b_0 = a_0 e^{i\phi_1}$. The explicit formulae of higher order terms of amplitude b_i are given in Section 2.2.4. This method is first applied to (CNLS) within the framework of analytic function spaces in [30] and of Sobolev spaces [34] for (CNLS). We remark that $(|a_0|^2, \nabla\phi_0)$ solves the compressible Euler equation (2.1.6).

In this chapter, we use the second method to justify the WKB approximation (2.1.2) for (CNLS), (SP), (H), and (L-NL) in a time interval which is small but independent of ε. We first clarify the difficulty and illustrate the general strategy of the proof in Section 2.2. Then, we consider (CNLS) in Section 2.3. We give a slightly different proof from [34] which is based on the modified energy method. Section 2.4 is devoted to the study of the equation with nonlocal nonlinearities, (SP) and (H). In final Section 2.5, we treat (L-NL).

2.2 General strategy and problem

In this paragraph, we show an outline for obtaining the WKB type approximation (2.1.2) of the solution of (2.1.1). No rigorous result is given through this Section 2.2, although we make some observation with calculations which we use in later sections.

We follow the approach by Grenier [34] (the second one introduced in Section 2.1.2) and work with a data in Sobolev space: We apply the modified Madelung transform (2.1.7) to (2.1.1) and consider the system (2.1.8) for amplitude a^ε and phase ϕ^ε. Let us introduce a new variable $v^\varepsilon = \nabla\phi^\varepsilon$. Differentiating the second equation of (2.1.8), we find

$$\begin{cases} \partial_t a^\varepsilon + (v^\varepsilon \cdot \nabla) a^\varepsilon + \dfrac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla N(|a^\varepsilon|) = 0, \\ \qquad\qquad (a^\varepsilon(0,x), v^\varepsilon(0,x)) = (A_0^\varepsilon, \nabla\Phi_0). \end{cases} \tag{SHS}$$

The main point of Grenier's idea is that this system can be regarded as a *symmetric hyperbolic system* with perturbation. We call this system as (SHS) in this respect. In this section, we give a general strategy to show that (SHS) admits a solution $(a^\varepsilon, v^\varepsilon)$ (Section 2.2.1); that the solution $(a^\varepsilon, \phi^\varepsilon)$ of (2.1.8) can be constructed from $(a^\varepsilon, v^\varepsilon)$ (Section 2.2.2); and that the solution is expanded in powers of ε (Section 2.2.3). Once we obtain this expansion of $(a^\varepsilon, \phi^\varepsilon)$, the WKB approximation (2.1.2) is an immediate consequence (Section 2.2.4).

2.2.1 Existence of the solution and the problem on the energy estimate

Our first step is to show that (SHS) has a solution. We try to obtain a solution $(a^\varepsilon, v^\varepsilon)$ in the class $C([0,T]; (H^s)^2)$. The main part of the proof is a priori estimate. Hence, we shall detail this part only. For other parts of proof, see [15, 50, 68]. Let us go along the classical energy method. Consider the energy

$$E(t) := \|a^\varepsilon\|_{H^s}^2 + \|v^\varepsilon\|_{H^s}^2 .$$

As a matter of fact, we cannot close the energy estimate with this energy. The purpose of this section is only to reveal what is wrong with this energy. In the concrete examples below, we modify this energy. These modifications are considered in Sections 2.3.1, 2.4.1, and 2.5.1.

Let us proceed with the standard energy argument as further as we can. Estimates in this section are quoted sometimes in forthcoming sections. We use the convention for the inner product in L^2:

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}^n} f\overline{g}dx.$$

We also denote $(1-\Delta)^{1/2}$ by Λ. Take $s \geqslant 0$. From the first line of (SHS), we have

$$\begin{aligned}
\frac{d}{dt}\|a^\varepsilon\|_{H^s}^2 &= 2\operatorname{Re}\langle \Lambda^s \partial_t a^\varepsilon, \Lambda^s a^\varepsilon \rangle_{L^2} \\
&= -2\operatorname{Re}\langle \Lambda^s((v^\varepsilon\cdot\nabla)a^\varepsilon), \Lambda^s a^\varepsilon\rangle_{L^2} - \operatorname{Re}\langle \Lambda^s(a^\varepsilon \nabla\cdot v^\varepsilon), \Lambda^s a^\varepsilon\rangle_{L^2} \\
&\quad + \operatorname{Re}\langle i\varepsilon\Lambda^s\Delta a^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2} \\
&=: I_1 + I_2 + I_3. \qquad (2.2.1)
\end{aligned}$$

By integration by parts, we see

$$I_3 = -\operatorname{Re}(i\varepsilon\|\nabla a^\varepsilon\|_{H^s}^2) = 0. \qquad (2.2.2)$$

This fact is one of the most remarkable point of modified Madelung's transform (2.1.7). It is very contrast with the fact that treatment of quantum pressure term often needs some care (such as $\rho^\varepsilon > 0$) when we employ Madlung's transform and work with (2.1.5) (see [72]). Moreover, I_1 is a good term: It writes

$$\begin{aligned}
I_1 &= -2\operatorname{Re}\langle (v^\varepsilon\cdot\nabla)\Lambda^s a^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2} - 2\operatorname{Re}\langle [\Lambda^s, v^\varepsilon\cdot\nabla]a^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2} \\
&= \operatorname{Re}\langle (\nabla\cdot v^\varepsilon)\Lambda^s a^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2} - 2\operatorname{Re}\langle [\Lambda^s, v^\varepsilon\cdot\nabla]a^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2}
\end{aligned}$$

by integration parts. By the Hölder inequality and the commutator estimate (Lemma A.2.2), we obtain

$$|I_1| \leqslant C(\|\nabla v^\varepsilon\|_{L^\infty}\|a^\varepsilon\|_{H^s}^2 + \|\nabla v^\varepsilon\|_{H^{s-1}}\|\nabla a^\varepsilon\|_{L^\infty}\|a^\varepsilon\|_{H^s}). \qquad (2.2.3)$$

Therefore, if we set $s > n/2+1$ then the right hand side is bounded by $C(\|a^\varepsilon\|_{H^s} + \|v^\varepsilon\|_{H^s})^3$. On the other hand, I_2 is rather bad in such a sense that it requires the bound of $(s+1)$-th derivative of v^ε. Indeed, extracting the main part, we have

$$I_2 = -\operatorname{Re}\langle a^\varepsilon\Lambda^s\nabla\cdot v^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2} - \operatorname{Re}\langle [\Lambda^s, a^\varepsilon]\nabla\cdot v^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2}.$$

Remark that integration by parts does not work so well as in the estimate of I_1. Therefore, we estimate as

$$\begin{aligned}|I_2| \leqslant& \|\nabla v^\varepsilon\|_{H^s} \|a^\varepsilon\|_{L^\infty} \|a^\varepsilon\|_{H^s} \\ &+ C(\|\nabla a^\varepsilon\|_{L^\infty} \|\nabla v^\varepsilon\|_{H^{s-1}} + \|\nabla v^\varepsilon\|_{L^\infty} \|\nabla a^\varepsilon\|_{H^{s-1}}) \|a^\varepsilon\|_{H^s}. \quad (2.2.4)\end{aligned}$$

Thus, we need $(s+1)$-th derivative of v^ε to be bounded in a suitable sense. For $s > n/2+1$, the right hand side is bounded by $C(\|a^\varepsilon\|_{H^s} + \|v^\varepsilon\|_{H^{s+1}})^3$. Alternatively, we have

$$\begin{aligned}&|I_2 + \operatorname{Re} \langle a^\varepsilon \Lambda^s \nabla \cdot v^\varepsilon, \Lambda^s a^\varepsilon \rangle_{L^2}| \\ &\quad\leqslant C(\|\nabla a^\varepsilon\|_{L^\infty} \|\nabla v^\varepsilon\|_{H^{s-1}} + \|\nabla v^\varepsilon\|_{L^\infty} \|\nabla a^\varepsilon\|_{H^{s-1}}) \|a^\varepsilon\|_{H^s}. \quad (2.2.5)\end{aligned}$$

If we are able to remove the bad part, then we only need s-time derivative of v^ε. We will see later that this fact is the key for (CNLS) case.

Now let us turn to the estimate of v^ε. We estimate H^s norm of v^ε, although the estimate actually required in (2.2.4) is the H^s norm of ∇v^ε. The estimate of H^s norm of ∇v^ε is an easy modification. From the second line of (SHS), we find

$$\begin{aligned}\frac{d}{dt} \|v^\varepsilon\|_{H^s}^2 &= 2 \operatorname{Re} \langle \Lambda^s \partial_t v^\varepsilon, \Lambda^s v^\varepsilon \rangle_{L^2} \\ &= -2 \operatorname{Re} \langle \Lambda^s((v^\varepsilon \cdot \nabla) v^\varepsilon), \Lambda^s v^\varepsilon \rangle_{L^2} - 2 \operatorname{Re} \langle \Lambda^s \nabla N(|a^\varepsilon|), \Lambda^s v^\varepsilon \rangle_{L^2} \\ &=: I_4 + I_5. \quad (2.2.6)\end{aligned}$$

I_4 is also a good term. The estimate of I_4 is similar to that of I_1: Since

$$\begin{aligned}I_4 &= -2 \operatorname{Re} \langle (v^\varepsilon \cdot \nabla) \Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon \rangle_{L^2} - 2 \operatorname{Re} \langle [\Lambda^s, v^\varepsilon \cdot \nabla] v^\varepsilon, \Lambda^s v^\varepsilon \rangle_{L^2} \\ &= \operatorname{Re} \langle (\nabla \cdot v^\varepsilon) \Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon \rangle_{L^2} - 2 \operatorname{Re} \langle [\Lambda^s, v^\varepsilon \cdot \nabla] v^\varepsilon, \Lambda^s v^\varepsilon \rangle_{L^2}\end{aligned}$$

by integration by parts, the Hölder and the commutator estimate (Lemma A.2.2) yield

$$|I_4| \leqslant C \|\nabla v^\varepsilon\|_{L^\infty} \|v^\varepsilon\|_{H^s}^2. \quad (2.2.7)$$

I_5 is the nonlinear term. The treatment of this term is the main difficulty. A straight forward calculation does not give any more than

$$|I_5| \leqslant C \|\nabla N(|a^\varepsilon|)\|_{H^s} \|v^\varepsilon\|_{H^s}. \quad (2.2.8)$$

Notice that the $(s+1)$-time derivative of the nonlinear term $N(|a^\varepsilon|)$ appears. Our naive hope is that this term might be bounded by the derivative of order $(s-1)$ for a^ε, which may enable us to close the energy estimate and obtain an estimate like

$$\frac{d}{dt}(\|a\|_{H^s}^2 + \|v^\varepsilon\|_{H^{s+1}}^2) \leqslant C(\|a\|_{H^s}^2 + \|v^\varepsilon\|_{H^{s+1}}^2).$$

However, of course it is impossible in general. In particular, when we consider the local nonlinearity such as $N(|a^\varepsilon|) = f(|a^\varepsilon|^2)$ with some smooth function f, we have $\nabla N(|a^\varepsilon|) = 2f'(|a^\varepsilon|^2)\operatorname{Re}(\overline{a^\varepsilon}\nabla a^\varepsilon)$, whose H^s norm seems not to be bounded by H^s norm of a^ε. So, we need to make some trick.

At the end of this section, we summarize the problem which we observed in this section:

1. The estimate of $\frac{d}{dt}\|a^\varepsilon\|_{H^s}^2$ requires the bound of derivative of order $(s+1)$ for v^ε through I_2.
2. The estimate of $\frac{d}{dt}\|v^\varepsilon\|_{H^s}^2$ requires the bound of derivative of order $(s+1)$ for the nonlinearity$N(|a^\varepsilon|)$ through I_5.

In forthcoming examples, we will see how to get over this difficulty and close the energy estimate. The required technique strongly depends on the nonlinearity $N(|a^\varepsilon|)$. In the local nonlinearity case, the key is the cancellation (Section 2.3.1), which is another formation of symmetrizability of (SHS). On the other hand, when the nonlinearity is nonlocal, we use the smoothing property of the nonlinearity (Section 2.4.1).

2.2.2 Construction procedures of the phase function

Once a solution $(a^\varepsilon, v^\varepsilon)$ of (SHS) is known, we can reconstruct a solution $(a^\varepsilon, \phi^\varepsilon)$ of (2.1.8). In this section, we discuss this integration procedures which we use to define ϕ^ε from v^ε so that $\nabla\phi^\varepsilon = v^\varepsilon$ and $\phi^\varepsilon(0,x) = \Phi_0(x)$. There are at least three possible ways to do this.

The Poicaré lemma

The first is the Poicaré lemma. We suppose that a solution $(a^\varepsilon, v^\varepsilon)$ of (SHS) and the initial data $\phi^\varepsilon(0,x) = \Phi_0(x)$ are known. If v^ε is irrotational, that is, if $\nabla\times v^\varepsilon = 0$, then there exist a function $\widetilde{\phi}^\varepsilon$ such that $\nabla\widetilde{\phi}^\varepsilon = v^\varepsilon$. At this step, there is a freedom of choice of a constant: Adding an arbitrary function $c = c(t)$ of time only, we see $\widetilde{\phi}^\varepsilon + c(t)$ also satisfies $\nabla(\widetilde{\phi}^\varepsilon + c(t)) = v^\varepsilon$. However, we can determine this function by $c(0) = \Phi(x) - \widetilde{\phi}^\varepsilon(0,x)$ and

$$c'(t) = \frac{1}{2}|\nabla\widetilde{\phi}^\varepsilon|^2 + N(|a^\varepsilon|) - \partial_t\widetilde{\phi}^\varepsilon.$$

Then, we see that $(a^\varepsilon, \widetilde{\phi}^\varepsilon + c)$ is a solution of (2.1.8). Note that the right hand side does not depend on space variable by the definition of $\widetilde{\phi}^\varepsilon$. Let us remind that this method requires the irrotational property of v^ε.

Direct definition

The second is to define directly by the equation. We suppose that we obtain $(a^\varepsilon, v^\varepsilon)$ and the uniqueness of (SHS) is known. Then, we can define ϕ^ε from its initial data by

$$\phi^\varepsilon(t,x) = \Phi_0(x) - \int_0^t \left(\frac{1}{2}|v^\varepsilon(s,x)|^2 + N(|a^\varepsilon|)(s,x) \right) ds.$$

One easily checks that $(a^\varepsilon, \phi^\varepsilon)$ solves (2.1.8) and $(a^\varepsilon, \nabla\phi^\varepsilon)$ solves (SHS). Then, by uniqueness, we conclude that $\nabla\phi^\varepsilon = v^\varepsilon$. This method requires the uniqueness of the solution to (SHS). Note that in this case the irrotational property immediately follows from the fact that v^ε is given by the gradient of ϕ^ε.

The Hardy-Littlewood-Sobolev inequality

The third is a consequence of the Hardy-Littlewood-Sobolev inequality, which can be found in [36, Th. 4.5.9] or [31, Lemma 7]:

Lemma 2.2.1. *If $\phi \in \mathcal{D}'(\mathbb{R}^n)$ is such that $\partial_j\phi \in L^p(\mathbb{R}^n)$, $j = 1, \dots, n$ for some $p \in (1, n)$, then there exists a constant c such that $\phi - c \in L^q(\mathbb{R}^n)$, with $1/p = 1/q + 1/n$.*

By this lemma, we can construct ϕ^ε as in the first case without the irrotational property of v^ε nor uniqueness of the solution to (SHS). However, for this method, we need the decay property of v^ε in the sense that it must belong $L^p(\mathbb{R}^n)$ for some $p < n$. Especially, this method is difficult to apply when $n = 2$ since the property $v^\varepsilon \in L^2(\mathbb{R}^n)$ is not sufficient: In space dimension $n = 2$, consider a function $f(x_1, x_2) = \log(1 + |\log(x_1^2 + x_2^2)|)$. One can check that $\nabla f \in H^\infty$, while $f \notin L^\infty$.

The first two methods are intended for giving ϕ^ε from its first derivative $v^\varepsilon = \nabla\phi^\varepsilon$. We will see later that, in some case, not the first derivative $v^\varepsilon = \nabla\phi^\varepsilon$ but the second derivative $\nabla v^\varepsilon = \nabla^2\phi^\varepsilon$ is first given as a source (see the proof of Theorem 2.4.2 below). The third can also be used to construct v^ε from ∇v^ε.

2.2.3 Expansion of the solution of the system

We have discussed the existence of the solution to (2.1.8) in the preceding two sections. As mentioned in Section 2.1.2, to obtain the WKB-type approximation (2.1.2) it suffices to expand the solution of (2.1.8) in powers of ε:

$$a^\varepsilon = a_0 + \varepsilon a_1 + \cdots + \varepsilon^k a_k + o(\varepsilon^k), \quad \phi^\varepsilon = \phi_0 + \varepsilon\phi_1 + \cdots + \varepsilon^k \phi_k + o(\varepsilon^k). \quad (2.2.9)$$

In this section, we turn to the method to obtain this expansion and illustrate a scheme for the justification of (2.2.9). It turns out that (a_0, ϕ_0) solves a system (2.1.8) with $\varepsilon = 0$ and (a_i, ϕ_i) solves a i-th linearized system of (2.1.8), and that the existence result for (2.1.8) can again be used to solve these systems and determine approximate solutions.

Let us describe our observation. We suppose that this expansion is given at the initial time, that is, there exists an integer $k \geqslant 1$ such that

$$A_0^\varepsilon = A_0 + \varepsilon A_1 + \cdots + \varepsilon^k A_k + o(\varepsilon^k) \quad \text{in } H^s. \tag{2.2.10}$$

Then, letting $\varepsilon = 0$ in (2.1.8), we obtain

$$\begin{cases} \partial_t a_0 + (\nabla \phi_0 \cdot \nabla) a_0 + \dfrac{1}{2} a_0 \Delta \phi_0 = 0, \\ \partial_t \phi_0 + \dfrac{1}{2} |\nabla \phi_0|^2 + N(|a_0|) = 0, \\ (a_0(0,x), \phi_0(0,x)) = (A_0, \Phi_0). \end{cases} \tag{2.2.11}$$

This system can be solved exactly the same way as in the case of (2.1.8), and moreover the existence time T can be chosen the same. This follows from the fact that the existence time T of solution $(a^\varepsilon, \phi^\varepsilon)$ of (2.1.8) depends only on the size of the initial data: If the initial data is bounded uniformly in ε then T can be independent of ε. Thus, we obtain $(a_0, \phi_0) := (a^\varepsilon, \phi^\varepsilon)_{|\varepsilon=0}$.

The zeroth order

We first prove that $(a^\varepsilon, \phi^\varepsilon)$ converges to (a_0, ϕ_0) as $\varepsilon \to 0$ in a suitable sense. This convergence immediately provides

$$a^\varepsilon = a_0 + o(1), \qquad \qquad \phi^\varepsilon = \phi_0 + o(1),$$

which is (2.2.9) with $k = 0$. The proof of this convergence again relies on the energy method. For example, we estimate time derivative of $\|a^\varepsilon - a_0\|_{H^s}^2 + \|v^\varepsilon - v_0\|_{H^s}^2$. At this step, again the problem is how to close the energy estimate (as in Section 2.2.1).

The first order

We next put $b_1^\varepsilon := (a^\varepsilon - a_0)/\varepsilon$, $\psi_1^\varepsilon := (\phi^\varepsilon - \phi_0)/\varepsilon$. Then the system for $(b_1^\varepsilon, \psi_1^\varepsilon)$ is very similar to the system (2.1.8) which $(a^\varepsilon, \phi^\varepsilon)$ solves. Indeed, that system becomes

$$\begin{cases} \partial_t b_1^\varepsilon + \varepsilon Q_1(b_1^\varepsilon, \psi_1^\varepsilon) + Q_1(b_1^\varepsilon, \phi_0) + Q_1(a_0, \psi_1^\varepsilon) = i\dfrac{\varepsilon}{2} \Delta b_1^\varepsilon + i\dfrac{1}{2} \Delta a_0, \\ \partial_t \psi_1^\varepsilon + \dfrac{\varepsilon}{2} |\nabla \psi_1^\varepsilon|^2 + \nabla \phi_0 \cdot \nabla \psi_1^\varepsilon + \dfrac{N(|a^\varepsilon|) - N(|a_0|)}{\varepsilon} = 0, \\ (b_1^\varepsilon(0,x), \psi_1^\varepsilon(0,x)) = \left(\dfrac{A_0^\varepsilon - A_0}{\varepsilon}, 0 \right), \end{cases} \tag{2.2.12}$$

where Q_1 denotes the quadratic term $Q_1(a, \phi) := (\nabla \phi \cdot \nabla) a + (1/2) a \Delta \phi$. Note that the main quadratic part of (2.2.12) is the same that of (2.1.8) up to a constant ε. Therefore, we can solve (2.2.12) in the same way as in

(2.1.8), although the existence of new linear terms and the nontrivial external force $(i/2)\Delta a_0$ cause loss of one-time derivative and two-time derivative, respectively. Note that if (2.2.10) is satisfied for $k \geqslant 1$ then the initial value $b^\varepsilon_{1|t=0}$ is uniformly bounded for $\varepsilon \in [0,1]$, which ensure that the existence time can be chosen independently of ε. Furthermore, it coincides with A_1 when $\varepsilon = 0$. We therefore obtain $(a_1, \phi_1) := (b^\varepsilon_1, \phi^\varepsilon_1)_{|\varepsilon=0}$ which solves

$$\begin{cases} \partial_t a_1 + Q_1(a_1, \phi_0) + Q_1(a_0, \phi_1) = i\dfrac{1}{2}\Delta a_0, \\ \partial_t \phi_1 + \nabla \phi_0 \cdot \nabla \phi_1 + N^{(1)} = 0, \\ (a_1(0,x), \phi_1(0,x)) = (A_1, 0). \end{cases} \tag{2.2.13}$$

Here, we denote

$$N^{(1)} = N^{(1)}(a_0, a_1) = \lim_{\varepsilon \to 0} \left(\frac{N(|a^\varepsilon|) - N(|a_0|)}{\varepsilon} \right) (a_0, a_1 = b^\varepsilon_{1|\varepsilon=0}).$$

Repeating the argument in the first step, we can claim that $(b^\varepsilon_1, \psi^\varepsilon_1)$ converges to (a_1, ϕ_1) as $\varepsilon \to 0$ by an energy estimate. This convergence implies

$$a^\varepsilon = a_0 + \varepsilon a_1 + o(\varepsilon), \qquad \phi^\varepsilon = \phi_0 + \varepsilon \phi_1 + o(\varepsilon),$$

which is (2.2.9) with $k = 1$.

The l-th order

We use an induction argument. For $l \leqslant k$, we put $b^\varepsilon_l := (a^\varepsilon - \sum_{j=0}^{l-1} \varepsilon^j a_j)/\varepsilon^l$, $\psi^\varepsilon_l := (\phi^\varepsilon - \sum_{j=0}^{l-1} \varepsilon^j \phi_j)/\varepsilon^l$. Then the system for $(b^\varepsilon_l, \psi^\varepsilon_l)$ is also similar to (2.1.8):

$$\begin{cases} \partial_t b^\varepsilon_l + \varepsilon^l Q_1(b^\varepsilon_l, \psi^\varepsilon_l) + Q_1(b^\varepsilon_l, \phi_0) \\ \qquad + Q_1(a_0, \psi^\varepsilon_l) + \displaystyle\sum_{i=1}^{l-1} Q_1(a_i, \phi_{l-i}) = i\frac{\varepsilon}{2}\Delta b^\varepsilon_l + i\frac{1}{2}\Delta a_{l-1}, \\ \partial_t \psi^\varepsilon_l + \dfrac{\varepsilon^l}{2}|\nabla \psi^\varepsilon_l|^2 + \nabla\phi_0 \cdot \nabla \psi^\varepsilon_l \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{i=1}^{l-1}(\nabla\phi_i \cdot \nabla\phi_{l-i}) + \frac{N(|a^\varepsilon|) - \sum_{j=0}^{l-1}\varepsilon^j N^{(j)}}{\varepsilon^l} = 0, \\ (b^\varepsilon_l(0,x), \psi^\varepsilon_l(0,x)) = \left(\dfrac{A^\varepsilon_0 - \sum_{j=0}^{l-1}\varepsilon^j A_j}{\varepsilon^l}, 0 \right). \end{cases} \tag{2.2.14}$$

Note that the main quadratic part of (2.2.14) is still the same that of (2.1.8) up to a constant ε^l. Therefore, we can solve (2.2.14) in the same way as in (2.1.8). In this step, we need the boundedness of $(i/2)\Delta a_{l-1}$. Therefore, we lose two-time derivative in each step. So long as $l \leqslant k$, we see from (2.2.10) that the initial value $b^{\varepsilon}_{l|t=0}$ is uniformly bounded for $\varepsilon \in [0,1]$. In particular, it coincides with A_l when $\varepsilon = 0$. We therefore obtain $(a_l, \phi_l) := (b^{\varepsilon}_l, \phi^{\varepsilon}_l)_{|\varepsilon=0}$, which solves

$$\begin{cases} \partial_t a_l + \displaystyle\sum_{i=0}^{l} Q_1(a_i, \phi_{l-i}) = i\frac{1}{2}\Delta a_{l-1}, \\ \partial_t \phi_l + \dfrac{1}{2}\displaystyle\sum_{i=0}^{l} (\nabla\phi_i \cdot \nabla\phi_{l-i}) + N^{(l)} = 0, \\ (a_l(0,x), \phi_l(0,x)) = (A_l, 0), \end{cases} \tag{2.2.15}$$

where $N^{(l)}$ is given inductively by $N^{(0)} = N(|a_0|)$ and

$$N^{(l)} = N^{(l)}(a_0, \dots, a_l) = \lim_{\varepsilon \to 0} \left(\frac{N(|a^{\varepsilon}|) - \sum_{j=0}^{l-1} \varepsilon^j N^{(j)}}{\varepsilon^l} \right).$$

The explicit form of $N^{(l)}$ is given in the following sections (see Remarks 2.3.6 and 2.4.10). As in the previous steps, we can claim that $(b^{\varepsilon}_l, \psi^{\varepsilon}_l)$ converges to (a_l, ϕ_l) as $\varepsilon \to 0$. This convergence implies

$$a^{\varepsilon} = \sum_{j=0}^{l} \varepsilon^j a_j + o(\varepsilon^l), \qquad \phi^{\varepsilon} = \sum_{j=0}^{l} \varepsilon^j \phi_j + o(\varepsilon^l),$$

which is (2.2.9) with $k = l$.

2.2.4 Nonlinear WKB approximation

We finally give a WKB type approximation (2.1.2) of the solution (2.1.1). Let us start our observation at the step where we obtain an ε-power expansion of the solution $(a^{\varepsilon}, \phi^{\varepsilon})$ to the system (2.1.8) such as

$$a^{\varepsilon} = a_0 + \varepsilon a_1 + \cdots + \varepsilon^k a_k + o(\varepsilon^k), \quad \phi^{\varepsilon} = \phi_0 + \varepsilon\phi_1 + \cdots + \varepsilon^k \phi_k + o(\varepsilon^k)$$

in a suitable topology, say in $C([0,T]; H^s)$ with $s > n/2+1$, for some integer $k \geqslant 1$. Recall that if $(a^{\varepsilon}, \phi^{\varepsilon})$ solves (2.1.8), then $u^{\varepsilon} = a^{\varepsilon} e^{i\frac{\phi^{\varepsilon}}{\varepsilon}}$ is an exact solution to (2.1.1). Now, we plug the above expansion to u^{ε} to have

$$u^{\varepsilon} = (a_0 + \cdots + \varepsilon^k a_k + o(\varepsilon^k)) \exp\left(i\frac{\phi_0}{\varepsilon} + i\phi_1 + \cdots + \varepsilon^{k-1} i\phi_k + o(\varepsilon^{k-1}) \right)$$

in $C([0,T]; H^s)$, which is written as a WKB type approximation

$$u^{\varepsilon} = e^{i\frac{\phi_0}{\varepsilon}} (\beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^{k-1}\beta_{k-1} + o(\varepsilon^{k-1})) \tag{2.2.16}$$

in $C([0,T];H^s)$. Remark that the leading order of the amplitude of u^ε is not a_0 but $\beta_0 = a_0 e^{i\phi_1}$. The important thing is that the ε^1-order term ϕ_1 have some influence on the leading order, and so that the ε^1-term of the initial amplitude A_1 is not negligible when we try to obtain the correct WKB estimate. This fact leads us to some instability results [2, 13, 15, 69] by the approach initiated in [8, 9, 21, 22, 41, 42]. We conclude this section with giving the explicit formulae of β_j for $j \geqslant 1$.

Notation 2.2.2. For a positive integer k, a set of positive integers P is called a partition of k if

$$P \in \bigcup_{l=1}^{k} \left\{ \alpha \in \mathbb{N}^l | 1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_l, \alpha_1 + \cdots + \alpha_l = k \right\}.$$

For a partition P of k, let $\sharp P$ be the integer L for which $P \in \mathbb{N}^L$ holds. Moreover, for a partition P of k, denote the components of P by P_l ($l = 1, 2, \ldots, \sharp P$).

Then, the amplitude β_j ($j = 1, 2, \ldots, k-1$) in (2.2.16) is given by

$$\beta_j = e^{i\phi_1} \left(\sum_{P:\text{partition of } j} i^{\sharp P} \left(a_0 \prod_{l=1}^{\sharp P} \phi_{1+P_l} + i^{-1} a_{P_1} \prod_{l=2}^{\sharp P} \phi_{1+P_l} \right) \right). \quad (2.2.17)$$

We see from the trivial partition $\{j\}$ of j that β_j contains ϕ_{j+1} in its definition. Therefore, we cannot define β_k from the source $\{(a_i, \phi_i)\}_{1 \leqslant i \leqslant k}$. Note that $\beta_j \in H^s$ as long as $\{(a_i, \phi_i)\}_{1 \leqslant i \leqslant j+1}$ is in $H^s \times H^s$.

2.3 Example 1: Local nonlinearity

In this section, we consider (CNLS). Then, the system (2.1.8) is

$$\begin{cases} \partial_t a^\varepsilon + (\nabla \phi^\varepsilon \cdot \nabla) a^\varepsilon + \dfrac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i \dfrac{\varepsilon}{2} \Delta a^\varepsilon, \\ \partial_t \phi^\varepsilon + \dfrac{1}{2} |\nabla \phi^\varepsilon|^2 + f(|a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0). \end{cases} \quad (2.3.1)$$

We introduce new unknown $v^\varepsilon := \nabla \phi^\varepsilon$ and consider

$$\begin{cases} \partial_t a^\varepsilon + (v^\varepsilon \cdot \nabla) a^\varepsilon + \dfrac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i \dfrac{\varepsilon}{2} \Delta a^\varepsilon, \\ \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla f(|a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), v^\varepsilon(0,x)) = (A_0^\varepsilon, \nabla \Phi_0), \end{cases} \quad (2.3.2)$$

which corresponds to the system (SHS).

2.3.1 Cancellation

For the model case $f(y) = y$, let us first observe how we overcome the difficulty in obtaining an energy estimate listed in Section 2.2.1. We keep the notation I_i ($i = 1, 2, \dots, 5$) in Section 2.2.1. Using $N(|a^\varepsilon|) = |a^\varepsilon|^2$, we have

$$\begin{aligned} I_5 = &-2\operatorname{Re}\langle \Lambda^s(2\operatorname{Re}(\overline{a^\varepsilon}\nabla a^\varepsilon)), \Lambda^s v^\varepsilon\rangle_{L^2} \\ = &-4\operatorname{Re}\langle \overline{a^\varepsilon}\Lambda^s\nabla a^\varepsilon, \Lambda^s v^\varepsilon\rangle_{L^2} - 4\operatorname{Re}\langle [\Lambda^s, \overline{a^\varepsilon}]\nabla a^\varepsilon, \Lambda^s v^\varepsilon\rangle_{L^2}. \end{aligned}$$

The first term of the right hand side is a bad term because it contains $(s+1)$-time derivative of a^ε. We now apply the integration by parts to obtain

$$\begin{aligned} I_5 = 4\operatorname{Re}\langle a^\varepsilon\Lambda^s\nabla\cdot v^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2} \\ + 4\operatorname{Re}\langle \nabla\overline{a^\varepsilon}\Lambda^s a^\varepsilon, \Lambda^s v^\varepsilon\rangle_{L^2} - 4\operatorname{Re}\langle [\Lambda^s, \overline{a^\varepsilon}]\nabla a^\varepsilon, \Lambda^s v^\varepsilon\rangle_{L^2}. \end{aligned}$$

This still contains a bad term and use of the Hölder inequality and commutator estimate (Lemma A.2.2) yield

$$|I_5| \leqslant C\,\|a^\varepsilon\|_{L^\infty}\|\nabla v^\varepsilon\|_{H^s}\|a^\varepsilon\|_{H^s} + C\,\|\nabla a^\varepsilon\|_{L^\infty}\|a^\varepsilon\|_{H^s}\|v^\varepsilon\|_{H^s}. \tag{2.3.3}$$

So far, it seems to be impossible to close the estimate. Indeed, plugging (2.2.3), (2.2.4), (2.2.2), (2.2.7), and (2.3.3) to (2.2.1) and (2.2.6), we obtain a bad estimate

$$\frac{d}{dt}(\|a^\varepsilon\|_{H^s}^2 + \|v^\varepsilon\|_{H^s}^2) \leqslant C(\|a^\varepsilon\|_{H^{s+1}}^2 + \|v^\varepsilon\|_{H^{s+1}}^2)^{\frac{3}{2}}.$$

However, one trick solves all problems at once. The remarkable fact is that the bad term of I_5 is the same as that of I_2 with different sign. In particular, we deduce that

$$|I_5 - 4\operatorname{Re}\langle a^\varepsilon\Lambda^s\nabla\cdot v^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2}| \leqslant C\,\|\nabla a^\varepsilon\|_{L^\infty}\|a^\varepsilon\|_{H^s}\|v^\varepsilon\|_{H^s} \tag{2.3.4}$$

and combining this estimate with (2.2.5) causes a cancellation:

$$\begin{aligned} \left|I_2 + \frac{1}{4}I_5\right| &\leqslant |I_2 + \operatorname{Re}\langle a^\varepsilon\Lambda^s\nabla\cdot v^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2}| + \frac{1}{4}|I_5 - 4\operatorname{Re}\langle a^\varepsilon\Lambda^s\nabla\cdot v^\varepsilon, \Lambda^s a^\varepsilon\rangle_{L^2}| \\ &\leqslant C(\|\nabla a^\varepsilon\|_{L^\infty}\|v^\varepsilon\|_{H^s} + \|\nabla v^\varepsilon\|_{L^\infty}\|a^\varepsilon\|_{H^s})\|a^\varepsilon\|_{H^s}. \end{aligned}$$

Namely, the sum of two terms which contain bad part becomes good, and so we conclude that

$$\frac{d}{dt}\left(\|a^\varepsilon\|_{H^s}^2 + \frac{1}{4}\|v^\varepsilon\|_{H^s}^2\right) \leqslant C\left(\|a^\varepsilon\|_{H^s}^2 + \frac{1}{4}\|v^\varepsilon\|_{H^s}^2\right)^{\frac{3}{2}}. \tag{2.3.5}$$

This cancellation is the heart in the case of local nonlinearity. Therefore, the sign of nonlinearity is essential, and this argument does not apply to the "focusing" case $f(y) = -y$. In the focusing case, we need analyticity of the data ([13, 30, 69]). In the original proof in [34], we construct a symmetrizer. Our cancellation can be regarded as another formulation of symmetrizability.

2.3.2 Existence result

We now state the result about the existence of the solution to (2.3.1).

Theorem 2.3.1 (Grenier [34])**.** *Let $f \in C^\infty(\mathbb{R}_+ : \mathbb{R}_+)$ with $f(0) = 0$ and $f' > 0$. Let $s > n/2 + 2$. Assume that $\Phi_0 \in X^{s+1}$, and that A_0^ε is uniformly bounded in H^s for $\varepsilon \in [0,1]$. Then, there exist $T > 0$ independent of $\varepsilon \in [0,1]$ and $s > n/2+2$, and $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ solution to* (CNLS) *on* $[0,T]$ *for* $\varepsilon \in (0,1]$*. Moreover, a^ε and ϕ^ε are the unique solution to* (2.3.1) *which are bounded in $C([0,T];H^s)$ and $C([0,T];X^{s+1})$, respectively, uniformly in $\varepsilon \in [0,1]$. Moreover, $\phi^\varepsilon - \Phi_0$ is bounded in $C([0,T];W^{s,1})$ uniformly in $\varepsilon \in [0,1]$.*

This result is extended to NLS with more general local nonlinearities in [3, 4, 20, 23, 28, 44] (see also [13]).

Remark 2.3.2. It is obvious from the following proof that if $\Phi_0 \in L^p(\mathbb{R}^n)$ for some $p \geqslant 1$ then $\phi^\varepsilon \in C([0,T];L^p(\mathbb{R}^n))$ for the same p. In particular, if $\Phi_0 \in H^{s+1}$ is assumed, as in the original proof in [34], then $\phi^\varepsilon \in C([0,T];H^{s+1})$.

Proof. The strategy is the same as in the case $f(y) = y$. We derive the cancellation of bad terms. For this purpose, set the energy $E(t)$ as

$$E(t) := \|a^\varepsilon\|_{H^s}^2 + \left\langle \frac{1}{4f'(|a^\varepsilon|^2)}\Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon \right\rangle_{L^2}.$$

Since A_0^ε and $\nabla\Phi_0$ are bounded in $H^s \hookrightarrow L^\infty$, there exists C_0 independent of $\varepsilon \in [0,1]$ such that $E(0)^{1/2} \leqslant C_0$. So long as $\|a^\varepsilon\|_{H^s} \leqslant 2C_0$, it holds that $\|a^\varepsilon\|_{L^\infty} \leqslant 2C_0$ and so there exist m and M such that

$$0 < m \leqslant \inf_{y\in[0,4C_0^2]} \frac{1}{f'(y)} \leqslant \sup_{y\in[0,4C_0^2]} \frac{1}{f'(y)} \leqslant M < \infty.$$

With these constants, it holds that

$$\frac{m}{4}\|v^\varepsilon\|_{H^s}^2 \leqslant \left\langle \frac{1}{4f'(|a^\varepsilon|^2)}\Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon \right\rangle_{L^2} \leqslant \frac{M}{4}\|v^\varepsilon\|_{H^s}^2.$$

We estimate $\frac{d}{dt}E(t)$. However, since the estimate of the time derivative of the Sobolev norm of a^ε is the same as in Section 2.2, we omit the detail. We estimate the time derivative of the second term of $E(t)$:

$$\begin{aligned}
\frac{d}{dt}\left\langle \frac{1}{4f'}\Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon \right\rangle &= \left\langle \frac{1}{2f'}\Lambda^s \partial_t v^\varepsilon, \Lambda^s v^\varepsilon \right\rangle + \left\langle \partial_t\left(\frac{1}{4f'}\right)\Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon \right\rangle \\
&= -\left\langle \frac{1}{2f'}\Lambda^s((v^\varepsilon\cdot\nabla)v^\varepsilon), \Lambda^s v^\varepsilon \right\rangle \\
&\quad -\left\langle \frac{1}{2f'}\Lambda^s(\nabla f), \Lambda^s v^\varepsilon \right\rangle + \left\langle \partial_t\left(\frac{1}{4f'}\right)\Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon \right\rangle \\
&=: \widetilde{I}_4 + \widetilde{I}_5 + \widetilde{I}_6.
\end{aligned} \tag{2.3.6}$$

The estimate of $\widetilde{I_4}$ is similar to (2.2.7):

$$\begin{aligned}\widetilde{I_4} = & -2\operatorname{Re}\left\langle \frac{1}{2f'}(v^\varepsilon\cdot\nabla)\Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon\right\rangle_{L^2} - 2\operatorname{Re}\left\langle \frac{1}{2f'}[\Lambda^s, v^\varepsilon\cdot\nabla]v^\varepsilon, \Lambda^s v^\varepsilon\right\rangle_{L^2} \\ = & \operatorname{Re}\left\langle \frac{1}{2f'}(\nabla\cdot v^\varepsilon)\Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon\right\rangle_{L^2} + \operatorname{Re}\left\langle \left(\nabla\frac{1}{2f'}\right)(\nabla\cdot v^\varepsilon)\Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon\right\rangle_{L^2} \\ & -2\operatorname{Re}\left\langle \frac{1}{2f'}[\Lambda^s, v^\varepsilon\cdot\nabla]v^\varepsilon, \Lambda^s v^\varepsilon\right\rangle_{L^2}.\end{aligned}$$

Since there exists a constant $C = C(f, C_0)$ such that

$$\left\|\nabla\frac{1}{f'(|a^\varepsilon|^2)}\right\|_{L^\infty} \leqslant \left(\sup_{y\in[0,4C_0^2]}\frac{|f''(y)|}{(f'(y))^2}\right)\left\|\nabla|a^\varepsilon|^2\right\|_{L^\infty} \leqslant CC_0\left\|\nabla a^\varepsilon\right\|_{L^\infty},$$

we see from (2.2.7) that

$$|\widetilde{I_4}| \leqslant C(M\left\|\nabla v^\varepsilon\right\|_{L^\infty} + C_0\left\|\nabla a^\varepsilon\right\|_{L^\infty})\left\|v^\varepsilon\right\|_{H^s}^2. \tag{2.3.7}$$

We next estimate $\widetilde{I_5}$. An elementary calculation shows

$$\begin{aligned}\widetilde{I_5} = & -\left\langle \frac{1}{2f'}\Lambda^s(2f'\operatorname{Re}(\overline{a^\varepsilon}\nabla a^\varepsilon)), \Lambda^s v^\varepsilon\right\rangle \\ = & \frac{1}{4}I_5 - \operatorname{Re}\left\langle \frac{1}{f'}[\Lambda^s, f']\overline{a^\varepsilon}\nabla a^\varepsilon, \Lambda^s v^\varepsilon\right\rangle,\end{aligned}$$

where I_5 is introduced in Section 2.3.1. Therefore, by (2.3.4), we have

$$\begin{aligned}& |\widetilde{I_5} - \operatorname{Re}\langle a^\varepsilon\Lambda^s\nabla v^\varepsilon, \Lambda^s a^\varepsilon\rangle| \\ & \leqslant \frac{1}{4}|I_5 - 4\operatorname{Re}\langle a^\varepsilon\Lambda^s\nabla v^\varepsilon, \Lambda^s a^\varepsilon\rangle| + \left|\left\langle \frac{1}{f'}[\Lambda^s, f']\overline{a^\varepsilon}\nabla a^\varepsilon, \Lambda^s v^\varepsilon\right\rangle\right| \\ & \leqslant C\left\|\nabla a^\varepsilon\right\|_{L^\infty}\left\|a^\varepsilon\right\|_{H^s}\left\|v^\varepsilon\right\|_{H^s} + \left|\left\langle \frac{1}{f'}[\Lambda^s, f']\overline{a^\varepsilon}\nabla a^\varepsilon, \Lambda^s v^\varepsilon\right\rangle\right| \end{aligned}\tag{2.3.8}$$

By the commutator estimate, we have

$$\begin{aligned}\left|\left\langle \frac{1}{f'}[\Lambda^s, f']\overline{a^\varepsilon}\nabla a^\varepsilon, \Lambda^s v^\varepsilon\right\rangle\right| \leqslant & CM(\left\|\nabla f'\right\|_{L^\infty}\left\|a^\varepsilon\nabla a^\varepsilon\right\|_{H^{s-1}} \\ & + \left\|\nabla f'\right\|_{H^{s-1}}\left\|a^\varepsilon\nabla a^\varepsilon\right\|_{L^\infty})\left\|v^\varepsilon\right\|_{H^s}.\end{aligned}$$

Here, we apply the estimate of composite function (Lemma A.2.4) to obtain

$$\begin{aligned}\left\|\nabla f'\right\|_{H^{s-1}} \leqslant & C(1+\left\|a^\varepsilon\right\|_{L^\infty}^2)^{\lceil s\rceil}\left(\sup_{y\in[0,4C_0^2],k\in[1,\lceil s\rceil+1]}|f^{(k)}(y)|\right)\left\|\nabla|a^\varepsilon|^2\right\|_{H^{s-1}} \\ \leqslant & C(1+4C_0^2)^{\lceil s\rceil}C_0\left\|a^\varepsilon\right\|_{H^s},\end{aligned}$$

where $\lceil s \rceil$ denotes the smallest integer bigger than or equal to s. Combining these estimates to (2.3.8), we find

$$|\widetilde{I}_5 - \operatorname{Re} \langle a^\varepsilon \Lambda^s \nabla v^\varepsilon, \Lambda^s a^\varepsilon \rangle | \leqslant C \|a^\varepsilon\|_{W^{1,\infty}} \|a^\varepsilon\|_{H^s} \|v^\varepsilon\|_{H^s} . \tag{2.3.9}$$

We finally estimate $\widetilde{I}_6$. Using the first equation of (2.3.2), we have

$$\begin{aligned}\left\| \partial_t \left(\frac{1}{f'} \right) \right\|_{L^\infty} &\leqslant \left(\sup_{y\in[0,4C_0^2]} \frac{|f''(y)|}{(f'(y))^2} \right) \left\| \partial_t |a^\varepsilon|^2 \right\|_{L^\infty} \\ &\leqslant CC_0(\|a^\varepsilon\|_{W^{1,\infty}} \|v^\varepsilon\|_{W^{1,\infty}} + \varepsilon \|\Delta a^\varepsilon\|_{L^\infty}).\end{aligned}$$

We hence obtain

$$|\widetilde{I}_6| \leqslant C(\|a^\varepsilon\|_{W^{1,\infty}} \|v^\varepsilon\|_{W^{1,\infty}} + \varepsilon \|\Delta a^\varepsilon\|_{L^\infty}) \|v^\varepsilon\|_{H^s}^2 . \tag{2.3.10}$$

The assumption $s > n/2 + 2$ comes from this point. We suppose this to ensure the Sobolev embedding $\|\Delta a^\varepsilon\|_{L^\infty} \leqslant C \|\Delta a^\varepsilon\|_{H^{s-2}} \leqslant C \|a^\varepsilon\|_{H^s}$. We also note that the term $\widetilde{I}_6$ does not appear if we assume f' is a constant as we have seen in Section 2.3.1. In this case we need only $s > n/2 + 1$.

We summarize estimates (2.2.1), (2.2.3) (2.2.5), (2.2.2), (2.3.6), (2.3.7), (2.3.9), and (2.3.10). Then,

$$\begin{aligned}\left| \frac{d}{dt} E(t) \right| \leqslant{}& \left| \sum_{i=1,2,3} I_i + \sum_{j=4,5,6} \widetilde{I}_j \right| \\ \leqslant{}& |I_1| + |I_2 + \operatorname{Re} \langle a^\varepsilon \Lambda^s \nabla \cdot v^\varepsilon, \Lambda^s a^\varepsilon \rangle_{L^2} | + 0 \\ &+ |\widetilde{I}_4| + |\widetilde{I}_5 - \operatorname{Re} \langle a^\varepsilon \Lambda^s \nabla \cdot v^\varepsilon, \Lambda^s a^\varepsilon \rangle_{L^2} | + |\widetilde{I}_6| \\ \leqslant{}& C(\|a^\varepsilon\|_{W^{1,\infty}} + \|v^\varepsilon\|_{W^{1,\infty}} + \|a^\varepsilon\|_{W^{1,\infty}} \|v^\varepsilon\|_{W^{1,\infty}} + \varepsilon \|\Delta a^\varepsilon\|_{L^\infty}) \\ &\times (\|a^\varepsilon\|_{H^s}^2 + \|v^\varepsilon\|_{H^s}^2) \\ \leqslant{}& C(E(t)^{\frac{3}{2}} + E(t)^2).\end{aligned}$$

Therefore, by the Gronwall lemma, for any $\delta > 0$ there exists $T = T(\delta) > 0$ such that $E(t) \leqslant 4E(0)$ holds for $t \in [0, T]$. For $t \in [0, T]$, it holds that $\|a^\varepsilon\|_{H^s} \leqslant E(t)^{1/2} \leqslant 2E(0)^{1/2} \leqslant 2C_0$, which ensures the above estimates. We obtain a priori estimate.

Uniqueness and construction of ϕ^ε

The uniqueness of $(a^\varepsilon, v^\varepsilon)$ also follows from the energy estimate. Let $(a_1^\varepsilon, v_1^\varepsilon)$ and $(a_2^\varepsilon, v_2^\varepsilon)$ be two solutions of (2.3.2) bounded in $C([0, T]; H^s)^2$. Then, denoting $(d_a^\varepsilon, d_v^\varepsilon) = (a_1^\varepsilon - a_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon)$, we have

$$\begin{cases} \partial_t d_a^\varepsilon + (d_v^\varepsilon \cdot \nabla) a_1^\varepsilon + (v_2^\varepsilon \cdot \nabla) d_a^\varepsilon + \dfrac{1}{2} d_a^\varepsilon \nabla \cdot v_1^\varepsilon + \dfrac{1}{2} a_2^\varepsilon \nabla \cdot d_v^\varepsilon = i\dfrac{\varepsilon}{2} \Delta d_a^\varepsilon, \\ \partial_t d_v^\varepsilon + (d_v^\varepsilon \cdot \nabla) v_1^\varepsilon + (v_2^\varepsilon \cdot \nabla) d_v^\varepsilon + 2f'(|a_2^\varepsilon|^2) \operatorname{Re}(\overline{d_a^\varepsilon} \nabla a_1^\varepsilon + \overline{a_2^\varepsilon} \nabla d_a^\varepsilon) \\ \quad + (d_a^\varepsilon \overline{a_1^\varepsilon} + a_2^\varepsilon \overline{d_a^\varepsilon}) \displaystyle\int_0^1 f''(|a_2^\varepsilon|^2 + \theta(|a_1^\varepsilon|^2 - |a_2^\varepsilon|^2)) d\theta \nabla |a_1^\varepsilon|^2 = 0, \\ (d_a^\varepsilon(0,x), d_v^\varepsilon(0,x)) = (0,0). \end{cases}$$

The bad terms are $(1/2)a_2^\varepsilon \nabla \cdot d_v^\varepsilon$ and $2f(|a_2^\varepsilon|^2) \operatorname{Re}(\overline{a_2^\varepsilon} \nabla d_a^\varepsilon)$ because the others do not include any derivative on $(d_a^\varepsilon, d_v^\varepsilon)$. To handle these term by cancellation, we consider

$$E_d(t) := \|d_a^\varepsilon\|_{L^2}^2 + \left\langle \frac{1}{4f'(|a_2^\varepsilon|^2)} d_v^\varepsilon, d_v^\varepsilon \right\rangle_{L^2}.$$

Then, mimicking the estimate for $E(t)$, we obtain

$$\frac{d}{dt} E_d(t) \leqslant C(\|a_i\|_{H^s}, \|v_i\|_{H^s}) E_d(t).$$

Therefore, by Gronwall's lemma, $E_d(t) = 0$ for $t \in [0,T]$ follows from $E_d(0) = 0$, and so $(a_1, v_1) = (a_2, v_2)$ holds. Once the uniqueness of (2.3.2) is known, along the argument in Section 2.2.2, we can construct ϕ^ε directly by

$$\phi^\varepsilon = \Phi_0 - \int_0^t \left(\frac{1}{2} |v^\varepsilon(s)|^2 + f(|a^\varepsilon(s)|^2) \right) ds.$$

Then, $(a^\varepsilon, \phi^\varepsilon)$ is a unique solution of (2.3.1). Since $a^\varepsilon, v^\varepsilon \in L^2 \cap L^\infty$ and $f(0) = 0$, we see the second term belongs to $L^1 \cap L^\infty$. Therefore, if Φ_0 belongs to L^p for some $p \in [1,\infty]$ then so is ϕ^ε, which ensures $\phi^\varepsilon \in X^{s+1}$ and completes the proof. Remark 2.3.2 also follows. $\square$

2.3.3 Justification of WKB approximation

We now prove the WKB approximation of the solution to (2.3.1).

Theorem 2.3.3. *Let f satisfy the same assumption as in Theorem 2.3.1. Let $k \geqslant 1$ be an integer and $s > n/2 + 2k + 4$ be a real number. Assume that $\Phi_0 \in X^{s+1}$, and that A_0^ε writes*

$$A_0^\varepsilon = \sum_{j=0}^k \varepsilon^j A_j + o(\varepsilon^k) \quad \text{in } H^s$$

for $\varepsilon \in [0,1]$*. Then, the unique solution* $(a^\varepsilon, \phi^\varepsilon)$ *of* (2.3.1) *has the following expansion:*

$$\begin{cases} a^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j a_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^{s-2k-2}), \\ \phi^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j \phi_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^{s-2k-1}). \end{cases} \tag{2.3.11}$$

Remark 2.3.4. Theorem 2.1.1 immediately follows from (2.3.11) by an argument in Section 2.2.4. Indeed, we obtain

$$u^\varepsilon = e^{i\frac{\phi_0}{\varepsilon}}(\beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^{k-1}\beta_{k-1} + o(\varepsilon^{k-1})) \quad \text{in } C([0,T]; H^{s-2k-2}),$$

where $\beta_0 = a_0 e^{i\phi_1}$ and β_j is given by the formula (2.2.17).

Remark 2.3.5. Recall that $\phi^\varepsilon - \phi_0 = (\phi^\varepsilon - \Phi_0) - (\phi_0 - \Phi_0) \in W^{s,1}$ while ϕ^ε and ϕ_0 belong to X^{s+1} and so they do not necessarily decay at spatial infinity. Similarly, the asymptotic of ϕ^ε in (2.3.11) holds in $C([0,T]; W^{s-2k-1,1})$.

Proof. The proof proceeds along the way which we show in Section 2.2.3. Instead of the asymptotic expansion of ϕ^ε itself, we consider the expansion of $v^\varepsilon = \nabla\phi^\varepsilon$:

$$v^\varepsilon = \sum_{j=0}^{k} \varepsilon^j v_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^s).$$

This is due to the following two reason: Firstly, it is rather easier to analyze the system for $(a^\varepsilon, v^\varepsilon)$ than the system for $(a^\varepsilon, \phi^\varepsilon)$ itself, and, secondly, once we obtain the above expansion of v^ε then it is easy to construct each ϕ_i from the corresponding v_i. Since A_0^ε is uniformly bounded and $A^\varepsilon_{0|\varepsilon=0} = A_0$, we see that (2.3.2) admits a solution even in the case $\varepsilon = 0$. We denote this solution by (a_0, ϕ_0), which solves

$$\begin{cases} \partial_t a_0 + (v_0 \cdot \nabla) a_0 + \dfrac{1}{2} a_0 \nabla \cdot v_0 = 0, \\ \partial_t v_0 + (v_0 \cdot \nabla) v_0 + \nabla f(|a_0|^2) = 0, \\ \qquad\qquad (a_0(0,x), v_0(0,x)) = (A_0, \nabla\Phi_0). \end{cases} \tag{2.3.12}$$

The zeroth order

We first prove $(a^\varepsilon, v^\varepsilon)$ converges to (a_0, v_0) as $\varepsilon \to 0$. Set $\widetilde{a}_0^\varepsilon = a^\varepsilon - a_0$ and $\widetilde{v}_0^\varepsilon = v^\varepsilon - v_0$. Then,

$$\begin{cases} \partial_t \widetilde{a}_0^\varepsilon + (\widetilde{v}_0^\varepsilon \cdot \nabla) a^\varepsilon + (v_0 \cdot \nabla) \widetilde{a}_0^\varepsilon + \dfrac{1}{2} \widetilde{a}_0^\varepsilon \nabla \cdot v^\varepsilon + \dfrac{1}{2} a_0 \nabla \cdot \widetilde{v}_0^\varepsilon = i \dfrac{\varepsilon}{2} \Delta \widetilde{a}_0^\varepsilon + i \dfrac{\varepsilon}{2} \Delta a_0, \\ \partial_t \widetilde{v}_0^\varepsilon + (\widetilde{v}_0^\varepsilon \cdot \nabla) v^\varepsilon + (v_0 \cdot \nabla) \widetilde{v}_0^\varepsilon + 2 f'(|a_0|^2) \operatorname{Re}(\overline{\widetilde{a}_0^\varepsilon} \nabla a^\varepsilon + \overline{a_0} \nabla \widetilde{a}_0^\varepsilon) \\ \quad + (\widetilde{a}_0^\varepsilon \overline{a^\varepsilon} + a_0 \overline{\widetilde{a}_0^\varepsilon}) \displaystyle\int_0^1 f''(|a_0|^2 + \theta(|a^\varepsilon|^2 - |a_0|^2)) d\theta \nabla |a^\varepsilon|^2 = 0, \\ (\widetilde{a}_0^\varepsilon(0,x), \widetilde{v}_0^\varepsilon(0,x)) = (A_0^\varepsilon - A_0, 0). \end{cases} \tag{2.3.13}$$

The bad terms are $(1/2) a_0 \nabla \cdot \widetilde{v}_0^\varepsilon$ and $2 f'(|a_0|^2) \operatorname{Re}(\overline{a_0} \nabla \widetilde{a}_0^\varepsilon)$. In order two derive cancellation between these two terms, we set

$$\widetilde{E}_0(t) := \|\widetilde{a}_0^\varepsilon\|_{H^s}^2 + \left\langle \frac{1}{4 f'(|a_0|^2)} \Lambda^s \widetilde{v}_0^\varepsilon, \Lambda^s \widetilde{v}_0^\varepsilon \right\rangle.$$

If $s > n/2 + 2$ then, in the way similar to the energy $E(t)$ in the proof of Theorem 2.3.1, we obtain

$$\begin{aligned} \frac{d}{dt} \widetilde{E}_0(t) &\leqslant C \widetilde{E}_0(t) + C \varepsilon \|a_0\|_{H^{s+2}} (\widetilde{E}_0(t))^{\frac{1}{2}} \\ &\leqslant C_1 \widetilde{E}_0(t) + C_2 \varepsilon. \end{aligned} \tag{2.3.14}$$

Note that the lower order term comes from the estimate of $\langle (i\varepsilon/2) \Lambda^s \Delta a_0, \Lambda^s \widetilde{a}_0^\varepsilon \rangle$, and that C_i depends on f, $\|a_0\|_{H^{s+2}}$, $\|v_0\|_{H^{s+1}}$, $\|a^\varepsilon\|_{H^{s+1}}$, and $\|v^\varepsilon\|_{H^{s+1}}$. Therefore, if $(a^\varepsilon, v^\varepsilon), (a_0, v_0) \in C([0,T]; H^{s+2})^2$ then Gronwall's lemma yields

$$\widetilde{E}_0(t) \leqslant \widetilde{E}_0(0) e^{C_1 t} + \varepsilon \frac{C_2}{C_1} (e^{C_1 t} - 1).$$

The right hand side converges to zero as $\varepsilon \to 0$ uniformly in $[0,T]$, which implies

$$a^\varepsilon = a_0 + o(1) \quad \text{in } C([0,T]; H^s), \qquad v^\varepsilon = v_0 + o(1) \quad \text{in } C([0,T]; H^s)$$

as $\varepsilon \to 0$.

The first order

We now put $b_1^\varepsilon := \widetilde{a}_0^\varepsilon / \varepsilon$ and $w_1^\varepsilon := \widetilde{v}_0^\varepsilon / \varepsilon$, and also set

$$E_1(t) := \|b_1^\varepsilon\|_{H^s}^2 + \left\langle \frac{1}{4 f'(|a_0|^2)} \Lambda^s w_1^\varepsilon, \Lambda^s w_1^\varepsilon \right\rangle.$$

Then, since $E_1(t) = \varepsilon^{-2}\widetilde{E}_0(t)$, we deduce from (2.3.14) that

$$\begin{aligned}\frac{d}{dt}E_1(t) &\leqslant CE_1(t) + C\left\|a_0\right\|_{H^{s+2}}(E_1(t))^{\frac{1}{2}} \\ &\leqslant C_1E_1(t) + C_2,\end{aligned}$$

and so that

$$\sup_{t\in[0,T]} E_1(t) \leqslant E_1(0)e^{C_1T} + \frac{C_2}{C_1}(e^{C_1T} - 1).$$

By assumption on the initial data, we know that $E_1(0) = \|(A_0^\varepsilon - A_0)/\varepsilon\|_{H^s}^2$ is uniformly bounded. Therefore, $\sup_{t\in[0,T]} E_1(t)$ is uniformly bounded. This implies that $(b_1^\varepsilon, w_1^\varepsilon) \in C([0,T];H^s)^2$ exists, where T is the existence time of $(a^\varepsilon, v^\varepsilon)$. Here, we remark that $(b_1^\varepsilon, w_1^\varepsilon) \in C([0,T];H^s)^2$ requires $a_0 \in C([0,T];H^{s+2})$. Similarly, we will see that we lose two-time derivative in each step.

Since $b^\varepsilon_{1|t=0,\varepsilon=0} = A_1$ by assumption, we can define $(a_1, v_1) \in C([0,T];H^s)^2$ by

$$(a_1, v_1) := (b_1^\varepsilon, w_1^\varepsilon)_{\varepsilon=0}.$$

One sees from (2.3.13) that the system for (a_1, v_1) is the following:

$$\begin{cases}\partial_t a_1 + (v_1\cdot\nabla)a_0 + (v_0\cdot\nabla)a_1 + \frac{1}{2}a_1\nabla\cdot v_0 + \frac{1}{2}a_0\nabla\cdot v_1 = i\frac{1}{2}\Delta a_0, \\ \partial_t v_1 + (v_1\cdot\nabla)v_0 + (v_0\cdot\nabla)v_1 + 2f'(|a_0|^2)\operatorname{Re}(\overline{a_1}\nabla a_0 + \overline{a_0}\nabla a_1) \\ \quad + (a_1\overline{a_0} + a_0\overline{a_1})f''(|a_0|^2)\nabla|a_0|^2 = 0, \\ (a_1(0,x), v_1(0,x)) = (A_1, 0).\end{cases} \tag{2.3.15}$$

As in the zeroth order, we then estimate the distance $\widetilde{a}_1^\varepsilon := b_1^\varepsilon - a_1$ and $\widetilde{v}_1^\varepsilon := w_1^\varepsilon - v_1$. From (2.3.13) and (2.3.15), we verify that the equation for $(\widetilde{a}_1^\varepsilon, \widetilde{v}_1^\varepsilon)$ is

$$\begin{cases}\partial_t\widetilde{a}_1^\varepsilon + (\widetilde{v}_1^\varepsilon\cdot\nabla)a^\varepsilon + (v_1\cdot\nabla)\widetilde{a}_0^\varepsilon + (v_0\cdot\nabla)\widetilde{a}_1^\varepsilon \\ \qquad + \frac{1}{2}\widetilde{a}_1^\varepsilon\nabla\cdot v^\varepsilon + \frac{1}{2}a_1\nabla\cdot\widetilde{v}_0^\varepsilon + \frac{1}{2}a_0\nabla\cdot\widetilde{v}_1^\varepsilon = i\frac{\varepsilon}{2}\Delta\widetilde{a}_1^\varepsilon + i\frac{\varepsilon}{2}\Delta a_1, \\ \partial_t\widetilde{v}_1^\varepsilon + (\widetilde{v}_1^\varepsilon\cdot\nabla)v^\varepsilon + (v_1\cdot\nabla)\widetilde{v}_0^\varepsilon + (v_0\cdot\nabla)\widetilde{v}_1^\varepsilon \\ + 2f'(|a_0|^2)\operatorname{Re}(\overline{\widetilde{a}_1^\varepsilon}\nabla a^\varepsilon + \overline{a_1}\nabla\widetilde{a}_0^\varepsilon + \overline{a_0}\nabla\widetilde{a}_1^\varepsilon) \\ + (\widetilde{a}_1^\varepsilon\overline{a^\varepsilon} + a_1\overline{\widetilde{a}_0^\varepsilon} + a_0\overline{\widetilde{a}_1^\varepsilon})\int_0^1 f''(|a_0|^2 + \theta(|a^\varepsilon|^2 - |a_0|^2))d\theta\nabla|a^\varepsilon|^2 \\ + (a_0\overline{a_1} + a_1\overline{a_0})(a^\varepsilon\overline{\widetilde{a}_0^\varepsilon} + \widetilde{a}_0^\varepsilon\overline{a_0}) \\ \times\int_0^1\int_0^1 \theta_1 f'''(|a_0|^2 + \theta_1\theta_2(|a^\varepsilon|^2 - |a_0|^2))d\theta_2 d\theta_1\nabla|a^\varepsilon|^2 = 0, \\ (\widetilde{a}_1^\varepsilon(0,x), \widetilde{v}_1^\varepsilon(0,x)) = \left(\frac{A_0^\varepsilon - A_0 - \varepsilon A_1}{\varepsilon}, 0\right).\end{cases} \tag{2.3.16}$$

The point is that bad terms are $(1/2)a_0\nabla\cdot\widetilde{v}_1^\varepsilon$ and $2f'(|a_0|^2)\,\mathrm{Re}(\overline{a_0}\nabla\widetilde{a}_1^\varepsilon)$, which is essentially same as in the previous (2.3.13). All other terms do not include any derivative on $(\widetilde{a}_1^\varepsilon,\widetilde{v}_1^\varepsilon)$ except for $(i\varepsilon/2)\Delta\widetilde{a}_1^\varepsilon$, which vanishes in the energy estimate. Setting the energy

$$\widetilde{E}_1(t) := \|\widetilde{a}_1^\varepsilon\|_{H^s}^2 + \left\langle \frac{1}{4f'(|a_0|^2)}\Lambda^s\widetilde{v}_1^\varepsilon, \Lambda^s\widetilde{v}_1^\varepsilon\right\rangle,$$

we obtain

$$\begin{aligned}\frac{d}{dt}\widetilde{E}_1(t) &\leqslant C\widetilde{E}_1(t) + C\varepsilon\,\|\Delta a_1\|_{H^s}\,(\widetilde{E}_1(t))^{\frac{1}{2}}\\ &\leqslant C_1\widetilde{E}_1(t) + C_2\varepsilon.\end{aligned}$$

The constant depends on the H^{s+2} norm of a_1 and H^s norm of v_1, a_0, v_0, a^ε, and v^ε. Using Gronwall's lemma and assumption, we see that

$$0 \leqslant \sup_{t\in[0,T]}\widetilde{E}_1(t) \leqslant \widetilde{E}_1(0)e^{C_1T} + \varepsilon\frac{C_2}{C_1}(e^{C_1T}-1) \to 0$$

as $\varepsilon\to 0$, which show

$$\begin{aligned}a^\varepsilon &= a_0 + \varepsilon a_1 + o(\varepsilon) \quad \text{in } C([0,T];H^s),\\ v^\varepsilon &= v_0 + \varepsilon v_1 + o(\varepsilon) \quad \text{in } C([0,T];H^s)\end{aligned}$$

as $\varepsilon\to 0$.

Higher order

We repeat the argument in the first order: Define $(b_2^\varepsilon, w_2^\varepsilon) := (\widetilde{a}_1^\varepsilon/\varepsilon, \widetilde{v}_1^\varepsilon/\varepsilon)$. Then, from the estimate of $\widetilde{E}_1(t)$, we see that

$$E_2(t) := \|b_2^\varepsilon\|_{H^s}^2 + \left\langle \frac{1}{4f'(|a_0|^2)}\Lambda^s w_2^\varepsilon, \Lambda^s w_2^\varepsilon\right\rangle$$

is uniformly bounded as long as so is $E_2(0)$. Therefore, $(b_2^\varepsilon, w_2^\varepsilon)$ exists as a function in $C([0,T];(H^s)^2)$ for all $\varepsilon\in[0,1]$. Set $(a_2,v_2) := (b_2^\varepsilon, w_2^\varepsilon)_{|\varepsilon=0}$. Then, $(\widetilde{a}_2^\varepsilon,\widetilde{v}_2^\varepsilon) = (b_2^\varepsilon - a_2, w_2^\varepsilon - v_2)$ solves a system similar to (2.3.16) and/or (2.3.13). Removing bad parts by the cancellation technique, we obtain the energy estimate on $(\widetilde{a}_2^\varepsilon,\widetilde{v}_2^\varepsilon)$ which ensures $(\widetilde{a}_2^\varepsilon,\widetilde{v}_2^\varepsilon)\to 0$ in $C([0,T];(H^s)^2)$ and so

$$\begin{aligned}a^\varepsilon &= a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + o(\varepsilon^2) \quad \text{in } C([0,T];H^s),\\ v^\varepsilon &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + o(\varepsilon^2) \quad \text{in } C([0,T];H^s)\end{aligned}$$

and so on.

Regularity counting

As we seen above, we construct three sequences from $(b_0^\varepsilon, w_0^\varepsilon) = (a^\varepsilon, v^\varepsilon)$ by

$$\begin{aligned}(a_l, v_l) &:= (b_l^\varepsilon, w_l^\varepsilon)_{|\varepsilon=0},\\ (\widetilde{a}_l^\varepsilon, \widetilde{v}_l^\varepsilon) &:= (b_l^\varepsilon - a_l, w_l^\varepsilon - v_l),\\ (b_{l+1}^\varepsilon, w_{l+1}^\varepsilon) &:= \left(\frac{\widetilde{a}_l^\varepsilon}{\varepsilon}, \frac{\widetilde{v}_l^\varepsilon}{\varepsilon}\right).\end{aligned}$$

By assumption, for some $k \geqslant 1$, A_0^ε is uniformly bounded in H^s for $s > n/2 + 2k + 2$, and moreover

$$A_0^\varepsilon = \sum_{j=0}^{k} \varepsilon^j A_j + o(\varepsilon^k) \quad \text{in } H^s.$$

For a while we denote $C([0,T]; (H^s)^2)$ by $C_T(H^s)^2$, for simplicity. By Theorem 2.3.1, $(b_0^\varepsilon, v_0^\varepsilon)$ exists in $C_T(H^s)^2$, which shows that (a_0, v_0) also belongs to this space. Then, we obtain $(\widetilde{a}_0^\varepsilon, \widetilde{v}_0^\varepsilon) \in C_T(H^{s-2})^2$ along the argument in the first order. Recall that for $(\widetilde{a}_0^\varepsilon, \widetilde{v}_0^\varepsilon) \in C_T(H^{s'})^2$ being true, we need $(a_0, v_0) \in C_T(H^{s'+2})^2$. At this step, we lose two-time derivative. Repeating this argument, we see inductively that

$$\begin{aligned}(a_l, v_l) \in C([0,T]; H^{s-2l})^2, \qquad (\widetilde{a}_l^\varepsilon, \widetilde{v}_l^\varepsilon) \in C([0,T]; H^{s-2l-2})^2,\\ (b_{l+1}^\varepsilon, w_{l+1}^\varepsilon) \in C([0,T]; H^{s-2l-2})^2.\end{aligned} \tag{2.3.17}$$

By assumption, $\widetilde{a}_{k|t=0}^\varepsilon = \varepsilon^{-k}(A^\varepsilon - \sum_{j=0}^{k} \varepsilon^j A_k)$ is uniformly bounded in H^s for $\varepsilon \in [0,1]$, and however, $b_{k+1|t=0}^\varepsilon = \varepsilon^{-k+1}(A^\varepsilon - \sum_{j=0}^{k} \varepsilon^j A_k)$ is not necessarily uniformly bounded. Therefore, the above induction argument stops with obtaining the uniform $C_T(H^{s-2k-2})^2$ bound of $(\widetilde{a}_k^\varepsilon, \widetilde{v}_k^\varepsilon)$. Thus, we conclude that

$$\begin{aligned}a^\varepsilon &= \sum_{j=0}^{k} \varepsilon^j a_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^{s-2k-2}),\\ v^\varepsilon &= \sum_{j=0}^{k} \varepsilon^j v_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^{s-2k-2}),\end{aligned} \tag{2.3.18}$$

where (a_j, v_j) $(j \geqslant 1)$ solves the system

$$
\begin{cases}
\partial_t a_j + \displaystyle\sum_{i_1+i_2=j} \left(\frac{1}{2}(v_{i_1} \cdot \nabla) a_{i_2} + a_{i_1} \nabla \cdot v_{i_2} \right) - \frac{i}{2} \Delta a_{j-1} = 0, \\
\partial_t v_j + \displaystyle\sum_{i_1+i_2=j} (v_{i_1} \cdot \nabla) v_{i_2} \\
\quad + \displaystyle\sum_{L=1}^{j} f^{(L)}(|a_0|^2) \sum_{\substack{J_1+\cdots+J_L=j \\ J_l \geqslant 1}} \prod_{l=1}^{L} \sum_{i_1+i_2=J_l} 2\,\mathrm{Re}(\overline{a_{i_1}} \nabla a_{i_2}) \\
\quad + \displaystyle\sum_{L=1}^{j} f^{(L+1)}(|a_0|^2)(\nabla |a_0|^2) \sum_{\substack{J_1+\cdots+J_L=j \\ J_l \geqslant 1}} \prod_{l=1}^{L} \sum_{i_1+i_2=J_l} (a_{i_1} \overline{a_{i_2}}) = 0, \\
\quad (a_j(0,x), v_j(0,x)) = (A_j(x), 0).
\end{cases}
$$

Expansion of ϕ^ε

So far, we obtain the expansion of the solution $(a^\varepsilon, v^\varepsilon)$ of (2.3.2). At the final step, we derive the expansion of $(a^\varepsilon, \phi^\varepsilon)$ of (2.3.1). To do this, it suffices to expand ϕ^ε. Recall that ϕ^ε is defined in the proof of Theorem 2.3.1 by the formula

$$\phi^\varepsilon = \Phi_0 - \int_0^t \left(\frac{1}{2} |v^\varepsilon|^2 + f(|a^\varepsilon|^2) \right) ds.$$

Substituting (2.3.18) to this formula, we see that

$$\phi^\varepsilon = \sum_{j=0}^{k} \varepsilon^j \phi_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^{s-2k-1})$$

as $\varepsilon \to 0$, where ϕ_j is given by the following formula:

$$\phi_0(t) = \Phi_0 - \int_0^t \left(\frac{1}{2} |v_0|^2 + f(|a_0|^2) \right) ds \tag{2.3.19}$$

and, for $j \geqslant 1$,

$$
\begin{aligned}
\phi_j(t) = & - \int_0^t \sum_{i_1+i_2=j} \frac{1}{2} v_{i_1} \cdot v_{i_2} ds \\
& - \int_0^t \sum_{L=1}^{j} f^{(L)}(|a_0|^2) \sum_{\substack{J_1+\cdots+J_L=j \\ J_l \geqslant 1}} \prod_{l=1}^{L} \sum_{i_1+i_2=J_l} (a_{i_1} \overline{a_{i_2}}) ds.
\end{aligned} \tag{2.3.20}
$$

By (2.3.17) and the assumption on Φ, it is easy to see that $\phi_0 \in C([0,T]; H^{s+1})$ with $\phi_0 - \Phi_0 \in C([0,T]; W^{s+1,1})$, and that

$$\phi_j \in C([0,T]; W^{s-2j+1,1}) \cap C([0,T]; H^{s-2j+1}).$$

□

Remark 2.3.6. The integrand of the second time integral in the right hand side of (2.3.20) is the explicit formula of $N^{(j)}$, introduced in Section 2.2.3. If the nonlinearity is cubic, say $f(y) = \lambda y$, then, this term is simplified as

$$\lambda \sum_{i_1+i_2=j} 2\operatorname{Re}(a_{i_1}\overline{a_{i_2}}).$$

2.4 Example 2: Nonlocal nonlinearities

Now we turn to the case of nonlocal nonlinearities. Our equations are the Schrödinger- Poisson system (SP) and the Hartree equation (H). As we observed in Section (2.2), we mainly analyze the system (2.1.8). If we consider (SP) and (H), then the corresponding systems are

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon \cdot \nabla)a^\varepsilon + \frac{1}{2}a^\varepsilon \Delta\phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t \phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda V_{\mathrm{P}}^\varepsilon = 0, \\ -\Delta V_{\mathrm{P}}^\varepsilon = |a^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x| \to \infty, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0), \end{cases} \tag{2.4.1}$$

and

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon \cdot \nabla)a^\varepsilon + \frac{1}{2}a^\varepsilon \Delta\phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t \phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0), \end{cases} \tag{2.4.2}$$

respectively. When $n \geqslant 3$, (SP) and (2.4.1) correspond to (H) and (2.4.2) with $\gamma = n-2$ since the Newtonian potential is written as $c_n|x|^{2-n}$ for $n \geqslant 3$ (see [33]). We therefore mainly treat (2.4.2) with $n \geqslant 3$.

2.4.1 Smoothing by the nonlocal nonlinearity

Let us first discuss how to manage to obtain an energy estimate in the case of nonlocal nonlinearity. Differentiation of the second equation of (2.4.2) yields

$$\begin{cases} \partial_t a^\varepsilon + (v^\varepsilon \cdot \nabla)a^\varepsilon + \frac{1}{2}a^\varepsilon \nabla \cdot v^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla)v^\varepsilon + \lambda\nabla(|x|^{-\gamma} * |a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), v^\varepsilon(0,x)) = (A_0^\varepsilon, \nabla\Phi_0), \end{cases} \tag{2.4.3}$$

which corresponds to (SHS). As presented in Section 2.2.1, the estimate of $\frac{d}{dt}\|a^\varepsilon\|_{H^s}^2$ involves $\|\nabla v^\varepsilon\|_{H^s}$: We deduce from (2.2.1), (2.2.3), (2.2.4), and (2.2.2) that

$$\frac{d}{dt}\|a^\varepsilon\|_{H^s}^2 \leqslant C(\|a^\varepsilon\|_{W^{1,\infty}} + \|\nabla v^\varepsilon\|_{L^\infty})(\|a^\varepsilon\|_{H^s}^2 + \|\nabla v^\varepsilon\|_{H^s}^2). \tag{2.4.4}$$

In the previous Section 2.3, we derive the cancellation which vanishes out both this bad term and another bad term in the nonlinearity of local type. Now, we are concerned with nonlocal nonlinearities involving integrals, and so it seems to be impossible to make this kind of cancellation. We hence accept the estimate of $\|\nabla v^\varepsilon\|_{H^s}$. Thus, as mentioned at the end of Section 2.2.1, the problem is that we have to estimate the $(s+2)$-time derivative of the nonlinearity by the s-time derivative of the amplitude a^ε. Fortunately, it is possible in the case of nonlocal nonlinearity. We use the following estimate used in [5, 15] to make the nonlocal nonlinearity produce two-time derivative gain. This is the key.

Lemma 2.4.1. *Let $n \geqslant 1$, $k \geqslant 0$, and $s \in \mathbb{R}$. Let $\gamma \in (0,n)$ satisfy $\frac{n}{2} - k < \gamma \leqslant n-k$. Then, there exists $C = C(n,k,p,s,\gamma)$ such that, for all $f \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$,*

$$\left\| |\nabla|^k (|x|^{-\gamma} * f) \right\|_{H^s} \leqslant C(\|f\|_{H^s} + \|f\|_{L^1}).$$

Proof. Since $\mathcal{F}|x|^{-\gamma} = C|\xi|^{-n+\gamma}$ for $\gamma \in (0,n)$, it holds that

$$\left\| |\nabla|^k (|x|^{-\gamma} * f) \right\|_{H^s} = C \left\| \langle \xi \rangle^s |\xi|^{-n+\gamma+k} \mathcal{F} f \right\|_{L^2}.$$

The high frequency part ($|\xi| > 1$) is bounded by $C\|f\|_{H^s}$ if $-n+\gamma+k \leqslant 0$. On the other hand, the low frequency part ($|\xi| \leqslant 1$) is bounded by

$$C\|\mathcal{F}f\|_{L^\infty} \left(\int_{|\xi|\leqslant 1} |\xi|^{2(-n+\gamma+k)} d\xi \right)^{\frac{1}{2}} \leqslant C\|f\|_{L^1}$$

if $2(-n+\gamma+k) > -n$, that is, if $\gamma > n/2 - k$. □

2.4.2 Existence result

Differentiating the second line of (2.4.3) again, we obtain

$$\begin{cases} \partial_t a^\varepsilon + (v^\varepsilon \cdot \nabla) a^\varepsilon + \frac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i\frac{\varepsilon}{2} \Delta a^\varepsilon, \\ \partial_t \nabla v^\varepsilon + \nabla (v^\varepsilon \cdot \nabla) v^\varepsilon + \lambda \nabla^2 (|x|^{-\gamma} * |a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), \nabla v^\varepsilon(0,x)) = (A_0^\varepsilon, \nabla^2 \Phi_0). \end{cases} \tag{2.4.5}$$

In the local nonlinearity case, we work with (2.3.2) corresponding to (SHS) which can be solved in the usual H^s framework with cancellation. However, it turns out that, in the nonlocal nonlinearity case, it is not (2.4.3) (corresponding to (SHS) and (2.3.2)) but (2.4.3) which can be solved in the usual H^s framework. Adding a L^∞-bound of v^ε, we obtain the solution to (2.4.3) in the Zhidkov space.

Theorem 2.4.2. *Let $n \geqslant 3$ and $\lambda \in \mathbb{R}$. Let γ be a positive number with $n/2-2 < \gamma \leqslant n-2$. Let $s > n/2+1$. Assume that $\Phi_0 \in C^3$ with $\nabla^2\Phi_0 \in H^s$, and that A_0^ε is uniformly bounded in H^s for $\varepsilon \in [0,1]$. Then, there exist $T > 0$ independent of $\varepsilon \in [0,1]$ and $s > n/2+1$, and $(a^\varepsilon, \phi^\varepsilon) \in C([0,T]; C^1 \times C^4)$ unique solution to (2.4.2) on $[0,T]$ for $\varepsilon \in [0,1]$. Moreover, a^ε and $\nabla\phi^\varepsilon$ are bounded in $C([0,T]; H^s)$ and $L^\infty([0,T] \times \mathbb{R}^n)$, respectively, uniformly in $\varepsilon \in [0,1]$. Moreover, with the notation $c_\infty := \lim_{|x|\to\infty} \nabla\Phi_0(x) \in \mathbb{R}^n$, ϕ^ε enjoys the following properties:*

- *(Tail estimates of ϕ^ε) It holds that*

$$\nabla\phi^\varepsilon(t,x) - \nabla\Phi_0(x - c_\infty t) \in (L^{\frac{n}{\gamma+1}+} \cap L^\infty)(\mathbb{R}^n), \tag{2.4.6}$$

$$\phi^\varepsilon(t,x) - \Phi_0(x) + \frac{1}{2}\int_0^t |\nabla\Phi_0(x - c_\infty s)|^2 ds \in (L^{\frac{n}{\gamma}+} \cap L^\infty)(\mathbb{R}^n). \tag{2.4.7}$$

  Furthermore, they are bounded in above norm uniformly in $t \in [0,T]$ and $\varepsilon \in [0,1]$.

- *If $\nabla\Phi$ decays at spacial infinity, that is, if $c_\infty = 0$, then $\nabla\phi^\varepsilon$ and $\phi^\varepsilon - \Phi_0$ are bounded in $C([0,T]; (L^{\frac{n}{\gamma+1}+} \cap L^\infty)(\mathbb{R}^n))$ and $C([0,T]; (L^{\frac{n}{\gamma}+} \cap L^\infty)(\mathbb{R}^n))$, respectively, uniformly in $\varepsilon \in [0,1]$.*

Remark 2.4.3. Since $n \geqslant 3$, by means of Lemma 2.2.1 and the Sobolev embedding, the assumption $\nabla^2\Phi_0 \in H^s$ implies the existence of a constant c_∞ such that $\nabla\Phi_0 - c_\infty \in L^p$ for $p \in [2n/(n-2), \infty]$. Similarly, $\nabla\phi^\varepsilon - c'_\infty \in L^p$ holds with some constant c'_∞ for $p \in [2n/(n-2), \infty]$. Thus, (2.4.7) is the asymptotics in such a sense that $c'_\infty = c_\infty$ and moreover $\nabla\phi^\varepsilon(t,x) - \nabla\Phi_0(x - c_\infty t) \in L^q$ for $q \in (n/(\gamma+1), \infty]$. Recall that $n/(\gamma+1) < 2n/(n-2) = n/(n/2-1)$ by assumption on γ.

Remark 2.4.4. In general, both ϕ^ε and Φ_0 are not bounded in any Lebesgue space. If $n \geqslant 5$ then Lemma 2.2.1 implies that there exist a real constant d such that $\Phi_0 - d - c_\infty \cdot x \in L^{2n/(n-4)} \cap L^\infty$. This is not true for $n \leqslant 4$, as shown by the following example; $f(x) = \log(1 + \log|x|)$, which is not bounded nor this form but satisfies $\nabla^2 f \in H^\infty(\mathbb{R}^n)$. Nevertheless, (2.4.7) shows the left hand side is always bounded and decays at spacial infinity. We also remark that

$$\begin{aligned}
&\nabla\left(\Phi_0(x) - \frac{1}{2}\int_0^t |\nabla\Phi_0(x - c_\infty s)|^2 ds\right) \\
&= \nabla\Phi_0(t - c_\infty x) - \int_0^t (\partial_t(\nabla\Phi_0(x - c_\infty t)))_{|t=s} ds - \frac{1}{2}\int_0^t \nabla|\nabla\Phi_0(x - c_\infty s)|^2 ds \\
&= \nabla\Phi_0(t - c_\infty x) - \int_0^t ((\nabla\Phi_0(x - c_\infty s) - c_\infty) \cdot \nabla)\nabla\Phi_0(x - c_\infty s) ds.
\end{aligned}$$

The first term of the right hand side is bounded and the difference of the first term and $\nabla\phi^\varepsilon$ belongs to $(L^{\frac{n}{\gamma+1}+}\cap L^\infty)$ as shown in (2.4.6). The second term is a "good tail" term belonging to L^r for $r\in[n/(n-1),\infty]$. Recall that $n/(\gamma+1)\geqslant n/(n-1)$ by assumption on γ.

Remark 2.4.5. If $\Phi_0\equiv 0$ then we have the uniform bound of $\|\phi^\varepsilon\|_{L^p}$ and $\|\nabla\phi^\varepsilon\|_{L^q}$ for $p\in(n/\gamma,\infty]$ and $q\in(n/(\gamma+1),\infty]$.

Remark 2.4.6. The decay property of ϕ^ε is different from in the case of local nonlinearities: The solution $(a_1^\varepsilon,\phi_1^\varepsilon)$ to (2.3.1) satisfies $\phi_1^\varepsilon-\Phi_0\in C([0,T];W^{s,1})$ (Theorem 2.3.1).

Remark 2.4.7. In the Schrödinger-Poisson case, the corresponding result follows by letting $\gamma=n-2$.

Proof. The proof is based on the classical energy method. We set a partial energy

$$E_{\text{part}}(t):=\|a^\varepsilon\|_{H^s}^2+\|\nabla v^\varepsilon\|_{H^s}^2.$$

Estimates similar to (2.2.7) and (2.2.8) give

$$\frac{d}{dt}\|\nabla v^\varepsilon\|_{H^s}^2\leqslant C(\|\nabla v^\varepsilon\|_{L^\infty}\|\nabla v^\varepsilon\|_{H^s}^2+\left\|\nabla^2(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{H^s}\|\nabla v^\varepsilon\|_{H^s}).$$

We apply above Lemma 2.4.1 with $k=2$. Then,

$$\begin{aligned}\left\|\nabla^2(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{H^s}&\leqslant C(\left\||a^\varepsilon|^2\right\|_{L^1}+\left\||a^\varepsilon|^2\right\|_{H^s})\\&\leqslant C(\|a^\varepsilon\|_{L^2}+\|a^\varepsilon\|_{L^\infty})\|a^\varepsilon\|_{H^s}.\end{aligned}$$

Therefore, we end up with

$$\frac{d}{dt}\|\nabla v^\varepsilon\|_{H^s}^2\leqslant C(\|\nabla v^\varepsilon\|_{L^\infty}+\|a^\varepsilon\|_{L^2}+\|a^\varepsilon\|_{L^\infty})(\|a^\varepsilon\|_{H^s}^2+\|\nabla v^\varepsilon\|_{H^s}^2).$$

Together with (2.4.4), this implies the desired (partial) energy estimate

$$\frac{d}{dt}E_{\text{part}}(t)\leqslant C(E_{\text{part}}(t))^{\frac{3}{2}}.$$

This estimate is partial in such a sense that we do not obtain any information about the boundedness of v^ε itself.

First integration and decay properties of v^ε

Let us add the bound of v^ε. By Lemma 2.2.1, there exists a function $F^\varepsilon(t)$ of time only such that $v^\varepsilon+F^\varepsilon(t)$ belongs to $C([0,T];L^{2n/(n-2)})$. It also follows from this lemma that $F^\varepsilon(0)$ is uniquely determined as a constant such that $\nabla\Phi_0+F^\varepsilon(0)\in L^{2n/(n-2)}$. Therefore, $F^\varepsilon(0)$ is independent of ε. Let us denote $c_\infty:=F^\varepsilon(0)$. We now use the Sobolev embedding and Lemma 2.2.1: For $n\geqslant 3$ and $\sigma>n/2$, if $f\to 0$ as $|x|\to\infty$ then

$$\|f\|_{L^\infty}\leqslant C\|\nabla f\|_{H^{\sigma-1}}.$$

This yields the uniform L^∞-bounds $\|v^\varepsilon + F^\varepsilon(t)\|_{L^\infty([0,T]\times\mathbb{R}^n)} < \infty$. Since $F^\varepsilon(t)$ is a time function, we also have $\|v^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^n)} < \infty$ for each fixed ε. Therefore, we obtain the (full) energy estimate

$$\frac{d}{dt}E(t) \leqslant C(E(t))^{\frac{3}{2}},$$

where $E(t) := \|a^\varepsilon\|_{H^s}^2 + \|v^\varepsilon\|_{L^\infty}^2 + \|\nabla v^\varepsilon\|_{H^s}^2$. From this estimate, we obtain a solution $(a^\varepsilon, v^\varepsilon) \in C([0,T]; H^s \times X^{s+1})$ of (2.4.3). For the detail of the proof of this part, see [5, 15]. Note that, at this step, we do not now whether v^ε is bounded in $L^\infty(\mathbb{R}^n)$ uniformly in ε or not, and so that the existence time T may depend on ε. Applying the Hölder and the Hardy-Littlewood-Sobolev inequalities to the second equation of (2.4.3), we have

$$\partial_t v^\varepsilon = -(v^\varepsilon \cdot \nabla)v^\varepsilon - \lambda\nabla(|x|^{-\gamma} * |a^\varepsilon|^2) \in L^{\max(2, \frac{n}{\gamma+1}+)}. \tag{2.4.8}$$

In particular, this implies $(F^\varepsilon)' \equiv 0$. Recall that $F^\varepsilon(0)$ is independent of ε. It turns out that $F^\varepsilon(t) = c_\infty$ as long as v^ε exists. Therefore, we can choose the existence time T independently of ε. We also have $v^\varepsilon - \nabla\Phi_0 \in L^{\max(2, \frac{n}{\gamma+1}+)}$ from (2.4.8).

Second integration and construction of ϕ^ε

Now, we shall define ϕ^ε. As mentioned in Section 2.2.2, the main step is to show the uniqueness of the solution of (2.4.3). Let $(a_i^\varepsilon, v_i^\varepsilon) \in C([0,T]; H^s \times X^{s+1})$ be two solutions to (2.4.3) with $v_i - \nabla\Phi_0 \in L^{\max(2, \frac{n}{\gamma+1}+)}$. Then, denoting $d_a^\varepsilon = a_1^\varepsilon - a_2^\varepsilon$ and $d_v^\varepsilon = v_1^\varepsilon - v_2^\varepsilon$, we find

$$\begin{cases} \partial_t d_a^\varepsilon + (d_v^\varepsilon \cdot \nabla)a_1^\varepsilon + (v_2^\varepsilon \cdot \nabla)d_a^\varepsilon + \frac{1}{2}d_a^\varepsilon \nabla \cdot v_1^\varepsilon + \frac{1}{2}a_2^\varepsilon \nabla \cdot d_v^\varepsilon = i\frac{\varepsilon}{2}\Delta d_a^\varepsilon, \\ \partial_t d_v^\varepsilon + (d_v^\varepsilon \cdot \nabla)v_1^\varepsilon + (v_2^\varepsilon \cdot \nabla)d_v^\varepsilon + \lambda\nabla(|x|^{-\gamma} * (d_a^\varepsilon \overline{a_1^\varepsilon} + a_2^\varepsilon \overline{d_a^\varepsilon}) = 0, \\ (d_a^\varepsilon(0,x), d_v^\varepsilon(0,x)) = (0,0). \end{cases}$$

We now define $E_{d,\mathrm{part}}(t) := \|d_a^\varepsilon\|_{L^2}^2 + \|\nabla d_v^\varepsilon\|_{L^2}^2$. Since

$$\frac{d}{dt}E_{d,\mathrm{part}}(t) \leqslant C(E_{d,\mathrm{part}}(t))^{\frac{3}{2}}$$

by the same calculation as in $E_{\mathrm{part}}(t)$, we see $a_1^\varepsilon - a_2^\varepsilon = \nabla(v_1^\varepsilon - v_2^\varepsilon) = 0$. Moreover, by assumption, we have $v_1^\varepsilon - v_2^\varepsilon = (v_1^\varepsilon - \nabla\Phi_0) - (v_2^\varepsilon - \nabla\Phi_0) \in L^{\max(2, \frac{n}{\gamma+1}+)}$. In particular, $v_1^\varepsilon - v_2^\varepsilon \to 0$ as $|x| \to \infty$. Thus, $a_1 = a_2$ and $v_1 = v_2$ holds, and hence the solution is unique. Once the uniqueness of the solution of (2.4.3) is deduced, we can use direct definition

$$\phi^\varepsilon(t,x) = \Phi_0(x) - \int_0^t \left(\frac{1}{2}|v^\varepsilon(s,x)|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2)(s,x)\right) ds \in C([0,T]; C^3).$$

Asymptotic behavior of ϕ^ε

Let us prove (2.4.6). A computation show

$$\begin{aligned}\partial_t(v^\varepsilon(t,x+c_\infty t)) &= (\partial_t v^\varepsilon)(t,x+c_\infty t) + ((c_\infty\cdot\nabla)v^\varepsilon)(t,x+c_\infty t)\\ &= -\big[((v^\varepsilon - c_\infty)\cdot\nabla)v^\varepsilon + \lambda\nabla(|x|^{-\gamma} * |a^\varepsilon|)\big](t,x+c_\infty t).\end{aligned}$$

Since $v^\varepsilon - c_\infty \to 0$ as $|x|\to\infty$, $\|v^\varepsilon - c_\infty\|_{L^p} \leqslant C\|\nabla v^\varepsilon\|_{H^s}$ is uniformly bounded for $p\in[2n/(n-2),\infty]$. Then, by the Hölder inequality,

$$((v^\varepsilon - c_\infty)\cdot\nabla)v^\varepsilon \in L^q(\mathbb{R}^n)$$

holds for $q\in[n/(n-1),\infty]$. Moreover, by the Sobolev and the Hardy-Littlewood-Sobolev inequalities,

$$\nabla(|x|^{-\gamma} * |a^\varepsilon|) \in L^r(\mathbb{R}^n)$$

for $r\in(n/(\gamma+1),\infty]$. Therefore,

$$v^\varepsilon(t,x+c_\infty t) - \nabla\Phi_0(x) = \int_0^t (\partial_t(v^\varepsilon(t,x+c_\infty t)))_{|t=s}ds \in L^r(\mathbb{R}^n)$$

for $r\in(n/(\gamma+1),\infty]$. Hence (2.4.6). We finally prove (2.4.7). By the definition of ϕ^ε,

$$\begin{aligned}&\phi^\varepsilon(t,x) - \Phi_0(x) + \frac{1}{2}\int_0^t |\nabla\Phi_0(x-c_\infty s)|^2 ds\\ =& \int_0^t \frac{1}{2}((\nabla\Phi_0(x-c_\infty s) - v^\varepsilon(s,x))\cdot\nabla\Phi_0(x-c_\infty s))ds\\ &+\int_0^t \frac{1}{2}(v^\varepsilon(s,x)\cdot(\nabla\Phi_0(x-c_\infty s) - v^\varepsilon(s,x)))ds\\ &-\int_0^t \lambda(|x|^{-\gamma} * |a^\varepsilon|^2)(s,x)ds.\end{aligned}$$

By the L^∞ bound of v^ε and the asymptotics (2.4.6), the first two terms of the right hand side belong to L^r for $r\in(n/(\gamma+1),\infty]$. On the other hand, by the Hardy-Littlewood-Sobolev inequality, the last term is in $L^{r'}$ for $r'\in(n/\gamma,\infty]$, which completes the proof. □

2.4.3 Justification of WKB approximation

Theorem 2.4.8. *Let $n\geqslant 3$ and $\lambda\in\mathbb{R}$. Let γ be a positive number with $n/2-2<\gamma\leqslant n-2$. Let k be a positive integer and let $s>n/2+2k+3$ be a real number. Assume that $\Phi_0\in C^{2k+5}$ with $\nabla^2\Phi_0\in H^s$, and that A_0^ε writes*

$$A_0^\varepsilon = \sum_{j=0}^k \varepsilon^j A_j + o(\varepsilon^k) \quad \textit{in } H^s$$

for $\varepsilon \in [0,1]$. Then, the unique solution $(a^\varepsilon, \phi^\varepsilon)$ of (2.4.2) has the following expansion:

$$\begin{cases} a^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j a_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^{s-2k-2}), \\ \phi^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j \phi_j + o(\varepsilon^k) \quad \text{in } C([0,T]; L^{\frac{n}{\gamma}+} \cap L^\infty). \\ \nabla\phi^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j \nabla\phi_j + o(\varepsilon^k) \quad \text{in } C([0,T]; X^{s-2k-1} \cap L^{\frac{n}{\gamma+1}+}). \end{cases}$$

Moreover, ϕ_0 has the same asymptotic behavior as ϕ^ε given in Theorem 2.4.2.

Remark 2.4.9. As in Theorem 2.4.2, ϕ^ε and ϕ_0 do not necessarily goes to zero as $|x|$ tends to infinity, while their distance satisfies $\phi^\varepsilon - \phi_0 \in L^{\frac{n}{\gamma}+} \cap L^\infty$ and $\nabla\phi^\varepsilon - \nabla\phi_0 \in L^{\frac{n}{\gamma+1}+} \cap L^\infty$.

Proof. We proceeds along in the similar way as in the proof of Theorem 2.3.3 (or as outlined in Section 2.2.3). Let $(a^\varepsilon, v^\varepsilon) = (a^\varepsilon, \nabla\phi^\varepsilon)$ be a solution to (2.4.3) given in the proof of Theorem 2.4.2. We first prove the expansion

$$\begin{cases} a^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j a_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^{s-2k-2}), \\ v^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j v_j + o(\varepsilon^k) \quad \text{in } C([0,T]; X^{s-2k-1} \cap L^{\frac{n}{\gamma+1}+}). \end{cases} \tag{2.4.9}$$

Since $A_0 = A^\varepsilon_{0|\varepsilon=0}$ exists, we obtain $(a_0, v_0) = (a^\varepsilon, v^\varepsilon)_{|\varepsilon=0}$ which solves

$$\begin{cases} \partial_t a_0 + (v_0 \cdot \nabla) a_0 + \dfrac{1}{2} a_0 \nabla \cdot v_0 = 0, \\ \partial_t v_0 + (v_0 \cdot \nabla) v_0 + \lambda \nabla(|x|^{-\gamma} * |a_0|^2) = 0, \\ (a_0(0,x), v_0(0,x)) = (A_0, \nabla\Phi_0). \end{cases} \tag{2.4.10}$$

The zeroth order

Introduce $(\widetilde{a}^\varepsilon_0, \widetilde{v}^\varepsilon_0) = (a^\varepsilon - a_0, v^\varepsilon - v_0)$. This solves the system

$$\begin{cases} \partial_t \widetilde{a}^\varepsilon_0 + (\widetilde{v}^\varepsilon_0 \cdot \nabla) a^\varepsilon + (v_0 \cdot \nabla) \widetilde{a}^\varepsilon_0 + \dfrac{1}{2} \widetilde{a}^\varepsilon_0 \nabla \cdot v^\varepsilon + \dfrac{1}{2} a_0 \nabla \cdot \widetilde{v}^\varepsilon_0 = i\dfrac{\varepsilon}{2} \Delta \widetilde{a}^\varepsilon_0 + i\dfrac{\varepsilon}{2} \Delta a_0, \\ \partial_t \widetilde{v}^\varepsilon_0 + (\widetilde{v}^\varepsilon_0 \cdot \nabla) v^\varepsilon + (v_0 \cdot \nabla) \widetilde{v}^\varepsilon_0 + \nabla(|x|^{-\gamma} * (\widetilde{a}^\varepsilon_0 \overline{a^\varepsilon} + a_0 \overline{\widetilde{a}^\varepsilon_0})) = 0, \\ (\widetilde{a}^\varepsilon_0(0,x), \widetilde{v}^\varepsilon_0(0,x)) = (A^\varepsilon_0 - A_0, 0). \end{cases} \tag{2.4.11}$$

Mimicking the energy estimate for $(a^\varepsilon, v^\varepsilon)$, we obtain

$$\begin{aligned}\frac{d}{dt}\widetilde{E}_0(t) &\leqslant C(\widetilde{E}_0(t))^{\frac{3}{2}} + C\varepsilon \|\Delta a_0\|_{H^s} (\widetilde{E}_0(t))^{\frac{1}{2}} \\ &\leqslant C_1(\widetilde{E}_0(t))^{\frac{3}{2}} + C_2\varepsilon,\end{aligned}$$

where $\widetilde{E}_0(t) := \|\widetilde{a}_0^\varepsilon\|_{H^s}^2 + \|\widetilde{v}_0^\varepsilon\|_{L^{\frac{n}{\gamma+1}+}}^2 + \|\widetilde{v}_0^\varepsilon\|_{L^\infty}^2 + \|\nabla\widetilde{v}_0^\varepsilon\|_{H^s}^2$ and C_i depends on $\|a_0\|_{H^{s+2}}$, $\|v_0\|_{X^{s+2}}$, $\|a^\varepsilon\|_{H^{s+1}}$, and $\|v^\varepsilon\|_{X^{s+2}}$. Note that by using the fact that $\widetilde{v}^\varepsilon_{0|t=0} \equiv 0$, we can add the term $\|\widetilde{v}_0^\varepsilon\|_{L^{\frac{n}{\gamma+1}+}}^2$ to the energy. This yields

$$\sup_{t\in[0,T]} \widetilde{E}_0(t) \leqslant \widetilde{E}_0(0)e^{C_1T} + \varepsilon\frac{C_2}{C_1}(e^{C_1T} - 1) \to 0$$

as $\varepsilon \to 0$, which ensures (2.4.9) for $k = 0$.

The first order

We put $(b_1^\varepsilon, w_1^\varepsilon) := (\widetilde{a}_0^\varepsilon/\varepsilon, \widetilde{v}_0^\varepsilon/\varepsilon)$ and $E_1(t) := \|b_1^\varepsilon\|_{H^s}^2 + \|w_1^\varepsilon\|_{L^\infty}^2 + \|\nabla w_1^\varepsilon\|_{H^s}^2$. Since $E_1(t) = \widetilde{E}_0(t)/\varepsilon$ and it is bounded at $t = 0$ uniformly in ε, we see that $\sup_{t\in[0,T]} E_1(t)$ is uniformly bounded. Therefore, we obtain $(b_1^\varepsilon, w_1^\varepsilon) \in C([0,T]; H^s \times X^{s+1} \cap L^{n/(\gamma+1)+})$ as a unique solution of a system similar to (2.4.11), provided $a_0 \in H^{s+2}$. Since $A_1 = b^\varepsilon_{1|t=0,\varepsilon=0}$ exists by assumption, we can define $(a_1, v_1) := (b_1^\varepsilon, w_1^\varepsilon)_{|\varepsilon=0}$, which solves

$$\begin{cases} \partial_t a_1 + (v_1\cdot\nabla)a_0 + (v_0\cdot\nabla)a_1 + \frac{1}{2}a_1\nabla\cdot v_0 + \frac{1}{2}a_0\nabla\cdot v_1 = i\frac{1}{2}\Delta a_0, \\ \partial_t v_1 + (v_1\cdot\nabla)v_0 + (v_0\cdot\nabla)v_1 + \operatorname{Re}\nabla(|x|^{-\gamma} * (a_1\overline{a_0} + a_0\overline{a_1})) = 0, \\ (a_1(0,x), v_1(0,x)) = (A_1, 0). \end{cases} \tag{2.4.12}$$

Let us estimate $(\widetilde{a}_1^\varepsilon, \widetilde{v}_1^\varepsilon) := (b_1^\varepsilon - a_1, w_1^\varepsilon - v_1)$. From (2.4.11) and (2.4.12), we see that

$$\begin{cases} \partial_t\widetilde{a}_1^\varepsilon + (\widetilde{v}_1^\varepsilon\cdot\nabla)a^\varepsilon + (v_1\cdot\nabla)\widetilde{a}_0^\varepsilon + (v_0\cdot\nabla)\widetilde{a}_1^\varepsilon \\ \qquad + \frac{1}{2}\widetilde{a}_1^\varepsilon\nabla\cdot v^\varepsilon + \frac{1}{2}a_1\nabla\cdot\widetilde{v}_0^\varepsilon + \frac{1}{2}a_0\nabla\cdot\widetilde{v}_1^\varepsilon = i\frac{\varepsilon}{2}\Delta\widetilde{a}_1^\varepsilon + i\frac{\varepsilon}{2}\Delta a_1, \\ \partial_t\widetilde{v}_1^\varepsilon + (\widetilde{v}_1^\varepsilon\cdot\nabla)v^\varepsilon + (v_1\cdot\nabla)\widetilde{v}_0^\varepsilon + (v_0\cdot\nabla)\widetilde{v}_1^\varepsilon \\ \qquad + \nabla(|x|^{-\gamma} * (\widetilde{a}_1^\varepsilon\overline{a^\varepsilon} + a_1\overline{\widetilde{a}_0^\varepsilon} + a_0\overline{\widetilde{a}_1^\varepsilon})) = 0, \\ (\widetilde{a}_1^\varepsilon(0,x), \widetilde{v}_1^\varepsilon(0,x)) = \left(\dfrac{A_0^\varepsilon - A_0 - \varepsilon A_1}{\varepsilon}, 0\right). \end{cases} \tag{2.4.13}$$

We use the energy method to obtain

$$\begin{aligned}\frac{d}{dt}\widetilde{E}_1(t) &\leqslant C(\widetilde{E}_1(t))^{\frac{3}{2}} + C\varepsilon \|\Delta a_1\|_{H^s} (\widetilde{E}_1(t))^{\frac{1}{2}} \\ &\leqslant C_1(\widetilde{E}_1(t))^{\frac{3}{2}} + C_2\varepsilon,\end{aligned}$$

where $\widetilde{E}_1(t) := \|\widetilde{a}_1^\varepsilon\|_{H^s}^2 + \|\widetilde{v}_1^\varepsilon\|_{L^{\frac{n}{\gamma+1}+}}^2 + \|\widetilde{v}_1^\varepsilon\|_{L^\infty}^2 + \|\nabla\widetilde{v}_1^\varepsilon\|_{H^s}^2$ and C_i depends on $\|a_1\|_{H^{s+2}}$. This yields

$$\sup_{t\in[0,T]} \widetilde{E}_1(t) \leqslant \widetilde{E}_1(0)e^{C_1T} + \varepsilon\frac{C_2}{C_1}(e^{C_1T}-1) \to 0$$

as $\varepsilon \to 0$, which gives (2.4.9) for $k=1$. Higher order estimate is similar, so we left the detail. The strategy is the induction argument as in the proof of Theorem 2.3.3. We only remark that the system for (a_j, v_j) $(j \geqslant 1)$ is given as

$$\begin{cases} \partial_t a_j + \displaystyle\sum_{i_1+i_2=j} \left(\frac{1}{2}(v_{i_1}\cdot\nabla)a_{i_2} + a_{i_1}\nabla\cdot v_{i_2}\right) - \frac{i}{2}\Delta a_{j-1} = 0, \\ \partial_t v_j + \displaystyle\sum_{i_1+i_2=j} \big((v_{i_1}\cdot\nabla)v_{i_2} + \lambda\nabla(|x|^{-\gamma} * (a_{i_1}\overline{a_{i_2}}))\big) = 0, \\ \qquad (a_j(0,x), v_j(0,x)) = (A_j(x), 0). \end{cases}$$

Expansion of ϕ^ε

We finally prove the expansion of ϕ^ε. Since ϕ^ε is given by the formula

$$\phi^\varepsilon = \Phi_0 - \int_0^t \left(\frac{1}{2}|v^\varepsilon|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2)\right) ds.$$

Now, we plug the expansion (2.4.9) to this. Then, this concludes

$$\phi^\varepsilon = \sum_{j=0}^k \varepsilon^j\phi_j + o(\varepsilon^k) \quad \text{in } L^\infty([0,T]\times\mathbb{R}^n),$$

where

$$\phi_0(t) = \Phi_0 - \int_0^t \left(\frac{1}{2}|v_0|^2 + \lambda(|x|^{-\gamma} * |a_0|^2)\right) ds \tag{2.4.14}$$

and, for $j \geqslant 1$,

$$\phi_j(t) = -\int_0^t \sum_{i_1+i_2=j} \left(\frac{1}{2}v_{i_1}\cdot v_{i_2} + \lambda(|x|^{-\gamma} * (a_{i_1}\overline{a_{i_2}}))\right) ds. \tag{2.4.15}$$

Notice that $(a_0, \phi_0) = (a^\varepsilon, \phi^\varepsilon)_{|\varepsilon=0}$ and so the asymptotic behavior of ϕ_0 is the same as for ϕ^ε given in Theorem 2.4.2. Furthermore, $\phi_j \in C([0,T]; L^{n/\gamma+} \cap L^\infty)$ for $j \geqslant 1$ by the Sobolev and the Hardy-Littlewood-Sobolev inequalities. $\square$

Remark 2.4.10. The integrand of the second time integral in the right hand side of (2.4.15) is the explicit formula of $N^{(j)}$, introduced in Section 2.2.3:

$$N^{(j)} = \sum_{i_1+i_2=j} \lambda(|x|^{-\gamma} * (a_{i_1}\overline{a_{i_2}}))$$

2.5 Example 3: Local and nonlocal nonlinearity

We next consider the presence of both the local and the nonlocal nonlinearities. We consider the following model, the nonlinear Schrödinger equation with local and nonlocal nonlinearity,

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f(|u^\varepsilon|^2) + \lambda(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon; \quad u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon), \tag{L-NL}$$

where f is supposed to be essentially cubic: $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $f \in C^\infty(\mathbb{R}_+)$, $f' > 0$, and $f(0) = 0$. Writing $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$, we find the system

$$\begin{cases} \partial_t a^\varepsilon + (v^\varepsilon \cdot \nabla) a^\varepsilon + \frac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t \phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + f(|a^\varepsilon|^2) + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0), \end{cases} \tag{2.5.1}$$

2.5.1 Cancellation versus smoothing

The difficulty of obtaining WKB approximation lies in obtaining energy estimate for the system for $(a^\varepsilon, \nabla\phi^\varepsilon)$ (Section 2.2). As we seen in Section 2.3, if the nonlinearity is of local type then we obtain the energy estimate by deriving the cancellation of bad terms. On the other hand, if the nonlinearity is of nonlocal type then we use the smoothing property of nonlinearity (Section 2.4). In this section, we shall observe what happens with the existence of both local and nonlocal nonlinearities. Let us introduce $v^\varepsilon := \nabla\phi^\varepsilon$ and consider

$$\begin{cases} \partial_t a^\varepsilon + (v^\varepsilon \cdot \nabla) a^\varepsilon + \frac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla f(|a^\varepsilon|^2) + \lambda\nabla(|x|^{-\gamma} * |a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0), \end{cases} \tag{2.5.2}$$

If we use the energy $E(t) := \|a^\varepsilon\|_{H^s}^2 + \|v^\varepsilon\|_{H^s}$, then the difficulty is the following two points (see Section 2.2.1):

1. The estimate of $\frac{d}{dt}\|a^\varepsilon\|_{H^s}^2$ requires the bound of $(s+1)$-time derivative of v^ε.
2. The estimate of $\frac{d}{dt}\|v^\varepsilon\|_{H^s}^2$ requires the bound of $(s+1)$-time derivative of the nonlinearity$f(|a^\varepsilon|^2) + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2)$.

It might be necessary to produce the cancellation by the local nonlinearity $\nabla f(|a^\varepsilon|^2)$ which solves the above two problem simultaneously. Otherwise, it would be difficult to handle the bad term coming from the local nonlinearity $f(|a^\varepsilon|^2)$, although we can accept the $(s+2)$-time derivative of $(|x|^{-\gamma} * |a^\varepsilon|)$

by the smoothing property of the nonlocal nonlinearity. Once the cancellation occurs, we do not need any longer to estimate $(s+2)$-time derivative of $(|x|^{-\gamma} * |a^\varepsilon|)$ by gaining two-time derivative. This causes the change of admissible range of γ. Thus, the nonlocal nonlinearity is almost a perturbation, however we can see the influence of the nonlocal nonlinearity in the tail estimate of ϕ^ε.

2.5.2 Existence result

Theorem 2.5.1. *Let $n \geqslant 2$ and $s > n/2+2$. Let $f \in C^\infty(\mathbb{R}_+ : \mathbb{R}_+)$ with $f(0) = 0$ and $f' > 0$. Let $\lambda \in \mathbb{R}$ and let γ be a positive number with $n/2-1 < \gamma \leqslant n-1$. Assume that $\Phi_0 \in X^{s+1}$ and A_0^ε is uniformly bounded in H^s for $\varepsilon \in [0,1]$. Then, there exist $T > 0$ independent of $\varepsilon \in [0,1]$ and $s > n/2+2$, and $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ solution to* (L-NL) *on $[0,T]$ for $\varepsilon \in (0,1]$. Moreover, $(a^\varepsilon, \phi^\varepsilon) \in C([0,T]; H^s \times X^{s+1})$ is the unique solution to* (2.5.1)*. Both a^ε and $\nabla\phi^\varepsilon$ are bonded in $C([0,T]; H^s)$ uniformly in $\varepsilon \in [0,1]$. Furthermore, $\phi^\varepsilon - \Phi_0$ is bounded in $C([0,T]; L^{n/\gamma+} \cap L^\infty)$ uniformly in $\varepsilon \in [0,1]$.*

Proof. We first show the existence of a unique solution to (2.5.2). Now, set the energy as

$$E(t) := \|a^\varepsilon\|_{H^s}^2 + \left\langle \frac{1}{4f'(|a^\varepsilon|^2)} \Lambda^s v^\varepsilon, \Lambda^s v^\varepsilon \right\rangle_{L^2},$$

where $s > n/2+1$ and $\Lambda = (1-\Delta)^{1/2}$. Take a constant C_0 so that $E(0) \leqslant (C_0)^{1/2}$. As long as $\|a^\varepsilon\|_{H^s} \leqslant 2C_0$, we obtain

$$\frac{d}{dt}E(t) \leqslant C(E(t))^{\frac{3}{2}} + C(E(t))^2 + \left| \left\langle \frac{\lambda}{4f'(|a^\varepsilon|^2)} \Lambda^s \nabla(|x|^{-\gamma} * |a^\varepsilon|^2), \Lambda^s v^\varepsilon \right\rangle \right|.$$

Estimates are the same as in the proof of Theorem 2.3.1. Lemma 2.4.1 with $k=1$ implies that

$$\left\|\nabla(|x|^{-\gamma} * |a^\varepsilon|^2)\right\|_{H^s} \leqslant C(\|a^\varepsilon\|_{L^\infty} \|a^\varepsilon\|_{H^s} + \|a^\varepsilon\|_{L^2}^2)$$

if $n/2-1 < \gamma \leqslant n-1$, and so that the third term of the right hand side is bounded by

$$(\|a^\varepsilon\|_{L^\infty} + \|a^\varepsilon\|_{L^2}) \|a^\varepsilon\|_{H^s} \|v^\varepsilon\|_{H^s} \leqslant C(E(t))^{\frac{3}{2}}$$

for such γ. Therefore, by Gronwall's lemma, there exists a time $T > 0$ depending only on $E(0)$ such that

$$\sup_{t\in[0,T]} E(t) \leqslant 4E(0) \leqslant (2C_0)^{\frac{1}{2}}.$$

This yields $\|a^\varepsilon\|_{H^s} \leqslant 2C_0$. Along the standard method, we see that the solution $(a^\varepsilon, v^\varepsilon) \in C([0,T]; H^s \times H^s)$ of (2.5.2) exists.

Uniqueness and construction of ϕ^ε

We next show the uniqueness of $(a^\varepsilon, v^\varepsilon)$. Let $(a_1^\varepsilon, v_1^\varepsilon)$ and $(a_2^\varepsilon, v_2^\varepsilon)$ be two solutions of (2.3.2) bounded in $C([0,T];H^s)^2$. Then, denoting $(d_a^\varepsilon, d_v^\varepsilon) = (a_1^\varepsilon - a_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon)$, we have

$$\begin{cases} \partial_t d_a^\varepsilon + (d_v^\varepsilon \cdot \nabla) a_1^\varepsilon + (v_2^\varepsilon \cdot \nabla) d_a^\varepsilon + \frac{1}{2} d_a^\varepsilon \nabla \cdot v_1^\varepsilon + \frac{1}{2} a_2^\varepsilon \nabla \cdot d_v^\varepsilon = i\frac{\varepsilon}{2}\Delta d_a^\varepsilon, \\ \partial_t d_v^\varepsilon + (d_v^\varepsilon \cdot \nabla) v_1^\varepsilon + (v_2^\varepsilon \cdot \nabla) d_v^\varepsilon + 2f'(|a_2^\varepsilon|^2)\,\mathrm{Re}(\overline{d_a^\varepsilon}\nabla a_1^\varepsilon + \overline{a_2^\varepsilon}\nabla d_a^\varepsilon) \\ \quad +(d_a^\varepsilon \overline{a_1^\varepsilon} + a_2^\varepsilon \overline{d_a^\varepsilon}) \displaystyle\int_0^1 f''(|a_2^\varepsilon|^2 + \theta(|a_1^\varepsilon|^2 - |a_2^\varepsilon|^2)) d\theta \nabla |a_1^\varepsilon|^2 \\ \qquad + \lambda \nabla(|x|^{-\gamma} * (d_a^\varepsilon \overline{a_1^\varepsilon} + a_2^\varepsilon \overline{d_a^\varepsilon})) = 0, \\ (d_a^\varepsilon(0,x), d_v^\varepsilon(0,x)) = (0,0). \end{cases}$$

We estimate

$$E_d(t) := \|d_a^\varepsilon\|_{L^s}^2 + \left\langle \frac{1}{4f'(|a_2^\varepsilon|^2)} d_v^\varepsilon, d_v^\varepsilon \right\rangle_{L^2}.$$

As in the proof of Theorem 2.3.1, we estimate

$$\frac{d}{dt}E_d(t) \leqslant C(\|a_i\|_{H^s}, \|a_i\|_{H^s}) E_d(t) + \left| \left\langle \frac{\lambda}{4f'(|a^\varepsilon|^2)} \nabla(|x|^{-\gamma} * (d_a^\varepsilon \overline{a_1^\varepsilon} + a_2^\varepsilon \overline{d_a^\varepsilon})), d_v^\varepsilon \right\rangle \right|.$$

By the use of the Hardy-Littlewood-Sobolev inequality and the Hölder inequality, the second term in the right hand side is bounded by

$$\begin{aligned} C \left\| \nabla(|x|^{-\gamma} * (d_a^\varepsilon \overline{a_1^\varepsilon} + a_2^\varepsilon \overline{d_a^\varepsilon})) \right\|_{L^2} \|d_v^\varepsilon\|_{L^2} \\ \leqslant C \left(\|a_1^\varepsilon\|_{L^{\frac{n}{n-(\gamma+1)}}} + \|a_2^\varepsilon\|_{L^{\frac{n}{n-(\gamma+1)}}} \right) \|d_a^\varepsilon\|_{L^2} \|d_v^\varepsilon\|_{L^2} \end{aligned}$$

for $n/2 - 1 < \gamma \leqslant n-1$, where we read $\frac{n}{n-(\gamma+1)} = \infty$ if $\gamma = n-1$. Therefore, we infer from Gronwall's lemma that

$$E_d(t) \leqslant C E_d(0) = 0$$

as long as (a_i, v_i) exists. Hence, the uniqueness holds. Then, using the argument in Section 2.2.2, we can determine ϕ^ε directly by

$$\phi^\varepsilon = \Phi_0 - \int_0^t \left(\frac{1}{2}|v^\varepsilon(s)|^2 + f(|a^\varepsilon(s)|^2) + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2) \right) ds \in C([0,T]; X^{s+1}).$$

One can easily check that $|v^\varepsilon|^2 \in L^1 \cap L^\infty$ and $f(|a^\varepsilon(s)|^2) \in L^1 \cap L^\infty$, and from the Hardy-Littlewood-Sobolev inequality and the Sobolev inequality that $(|x|^{-\gamma} * |a^\varepsilon|^2) \in L^{n/\gamma+} \cap L^\infty$. Therefore,

$$\phi^\varepsilon(t) - \Phi_0 \in C([0,T]; L^{\frac{n}{\gamma}+} \cap L^\infty).$$

$\square$

2.5.3 Justification of WKB estimate

Theorem 2.5.2. *Let $n \geqslant 2$. Let f satisfy the same assumption as in Theorem 2.5.1. Suppose that Let $k \geqslant 1$ be an integer and $s > n/2 + 2k + 4$ be a real number. Assume that $\Phi_0 \in X^{s+1}$ and that A_0^ε writes*

$$A_0^\varepsilon = \sum_{j=0}^{k} \varepsilon^j A_j + o(\varepsilon^k) \quad \text{in } H^s$$

for $\varepsilon \in [0,1]$. Then, the unique solution $(a^\varepsilon, \phi^\varepsilon)$ of (2.5.1) has the following expansion:

$$\begin{cases} a^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j a_j + o(\varepsilon^k) \quad \text{in } C([0,T]; H^{s-2k-2}), \\ \phi^\varepsilon = \displaystyle\sum_{j=0}^{k} \varepsilon^j \phi_j + o(\varepsilon^k) \quad \text{in } C([0,T]; L^{\frac{n}{\gamma}+} \cap X^{s-2k-1}). \end{cases} \tag{2.5.3}$$

We note that the expansion of ϕ^ε never holds in $C([0,T]; W^{s-2k-1,1})$. This part is different from (CNLS) case, and due to the presence of the nonlocal nonlinearity. The proof is similar to that for Theorem 2.3.3. We hence omit the details.

Chapter 3

Analysis of Classical trajectories

3.1 Introduction

In previous Chapter 2, we consider the solution to the semiclassical nonlinear Schrödinger equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = N(|u^\varepsilon|)u^\varepsilon, \quad u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon). \tag{3.1.1}$$

and give an approximate solution of phase-amplitude form

$$u^\varepsilon(t,x) \sim e^{i\frac{\phi_0(t,x)}{\varepsilon}}(b_0(t,x) + \varepsilon b_1(t,x) + \varepsilon b_2(t,x) + \cdots) \tag{3.1.2}$$

for small time. Our next problem is whether we can extend this approximation for large time or not. In general, there exists a critical time $t_c < \infty$ such that the approximation (3.1.2) breaks down at $t = t_c$. This is due to the fact that ϕ_0 makes singularity in finite time and so that the right hand side of (3.1.2) is not defined globally in time. A set of singular point of ϕ_0 is called *caustic set*. At the caustic, the approximation of the form (3.1.2) ceases to be valid. The analysis of the asymptotic behavior of the solution near and after the caustic is one of the most interesting problem of semiclassical analysis. In this chapter, we investigate with the model case when ϕ_0 exists globally in time (global existence, *GE*) and when ϕ_0 breaks down in finite time with the formulation of singularity (finite-time breakdown, *FB*). More explicitly, we consider the compressible Euler-Poisson equations as the model case:

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ v_t + v\cdot\nabla v + \lambda\nabla V_{\mathrm{P}} = 0, \\ -\Delta V_{\mathrm{P}} = \rho - b, \\ (\rho, v)(0,x) = (\rho_0, v_0)(x), \quad \rho_0 \geqslant 0, \end{cases} \tag{EP$_b$}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, λ is given physical constant, and b is the *background* or *impurity*. We assume b is a nonnegative constant.

In this chapter, we assume that the unknowns have radial symmetry and concentrate on the multi-dimensional isotropic model:

$$\begin{cases} r^{n-1}\rho_t + \partial_r(r^{n-1}\rho v) = 0, \\ v_t + v\partial_r v + \lambda \partial_r V_{\mathrm{P}} = 0, \\ -\partial_r(r^{n-1}\partial_r V_{\mathrm{P}}) = r^{n-1}(\rho - b), \\ (\rho, v)(0,r) = (\rho_0, v_0)(r), \quad \rho_0 \geqslant 0 \end{cases} \tag{rEP$_b$}$$

for $(t,r) \in \mathbb{R}_+ \times \mathbb{R}_+$ with initial data Here, $r \geqslant 0$ denotes the distance from the origin. Now, the unknowns are $\rho = \rho(t,r)$ and $v = v(t,r)$. $V_{\mathrm{P}} = V_{\mathrm{P}}(t,r)$ is defined by

$$V_{\mathrm{P}}(t,r) = V_{\mathrm{P}}(t,r_0) + \int_{r_0}^{r} \frac{1}{r_1^{n-1}} \left(\int_0^{r_1} r_2^{n-1}(\rho(t,r_2) - b) dr_2 \right) dr_1.$$

We suppose suitable boundary condition such as $V_{\mathrm{P}}(t,\infty) = 0$.

The Euler-Poisson equations arise in many physical problems such as fluid mechanics, plasma physics, gaseous stars, quantum gravity and semiconductors, etc. There is a large amount of literature available on the global behavior of Euler-Poisson and related problem, from local existence in the small H^s-neighborhood of a steady state [27, 51, 53] to global existence of weak solution with geometrical symmetry [19]. For the two-carrier types in one dimension, see [71]. The relaxation limit for the weak entropy solution, consult [54] for isentropic case, and [39] for isothermal case. The global existence for some large class of initial data near a steady state is obtained by Guo [35] assuming the flow is irrotational.

For isotropic model, the finite time blowup for three dimensional case with the attractive force, pressure, and compactly supported mass density is obtained in [52], and the blowup for the repulsive case in the similar settings is deduced in [63] (see also [24, 64]). In [25], the global existence/finite-time breakdown of the strong solution is studied from the view point of critical threshold. They give a complete criterion in one-dimensional case without spatial symmetry and with spatial symmetry in one and four dimension. A sufficient condition for finite-time breakdown without spatial symmetry is obtained in [17, 18], and the complete description of the critical threshold phenomenon for the two-dimensional restricted Euler-Poisson equations is given in [47]. In [67], the similar issue is treated with pressure term.

In this chapter, applying the method in [25], we discuss the necessary and sufficient conditions for the global existence of the solution to the Euler-Poisson equations with spatial symmetry (rEP$_b$) in multi-dimensional case. One of the main result is Theorem 3.3.14, which is used in Chapter 4. The results are too much to state them all here, we only quote them.

The $b = 0$ case The necessary and sufficient conditions for the global existence are given in Theorem 3.3.1 for attractive ($\lambda < 0$) case, and in Theorems 3.3.2, 3.3.3, 3.3.7, and 3.3.12 for repulsive ($\lambda > 0$) case.

The $b > 0$ case The necessary and sufficient conditions for the global existence are given in Theorems 3.4.1, 3.4.2, and 3.4.3 for attractive ($\lambda < 0$) case, and in Theorems 3.4.4, 3.4.5, and 3.4.7 for repulsive ($\lambda > 0$) case.

Section 3.5 is devoted to the study of the limit $b \to 0$. This limit reveals the feature of two dimensional case.

3.1.1 Semiclassical analysis and Euler equation

As we seen in Section 2.1.2, there is at least two approach to obtain a WKB approximation (3.1.2) of the solution to (3.1.1). Let us now recall briefly. First is to apply the Madelung transform $u^\varepsilon(t,x) = \sqrt{\rho^\varepsilon(t,x)}e^{i\frac{S^\varepsilon(t,x)}{\varepsilon}}$ and work with the quantum Euler equation

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon \nabla S^\varepsilon) = 0, \\ \partial_t \nabla S^\varepsilon + (\nabla S^\varepsilon \cdot \nabla)\nabla S^\varepsilon + \nabla N(\sqrt{\rho^\varepsilon}) = \varepsilon^2 \nabla \left(\dfrac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right), \\ (\rho^\varepsilon(0,x), \nabla S^\varepsilon(0,x)) = (|A_0^\varepsilon|^2, \nabla(\Phi_0 + \varepsilon \arg A_0^\varepsilon)). \end{cases} \tag{3.1.3}$$

The second is employing the modified Madelung transform $u^\varepsilon = a^\varepsilon e^{i\frac{\phi^\varepsilon}{\varepsilon}}$ and considering the system

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon \cdot \nabla) a^\varepsilon + \dfrac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t \phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + N(|a^\varepsilon|) = 0, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0). \end{cases} \tag{3.1.4}$$

Either way we take, we encounter the compressible Euler equation: Set $(\rho_1, v_1) := (\rho^\varepsilon, \nabla S^\varepsilon)_{|\varepsilon=0}$ and $(\rho_2, v_2) := (|a^\varepsilon|, \nabla\phi^\varepsilon)_{|\varepsilon=0}$. Then, one sees that both (ρ_1, v_1) and (ρ_2, v_2) solve, at least formally, the system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + (v \cdot \nabla) v + \nabla N(\sqrt{\rho}) = 0, \\ (\rho(0,x), v(0,x)) = (|A_0|^2, \nabla\Phi_0). \end{cases} \tag{3.1.5}$$

In Theorems 2.1.1, 2.1.2, 2.1.3, and 2.1.4, we actually justify the WKB type approximation

$$u^\varepsilon = e^{i\frac{\phi_0}{\varepsilon}}(\beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^{k-1}\beta_{k-1} + o(\varepsilon^{k-1})) \tag{3.1.6}$$

of the solutions to (CNLS), (SP), (H), and (L-NL), respectively, by analyzing the system (3.1.4)[1]. Let us give some examples of (3.1.5):

- If (3.1.1) has the defocusing nonlinearity of power type, that is, if $N(y) = y^{p-1}$ then (3.1.1) is the power-type nonlinear Schrödinger equation
$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{p-1}u^\varepsilon \tag{NLS}$$
and (3.1.5) becomes the *compressible Euler equation with pressure*:
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + (v\cdot\nabla)v + \nabla(\rho^{\frac{p}{2}}) = 0, \\ (\rho(0,x), v(0,x)) = (|A_0|^2, \nabla\Phi_0). \end{cases} \tag{3.1.7}$$
We justify the small time WKB approximation of the solution in Section 2.3 for the cubic case $p = 3$.

- If (3.1.1) is the Schrödinger-Poisson system
$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda V_P u^\varepsilon, \quad -\Delta V_P = |u^\varepsilon|^2, \tag{SP}$$
then (3.1.5) becomes the *compressible Euler-Poisson equations*:
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + (v\cdot\nabla)v + \lambda\nabla V_{\mathrm{P}} = 0, \\ -\Delta V_{\mathrm{P}} = \rho, \\ (\rho(0,x), v(0,x)) = (|A_0|^2, \nabla\Phi_0). \end{cases} \tag{EP}$$
We justify small time WKB approximation in Section 2.4. In this chapter, we consider this equations in the presence of background (EP_b).

- If the nonlinearity of (3.1.1) is the sum of above two nonlinearities like
$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda V_P u^\varepsilon + |u^\varepsilon|^{p-1}u^\varepsilon, \quad \Delta V_P = |u^\varepsilon|^2 \tag{LNL}$$
(3.1.5) becomes the compressible Euler-Poisson equations with pressure:
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + (v\cdot\nabla)v + \lambda\nabla V_{\mathrm{P}} + \nabla(\rho^{\frac{p}{2}}) = 0, \\ \Delta V_{\mathrm{P}} = \rho, \\ (\rho(0,x), v(0,x)) = (|A_0|^2, \nabla\Phi_0). \end{cases} \tag{3.1.8}$$

[1] In the approximation solution (3.1.6), the main amplitude β_0 is not $a_0 = \lim_{\varepsilon\to 0} a^\varepsilon$ but $a_0 e^{i\phi_1}$, where $(a^\varepsilon, \phi^\varepsilon)$ is a solution to (3.1.4) and ϕ_1 is ε^1-order term of ϕ^ε (see Section 2.2.4). However, $|a_0|^2 = |a_0 e^{i\phi_1}|^2 = |\beta_0|^2$ and so both $(|a_0|^2, \nabla\phi_0)$ and $(|\beta_0|^2, \nabla\phi_0)$ solve (3.1.5), where $\phi_0 = \lim_{\varepsilon\to 0}\phi^\varepsilon$.

In Section 2.5, the WKB approximation of the solution to this equation is deduced.

3.1.2 Classical trajectory

Let us introduce the notion of classical trajectory.

Linear case

We first briefly recall the linear case $N \equiv 0$. Substitution of $u^\varepsilon = a^\varepsilon e^{i\phi_0/\varepsilon}$ to (3.1.1) suggests that $\phi_0 = \phi_{\text{eik}}$, where ϕ_{eik} solves the *eikonal equation*:

$$\partial_t \phi_{\text{eik}} + \frac{1}{2}|\nabla\phi_{\text{eik}}|^2 = 0, \quad \phi_{\text{eik}}(0,x) = \Phi_0(x). \tag{3.1.9}$$

The term "eikonal" comes from the theory of geometric optics: The solution to this equation determines the set where light is propagated. We remark that the equation (3.1.9) is regarded as a Hamilton-Jacobi system. One can solve this equation by a characteristic curve $X = X(t,y) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ defined by and ordinal differential equation

$$\frac{d}{dt}X_{\text{eik}}(t,y) = \nabla\phi_{\text{eik}}(t, X_{\text{eik}}(t,y)), \quad X_{\text{eik}}(0,y) = y.$$

$X_{\text{eik}}(t,y)$ is called *classical trajectory*, or *ray*. With this notation, (3.1.9) is simply $\frac{d^2}{dt^2}X_{\text{eik}}(t,y) = 0$. Therefore, in the linear case (without external potential), the classical trajectory X_{eik} is a straight line

$$X_{\text{eik}}(t,y) = y + t\nabla\Phi_0(y).$$

For more detail, see [13, Section1.3] and references therein.

Nonlinear case

Now we turn to the nonlinear equation (3.1.1). We now consider general nonlinearity. As shown in Chapter 2, the principal phase ϕ_0 of WKB approximation (3.1.6) solves the system

$$\begin{cases} \partial_t a_0 + (\nabla\phi_0\cdot\nabla)a_0 + \frac{1}{2}a_0\Delta\phi_0 = 0, \\ \partial_t\phi_0 + \frac{1}{2}|\nabla\phi_0|^2 + N(|a_0|) = 0, \\ (a_0(0,x), \phi_0(0,x)) = (A_0, \Phi_0) \end{cases} \tag{3.1.10}$$

whose second equation is the eikonal equation (3.1.9) with nonlinear interaction term. We now suppose that the solution ϕ_0 exists with a certain regularity and that the characteristic curve $X(t,y)$ can be defined by an ordinal differential equation

$$\frac{d}{dt}X(t,y) = \nabla\phi_0(t,X(t,y)), \quad X(0,y) = 0. \tag{3.1.11}$$

Although the system (3.1.10) is not always regarded as a Hamilton-Jacobi system, let us call $X(t,y)$ as *classical trajectory* in this article, by an analogy of the linear case. Unlike X_{eik}, the classical trajectory X defined from (3.1.11) is not always a straight line even without the presence of external force. This is because $\frac{d^2}{dt^2}X(t,y) = N(|a_0|)(t,X(t,y))$ is not zero in general. This reflects the fact that there are interactions, represented by the nonlinear term N, which bends the classical trajectory. When we consider (rEP_b), the use of classical trajectory works extremely well, and the equation can be reduced to an ordinal differential equation of classical trajectory. Remark that the analysis of the classical trajectories does not always yield a good analysis of ϕ_0. This strongly depends of the nonlinearity and the geometry of considering space. Nevertheless, the classical trajectory X has a general property: This traces the flow of the "mass". We conclude this section with this property.

Proposition 3.1.1. *Let (a_0, ϕ_0) be a smooth solution of (3.1.10) on $[0,T]$ and X be a classical trajectory on $[0,T]$ defined by (3.1.11). Let Ω be a bounded set in $\mathbb{R}^n$ and define a set $\Omega_t := \{X(t,y) \in \mathbb{R}^n | y \in \Omega\}$ for $t \in [0,T]$. Then, for $\rho(t,x) = |a_0(t,x)|^2$, we have*

$$\int_{\Omega_t} \rho(t,x)dx = \int_{\Omega} \rho(0,x)dx$$

for all $t \in [0,T]$.

Proof. Let $X(t,y)$ be the classical trajectory. Then, by the change of variable $x = X(t,y)$, we have

$$\int_{\Omega_t} \rho(t,x)dx = \int_{\Omega} \rho(t,X(t,y))\big(\det(\nabla^2\phi_0)\big)(t,X(t,y))dy,$$

where $\nabla^2\phi_0$ is the $n\times n$ matrix $(\partial_i\partial_j\phi_0)_{i,j}$. The time derivative of the right hand side is equal to

$$\int_{\Omega}(\partial_t\rho + \nabla\phi_0\cdot\nabla\rho)(t,X(t,y))\big(\det(\nabla^2\phi_0)\big)(t,X(t,y))dy \\ + \int_{\Omega}\rho(t,X(t,y))\frac{d}{dt}\Big[\big(\det(\nabla^2\phi_0)\big)(t,X(t,y))\Big]dy. \tag{3.1.12}$$

We now claim

$$\frac{d}{dt}\Big[\big(\det(\nabla^2\phi_0)\big)(t,X(t,y))\Big] = (\Delta\phi_0)(t,X(t,y))\big(\det(\nabla^2\phi_0)\big)(t,X(t,y)). \tag{3.1.13}$$

If this is true, then, plugging to (3.1.12), we deduce from the first equation of (3.1.10) that

$$\int_\Omega (\partial_t \rho + \nabla\phi_0 \cdot \nabla\rho + \rho\Delta\phi_0)(t, X(t,y))\big(\det(\nabla^2\phi_0)\big)(t, X(t,y))dy = 0,$$

which shows the proposition. Hence, it suffices to prove (3.1.13). By the relation $\frac{d}{dt}\partial_j X_i = \sum_{k=1}^n \partial_k v_i(t,X)\partial_j X_k$, we have

$$\begin{aligned}
&\frac{d}{dt}\Big[\big(\det(\nabla^2\phi_0)\big)(t, X(t,y))\Big] \\
&= \frac{d}{dt}\sum_{\sigma\in S_n} \operatorname{sign}\sigma \prod_{i=1}^n \partial_i X_{\sigma(i)}(t,X) \\
&= \sum_{\sigma\in S_n} \operatorname{sign}\sigma \sum_{j=1}^n \left(\sum_{k=1}^n \partial_k v_{\sigma(j)}(t,X)\partial_j X_k\right) \prod_{i=1, i\neq j}^n \partial_i X_{\sigma(i)}(t,X) \\
&= \sum_{\sigma\in S_n} \operatorname{sign}\sigma \sum_{j=1}^n \left(\sum_{k=1, k\neq\sigma(j)}^n \partial_k v_{\sigma(j)}(t,X)\partial_j X_k\right) \prod_{i=1, i\neq j}^n \partial_i X_{\sigma(i)}(t,X) \\
&\quad + \big(\Delta\phi_0 \det(\nabla^2\phi_0)\big)(t,X),
\end{aligned}$$

where σ is a permutation and S_n denotes the symmetric group. Let us now prove that the first term of the right hand side is zero. For fixed $\sigma \in S_n$, j, and $k \neq \sigma(j)$, we can choose $\sigma' \in S_n$ so that

$$\sigma'(j) = k, \quad \sigma'(\sigma^{-1}(k)) = \sigma(j), \quad \sigma'(i) = \sigma(i) \quad [1,n] \ni \forall i \neq j, \sigma^{-1}(k).$$

Then, it holds that $\operatorname{sign}\sigma' = -\operatorname{sign}\sigma$. We put $j' = \sigma^{-1}(k)$ and $k' = k$. Note that $((\sigma')', (j')', (k')') = (\sigma, j, k)$ and so that the correspondence $(\sigma, j, k) \mapsto (\sigma', j', k')$ is a bijection on $\{(\sigma, j, k) \in S_n \times [1,n] \times [1,n] | |k \neq \sigma(j)\}$. Furthermore, one verifies that

$$\operatorname{sign}\sigma \partial_k v_{\sigma(j)} \partial_j X_k \prod_{i=1, i\neq j}^n \partial_i X_{\sigma(i)} + \operatorname{sign}\sigma' \partial_{k'} v_{\sigma'(j')} \partial_{j'} X_{k'} \prod_{i'=1, i'\neq j'}^n \partial_{i'} X_{\sigma'(i')} = 0$$

Hence, we obtain (3.1.13). □

3.2 Preliminary results

The method of characteristic curve (the use of classical trajectory introduced in Section 3.1.2) does not necessarily give a good analysis of compressible Euler equations. However, in the radial Euler-Poisson case, there is an amazing transform which reduces the system to an ODE of the classical trajectory X. Let us first describe this reduction of (rEP$_b$) which is introduced in [25] (Section 3.2.1). Then, Section 3.2.2 is devoted to the study of local existence of classical solution to (rEP$_b$) by the analysis of classical trajectories. The criterion of global existence/finite-time breakdown is also translated in words of classical trajectory (Section 3.2.3). We introduce the notion of pointwise condition for finite-time breakdown in Definition 3.2.8, which is introduced in [59].

3.2.1 Reduction of Euler-Poisson equations to an ODE of classical trajectories

Let us recall the radial Euler-Poisson equations:

$$
\begin{aligned}
&r^{n-1}\rho_t + \partial_r(r^{n-1}\rho v) = 0, && (3.2.1)\\
&v_t + v\partial_r v + \lambda \partial_r V_{\mathrm{P}} = 0, && (3.2.2)\\
&-\partial_r(r^{n-1}\partial_r V_{\mathrm{P}}) = r^{n-1}(\rho - b) && (3.2.3)\\
&(\rho, v)(0,r) = (\rho_0, v_0)(r), \quad \rho_0 \geqslant 0. && (3.2.4)
\end{aligned}
$$

Note that (3.2.1)–(3.2.4) is equal to (rEP$_b$). Let X be a classical trajectory defined by

$$\frac{d}{dt}X(t,R) = v(t, X(t,R)), \quad X(0,R) = R.$$

We also introduce the "mass"

$$m(t,r) := \int_0^r \rho(t,s)s^{n-1}ds.$$

Then, an integration of (3.2.1) yields

$$\partial_t m + v\partial_r m = 0, \tag{3.2.5}$$

which is written as

$$\frac{d}{dt}m(t, X(t,R)) = 0. \tag{3.2.6}$$

Note that (3.2.5) implies that the mass is conserved along the characteristic curve. This property holds for general nonlinearity without symmetry (see Proposition 3.1.1). Integrating (3.2.3) and combining with (3.2.2), we also have

$$\frac{d^2}{dt^2}X(t,R) = \frac{d}{dt}v(t,X(t,R)) = \frac{\lambda m(t,X(t,R))}{(X(t,R))^{n-1}} - \frac{\lambda}{n}bX(t,R). \tag{3.2.7}$$

Thus, it turns out that the system (3.2.1)–(3.2.4) is reduce to an ODE for X:

$$X''(t,R) = \frac{\lambda m_0(R)}{X(t,R)^{n-1}} - \frac{\lambda}{n}bX(t,R), \quad X'(0,R) = v_0(R), \quad X(0,R) = R, \tag{3.2.8}$$

where m_0 is the "initial mass" $m_0(R) = \int_0^R \rho_0(s)s^{n-1}ds$. This reduction is the key for our analysis in this chapter.

We also introduce the integral form of this equation. Multiply both sides by X' to obtain

$$(X'(t,R))^2 = v_0(R)^2 + \frac{2\lambda m_0(R)}{(n-2)R^{n-2}} + \frac{\lambda}{n} bR^2 - \frac{2\lambda m_0(R)}{(n-2)X(t,R)^{n-2}} - \frac{\lambda}{n} bX(t,R)^2 \tag{3.2.9}$$

if $n \geqslant 3$ and

$$(X'(t,R))^2 = v_0(R)^2 + \frac{\lambda}{2} b(R^2 - X(t,R)^2) + 2\lambda m_0(R) \log \frac{X(t,R)}{R} \tag{3.2.10}$$

if $n = 2$. They are also useful.

3.2.2 Local existence of classical solution

In this section, we shall show the local existence of classical solution to (rEP$_b$) by using the classical trajectories. The strategy is the following: We first show the existence of classical trajectory X which solves the ODE (3.2.8) (Proposition 3.2.1). Then, the solution of (rEP$_b$) is defined from X by an explicit formula (Proposition 3.2.3).

Local existence of classical trajectory

Let us begin with the local existence of X. We regard $X(t,R)$ as a function $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. For a nonnegative integer k, we define

$$D^k := \begin{cases} C([0,\infty)) & \text{if } k = 0, \\ C([0,\infty)) \cap C^k((0,\infty)) & \text{if } k > 0. \end{cases} \tag{3.2.11}$$

For nonnegative integers k_1, k_2 and intervals I_1, I_2, we define

$$\begin{aligned} C^{k_1,k_2}(I_1 \times I_2) = \{ f(t,x) : I_1 \times I_2 \to \mathbb{R} | \partial_t^a \partial_x^b f \in C(I_1 \times I_2), \\ \forall a \in [0,k_1], \forall b \in [0,k_2] \}. \end{aligned}$$

Proposition 3.2.1 (Existence of solution of (3.2.8))**.** *Suppose that $n \geqslant 1$, $\lambda \in \mathbb{R}$, and $b \geqslant 0$. Let k be a nonnegative integer and assume $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ with $v_0(0) = 0$. Then, $m_0 \in D^{k+1}$ holds, and for any $R > 0$ there exists $t(R) > 0$ such that $X(t,R)$ is uniquely defined from the ODE* (3.2.8) *in an interval $[0, t(R))$. Moreover, if there exists $T > 0$ such that $X(t,R) > 0$ holds for all $(t,R) \in [0,T) \times (0,\infty)$, then we have*

$$X \in C^{2,k+1}([0,T) \times (0,\infty)) \cap C^{\infty,k+1}((0,T) \times (0,\infty)).$$

This proposition follows by applying a general theory of ordinal differential equations for each fixed R. We omit the detail.

Local existence of the solution to (rEP$_b$)

Let us turn to the local existence of the classical solution to (rEP$_b$). We introduce the indicator function

$$\Gamma(t,R) := \exp\left(\int_0^t \partial_r u(s,X(s,R))ds\right). \tag{3.2.12}$$

The interpretation of $\Gamma(t,R)$ will be clear from the following lemma and Proposition 3.2.6, below.

Lemma 3.2.2. *Consider the Euler-Poisson equations* (rEP$_b$)*. Let X be the classical trajectory, then*

$$\Gamma(t,R) = \partial_R X(t,R).$$

Moreover, the solution of (rEP$_b$) *is given by*

$$v(t,X(t,R)) = \frac{d}{dt}X(t,R), \tag{3.2.13}$$

$$\rho(t,X(t,R)) = \frac{R^{n-1}\rho_0(R)}{X^{n-1}\Gamma(t,R)}, \tag{3.2.14}$$

$$\partial_r v(t,X(t,X)) = \frac{\partial_t \Gamma(t,R)}{\Gamma(t,R)}. \tag{3.2.15}$$

Even if it is possible to determine a function X which solves the ODE (3.2.8) for large time, we can define the solution to the Euler-Poisson equations (rEP$_b$) by Lemma 3.2.2 as long as X and $\Gamma = \partial_R X$ are positive.

Proposition 3.2.3 (Local existence of the solution of (rEP$_b$))**.** *Suppose that $n \geqslant 1$, $\lambda \in \mathbb{R}$, and $b \geqslant 0$. Let k be a nonnegative integer and assume $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ with $v_0(0) = 0$. Let X be the solution of* (3.2.8) *given by Proposition 3.2.1. Define Γ by* (3.2.12)*. If $X(t,R) > 0$ and $\Gamma(t,R) > 0$ hold for all $R > 0$ and $t \in [0,T)$ and if $\liminf_{R\to 0}\Gamma(t,R) > 0$ for $t \in [0,T)$, then $X(t,0) = 0$ for $t \in [0,T)$ and* (rEP$_b$) *has a unique solution*

$$\rho \in C^2([0,T),D^k) \cap C^\infty((0,T),D^k),$$
$$v \in C^1([0,T),D^{k+1}) \cap C^\infty((0,T),D^{k+1}).$$

Remark 3.2.4. In above proposition, if $s = 0$ then ρ is not spatially differentiable. In that case, we use the mass m instead of ρ and consider the modified equations (3.2.5) and (3.2.2)–(3.2.4) instead of (rEP$_b$).

Proof. We first show $X(t,0) = 0$ for $t \in [0,T)$. Since (3.2.8) can be solved explicitly, one easily checks this if $n = 1$. Let us consider $n \geqslant 2$. By $\liminf_{R\to 0}\Gamma(t,R) > 0$, we have $R \leqslant CX(t,R)$ for small R. Plugging this to (3.2.9) and (3.2.10), we deduce from $m_0(R) = O(R^n)$ as $R \to 0$ that

$$\lim_{R\to 0}\left(X'(t,R)^2 + \frac{\lambda b}{n}X^2(t,R)\right) = 0.$$

If $\lambda > 0$ then, it immediately follows that $X'(t,0) = X(t,0) = 0$ for $t \in [0,T)$. On the other hand, if $\lambda < 0$ then we have

$$|X(t,0)|' \leqslant |X'(t,0)| = \sqrt{\frac{|\lambda|b}{n}}|X(t,0)|$$

and so $|X(t,0)| \leqslant |X(0,0)|e^{t\sqrt{|\lambda|b/n}} = 0$. Hence, $X(t,0) = 0$ for $t \in [0,T)$. It gives the continuities of X, X', and X'' (and higher time derivatives) around $R = 0$:

$$\begin{aligned} X \in C^{2,0}([0,T) \times [0,\infty)) \cap C^{2,k+1}([0,T) \times (0,\infty)) \\ \cap C^{\infty,0}((0,T) \times [0,\infty)) \cap C^{\infty,k+1}((0,T) \times (0,\infty)). \end{aligned}$$

Then, the existence part is an immediate consequence of Lemma 3.2.2.

We prove the uniqueness. It suffices to show in the case $k = 0$. Let (ρ_i, v_i) $(i = 1, 2)$ be two solutions to (3.2.5) and (3.2.2)–(3.2.4) which satisfy

$$\begin{aligned} \rho_i \in C^2([0,T), D^0), \\ v_i \in C^1([0,T), D^1). \end{aligned}$$

Now, solving $\frac{d}{dt}X_i(t,R) = v_i(t, X(t,R))$, we can define the classical trajectories X_1 and X_2, and the indicator functions Γ_1 and Γ_2. Then, we have

$$X_i \in C^2([0,T), D^1), \qquad \Gamma_i \in C^2([0,T), C((0,\infty))).$$

Since two solutions exist until $t < T$, for all $R > 0$ and $\delta > 0$ there exist positive constants $c_1 = c_1(R,\delta)$ and $c_2 = c_2(R,\delta)$ such that

$$X_i(t,R) \geqslant c_1 > 0 \quad \text{and} \quad \Gamma_i(t,R) \geqslant c_2 > 0, \quad \forall t \in [0, T-\delta].$$

Recall that both X_1 and X_2 solve

$$X''(t,R) = \frac{\lambda m_0(R)}{X(t,R)^{n-1}} - \frac{\lambda b}{n}X(t,R), \quad X'(0,R) = v_0(R), \quad X(0,R) = R$$

We fix $R > 0$ and $\delta > 0$. Using the fact that

$$\left| \frac{1}{X_1(t,R)^{n-1}} - \frac{1}{X_2(t,R)^{n-1}} \right| \leqslant \frac{n-1}{c_1^n}|X_1(t,R) - X_2(t,R)|$$

for all $t \in [0, T-\delta]$, and applying Gronwall's lemma to

$$\begin{cases} X(t,R) = R + \displaystyle\int_0^t X'(\tau)d\tau, \\ X'(t,R) = v_0(R) + \displaystyle\int_0^t \left(\frac{\lambda m_0(R)}{X(\tau,R)^{n-1}} - \frac{\lambda b}{n}X(\tau,R) \right) d\tau, \end{cases}$$

we deduce that $X_1'(t,R) = X_2'(t,R)$ and $X_1(t,R) = X_2(t,R)$ hold for $t \in [0, T-\delta]$. Since $R > 0$ is arbitrary, we also have $X_1(t,0) = X_2(t,0)$ for all $t \in [0, T-\delta]$ by continuity. Thus, we see that $X_1(t,R) = X_2(t,R)$ for all $R \geqslant 0$ and $t \in [0,T)$ since $\delta > 0$ is also arbitrary. Applying Lemma 3.2.2, we conclude that $\rho_1 = \rho_2$ and $v_1 = v_2$. □

So far, we obtain the unique solution to (rEP$_b$). We finally confirm that this solution solves the original equation in the distribution sense.

Proposition 3.2.5. *Suppose $n \geqslant 1$, $\lambda \in \mathbb{R}$, and $b \geqslant 0$. Assume $\rho_0 \in D^0$ and $v_0 \in D^1$ with $v_0(0) = 0$. Let (ρ, v) be a solution to* (rEP$_b$) *given in Proposition 3.2.3. Then, $\mathbf{r}(t,x) := \rho(t,|x|)$ and $\mathbf{v}(t,x) = (x/|x|)v(t,|x|)$ solve the Euler-Poisson equations*

$$\begin{aligned} \mathbf{r}_t + \operatorname{div}(\mathbf{r}\mathbf{v}) &= 0, \\ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \lambda \nabla \mathbf{V}_{\mathrm{P}} &= 0, \\ -\Delta \mathbf{V}_{\mathrm{P}} &= \mathbf{r} - b. \end{aligned}$$

in the distribution sense.

Proof. Suppose that the solution of (rEP$_b$) exists for $t < T$. Since (m, v) solves (3.2.5) and (3.2.2)–(3.2.4) in the classical sense, and since moreover it is continuous at $x = 0$ with $v(0) = 0$, the pair $(\mathbf{r}, \mathbf{v})$ solves the (EP$_b$) in the distribution sense. □

3.2.3 Pointwise condition for finite-time breakdown

Let us proceed to the discussion on global existence. By means of Lemma 3.2.2, it is clear that the existence of $t_c > 0$ such that $\Gamma(t_c, R) = 0$ implies the finite-time breakdown of the solution.

Proposition 3.2.6. *The smooth solution to the radial Euler-Poisson equations* (3.2.1)–(3.2.4) *is global if and only if $\Gamma(t,R)$ is positive for all $t \geqslant 0$ and $R \geqslant 0$. On the other hand, the smooth solution to the Euler-Poisson equations breaks down at $t = t_c$ if and only if the following equivalent condition is met for some $R = R_c$:*

1. $\int_0^{t_c} \partial_r v(\tau, X(\tau, R_c)) d\tau = -\infty$;
2. $\Gamma(t_c, R_c) = 0$;
3. $\partial_R X(t_c, R_c) = 0$.

The next elementary lemma suggests that the existence of $t_0 > 0$ such that $X(t_0, R_0) = 0$ holds for some $R_0 > 0$ also leads to the same situation.

Lemma 3.2.7. *Let X be a characteristic curve. If $X(t_0, R_1) = X(t_0, R_2)$ for some $t_0 > 0$ and $0 \leqslant R_1 < R_2$, then there exist $t \in [0, t_0]$ and $R \in [R_1, R_2]$ such that $\Gamma(t, R) = 0$. In particular, if $X(t_0, R_0) = 0$ for some $t_0 > 0$ and $R_0 > 0$, then there exist $t \leqslant t_0$ and $R \leqslant R_0$ such that $\Gamma(t, R) = 0$.*

By Proposition 3.2.6, to ensure the existence of the global regular solution, it suffices to start with the initial data for which

$$X(t, R) > 0, \quad \forall R > 0 \quad \text{and} \quad \Gamma(t, R) > 0, \quad \forall R \geqslant 0$$

hold for all $t > 0$. Now, we introduce the notion of *pointwise condition for finite-time breakdown.*

Definition 3.2.8. *For fixed $R > 0$, we call a necessary and sufficient condition for the existence of $t_c \in (0, \infty)$ such that $X(t_c, R) = 0$ or $\Gamma(t_c, R) = 0$ hold as a pointwise condition for finite-time breakdown. In the case of $R = 0$, we regard a necessary and sufficient condition for the existence of $t_c \in (0, \infty)$ such that $\Gamma(t_c, 0) = 0$ as a pointwise condition for finite-time breakdown. We denote PCFB, for short.*

With this notion, Propositions 3.2.6 is reduced as follows:

Proposition 3.2.9. *The local solution to the radial Euler-Poisson equations* (3.2.1)–(3.2.4) *given in Proposition 3.2.3 breaks down in finite time if and only if there exist some $R \geqslant 0$ such that the PCFB is met.*

3.3 Global existence of classical solutions to radial Euler-Poisson equations 1: without background

In this section, we give a necessary and sufficient condition for global existence/finite-time breakdown of the classical solution to the radial Euler-Poisson equation without background:

$$\begin{cases} r^{n-1}\rho_t + \partial_r(r^{n-1}\rho v) = 0, \\ v_t + v\partial_r v + \lambda \partial_r V_{\mathrm{P}} = 0, \\ -\partial_r(r^{n-1}\partial_r V_{\mathrm{P}}) = r^{n-1}\rho, \\ (\rho, v)(0, r) = (\rho_0, v_0)(r), \quad \rho_0 \geqslant 0 \end{cases} \tag{rEP$_0$}$$

for $(t, r) \in \mathbb{R}_+ \times \mathbb{R}_+$, which is a radial model of

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ v_t + v \cdot \nabla v + \lambda \nabla V_{\mathrm{P}} = 0, \\ -\Delta V_{\mathrm{P}} = \rho, \\ (\rho, v)(0, x) = (\rho_0, v_0)(x), \quad \rho_0 \geqslant 0. \end{cases} \tag{EP$_0$}$$

As seen in the previous section, the problem boils down to the analysis of the classical trajectory $X(t,R)$ which satisfies

$$X''(t,R) = \frac{\lambda m_0(R)}{X(t,R)^{n-1}}, \quad X'(0,R) = v_0(R), \quad X(0,R) = R. \tag{3.3.1}$$

It turns out that, for $n \geqslant 3$, the use of the quantities

$$A(r) := \frac{\lambda m_0(r)}{(n-2)}, \quad C(r) := v_0(r)^2 + \frac{A(r)}{r^{n-2}} \tag{3.3.2}$$

makes the description of the condition clearer. We also note that the (3.2.9) is written as

$$(X'(t,R))^2 = C(R) - \frac{A(R)}{X(t,R)^{n-1}}. \tag{3.3.3}$$

For $n = 2$, we use another quantity

$$\mathcal{A}(r) := 2\lambda m_0(r), \quad \mathcal{C}(r) := v_0(r)^2 - A(r)\log r \tag{3.3.4}$$

which enables us to translate (3.2.10) into

$$(X'(t,R))^2 = \mathcal{C}(R) + \mathcal{A}(R)(\log X(t,R)). \tag{3.3.5}$$

3.3.1 Attractive case

Let us begin with the attractive case $\lambda < 0$. In this case, $A(R) \leqslant 0$.

Theorem 3.3.1. *Suppose $\lambda < 0$, $n \geqslant 1$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$.*

1. *If $n = 1$ or 2 then the solution to* (rEP$_0$) *is global if and only if $\rho_0(r) = 0$, $v_0(r) \geqslant 0$, and $\partial_r v_0(r) \geqslant 0$ holds for all $r \geqslant 0$. In particular, if $\rho_0 \not\equiv 0$ then the solution breaks down in finite time.*

2. *If $n \geqslant 3$ then the solution is global if and only if*

$$v_0(r) \geqslant 0, \quad C(r) \geqslant 0, \quad \text{and} \quad \partial_r C(r) \geqslant 0$$

   hold for all $r \geqslant 0$.

Let k be a nonnegative integer. If $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ satisfy the condition for global existence, then the corresponding solution of (rEP$_0$) *satisfies*

$$\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}),$$

The solution is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_0$) *in the distribution sense.*

Proof. **Step 1.**

We begin with the one-dimensional case. If ρ_0 is not identically zero, then we can choose R_0 so that $m_0(R_0) > 0$. Twice integration of (3.3.1) yields $X(t, R_0) = R_0 + v_0(R_0)t - (|\lambda| m_0(R_0)/2)t^2$. Therefore, we can find t_0 such that $X(t_0, R_0) = 0$, which leads to the finite-time breakdown of the solution. On the other hand, if $\rho_0 \equiv 0$ then $X(t, R) = R + v_0(R)t$ and $\Gamma(t, R) = 1 + v_0'(R)t$. Hence, the solution is global if and only if $v_0(R) \geqslant 0$ and $v_0'(R) \geqslant 0$ holds for all $R \geqslant 0$..

Step 2.

We next treat the two-dimensional case. If ρ_0 is not identically zero, then we can choose R_0 so that $m_0(R_0) > 0$. Recall that X solves

$$(X'(t, R_0))^2 = v_0(R_0)^2 - 2|\lambda| m_0(R_0) \log\left(\frac{X(t, R_0)}{R_0}\right). \tag{3.2.10}$$

Since the left hand side is nonnegative, we obtain the upper bound of X:

$$X(t, R_0) \leqslant R_0 \exp\left(\frac{v_0(R_0)^2}{2|\lambda| m_0(R_0)}\right) =: X_{\mathrm{ub}} > 0.$$

Plugging this to (3.3.1), we see that

$$X''(t, R_0) \leqslant -\frac{|\lambda| m_0(R_0)}{X_{\mathrm{ub}}} < 0.$$

Therefore, there exists t_0 such that $X(t_0, R_0) = 0$. In the case where $\rho \equiv 0$, by the same argument as in the one-dimensional case, we see that the solution is global if and only if $v_0(R) \geqslant 0$ and $v_0'(R) \geqslant 0$ hold for all $R \geqslant 0$.

Step 3.

Let us proceed to $n \geqslant 3$ case. Let A and C be as in (3.3.2). We first note that $v_0 \geqslant 0$ is necessary for global existence. Indeed, if $v_0(R_0) < 0$ for some $R_0 > 0$, then $X''(t, R_0) \leqslant 0$ follows from (3.3.1) and so $X'(t, R) \leqslant X'(0, R) = v_0(R) < 0$ for $t \geqslant 0$. Hence, there exists t_0 such that $X(t_0, R_0) = 0$. We next show that $C \geqslant 0$ is also necessary for global existence. Assume that there exists R_0 such that $C(R_0) < 0$. In this case, $A(R_0) < -R_0^{n-2} v_0(R_0)^2 \leqslant 0$ by definition of C. Then, from (3.3.3), we have

$$0 \leqslant (X'(t, R_0))^2 = -|C(R_0)| + \frac{|A(R_0)|}{X(t, R_0)^{n-2}}.$$

This yields an upper bound of X:

$$X(t, R_0) \leqslant \left|\frac{C(R_0)}{A(R_0)}\right|^{\frac{1}{n-2}}.$$

Then, the same argument as in the two-dimensional case shows the existence of t_0 such that $X(t_0, R_0) = 0$. Therefore, $C \geqslant 0$ is necessary for global existence.

In the followings, we suppose $v_0 \geqslant 0$ and $C \geqslant 0$ are satisfied. Under this restriction, we show that the solution is global if and only if $\partial_r C(R) \geqslant 0$ holds for all $R \geqslant 0$. Namely, what to show is that

$$\partial_r C(R) \geqslant 0 \Longleftrightarrow \Gamma(t, R) > 0, \quad \forall t \geqslant 0 \tag{3.3.6}$$

under the assumption $C(R) \geqslant 0$ and $v_0(R) \geqslant 0$. We first consider the case $v_0(R) > 0$. Then, $C(R) > 0$ or $A(R) < 0$ holds. Moreover, $X(t, R) \to \infty$ as $t \to \infty$ since $X''(t, R) \geqslant 0$ and so $X'(t, R) \geqslant X'(0, R) = v_0(R) > 0$ for all $t \geqslant 0$. In this case, by (3.3.3)

$$X'(t, R) = \sqrt{C(R) - \frac{A(R)}{X(t, R)^{n-2}}} > 0,$$

and so

$$\int_R^{X(t,R)} \frac{dy}{\sqrt{C(R) - A(R) y^{-(n-2)}}} = t.$$

Differentiate with respect to R to obtain

$$\frac{\Gamma(t, R)}{X'(t, R)} - \frac{1}{v_0(R)} - \frac{1}{2} \int_R^{X(t,R)} \frac{\partial_r C(R) - \partial_r A(R) y^{-(n-2)}}{\left(C(R) - A(R) y^{-(n-2)}\right)^{3/2}} dy = 0.$$

We put

$$B(t, R) := \frac{\Gamma(t, R)}{X'(t, R)} = \frac{1}{v_0(R)} + \frac{1}{2} \int_R^{X(t,R)} \frac{\partial_r C(R) - \partial_r A(R) y^{-(n-2)}}{\left(C(R) - A(R) y^{-(n-2)}\right)^{3/2}} dy.$$

Two quantity B and Γ have the same sign. Notice that

$$\partial_r A(R) = \frac{2\lambda}{n-2} \rho_0(R) R^{n-1} \leqslant 0$$

and the denominator in the last integral is always positive. Therefore, if $\partial_r C(R) \geqslant 0$ then the above integral is positive, and so $B(t, R)$ stays positive for all $t \geqslant 0$. On the other hand, if $\partial_r C(R) < 0$ then the integral in $B(t, R)$ tends to $-\infty$ as $t \to \infty$. This is because, choosing X_0 so large that $-|\partial_r C(R)| + |\partial_r A(R)| X_0^{-(n-2)} < -|\partial_r C(R)|/2$, we have

$$\int_{X_0}^{X(t,R)} \frac{-|\partial_r C(R)| + |\partial_r A(R)| y^{-(n-2)}}{\left(C(R) + |A(R)| y^{-(n-2)}\right)^{3/2}} dy < -\int_{X_0}^{X(t,R)} \frac{|\partial_r C(R)| y^{3(n-2)/2}}{2|A(R)|^{3/2}} dy$$

if $A(R) < 0$ and

$$\int_{X_0}^{X(t,R)} \frac{-|\partial_r C(R)| + |\partial_r A(R)| y^{-(n-2)}}{\left(C(R) + |A(R)| y^{-(n-2)}\right)^{3/2}} dy < -\int_{X_0}^{X(t,R)} \frac{|\partial_r C(R)|}{2C(R)^{3/2}} dy$$

if $C(R) > 0$. The right hand sides of both inequalities tend to $-\infty$ as $t \to \infty$ ($X(t,R) \to \infty$). Therefore, we can choose t_c such that $\Gamma(t_c, R) = 0$. We finally discuss the case where $v_0(R) = 0$. In this case, since $C(R) \geqslant 0$, we have $C(R) = 0$ and so $A(R) = 0$ ($m_0(R) = 0$) by the definition of C. It implies that $\rho(r) = 0$ for all $r \leqslant R$ and so that, for all $r \leqslant R$, $X'(t,r) \equiv 0$ and $X(t,r) \equiv r$. Hence, by continuity of Γ, one verifies that $\Gamma(t,R) = \lim_{r\uparrow R} \partial_R X(t,r) = 1 > 0$ for all $t \geqslant 0$. Note that $\partial_r C(R) = 0$ since $\rho(R) = 0$. Thus, (3.3.6) is justified. □

3.3.2 Repulsive case 1: $n = 1$

Theorem 3.3.2 (Critical thresholds in 1D case [25])**.** *Suppose $n = 1$, $\lambda > 0$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. Then, the classical solution to* (rEP_0) *is global if and only if*

$$v_0(R) > -\sqrt{2\lambda R m_0(R)} \quad \textit{and} \quad v_0'(R) > -\sqrt{2\lambda\rho_0(R)}, \quad \forall R > 0, \qquad (3.3.7)$$

where, in both inequalities, we allow the case where the both sides equal zero. Let k be a nonnegative integer. If $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ satisfy $v_0(0) = 0$ and (3.3.7) *then the corresponding solution of* (rEP_0) *satisfies*

$$\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).$$

The solution is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP_0) *in the distribution sense.*

Proof. Integrating (3.3.1) twice, we immediately obtain

$$X(t,R) = R + v_0(R)t + \frac{\lambda m_0(R)}{2} t^2$$

and so

$$\Gamma(t,R) = 1 + v_0'(R)t + \frac{\lambda\rho_0(R)}{2} t^2.$$

The solution is global if and only if these two values stay positive for all positive time. $X(t,R) > 0$ holds for all $t > 0$ if and only if $v_0(R) \geqslant 0$ or $v_0(R)^2 - |\lambda| R m_0(R)/2 < 0$, and $X(t,R) > 0$ holds for all $t > 0$ if and only if $v_0'(R) \geqslant 0$ or $(v_0'(R))^2 - \lambda\rho_0(R)/2 < 0$. Therefore, the solution is global if and only if

$$v_0(R) > -\sqrt{2\lambda R m_0(R)}, \quad \text{and} \quad v_0'(R) > -\sqrt{2\lambda\rho_0(R)}$$

holds for all $R > 0$. Moreover, it is easy to check that the case $v_0(R) = m_0(R) = 0$ and the case $v_0'(R) = \rho(R) = 0$ is also admissible. □

3.3.3 Repulsive case 2: $n \geqslant 3$

We first consider the special case $n = 4$.

Theorem 3.3.3 (Critical thresholds in 4D case [25]). *Suppose $n = 4$, $\lambda > 0$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. Let $C(r)$ be defined in* (3.3.2). *The classical solution to* (rEP$_0$) *is global if and only if both of the following conditions hold for all $R > 0$:*

1. $v_0(R) \geqslant 0$ *or* $\int_0^R \rho_0(s)s^3 ds > 0$;
2. $\partial_r C(R) \geqslant 0$ *and* $v_0(R) + Rv_0'(R) > -\sqrt{2R\partial_r C(R)}$;

where, in the last inequality, we allow the case where the both sides equal zero. Let k be a nonnegative integer. If $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ satisfy the above condition then the corresponding solution of (rEP$_0$) *satisfies*

$$\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^s),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).$$

The solution is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_0$) *in the distribution sense.*

In the general $n \geqslant 3$ case, things are not so simple. We rely on Proposition 3.2.6. Then, our task is to determine the PCFB introduced in Definition 3.2.8. Now let us give a complete description.

Definition 3.3.4 (PCFB for $v_0 > 0$). *Suppose $\lambda > 0$ and $n \geqslant 3$. The PCFB under $v_0(R) > 0$ is that either one of following three conditions holds:*

1. $\partial_r C(R) < 0$;
2. $\partial_r C(R) = 0$ *and*
$$\frac{1}{v_0(R)} - \frac{\partial_r A(R)}{2} \int_R^\infty \frac{y^{-(n-2)}}{(C(R) - A(R)y^{-(n-2)})^{3/2}} dy < 0;$$
3. $0 < \partial_r C(R) < \partial_r A(R) R^{-(n-2)}$ *and*
$$\frac{1}{v_0(R)} + \frac{1}{2} \int_R^{\left(\frac{\partial_r A(R)}{\partial_r C(R)}\right)^{\frac{1}{n-2}}} \frac{\partial_r C(R) - \partial_r A(R) y^{-(n-2)}}{(C(R) - A(R)y^{-(n-2)})^{3/2}} dy \leqslant 0.$$

Definition 3.3.5 (PCFB for $v_0 = 0$). *Suppose $\lambda > 0$ and $n \geqslant 3$. The PCFB under $v_0(R) = 0$ is that either one of following three conditions holds:*

1. $\partial_r C(R) < 0$;
2. $\partial_r C(R) = 0$ *and*

(a) $n = 3$;

(b) $n = 4$ and $v_0'(R)R < 0$;

(c) $n \geqslant 5$ and $v_0'(R)R < -\frac{(n-2)\sqrt{C(R)}}{2}(1 - \mathcal{I}_n)$,

where I_n is a constant given by

$$\mathcal{I}_n := \int_1^\infty \left((1 - y^{-2})^{-\frac{1}{n-2}} - 1\right) dy < 1;$$

3. $\partial_r C(R) > 0$ *and*

(a) $n = 3$ and

$$\begin{aligned} v_0' R \leqslant & -\frac{3}{4}\sqrt{C + R\partial_r C} \\ & + \frac{\sqrt{C}}{2}\left(1 - \frac{R\partial_r C}{2C}\right) \log\left(\frac{\sqrt{C} + \sqrt{C + R\partial_r C}}{\sqrt{R\partial_r C}}\right); \end{aligned}$$

(b) $n = 4$ and $v_0' R \leqslant -\sqrt{2R\partial_r C}$;

(c) $n \geqslant 5$ and

$$\begin{aligned} v_0' R \leqslant & -\frac{(n-2)^{\frac{1}{2}}(R\partial_r C)^{\frac{3}{2}}}{4C}\left(1 + \frac{(n-2)C}{R\partial_r C}\right)^{\frac{n}{2(n-2)}} \\ & - \frac{(n-2)C^{\frac{1}{2}}}{2}\left(1 - \frac{R\partial_r C}{2C}\right) \\ & \times \left[\left(1 + \frac{R\partial_r C}{(n-2)C}\right)^{\frac{1}{2}} - \int_{\left(1+\frac{R\partial_r C}{(n-2)C}\right)^{\frac{1}{2}}}^\infty \left((1 - y^{-2})^{-\frac{1}{n-2}} - 1\right) dy\right]. \end{aligned}$$

Here, we omit R variable in C, $\partial_r C$, and v_0', for simplicity.

Definition 3.3.6 (PCFB for $v_0 < 0$)**.** *Suppose $\lambda > 0$ and $n \geqslant 3$. The PCFB under $v_0(R) < 0$ is that $A(R) = 0$ or either one of following five conditions holds:*

1. $\partial_r C(R) < 0$;

2. $\partial_r C(R) = 0$ *and*

$$\frac{1}{|v_0(R)|} - \frac{1}{2}\int_R^\infty \frac{\partial_r A(R) y^{-(n-2)}}{(C(R) - A(R)y^{-(n-2)})^{3/2}} dy < 2\partial_r t_*(R);$$

3. $0 < \partial_r C(R) \leqslant \partial_r A(R) R^{-(n-2)}$ *and*

$$\frac{1}{|v_0(R)|} + \frac{1}{2}\int_R^{\left(\frac{\partial_r A(R)}{\partial_r C(R)}\right)^{\frac{1}{n-2}}} \frac{\partial_r C(R) - \partial_r A(R) y^{-(n-2)}}{(C(R) - A(R)y^{-(n-2)})^{3/2}} dy \leqslant 2\partial_r t_*(R);$$

4. $\partial_r A(R)R^{-(n-2)} < \partial_r C(R) < \partial_r A(R)(R^{-(n-2)} + v_0(R)^2/A(R))$ *and*

$$\frac{1}{|v_0(R)|}+\frac{1}{2}\int_R^{(\frac{\partial_r A(R)}{\partial_r C(R)})^{\frac{1}{n-2}}} \frac{\partial_r C(R) - \partial_r A(R)y^{-(n-2)}}{(C(R) - A(R)y^{-(n-2)})^{3/2}}dy \leqslant \max(0, 2\partial_r t_*(R));$$

5. $\partial_r A(R)(R^{-(n-2)} + v_0(R)^2/A(R)) \leqslant \partial_r C(R)$,

where

$$t_*(R) := \left(A(R)C(R)^{-\frac{n}{2}}\right)^{\frac{1}{n-2}} \int_1^{R\left(\frac{A(R)}{C(R)}\right)^{-\frac{1}{n-2}}} \frac{dz}{\sqrt{1 - z^{-(n-2)}}}.$$

Theorem 3.3.7. *Suppose $\lambda > 0$, $n \geqslant 3$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. Then, the classical solution of* (rEP$_0$) *breaks down in finite time if and only if there exists R such that the one of the PCFB given in Definitions 3.3.4, 3.3.5, and 3.3.6 is met. On the other hand, the classical solution is global if and only if, for all $r > 0$, the PCFB does not hold. Moreover, if $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ ($k \geqslant 0$) satisfy the condition for global existence, then the corresponding solution satisfies*

$$\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).$$

Furthermore, it is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_0$) *in the distribution sense.*

Proof. **Case 1: $v_0 > 0$.**

We first note that, by (3.3.1) and the assumption $\lambda < 0$, $X''(t,R) > 0$ holds as long as $X(t,R) > 0$. Since $X'(0,R) = v_0(R) > 0$, we have $X'(t,R) > 0$, at least for small time $t \in [0,T_0]$. Note that $X'(t,R) > 0$ for $t \in [0,T_0]$ implies that, for $t \in [0,T_0]$, $X(t,R) \geqslant X(0,R) = R > 0$ and so $X''(t,R) > 0$. Then, it means that X' is also increasing for $t \in [0,T_0]$. Thus, we can choose T_0 arbitrarily large, that is, $X'(t,R) > 0$ for all $t \geqslant 0$. Then, for all $t \geqslant 0$, it follows from (3.3.3) that

$$\int_R^{X(t,R)} \frac{dy}{\sqrt{C(R) - A(R)y^{-(n-2)}}} = t.$$

This identity tells us that $X(t,R) \to \infty$ as $t \to \infty$ (This also follows from the fact that $X'(t,R) \geqslant X'(0,R) = v_0(R) > 0$). For simplicity, we omit the R variable in the followings. Differentiate with respect to R to obtain

$$\frac{\Gamma(t)}{\sqrt{C - AX(t)^{-(n-2)}}} - \frac{1}{v_0} - \frac{1}{2}\int_R^{X(t)} \frac{\partial_r C - \partial_r A y^{-(n-2)}}{\left(C - Ay^{-(n-2)}\right)^{3/2}}dy = 0.$$

We put

$$B(t) := \frac{\Gamma(t)}{\sqrt{C - AX^{-(n-2)}}} = \frac{1}{v_0} + \frac{1}{2}\int_R^{X(t)} \frac{\partial_r C - \partial_r A y^{-(n-2)}}{\left(C - Ay^{-(n-2)}\right)^{3/2}} dy.$$

Assume $\partial_r C(R) < 0$. Then, since $X(t) \to \infty$ as $t \to \infty$,

$$\frac{d}{dt}B(t) = \frac{\partial_r C - \partial_r A X(t)^{-(n-2)}}{2\left(C - AX(t)^{-(n-2)}\right)^{3/2}} X'(t) < \frac{\partial_r C}{2C^{3/2}} v_0 < 0$$

holds for sufficiently large t. Hence, we have $B(t) \to -\infty$ as $t \to \infty$, and so there always exists a time $t_0 \geqslant 0$ such that $B(t_0) \leqslant 0$. We see that $\partial_r C(R) < 0$ is a sufficient condition for finite-time breakdown.

Next we assume $\partial_r C(R) = 0$. Then, $B(t)$ is monotone decreasing because

$$\frac{d}{dt}B(t) = -\frac{\partial_r A X(t)^{-(n-2)}}{2\left(C - AX(t)^{-(n-2)}\right)^{3/2}} X'(t) \leqslant 0.$$

Therefore, there exists a time $t_0 \geqslant 0$ such that $B(t_0) \leqslant 0$ if and only if $\lim_{t\to\infty} B(t) < 0$ (including the case $\lim_{t\to\infty} B(t) = -\infty$). This condition is equivalent to

$$\frac{1}{v_0} - \frac{1}{2}\int_R^{\infty} \frac{\partial_r A y^{-(n-2)}}{\left(C - Ay^{-(n-2)}\right)^{3/2}} dy < 0.$$

We finally assume $\partial_r C(R) > 0$. We first consider the case $(\frac{\partial_r A}{\partial_r C})^{\frac{1}{n-2}} > R$. Then, $B(t)$ takes it minimum at a time $t = t_1 \geqslant 0$ such that

$$X(t_1, R) = \left(\frac{\partial_r A}{\partial_r C}\right)^{\frac{1}{n-2}} > R$$

because $\frac{d}{dt}B(t)$ is as above and t_1 is the time such that $\frac{d}{dt}B(t_1) = 0$. Therefore, there exists a time t_0 such that $B(t_0) \leqslant 0$ if and only if

$$B(t_1) = \frac{1}{v_0} + \frac{1}{2}\int_R^{(\frac{\partial_r A}{\partial_r C})^{\frac{1}{n-2}}} \frac{\partial_r C - \partial_r A y^{-(n-2)}}{(C - Ay^{-(n-2)})^{3/2}} dy \leqslant 0.$$

We finally consider the case $(\frac{\partial_r A}{\partial_r C})^{\frac{1}{n-2}} \leqslant R$. However, in this case, B is monotone increasing. Therefore, $B \geqslant B(0) = 1/v_0 > 0$ for all $t \geqslant 0$.

Case 2: $v_0 = 0$**.**

First note that we have, at least in a small time interval, $X(t, R) > 0$ because $X(0, R) = R > 0$. Since $X''(t, R) > 0$ holds as long as $X(t, R) > 0$ by (3.3.1), we can find a time $t_0 > 0$ such that $X'(t_0, R) > X'(0, R) = v_0(R) = 0$. Note that t_0 can be chosen arbitrarily small. Then, repeating the argument as in the previous case, we see that, $X'(t,R) \geqslant X'(t_0,R) > 0$ for all $t \geqslant t_0$, which shows $X'(t,R) > 0$ for all $t > 0$ and $X(t,R) \to \infty$ as $t \to \infty$. Moreover, $X(t,R) \sim C(R)^{1/2}t$ for sufficiently large t since $X'(t,R) \to C(R)^{1/2}$ as $t \to \infty$. It reveals that if $\partial_r C(R) < 0$ then the characteristic curves must cross and so the solution breaks down in finite time by Lemma 3.2.7.

We now suppose $\partial_r C(R) \geqslant 0$. We omit R variable in the followings. Since $X'(t) \geqslant 0$ for all $t \geqslant 0$, an integration of (3.3.3) gives

$$\int_R^{X(t)} \frac{dy}{\sqrt{C - Ay^{-(n-2)}}} = t.$$

By a change of variable $z = y/R$, the left hand side is equal to

$$\int_1^{X(t)/R} \frac{Rdz}{\sqrt{C - AR^{-(n-2)}z^{-(n-2)}}}.$$

We temporally assume that $v_0 > 0$ and take the limit $v_0 \downarrow 0$ later. This computation is justified, for example, by replacing v_0 by $X'(\varepsilon R, R) > 0$ with small $\varepsilon > 0$ and taking the limit $\varepsilon \downarrow 0$. Differentiation with respect R yields

$$\begin{aligned}0 =& \frac{R\partial_R(X(t)/R)}{\sqrt{C - AX(t)^{-(n-2)}}} + \int_1^{X(t)/R} \frac{dz}{\sqrt{C - AR^{-(n-2)}z^{-(n-2)}}} \\ &- R\int_1^{X(t)/R} \frac{\partial_r C - (\partial_r AR^{-(n-2)} - (n-2)AR^{-(n-1)})z^{-(n-2)}}{2\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz.\end{aligned}$$

For simplicity, we omit t variable in X and $\partial_R X$ for a while because the following computations do not include any differentiation. An elementary calculation shows

$$
\begin{aligned}
0 =& \frac{\partial_R X}{\sqrt{C - AX^{-(n-2)}}} - \frac{X}{R\sqrt{C - AX^{-(n-2)}}} + \int_1^{X/R} \frac{dz}{\sqrt{C - AR^{-(n-2)}z^{-(n-2)}}} \qquad (3.3.8)\\
&- \frac{R\partial_r C}{2C} \int_1^{X/R} \frac{C - AR^{-(n-2)}z^{-(n-2)}}{\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz\\
&- \frac{R\partial_r C}{2C} \int_1^{X/R} \frac{AR^{-(n-2)}z^{-(n-2)}}{\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz\\
&+ \int_1^{X/R} \frac{R(\partial_r AR^{-(n-2)} - (n-2)AR^{-(n-1)})z^{-(n-2)}}{2\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz\\
=& \frac{\partial_R X}{\sqrt{C - AX^{-(n-2)}}} - \frac{X}{R\sqrt{C - AX^{-(n-2)}}} + \int_1^{X/R} \frac{dz}{\sqrt{C - AR^{-(n-2)}z^{-(n-2)}}}\\
&- \frac{R\partial_r C}{2C} \int_1^{X/R} \frac{dz}{\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{1/2}}\\
&+ \frac{(-\partial_r CAR + C\partial_r AR - (n-2)AC)}{2CR^{n-2}} \int_1^{X/R} \frac{z^{-(n-2)}}{\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz
\end{aligned}
$$

Now, it also holds that

$$
\frac{-\partial_r CAR + C\partial_r AR - (n-2)AC}{2CR^{n-2}} = \left(-\frac{v_0' A}{CR^{n-3}} + \frac{\partial_r AR - (n-2)A}{2CR^{n-2}} v_0\right) v_0.
$$

Now, let us show that

$$
\lim_{v_0 \downarrow 0} v_0 \int_1^{X/R} \frac{z^{-(n-2)}}{\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz = \frac{2}{AR^{-(n-2)}(n-2)}. \qquad (3.3.9)
$$

Fix a small $\varepsilon > 0$. Then, we have

$$
\lim_{v_0 \downarrow 0} v_0 \int_{1+\varepsilon}^{X/R} \frac{z^{-(n-2)}}{\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz = 0,
$$

since the integral is uniformly bounded with respect to v_0. Moreover,

$$
\begin{aligned}
& v_0 \int_1^{1+\varepsilon} \frac{z^{-(n-2)}}{\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz\\
\leqslant & \frac{2v_0(1+\varepsilon)}{AR^{-(n-2)}(n-2)} \int_1^{1+\varepsilon} \frac{AR^{-(n-2)}(n-2)z^{-(n-1)}}{2\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz\\
\leqslant & \frac{2v_0(1+\varepsilon)}{AR^{-(n-2)}(n-2)} \left[\left(C - AR^{-(n-2)}\right)^{-\frac{1}{2}} - \left(C - AR^{-(n-2)}(1+\varepsilon)^{-(n-2)}\right)^{-\frac{1}{2}}\right]\\
\to & \frac{2(1+\varepsilon)}{AR^{-(n-2)}(n-2)}
\end{aligned}
$$

as $v_0 \to 0$. Similarly,

$$\begin{aligned}
&v_0 \int_1^{1+\varepsilon} \frac{z^{-(n-2)}}{\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz \\
&\geqslant \frac{2v_0}{AR^{-(n-2)}(n-2)} \int_1^{1+\varepsilon} \frac{AR^{-(n-2)}(n-2)z^{-(n-1)}}{2\left(C - AR^{-(n-2)}z^{-(n-2)}\right)^{3/2}} dz \\
&\to \frac{2}{AR^{-(n-2)}(n-2)}
\end{aligned}$$

as $v_0 \to 0$. Since $\varepsilon > 0$ is arbitrary, we obtain (3.3.9). Then, taking the limit $v_0 \downarrow 0$ in (3.3.8),

$$\begin{aligned}
0 =& \frac{\partial_R X}{C^{1/2}\sqrt{1-(R/X)^{n-2}}} - \frac{X}{RC^{1/2}\sqrt{1-(R/X)^{n-2}}} + \int_1^{X/R} \frac{dz}{C^{1/2}\sqrt{1-z^{-(n-2)}}} \\
&- \frac{R\partial_r C}{2C^{3/2}} \int_1^{X/R} \frac{dz}{\sqrt{1-z^{-(n-2)}}} - \frac{2v_0' R}{(n-2)C}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{\partial_R X(t)}{\sqrt{1-(R/X(t))^{n-2}}} =& \frac{X(t)}{R\sqrt{1-(R/X(t))^{n-2}}} - \int_1^{X(t)/R} \frac{dz}{\sqrt{1-z^{-(n-2)}}} \\
&+ \frac{R\partial_r C}{2C} \int_1^{X(t)/R} \frac{dz}{\sqrt{1-z^{-(n-2)}}} + \frac{2v_0' R}{(n-2)C^{1/2}}.
\end{aligned}$$

We denote this by $B(t)$.

Case 2-a.

We first assume that $\partial_r C(R) = 0$. We put

$$\mathcal{G}(s) := \frac{s}{\sqrt{1-s^{-(n-2)}}} - \int_1^s \frac{dz}{\sqrt{1-z^{-(n-2)}}}$$

An elementary calculation shows, for $s > 1$,

$$\mathcal{G}'(s) = -\frac{(n-2)s^{-(n-2)}}{2(1-s^{-(n-2)})^{3/2}} < 0,$$

and so $\mathcal{G}$ is monotone decreasing. Moreover, considering the inverse map of $z \mapsto (1-z^{-(n-2)})^{-1/2}$, we have

$$\int_1^s \frac{dz}{\sqrt{1-z^{-(n-2)}}} = \frac{s-1}{\sqrt{1-s^{-(n-2)}}} + \int_{(1-s^{-(n-2)})^{-\frac{1}{2}}}^{\infty} \left((1-y^{-2})^{-\frac{1}{n-2}} - 1\right) dy.$$

Therefore,

$$\mathcal{G}(s) = \frac{1}{\sqrt{1 - s^{-(n-2)}}} - \int_{(1-s^{-(n-2)})^{-\frac{1}{2}}}^{\infty} \left((1 - y^{-2})^{-\frac{1}{n-2}} - 1 \right) dy.$$

One verifies that if $n = 3$ then $\lim_{s \to \infty} \mathcal{G}(s) = -\infty$. We now put, for $n \geqslant 4$,

$$\mathcal{I}_n := \int_1^{\infty} \left((1 - y^{-2})^{-\frac{1}{n-2}} - 1 \right) dy.$$

For any $m > l \geqslant 4$ and $y \in (1, \infty)$, it holds that $(1 - y^{-2})^{-\frac{1}{m-2}} < (1 - y^{-2})^{-\frac{1}{l-2}}$. This leads to $\mathcal{I}_m < \mathcal{I}_l$ for $m > l \geqslant 4$. If $n = 4$ then

$$\begin{aligned} \mathcal{I}_4 &= \lim_{N \to \infty} \int_1^N \left((1 - y^{-2})^{-\frac{1}{2}} - 1 \right) dy \\ &= \lim_{N \to \infty} \left((N^2 - 1)^{\frac{1}{2}} - (N - 1) \right) = 1. \end{aligned}$$

Thus, we obtain

$$\lim_{s \to \infty} \mathcal{G}(s) = \begin{cases} -\infty & \text{if } n = 3, \\ 0 & \text{if } n = 4, \\ 1 - \mathcal{I}_n > 0 & \text{if } n \geqslant 5. \end{cases}$$

Since

$$B(t) = \mathcal{G}(X(t)/R) - \frac{2v_0' R}{(n-2)C^{1/2}},$$

we conclude that there exists $t_0 \in [0, \infty)$ such that $\Gamma(t_0) \leqslant 0$ if and only if

1. $n = 3$;
2. $n = 4$ and $v_0' R < 0$;
3. $n \geqslant 5$ and $v_0' R < -\frac{(n-2)\sqrt{C}}{2}(1 - \mathcal{I}_n)$.

Case 2-b.

We assume that $\partial_r C(R) > 0$. We write $B(t) = \mathcal{H}(X(t)/R)$. Then, it holds that

$$\frac{d}{ds}\mathcal{H}(s) = -\frac{(n-2)s^{-(n-2)}}{2(1 - s^{-(n-2)})^{3/2}} + \frac{R\partial_r C}{2C(1 - s^{-(n-2)})^{1/2}}.$$

Therefore, the minimum of $\mathcal{H}$, hence of B, is

$$\mathcal{H}\left(\left(1 + \frac{(n-2)C}{R\partial_r C} \right)^{\frac{1}{n-2}} \right).$$

The solution breaks down in finite time if and only if this value is less than or equal to zero. This gives the condition

$$\begin{aligned} v_0'R \leqslant & -\frac{\sqrt{(n-2)R\partial_r C}}{2}\left(1+\frac{(n-2)C}{R\partial_r C}\right)^{\frac{n}{2(n-2)}} \\ & -\frac{n-2}{2}C^{\frac{1}{2}}\left(\frac{R\partial_r C}{2C}-1\right)\int_1^{\left(1+\frac{(n-2)C}{R\partial_r C}\right)^{\frac{1}{n-2}}}\frac{dz}{\sqrt{1-z^{-(n-2)}}}. \end{aligned}$$

Using the identity

$$\begin{aligned} \int_1^{\left(1+\frac{(n-2)C}{R\partial_r C}\right)^{\frac{1}{n-2}}}\frac{dz}{\sqrt{1-z^{-(n-2)}}} = & \left(\left(1+\frac{(n-2)C}{R\partial_r C}\right)^{\frac{1}{n-2}}-1\right)\left(1+\frac{R\partial_r C}{(n-2)C}\right)^{\frac{1}{2}} \\ & +\int_{\left(1+\frac{R\partial_r C}{(n-2)C}\right)^{\frac{1}{2}}}^{\infty}\left((1-y^{-2})^{-\frac{1}{n-2}}-1\right)dy, \end{aligned}$$

we obtain the equivalent condition

$$\begin{aligned} v_0'R \leqslant & -\frac{(n-2)^{\frac{1}{2}}(R\partial_r C)^{\frac{3}{2}}}{4C}\left(1+\frac{(n-2)C}{R\partial_r C}\right)^{\frac{n}{2(n-2)}} \\ & -\frac{(n-2)C^{\frac{1}{2}}}{2}\left(1-\frac{R\partial_r C}{2C}\right) \\ & \times\left[\left(1+\frac{R\partial_r C}{(n-2)C}\right)^{\frac{1}{2}}-\int_{\left(1+\frac{R\partial_r C}{(n-2)C}\right)^{\frac{1}{2}}}^{\infty}\left((1-y^{-2})^{-\frac{1}{n-2}}-1\right)dy\right]. \end{aligned}$$

In particular, if $n=3$ or 4, then the above integral is computable, and we have more explicit condition

$$v_0'R \leqslant -\frac{3}{4}\sqrt{C+R\partial_r C}+\frac{\sqrt{C}}{2}\left(1-\frac{R\partial_r C}{2C}\right)\log\left(\frac{\sqrt{C}+\sqrt{C+R\partial_r C}}{\sqrt{R\partial_r C}}\right)$$

if $n=3$ and

$$v_0'R \leqslant -\sqrt{2R\partial_r C}$$

if $n=4$.

Case 3: $v_0 < 0$.

We first note that if $A(R)=0$, then $X'(t,R)\equiv v_0(R)<0$. Therefore, the solution breaks down no latter than $t=R/|v_0(R)|$ by Lemma 3.2.7. Hence, we assume $A(R)>0$. Then, since $X'(0,R)=v_0(R)<0$, we deduce from

(3.3.3) that $X'(t,R) = -\sqrt{C - AX(t)^{-(n-2)}}$ as long as $X'(t,R) \leqslant 0$. Take

$$t_* = \int_{\left(\frac{A}{C}\right)^{\frac{1}{n-2}}}^{R} \frac{dy}{\sqrt{C - Ay^{-(n-2)}}}$$
$$= \left(AC^{-\frac{n}{2}}\right)^{\frac{1}{n-2}} \int_1^{R\left(\frac{A}{C}\right)^{-\frac{1}{n-2}}} \frac{dz}{\sqrt{1 - z^{-(n-2)}}}.$$

We see that, for all $t \in [0, t_*)$, $X(t,R) > X(t_*, R) = (A(R)/C(R))^{1/(n-2)} > 0$ and $X'(t,R) < X'(t_*, R) = 0$. Since $X''(t_*, R) > 0$ by (3.3.1), repeating the same argument as in the previous two cases, we have $X'(t,R) \geqslant 0$ for all $t \geqslant t_*$ and so

$$X'(t,R) = \begin{cases} -\sqrt{C(R) - A(R)X(t,R)^{-(n-2)}}, & \text{for } t \leqslant t_*, \\ \sqrt{C(R) - A(R)X(t,R)^{-(n-2)}}, & \text{for } t \geqslant t_*. \end{cases}$$

We also obtain $X(t,R) \to \infty$ as $t \to \infty$. In the followings, we omit R variable. For sufficient large t, $X(t) \sim C^{1/2}t$ holds since $X'(t) \to C^{1/2}$ as $t \to \infty$. It implies that if $\partial_r C(R) < 0$ then the characteristic curves must cross and so the solution breaks down in finite time by Lemma 3.2.7. Differentiation of $X(t_*, R) = (A/C)^{1/(n-2)}$ with respect to R gives

$$(\partial_r t_*) X'(t_*, R) + \partial_R X(t_*, R) = \partial_r \left(\frac{A}{C}\right)^{\frac{1}{n-2}}.$$

Using the fact that $X'(t_*) = 0$, we obtain

$$\partial_R X(t_*, R) = \partial_r \left(\frac{A}{C}\right)^{\frac{1}{n-2}}.$$

Hence, if $\partial_r (A/C)^{1/(n-2)} \leqslant 0$ then the solution breaks down no latter than t_*.

Thus, we assume $\partial_r C(R) \geqslant 0$ and $(\partial_r (A/C)(R))^{1/(n-2)} > 0$ in the followings. Notice that the latter condition is equivalent to the following two conditions:

$$\partial_r C < \partial_r A(R^{-(n-2)} + v_0^2/A), \quad \left(\frac{A}{C}\right)^{\frac{1}{n-2}} < \left(\frac{\partial_r A}{\partial_r C}\right)^{\frac{1}{n-2}}.$$

Step 1. We determine the condition that solution can be extended to time $t = t_*$. For $t \leqslant t_*$, we have

$$\int_{X(t)}^{R} \frac{dy}{\sqrt{C - Ay^{-(n-2)}}} = t.$$

Differentiation with respect to R yields

$$\frac{1}{\sqrt{C-AR^{-(n-2)}}}-\frac{\Gamma(t)}{\sqrt{C-AX(t)^{-(n-2)}}}-\frac{1}{2}\int_{X(t)}^{R}\frac{\partial_r C-\partial_r A y^{-(n-2)}}{(C-Ay^{-(n-2)})^{3/2}}dy=0.$$

For $0\leqslant t<t_*$, it holds that

$$0<\sqrt{C-AX(t)^{-(n-2)}}\leqslant\sqrt{C-AR^{-(n-2)}}=|v_0|$$

Therefore, $\Gamma(t)$ has the same sign as

$$B_1(t):=\frac{\Gamma(t)}{\sqrt{C-AX^{-(n-2)}}}=\frac{1}{|v_0|}-\frac{1}{2}\int_{X(t)}^{R}\frac{\partial_r C-\partial_r A y^{-(n-2)}}{(C-Ay^{-(n-2)})^{3/2}}dy.$$

Taking time derivative, one verifies that B_1 takes it minimum at $t=t_1\in[0,t_*)$ such that

$$X(t_1,R)=\min\left(R,\left(\frac{\partial_r A}{\partial_r C}\right)^{\frac{1}{n-2}}\right).$$

Note that $(A/C)^{1/(n-2)}<X(t_1)$ by assumption, and that $(\partial_r A/\partial_r C)^{1/(n-2)}<R$ is equivalent to $\partial_r C>\partial_r AR^{-(n-2)}$. Since we have already known that $\Gamma(0)=1>0$, the solution can be extended to the time $t=t_*$ UNLESS $\partial_r C>\partial_r AR^{-(n-2)}$ and

$$B_1(t_1)=\frac{1}{|v_0|}-\frac{1}{2}\int_{\left(\frac{\partial_r A}{\partial_r C}\right)^{\frac{1}{n-2}}}^{R}\frac{\partial_r C-\partial_r A y^{-(n-2)}}{(C-Ay^{-(n-2)})^{3/2}}dy\leqslant 0$$

is satisfied. Notice that this condition is a sufficient condition for finite-time breakdown.

Step 2. We consider the condition that the solution can be extended from the time $t=t_*$ to $t=\infty$. For simplicity, we suppose that solutions are extended to time $t=t_*$ (we keep assuming $0\leqslant\partial_r C<\partial_r A(R^{-(n-2)}+v_0^2/A)$ holds). Recall that, for $t\geqslant t_*$, $X'(t)=\sqrt{C-AX(t)^{-(n-2)}}\geqslant 0$. As in the case $v_0=0$, this inequality with $X''(t)>0$ gives $X(t)\sim C^{1/2}t\to\infty$ as $t\to\infty$.

We define t_{**} as the time that $t_{**}>t_*$ and $X(t_{**})=R$. Then, we have

$$t_{**}-t_*=\int_{\left(\frac{A}{C}\right)^{\frac{1}{n-2}}}^{R}\frac{dy}{\sqrt{C-Ay^{-(n-2)}}}=t_*.$$

Therefore, $t_{**}=2t_*$ and

$$\int_R^{X(t)}\frac{dy}{\sqrt{C-Ay^{-(n-2)}}}=t-2t_*$$

for all $t \geqslant t_*$. As in the previous step, we set

$$B_2(t) := \frac{\Gamma(t)}{\sqrt{C - AX(t)^{-(n-2)}}} = \frac{1}{|v_0|} + \frac{1}{2}\int_R^{X(t)} \frac{\partial_r C - \partial_r A y^{-(n-2)}}{(C - Ay^{-(n-2)})^{3/2}} dy - 2\partial_r t_*.$$

$B_2(t)$ and $\Gamma(t)$ has the same sign for $t \geqslant t_*$. We also note that $B_2(t) \to \infty$ as $t \downarrow t_*$ because $\Gamma(t_*) > 0$ and $\sqrt{C - AX(t)^{-(n-2)}} \to 0$ as $t \downarrow t_*$. It holds that

$$\frac{d}{dt} B_2(t) = \frac{\partial_r C - \partial_r AX(t)^{-(n-2)}}{2(C - AX(t)^{-(n-2)})^{3/2}} X'(t).$$

1. If $\partial_r C(R) = 0$ then B_2 is monotone decreasing because $\frac{d}{dt} B_2(t) \leqslant 0$. Therefore, solution can be extended to $t = \infty$ if and only if

$$\lim_{t \to \infty} B_2(t) = \frac{1}{|v_0|} - \frac{1}{2}\int_R^{\infty} \frac{\partial_r A y^{-(n-2)}}{(C - Ay^{-(n-2)})^{3/2}} dy - 2\partial_r t_* \geqslant 0.$$

2. If $\partial_r C(R) > 0$ then B_2 takes it minimum at $t = t_2$ such that $X(t_2) = (\partial_r A / \partial_r C)^{1/(n-2)}$. Therefore, solution can be extended to $t = \infty$ if and only if

$$B_2(t_2) = \frac{1}{|v_0|} + \frac{1}{2}\int_R^{\left(\frac{\partial_r A}{\partial_r C}\right)^{\frac{1}{n-2}}} \frac{\partial_r C - \partial_r A y^{-(n-2)}}{(C - Ay^{-(n-2)})^{3/2}} dy - 2\partial_r t_* > 0.$$

□

A remark

Before proceeding to the two-dimensional case, let us see that Theorem 3.3.7 gives the same criterion as in Theorem 3.3.3 if $n = 4$.

Corollary 3.3.8. *If $n = 4$, the PCFB given in Definitions 3.3.4, 3.3.5, and 3.3.6 is reduced to the following condition:*

1. $A(R) = 0$ *and* $v_0(R) < 0$.
2. $\partial_r C(R) < 0$;
3. $\partial_r C(R) = 0$ *and* $v_0(R) + v_0'(R)R < 0$;
4. $\partial_r C(R) > 0$ *and* $v_0(R) + v_0'(R)R \leqslant -\sqrt{2R\partial_r C(R)}$.

Proof. Before the proof, we prepare some elementary computations. We note that

$$
\begin{aligned}
&\int_R^{\sqrt{\frac{\partial_r A}{\partial_r C}}} \frac{\partial_r C - \partial_r A y^{-2}}{(C - Ay^{-2})^{3/2}} dy \\
&= \frac{\partial_r C}{C} \int_R^{\sqrt{\frac{\partial_r A}{\partial_r C}}} \frac{y}{(Cy^2 - A)^{1/2}} dy + \frac{A\partial_r C - C\partial_r A}{C} \int_R^{\sqrt{\frac{\partial_r A}{\partial_r C}}} \frac{y}{(Cy^2 - A)^{3/2}} dy \\
&= \frac{\partial_r C}{C^2} \left[\left(C\frac{\partial_r A}{\partial_r C} - A \right)^{\frac{1}{2}} - (CR^2 - A)^{\frac{1}{2}} \right] \\
&\quad + \frac{A\partial_r C - C\partial_r A}{C^2} \left[(CR^2 - A)^{-\frac{1}{2}} - \left(C\frac{\partial_r A}{\partial_r C} - A \right)^{-\frac{1}{2}} \right] \\
&= \frac{2(\partial_r C)^{\frac{1}{2}}}{C^2} (C\partial_r A - A\partial_r C)^{\frac{1}{2}} - \frac{|v_0|R}{C^2} \partial_r C - \frac{C\partial_r A - A\partial_r C}{C^2 |v_0| R},
\end{aligned}
$$

and that

$$
\begin{aligned}
&\frac{1}{|v_0|} - \frac{|v_0|R}{2C^2}\partial_r C - \frac{C\partial_r A - A\partial_r C}{2C^2|v_0|R} \\
&= \left(\frac{1}{|v_0|} + \frac{v_0^2 R^2 + A}{2C^2|v_0|R}\partial_r C - \frac{\partial_r A}{2C|v_0|R} \right) - \frac{|v_0|R}{C^2}\partial_r C \\
&= \frac{2CR + R^2\partial_r C - \partial_r A}{2C|v_0|R} - \frac{|v_0|R}{C^2}\partial_r C \\
&= (\operatorname{sign} v_0) \left(\frac{v_0 + Rv_0'}{C} - \frac{v_0 R}{C^2}\partial_r C \right) \\
&= (\operatorname{sign} v_0) \partial_r \left(\frac{v_0 R}{C} \right),
\end{aligned}
$$

where we have used $v_0^2 R^2 + A = CR^2$ and

$$
\begin{aligned}
&2CR + R^2\partial_r C - \partial_r A \\
&= \left(2v_0^2 R + \frac{2\lambda m_0}{R} \right) + \left(2v_0 v_0' R^2 + \partial_r A - \frac{2\lambda m_0}{R} \right) - \partial_r A \\
&= 2v_0 R(v_0 + Rv_0').
\end{aligned}
$$

It also holds that

$$
\begin{aligned}
t_* &= \sqrt{AC^{-2}} \int_1^{R\left(\frac{A}{C}\right)^{-\frac{1}{2}}} \frac{dz}{\sqrt{1 - z^{-2}}} = \sqrt{AC^{-2}} \int_1^{R^2\left(\frac{C}{A}\right)} \frac{dz}{2\sqrt{z - 1}} \\
&= \sqrt{AC^{-2}} \sqrt{\frac{R^2 C}{A} - 1} = \frac{|v_0|R}{C}.
\end{aligned}
$$

From Definitions 3.3.4, 3.3.5, and 3.3.6, we see that $\partial_r C < 0$ is the sufficient condition for blow-up. Moreover, the PCFB in the case $\partial_r C = 0$ is

$$\frac{(\partial_r C)^{\frac{1}{2}}}{C^2}(C\partial_r A - A\partial_r C)^{\frac{1}{2}} + \partial_r\left(\frac{v_0 R}{C}\right) = \frac{v_0 + Rv_0'}{C} < 0$$

if $v_0 > 0$,

$$Rv_0' < 0$$

if $v_0 = 0$, and

$$\frac{(\partial_r C)^{\frac{1}{2}}}{C^2}(C\partial_r A - A\partial_r C)^{\frac{1}{2}} - \partial_r\left(\frac{v_0 R}{C}\right) + 2\partial_r\left(\frac{v_0 R}{C}\right) = \frac{v_0 + Rv_0'}{C} < 0$$

if $v_0 < 0$. Hence, the PCFB can be summarized as $v_0 + Rv_0' < 0$.

Let us proceed to the case $\partial_r C > 0$. If $v_0 > 0$ then Definition 3.3.4 implies that the PCFB is $\partial_r C < \partial_r AR^{-2} \Leftrightarrow v_0 + Rv_0' < C/v_0$ and

$$\frac{(\partial_r C)^{\frac{1}{2}}}{C^2}(C\partial_r A - A\partial_r C)^{\frac{1}{2}} + \partial_r\left(\frac{v_0 R}{C}\right) \leqslant 0. \tag{3.3.10}$$

We put $\alpha = v_0 + Rv_0'$, $\beta = v_0 R\partial_r C/C > 0$, and $\gamma = \partial_r C(C\partial_r A - A\partial_r C)/C^2$. Note that, by assumption, we have $0 < A/C < R^2 < \partial_r A/\partial_r C$, which implies $\gamma > 0$. Then, (3.3.10) can be written as $\alpha \leqslant \beta - \sqrt{\gamma}$. We make this condition clearer. An elementary computation shows that $\delta := \gamma + 2\alpha\beta - \beta^2 = 2R\partial_r C > 0$, and that $\delta - 2\alpha\beta = \frac{R\partial_r C(-\partial_r C + \partial_r AR^{-2})}{C} > 0$. The latter one means $\beta^2 < \gamma$. Thus, the inequality $\alpha \leqslant \beta - \sqrt{\gamma} < 0$ is reduced to $\alpha \leqslant -\sqrt{\gamma + 2\alpha\beta - \beta^2} = -\sqrt{\delta}$, that is, $v_0 + Rv_0' \leqslant -\sqrt{2R\partial_r C}$. This condition is stronger than $\partial_r C < \partial_r AR^{-2} \Leftrightarrow v_0 + Rv_0' < C/v_0$.

If $\partial_r C > 0$ and $v_0 = 0$, then it immediately follows from Definition 3.3.5 that $\alpha \leqslant -\sqrt{\delta}$ is the PCFB.

We next consider the case $\partial_r C > 0$ and $v_0 < 0$. Definition 3.3.6 gives the PCFB. If $\partial_r C \leqslant \partial_r AR^{-2}$, then the condition is

$$\frac{(\partial_r C)^{\frac{1}{2}}}{C^2}(C\partial_r A - A\partial_r C)^{\frac{1}{2}} - \partial_r\left(\frac{v_0 R}{C}\right) + 2\partial_r\left(\frac{v_0 R}{C}\right) \leqslant 0.$$

We keep the above notations α, β, γ, and δ. Then, this is written as $\alpha \leqslant \beta - \sqrt{\gamma}$. Note that the right hand side is negative. By the same argument as above, it is also written as $\alpha \leqslant -\sqrt{\delta}$. If $\partial_r AR^{-2} < \partial_r C \leqslant \partial_r A(R^{-2} + v_0^2/A)$, then the condition is

$$\frac{(\partial_r C)^{\frac{1}{2}}}{C^2}(C\partial_r A - A\partial_r C)^{\frac{1}{2}} \leqslant \left|\partial_r\left(\frac{v_0 R}{C}\right)\right|,$$

which is written as $\sqrt{\gamma} \leqslant |\alpha - \beta|$. Note that $\partial_r C < \partial_r A(R^{-2} + v_0^2/A) = C\partial_r A/A$ is equivalent to $\gamma > 0$. By assumption, we also have $\beta < 0$ and

$\gamma - \beta^2 = \delta - 2\alpha\beta < 0$. We now show that $\alpha \geqslant \beta$ leads the contradiction. In this case, $\sqrt{\gamma} \leqslant |\alpha - \beta|$ is equivalent to $\alpha \geqslant \beta + \sqrt{\gamma}$. However, this is also written as

$$0 < \sqrt{\gamma} \leqslant |\alpha - \beta| = \alpha - \beta \Longleftrightarrow \alpha^2 \geqslant \gamma + 2\alpha\beta - \beta^2 = \delta > 0$$
$$\Longleftrightarrow \alpha \geqslant \sqrt{\delta} \text{ or } \alpha \leqslant -\sqrt{\delta}.$$

The last inequalities cannot be iquivalent to $\alpha \geqslant \beta + \sqrt{\gamma}$ since $\sqrt{\delta} > 0$ and $\beta + \sqrt{\gamma} < 0$. This is the contradiction. Hence, $\beta \geqslant \alpha$. Then, $\sqrt{\gamma} \leqslant |\alpha - \beta| = \beta - \alpha$ corresponds to $\alpha \leqslant -\sqrt{\delta}$.

We finally treat the case $\partial_r C \geqslant \partial_r A(R^{-2} + v_0^2/A)$. We prove this condition is stronger than $\alpha \leqslant -\sqrt{\delta}$. An elementary computation show that $\partial_r C \geqslant \partial_r A(R^{-2} + v_0^2/A)$ implies

$$\alpha \leqslant \frac{C}{v_0} + \frac{v_0 R \partial_r A}{2A} < 0.$$

Moreover, introducing the function $P(t) = \partial_r C t^2 + 2\alpha t + 2R$, we see that

$$\begin{aligned}
\frac{\delta - \alpha^2}{\partial_r C} = \min_t P(t) \leqslant P\left(-\frac{v_0 R}{C}\right) &= \partial_r C \left(-\frac{v_0 R}{C}\right)^2 + 2\alpha\left(-\frac{v_0 R}{C}\right) + 2R \\
&= \frac{1}{C^2}\Bigg[\left(2v_0 v_0' + \frac{\partial_r A}{R^2} - \frac{2A}{R^3}\right) v_0^2 R^2 \\
&\quad - 2(v_0 + R v_0') v_0 R \left(v_0^2 + \frac{A}{R^2}\right) + 2R\left(v_0^2 + \frac{A}{R^2}\right)^2\Bigg] \\
&= \frac{1}{C^2}\left(v_0^2 \partial_r A - 2v_0 v_0' A + \frac{2A^2}{R^3}\right) \\
&= \frac{A}{C^2}\left(\partial_r A\left(R^{-2} + \frac{v_0^2}{A}\right) - \partial_r C\right) \leqslant 0.
\end{aligned}$$

□

3.3.4 Repulsive case 3: $n = 2$

We finally consider the two-dimensional case. Though we can calculate the characteristic curve in an implicit way ([25]), we use the argument similar to the previous $n \geqslant 3$ case. Let us recall the ODE which we analyze:

$$X''(t,R) = \frac{\lambda m_0(R)}{X(t,R)}, \quad X'(0,R) = v_0(R), \quad X(0,R) = R. \tag{3.3.1}$$

and its integral form

$$(X'(t,R))^2 = \mathcal{C}(R) + \mathcal{A}(R)(\log X(t,R)). \tag{3.3.5}$$

Let us first describe the PCFBs with $\mathcal{A}$ and $\mathcal{C}$ introduced as

$$\mathcal{A}(r) := 2\lambda m_0(r), \quad \mathcal{C}(r) := v_0(r)^2 - A(r)\log r \tag{3.3.4}$$

Definition 3.3.9 (PCFB for $v_0 > 0$)**.** *Suppose $\lambda > 0$ and $n = 2$. The PCFB under $v_0(R) > 0$ is that*

$$v_0'(R) < \frac{\mathcal{A}(R)}{2Rv_0(R)}$$

($\Leftrightarrow \exp(-\partial_r\mathcal{C}(R)/\partial_r\mathcal{A}(R)) > R$) and either one of following conditions hold:

1. $\rho_0(R) = 0$ *($\partial_r\mathcal{A}(R) = 0$);*
2. $\partial_r\mathcal{A}(R) > 0$ *and*

$$\frac{1}{v_0(R)} + \frac{1}{2}\int_R^{\exp\left(-\frac{\partial_r\mathcal{C}(R)}{\partial_r\mathcal{A}(R)}\right)} \frac{\partial_r\mathcal{C}(R) + \partial_r\mathcal{A}(R)\log y}{(\mathcal{C}(R) + \mathcal{A}(R)\log y)^{3/2}}dy \leqslant 0.$$

Definition 3.3.10 (PCFB for $v_0 = 0$)**.** *Suppose $\lambda > 0$ and $n = 2$. The PCFB under $v_0(R) = 0$ is that $\mathcal{A}(R) > 0$ and either one of following conditions hold:*

1. $\rho_0(R) = 0$ *($\partial_r\mathcal{A}(R) = 0$);*
2. $\partial_r\mathcal{A}(R) > 0$ *and*

$$Rv_0'(R) \leqslant -\frac{\sqrt{\mathcal{A}(R)R\partial_r\mathcal{A}(R)}}{2}e^{\frac{\mathcal{A}(R)}{R\partial_r\mathcal{A}(R)}} + \frac{2\mathcal{A}(R) - R\partial_r\mathcal{A}(R)}{4}\int_1^{e^{\frac{\mathcal{A}(R)}{R\partial_r\mathcal{A}(R)}}} \frac{dz}{\sqrt{\log z}}.$$

Definition 3.3.11 (PCFB for $v_0 < 0$)**.** *Suppose $\lambda > 0$ and $n = 2$. The PCFB under $v_0(R) < 0$ is that $\mathcal{A}(R) = 0$ or either one of following conditions holds (we omit all R variables, for simplicity):*

1. $\rho_0 = 0$ *($\partial_r\mathcal{A} = 0$);*
2. $\partial_r\mathcal{A} > 0$ *and*
   (a) $\partial_r(v_0^2) \geqslant \mathcal{A}/R + (\partial_r\mathcal{A})\log(Re^{\mathcal{A}/\mathcal{C}})$*;*
   (b) $\mathcal{A}/R \leqslant \partial_r(v_0^2) < \mathcal{A}/R + (\partial_r\mathcal{A})\log(Re^{\mathcal{A}/\mathcal{C}})$ *and*

$$\frac{1}{|v_0|} + \frac{1}{2}\int_R^{\exp\left(-\frac{\partial_r\mathcal{C}}{\partial_r\mathcal{A}}\right)} \frac{\partial_r\mathcal{C} + \partial_r\mathcal{A}\log y}{(\mathcal{C} + \mathcal{A}\log y)^{3/2}}dy \leqslant \max(0, 2\partial_r t_*);$$

   (c) $\partial_r(v_0^2) < \mathcal{A}/R$ *and*

$$\frac{1}{|v_0|} + \frac{1}{2}\int_R^{\exp\left(-\frac{\partial_r\mathcal{C}}{\partial_r\mathcal{A}}\right)} \frac{\partial_r\mathcal{C} + \partial_r\mathcal{A}\log y}{(\mathcal{C} + \mathcal{A}\log y)^{3/2}}dy \leqslant 2\partial_r t_*,$$

where

$$t_* = t_*(R) := \frac{R}{\mathcal{A}(R)^{1/2} e^{v_0(R)^2/\mathcal{A}(R)}} \int_1^{e^{v_0(R)^2/\mathcal{A}(R)}} \frac{dz}{\sqrt{\log z}}.$$

Theorem 3.3.12. *Suppose $\lambda > 0$, $n = 2$, and $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. Then, the classical solution of* (rEP$_0$) *breaks down in finite time if and only if there exists R such that one of the PCFB given in Definitions 3.3.9, 3.3.10, and 3.3.11 is met. On the other hand, the classical solution is global if and only if, for all $r > 0$, the PCFB does not hold. Moreover, if $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ ($k \geqslant 0$) satisfy the condition for global existence, then the corresponding solution satisfies*

$$\rho \in C^2([0,\infty), D^s) \cap C^\infty((0,\infty), D^s),$$
$$v \in C^1([0,\infty), D^{s+1}) \cap C^\infty((0,\infty), D^{s+1}).$$

Furthermore, it is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_0$) *in the distribution sense.*

Proof. **Case 1: $v_0 > 0$.**

We first note that $X'(t,R) \geqslant v_0(R) > 0$, $\forall t \geqslant 0$ follows from the same argument as in the Case 1 of the proof of Theorem 3.3.7. Then, $X(t,R) \to \infty$ as $t \to \infty$, and, by (3.3.5),

$$\int_R^{X(t,R)} \frac{dy}{\sqrt{v_0(R)^2 + \mathcal{A}(R)\log(y/R)}} = t.$$

for all $t \geqslant 0$. For simplicity, we omit the R variable in the followings. Differentiate this with respect to R to get

$$\frac{\Gamma(t)}{X'(t)} - \frac{1}{v_0} - \frac{1}{2}\int_R^{X(t)} \frac{2v_0v_0' - \mathcal{A}/R + \partial_r\mathcal{A}\log(y/R)}{(v_0^2 + \mathcal{A}\log(y/R))^{3/2}} dy = 0.$$

We put

$$B(t) := \frac{\Gamma(t)}{X'(t)} = \frac{1}{v_0} + \frac{1}{2}\int_R^{X(t)} \frac{2v_0v_0' - \mathcal{A}/R + \partial_r\mathcal{A}\log(y/R)}{(v_0^2 + \mathcal{A}\log(y/R))^{3/2}} dy.$$

Since $X'(t) > 0$ for all $t \geqslant 0$, $B(t)$ and $\Gamma(t)$ has the same sign. Since $\partial_r\mathcal{A} \geqslant 0$ by definition, the right hand side is positive for all time if $2v_0v_0' - \mathcal{A}/R \geqslant 0$. Now, we suppose $2v_0v_0' - \mathcal{A}/R < 0$. Recall that $X(t) \to \infty$ as $t \to \infty$ and that $\mathcal{A}$ and v_0 are independent of time. If $\partial_r\mathcal{A} = 0$ then one sees that there exist $t_0 > 0$ such that

$$\int_R^{X(t_0)} \frac{dy}{(v_0^2 + \mathcal{A}\log(y/R))^{3/2}} = \frac{2}{v_0|2v_0v_0' - \mathcal{A}/R|}$$

since $\int_R^{X(t)}(v_0^2+\mathcal{A}\log(y/R))^{-3/2}dy \to \infty$ as $t \to \infty$. This implies $\Gamma(t_0) = B(t_0) = 0$, which lead to finite-time breakdown. Let us proceed to the case $\partial_r \mathcal{A} > 0$. An elementary computation shows that the minimum of B is $B\left(e^{-\frac{\partial_r \mathcal{C}}{\partial_r \mathcal{A}}}\right)$. Therefore, under the assumption $v_0 v_0' - \mathcal{A}/R < 0$ and $\partial_r \mathcal{A} > 0$, there exists a time t_0 such that $\Gamma(t_0) \leqslant 0$ if and only if

$$\frac{1}{v_0} + \frac{1}{2}\int_R^{\exp\left(-\frac{\partial_r \mathcal{C}}{\partial_r \mathcal{A}}\right)} \frac{\partial_r \mathcal{C} + \partial_r \mathcal{A}\log y}{(\mathcal{C}+\mathcal{A}\log y)^{3/2}} dy \leqslant 0.$$

Case 2: $v_0 = 0$.

Let us begin with pointing out that the exactly same argument as in the Case 2 of the proof of Theorem 3.3.7 shows that $X'(t,R) > 0$ for all $t > 0$ and $X(t,R) \to \infty$ as $t \to \infty$. We omit R variable in the followings. Let us temporarily suppose that $v_0 > 0$ and let $v_0 \to 0$ later. Integration of (3.2.10) gives

$$\int_R^{X(t)} \frac{dy}{\sqrt{v_0^2 + \mathcal{A}\log(y/R)}} = t.$$

By a change of variable $z = y/R$, the left hand side is equal to

$$\int_1^{X(t)/R} \frac{R dz}{\sqrt{v_0^2 + \mathcal{A}\log z}}.$$

Hence, differentiation with respect R yields

$$\begin{aligned} 0 =& \frac{R\partial_R (X(t)/R)}{\sqrt{v_0^2 + \mathcal{A}\log(X(t)/R)}} + \int_1^{X(t)/R} \frac{dz}{\sqrt{v_0^2 + \mathcal{A}\log z}} \\ &- R\int_1^{X(t)/R} \frac{\partial_r v_0^2 + \partial_r \mathcal{A}\log z}{2\left(v_0^2+\mathcal{A}\log z\right)^{3/2}} dz. \end{aligned}$$

For a while, we omit also t variable. An elementary calculation shows

$$\begin{aligned} 0 =& \frac{\partial_R X}{\sqrt{v_0^2 + \mathcal{A}\log(X/R)}} - \frac{X}{R\sqrt{v_0^2 + \mathcal{A}\log(X/R)}} + \int_1^{X/R} \frac{dz}{\sqrt{v_0^2 + \mathcal{A}\log z}} \\ &- \frac{R\partial_r \mathcal{A}}{2\mathcal{A}} \int_1^{X/R} \frac{dz}{\sqrt{v_0^2 + \mathcal{A}\log z}} + \frac{R v_0^2 \partial_r \mathcal{A}}{2\mathcal{A}} \int_1^{X/R} \frac{dz}{\left(v_0^2+\mathcal{A}\log z\right)^{3/2}} \\ &- R v_0 v_0' \int_1^{X/R} \frac{dz}{\left(v_0^2+\mathcal{A}\log z\right)^{3/2}}. \end{aligned} \tag{3.3.11}$$

We now show that

$$\lim_{v_0 \downarrow 0} v_0 \int_1^{X/R} \frac{dz}{\left(v_0^2+\mathcal{A}\log z\right)^{3/2}} = \frac{2}{\mathcal{A}}. \tag{3.3.12}$$

Fix a small $\varepsilon > 0$. Then, we have

$$\lim_{v_0 \downarrow 0} v_0 \int_{1+\varepsilon}^{X/R} \frac{dz}{\left(v_0^2 + \mathcal{A}\log z\right)^{3/2}} = 0,$$

since the integral is uniformly bounded with respect to v_0. Moreover,

$$\begin{aligned}
&v_0 \int_1^{1+\varepsilon} \frac{dz}{\left(v_0^2 + \mathcal{A}\log z\right)^{3/2}} \\
&\leqslant \frac{2v_0(1+\varepsilon)}{\mathcal{A}} \int_1^{1+\varepsilon} \frac{\mathcal{A}}{2z(v_0^2 + \mathcal{A}\log z)^{3/2}} dz \\
&\leqslant \frac{2v_0(1+\varepsilon)}{\mathcal{A}} \left[\frac{1}{v_0} - \left(v_0^2 + \mathcal{A}\log(1+\varepsilon)\right)^{-\frac{1}{2}}\right] \to \frac{2(1+\varepsilon)}{\mathcal{A}}
\end{aligned}$$

as $v_0 \to 0$. Similarly,

$$v_0 \int_1^{1+\varepsilon} \frac{dz}{\left(v_0^2 + \mathcal{A}\log z\right)^{3/2}} \geqslant \frac{2v_0}{\mathcal{A}} \int_1^{1+\varepsilon} \frac{\mathcal{A}}{2z(v_0^2 + \mathcal{A}\log z)^{3/2}} dz \to \frac{2}{\mathcal{A}}$$

as $v_0 \to 0$. It proves (3.3.12) since $\varepsilon > 0$ is arbitrary. Taking the limit $v_0 \downarrow 0$ in (3.3.11),

$$\begin{aligned}
0 = &\frac{\partial_R X}{\mathcal{A}^{1/2}\sqrt{\log(X/R)}} - \frac{X/R}{\mathcal{A}^{1/2}\sqrt{\log(X/R)}} + \frac{1}{\mathcal{A}^{1/2}} \int_1^{X/R} \frac{dz}{\sqrt{\log z}} \\
&- \frac{R\partial_r \mathcal{A}}{2\mathcal{A}^{3/2}} \int_1^{X/R} \frac{dz}{\sqrt{\log z}} - \frac{2Rv_0'}{\mathcal{A}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
B(t) := \frac{\partial_R X(t)}{\sqrt{\log(X(t)/R)}} = &\frac{X(t)/R}{\sqrt{\log(X(t)/R)}} - \int_1^{X(t)/R} \frac{dz}{\sqrt{\log z}} \\
&+ \frac{R\partial_r \mathcal{A}}{2\mathcal{A}} \int_1^{X(t)/R} \frac{dz}{\sqrt{\log z}} + \frac{2Rv_0'}{\mathcal{A}^{1/2}}
\end{aligned}$$

Case 2-a.

We first assume that $\partial_r \mathcal{A} = 0$. Put

$$\mathcal{G}(s) := \frac{s}{\sqrt{\log s}} - \int_1^s \frac{dz}{\sqrt{\log z}}$$

An elementary calculation shows $\mathcal{G}'(s) = -(1/2)(\log s)^{-3/2} < 0$ for $s > 1$, and so $\mathcal{G}$ is monotone decreasing. We also see that $\mathcal{G}'$ is not integrable, and so that $\lim_{s\to\infty} G(s) = -\infty$. Since

$$B(t) = \mathcal{G}(X(t)/R) + \frac{2Rv_0'}{\mathcal{A}^{1/2}},$$

we conclude that there always exists $t_0 \in (0, \infty)$ such that $\Gamma(t_0) = 0$.

Case 2-b.

We next assume that $\partial_r \mathcal{A} > 0$. Write $B(t) =: \mathcal{H}(X(t)/R)$. Then, it holds that

$$\frac{d}{ds}\mathcal{H}(s) = -\frac{1}{2(\log s)^{3/2}} + \frac{R\partial_r \mathcal{A}}{2\mathcal{A}(\log s)^{1/2}}.$$

Therefore, the minimum of $\mathcal{H}$, hence of B, is $\mathcal{H}(e^{\frac{\mathcal{A}}{R\partial_r \mathcal{A}}})$. The solution breaks down in finite time if and only if this value is less than or equal to zero. This leads to the condition

$$v_0' R \leqslant -\frac{\sqrt{R\partial_r \mathcal{A}}}{2} e^{\frac{\mathcal{A}}{R\partial_r \mathcal{A}}} + \left(\sqrt{\mathcal{A}} - \frac{R\partial_r \mathcal{A}}{2\sqrt{\mathcal{A}}}\right) \int_0^{\sqrt{\frac{\mathcal{A}}{R\partial_r \mathcal{A}}}} e^{x^2} dx.$$

Case 3: $v_0 < 0$.

If $A(R) = 0$, then $X'(t, R) = v_0(R) < 0$ for all $t \geqslant 0$. Therefore, we deduce from Lemma 3.2.7 that the solution breaks down no latter than $t = R/|v_0(R)|$. Hence, we assume $\mathcal{A}(R) > 0$. Then, since $X'(0, R) = v_0(R) < 0$, $X'(t) = -\sqrt{v_0(R)^2 + \mathcal{A}(R)\log(X(t, R)/R)}$ as long as $X'(t, R) \leqslant 0$. Put

$$t_* = \int_{Re^{-v_0^2/\mathcal{A}}}^{R} \frac{dy}{\sqrt{v_0^2 + \mathcal{A}\log(y/R)}} = \frac{R}{\mathcal{A}^{1/2} e^{v_0^2/\mathcal{A}}} \int_1^{e^{v_0^2/\mathcal{A}}} \frac{dz}{\sqrt{\log z}}.$$

Then, one sees that, for $t \in [0, t_*)$, $X(t, R) > X(t_*, R) = Re^{-v_0^2/\mathcal{A}} > 0$ and $X'(t, R) < X'(t_*, R) = 0$. Since $X''(t_*, R) > 0$, the same argument as in the Case 3 of the proof of Theorem 3.3.7 shows that $X'(t, R) \geqslant 0$ for all $t \geqslant t_*$ and so that

$$X'(t, R) = \begin{cases} -\sqrt{v_0(R)^2 + \mathcal{A}(R)\log(X(t, R)/R)}, & \text{for } t \leqslant t_*, \\ \sqrt{v_0(R)^2 + \mathcal{A}(R)\log(X(t, R)/R)}, & \text{for } t \geqslant t_*. \end{cases}$$

$X(t, R) \to \infty$ as $t \to \infty$ is also deduced. We omit R variable in the followings. Differentiation of the identity $X(t_*, R) = Re^{-v_0^2/\mathcal{A}}$ with respect to R gives

$$\partial_R X(t_*) = e^{-v_0^2/\mathcal{A}} \left(1 - R\partial_r \left(\frac{v_0^2}{\mathcal{A}}\right)\right).$$

Hence, if $R\partial_r(v_0^2/\mathcal{A}) \geqslant 1$ then the solution breaks down no latter than $t = t_*$. Thus, we assume $R\partial_r(v_0^2/\mathcal{A}) < 1$ in the followings. This is equivalent to $\partial_r v_0^2 < \mathcal{A}/R + (v_0^2/\mathcal{A})\partial_r \mathcal{A}$ and to $-\mathcal{C}/\mathcal{A} < -\partial_r \mathcal{C}/\partial_r \mathcal{A}$.

Step 1. We first consider the condition that solution can be extended to time $t = t_*$. For $t \leqslant t_*$, we have

$$\int_{X(t)}^{R} \frac{dy}{\sqrt{\mathcal{C} + \mathcal{A}\log y}} = t.$$

Differentiation with respect to R yields

$$\frac{1}{\sqrt{\mathcal{C}+\mathcal{A}\log R}}-\frac{\Gamma(t)}{\sqrt{\mathcal{C}+\mathcal{A}\log X(t)}}-\frac{1}{2}\int_{X(t)}^{R}\frac{\partial_r\mathcal{C}+\partial_r\mathcal{A}\log y}{(\mathcal{C}+\mathcal{A}\log y)^{3/2}}dy=0.$$

For $0\leqslant t<t_*$,

$$0<\sqrt{\mathcal{C}+\mathcal{A}\log X(t)}\leqslant\sqrt{\mathcal{C}+\mathcal{A}\log R}=|v_0|$$

holds. Therefore, $\Gamma(t)$ has the same sign as

$$B_1(t):=\frac{\Gamma(t)}{\sqrt{\mathcal{C}+\mathcal{A}\log X(t)}}=\frac{1}{|v_0|}-\frac{1}{2}\int_{X(t)}^{R}\frac{\partial_r\mathcal{C}+\partial_r\mathcal{A}\log y}{(\mathcal{C}+\mathcal{A}\log y)^{3/2}}dy.$$

Taking time derivative, one verifies that B_1 takes it minimum at $t=t_1\in[0,t_*)$ such that

$$X(t_1)=\min\left(R,\exp\left(-\frac{\partial_r\mathcal{C}}{\partial_r\mathcal{A}}\right)\right).$$

Here, note that $X(t_*)=\exp(-\mathcal{C}/\mathcal{A})<X(t_1)$ by assumption. Also note that

$$\exp\left(-\frac{\partial_r\mathcal{C}}{\partial_r\mathcal{A}}\right)=R\exp\left(-\frac{\partial_r v_0^2-\mathcal{A}/R}{\partial_r\mathcal{A}}\right).$$

Since we have already known that $\Gamma(0)=1>0$, the solution can be extended to the time $t=t_*$ UNLESS $\partial_r v_0^2>\mathcal{A}/R$ and

$$B_1(t_1)=\frac{1}{|v_0|}-\frac{1}{2}\int_{\exp\left(-\frac{\partial_r\mathcal{C}}{\partial_r\mathcal{A}}\right)}^{R}\frac{\partial_r\mathcal{C}+\partial_r\mathcal{A}\log y}{(\mathcal{C}+\mathcal{A}\log y)^{3/2}}dy\leqslant 0$$

is satisfied. Notice that this condition is a sufficient condition for finite-time breakdown.

Step 2. We next consider the condition that the solution can be extended from the time $t=t_*$ to $t=\infty$. For simplicity, we suppose that solutions are extended to time $t=t_*$ (we keep assuming $\partial_r v_0^2<\mathcal{A}/R+(v_0^2/\mathcal{A})\partial_r\mathcal{A}$ holds). Recall that, for $t>t_*$, $X'(t)=\sqrt{\mathcal{C}+\mathcal{A}\log X(t)}>0$.

We define t_{**} as a time such that $t_{**}>t_*$ and $X(T_{**})=R$. Then, we have

$$t_{**}-t_*=\int_{Re^{-v_0^2/\mathcal{A}}}^{R}\frac{dy}{\sqrt{\mathcal{C}+\mathcal{A}\log y}}=t_*,$$

and so $t_{**}=2t_*$. Thus,

$$\int_{R}^{X(t)}\frac{dy}{\sqrt{\mathcal{C}+\mathcal{A}\log y}}=t-2t_*$$

for all $t \geqslant t_*$. As in the previous step, we set

$$B_2(t) := \frac{\Gamma(t)}{\sqrt{\mathcal{C} + \mathcal{A}\log X(t)}} = \frac{1}{|v_0|} + \frac{1}{2}\int_R^{X(t)} \frac{\partial_r \mathcal{C} + \partial_r \mathcal{A}\log y}{(\mathcal{C} + \mathcal{A}\log y)^{3/2}} dy - 2\partial_r t_*$$
$$= \frac{1}{|v_0|} + \frac{1}{2}\int_R^{X(t)} \frac{\partial_r v_0^2 - (\mathcal{A}/R) + \partial_r \mathcal{A}\log(y/R)}{(\mathcal{C} + \mathcal{A}\log y)^{3/2}} dy - 2\partial_r t_*$$

$B_2(t)$ and $\Gamma(t)$ has the same sign for $t > t_*$. We also note that $B_2 \to \infty$ as $t \downarrow t_*$ because $\Gamma(t_*) > 0$ and $\sqrt{\mathcal{C} + \mathcal{A}\log X} \to 0$ as $t \downarrow t_*$. It holds that

$$\frac{d}{dt} B_2(t) = \frac{\partial_r v_0^2 - (\mathcal{A}/R) + \partial_r \mathcal{A}\log(X(t)/R)}{(\mathcal{C} + \mathcal{A}\log X(t))^{3/2}} X'(t).$$

If $\partial_r \mathcal{A}(R) = 0$ then B_2 is monotone decreasing by assumption $\partial_r v_0^2 - \mathcal{A}/R < 0$. Moreover, $\frac{d}{dt}B_2(t)$ is uniformly bounded by $(\partial_r v_0^2 - (\mathcal{A}/R))/|v_0| < 0$ from above, and so there exists time t_2 such that $B_2(t_2) = 0$. Therefore, now we suppose $\partial_r \mathcal{A}(R) > 0$.

B_2 takes it minimum at $t = t_2$ such that $X(t_2) = \exp(-\partial_r \mathcal{C}/\partial_r \mathcal{A})$. Therefore, the solution can be extended to $t = \infty$ if and only if

$$B_2(t_2) = \frac{1}{|v_0|} + \frac{1}{2}\int_R^{\exp\left(-\frac{\partial_r \mathcal{C}}{\partial_r \mathcal{A}}\right)} \frac{\partial_r \mathcal{C} + \partial_r \mathcal{A}\log y}{(\mathcal{C} + \mathcal{A}\log y)^{3/2}} dy - 2\partial_r t_* > 0.$$

□

3.3.5 Applications

Example 3.3.13. In the following cases, (rEP$_0$) has a unique global solution, and the solution solves (EP$_0$) in the distribution sense.

1. $n = 1$, $\lambda > 0$, and

$$\rho_0(r) = e^{-r}, \quad v_0(r) = \sqrt{\frac{\lambda}{e^r + e^{1/r}}} \sin r.$$

2. $n = 2$, $\lambda > 0$, and

$$\rho_0(r) = \frac{1}{1 + r^2}, \quad v_0(r) = \sqrt{\lambda r}.$$

Theorem 3.3.14. *Let $\lambda < 0$ or $n \geqslant 3$. Suppose $\rho_0 \in D^0 \cap L^1((0,\infty), r^{n-1}dr)$ is not identically zero and $v_0 \in D^1$ satisfies $v_0(0) = 0$ and $v_0 \to 0$ as $r \to \infty$. Then, the solution of* (rEP$_0$) *is global if and only if $\lambda < 0$ and $n \geqslant 3$, and the initial data is of particular form*

$$v_0(r) = \sqrt{\frac{2|\lambda|}{(n-2)r^{n-2}} \int_0^r \rho_0(s) s^{n-1} ds}.$$

Suppose $\lambda < 0$ and $n \geqslant 3$. If $\rho_0 \in D^k \cap L^1((0,\infty), r^{n-1}dr)$ for $k \geqslant 0$ and if v_0 is as above, then $v_0 \in D^{k+1}$ and the corresponding solution is

$$\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).$$

given explicitly by

$$\rho(t, X(t,R)) = \rho_0(R)\left(1 + \frac{nv_0(R)}{2R}t\right)^{-1}\left(1 + \frac{2|\lambda|R\rho_0(R)}{(n-2)v_0(R)}t\right)^{-1},$$
$$v(t, X(t,R)) = v_0(R)\left(1 + \frac{nv_0(R)}{2R}t\right)^{1-\frac{2}{n}},$$

where $X(t,R) = R(1+\frac{nv_0(R)}{2R}t)^{2/n}$. Furthermore, it is unique in $C^2([0,\infty), D^0)\times C^1([0,\infty), D^1)$ and also solves (EP_0) *in the distribution sense.*

Proof. In the case where $n = 1, 2$ and $\lambda < 0$, we deduce from Theorem 3.3.1 that the solution breaks down in finite time because ρ_0 is nontrivial. Let $n \geqslant 3$, then the assumptions $\rho_0 \in L^1((0,\infty), r^{n-1}dr)$ and $v_0 \to 0$ as $r \to \infty$ lead to $\mathcal{C}(R) \to 0$ as $R \to \infty$, where

$$\mathcal{C}(R) = v_0(R)^2 - \frac{2\lambda m_0(R)}{(n-2)R^{n-2}}.$$

Since $\mathcal{C}(0) = 0$ by assumption, we see from Theorems 3.3.1 and 3.3.7 that the solution is global only if $\mathcal{C} \equiv 0$. In the case $\lambda > 0$, $C \equiv 0$ implies $\rho_0 \equiv 0$, which is excluded by assumption. In the case $\lambda < 0$, the solution is global if we take the positive root $v_0(R) = \sqrt{\frac{2|\lambda|}{(n-2)R^{n-2}}\int_0^R \rho_0(s)s^{n-1}ds}$. In this case, $C \equiv 0$ and so X satisfies the equation

$$X'(t,R) = \sqrt{\frac{2|\lambda|m_0(R)}{(n-2)X(t,R)^{n-2}}}, \quad X(0,R) = R.$$

By separation of variables, we obtain

$$X(t,R) = \left(R^{\frac{n}{2}} + \frac{n}{2}\sqrt{\frac{2|\lambda|m_0(R)}{n-2}}t\right)^{\frac{2}{n}} = R\left(1 + \frac{nv_0(R)}{2R}\right)^{\frac{2}{n}}.$$

Then, Lemma 3.2.2 gives the solution to (rEP_0). $\square$

Remark 3.3.15. In this theory, the case $\lambda > 0$, $n = 1$ and the case $\lambda > 0$, $n = 2$ are excluded. If $\lambda > 0$ and $n = 2$ then it is not clear whether or not the assumption of Theory 3.3.14 leads to nonexistence of global solution, but following another non-existence result holds. On the other hand, the case where $\lambda > 0$ and $n = 1$ must be excluded since the first example in Example 3.3.13 is a counter example. This example also suggests that the following different version also fails if $n = 1$.

Theorem 3.3.16. *Let $\lambda > 0$ and $n \geqslant 2$. Suppose $\rho_0 \in D^0$ is not identically zero and $v_0 \in D^1$ satisfies $v_0(0) = 0$. Suppose, in addition, that there exists a sequence $\{r_j\}_{j\geqslant 1}$ with $r_j \to \infty$ as $j \to \infty$ such that $v_0(r_j) = 0$ for all $j \geqslant 1$, $\limsup_{j\to\infty} r_j v_0'(r_j) < \infty$, and $r_j^n \rho_0(r_j) \to 0$ as $j \to \infty$. Then, the solution of* (rEP$_0$) *breaks down in finite time.*

Proof. In the $n \geqslant 3$ case, $v_0(r_j) = 0$ leads to

$$\partial_r C(r_j) = \frac{2\lambda(r_j^n \rho_0(r_j) - (n-2)\int_0^{r_j} \rho_0(s)s^{n-1}ds)}{(n-2)r_j^{n-1}}$$

Since ρ_0 is nontrivial, $\int_0^{r_j} \rho_0(s)s^{n-1}ds > 0$ for large j. Moreover, $r_j^n \rho_0(r_j) \to 0$ as $j \to \infty$ by assumption. Hence, we conclude that $\partial_r C(r_j) < 0$ for large j, which is a sufficient condition for finite-time breakdown.

Let us proceed to the two dimensional case. We now show that, if j is sufficiently large, then the PCFB (given in Definition 3.3.10) is satisfied at $R = r_j$ and so the solution breaks down in finite time. Since ρ_0 is nontrivial, we can suppose $\mathcal{A}(r_j) = 2\lambda \int_0^{r_j} \rho_0(s)sds > 0$. The case $\rho_0(r_j) = 0$ is trivial and so we now suppose $\partial_r \mathcal{A}(r_j) > 0$. It suffices to prove that the inequality

$$\begin{aligned} r_j v_0'(r_j) \leqslant & -\frac{\sqrt{\mathcal{A}(r_j) r_j \partial_r \mathcal{A}(r_j)}}{2} e^{\frac{\mathcal{A}(r_j)}{r_j \partial_r \mathcal{A}(r_j)}} \\ & + \frac{2\mathcal{A}(r_j) - r_j \partial_r \mathcal{A}(r_j)}{4} \int_1^{e^{\frac{\mathcal{A}(r_j)}{r_j \partial_r \mathcal{A}(r_j)}}} \frac{dz}{\sqrt{\log z}}. \end{aligned} \tag{3.3.13}$$

is true for some j. Since the left hand side is upper bounded for large j, by assumption, it suffices to show that the right hand side is arbitrarily large for large j. Notice that the right hand side of (3.3.13) can be written as $(\mathcal{A}(r_j)/2) f(\mathcal{A}(r_j)/r_j \partial_r \mathcal{A}(r_j))$, where

$$f(x) = -\frac{1}{\sqrt{x}} e^x + \int_1^{e^x} \frac{dz}{\sqrt{\log z}} - \frac{1}{2x} \int_1^{e^x} \frac{dz}{\sqrt{\log z}}.$$

Since $f(1/2) = -\sqrt{2e}$ and $f'(x) = (2x^2)^{-1} \int_1^{e^x} (\log z)^{-1/2} dz \geqslant (2x^{5/2})^{-1}(e^x - 1)$, we see that $f(x) \to \infty$ as $x \to \infty$. By assumption, $A(r_j)/r_j \partial_r A(r_j) = \int_0^{r_j} \rho_0(s)sds / r_j \rho_0(r_j) \to \infty$ as $j \to \infty$. Thus, the right hand side of (3.3.13) goes to infinity as $j \to \infty$. □

Corollary 3.3.17. *Suppose $n \geqslant 1$, $\rho_0 \equiv \rho_c > 0$ is a constant, and $v_0 \equiv 0$. Then, the solution of* (rEP$_0$) *is global if and only if $\lambda > 0$. If $\lambda > 0$ then the corresponding solution satisfies*

$$\rho \in C^2([0,\infty), D^\infty) \cap C^\infty((0,\infty), D^\infty),$$
$$v \in C^1([0,\infty), D^\infty) \cap C^\infty((0,\infty), D^\infty).$$

Furthermore, it is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_0$) *in the distribution sense.*

Proof. We first consider negative λ case. Since ρ_0 is not zero, solution breaks down if $n = 1, 2$. In the case $n \geqslant 3$, we have $C(R) < 0$ for all $R > 0$, which immediately leads to finite time breakdown.

Let us show that the solution is global if $\lambda > 0$. The one-dimensional case is obvious from Theorem 3.3.2. In the two-dimensional case, we apply Theorem 3.3.12. The PCFB is given by Definition 3.3.10 for all $R > 0$ because $v_0 \equiv 0$. Notice that $\mathcal{A}(R) = \lambda\rho_c R^2 > 0$ and $\partial_r\mathcal{A}(R) = 2\lambda\rho_c R > 0$ for all $R > 0$. Therefore, in the end, we see that the solution breaks down if and only if there exists $R_0 > 0$ such that

$$R_0 v_0'(R_0) \leqslant -\frac{\sqrt{\mathcal{A}(R_0)R_0\partial_r\mathcal{A}(R_0)}}{2}e^{\frac{\mathcal{A}(R_0)}{R_0\partial_r\mathcal{A}(R_0)}} + \frac{2\mathcal{A}(R_0) - R_0\partial_r\mathcal{A}(R_0)}{4}\int_1^{e^{\frac{\mathcal{A}(R_0)}{R_0\partial_r\mathcal{A}(R_0)}}}\frac{dz}{\sqrt{\log z}}.$$

However, the left hand side is zero, and the second term of the right hand side is also zero by the relation $2\mathcal{A}(R) - R\partial_r\mathcal{A}(R) \equiv 0$. Since the first term in the right side is negative, such R_0 does not exist and so the solution to (rEP$_0$) is global.

We proceed to the case $n \geqslant 3$. The proof is the same as in two-dimensional case. Notice that $\partial_r C(R) = 4\lambda\rho_c R/n(n-2) > 0$ and so that the PCFB is given in Definition 3.3.5. In the case $n = 4$, it is obvious that there does not exist R_0 such that $v_0'(R_0)R_0 \leqslant -\sqrt{2R_0\partial_r C(R_0)}$. In the cases $n = 3$ and $n \geqslant 5$, by using the fact that $C(R) = 2\lambda\rho_c R^2/n(n-2)$ and so $R\partial_r C/2C \equiv 1$, we verify nonexistence of R_0 for which the PCFB holds. □

3.4 Global existence of classical solutions to radial Euler-Poisson equations 2: existence of constant background

In this section, we consider the effect from the presence of background constant. Our equation is the following:

$$\begin{cases} r^{n-1}\rho_t + \partial_r(r^{n-1}\rho v) = 0, \\ v_t + v\partial_r v + \lambda\partial_r V_{\mathrm{P}} = 0, \\ -\partial_r(r^{n-1}\partial_r V_{\mathrm{P}}) = r^{n-1}(\rho - b), \\ (\rho, v)(0, r) = (\rho_0, v_0)(r), \quad \rho_0 \geqslant 0 \end{cases} \tag{rEP$_b$}$$

with $b > 0$ for $(t, r) \in \mathbb{R}_+ \times \mathbb{R}_+$, which is a radial model of

$$\begin{cases} \rho_t + \mathrm{div}(\rho v) = 0, \\ v_t + v \cdot \nabla v + \lambda\nabla V_{\mathrm{P}} = 0, \\ -\Delta V_{\mathrm{P}} = \rho - b, \\ (\rho, v)(0, x) = (\rho_0, v_0)(x), \quad \rho_0 \geqslant 0. \end{cases} \tag{EP$_b$}$$

We analyze the classical trajectory which solves

$$X''(t,R) = G(X(t,R)), \quad X'(0,R) = v_0(R), \quad X(0,R) = R, \tag{3.4.1}$$

where

$$G(x) := \frac{\lambda m_0}{x^{n-1}} - \frac{\lambda b}{n} x.$$

If $n \geqslant 2$ then the function $G(x)$ is monotone increasing for $\lambda < 0$ and monotone decreasing for $\lambda > 0$. In particular, if $b > 0$ then $G(x)$ has only one zero point

$$x = X_D(R) := \left(\frac{n m_0(R)}{b} \right)^{1/n}.$$

Multiply the both side of (3.4.1) by X' to obtain

$$(X'(t,R))^2 = D(R) - F(X(t,R)), \quad D(R) := v_0(R)^2 + F(R) \tag{3.4.2}$$

if $n \geqslant 3$, where

$$F(x) = F_R(x) := \frac{A(R)}{x^{n-2}} + Bx^2, \quad A(R) := \frac{2\lambda m_0(R)}{n-2}, \quad B := \frac{\lambda b}{n}. \tag{3.4.3}$$

Similarly, if $n = 2$ then (3.4.2) holds with

$$F(x) = F_R(x) := -\mathcal{A}(R) \log x + Bx^2, \quad \mathcal{A}(R) := 2\lambda m_0(R). \tag{3.4.4}$$

The function F defined (3.4.3) or (3.4.4) satisfies $F' = -2G$. In particular, at $x = X_D$, F takes its minimum if $\lambda > 0$. and takes its maximum if $\lambda < 0$. We denote them by $F_{\min}$ and $F_{\max}$, respectively.

3.4.1 Attractive case 1: $n = 1$.

We first consider the case $\lambda < 0$.

Theorem 3.4.1. *Suppose $n = 1$, $\lambda < 0$, $b > 0$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. Then, the classical solution of* (rEP$_b$) *is global if and only if*

$$v_0(R) \geqslant \sqrt{|\lambda|} \left(\frac{m_0(R)}{\sqrt{b}} - \sqrt{b} R \right), \qquad v_0'(R) \geqslant \sqrt{|\lambda|} \left(\frac{\rho_0(R)}{\sqrt{b}} - \sqrt{b} \right)$$

holds for all $R > 0$. Let k be a nonnegative integer. If $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ satisfy the condition for global existence then the corresponding solution of (rEP$_b$) *satisfies*

$$\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).$$

The solution is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_b$) *in the distribution sense.*

Proof. Since (3.4.1) is solved explicitly:

$$X(t,R) = \frac{m_0(R)}{b} + \left(R - \frac{m_0(R)}{b}\right)\cosh\sqrt{|\lambda|b}t + \frac{v_0(R)}{\sqrt{\lambda b}}\sinh\sqrt{|\lambda|b}t.$$

We have

$$\Gamma(t,R) = \frac{\rho_0(R)}{b} + \left(1 - \frac{\rho_0(R)}{b}\right)\cosh\sqrt{|\lambda|b}t + \frac{v_0'(R)}{\sqrt{\lambda b}}\sinh\sqrt{|\lambda|b}t.$$

Fix $R > 0$. Since $X(t,R)$ and $\Gamma(t,R)$ are positive at $t = 0$, they stay positive for all $t \geqslant 0$ if and only if

$$\left(R - \frac{m_0(R)}{b}\right) + \frac{v_0(R)}{\sqrt{|\lambda|b}} \geqslant 0, \qquad \left(1 - \frac{\rho_0(R)}{b}\right) + \frac{v_0'(R)}{\sqrt{|\lambda|b}} \geqslant 0,$$

respectively. Similarly, $\liminf_{R\to 0}\Gamma(t,R) > 0$ for all $t \geqslant 0$ if and only if $\liminf_{R\to 0} v_0'(R) \geqslant -\sqrt{|\lambda|b}$. Hence, the theorem. □

3.4.2 Attractive case 2: $n \geqslant 3$.

Theorem 3.4.2. *Suppose $n \geqslant 3$, $\lambda < 0$, $b > 0$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. The classical solution to* (rEP$_b$) *is global in time if and only if the one of the following four conditions is satisfied for each $R > 0$:*

1. $m_0(R) = 0$, $v_0(R) \geqslant -R|B|$, *and* $v_0'(R) \geqslant -|B|$;
2. $m_0(R) > 0$, $v_0(R)^2 > F_{\max} - F(R)$, $v_0(R) > 0$, *and*
$$\partial_r D(R) \geqslant -\frac{2v_0(R)^{-1} - \partial_r A(R)\int_R^\infty y^{-(n-2)}(D(R) - F(y))^{-3/2}dy}{\int_R^\infty (D(R) - F(y))^{-3/2}dy};$$
3. $m_0(R) > 0$, $v_0(R)^2 = F_{\max} - F(R)$, *and either*
   (a) $R < X_D(R)$, $v_0(R) > 0$, *and* $\partial_r D(R) \geqslant -\partial_r A(R) X_D(R)^{-(n-2)}$,
   (b) $R = X_D(R)$,
   (c) $R > X_D(R)$ *and either*
      i. $v_0(R) < 0$ *and* $\partial_r D(R) \leqslant -\partial_r A(R) X_D(R)^{-(n-2)}$,
      ii. $v_0(R) > 0$ *and*
$$\partial_r D(R) < -\frac{2v_0(R)^{-1} - \partial_r A(R)\int_R^\infty y^{-(n-2)}(D(R) - F(y))^{-3/2}dy}{\int_R^\infty (D(R) - F(y))^{-3/2}dy};$$
4. $m_0(R) > 0$, $v_0(R)^2 < F_{\max} - F(R)$, $R > X_D(R)$, *and either*

(a) $v_0(R) > 0$ *and*

$$\partial_r D(R) < -\frac{2v_0(R)^{-1} - \partial_r A(R) \int_R^\infty y^{-(n-2)}(D(R) - F(y))^{-3/2} dy}{\int_R^\infty (D(R) - F(y))^{-3/2} dy},$$

(b) $v_0(R) = 0$ *and*

$$v_0' \geqslant -\frac{G(R)}{R\sqrt{|B|}} - G(R)\left(\frac{R\partial_r A(R) - nA(R)}{2R}\right)\int_R^\infty \frac{R^{-(n-2)} - y^{-(n-2)}}{(F(R) - F(y))^{3/2}} dy,$$

(c) $v_0(R) < 0$, $\partial_r D(R) - \partial_r A(R)\xi_2^{-(n-2)} \leqslant 0$, *and*

$$\frac{1}{|v_0|} + \frac{1}{2}\int_R^\infty \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(F(\xi_2) - F(y))^{3/2}} - 2\partial_r t_* \geqslant 0,$$

where ξ_2 *is the root of* $F(\xi) = F(R) + v_0(R)^2$ *bigger than* $X_D(R)$ *and*

$$t_* := \int_R^{\xi_2} \frac{dy}{\sqrt{F(\xi_2) - F(y)}}.$$

Let k *be a nonnegative integer. If* $\rho_0 \in D^k$ *and* $v_0 \in D^{k+1}$ *satisfy the condition for global existence then the corresponding solution of* (rEP$_b$) *satisfies*

$$\begin{aligned}
&\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k),\\
&v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).
\end{aligned}$$

The solution is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ *and also solves* (EP$_b$) *in the distribution sense.*

Proof. We first consider the special case $m_0(R) = 0$. In this case, $A(R) = 0$ and so (3.4.1) becomes $X''(t,R) = BX(t,R)$. Therefore, we obtain

$$\begin{aligned}
X(t,R) &= \frac{R + v_0(R)/|B|}{2} e^{|B|t} + \frac{R - v_0(R)/|B|}{2} e^{-|B|t},\\
\Gamma(t,R) &= \frac{1 + v_0'(R)/|B|}{2} e^{|B|t} + \frac{1 - v_0'(R)/|B|}{2} e^{-|B|t}.
\end{aligned}$$

Now, the situation is the same as in the one dimensional case, and so one easily verifies that they stay positive for all positive time if and only if $R + v_0(R)/|B| \geqslant 0$ and $1 + v_0'(R)/|B| \geqslant 0$ hold.

Case 1: $v_0(R)^2 > F_{\max} - F(R)$.

We first note that $v_0(R)^2 > 0$ and $(X'(t,R))^2 = D(R) - F(X(t,R)) \geqslant D(R) - F_{\max} = v_0(R)^2 + F(R) - F_{\max} > 0$ hold. In particular, by continuity, $X'(t,R)$ does not change its sign. If $v_0(R) < 0$ then we have $X'(t,R) \leqslant$

$-\sqrt{D(R)-F_{\max}}<0$, which leads to $X \leqslant 0$ for large t and so to the finite-time breakdown. If $v_0(R)>0$ then $X'(t,R) \geqslant \sqrt{D(R)-F_{\max}}>0$ holds and so $X(t,R) \to \infty$ as $t \to \infty$. By a differentiation of the equality

$$\int_R^X (D(R)-F(y))^{-1/2} dy = t$$

with respect to R, we obtain

$$\frac{\Gamma(t,R)}{X'(t,R)} - \frac{1}{v_0(R)} - \frac{1}{2}\int_R^{X(t,R)} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R)-F(y))^{3/2}} dy = 0.$$

We put

$$\mathcal{B}(t) := \frac{\Gamma(t,R)}{X'(t,R)} = \frac{1}{v_0(R)} + \frac{1}{2}\int_R^{X(t,R)} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R)-F(y))^{3/2}} dy.$$

Since $X'>0$, $\mathcal{B}$ and Γ have the same sign. If $\partial_r D(R) \geqslant 0$ then $\mathcal{B}(t)$ stays positive for all time because $\partial_r A \leqslant 0$ by definition. Now, we suppose $\partial_r D < 0$. An elementary calculation shows that there exists t_0 such that $\mathcal{B}$ is monotone increasing for $t<t_0$ and monotone decreasing for $t>t_0$. Since $\mathcal{B}(0)>0$, $\mathcal{B}$ stays positive for all time if and only if

$$\lim_{t\to\infty} \mathcal{B}(t) = \frac{1}{v_0(R)} + \frac{1}{2}\int_R^{\infty} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R)-F(y))^{3/2}} dy \geqslant 0, \qquad (3.4.5)$$

which implies the stated condition.

Case 2: $v_0(R)^2 = F_{\max} - F(R)$**.**

In this case, $(X'(t,R))^2 = D(R)-F(X(t,R)) = v_0(R)^2 + F(R) - F(X(t,R)) = F_{\max} - F(X(t,R))$. Note that the right hand side is nonnegative and $O((X-X_D)^2)$ as $X \to X_D$.

We first consider the case $R < X_D(R)$. If $v_0(R)<0$ then, the finite-time breakdown is straightforward as in the previous case, and so we omit the detail. Let us assume $v_0(R)>0$. An integration gives, for $R<X_D(R)$,

$$\int_R^{X(t,R)} \frac{dy}{\sqrt{F_{\max}-F(y)}} = t.$$

The left hand side tends to infinity as $X(t,R) \uparrow X_D(R)$ because the integrand is order $O((X_D-y)^{-1})$. Hence, we see that $X(t,R) \to X_D(R)$ as $t \to \infty$. Differentiate the above identity to obtain

$$\frac{\Gamma(t,R)}{X'(t,R)} - \frac{1}{v_0(R)} - \frac{1}{2}\int_R^{X(t,R)} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(F_{\max}-F(y))^{3/2}} dy = 0.$$

We put

$$\mathcal{B}(t) := \frac{\Gamma(t,R)}{X'(t,R)} = \frac{1}{v_0(R)} + \frac{1}{2}\int_R^{X(t,R)} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(F_{\max} - F(y))^{3/2}} dy.$$

Since $X' > 0$, $\mathcal{B}$ and Γ have the same sign. Here, $\partial_r D(R) - \partial_r A(R) y^{-(n-2)}$ is nonincreasing (recall $\partial_r A \leqslant 0$), and so $\mathcal{B}$ stays positive for all time if

$$\partial_r D(R) - \frac{\partial_r A(R)}{X_D(R)^{n-2}} \geqslant 0.$$

On the other hand, if this condition fails then $\mathcal{B}(t) \to -\infty$ as $t \to \infty$.

We next consider $R = X_D(R)$. In this case, $v_0(R)^2 = F_{\max} - F(X_D(R)) = 0$ and so $X'(t,R) = F_{\max} - F(X(t,R))$. Since holds $X' = X'' = 0$ at $X = X_D$, we have $X(t,R) \equiv R$.

Let us proceed to the third case $R > X_D(R)$. Note that $v_0(R)^2 > 0$ and so $v_0(R) \neq 0$. Let us first suppose $v_0(R) > 0$. By the differentiation of

$$\int_R^{X(t,R)} (D(R) - F(y))^{-1/2} dy = t$$

with respect to R, we obtain

$$\frac{\Gamma(t,R)}{X'(t,R)} - \frac{1}{v_0(R)} - \frac{1}{2}\int_R^{X(t,R)} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(F_{\max} - F(y))^{3/2}} dy = 0.$$

Recall that $D(R) = F_{\max}$ holds in this case. An analysis of the sign of $\mathcal{B}(t) := \Gamma(t,R)/X'(t,R)$ gives the condition as in the case $v_0(R)^2 > F_{\max} - F(R)$ and $v_0(R) > 0$, and so we left the detail. We consider $v_0(R) < 0$. As in the case $R < X_D(R)$ and $v_0(R) > 0$, one verifies $X(t,R) \downarrow X_D(R)$ as $t \to \infty$. The differentiation of

$$\int_{X(t,R)}^R (D(R) - F(y))^{-1/2} dy = t$$

with respect to R yields

$$\frac{\Gamma(t,R)}{|X'(t,R)|} = \frac{1}{|v_0(R)|} - \frac{1}{2}\int_X^R \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(F_{\max} - F(y))^{3/2}} dy.$$

We put $G(t) := \Gamma(t,R)/|X'(t,R)|$. Note that $\partial_r D - \partial_r A y^{-(n-2)}$ is nonincreasing in y and $X(t,R) \downarrow X_D(R)$ as $t \to \infty$. If $\partial_r D(R) + \partial_r A(R) X_D(R)^{-(n-2)} \leqslant 0$ stays positive for all time. On the other hand, if this value is positive then $\lim_{t\to\infty} B(t) = -\infty$ since $(F(y) - F_{\min})^{-3/2}$ is not integrable in $(X_D, R]$.

Case 3: $v_0(R)^2 < F_{\max} - F(R)$**.**

We first show that $R < X_D(R)$ leads to the finite-time breakdown. Since $D(R) - F(R) = v_0^2 \geqslant 0$ and $D - F(X_D) < 0$ by assumption, there exists $\xi_1 \in [R, X_D)$ such that $D - F(\xi_1) = 0$. Then, $D - F(X) = (X')^2 \geqslant 0$ provides the upper bound of X; $X \leqslant \xi_1 < X_D$. Recall that G is monotone increasing and that X_D is the only zero point of G. Hence, we have $X''(t, R) = G(X(t, R)) \leqslant G(\xi_1) < 0$. Integration twice gives $X(t, R) \leqslant R + v_0(R)t + (G(\xi_1)/2)t^2$, which implies $X \leqslant 0$ for large t.

Now, let us suppose $R > X_D(R)$. In this case, one sees that $X(t, R) \to \infty$ as $t \to \infty$. Since $D(t, R) - F(R) = v_0^2 \geqslant 0$ and $D - F(X_D) < 0$ by assumption, there exists $\xi_2 \in (X_D, R]$ such that $D - F(\xi_2) = 0$.

Case 3-a: $v_0(R) > 0$**.**

If $v_0 > 0$ then a differentiation of

$$\int_R^X \frac{dy}{\sqrt{D - F(y)}} = t \tag{3.4.6}$$

with respect to R gives

$$\frac{\Gamma(t, R)}{X'(t, R)} = \frac{1}{v_0(R)} + \frac{1}{2} \int_R^{X(t,R)} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R) - F(y))^{3/2}}.$$

As in the previous cases, we obtain the stated condition.

Case 3-b: $v_0(R) = 0$**.**

We next consider the case $v_0(R) = 0$. We would like to perform the similar analysis with differentiation of (3.4.6) with respect to R, as above. The point is that we must first differentiate before letting $v_0(R) = 0$ in (3.4.6) in order to obtain correct identities. For this purpose, we temporary assume $v_0(R) > 0$. Then, we change the variable in (3.4.6);

$$t = \int_R^{X(t,R)} \frac{dy}{\sqrt{D(R) - F(y)}} = \int_1^{X(t,R)/R} \frac{R dz}{\sqrt{D(R) - F(Rz)}}.$$

We differentiate with respect R to obtain

$$
\begin{aligned}
0 =& \frac{R\partial_R(X/R)}{\sqrt{D-F(X)}} + \int_1^{X/R} \frac{dz}{\sqrt{D-F(Rz)}} \\
&- \frac{R}{2}\int_1^{X/R} \frac{\partial_r D - 2zG(Rz) - \partial_r A(Rz)^{-(n-2)}}{(D-F(Rz))^{3/2}} \\
=& \frac{\Gamma}{X'} - \frac{X}{RX'} + \int_R^X \frac{dy}{R\sqrt{v_0^2 - F(y) + F(R)}} \\
&- \frac{1}{2}\int_R^X \frac{\partial_r D - \partial_r A y^{-(n-2)} - 2(y/R)G(y)}{(v_0^2 - F(y) + F(R))^{3/2}} dy,
\end{aligned}
$$

where we omit all t and R variables for simplicity. Here, we shall take the limit $v_0 \downarrow 0$ in this identity. However, $F(R) - F(y) = O(y-R)$ as $y \to R$ and so an integral $\int_R^X (v_0^2 + F(y) - F(R))^{-3/2} dy$ diverges as $v_0 \downarrow 0$. Here, we employ the following limit identity;

$$
\lim_{v_0 \downarrow 0} v_0 \int_R^X (v_0^2 - F(y) + F(R))^{-3/2} dy = \frac{1}{G(R)}.
$$

An elementary calculation shows that

$$
\begin{aligned}
&\frac{1}{2}\int_R^X \frac{\partial_r D - \partial_r A y^{-(n-2)} + 2(y/R)G(y)}{(v_0^2 - F(y) + F(R))^{3/2}} \\
=& v_0' \left(v_0 \int_R^X (v_0^2 - F(y) + F(R))^{-3/2} dy \right) \\
&+ \left(\frac{\partial_r A}{2} - \frac{(n-2)A}{2R} \right) \int_R^X \frac{R^{-(n-2)} - y^{-(n-2)}}{(v_0^2 - F(y) + F(R))^{3/2}} dy \\
&- \frac{B}{R}\int_R^X \frac{y^2 - R^2}{(v_0^2 - F(y) + F(R))^{3/2}} dy \\
=& v_0' \left(v_0 \int_R^X (v_0^2 - F(y) + F(R))^{-3/2} dy \right) \\
&- \left(\frac{\partial_r A}{2A} - \frac{n-2}{2R} \right) v_0^2 \int_R^X (v_0^2 - F(y) + F(R))^{-3/2} dy \\
&+ \left(\frac{\partial_r A}{2A} - \frac{n-2}{2R} \right) \int_R^X \frac{dy}{\sqrt{v_0^2 - F(y) + F(R)}} \\
&+ \left(\frac{\partial_r A}{2A} - \frac{n}{2R} \right) B \int_R^X \frac{y^2 - R^2}{(v_0^2 - F(y) + F(R))^{3/2}} dy,
\end{aligned}
$$

where we have used the relation

$$
R^{-(n-2)} - y^{-(n-2)} = \frac{(v_0^2 - F(y) + F(R)) - v_0^2 + B(y^2 - R^2)}{A}.
$$

Combining these identities and taking the limit $v_0 \to 0$, we obtain

$$\frac{\Gamma}{X'} = \frac{X}{RX'} + \frac{v_0'}{G(R)} + \left(\frac{\partial_r A}{2A} - \frac{n}{2R}\right)\int_R^X \frac{dy}{\sqrt{F(R)-F(y)}} + \left(\frac{\partial_r A}{2A} - \frac{n}{2R}\right) B \int_R^X \frac{y^2-R^2}{(F(R)-F(y))^{3/2}}dy \quad (3.4.7)$$

for $t > 0$. This manipulation is justified by considering $X'(\varepsilon R, R) > 0$ and taking the limit $\varepsilon \downarrow 0$. Note that the last integral is finite thanks to the factor $y^2 - R^2$. We put $\mathcal{B}(t) := \Gamma(t,R)/X'(t,R)$. Since $X' > 0$ for positive time, $\mathcal{B}$ and Γ have the same sign for positive time. We have

$$\begin{aligned}
\mathcal{B}'(t) =& \frac{1}{R} - \frac{XX''}{R(X')^2} + \left(\frac{\partial_r A}{2A} - \frac{n}{2R}\right) + \left(\frac{\partial_r A}{2A} - \frac{n}{2R}\right)\frac{B(X^2-R^2)}{(X')^2} \\
=& \frac{F(R) - F(X) - XG(X)}{R(X')^2} - \left(\frac{R\partial_r A}{2A} - \frac{n}{2}\right)\frac{AX^{-(n-2)} - AR^{-(n-2)}}{R(X')^2} \\
=& \frac{1}{R(X')^2}\left[F(R) - \frac{R\partial_r A}{2X^{n-2}} + \left(\frac{R\partial_r A}{2A} - \frac{n}{2}\right)\frac{A}{R^{n-2}}\right] \\
=& \frac{1}{R(X')^2}\left[\frac{R\partial_r A}{2}\left(\frac{1}{R^{n-2}} - \frac{1}{X^{n-2}}\right) - RG(R)\right] < 0
\end{aligned}$$

for all $t > 0$ since $R > X_D$, $G(R) > G(X_D) = 0$, and $\partial_r A \leqslant 0$. Therefore, $\mathcal{B}$ is positive for all time if and only if $\lim_{t\to\infty} \mathcal{B}(t) \geqslant 0$, which is equivalent to

$$\frac{1}{R\sqrt{|B|}} + \frac{v_0'}{G(R)} + \left(\frac{\partial_r A}{2} - \frac{nA}{2R}\right)\int_R^\infty \frac{R^{-(n-2)} - y^{-(n-2)}}{(F(R)-F(y))^{3/2}}dy \geqslant 0,$$

where we have used $X'(t,R)^2/X(t.R)^2 \to -B > 0$ as $t \to \infty$.

Case 3-c: $v_0(R) < 0$.

We finally treat the case $v_0(R) < 0$. Recall that $X \geqslant \xi_2$ holds, where ξ_2 is the zero of $D(R) - F(x)$ bigger than $x = X_D$. Therefore, there exists a time $t_* = t_*(R) \in (0,\infty)$ such that $X(t_*, R) = \xi_2$, $X'(t_*, R) = 0$, and

$$X'(t,R) = \begin{cases} -\sqrt{D - F(X(t,R))} & \text{for } t < t_*, \\ \sqrt{D - F(X(t,R))} & \text{for } t > t_*. \end{cases}$$

Moreover, t_* is described by

$$t_* = \int_{\xi_2}^R \frac{dy}{\sqrt{D-F(y)}} = \int_1^{R/\xi_2} \frac{\xi_2 dz}{\sqrt{D - F(\xi_2 z)}}.$$

Let us restrict our attention to the case $t \leqslant t_*$ and derive the condition for the existence of solution in $[0, t_*]$. For $t < t_*$, we have $X(t,R) > \xi_2$ and

$$\int_{X(t,R)}^R \frac{dy}{\sqrt{D-F(y)}} = t.$$

Differentiation in R gives

$$\frac{1}{|v_0|} - \frac{\Gamma}{|X'|} - \frac{1}{2}\int_{X(t,R)}^{R} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R) - F(y))^{3/2}} dy = 0.$$

We put $\mathcal{B} := \Gamma/|X'|$. Since $\partial_r D(R) - \partial_r A(R) y^{-(n-2)}$ is nonincreasing and $\int_{X(t,R)}^{R} (D - F(y))^{-3/2} dy$ tends to infinity as $X(t,R) \downarrow \xi_2$, $\mathcal{B}$ stays positive for all $t < t_*$ if and only if

$$\partial_r D(R) - \partial_r A(R) \xi_2^{-(n-2)} \leqslant 0.$$

Note that this condition is equivalent to $\partial_R X(t_*) \geqslant 0$ and to $\partial_r \xi_2 \geqslant 0$ because one deduces from the differentiation of identity $X(t_*, R) = \xi_2$ that

$$\partial_r \xi_2 = X'(t_*, R) \partial_r t_* + \partial_R X(t_*, R) = \partial_R X(t_*, R),$$

and from the identities $D - F(\xi_2) = 0$ and $F' = -2G$ that

$$\partial_r \xi_2 = -\frac{\partial_r D - \partial_r A \xi_2^{-(n-2)}}{2G(\xi_2)}$$

by assumption $G(\xi_2) > G(X_D) = 0$.

Now, let us find the condition that the solution which can be extended up to time $t = t_*$ is global. For simplicity, we suppose that all solution can be extended up to time $t = t_*$ since we have already known the condition for existence of the solution in $[0, t_*]$. Let t_{**} be a time bigger than t_* such that $X(t_{**}, R) = R$. Since for any $t_* < t_0 < t_{**}$, it holds that

$$\int_{X(t_0)}^{R} \frac{dy}{\sqrt{D - F(y)}} = t_{**} - t_0.$$

letting $t_0 \downarrow t_*$, we obtain $t_{**} = 2t_*$. Therefore, for all $t > t_*$ it holds that

$$\int_{R}^{X} \frac{dy}{\sqrt{D - F(y)}} = t - 2t_*.$$

Note that $X'(2t_*, R) = |v_0| > 0$. Thus, the same analysis as in the case $v_0 > 0$ tells us that Γ stays positive for all $t > t_*$ if and only if

$$\frac{1}{|v_0|} + \frac{1}{2}\int_{R}^{\infty} \frac{\partial_r D - \partial_r A y^{-(n-2)}}{(D - F(y))^{3/2}} - 2\partial_r t_* \geqslant 0.$$

□

3.4.3 Attractive case 3: $n = 2$.

We consider two-dimensional case. We recall that X solves (3.4.1) and (3.4.2) with (3.4.4).

Theorem 3.4.3. *Suppose $n = 2$, $\lambda < 0$, $b > 0$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. The classical solution to* (rEP$_b$) *is global in time if and only if one the following four conditions is satisfied for each $R > 0$:*

1. *$m_0(R) = 0$, $v_0(R) \geqslant -R|\lambda|b/2$, and $v_0'(R) \geqslant -|\lambda|b/2$;*
2. *$m_0(R) > 0$, $v_0(R)^2 > F_{\max} - F(R)$, $v_0(R) > 0$, and*
$$\partial_r D(R) \geqslant -\frac{2v_0(R)^{-1} + \partial_r \mathcal{A}(R) \int_R^\infty \log y (D(R) - F(y))^{-3/2} dy}{\int_R^\infty (D(R) - F(y))^{-3/2} dy};$$
3. *$m_0(R) > 0$, $v_0(R)^2 = F_{\max} - F(R)$, and either*
    (a) *$R < X_D(R)$, $v_0(R) > 0$, and $\partial_r D(R) \geqslant \partial_r \mathcal{A}(R) \log X_D(R)$,*
    (b) *$R = X_D(R)$,*
    (c) *$R > X_D(R)$ and either*
        i. *$v_0(R) < 0$ and $\partial_r D(R) \leqslant \partial_r \mathcal{A}(R) \log X_D(R)$,*
        ii. *$v_0(R) > 0$ and*
$$\partial_r D(R) < -\frac{2v_0(R)^{-1} + \partial_r \mathcal{A}(R) \int_R^\infty \log y (D(R) - F(y))^{-3/2} dy}{\int_R^\infty (D(R) - F(y))^{-3/2} dy};$$
4. *$m_0(R) > 0$, $v_0(R)^2 < F_{\max} - F(R)$, $R > X_D(R)$. and either*
    (a) *$v_0(R) > 0$ and*
$$\partial_r D(R) < -\frac{2v_0(R)^{-1} + \partial_r \mathcal{A}(R) \int_R^\infty \log y (D(R) - F(y))^{-3/2} dy}{\int_R^\infty (D(R) - F(y))^{-3/2} dy},$$
    (b) *$v_0(R) = 0$ and*
$$v_0' \geqslant -\frac{G(R)}{R\sqrt{|B|}} - G(R) \left(\frac{\partial_r \mathcal{A}(R)}{2} - \frac{\mathcal{A}(R)}{R} \right) \int_R^\infty \frac{\log(y/R)}{(F(R) - F(y))^{3/2}} dy,$$
    (c) *$v_0(R) < 0$, $\partial_r D(R) + \partial_r \mathcal{A}(R) \log \xi_2 \leqslant 0$, and*
$$\frac{1}{|v_0|} + \frac{1}{2} \int_R^\infty \frac{\partial_r D(R) + \partial_r \mathcal{A}(R) \log y}{(F(\xi_2) - F(y))^{3/2}} - 2\partial_r t_* \geqslant 0,$$
    where ξ_2 is the root of $F(\xi) = F(R) + v_0(R)^2$ bigger than $X_D(R)$ and
$$t_* := \int_R^{\xi_2} \frac{dy}{\sqrt{F(\xi_2) - F(y)}}.$$

Let k be a nonnegative integer. If $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ satisfy the condition for global existence then the corresponding solution of (rEP$_b$) *satisfies*

$$\begin{aligned}
&\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k), \\
&v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).
\end{aligned}$$

The solution is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_b$) *in the distribution sense.*

The proof is the exactly the same as Theorem 3.4.2. The only difference is that F is defined by not (3.4.3) but (3.4.4). The F defined by (3.4.4) is the same as the one defined by (3.4.3) in the following respect: $F' = -2G$; it takes it maximum at $x = X_D$ if $\lambda < 0$.

3.4.4 Repulsive case 1: $n = 1$.

We consider the case $\lambda > 0$. In this case, the classical trajectory (and the solution) becomes time-periodic.

Theorem 3.4.4 ([25])**.** *Suppose $n = 1$, $\lambda > 0$, $b > 0$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. The classical solution of* (rEP$_b$) *is global if and only if*

$$|v_0(R)| < \sqrt{\lambda(2Rm_0(R) - bR^2)}, \qquad |v_0'(R)| < \sqrt{\lambda(2\rho_0(R) - b)}$$

holds for all $R \geqslant 0$. In particular, if $\rho_0(R_0) \leqslant b/2$ holds for some $R_0 \geqslant 0$ then the solution breaks down in finite time.

Proof. Since (3.4.1) is solved explicitly:

$$X(t,R) = \frac{m_0(R)}{b} + \left(R - \frac{m_0(R)}{b}\right)\cos\sqrt{\lambda b}t + \frac{v_0(R)}{\sqrt{\lambda b}}\sin\sqrt{\lambda b}t.$$

We have

$$\Gamma(t,R) = \frac{\rho_0(R)}{b} + \left(1 - \frac{\rho_0(R)}{b}\right)\cos\sqrt{\lambda b}t + \frac{v_0'(R)}{\sqrt{\lambda b}}\sin\sqrt{\lambda b}t.$$

Therefore,

$$\begin{aligned}
\min_{t\geqslant 0} X(t,R) &= \frac{m_0(R)}{b} - \sqrt{\left(R - \frac{m_0(R)}{b}\right)^2 + \frac{v_0(R)^2}{\lambda b}} \\
\min_{t\geqslant 0} \Gamma(t,R) &= \frac{\rho_0(R)}{b} - \sqrt{\left(1 - \frac{\rho_0(R)}{b}\right)^2 + \frac{v_0'(R)^2}{\lambda b}}.
\end{aligned}$$

These values stay positive for all $t \geqslant 0$ if and only if

$$|v_0(R)| < \sqrt{\lambda(2Rm_0(R) - bR^2)}, \qquad |v_0'(R)| < \sqrt{\lambda(2\rho_0 - b)}$$

holds. Let k be a nonnegative integer. If $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ satisfy the condition for global existence then the corresponding solution of (rEP$_b$) satisfies

$$\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).$$

The solution is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_b$) in the distribution sense. □

3.4.5 Repulsive case 2: $n \geqslant 3$.

We proceed to the case $n \geqslant 3$.

Theorem 3.4.5. *Suppose $n \geqslant 3$, $\lambda > 0$, $b > 0$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. The solution to* (rEP$_b$) *is global if and only if $m_0(R) > 0$ for all $R > 0$, the value*

$$T_* = T_*(R) := \int_{\xi_1}^{\xi_2} \frac{dy}{\sqrt{D(R) - F(y)}}$$

is a universal constant for $R \in \{R \in \mathbb{R}_+ | v_0(R) \neq 0$ or $R \neq X_D\}$, and the following condition holds for all $R \in \{R \in \mathbb{R}_+ | v_0(R) \neq 0$ or $R \neq X_D\}$:

$$\xi_1 < \left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{\frac{1}{n-2}} < \xi_2$$

and either

- *$v_0(R) \neq 0$ and*

$$\frac{1}{|v_0(R)|} + \frac{1}{2} \int_R^{\left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{1/(n-2)}} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R) - F(y))^{3/2}} dy > \max(0, -2\partial_R t_1);$$

- *$v_0(R) = 0$ and*

$$\left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{\frac{1}{n-2}} \frac{1}{R\sqrt{F(R) - F\left(\left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{\frac{1}{n-2}}\right)}} + \frac{v_0'}{G(R)}$$
$$+ \left(\frac{\partial_r A(R)}{2} - \frac{nA(R)}{2R}\right) \int_R^{\left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{1/(n-2)}} \frac{R^{-(n-2)} - y^{-(n-2)}}{(F(R) - F(y))^{3/2}} dy > 0,$$

where ξ_1 and ξ_2 are two roots of $D(R) - F(\xi) = 0$ with $\xi_1 < \xi_2$ and

$$t_1 = \int_R^{\xi_2} \frac{dy}{\sqrt{D(R) - F(y)}}.$$

Let k be a nonnegative integer. If $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ satisfy the condition for global existence then the corresponding solution of (rEP$_b$) *satisfies*

$$\rho \in C^2([0,\infty), D^k) \cap C^\infty((0,\infty), D^k),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1}).$$

The solution is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and also solves (EP$_b$) *in the distribution sense.*

Remark 3.4.6. The time T_* is the period: If $X(t,R)$ exists with $T_* \in (0,\infty)$ then it satisfies $X(t+T_*,R) = X(t,R)$. In four dimensional case, $T_* = \pi/2\sqrt{B}$ since

$$T_* = \int_{\xi_1}^{\xi_2} \frac{dy}{\sqrt{A\xi_1^{-2} + B\xi_1^2 - Ay^{-2} - By^2}} = \frac{1}{2\sqrt{B}} \int_{\xi_1^2}^{\xi_2^2} \frac{dz}{\sqrt{(z-\xi_1^2)(\xi_2^2 - z)}}$$
$$= \frac{1}{2\sqrt{B}} \int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} dt = \frac{\pi}{2\sqrt{B}}.$$

Therefore, T_* is independent of R.

Proof. We first consider the two special cases: First is the case where $m_0(R) = 0$. In this case, $X''(t,R) = -BX(t,R)$ by (3.4.1), and so

$$X(t,R) = R\cos Bt + \frac{v_0(R)}{B}\sin Bt.$$

Hence, $X(t,R) = 0$ holds in finite time, which is a sufficient condition for finite-time breakdown (Lemma 3.2.7). Second is the case where $v_0(R) = 0$ and $X_D(R) = R$. In this case, by (3.4.2), we have

$$0 \leqslant (X'(t,R))^2 = D(R) - F(X(t,R)) = F(X_D) - F(X(t,R)) \leqslant 0$$

since $F(X_D) = \min_{x\geqslant 0} F(x)$. Therefore, $X'(t,R) \equiv 0$ and $X(t,R) \equiv R$. We also have $\Gamma(t,R) \equiv 1 > 0$. Thus, this case is admissible.

In the followings, we assume that $m_0(R) > 0$ for all $R > 0$, and that either $v_0(R) \neq 0$ or $X_D(R) \neq R$ holds. These conditions leads to $\max_{x>0}(D(R) - F(x)) = v_0(R)^2 + F(R) - F_{\min} > 0$. Recall that F' is monotone increasing and that $\lim_{x\to 0} F(x) = \lim_{x\to\infty} F(x) = \infty$. Therefore, there exist two roots ξ_1 and ξ_2 satisfying

$$D(R) - F(\xi_i) = 0, \quad 0 < \xi_1 < X_D(R) < \xi_2 < \infty.$$

We remark that X solves (3.4.2) and so $X(t,R)$ satisfies $0 < \xi_1 \leqslant X(t,R) \leqslant \xi_2 < \infty$ for all $t \geqslant 0$.

Step 1. We now claim that $X(t,R)$ is periodic in time and the period is given by

$$T_* = 2\int_{\xi_1}^{\xi_2} \frac{dy}{\sqrt{D(R)-F(y)}}. \tag{3.4.8}$$

We first note that $T_* < \infty$: Since $F'(\xi_i) = -2G(\xi_i) \neq 0$, we see that

$$\left| \frac{\xi_i - y}{D(R)-F(y)} \right| < \infty \quad \text{ as } y \to \xi_i$$

for $i = 1,2$, and so that (3.4.8) is integrable. We first consider the case $v_0(R) > 0$. Because $(X')^2 = D(R) - F(X) > 0$ as long as $X \in (\xi_1, \xi_2)$, $X' = \sqrt{D(R)-F(X)} > 0$ holds before X reaches to ξ_2. Since

$$\int_R^{X(t,R)} \frac{dy}{\sqrt{D(R)-F(y)}} = t$$

holds for such time. The left hand side is integrable and so there exists a time $t_1 < \infty$ such that $X(t_1,R) = \xi_2$. Note that t_1 is given by

$$t_1 = \int_R^{\xi_2} \frac{dy}{\sqrt{D(R)-F(y)}} \in (0, T_*/2),$$

where T_* is given by (3.4.8). By (3.4.2) and (3.4.1), we see that $X'(t_1,R) = 0$ and $X''(t_1,R) < 0$, respectively. Therefore, for a time such that $0 < t - t_1 \ll 1$, we have $X'(t,R) < 0$. We put $t_2 = t_1 + T_*/2 \in (T_*/2, T_*)$. Repeating the above argument, one sees that $X'(t,R) = -\sqrt{D(R)-F(X)} < 0$ and

$$\int_{X(t,R)}^{\xi_2} \frac{dy}{\sqrt{D(R)-F(y)}} + t_1 = t \tag{3.4.9}$$

hold for $t \in (t_1, t_2)$, and that, at $t = t_2$,

$$X(t_2,R) = \xi_1, \quad X'(t_2,R) = 0, \quad X''(t_2,R) > 0.$$

Now, we remark that, letting $X(t,R) = R$ in (3.4.9), we obtain $X(2t_1,R) = R$ and $X'(2t_1,R) = -v_0(R)$. Similarly, for $t \in (t_2, t_2 + T_*/2)$, we have

$$\int_{\xi_1}^{X(t,R)} \frac{dy}{\sqrt{D(R)-F(y)}} + t_2 = t.$$

By the definitions of t_1 and t_2, we can rewrite this identity as

$$\int_R^{X(t,R)} \frac{dy}{\sqrt{D(R)-F(y)}} + T_* = t.$$

Therefore, we obtain $X(T_*,R) = R$. Then, (3.4.2) gives $X'(T_*,R) = v_0(R)$. By uniqueness of X (Proposition 3.2.1), we conclude that $X(t+T_*,R) = X(t,R)$.

The case $v_0(R) < 0$ can be handled in the same way. If $v_0(R) = 0$ then either $R = \xi_1$ or $R = \xi_2$ holds and so we obtain either

$$X(0,R) = \xi_1, \quad X'(0,R) = 0, \quad X''(0,R) > 0$$

or

$$X(0,R) = \xi_2, \quad X'(0,R) = 0, \quad X''(0,R) < 0.$$

Hence, we can show $X(t + T_*, R) = X(t,R)$ also in the same way.

Step 2. We next prove that $\partial_R T_* = 0$ is a necessary condition for global existence. We suppose $v_0(R) \neq 0$. Let m be a positive integer. By periodicity, we have

$$\begin{aligned} &X(mT_*, R) = R, && X'(mT_*, R) = v_0(R), \\ &X\left(mT_* + 2t_1, R\right) = R, && X'\left(mT_* + 2t_1, R\right) = -v_0(R). \end{aligned}$$

Hence, we obtain

$$1 = \partial_R(X(mT_*, R)) = m(v_0(R)\partial_R T_*) + \Gamma(mT_*, R).$$

Similarly,

$$1 = -m(v_0(R)\partial_R T_*) - 2v_0\partial_R t_1 + \Gamma\left(mT_* + 2t_1, R\right).$$

Therefore, if $\partial_R T_* \neq 0$ then either $\Gamma(mT_*, R) < 0$ or $\Gamma(mT_* + 2t_1, R) < 0$ holds for large m. Even if $v_0(R) = 0$, since $X'(\varepsilon, R) \neq 0$ holds for any small $\varepsilon > 0$, the above argument is applicable. Thus, $\partial_R T_* = 0$ is a necessary condition for global existence.

Step 3. Under the restriction $\partial_R T_* = 0$, we derive the condition on initial data which ensures $\Gamma(t,R) > 0$ for all $t \geqslant 0$. We first consider the case $v_0(R) > 0$. Let t_1 be defined in (3.4.9). Differentiating the equalities $X(t_1, R) = \xi_2$ and $D(R) - F(\xi_2) = 0$, we see that

$$\Gamma(t_1, R) = \partial_R \xi_2 = -\frac{\partial_r D(R) - \partial_r A \xi_2^{-(n-2)}}{2G(\xi_2)}.$$

Similarly, we obtain

$$\Gamma(t_1 + T_*/2, R) = \partial_R \xi_1 = -\frac{\partial_r D(R) - \partial_r A \xi_1^{-(n-2)}}{2G(\xi_1)}.$$

These two values are positive if

$$\xi_1 < \left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{\frac{1}{n-2}} < \xi_2. \tag{3.4.10}$$

This condition is necessary for global existence, so let us assume this. Let t_1 be as in (3.4.9). However $t_1 - T_*/2 < 0$, by periodicity, we consider $t \in (t_1 - T_*/2, t_1)$. In this case, $X'(t,R) = \sqrt{D(R) - F(X(t,R)} > 0$ and so

$$t = \int_R^{X(t,R)} \frac{dy}{\sqrt{D(R) - F(y)}}.$$

Differentiate with respect to R to obtain

$$0 = \frac{\Gamma(t,R)}{X'(t,R)} - \frac{1}{v_0(R)} - \frac{1}{2}\int_R^{X(t,R)} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R) - F(y))^{3/2}} dy.$$

Put $\mathcal{B}(t) = \Gamma(t,R)/X'(t,R)$. An elementary calculation shows that $\mathcal{B}(t)$ takes it minimum at $t = t_0$ such that $X(t_0, R) = (\partial_r A(R)/\partial_r D(R))^{1/(n-2)}$. Since $X' > 0$, Γ stays for all $t \in (t_1 - T_*/2, t_1)$ if and only if

$$\mathcal{B}(t_0) = \frac{1}{v_0(R)} + \frac{1}{2}\int_R^{\left(\frac{\partial_r A(r)}{\partial_r D(R)}\right)^{1/(n-2)}} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R) - F(y))^{3/2}} dy > 0.$$

Let us proceed to the case $t \in (t_1, t_1 + T_*/2)$. Since $X'(t,R) = -\sqrt{D(R) - F(y)} < 0$ and so

$$t - 2t_1 = \int_{X(t,R)}^R \frac{dy}{\sqrt{D(R) - F(y)}},$$

we have

$$\begin{aligned}\min_{t \in (t_1, t_1 + T_*/2)} \frac{\Gamma(t,R)}{|X'(t,R)|} &= 2\partial_R t_1 + \frac{1}{v_0(R)} \\ &\quad + \frac{1}{2}\int_R^{\left(\frac{\partial_r A(r)}{\partial_r D(R)}\right)^{1/(n-2)}} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R) - F(y))^{3/2}} dy\end{aligned}$$

in the similar way. Hence, we obtain stated condition.

We next suppose $v_0(R) < 0$. Mimicking the previous case, we deduce from the identity

$$t = \int_{X(t,R)}^R \frac{dy}{\sqrt{D(R) - F(y)}}, \quad t \in (-t_1, T_*/2 - t_1)$$

that $\Gamma(t,R) > 0$ holds for all $t \in (-t_1, T_*/2 - t_1)$ if and only if

$$\frac{1}{|v_0(R)|} + \frac{1}{2}\int_R^{\left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{1/(n-2)}} \frac{\partial_r D(R) - \partial_r A(R) y^{-(n-2)}}{(D(R) - F(y))^{3/2}} dy > 0.$$

Furthermore, we see from

$$t - 2(T_*/2 - t_1) = \int_R^{X(t,R)} \frac{dy}{\sqrt{D(R) - F(y)}}, \quad t \in (T_*/2 - t_1, T_* - t_1)$$

that $\Gamma(t,R)>0$ holds for all $t\in(T_*/2-t_1,T_*-t_1)$ if and only if

$$2\partial_R t_1+\frac{1}{|v_0(R)|}+\frac{1}{2}\int_R^{\left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{1/(n-2)}}\frac{\partial_r D(R)-\partial_r A(R)y^{-(n-2)}}{(D(R)-F(y))^{3/2}}dy>0.$$

We finally treat the case $v_0(R)=0$. In this case, either $R=\xi_1$ or $R=\xi_2$ holds. Suppose $R=\xi_1$. For $t\in(0,T_*/2)$, we have

$$t=\int_R^{X(t,R)}\frac{dy}{\sqrt{F(R)-F(y)}}.$$

We set $\mathcal{B}(t):=\Gamma(t,R)/X'(t,R)$. By the same calculation as in Case 3-b of the proof of Theorem 3.4.2, we see that $\mathcal{B}(t)$ is given by (3.4.7). Therefore,

$$\frac{d}{dt}\mathcal{B}(t)=\frac{1}{R(X')^2}\left[\frac{R\partial_r A}{2}\left(\frac{1}{R^{n-2}}-\frac{1}{X^{n-2}}\right)-RG(R)\right].$$

Since $G(R)>0$ and $\partial_r A(R)>0$, we see that $\mathcal{B}$ takes it minimum at $t=t_0$, where t_0 satisfies

$$X(t_0,R)=\left(\frac{1}{R^{n-2}}-\frac{2G(R)}{\partial_r A}\right)^{-\frac{1}{n-2}}=\left(\frac{\partial_r A(R)}{\partial_r D(R)}\right)^{\frac{1}{n-2}}.$$

Therefore, $\Gamma(t,R)>0$ holds for $t\in(0,T_*/2)$ if and only if $\mathcal{B}(t_0)>0$. For $t\in(T_*/2,T_*)$, it holds that

$$T_*-t=\int_R^{X(t,R)}\frac{dy}{\sqrt{F(R)-F(y)}}.$$

Since the left hand side is independent of R, we can repeat the same analysis. We next suppose $R=\xi_2$. In this case,

$$-t=\int_R^{X(t,R)}\frac{dy}{\sqrt{F(R)-F(y)}}$$

holds for $t\in(0,T_*/2)$ and

$$t-T_*=\int_R^{X(t,R)}\frac{dy}{\sqrt{F(R)-F(y)}}$$

holds for $t\in(T_*/2,T_*)$. Therefore, the condition becomes the same form as in the previous $R=\xi_1$ case. $\square$

3.4.6 Repulsive case 3: $n = 2$

Theorem 3.4.7. *Suppose $n = 2$, $\lambda > 0$, $b > 0$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. The solution to* (rEP$_b$) *is global if and only if*

$$T_* = T_*(R) := \int_{\xi_1}^{\xi_2} \frac{dy}{\sqrt{D(R) - F(y)}}$$

is independent of $R \in \{R \in \mathbb{R}_+ | v_0(R) \neq 0 \text{ or } R \neq X_D\}$, and the following condition holds for all $R \in \{R \in \mathbb{R}_+ | v_0(R) \neq 0 \text{ or } R \neq X_D\}$:

$$\xi_1 < \exp\left(-\frac{\partial_r D(R)}{\partial_r \mathcal{A}(R)}\right) < \xi_2$$

and

- $v_0(R) \neq 0$ *and*

$$\frac{1}{|v_0(R)|} + \frac{1}{2}\int_R^{\left(\frac{\partial_r \mathcal{A}(R)}{\partial_r D(R)}\right)^{1/(n-2)}} \frac{\partial_r D(R) + \partial_r \mathcal{A}(R) \log y}{(D(R) - F(y))^{3/2}} dy > \max(0, -2\partial_R t_1),$$

- $v_0(R) = 0$ *and*

$$\frac{e^{-\frac{\partial_r D(R)}{\partial_r \mathcal{A}(R)}}}{R\sqrt{F(R) - F\left(e^{-\frac{\partial_r D(R)}{\partial_r \mathcal{A}(R)}}\right)}} + \frac{v_0'}{G(R)}$$
$$+ \left(\frac{\partial_r \mathcal{A}(R)}{2} - \frac{\mathcal{A}(R)}{R}\right) \int_R^{e^{-\frac{\partial_r \mathcal{A}(R)}{\partial_r D(R)}}} \frac{\log(y/R)}{(F(R) - F(y))^{3/2}} dy > 0,$$

where ξ_1 and ξ_2 are two roots of $D(R) - F(\xi) = 0$ with $\xi_1 < \xi_2$ and

$$t_1 = \int_R^{\xi_2} \frac{dy}{\sqrt{D(R) - F(y)}}.$$

Let k be a nonnegative integer. If $\rho_0 \in D^k$ and $v_0 \in D^{k+1}$ satisfy the condition for global existence then the corresponding solution of (rEP$_b$) *satisfies*

$$\rho \in C^2([0, \infty), D^k) \cap C^\infty((0, \infty), D^k),$$
$$v \in C^1([0, \infty), D^{k+1}) \cap C^\infty((0, \infty), D^{k+1}).$$

The solution is unique in $C^2([0, \infty), D^0) \times C^1([0, \infty), D^1)$ and also solves (EP$_b$) *in the distribution sense.*

3.5 The zero background limit

At the end of this chapter, we observe the correspondence of the results in Section 3.4 to the results in Section 3.3 in the *zero background limit*. More precisely, we check whether or not the necessary and sufficient condition for global existence of the classical solution to (rEP$_b$) coincides with that for the classical solution (rEP$_0$) in the limit $b \downarrow 0$. In the one-dimensional case with $v_0(R) > 0$, this limit is considered in [25]. This limit especially reveals that the two-dimensional case is special.

3.5.1 Attractive case.

It turns out that, in the attractive case, the answer is yes.

Theorem 3.5.1. *By letting $b \to 0$, the conditions in Theorems 3.4.1, 3.4.2, and 3.4.3 becomes the same as in Theorem 3.3.1.*

One dimensional case.

In the one dimensional case, one necessary condition for global existence is that

$$v_0(R) \geqslant \sqrt{|\lambda|}\left(\frac{m_0(R)}{\sqrt{b}} - \sqrt{b}R\right)$$

holds for all $R > 0$ (Theorem 3.4.1). If $m_0(R_0) > 0$ holds for some R_0 then, for such R_0, the right hand side tends to infinity as $b \downarrow 0$ and hence this condition breaks. On the other hand, once we have $m_0(R) \equiv 0$, the condition becomes

$$v_0(R) \geqslant -\sqrt{|\lambda|b}R, \quad v_0' \geqslant -\sqrt{|\lambda|b},$$

which is identical with the condition in Theorem 3.3.1 in the limit $b \downarrow 0$.

Two dimensional case.

The two dimensional case is special. The necessary and sufficient condition for the global existence for $b > 0$ case given in Theorem 3.4.3 is very similar to the condition for the $n \geqslant 3$ case established in Theorem 3.4.2. This is due to the fact that the functions $G(x)$ and $F(x)$ appearing in (3.4.1) and (3.4.2), respectively, have the similar shape in the $n = 2$ and the $n \geqslant 3$ cases. However, as far as the limit equation (rEP$_0$) is concerned, the situation is quite different from the $n \geqslant 3$ case, and rather similar to the previous one dimensional case. We deduce from Theorem 3.3.1 that if $n = 2$ then nonzero initial density ρ_0 never admit the global solution as in one dimensional case, while it does in the $n \geqslant 3$ case as Theorem 3.3.14 suggests. The zero background limit $b \downarrow 0$ clarifies this difference: We fix $R > 0$ and suppose $m_0(R) > 0$. Then, for $n = 2$, it holds that

$$\begin{aligned}\lim_{b\downarrow 0}\max_{x>0} F(x) &= \lim_{b\downarrow 0} F(X_D) \\ &= \lim_{b\downarrow 0}\left(-\lambda m_0(R)\log\left(\frac{2m_0(R)}{b}\right)+\lambda\left(\frac{bm_0(R)}{2}\right)^{\frac{1}{2}}\right) = +\infty,\end{aligned}$$

while the same limit is

$$\begin{aligned}\lim_{b\downarrow 0}\max_{x>0} F(x) &= \lim_{b\downarrow 0} F(X_D) \\ &= \lim_{b\downarrow 0}\left(\frac{2\lambda m_0(R)^{\frac{n-1}{n}} b^{\frac{n-2}{n}}}{(n-2)n^{\frac{1}{n}}}+\frac{\lambda b^{\frac{n-1}{n}} m_0(R)^{\frac{1}{n}}}{n^{\frac{n-1}{n}}}\right) = 0\end{aligned}$$

if $n \geqslant 3$. Therefore, in Theorem 3.4.3, the cases 2 and 3 do not happen for sufficiently small b. Moreover, since $X_D(R) = (nm_0(R)/b)^{1/n} \to +\infty$ as $b \downarrow 0$ for $n \geqslant 2$, we have $X_D(R) \geqslant R$ for small b and so the conditions in the case 4 is not fulfilled, neither. Hence, the only possibility for admitting the global classical solution is that the condition in the case 1 is true for all $R > 0$, that is, $m_0(R) \equiv 0$ and

$$v_0(R) \geqslant -\frac{R|\lambda|b}{2}, \quad v_0'(R) \geqslant -\frac{|\lambda|b}{2}$$

holds for all $R > 0$. Hence, in the limit $b \downarrow 0$, the condition is identical to the one given by Theorem 3.3.1.

Three and higher dimensional cases.

As mentioned above, we have $\lim_{b\downarrow 0}\max_{x>0} F(x) = 0$ and $X_D(R) \to 0$ as $b \downarrow 0$. Therefore, in Theorem 3.4.2, the three cases 1, 2, and 3-(a) can happen for sufficiently small b. The condition in the case 1 becomes

$$m_0(R) = 0, \quad v_0(R) \geqslant 0, \quad v_0'(R) \geqslant 0$$

in the limit $b \downarrow 0$. Similarly, one verifies that the condition in the case 2 tends to

$$m_0(R) > 0, \quad v_0(R)^2 > -F(R), \quad v_0(R) > 0, \quad \partial_r C(R) \geqslant 0,$$

and the condition in the case 3-(a) to

$$m_0(R) > 0, \quad v_0(R)^2 = -F(R), \quad v_0(R) > 0, \quad \partial_r C(R) \geqslant 0,$$

where C is defined in (3.3.2). Note that $v_0(R)^2 + F(R) = C(R)$. Hence, we conclude that the above three conditions can be unified into

$$v_0(R) \geqslant 0, \quad C(R) \geqslant 0, \quad \partial_r C(R) \geqslant 0,$$

which is the one in Theorem 3.3.1.

3.5.2 Repulsive case

In the Repulsive case, the results given in Theorems 3.4.4, 3.4.5, and 3.4.7 are not the same as the result given in Theorems 3.3.2, 3.3.7, and 3.3.12, respectively, in the zero background limit. This is closely related to the periodicity of the solution.

We consider the simplest case $n = 1$. The higher dimensional case is similar. If we let $b \downarrow 0$, Theorem 3.4.4 suggests that the classical solution to (rEP_0) is global if and only if

$$|v_0(R)| < \sqrt{2\lambda m_0(R)}, \quad |v_0'(R)| < \sqrt{2\lambda \rho_0(R)} \tag{3.5.1}$$

holds for all $R > 0$. However, as Theorem 3.3.7 shows, the correct condition is not (3.5.1) but

$$v_0(R) > -\sqrt{2\lambda m_0(R)}, \quad v_0'(R) > -\sqrt{2\lambda \rho_0(R)}. \tag{3.5.2}$$

The question is the followings: What produces this difference, and what happens if $v_0(R) \geqslant \sqrt{2\lambda m_0(R)}$ and/or $v_0'(R) \geqslant \sqrt{2\lambda \rho_0(R)}$?

We can answer all these questions by the periodicity. Let us assume that $m_0(R) > 0$ and $v_0(R) \geqslant \sqrt{2\lambda m_0(R)}$. For $b > 0$, we deduce from (3.4.1) that

$$X(t, R) = X(t, R; b) = \frac{m_0(R)}{b} + \left(R - \frac{m_0(R)}{b}\right) \cos \sqrt{\lambda b} t + \frac{v_0(R)}{\sqrt{\lambda b}} \sin \sqrt{\lambda b} t.$$

We consider $t \in [0, 2\pi/\sqrt{\lambda b})$ because X is $(2\pi/\sqrt{\lambda b})$-periodic in time. We fix b so small that $2m_0(R) - bR > 0$. Since $v_0(R) \geqslant \sqrt{\lambda(2Rm_0(R) - bR)}$, we can choose t_0 so that $X(t_0, R; b) = 0$. Therefore, the classical solution to (rEP_b) breaks down in finite time. However, we note that $t_0 \in (\pi/\sqrt{\lambda b}, 2\pi/\sqrt{\lambda b})$. This is because, for $t \in [0, \pi/\sqrt{\lambda b}]$, we have

$$X(t, R; b) \geqslant \frac{m_0(R)}{b} - \left| R - \frac{m_0(R)}{b} \right| = \min\left(R, \frac{2m_0(R)}{b} - R\right) > 0.$$

Hence, we can choose a finite time $t_0 < \infty$ so that $X(t_0, R) = 0$ for all $b > 0$ but cannot in the limit case $b = 0$. As it were, such t_0 goes "beyond the infinity" as $b \downarrow 0$. Thus, once we deduce the condition which ensures $X(t, R) > 0$ $\Gamma(t, R) > 0$ for $t \in [0, \pi/\sqrt{\lambda b}]$, this is sufficient to claim that the solution to the limit equation exists for all $t \in \mathbb{R}_+$. The condition (3.5.2) corresponds to nothing but this limit condition.

On the other hand, the condition (3.5.2) is the condition for the existence of classical solution to (rEP_0) for all time $t \in \mathbb{R}$ including also the negative time. Let $b > 0$ be sufficiently small. By periodicity of X, if $m_0(R) > 0$ and $v_0(R) \geqslant \sqrt{2\lambda m_0(R)}$ then we have $X(t_0 - 2\pi/\sqrt{\lambda b}, R; b) = 0$. Now we remark that $\lim_{b \to 0}(t_0 - 2\pi/\sqrt{\lambda b}) > -\infty$ follows from the fact that $X(t, R)$

has at most two zero in the interval $(-\pi/\sqrt{\lambda b}, 0)$ for all $b > 0$, and that these zeros tend to $2R/(v_0(R) \pm \sqrt{v_0(R)^2 - 2\lambda R m_0(R)})$ as $b \downarrow 0$ because

$$\begin{aligned}\lim_{b\downarrow 0} X(t,R;b) &= R\left(\lim_{b\downarrow 0}\cos\sqrt{\lambda b}t\right) + v_0(R)t\left(\lim_{b\downarrow 0}\frac{\sin\sqrt{\lambda b}t}{\sqrt{\lambda b}t}\right)\\ &\quad + \lambda m_0(R)t^2\left(\lim_{b\downarrow 0}\frac{1-\cos\sqrt{\lambda b}t}{\lambda b t^2}\right)\\ &= R + v_0(R)t + \frac{\lambda m_0(R)}{2}t^2.\end{aligned}$$

Therefore, the limit solution breaks down in finite (negative) time. Similarly, we have

$$\begin{aligned}\lim_{b\downarrow 0} \Gamma(t,R;b) &= \left(\lim_{b\downarrow 0}\cos\sqrt{\lambda b}t\right) + v_0'(R)t\left(\lim_{b\downarrow 0}\frac{\sin\sqrt{\lambda b}t}{\sqrt{\lambda b}t}\right)\\ &\quad + \lambda \rho_0(R)t^2\left(\lim_{b\downarrow 0}\frac{1-\cos\sqrt{\lambda b}t}{\lambda b t^2}\right)\\ &= 1 + v_0'(R)t + \frac{\lambda \rho_0(R)}{2}t^2.\end{aligned}$$

We see that if (3.5.1) is satisfied then $X(t,R;0) > 0$ and $\Gamma(t,R;0) > 0$ hold for all $t \in \mathbb{R}$.

Chapter 4

Large time WKB analysis for Schrödigner-Poisson system

4.1 Introduction and main result

In this chapter, we back to a phase-amplitude approximation of the semi-classical nonlinear Schrödinger equation. We consider Schrödinger-Poisson system

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda V_{\mathrm{P}}^\varepsilon u^\varepsilon, \\ -\Delta V_{\mathrm{P}}^\varepsilon = |u^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x| \to \infty, \\ u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon), \end{cases} \tag{SP}$$

In Chapter 2, we justify the WKB approximation of the solution

$$u^\varepsilon = e^{i\frac{\phi_0}{\varepsilon}}(\beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^k\beta_k + o(\varepsilon^k)) \tag{4.1.1}$$

in a time interval $[0,T]$ for a data in Sobolev space (Theorem 2.1.2). The aim of this chapter is to show the asymptotics (4.1.1) for a large time interval. In [46], this kind of result is established in one-dimension case. Now, we generalize to the $n \geqslant 3$ case. This is done by a combination of the results in Chapters 2 and 3. Let us describe the outline of proof. In Chapter 2, the asymptotics (4.1.1) is established for small time. We apply the modified Madelung transform $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ and consider the system

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon \cdot \nabla)a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t\phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda V_{\mathrm{P}}^\varepsilon = 0, \\ -\Delta V_{\mathrm{P}}^\varepsilon = |a^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x| \to \infty, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0). \end{cases} \tag{4.1.2}$$

Then, we show the asymptotic expansion

$$a^{\varepsilon} = a_0 + \sum_{j=1}^{k+1} \varepsilon^j a_j + o(\varepsilon^{k+1}), \qquad \phi^{\varepsilon} = \phi_0 + \sum_{j=1}^{k+1} \varepsilon^j \phi_j + o(\varepsilon^{k+1}),$$

which leads to the desired WKB approximate solution (4.1.1). Here we note that (a_0, ϕ_0) is the solution to

$$\begin{cases} \partial_t a_0 + (\nabla\phi_0 \cdot \nabla) a_0 + \dfrac{1}{2} a_0 \Delta \phi_0 = 0, \\ \partial_t \phi_0 + \dfrac{1}{2}|\nabla \phi_0|^2 + \lambda V_{\mathrm{P}} = 0, \\ -\Delta V_{\mathrm{P}} = |a_0|^2, \quad V_{\mathrm{P}} \to 0 \text{ as } |x| \to \infty, \\ (a_0(0,x), \phi_0(0,x)) = (A_0(x), \Phi_0(x)). \end{cases} \tag{4.1.3}$$

Then, we see that $\rho = |a_0|^2$ and $v = \nabla\phi_0$ solve the Euler-Poisson equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ v_t + v \cdot \nabla v + \lambda \nabla V_{\mathrm{P}} = 0, \\ -\Delta V_{\mathrm{P}} = \rho, \quad V_{\mathrm{P}} \to 0 \text{ as } |x| \to \infty, \\ (\rho, v)(0,x) = (|A_0|^2, \nabla\Phi_0)(x). \end{cases} \tag{4.1.4}$$

In Chapter 3, we derive the necessary and sufficient condition on the initial data for global existence of a classical solution to (4.1.4) under the radial symmetry (Theorems 3.3.1, 3.3.2, 3.3.7, and 3.3.12). In particular, Theorem 3.3.14 shows that there actually exists an example of initial data $(|A_0|^2, \nabla\Phi_0)$ for $n \geqslant 3$ and $\lambda < 0$ which admits a global solution of (4.1.4). For such initial data, it turns out that the functions (a_j, ϕ_j) in the expansion of $(a^{\varepsilon}, \phi^{\varepsilon})$ are defined globally in time. Then, applying the analysis in Chapter 2, we can conclude that (4.1.1) is extended to an arbitrarily large interval. The size of the interval in which (4.1.1) is valid depends on the parameter ε, and tends to infinity as $\varepsilon \to 0$. Since the assumptions for the main theorem is complicated, here we only state the theorem.

Theorem 4.1.1. *Let Assumption 4.5.3 be satisfied. Let $(a^{\varepsilon}, \phi^{\varepsilon})$ be the solution to (4.1.2) given by Theorem 4.4.2 and (a_0, ϕ_0) be the global solution to (4.1.3) given by Theorem 4.2.1. Then, there exist*

$$(a_j, \phi_j) \in C([0,\infty); H^{s-2j+3} \times (X^{s-2j+5} \cap L^{\frac{n}{n-2}+}))$$

$(1 \leqslant j \leqslant k)$ and constant C_s depending only on n and s such that, for any $T > 0$, it holds that

$$u^{\varepsilon} = e^{i\frac{\phi_0}{\varepsilon}}(\beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^{k-1}\beta_{k-1} + O(\varepsilon^k)) \quad \text{in } L^{\infty}([0,T]; H^{s-2k+1}) \tag{4.1.5}$$

for $\varepsilon \leqslant C\eta(T)e^{-3C_s\eta(T)T}$, where $\eta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function defined in (4.5.1).

This chapter is organized as follows: In Section 4.2, we first construct the solution (a_0, ϕ_0) of (4.1.3) time-globally by using Theorem 3.3.14. Then, Section 4.3 is devoted to the study of the regularity of this global solution. Namely, we investigate in which function space this solution lies. Then, it turns out that the above global solution is out of reach of the framework in Section 2.4. Hence, we choose the function space as something like Sobolev space so that we can apply the analysis of Chapter 2, and make some modification in Section 4.4. In the end, we prove Theorem 4.1.1 in Section 4.5 by modifying the argument in [46].

4.2 Global existence of limit solution

We first construct the global (classical) solution of (4.1.3). Theorem 3.3.14 suggests that there is only one global solution (a_0, ϕ_0) of (4.1.3) under certain restriction such as radial symmetry. We consider a radial version of the equation:

$$\begin{cases} \partial_t \mathbf{a} + \partial_r \mathbf{\Phi} \partial_r \mathbf{a} + \dfrac{\mathbf{a}}{2r^{n-1}} \partial_r (r^{n-1} \partial_r \mathbf{\Phi}) = 0, & \mathbf{a}_{|t=0}(r) = \mathbf{A}_0(r); \\ \partial_t \mathbf{\Phi} + \dfrac{1}{2} (\partial_r \mathbf{\Phi})^2 + \lambda V_{\mathrm{P}} = 0, & \mathbf{\Phi}_{|t=0}(r) = \mathbf{\Phi}_0(r); \\ -\partial_r (r^{n-1} \partial_r V_{\mathrm{P}}) = r^{n-1} |\mathbf{a}|^2, & V_{\mathrm{P}} \to 0 \text{ as } r \to \infty, \end{cases} \tag{4.2.1}$$

where unknowns $\mathbf{a}$ and $\mathbf{\Phi}$ are the function of $(t, r) \in \mathbb{R}_+^{1+1}$, and take complex value and real value, respectively. We have removed the index "0" from a_0 and ϕ_0, for simplicity. Through this Section 4.2 and next Section 4.3, we use the bold style characters to denote the radial functions; $\mathbf{a}$ for a, $\mathbf{A}_0$ for A_0, $\mathbf{\Phi}$ for ϕ, $\mathbf{\Phi}_0$ for Φ_0, etc. We use the function space D^k introduced in (3.2.11). Let us introduce three more function spaces:

$$\begin{aligned} D_\rho^k &:= D^k \cap L^1((0, \infty), r^{n-1} dr), \\ D_a^k &:= D^k \cap L^2((0, \infty), r^{n-1} dr) \end{aligned}$$

and

$$D_\phi^k := \begin{cases} C^1([0, \infty)) & \text{if } k = 1, \\ C^1([0, \infty)) \cap C^k((0, \infty)) & \text{if } k > 1. \end{cases}$$

The main result of this section is the following:

Theorem 4.2.1. *Suppose $\lambda < 0$ or $n \geqslant 3$. Suppose $\mathbf{A}_0 \in D_a^1$ is not identically zero and $\mathbf{\Phi}_0 \in D_\phi^3$ satisfies $\partial_r \mathbf{\Phi}_0(0) = 0$ and $\partial_r \mathbf{\Phi}_0(r) \to 0$ as $r \to \infty$. Then, the solution of (4.2.1) is global if and only if $\lambda < 0$ and $n \geqslant 3$, and the initial data is of particular form*

$$\mathbf{\Phi}_0(r) = \int_0^r \sqrt{\frac{2|\lambda|}{(n-2)s^{n-2}} \int_0^s |\mathbf{A}_0(\sigma)|^2 \sigma^{n-1} d\sigma ds + \text{const.}} \tag{4.2.2}$$

Moreover, if $\mathbf{A}_0 \in D_a^k$ *for some* $m \geqslant 1$ *and* $\mathbf{\Phi}_0$ *is given by* (4.2.2)*, then* $\mathbf{\Phi}_0 \in D_\phi^{k+2}$ *holds and the corresponding global solution*

$$\mathbf{a} \in C^2([0,\infty), D_a^k) \cap C^\infty((0,\infty), D_a^k)$$
$$\mathbf{\Phi} \in C^1([0,\infty), D_\phi^{k+2}) \cap C^\infty((0,\infty), D_\phi^{k+2})$$

are given explicitly in terms of $\mathbf{v}_0(r) = \partial_r \mathbf{\Phi}_0(r)$ *as*

$$\mathbf{a}(t, X(t,R)) = \mathbf{A}_0(R) \left(1 + \frac{n\mathbf{v}_0(R)}{2R}t\right)^{-\frac{1}{2}} \left(1 + \frac{2|\lambda| R |\mathbf{A}_0(R)|^2}{(n-2)\mathbf{v}_0(R)}t\right)^{-\frac{1}{2}},$$
$$\mathbf{\Phi}(t, X(t,R)) = \mathbf{\Phi}_0(R) + \frac{t}{2}\left(\mathbf{v}_0(R)^2 + \frac{n-2}{2}\int_0^R \frac{\mathbf{v}_0(r)^2}{r} dr\right) + g(t),$$

where $X(t,R) = R(1 + \frac{n\mathbf{v}_0(R)}{2R}t)^{2/n}$ *and* g *is a function of time given by*

$$g(t) = \begin{cases} \dfrac{2\lambda}{(n-2)(n-4)} \displaystyle\int_0^\infty \frac{|\mathbf{A}_0(r)|^2 r^2}{\mathbf{v}_0(r)} \left[\left(1 + \frac{n\mathbf{v}_0(r)}{2r}t\right)^{\frac{4}{n}-1} - 1\right] dr, & \text{if } n \neq 4, \\ -\lambda \displaystyle\int_0^\infty \frac{|\mathbf{A}_0(r)|^2 r^2}{\mathbf{v}_0(r)} \log\left(1 + \frac{2\mathbf{v}_0(r)}{r}t\right) dr, & \text{if } n = 4. \end{cases}$$

Furthermore, the solution is unique and $(a, \phi)(t,x) = (\mathbf{a}, \mathbf{\Phi})(t, |x|)$ *solves* (4.1.3) *in the distribution sense.*

Remark 4.2.2. If $n = 3$ or 4, then the integral in (4.2.2) is not integrable over $(0,\infty)$ and so $\mathbf{\Phi}_0, \mathbf{\Phi} \notin L^\infty$.

The key is Theorem 3.3.14. Let us quote it. For the equation

$$\begin{cases} \mathbf{r}_t + r^{-(n-1)}\partial_r(r^{n-1}\mathbf{r}\mathbf{v}) = 0, & \mathbf{r}(0,r) = |\mathbf{A}_0(r)|^2; \\ \mathbf{v}_t + \mathbf{v}\partial_r\mathbf{v} + \lambda\partial_r V_{\mathrm{P}} = 0, & \mathbf{v}(0,r) = \partial_r\mathbf{\Phi}_0(r); \\ -r^{-(n-1)}\partial_r(r^{n-1}V_{\mathrm{P}}) = \mathbf{r}, & V_{\mathrm{P}} \to 0 \text{ as } r \to \infty, \end{cases} \tag{4.2.3}$$

we have showed the following theorem in Chap 3:

Theorem 4.2.3 (Theorem 3.3.14)**.** *Let* $\lambda < 0$ *or* $n \geqslant 3$*. Suppose* $\mathbf{A}_0 \in D_a^0$ *is not identically zero and* $\mathbf{\Phi}_0 \in D_\phi^2$ *satisfies* $\partial_r\mathbf{\Phi}_0(0) = 0$ *and* $\partial_r\mathbf{\Phi}_0 \to 0$ *as* $r \to \infty$*. Then, the solution of* (4.2.3) *is global if and only if* $\lambda < 0$ *and* $n \geqslant 3$*, and the initial data is of particular form*

$$\mathbf{v}_0(r) := \partial_r\mathbf{\Phi}_0(r) = \sqrt{\frac{2|\lambda|}{(n-2)r^{n-2}} \int_0^r |\mathbf{A}_0(s)|^2 s^{n-1} ds}.$$

Suppose $\lambda < 0$ and $n \geqslant 3$. If $\mathbf{A}_0 \in D_a^k$ for $k \geqslant 0$ and if $\mathbf{v}_0$ is as above, then $\mathbf{v}_0 \in D^{k+1}$ and the corresponding solution is

$$\mathbf{r} \in C^2([0,\infty), D_\rho^k) \cap C^\infty((0,\infty), D_\rho^k),$$
$$\mathbf{v} \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1})$$

and given explicitly by

$$\mathbf{r}(t, X(t,R)) = |\mathbf{A}_0(R)|^2 \left(1 + \frac{n\mathbf{v}_0(R)}{2R}t\right)^{-1} \left(1 + \frac{2|\lambda|R|\mathbf{A}_0(R)|^2}{(n-2)\mathbf{v}_0(R)}t\right)^{-1},$$
$$\mathbf{v}(t, X(t,R)) = \mathbf{v}_0(R) \left(1 + \frac{n\mathbf{v}_0(R)}{2R}t\right)^{1-\frac{2}{n}},$$

where $X(t,R) = R(1+\frac{n\mathbf{v}_0(R)}{2R}t)^{2/n}$. Furthermore, it is unique in $C^2([0,\infty), D^0) \times C^1([0,\infty), D^1)$ and $(\rho(t,x), v(t,x)) = (\mathbf{r}(t,|x|), (x/|x|)\mathbf{v}(t,|x|)))$ solves (4.1.4) in the distribution sense.

4.2.1 Auxiliary system

To prove Theorem 4.2.1, we introduce following auxiliary system:

$$\begin{cases} \partial_t \mathbf{a} + \mathbf{v}\partial_r \mathbf{a} + \dfrac{\mathbf{a}}{2r^{n-1}} \partial_r (r^{n-1}\mathbf{v}) = 0, & \mathbf{a}_{|t=0} = \mathbf{A}_0, \\ \partial_t \mathbf{v} + \mathbf{v}\partial_r \mathbf{v} + \lambda \partial_r V_{\mathrm{P}} = 0, & \mathbf{v}_{|t=0} = \partial_r \mathbf{\Phi}_0, \\ -\partial_r (r^{n-1}\partial_r V_{\mathrm{P}}) = r^{n-1}|\mathbf{a}|^2, & V_{\mathrm{P}} \to 0 \text{ as } r \to \infty, \end{cases} \tag{4.2.4}$$

This is the radial version of the following system:

$$\begin{cases} \partial_t a + v \cdot \nabla a + \dfrac{1}{2} a \nabla \cdot v = 0, & a_{|t=0}(x) = A_0(x), \\ \partial_t v + v \cdot \nabla v + \lambda \nabla V_{\mathrm{P}} = 0, & v_{|t=0}(x) = \nabla \Phi_0(x), \\ -\Delta V_{\mathrm{P}} = |a|^2, & V_{\mathrm{P}} \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{4.2.5}$$

Now we have the following lemma.

Lemma 4.2.4. *Let $\mathbf{A}_0 \in D_a^k$ and $\mathbf{\Phi}_0 \in D_\phi^{k+2}$ for some $k \geqslant 0$. Then, the following three statements are equivalent;*

1. *the system (4.2.1) has a unique solution $(\mathbf{a}, \mathbf{\Phi}) \in C([0,T), D_a^k \times D_\phi^{k+2}) \cap C^1((0,T), D_a^k \times D_\phi^{k+2})$ with initial data $(\mathbf{a}, \mathbf{\Phi})_{|t=0} = (\mathbf{A}_0, \mathbf{\Phi}_0)$;*

2. *the system (4.2.4) has a unique solution $(\mathbf{a}, \mathbf{v}) \in C([0,T), D_a^k \times D^{k+1}) \cap C^1((0,T), D_a^k \times D^{k+1})$ with initial data $(\mathbf{a}, \mathbf{v})_{|t=0} = (\mathbf{A}_0, \mathbf{v}_0 := \partial_r \mathbf{\Phi}_0)$;*

3. *the radial Euler-Poisson equations (4.2.3) has a unique solution $(\mathbf{r}, \mathbf{v}) \in C([0,T), D_\rho^k \times D^{k+1}) \cap C^1((0,T), D_\rho^k \times D^{k+1})$ with initial data $(\mathbf{r}, \mathbf{v})_{|t=0} = (\mathbf{r}_0 := |\mathbf{A}_0|^2, \mathbf{v}_0)$.*

Moreover, the maximal existence times of $(\mathbf{a}, \mathbf{\Phi})$*, of* $(\mathbf{a}, \mathbf{v})$*, and of* $(\mathbf{r}, \mathbf{v})$ *are the same.*

Remark 4.2.5. It seems that the information on the amplitude of $\mathbf{A}_0$ is lost in Euler-Poisson equations (4.2.3). However, we can recover. This is due to the fact that the classical trajectory, which is defined only by the modulus of $\mathbf{A}_0$, propagates the information of the amplitude.

Proof. We first prove $2 \Rightarrow 1$. Let $(\mathbf{a}, \mathbf{v})$ be a unique solution of (4.2.4). We define

$$\mathbf{\Phi}_{\text{tmp}}(t,r) := \int_0^r \mathbf{v}(t,s)ds + \mathbf{\Phi}_0(0).$$

Note that $\mathbf{\Phi}_{\text{tmp}} \in C([0,T), D_\phi^{k+2}) \cap C^1((0,T), D_\phi^{k+2})$ and $\mathbf{\Phi}_{\text{tmp}}(0,r) = \mathbf{\Phi}_0(r)$. Integration of the second equation of (4.2.4) over $[0,r]$ gives

$$\partial_t \mathbf{\Phi}_{\text{tmp}}(t,r) = -\frac{1}{2}\mathbf{v}(t,r)^2 - \lambda V_{\text{P}}(t,r) + \frac{1}{2}\mathbf{v}(t,0)^2 + \lambda V_{\text{P}}(t,0).$$

Hence, $\mathbf{\Phi}(t,r) = \mathbf{\Phi}_{\text{tmp}}(t,r) - \int_0^t (\frac{1}{2}\mathbf{v}(s,0)^2 + \lambda V_{\text{P}}(s,0))ds$ solves the second equation of (4.2.1), so the pair $(\mathbf{a}, \mathbf{\Phi})$ is the solution of (4.2.1). Let us proceed to the uniqueness. Let $(\widetilde{\mathbf{a}}, \widetilde{\mathbf{\Phi}}) \in C([0,T), D_a^k \times D_\phi^{k+2}) \cap C^1((0,T), D_a^k \times D_\phi^{k+2})$ be another solution of (4.2.1) with initial data $(\widetilde{\mathbf{a}}, \widetilde{\mathbf{\Phi}})_{|t=0} = (\mathbf{A}_0, \mathbf{\Phi}_0)$. Then, $(\widetilde{\mathbf{a}}, \partial_r \widetilde{\mathbf{\Phi}}) \in C([0,T), D_a^k \times D^{k+1}) \cap C^1((0,T), D_a^k \times D^{k+1})$ solves (4.2.4) with initial data $(\widetilde{\mathbf{a}}, \partial_r \widetilde{\mathbf{\Phi}})_{|t=0} = (\mathbf{A}_0, \partial_r \mathbf{\Phi}_0)$. By the uniqueness of the solution to (4.2.4), we see that $(\widetilde{\mathbf{a}}, \partial_r \widetilde{\mathbf{\Phi}}) = (\mathbf{a}, \mathbf{v})$. Moreover, by definition of $\mathbf{\Phi}$,

$$\begin{aligned}
\mathbf{\Phi}(t,x) &= \int_0^r \partial_r \widetilde{\mathbf{\Phi}}(t,s)ds + \mathbf{\Phi}_0(0) - \int_0^t \left(\frac{1}{2}\partial_r \widetilde{\mathbf{\Phi}}(s,0)^2 + \lambda V_{\text{P}}(s,0)\right) ds \\
&= \widetilde{\mathbf{\Phi}}(t,r) - \widetilde{\mathbf{\Phi}}(t,0) + \widetilde{\mathbf{\Phi}}(0,0) - \int_0^t \left(\frac{1}{2}\partial_r \widetilde{\mathbf{\Phi}}(s,0)^2 + \lambda V_{\text{P}}(s,0)\right) ds \\
&= \widetilde{\mathbf{\Phi}}(t,r) - \int_0^t \left(\partial_t \widetilde{\mathbf{\Phi}}(s,0) + \frac{1}{2}\partial_r \widetilde{\mathbf{\Phi}}(s,0)^2 + V_{\text{P}}(s,0)\right) ds = \widetilde{\mathbf{\Phi}}(t,r).
\end{aligned}$$

Hence, we see that the solution of (4.2.1) is unique. This also show that the maximal existence time of $(\mathbf{a}, \mathbf{\Phi})$ is larger than or equal to that of $(\mathbf{a}, \mathbf{v})$. A similar argument shows $1 \Rightarrow 2$ and the maximal existence time of $(\mathbf{a}, \mathbf{v})$ is larger than or equal to that of $(\mathbf{a}, \mathbf{\Phi})$. We omit the details.

We next show $3 \Rightarrow 2$. Suppose that (4.2.3) has a unique solution $(\mathbf{r}, \mathbf{v}) \in C([0,T), D_\rho^k \times D^{k+1}) \cap C^1((0,T), D_\rho^k \times D^{k+1})$ with initial data $(\mathbf{r}, \mathbf{v})_{|t=0} = (\mathbf{r}_0, \mathbf{v}_0)$. We define $\mathbf{a} \in C([0,T), D_a^k) \cap C^1((0,T), D_a^k)$ by

$$\mathbf{a}(t, X(t,R)) := \mathbf{A}_0(R) \exp\left(\int_0^t \partial_r \mathbf{v}(\tau, X(\tau, r))d\tau\right)$$

with a classical trajectory X defined by

$$\frac{d}{dt}X(t,R) = \mathbf{v}(t, X(t,R)), \quad X(0,R) = 0.$$

Then, one sees that the pair $(\mathbf{a}, \mathbf{v})$ is a solution of (4.2.4) and that $|\mathbf{a}|^2 = \mathbf{r}$. We prove the uniqueness of (4.2.4). Let $(\widetilde{\mathbf{a}}, \widetilde{\mathbf{v}}) \in C([0,T), D_a^k \times D^{k+1}) \cap C^1((0,T), D_a^k \times D^{k+1})$ be another solution of (4.2.4). Then, $(\widetilde{\mathbf{r}} := |\widetilde{\mathbf{a}}|^2, \widetilde{\mathbf{v}}) \in C([0,T), D_\rho^k \times D^{k+1}) \cap C^1((0,T), D_a^k \times D^{k+1})$ solves (4.2.3) with initial data $(\widetilde{\mathbf{r}}, \widetilde{\mathbf{v}})_{|t=0} = (\mathbf{r}_0, \mathbf{v}_0)$. Note that $\widetilde{\mathbf{a}}$ is given by

$$\widetilde{\mathbf{a}}(t, \widetilde{X}(t,R)) := \mathbf{A}_0(R) \exp\left(\int_0^t \partial_r \widetilde{\mathbf{v}}(\tau, \widetilde{X}(\tau, r)) d\tau \right)$$

with a classical trajectory $\widetilde{X}$ defined by

$$\frac{d}{dt}\widetilde{X}(t,R) = \widetilde{\mathbf{v}}(t, \widetilde{X}(t,R)), \quad \widetilde{X}(0,R) = 0,$$

Since $(\widetilde{\mathbf{r}}, \widetilde{\mathbf{v}}) = (\mathbf{r}, \mathbf{v})$ by uniqueness, we see that $\widetilde{X} = X$ and so $\widetilde{\mathbf{a}} = \mathbf{a}$ holds. Hence, the solution to (4.2.4) is unique. A similar argument shows $2 \Rightarrow 3$ and the maximal existence time of $(\mathbf{r}, \mathbf{v})$ is larger than or equal to that of $(\mathbf{a}, \mathbf{v})$. We omit the details. $\square$

As a byproduct, we have the following theorem:

Theorem 4.2.6. *Suppose $\lambda < 0$ or $n \geqslant 3$. Suppose $\mathbf{A}_0 \in D_a^0$ is not identically zero and $\mathbf{v}_0 \in D^1$ satisfies $\mathbf{v}_0(0) = 0$ and $\mathbf{v}_0 \to 0$ as $r \to \infty$. Then, the solution of (4.2.4) is global if and only if $\lambda < 0$ and $n \geqslant 3$, and the initial data is of particular form*

$$\mathbf{v}_0(r) = \sqrt{\frac{2|\lambda|}{(n-2)r^{n-2}} \int_0^r |\mathbf{A}_0(s)|^2 s^{n-1} ds}.$$

Moreover, if $\mathbf{A}_0 \in D_a^k$ for some $k \geqslant 0$, then the above $\mathbf{v}_0$ belongs to D^{k+1} with $\mathbf{v}_0(r) = O(r)$ as $r \to 0$ and $\mathbf{v}_0(r) = O(r^{1-n/2})$ as $r \to \infty$, and the corresponding global solution

$$\mathbf{a} \in C^2([0,\infty), D_a^k) \cap C^\infty((0,\infty), D_a^k)$$
$$\mathbf{v} \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1})$$

are given explicitly by

$$\mathbf{a}(t, X(t,R)) = \mathbf{A}_0(R) \left(1 + \frac{n\mathbf{v}_0(R)}{2R} t\right)^{-\frac{1}{2}} \left(1 + \frac{2\lambda R |\mathbf{A}_0(R)|^2}{(n-2)\mathbf{v}_0(R)} t\right)^{-\frac{1}{2}},$$

$$\mathbf{v}(t, X(t,R)) = \mathbf{v}_0(R) \left(1 + \frac{n\mathbf{v}_0(R)}{2R} t\right)^{1-\frac{2}{n}},$$

where $X(t,R) = R(1 + \frac{n\mathbf{v}_0(R)}{2R} t)^{2/n}$. Furthermore, the solution is unique and also solves (4.2.5) in the distribution sense.

Proof. This is an immediate consequence of Theorem 4.2.3 and Lemma 4.2.4. We note two points: Firstly, $\mathbf{a}$ is given explicitly by the classical trajectory X and the indicator function Γ as

$$\mathbf{a}(t, X(t,R)) := \mathbf{A}_0(R) \exp\left(\int_0^t \partial_r \mathbf{v}(\tau, X(\tau, R)) d\tau\right) = \frac{\mathbf{A}_0(R) R^{\frac{n-1}{2}}}{X(t,R)^{\frac{n-1}{2}} \sqrt{\Gamma(t,R)}}.$$

Secondly, the solution to (4.2.5) is given by

$$(a(t,x), v(t,x)) = (\mathbf{a}(t,|x|), (x/|x|)\mathbf{v}(t,|x|)).$$

This solves (4.2.5) for $x \neq 0$ and it is continuous at $x = 0$ for all time, and so it solves (4.2.5) in the distribution sense. □

4.2.2 Proof of the theorem

We now in a position to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. The result follows from Theorem 4.2.3 and Lemma 4.2.4. The solution is global if and only if $\mathbf{\Phi}_0(r)$ is such that

$$\partial_r \mathbf{\Phi}_0(r) = \sqrt{\frac{2\lambda}{(n-2)r^{n-2}} \int_0^r |a_0(s)|^2 s^{n-1} ds}.$$

Note that, then, $a(t,r)$ is given as in the proof of Theorem 4.2.6, and that it solves (4.1.3) in the distribution sense because it solves (4.1.3) for $x \neq 0$ and it is continuous at $x = 0$. We conclude with the construction of $\mathbf{\Phi}$. As in the former part of the proof of Lemma 4.2.4, $\mathbf{\Phi}$ is given by

$$\mathbf{\Phi}(t,r) = \int_0^r \mathbf{v}(t,s) ds + \mathbf{\Phi}_0(0) - \int_0^t \left(\frac{1}{2}|\mathbf{v}(s,0)|^2 + \lambda \mathbf{\Phi}(s,0)\right) ds.$$

Combining the explicit formulae of $\mathbf{v}(t, X(t,R))$ and $X(t,R)$ in Theorem 4.2.3 (or Theorem 4.2.6) and letting $r = X(t,R)$, we see that the first term of the right hand side equals to

$$\begin{aligned}
&\int_0^R \mathbf{v}(t, X(t,s)) \partial_R X(t,s) ds \\
&= \int_0^R \left(\mathbf{v}_0(s)\left(1 + \frac{n\mathbf{v}_0(s)}{2s} t\right) + \mathbf{v}_0(s) s \partial_s \left(\frac{\mathbf{v}_0(s)}{s}\right) t\right) ds, \\
&= \int_0^R \left(\mathbf{v}_0(s) + \frac{t}{2} s^{\frac{2-n}{2}} \partial_s (s^{\frac{n-2}{2}} \mathbf{v}_0(s)^2)\right) ds \\
&= \mathbf{\Phi}_0(R) - \mathbf{\Phi}_0(0) + \frac{t}{2}\left(\mathbf{v}_0(R)^2 + \frac{n-2}{2}\int_0^R \frac{\mathbf{v}_0(s)^2}{s} ds\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathbf{\Phi}(t, X(t,R)) = & \int_0^R \mathbf{v}(t, X(t,r))\partial_R X(t,r)dr \\
& + \mathbf{\Phi}_0(0) - \int_0^t \left(\frac{1}{2}|\mathbf{v}(\tau,0)|^2 + \lambda V_{\mathrm{P}}(\tau,0)\right) d\tau \\
= & \mathbf{\Phi}_0(R) + \frac{t}{2}\left(\mathbf{v}_0(R)^2 + \frac{n-2}{2}\int_0^R \frac{\mathbf{v}_0(r)^2}{r}dr\right) - \lambda \int_0^t V_{\mathrm{P}}(\tau,0)d\tau.
\end{aligned}
$$

Notice that $\mathbf{v}(t,0) \equiv 0$. We denote $g(t) := -\lambda \int_0^t V_{\mathrm{P}}(\tau,0)d\tau$. Then, by the boundary condition $V_{\mathrm{P}} \to 0$ as $r \to \infty$, we have

$$
V_{\mathrm{P}}(t,r) = -\int_r^\infty \frac{1}{s^{n-1}} \int_0^s |\mathbf{a}(t,\sigma)|^2 \sigma^{n-1} d\sigma ds
$$

and so $g(t)$ is equal to

$$
\begin{aligned}
& \lambda \int_0^t \int_0^\infty \frac{1}{r^{n-1}} \int_0^r |\mathbf{a}(\tau,s)|^2 s^{n-1} ds\, dr\, d\tau \\
& = -\frac{\lambda}{n-2}\int_0^t\int_0^\infty s|\mathbf{a}(t,s)|^2 ds\, d\tau = -\frac{\lambda}{n-2}\int_0^t\int_0^\infty \frac{|\mathbf{A}_0(r)|^2 r^{n-1}}{X(t,r)^{n-2}} dr\, d\tau \\
& = -\frac{\lambda}{n-2}\int_0^\infty |\mathbf{A}_0(r)|^2 r \left(\int_0^t \left(1 + \frac{n\mathbf{v}_0(r)}{2r}\tau\right)^{\frac{4}{n}-2} d\tau\right) dr \\
& = \begin{cases} \dfrac{2\lambda}{(n-2)(n-4)} \displaystyle\int_0^\infty \frac{|\mathbf{A}_0(r)|^2 r^2}{\mathbf{v}_0(r)} \left[\left(1 + \frac{n\mathbf{v}_0(r)}{2r}t\right)^{\frac{4}{n}-1} - 1\right] dr, & \text{if } n \neq 4, \\ -\lambda \displaystyle\int_0^\infty \frac{|\mathbf{A}_0(r)|^2 r^2}{\mathbf{v}_0(r)} \log\left(1 + \frac{2\mathbf{v}_0(r)}{r}t\right) dr, & \text{if } n = 4, \end{cases}
\end{aligned}
$$

which completes the proof. □

4.3 Regularity of limit solution

As performed in Chapter 2, to obtain the solution $(a^\varepsilon, \phi^\varepsilon)$ of (4.1.2) and its ε-power expansion, we introduce the velocity $v^\varepsilon = \nabla\phi^\varepsilon$ and analyze the system

$$
\begin{cases} \partial_t a^\varepsilon + (v^\varepsilon \cdot \nabla)a^\varepsilon + \frac{1}{2}a^\varepsilon \nabla \cdot v^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla)v^\varepsilon + \lambda \nabla V_{\mathrm{P}}^\varepsilon = 0, \\ -\Delta V_{\mathrm{P}}^\varepsilon = |a^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x| \to \infty, \\ (a^\varepsilon(0,x), v^\varepsilon(0,x)) = (A_0^\varepsilon, \nabla\Phi_0) \end{cases} \tag{4.3.1}
$$

by regarding this as a symmetric hyperbolic system. We see in Theorem 4.2.6 that the limit radial system (4.2.4) of (4.3.1) has time-global solution if the velocity is of particular form. Namely, we would like to choose the initial data of (4.3.1) as

$$A_0^\varepsilon(x) := \mathbf{A}_0(|x|), \quad v_0(x) := \frac{x}{|x|}\mathbf{v}_0(|x|) \tag{4.3.2}$$

with $\mathbf{A}_0(r) \in D_a^k$ and

$$\mathbf{v}_0(r) = \sqrt{\frac{2|\lambda|}{(n-2)r^{n-2}}\int_0^r |\mathbf{A}_0(s)|^2 ds} \in D^{k+1}.$$

4.3.1 Choice of the function space and regularity theorem

Let us consider (4.3.1) with the data (4.3.2). Then, it holds that $A_0^\varepsilon \in L^2$ by assumption on $\mathbf{A}_0$. Since $\mathbf{v}_0(r) = O(r)$ as $r \to 0$ and $\mathbf{v}_0(r) = O(r^{1-\frac{n}{2}})$ as $r \to \infty$, we see that

$$v_0(x) \in L^p(\mathbb{R}) \quad \text{for } p \in (2^*, \infty],$$

where $r^* = nr/(n-r)$ for $r < n$. Note that, no matter how fast $\mathbf{A}_0$ decays at spatial infinity (such as $\mathbf{A}_0 \in \mathcal{S}$ or $\mathbf{A}_0 \in C_0^\infty$), the decay rate of $\mathbf{v}_0(r)$ is the same as long as $\mathbf{A}_0$ is nontrivial (and in D_a^0). Similarly, we will see in Proposition 4.3.4 that

$$\nabla v_0(x) \in L^q(\mathbb{R}) \quad \text{for } q \in (2, \infty].$$

Hence, $\nabla v_0 (= \nabla^2 \Phi_0)$ never belongs to the Sobolev space H^s and so we cannot apply Theorem 2.1.2. This lack of decay of v_0 is one of the main obstacle of this chapter.

Nevertheless, we can verify $\nabla^2 v_0 \in L^2$. According to this fact, we introduce the following function space.

Definition 4.3.1. *For $n \geqslant 3$, $s > n/2 + 1$, $p \in [1, \infty]$, and $q \in [1, \infty]$, we define a function space $Y_{p,q}^s(\mathbb{R}^n)$ by*

$$Y_{p,q}^s(\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{Y_{p,q}^s(\mathbb{R}^n)}} \tag{4.3.3}$$

with norm

$$\|\cdot\|_{Y_{p,q}^s(\mathbb{R}^n)} := \|\cdot\|_{L^p(\mathbb{R}^n)} + \|\nabla\cdot\|_{L^q(\mathbb{R}^n)} + \left\|\nabla^2\cdot\right\|_{H^{s-2}(\mathbb{R}^n)}. \tag{4.3.4}$$

We denote $Y_{p,q}^s = Y_{p,q}^s(\mathbb{R}^n)$, for short.

For $q < n$, we use the notation $q^* = nq/(n-q)$. This space $Y_{p,q}^s$ is a modification of the Zhidkov space X^s, which is defined, for $s > n/2$, by $X^s(\mathbb{R}^n) := \{f \in L^\infty(\mathbb{R}^n) | \nabla f \in H^{s-1}(\mathbb{R}^n)\}$. The Zhidkov space was introduced in [74] (see, also [26]). Roughly speaking, the exponents p and q

in $Y^s_{p,q}$ indicate the decay rates at spatial infinity of the function and of its first derivative, respectively. Moreover, the Zhidkov space X^s corresponds to $Y^s_{\infty,2}$ in a sense, if $n \geqslant 3$ (see, [62]). The difference is that all function in $Y^s_{\infty,2}$ decays at spatial infinity. We also note that $Y^s_{2,2}$ is the usual Sobolev space H^s. We use the following notation:

$$Y^\infty_{p,q} := \cap_{s>0} Y^s_{p,q}, \qquad Y^s_{I,q} := \cap_{p\in I} Y^s_{p,q}, \qquad Y^s_{p,I} := \cap_{q\in I} Y^s_{p,q},$$

where I is an an interval of $\mathbb{R}$. These notations are sometimes used simultaneously, for example $Y^\infty_{I_1,I_2} := \cap_{s>0,p\in I_1,q\in I_2} Y^s_{p,q}$. Recall that all functions in $Y^s_{p,q}$ decays at spatial infinity, by definition. Hence, if $f \in Y^s_{p,q}$ for some $s > n/2+1$, $p \geqslant 1$, and $q \in [1,\infty)$, then $\|\nabla f\|_{L^\infty} \leqslant \left\|\nabla^2 f\right\|_{H^{s-1}} < \infty$ by Sobolev embedding and so it holds that $Y^s_{p,q} = Y^s_{p,[q,\infty]}$. Similarly, we have

$$Y^s_{p,q} = Y^s_{p,[q,\infty]\cup[2^*,\infty]}, \qquad Y^s_{p,q} \subset Y^s_{q^*,q} \quad \text{if } q < n.$$

We now state the main theorem of this section:

Theorem 4.3.2. *Let $\lambda < 0$ and $n \geqslant 3$. Let $s > n/2+1$ and $\lceil s \rceil$ be the minimum integer larger than or equal to s. Let $\mathbf{A}_0 \in C^{\lceil s\rceil}([0,\infty))$ be nontrivial function satisfying*

$$\begin{aligned} &r^{j-\lceil s\rceil}\partial_r^j \mathbf{A}_0 \in L^2((0,\infty), r^{n-1}dr) && 1 \leqslant j \leqslant \lceil s \rceil, \\ &\partial_r^j \mathbf{A}_0 \in L^2((0,\infty), r^{n-1}dr) && 0 \leqslant j \leqslant \lceil s \rceil, \\ &\partial_r^j \mathbf{A}_0 = O(r^{-n/2}) \quad \text{as } r \to \infty && 0 \leqslant j \leqslant \lceil s \rceil, \end{aligned}$$

and that there exits $k_0 \geqslant \lceil s \rceil - \lceil (n-1)/2 \rceil$ such that

$$(\partial_r^j \mathbf{A}_0)(0) = 0 \quad \text{for } j \in [0, k_0 - 1], \quad (\partial_r^{k_0} \mathbf{A}_0)(0) \neq 0.$$

Define $\mathbf{\Phi}_0$ by (4.2.2). Let $(\mathbf{a}, \mathbf{\Phi})$ be the unique global solution of (4.2.1) given in Theorem 4.2.1. Then, $(a,\phi)(t,x) = (\mathbf{a},\mathbf{\Phi})(t,|x|)$ is a global solution of (4.1.3) satisfying $a(t) \in H^s$, $\nabla\phi(t) \in Y^{s+1}_{(2^*,\infty],(2,\infty]}$ for all $t \geqslant 0$. Moreover, $\phi(t) \in L^\infty$ for all $t \geqslant 0$ if and only if $n \geqslant 5$.

In the rest of this section, we prove this theorem: We confirm that $v_0 \in Y^s_{(2^*,\infty],(2,\infty]}$ holds for a good A_0 (Section 4.3.3), and moreover the corresponding global solution (a,v) of (4.2.5) given explicitly in Theorem 4.2.6 enjoys the same property for all $t \geqslant 0$ (Section 4.3.4).

4.3.2 Preliminary lemma

Before the proof of Theorem 4.3.2, we state a preliminary lemma. This is the key tool for investigation of higher derivatives of ϕ. This reflects the special form of $\mathbf{v}_0$.

Lemma 4.3.3. *Let $n \geqslant 3$ and $\lambda < 0$. Suppose $\mathbf{A}_0 \in C^m([0,\infty))$ for large m. Define*

$$\mathbf{v}_0(r) = \sqrt{\frac{2|\lambda|}{(n-2)r^{n-2}} \int_0^r |\mathbf{A}_0(s)|^2 s^{n-1} ds.}$$

Then, the following equality holds for $k \in [1, m+1]$:

$$\sum_{j=0}^{k} \alpha_{j,k} r^j \mathbf{v}_0^{(j)} = \sum_{l=1}^{k} \sum_{m \in (\mathbb{N}\cup\{0\})^l, |m| \leqslant k-l} \beta_{l,m_i,k} \frac{\left(\prod_{i=1}^{l} \rho_0^{(m_i)}\right) r^{2l+|m|}}{\mathbf{v}_0^{2l-1}}, \quad (4.3.5)$$

where $\rho_0 = |\mathbf{A}_0|^2$, $g^{(m)}$ denotes the m-th derivative of g with $g^{(0)} = g$, and α and β are real constants with $\alpha_{k,k} = 1$. Moreover, $\alpha_{0,k} = 0$ for $k \geqslant 2$.

Proof. By definition, $\mathbf{v}_0$ satisfies

$$\frac{n-2}{2}\mathbf{v}_0 + r\mathbf{v}_0' = \frac{|\lambda|}{n-2} \frac{\rho_0 r^2}{\mathbf{v}_0}. \quad (4.3.6)$$

This implies that (4.3.5) holds if $k = 1$. Suppose (4.3.5) is true for $k = k_0 < m$. Differentiate (4.3.5) with respect to R and multiply by R. Then, the left hand side becomes

$$\sum_{j=0}^{k_0} \alpha_{j,k_0} j r^j \mathbf{v}_0^{(j)} + \sum_{j=1}^{k_0+1} \alpha_{j-1,k_0} r^j \mathbf{v}_0^{(j)},$$

which can be expressed as $\sum_{j=0}^{k_0+1} \alpha_{j,k_0+1} r^j \mathbf{v}_0^{(j)}$. Note that $\alpha_{k_0+1,k_0+1} = \alpha_{k_0,k_0} = 1$ and $\alpha_{0,k_0+1} = 0$. Moreover, the same operation makes the right hand side as

$$\begin{aligned}\sum_{l=1}^{k_0} \sum_{m \in (\mathbb{N}\cup\{0\})^l, |m| \leqslant k_0-l} \beta_{l,m_i,k_0} \Bigg[&\frac{\sum_{j=1}^{l} \left(\prod_{i=1}^{l} \rho_0^{(m_i+\delta_{ij})}\right) r^{2l+(|m|+1)}}{\mathbf{v}_0^{2l-1}} \\ &+ (2l+|m|) \frac{\left(\prod_{i=1}^{l} \rho_0^{(m_i)}\right) r^{2l+|m|}}{\mathbf{v}_0^{2l-1}} \\ &+ (1-2l) \frac{\left(\prod_{i=1}^{l} \rho_0^{(m_i)}\right) r^{2l+|m|}}{\mathbf{v}_0^{2l}} \left(-\frac{n-2}{2}\mathbf{v}_0 + \frac{|\lambda|}{n-2} \frac{\rho_0 r^2}{\mathbf{v}_0} \right) \Bigg],\end{aligned}$$

where we have used (4.3.6). This can be written as

$$\sum_{l=1}^{k_0+1} \sum_{m \in (\mathbb{N}\cup\{0\})^l, |m| \leqslant k_0+1-l} \beta_{l,m_i,k_0+1} \frac{\left(\prod_{i=1}^{l} \rho_0^{(m_i)}\right) r^{2l+|m|}}{\mathbf{v}_0^{2l-1}}.$$

Hence, (4.3.5) holds by induction. $\square$

4.3.3 Regularity at the initial time

Proposition 4.3.4. *Let $s > n/2 + 1$ and $\lceil s \rceil$ denote the minimum integer bigger than or equal to s. Assume $\mathbf{A}_0 \in C^{\lceil s \rceil}([0,\infty))$ is nontrivial and satisfies*

$$\begin{aligned} &r^{j-\lceil s \rceil}\partial_r^j \mathbf{A}_0 \in L^2((0,\infty), r^{n-1}dr) && 1 \leqslant j \leqslant \lceil s \rceil\,, \\ &\partial_r^j \mathbf{A}_0 \in L^2((0,\infty), r^{n-1}dr) && 0 \leqslant j \leqslant \lceil s \rceil\,, \\ &\partial_r^j \mathbf{A}_0 = O(r^{-n/2}) \quad \text{as } r \to \infty && 0 \leqslant j \leqslant \lceil s \rceil\,. \end{aligned}$$

Assume that there exits $k_0 \geqslant \lceil s \rceil - \lceil (n-1)/2 \rceil$ such that

$$(\partial_r^{k_0}\mathbf{A}_0)(0) \neq 0, \quad (\partial_r^j \mathbf{A}_0)(0) = 0 \quad \text{for } j \in [0, k_0 - 1].$$

Let

$$\mathbf{v}_0(r) = \sqrt{\frac{2\lambda}{(n-2)r^{n-2}} \int_0^r |\mathbf{A}_0(s)|^2 s^{n-1} ds}$$

and (A_0^ε, v_0) be defined as in (4.3.2). Then, $A_0^\varepsilon \in H^s(\mathbb{R}^n)$ and $v_0 \in Y^{s+1}_{(2^*,\infty],(2,\infty]}(\mathbb{R}^n)$. In particular, it holds that

$$\mathbf{v}_0(r) = \begin{cases} O\left(r^{k_0+1}\right) & \text{as } r \to 0, \\ O\left(r^{1-n/2}\right) & \text{as } r \to \infty, \end{cases} \tag{4.3.7}$$

and so that $\mathbf{v}_0 \in L^p((0,\infty), r^{n-1}dr)$ for all $p \in (2^*, \infty]$. Moreover,

$$\partial_r \mathbf{v}_0(r) = \begin{cases} O\left(r^{k_0}\right) & \text{as } r \to 0, \\ O\left(r^{-n/2}\right) & \text{as } r \to \infty, \end{cases} \tag{4.3.8}$$

and so $\partial_r \mathbf{v}_0 \in L^q((0,\infty), r^{n-1}dr)$ for all $q \in (2,\infty]$. Furthermore, for all $k \in [2, \lceil s \rceil + 1]$, it holds that $\partial_r^k \mathbf{v}_0 \in L^2((0,\infty), r^{n-1}dr)$ and that

$$\partial_r^k \mathbf{v}_0(r) = \begin{cases} O\left(r^{k_0+1-k}\right) & \text{as } r \to 0, \\ O\left(r^{-n/2}\right) & \text{as } r \to \infty. \end{cases} \tag{4.3.9}$$

Proof. We first note $A_0^\varepsilon \in L^2(\mathbb{R}^n)$ follows from $\mathbf{A}_0 \in L^2((0,\infty), r^{n-1}dr)$. Moreover, for $k \in [1, \lceil s \rceil]$, we have

$$\begin{aligned} \left\|\nabla^k A_0^\varepsilon\right\|_{L^2(\mathbb{R}^n)} &\leqslant C \sum_{j=1}^k \left\| |\cdot|^{j-k} (\partial_r^j \mathbf{A}_0)(|\cdot|)\right\|_{L^2((0,\infty),r^{n-1}dr)} \\ &\leqslant C \sum_{j=1}^k \left\| |\cdot|^{j-\lceil s \rceil} (\partial_r^j \mathbf{A}_0)(|\cdot|)\right\|_{L^2((0,1),r^{n-1}dr)} \\ &\quad + C \sum_{j=1}^k \left\| (\partial_r^j \mathbf{A}_0)(|\cdot|)\right\|_{L^2((1\infty),r^{n-1}dr)} < \infty. \end{aligned}$$

Therefore, $A_0 \in H^{\lceil s \rceil}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$. Let us prove that $v_0 \in Y^{s+1}_{(2^*,\infty],(2,\infty]}(\mathbb{R}^n)$ and (4.3.7)–(4.3.9) hold. The proof proceeds in three steps.

Step 1. We first show (4.3.7) and $v_0 \in L^p(\mathbb{R}^n)$ for $p \in (2^*, \infty]$. Since $|\mathbf{A}_0(r)|^2 = O(r^{2k_0})$ as $r \to 0$ by assumption, $\mathbf{v}_0(r) = O(r^{k_0+1})$ as $r \to 0$. On the other hand, since $\mathbf{A}_0$ is nontrivial and in $L^2((0,\infty), r^{n-1}dr)$, there exist two positive constants c and C such that

$$cr^{1-\frac{n}{2}} \leqslant \mathbf{v}_0(r) \leqslant Cr^{1-\frac{n}{2}} \tag{4.3.10}$$

for large r. Therefore, $\sup_{r\geqslant 0} \mathbf{v}_0(r) < \infty$ and

$$\int_0^\infty |\mathbf{v}_0(s)|^p s^{n-1} ds < \infty$$

if $p(1-n/2) + n - 1 < -1$, that is, if $p > 2^*$. Hence, (4.3.7) holds and $v_0 \in L^p(\mathbb{R}^n)$ for $p \in (2^*, \infty]$.

Step 2. We next show (4.3.8) and $\nabla v_0 \in L^q(\mathbb{R}^n)$ for $q \in (2, \infty]$. Note that

$$\|\nabla v_0\|^q_{L^q(\mathbb{R}^n)} \leqslant C \sum_{j=0}^{1} \int_0^\infty \left| r^{j-1} \partial_r^j \mathbf{v}_0(r) \right|^q r^{n-1} dr.$$

Since $r^{-1}\mathbf{v}_0(r)$ is $O(r^{k_0})$ as $r \to 0$ and $O(r^{-n/2})$ as $r \to \infty$ by (4.3.7), we have $\int_0^\infty |s^{-1}\mathbf{v}_0(s)|^q s^{n-1} ds < \infty$ for $q(-n/2)+n-1 < -1$, that is, for $q > 2$. It also holds that $\sup_{r\geqslant 0}(\mathbf{v}_0(r)/r) < \infty$. We now use (4.3.6) to obtain

$$\mathbf{v}_0'(r) = \alpha \frac{\mathbf{v}_0(r)}{r} + \beta \frac{|\mathbf{A}_0(r)|^2 r}{\mathbf{v}_0(r)}$$

with suitable constants α and β. As shown above, the first term belongs to $L^q((0,\infty), r^{n-1}dr)$ for $q \in (2,\infty]$. The second term in the right hand side is $O(r^{k_0})$ as $r \to 0$. Moreover, $|\mathbf{A}_0(r)|^2 r^n$ is bounded for large r by assumption, and so the second term is $O(r^{-n/2})$ as $r \to \infty$. Thus, $\mathbf{v}_0'(r)$ has the same decay order as $\mathbf{v}_0(r)/r$ and so belongs to $L^q((0,\infty), r^{n-1}dr)$ for $q \in (2,\infty]$, which completes the proof of (4.3.8) and of $\nabla v_0 \in L^q(\mathbb{R}^n)$ for $q \in (2,\infty]$.

Step 3. We finally show (4.3.9) and $\nabla^k v_0 \in L^2(\mathbb{R}^n)$ for all $k \in [2, \lceil s \rceil + 1]$. An elementary computation shows that

$$\left\| \nabla^k v_0 \right\|_{L^2(\mathbb{R}^n)} \leqslant C \sum_{j=0}^{k} \left\| \frac{\partial_r^j \mathbf{v}_0(r)}{r^{k-j}} \right\|_{L^2((0,\infty), r^{n-1}dr)}.$$

Hence, it suffices to show that

$$\partial_r^j \mathbf{v}_0 \in L^2((1,\infty), r^{n-1}dr) \quad \text{for } j \in [2, \lceil s \rceil + 1], \tag{4.3.11}$$

$$|\cdot|^{j-\lceil s\rceil-1}\partial_r^j \mathbf{v}_0 \in L^2((0,1),r^{n-1}dr) \quad \text{for } j\in[0,\lceil s\rceil+1], \tag{4.3.12}$$

and

$$\partial_r^j \mathbf{v}_0(r) = O(r^{k_0+1-j}) \quad \text{as } r\to 0, \tag{4.3.9a}$$

$$\partial_r^j \mathbf{v}_0(r) = O(r^{-n/2}) \quad \text{as } r\to \infty \tag{4.3.9b}$$

for all $j\in[2,\lceil s\rceil+1]$.

We now prove (4.3.11) and (4.3.9b) by induction. In the followings, we denote $|\mathbf{A}_0|^2$ by ρ_0, for simplicity. It follows from (4.3.5) with $k=2$ that

$$\mathbf{v}_0''(r) = -\alpha_{1,2}\frac{\mathbf{v}_0'(r)}{r} + \beta_1\frac{\rho_0(r)}{\mathbf{v}_0(r)} + \beta_2\frac{\rho_0'(r)r}{\mathbf{v}_0(r)} + \beta_3\frac{\rho_0(r)^2r^2}{\mathbf{v}_0(r)^3}$$

with suitable coefficients. It follows from (4.3.8) that $\mathbf{v}_0'/r\in L^2((1,\infty),r^{n-1}dr)$ and $\mathbf{v}_0'/r = O(r^{-n/2})$ as $r\to\infty$. By (4.3.10), we also have

$$\begin{aligned}
\int_1^\infty \frac{\rho_0(r)^2}{\mathbf{v}_0(r)^2}r^{n-1}dr &\leqslant C\int_1^\infty \rho_0(r)^2r^{2n-3}dr\\
&\leqslant C\sup_{r\geqslant 1}(\rho_0(r)r^n)\int_1^\infty \rho_0(r)r^{n-1}dr<\infty,\\
\frac{\rho_0(r)}{\mathbf{v}_0(r)}r^{\frac{n}{2}} &= O\left(\frac{\rho_0(r)r^n}{r}\right) = O(r^{-1}) \quad (r\to\infty),
\end{aligned}$$

$$\begin{aligned}
\int_1^\infty \frac{\rho_0'(r)^2r^2}{\mathbf{v}_0(r)^2}r^{n-1}dr &\leqslant C\int_1^\infty \rho_0'(r)^2r^{2n-1}dr\\
&\leqslant C\sup_{r\geqslant 1}(|\rho_0'(r)|r^n)\int_1^\infty |\rho_0'(r)|r^{n-1}dr<\infty,\\
\frac{\rho_0'(r)r}{\mathbf{v}_0(r)}r^{\frac{n}{2}} &= O\left(\rho_0'(r)r^n\right) = O(1) \quad (r\to\infty),
\end{aligned}$$

and

$$\begin{aligned}
\int_1^\infty \frac{\rho_0(r)^4r^4}{\mathbf{v}_0(r)^6}r^{n-1}dr &\leqslant C\int_1^\infty \rho_0(r)^4r^{4n-3}dr\\
&\leqslant C(\sup_{r\geqslant 1}(\rho_0(r)r^n))^3\int_1^\infty \rho_0(r)r^{n-1}dr<\infty,\\
\frac{\rho_0(r)^2r^2}{\mathbf{v}_0(r)^3}r^{\frac{n}{2}} &= O\left(\frac{(\rho_0(r)r^n)^2}{r}\right) = O(r^{-1}) \quad (r\to\infty).
\end{aligned}$$

Therefore, $\mathbf{v}_0''\in L^2((1,\infty),r^{n-1}dr)$ and $\mathbf{v}_0''=O(r^{-n/2})$ as $r\to\infty$.

We now take $j_0\in[2,\lceil s\rceil]$ and suppose for induction that $\mathbf{v}_0^{(j)}$ is in $L^2((1,\infty),r^{n-1}dr)$ and $\mathbf{v}_0^{(j)}=O(r^{-n/2})$ as $r\to\infty$ hold for all $j\in[2,j_0]$.

By (4.3.5) with $k = j_0 + 1$, we have

$$\mathbf{v}_0^{(j_0+1)}(r) = -\sum_{j=1}^{j_0} \alpha_{j,j_0+1} \frac{\mathbf{v}_0^{(j)}(r)}{r^{j_0+1-j}} + \sum_{l=1}^{j_0+1} \sum_{m\in(\mathbb{N}\cup\{0\})^l, |m|\leqslant j_0+1-l} \beta_{l,m_i,j_0+1} \frac{\left(\prod_{i=1}^l \rho_0^{(m_i)}(r)\right) r^{2l+|m|-j_0-1}}{\mathbf{v}_0(r)^{2l-1}}.$$

Notice that $r^{-j_0}\mathbf{v}_0' \in L^2((1,\infty), r^{n-1}dr)$ and $r^{-j_0}\mathbf{v}_0' = O(r^{-n/2})$ as $r \to \infty$ follow as in the previous $j_0 = 2$ case. Similarly, since $\mathbf{v}_0^{(j)} \in L^2((1,\infty), r^{n-1}dr)$ and $\mathbf{v}_0^{(j)} = O(r^{-n/2})$ as $r \to \infty$ hold for $j \in [2, j_0]$ by assumption of induction, $\mathbf{v}_0^{(j)} r^{-j_0-1+j}$ also belongs to $L^2((1,\infty), r^{n-1}dr)$ and is order $O(r^{-n/2})$ as $r \to \infty$ for $j \in [2, j_0]$. Moreover, for all $l \in [1, j_0+1]$ and $m \in (\mathbb{N}\cup\{0\})^l$ with $|m| \leqslant j_0 + 1 - l$, it holds that

$$\frac{\left(\prod_{i=1}^l \rho_0^{(m_i)}(r)\right) r^{2l+|m|-j_0-1}}{\mathbf{v}_0(r)^{2l-1}} r^{\frac{n}{2}} = O\left(\left(\prod_{i=1}^l \rho_0^{(m_i)}(r) r^n\right) r^{|m|-j_0}\right) = O(1)$$

and that

$$\begin{aligned}
&\int_1^\infty \frac{\left(\prod_{i=1}^l \left(\rho_0^{(m_i)}(r)\right)^2\right) r^{4l+2|m|-2j_0-2}}{\mathbf{v}_0(r)^{4l-2}} r^{n-1} dr \\
&\leqslant C \int_1^\infty \left(\prod_{i=1}^l \left(\rho_0^{(m_i)}\right)^2\right) r^{2-2l} r^{(2l-1)n} r^{n-1} dr \\
&\leqslant C \left(\int_1^\infty |\rho_0^{(m_1)}| r^{n-1} dr\right) \left(\sup_{r\geqslant 1} \left(|\rho_0^{(m_1)}| r^n\right)\right) \prod_{i=2}^l \left(\sup_{r\geqslant 1} \left(|\rho_0^{(m_i)}| r^n\right)\right)^2 < \infty,
\end{aligned}$$

where we have used (4.3.10). Hence, (4.3.11) and (4.3.9b) hold by induction.

Let us proceed to the proof of (4.3.12) and (4.3.9a). (4.3.9a) is equivalent to

$$\frac{\partial_r^j \mathbf{v}_0(r)}{r^{\lceil s\rceil+1-j}} = O(r^{k_0-\lceil s\rceil}) \quad \text{as } r \to 0 \quad \text{for } j \in [2, \lceil s\rceil + 1]. \tag{4.3.13}$$

Recall that $\mathbf{v}_0(r) = O(r^{k_0+1})$ and $\mathbf{v}_0'(r) = O(r^{k_0})$ as $r \to 0$, and so that (4.3.13) is true for $j = 0, 1$. Then, by the assumption that $k_0 - \lceil s\rceil > -n/2$, (4.3.12) immediately follows from (4.3.13). Hence, it suffices to prove (4.3.13). Take $j_1 \in [1, \lceil s\rceil]$ and suppose that (4.3.13) holds for all $j \in [0, j_1]$.

Then, by (4.3.5) with $k = j_1 + 1$, we have

$$\frac{\mathbf{v}_0^{(j_1+1)}(r)}{r^{\lceil s\rceil - j_1}} = -\sum_{j=1}^{j_1} \alpha_{j,j_1+1} \frac{\mathbf{v}_0^{(j)}(r)}{r^{\lceil s\rceil + 1 - j}}$$
$$+ \sum_{l=1}^{j_1+1} \sum_{m\in(\mathbb{N}\cup\{0\})^l, |m|\leqslant j_1+1-l} \beta_{l,m_i,j_1+1} \frac{\left(\prod_{i=1}^l \rho_0^{(m_i)}(r)\right) r^{2l+|m|-\lceil s\rceil - 1}}{\mathbf{v}_0(r)^{2l-1}}.$$

The first sum is $O(r^{k_0-\lceil s\rceil})$ as $r \to 0$ by assumption of induction. Recall that $\rho_0(r) = O(r^{2k_0})$ as $r \to 0$. This implies $r^j \rho_0^{(j)}(r) = O(r^{2k_0})$ as $r \to 0$ for all $j \geqslant 0$. By this fact, the second sum is also $O(r^{k_0-\lceil s\rceil})$ as $r \to 0$ because, for each $l \in [1, j_1 + 1]$ and $m \in (\mathbb{N} \cup \{0\})^l$ with $|m| \leqslant j_1 + 1 - l$, we have

$$\frac{\left(\prod_{i=1}^l \rho_0^{(m_i)}(r)\right) r^{2l+|m|-\lceil s\rceil-1}}{\mathbf{v}_0(r)^{2l-1}} = \frac{\left(\prod_{i=1}^l r^{m_i}\rho_0^{(m_i)}(r)\right) r^{2l-\lceil s\rceil-1}}{\mathbf{v}_0(r)^{2l-1}} = O(r^{k_0-\lceil s\rceil})$$

as $r \to 0$. Therefore, (4.3.13) holds by induction. □

4.3.4 Persistence of the regularity

We next show that the (4.2.4) keeps the same regularity as the initial data for all positive time, thanks to its explicit representation.

Proposition 4.3.5. *Under the same assumption as in Proposition 4.3.4, let* $(\mathbf{A}(t,r), \mathbf{v}(t,r))$ *be the global solution of* (4.2.4) *given by Theorem 4.2.6:*

$$\mathbf{A}(t, X(t,R)) = \mathbf{A}_0(R)\left(1 + \frac{n\mathbf{v}_0(R)}{2R}t\right)^{-\frac{1}{2}}\left(1 + \frac{2\lambda|\mathbf{A}_0(R)|^2 R}{(n-2)\mathbf{v}_0(R)}t\right)^{-\frac{1}{2}},$$
$$\mathbf{v}(t, X(t,R)) = \mathbf{v}_0(R)\left(1 + \frac{n\mathbf{v}_0(R)}{2R}t\right)^{1-\frac{2}{n}}$$

with $X(t,R) = R(1 + \frac{n\mathbf{v}_0(R)}{2R}t)^{2/n}$. *Then, the corresponding global solution*

$$a(t,x) = \mathbf{A}(t,|x|), \quad v(t,x) = \frac{x}{|x|}\mathbf{v}(t,|x|) \tag{4.3.14}$$

of (4.2.5) *belongs to the space* $H^s(\mathbb{R}^n) \times Y^{s+1}_{(2^*,\infty],(2,\infty]}(\mathbb{R}^n)$ *for all* $t \geqslant 0$.

Proof. First of all, we put

$$F(R) := \frac{n\mathbf{v}_0(R)}{2R} \geqslant 0, \quad G(R) := \frac{2|\lambda||\mathbf{A}_0(R)|^2 R}{(n-2)\mathbf{v}_0(R)} = \mathbf{v}_0'(R) + \frac{(n-2)\mathbf{v}_0(R)}{2R} \geqslant 0.$$

Then, it simplifies the notations into

$$\begin{aligned}\mathbf{A}(t,X(t,R)) &= \mathbf{A}_0(R)(1+F(R)t)^{-1/2}(1+G(R)t)^{-1/2},\\ \mathbf{v}(t,X(t,R)) &= \mathbf{v}_0(R)(1+F(R)t)^{1-2/n},\\ X(t,R) &= R(1+F(R)t)^{2/n},\\ \Gamma(t,R) := \partial_R X(t,R) &= (1+F(R)t)^{2/n-1}(1+G(R)t).\end{aligned}$$

Moreover, (4.3.7)–(4.3.9) give

$$\partial_R^j F(R) = O(R^{k_0-j}), \quad \partial_R^j G(R) = O(R^{k_0-j}) \tag{4.3.15}$$

as $R\to 0$ for all $j\in[0,\lceil s\rceil+1]$. We also have

$$\partial_R^j F(R) = O\left(R^{-\frac{n}{2}-\min(j,1)}\right) \qquad \partial_R^j G(R) = O\left(R^{-\frac{n}{2}}\right) \tag{4.3.16}$$

as $R\to\infty$.

Step 1. We show $a(t,\cdot)\in H^s(\mathbb{R}^n)$. Let us claim that

$$\begin{aligned}(\partial_r^k\mathbf{A})(t,X(t,R)) = &\sum_{l_i\geqslant 0, l_1+l_2+l_3\leqslant k}\ \sum_{\substack{m_i\in(\mathbb{N}\cup\{0\})^{l_i},\\ |m_1|+|m_2|=k-l_1-l_2-l_3}}\\ &C_{k,l_i,m_1,m_2}(\partial_r^{l_3}\mathbf{A}_0)(R)\big(1+F(R)t\big)^{-\frac12+k\left(1-\frac{2}{n}\right)-l_1}\big(1+G(R)t\big)^{-\frac12-k-l_2}t^{l_1+l_2}\\ &\prod_{i_1=1}^{l_1}\partial_r^{1+m_{1i_1}}F(R)\prod_{i_2=1}^{l_2}\partial_r^{1+m_{2i_2}}G(R),\end{aligned} \tag{4.3.17}$$

where we let $\prod_{i_1=1}^{l_1}\partial_r^{1+m_{1i_1}}F(R)=1$ and $|m_1|=0$ if $l_1=0$, the similar rule is applied to the case $l_2=0$. By definition of $\mathbf{A}$, (4.3.17) is true if $k=0$. Then, differentiate with respect to r and multiply by Γ^{-1} to obtain

$$\begin{aligned}(\partial_r\mathbf{A})(t,X(t,R)) = &\,\partial_r\mathbf{A}_0(R)(1+F(R)t)^{\frac12-\frac2n}(1+G(R)t)^{-\frac32}\\ &-\frac12\mathbf{A}_0(R)(1+F(R)t)^{-\frac12-\frac2n}(1+G(R)t)^{-\frac32}\partial_r F(R)t\\ &-\frac12\mathbf{A}_0(R)(1+F(R)t)^{\frac12-\frac2n}(1+G(R)t)^{-\frac52}\partial_r G(R)t.\end{aligned}$$

Repeating this operation, we obtain (4.3.17) by induction. Since $\mathbf{A}_0(t,X(t,R))$ is written as $\mathbf{A}_0(R)R^{n-1}/X(t,R)^{n-1}\Gamma(t,R)$, one verifies that the L^2 norm is conserved: $\|a(t)\|_{L^2(\mathbb{R}^n)}=\|a_0\|_{L^2(\mathbb{R}^n)}$. For $k\geqslant 1$, we have

$$\left\|\nabla^k a(t)\right\|_{L^2(\mathbb{R}^n)}^2 \leqslant C\sum_{j=1}^k\int_0^\infty\left|\frac{(\partial_r^j\mathbf{A})(t,X(t,R))}{X(t,R)^{k-j}}\right|^2 X(t,R)^{n-1}\Gamma(t,R)dR. \tag{4.3.18}$$

Recall that $F, G \geqslant 0$ and $\sup_{r\geqslant 0}(|F(r)| + |G(r)|) < \infty$. Hence, it suffices to show that

$$\int_0^\infty \frac{|\partial_r^{l_3}\mathbf{A}_0(R)|^2}{R^{2k-2j-n+1}} \prod_{i_1=1}^{l_1} (\partial_r^{1+m_{1i_1}} F)(R)^2 \prod_{i_2=1}^{l_2} (\partial_r^{1+m_{2i_2}} G)(R)^2 dR < \infty \quad (4.3.19)$$

for each $k \in [1, \lceil s \rceil]$, $j \in [1, k]$, $l_i \geqslant 0$ ($i = 1, 2, 3$) with $l_1 + l_2 + l_3 \leqslant j$, and $m_i \in (\mathbb{N} \cup \{0\})^{l_i}$ ($i = 1, 2$) with $|m_1| + |m_2| = j - l_1 - l_2 - l_3$. By (4.3.15),

$$\frac{|\partial_r^{l_3}\mathbf{A}_0|^2}{r^{2k-2j-n+1}} \prod_{i_1=1}^{l_1} |\partial_r^{1+m_{1i_1}} F|^2 \prod_{i_2=1}^{l_2} |\partial_r^{1+m_{2i_2}} G|^2 = O(r^{2k_0-2\lceil s\rceil+n-1})$$

as $r \to 0$ and so this is integrable around $r = 0$ by the choice of k_0. On the other hand, (4.3.16) gives

$$\sup_{r\geqslant 1} r^{-2k+2j} \prod_{i_1=1}^{l_1} (\partial_r^{1+m_{1i_1}} F)(r)^2 \prod_{i_2=1}^{l_2} (\partial_r^{1+m_{2i_2}} G)(r)^2 < \infty,$$

and so we conclude that (4.3.19) follows from the integrability property of $\mathbf{A}_0$ (Proposition 4.3.4).

Step 2. We show $v \in Y^{s+1}_{(2^*,\infty].(2,\infty]}(\mathbb{R}^n)$. Since $\sup_{r\geqslant 0}(|F(r)| + |G(r)|) < \infty$, we see that $\|v(t)\|_{L^p(\mathbb{R}^n)} \leqslant C_t \|v(0)\|_{L^p(\mathbb{R}^n)} < \infty$ for all $t \geqslant 0$ and $p \in (2^*, I]$. Similarly, we have

$$\begin{aligned}\|\nabla v(t)\|^q_{L^q(\mathbb{R}^n)} &\leqslant C \int_0^\infty \left|\frac{\mathbf{v}(t,X)}{X}\right|^q X^{n-1}\Gamma dR + C\int_0^\infty |\mathbf{v}'(t,X)|^q X^{n-1}\Gamma dR \\ &\leqslant C_t \int_0^\infty |\mathbf{v}_0(R)|^q R^{n-q-1} dR + C_t \int_0^\infty |\mathbf{v}_0'(R)|^q R^{n-1} dR \\ &\quad + C_t \int_0^\infty |\mathbf{v}_0(R)|^q |\partial_r F(R)|^q R^{n-1} dR.\end{aligned}$$

By (4.3.7), (4.3.8), (4.3.15), and (4.3.16), one sees that the right hand side is finite. Now, let us prove $\nabla^k v \in L^2(\mathbb{R}^n)$ for $k \in [2, \lceil s \rceil + 1]$. Note that

$$\left\|\nabla^k v\right\|^2_{L^2(\mathbb{R}^n)} \leqslant C \sum_{j=0}^k \int_0^\infty \left|\frac{\partial_r^j \mathbf{v}(t, X(t,R))}{X(t,R)^{k-j}}\right|^2 X^{n-1}(t,R)\Gamma(t,R) dR. \quad (4.3.20)$$

The same calculation as in (4.3.17) shows the following identity:

$$\begin{aligned}(\partial_r^k \mathbf{v})(t, X(t,R)) = &\sum_{l_i\geqslant 0, l_2<k, l_1+l_2+l_3\leqslant k} \; \sum_{m_i\in(\mathbb{N}\cup\{0\})^{l_i}, |m_1|+|m_2|=k-l_1-l_2-l_3} \\ &C_{k,l_i,m_1,m_2} (\partial_r^{l_3}\mathbf{v}_0)(R)(1+F(R)t)^{(k+1)\left(1-\frac{2}{n}\right)-l_1}(1+G(R)t)^{-k-l_2} t^{l_1+l_2} \\ &\prod_{i_1=1}^{l_1} \partial_r^{1+m_{1i_1}} F(R) \prod_{i_2=1}^{l_2} \partial_r^{1+m_{2i_2}} G(R). \quad (4.3.21)\end{aligned}$$

Therefore, our task is to prove that

$$\int_0^\infty \frac{|\partial_r^{l_3}\mathbf{v}_0(R)|^2}{R^{2k-2j-n+1}} \prod_{i_1=1}^{l_1} (\partial_r^{1+m_{1i_1}} F)(R)^2 \prod_{i_2=1}^{l_2} (\partial_r^{1+m_{2i_2}} G)(R)^2 dR < \infty \quad (4.3.22)$$

for each $k \in [2, \lceil s \rceil]$, $j \in [0, k]$, $l_i \geqslant 0$ $(i = 1, 2, 3)$ with $l_2 < j$ and $l_1+l_2+l_3 \leqslant j$, and $m_i \in (\mathbb{N} \cup \{0\})^{l_i}$ $(i = 1, 2)$ with $|m_1| + |m_2| = j - l_1 - l_2 - l_3$. We divide $\int_0^\infty = \int_0^1 + \int_1^\infty$ and denote the left hand side of (4.3.22) as $I_1 + I_2$. By (4.3.7)–(4.3.9) and (4.3.15), the integrand of I_1 is

$$\begin{aligned}
&O\left((r^{k_0+1-l_3})^2 r^{-2k+2j+n-1} \prod_{i_1=1}^{l_1} (r^{k_0-1-m_{1i_1}})^2 \prod_{i_2=1}^{l_2} (r^{k_0-1-m_{2i_2}})^2 \right) \\
&= O(r^{2k_0-2(k-1)+n-1+2k_0(l_1+l_2)}) = O(r^{2k_0-2\lceil s \rceil+n-1})
\end{aligned}$$

as $r \to 0$, and so this is integrable near $r = 0$ thanks to the choice of k_0. Hence, $I_1 < \infty$. We finally show that $I_2 < \infty$. Suppose $l_3 = 0$ or 1. Then, it automatically holds that $l_1 + l_2 \geqslant 1$ because, otherwise $0 = |m_1| + |m_2| \neq j - l_1 - l_2 - l_3 \geqslant 1$. Then, by (4.3.7) and (4.3.16), we deduce that the integrand of I_2 is

$$\begin{aligned}
&O\left((r^{1-\frac{n}{2}-l_3})^2 r^{-2k+2j+n-1} \prod_{i_1=1}^{l_1} (r^{-\frac{n}{2}-1})^2 \prod_{i_2=1}^{l_2} (r^{-\frac{n}{2}})^2 \right) \\
&\qquad = O(r^{-n+1-2(k-j)-2(l_1+l_3)-n(l_1+l_2-1)}) = O(r^{-n+1})
\end{aligned}$$

as $r \to \infty$. Hence, $I_2 < 0$. We next suppose $l_3 \geqslant 2$. In this case, we have

$$\begin{aligned}
I_2 \leqslant \sup_{r \geqslant 1} &\left(r^{-2(k-j)} \prod_{i_1=1}^{l_1} (\partial_r^{1+m_{1i_1}} F)(r)^2 \prod_{i_2=1}^{l_2} (\partial_r^{1+m_{2i_2}} G)(r)^2 \right) \\
&\qquad \times \int_1^\infty |\partial_r^{l_3}\mathbf{v}_0(R)|^2 R^{n-1} dR < \infty.
\end{aligned}$$

by (4.3.16) and by (4.3.11) in the proof of Proposition 4.3.4. $\square$

4.4 Local existence with slowly decaying data

In previous Sections 4.2 and 4.3, we see that if $\lambda < 0$ and $n \geqslant 3$ then there exists an example of global solution of (4.1.3). That solution can be constructed so that

$$a(t) \in H^s(\mathbb{R}^n), \quad \phi(t) \in C^3(\mathbb{R}^n), \quad \nabla\phi(t) \in Y^{s+1}_{(2^*,\infty],(2,\infty]}, \quad \forall t \geqslant 0$$

for some $s > n/2 + 1$. Remark that $\nabla^2\phi \notin L^2(\mathbb{R}^n)$ and so that we cannot apply Theorem 2.1.2. In this section, we adapt the results in Chapter 2 so that the system

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon \cdot \nabla) a^\varepsilon + \dfrac{1}{2} a^\varepsilon \Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t \phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda V_{\mathrm{P}}^\varepsilon = 0, \\ -\Delta V_{\mathrm{P}}^\varepsilon = |a^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x| \to \infty, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0) \end{cases} \tag{4.1.2}$$

(and (4.1.3)) can be solved for such a initial data. For this purpose, we prove that there exists a unique solution to (4.1.2) for an initial data satisfies $A_0^\varepsilon \in H^{s+1}(\mathbb{R}^n)$, $\Phi_0 \in C^4$, and $\nabla\Phi_0 \in Y_{p,q}^{s+2}$ ($s > n/2+1$, $p > 2^*$, and $q > 2$). We remove the radial symmetry and forget the specific definition of $\nabla\Phi_0$. We generalize nonlinearity and work with the Hartree type nonlinearity: We replace (SP) by

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon; \quad u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp(i\Phi_0(x)/\varepsilon) \tag{4.4.1}$$

and (4.1.2) by

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon \cdot \nabla) a^\varepsilon + \dfrac{1}{2} a^\varepsilon \Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t \phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), \phi^\varepsilon(0,x)) = (A_0^\varepsilon, \Phi_0). \end{cases} \tag{4.4.2}$$

The case $\gamma = n-2$ corresponds to the Scrhrödinger-Poisson system because the Newtonian potential is given by $c_n|x|^{-(n-2)}$ for $n \geqslant 3$. This generalization clarifies the required smoothing property of the nonlocal nonlinearities slightly. We also assume that λ is not necessarily negative. The main result of this section is Theorem 4.4.2 in Section 4.4.3

4.4.1 Lack of the decay of the phase function

In Section 2.4, we have establish the WKB approximation of the solution of (4.4.1) (and (SP)) for a data $(A_0^\varepsilon, \Phi_0)$ such that $A_0^\varepsilon \in H^s$ and $\nabla\Phi_0 \in X^{s+1}$. It is convenient to employ the velocity $v^\varepsilon := \nabla\phi^\varepsilon$ and consider the system

$$\begin{cases} \partial_t a^\varepsilon + (v^\varepsilon \cdot \nabla) a^\varepsilon + \dfrac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\ \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \lambda\nabla(|x|^{-\gamma} * |a^\varepsilon|^2) = 0, \\ (a^\varepsilon(0,x), v^\varepsilon(0,x)) = (A_0^\varepsilon, \nabla\Phi_0) \end{cases} \tag{4.4.3}$$

because this system can be regarded as a symmetric hyperbolic system with perturbation. We observe in Section 2.2 that the general strategy and the common problem in solving this system, and in Section 2.4.1 that the key for treating the nonlocal nonlinearity is to derive the smoothing property from the nonlinearity. The main step of the proof is to obtain a priori bound by the energy method. For the proof of Theorems 2.1.3 and 2.1.2, we have chosen the energy $E(t) := \|a^\varepsilon\|_{H^s}^2 + \|\nabla v\|_{H^s}^2$. We select this energy by the following respects:

1. When we estimate $\frac{d}{dt}\|a^\varepsilon(t)\|_{H^s}^2$, we need to bound the $s+1$-time derivative of the v^ε, such as $\|\nabla v^\varepsilon\|_{H^s}$.
2. When we estimate $\frac{d}{dt}\|\nabla v^\varepsilon\|_{H^s}^2$, we need to bound the $s+2$-time derivative of the nonlinear term. At this step, we can gain two-time derivative from the nonlocal nonlinearity by using Lemma 2.4.1, and bound it with $\|a^\varepsilon(t)\|_{H^s}$.

However, we cannot go along this scenario any more because we are now considering the data such that $\nabla v^\varepsilon \notin L^2$. According to the fact that $\nabla^2 v^\varepsilon \in L^2$, we modify the energy as

$$E(t) := \|a^\varepsilon\|_{H^{s+1}}^2 + \left\|\nabla^2 v^\varepsilon\right\|_{H^s}^2 + \|\nabla v^\varepsilon\|_{L^q}^2 + \|v^\varepsilon\|_{L^p}^2 \tag{4.4.4}$$

and change the above strategy as follows:

1. When we estimate $\frac{d}{dt}\|a^\varepsilon(t)\|_{H^{s+1}}^2$, we bound it with not $\|\nabla v^\varepsilon\|_{H^{s+1}}$ but $\|\nabla^2 v^\varepsilon\|_{H^s}$.
2. We estimate $\frac{d}{dt}\|\nabla^2 v^\varepsilon\|_{H^s}^2$ by $\|a^\varepsilon\|_{H^{s+1}}$ with the two-time derivative gain, as above.
3. Since $p, q > 2$, we cannot estimate $\|\nabla v^\varepsilon\|_{L^q}$ and $\|v^\varepsilon\|_{L^p}$ by the energy method. Hence, we try to obtain this bound by using the equation.

4.4.2 Modified energy estimate

At this section we perform the energy estimate, along the strategy given in the previous section. We would like to choose the energy defined by (4.4.4). As a first step, we shall show the following proposition.

Proposition 4.4.1. *Let $n \geqslant 3$ and $\lambda \in \mathbb{R}$. Assume $s > n/2+1$ and $\gamma > 0$ satisfies $n/2-2 < \gamma \leqslant n-2$. Also assume $p \in (2^*, \infty]$ and $q \in (2, \infty)$. If $(a^\varepsilon, v^\varepsilon) \in H^{s+1} \times Y_{p,q}^{s+2}$ solves (4.4.3), then the "partial energy" $E_{\mathrm{part}}(t) := \|a^\varepsilon\|_{H^{s+1}}^2 + \left\|\nabla^2 v^\varepsilon\right\|_{H^s}^2$ satisfies*

$$\frac{d}{dt}E_{\mathrm{part}}(t) \leqslant CE_{\mathrm{part}}(t)^{\frac{3}{2}}.$$

Proof. We first estimate the H^{s+1} norm of a^ε. As in Chapter 2, we use the following convention for the scalar product in L^2:

$$\langle \varphi, \psi \rangle := \int_{\mathbb{R}^n} \varphi(x)\overline{\psi(x)}dx.$$

We use the notation $\Lambda = (I - \Delta)^{1/2}$. We have

$$\frac{d}{dt} \|a^\varepsilon\|_{H^{s+1}}^2 = 2 \operatorname{Re} \left\langle \partial_t \Lambda^{s+1} a^\varepsilon, \Lambda^{s+1} a^\varepsilon \right\rangle.$$

Let us bound the right hand side. The point is that we cannot use $\|\nabla v^\varepsilon\|_{L^2}$ as a bound. By commuting Λ^{s+1} with the equation for a^ε, we find:

$$\partial_t \Lambda^{s+1} a^\varepsilon + \Lambda^{s+1}(v^\varepsilon \cdot \nabla a^\varepsilon) + \frac{1}{2}\Lambda^{s+1}(a^\varepsilon \nabla \cdot v^\varepsilon) - i\frac{\varepsilon}{2}\Delta\Lambda^{s+1} a^\varepsilon = 0. \quad (4.4.5)$$

The coupling of the second term and $\Lambda^{s+1}a^\varepsilon$ is written as

$$\begin{aligned}\left\langle \Lambda^{s+1}(v^\varepsilon \cdot \nabla a^\varepsilon), \Lambda^{s+1} a^\varepsilon \right\rangle = &\left\langle v^\varepsilon \cdot \nabla \Lambda^{s+1} a^\varepsilon, \Lambda^{s+1} a^\varepsilon \right\rangle \\ &+ \left\langle [\Lambda^{s+1}, v_\delta^\varepsilon] \cdot \nabla a^\varepsilon, \Lambda^{s+1} a^\varepsilon \right\rangle.\end{aligned}$$

We see from the integration by parts that

$$\left| \operatorname{Re} \left\langle v^\varepsilon \cdot \nabla \Lambda^{s+1} a^\varepsilon, \Lambda^{s+1} a^\varepsilon \right\rangle \right| \leqslant \frac{1}{2} \|\nabla v^\varepsilon\|_{L^\infty} \|a^\varepsilon\|_{H^{s+1}}^2. \quad (4.4.6)$$

Moreover, the commutator estimate (Lemma A.2.2) with $k = 2$ shows that

$$\begin{aligned}&\left| \operatorname{Re} \left\langle [\Lambda^{s+1}, v_\delta^\varepsilon] \cdot \nabla a^\varepsilon, \Lambda^{s+1} a^\varepsilon \right\rangle \right| \\ &\qquad \leqslant C(\|\nabla v^\varepsilon\|_{L^\infty} \|\nabla a^\varepsilon\|_{H^s} + \left\|\nabla^2 v^\varepsilon\right\|_{H^{s-1}} \|\nabla a^\varepsilon\|_{L^\infty}) \|a^\varepsilon\|_{H^{s+1}}. \quad (4.4.7)\end{aligned}$$

We estimate the third term of (4.4.5) by (A.2.2) as

$$\begin{aligned}&\left| \operatorname{Re} \left\langle \Lambda^{s+1}(a^\varepsilon \nabla \cdot v^\varepsilon), \Lambda^{s+1} a^\varepsilon \right\rangle \right| \\ &\qquad \leqslant C(\|a\|_{H^{s+1}} \|\nabla v^\varepsilon\|_{L^\infty} + \|a^\varepsilon\|_{L^\infty} \left\|\nabla^2 v^\varepsilon\right\|_{H^s}) \|a^\varepsilon\|_{H^{s+1}}. \quad (4.4.8)\end{aligned}$$

Recall that this part is the bad term: This is the only term which contains the $(s+2)$-time derivative of v^ε. The last term vanishes as in (2.2.2):

$$\operatorname{Re} \left\langle -i\Delta\Lambda^{s+1} a, \Lambda^{s+1} a \right\rangle = \operatorname{Re} i \|\nabla a\|_{H^{s+1}}^2 = 0. \quad (4.4.9)$$

Therefore, summarizing (4.4.5)–(4.4.9), we have

$$\frac{d}{dt} \|a^\varepsilon\|_{H^{s+1}}^2 \leqslant C(\|a\|_{W^{1,\infty}} + \|\nabla v^\varepsilon\|_{L^\infty})(\|a^\varepsilon\|_{H^{s+1}} + \left\|\nabla^2 v^\varepsilon\right\|_{H^s}) \|a^\varepsilon\|_{H^{s+1}}.$$

Recall that $\nabla v^\varepsilon \in L^q$ ($q < \infty$) and so $\nabla v^\varepsilon \to 0$ as $|x| \to \infty$. Hence, by the Sobolev embedding $\|\nabla v^\varepsilon\|_{L^\infty} \leqslant C \left\|\nabla^2 v^\varepsilon\right\|_{H^s}$, we end up with

$$\frac{d}{dt} \|a\|_{H^{s+1}}^2 \leqslant C E_{\mathrm{part}}(t)^{\frac{3}{2}}. \quad (4.4.10)$$

Let us proceed to the estimate of v^ε. We denote the operator $\Lambda^s\nabla^2$ by Q. From the equation for v^ε, we have

$$\partial_t Qv^\varepsilon + Q(v^\varepsilon \cdot \nabla v^\varepsilon) + Q\nabla(|x|^{-\gamma} * |a_\delta^\varepsilon|^2) = 0 \tag{4.4.11}$$

We consider the coupling of this equation and Qv^ε. The second term can be written as

$$\begin{aligned}\langle Q(v^\varepsilon \cdot \nabla v^\varepsilon), Qv^\varepsilon\rangle &= \langle v^\varepsilon \cdot \nabla Qv^\varepsilon, Qv^\varepsilon\rangle + \left\langle [\Lambda^s\nabla, v^\varepsilon] \cdot \nabla^2 v^\varepsilon, Qv^\varepsilon\right\rangle \\ &\quad + \langle \Lambda^s\nabla(\nabla v^\varepsilon \cdot \nabla v^\varepsilon), Qv^\varepsilon\rangle .\end{aligned}$$

As the previous case, integration by parts shows

$$|\operatorname{Re}\langle v^\varepsilon \cdot \nabla Qv^\varepsilon, Qv^\varepsilon\rangle| \leqslant \frac{1}{2}\|\nabla v^\varepsilon\|_{L^\infty}\left\|\nabla^2 v^\varepsilon\right\|_{H^s}, \tag{4.4.12}$$

and the commutator estimate with $k=1$ also shows

$$\begin{aligned}&|\operatorname{Re}\left\langle [\Lambda^s\nabla, v_\delta^\varepsilon] \cdot \nabla^2 v^\varepsilon, Qv^\varepsilon\right\rangle| \\ &\leqslant C(\|\nabla v^\varepsilon\|_{L^\infty}\left\|\nabla^2 v^\varepsilon\right\|_{H^s} + \left\|\nabla^2 v^\varepsilon\right\|_{H^{s-1}}\left\|\nabla^2 v^\varepsilon\right\|_{L^\infty})\left\|\nabla^2 v^\varepsilon\right\|_{H^s}.\end{aligned} \tag{4.4.13}$$

For the estimate of the Hartree nonlinearity, we use Lemma 2.4.1 with $p=\infty$ and $k=2$ to obtain

$$\begin{aligned}\left\|\lambda\nabla^3(|x|^{-\gamma} * |a^\varepsilon|^2)\right\|_{H^s} &\leqslant C\left\|\nabla^2(|x|^{-\gamma} * |a^\varepsilon|^2)\right\|_{H^{s+1}} \\ &\leqslant C(\|a^\varepsilon\|_{L^\infty}\|a^\varepsilon\|_{H^{s+1}} + \|a^\varepsilon\|_{L^2}^2).\end{aligned} \tag{4.4.14}$$

Sum up (4.4.11)–(4.4.14) to have

$$\frac{d}{dt}\left\|\nabla^2 v^\varepsilon\right\|_{H^s}^2 \leqslant CE_{\text{part}}(t)^{\frac{3}{2}}, \tag{4.4.15}$$

which completes the proof. □

4.4.3 Existence result

We now show our main result in this section.

Theorem 4.4.2. *Let $n \geqslant 3$ and $\lambda \in \mathbb{R}$. Let γ be a positive number with $n/2-2 < \gamma \leqslant n-2$. Let $s > n/2+1$. Assume that $\Phi_0 \in C^4$ with $\nabla\Phi_0 \in Y_{p,q}^{s+2}$ for $p \in (2*, \infty]$ and $q \in (2, \infty)$ with $p \geqslant q$. Also assume that A_0^ε is uniformly bounded in H^{s+1} for $\varepsilon \in [0,1]$. Then, there exist $T > 0$ independent of $\varepsilon \in [0,1]$ and $s > n/2+1$, and $(a^\varepsilon, \phi^\varepsilon) \in C([0,T]; C^2 \times C^4)$ unique solution to (4.4.2) on $[0,T]$ for $\varepsilon \in [0,1]$. Moreover, a^ε is bounded in $C([0,T]; H^{s+1})$ uniformly in $\varepsilon \in [0,1]$. Furthermore, ϕ^ε is bounded in $L^\infty([0,T] \times \mathbb{R}^n)$ uniformly in $\varepsilon \in [0,1]$ if $n \geqslant 5$. $\phi^\varepsilon - \Phi_0$ and $\nabla(\phi^\varepsilon - \Phi_0)$ are bounded in $C([0,T]; (L^{\max\left(\frac{p}{2}, \frac{n}{\gamma}+\right)} \cap L^\infty)(\mathbb{R}^n))$ and $C([0,T]; (L^{\max\left(\frac{pq}{p+q}, \frac{n}{\gamma+1}+\right)} \cap L^\infty)(\mathbb{R}^n))$, respectively, uniformly in $\varepsilon \in [0,1]$.*

Remark 4.4.3. In above theorem, the case $\gamma = n-2$ is admissible. Therefore, we immediately obtain the same results for (4.1.2).

Proof. As in the proof of Theorem 2.4.2, we first obtain the solution $(a^\varepsilon, v^\varepsilon)$ of (4.4.3) and then integrate v^ε to construct ϕ^ε.

Local existence of the solution to (4.4.3)**.**

Let us obtain the solution $(a^\varepsilon, v^\varepsilon) \in C([0,T]; H^{s+1} \times Y^{s+2}_{p,q})$ for small $T > 0$. The proof of this part proceeds along the classical energy method, and so it suffices to establish a priori bound of the solution. We first deduce from Proposition 4.4.1 that

$$\frac{d}{dt} E_{\text{part}}(t) \leqslant C E_{\text{part}}(t)^{\frac{3}{2}},$$

where $E_{\text{part}}(t) = \|a^\varepsilon\|^2_{H^{s+1}} + \|\nabla^2 v^\varepsilon\|^2_{H^s}$. Therefore, by Gronwall's lemma, there exist T and C such that

$$\sup_{t\in[0,T]} E_{\text{part}}(t) \leqslant C(E_{\text{part}}(0)). \tag{4.4.16}$$

Next we estimate v^ε and ∇v^ε. Let $E(t)$ be as in (4.4.4). By the second equation of (4.4.3), we obtain

$$v^\varepsilon(t) = \nabla \Phi_0 - \int_0^t \big((v^\varepsilon \cdot \nabla) v^\varepsilon + \lambda \nabla(|x|^{-\gamma} * |a^\varepsilon|^2) \big)\, ds.$$

Therefore, we have

$$\begin{aligned}\|v^\varepsilon\|_{L^\infty([0,T];L^p)} \leqslant\ & \|\nabla\Phi_0\|_{L^p} + T \|v^\varepsilon\|_{L^\infty([0,T];L^p)} \|\nabla v^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^n)} \\ & + T|\lambda| \left\|\nabla(|x|^{-\gamma} * |a^\varepsilon|^2)\right\|_{L^\infty([0,T];L^p)}.\end{aligned}$$

and

$$\begin{aligned}\|\nabla v^\varepsilon\|_{L^\infty([0,T];L^q)} \leqslant\ & \left\|\nabla^2\Phi_0\right\|_{L^q} + T \|\nabla v^\varepsilon\|_{L^\infty([0,T];L^q)} \|\nabla v^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^n)} \\ & + T \|v^\varepsilon\|_{L^\infty([0,T];L^p)} \left\|\nabla^2 v^\varepsilon\right\|_{L^\infty([0,T];L^{\frac{pq}{p-q}})} \\ & + T|\lambda| \left\|\nabla^2(|x|^{-\gamma} * |a^\varepsilon|^2)\right\|_{L^\infty([0,T];L^q)}.\end{aligned}$$

We have $H^s \hookrightarrow L^2 \cap L^\infty \hookrightarrow L^{\frac{pq}{p-q}}$ since $\frac{pq}{p-q} \in (2,\infty]$ holds by assumption $p \geqslant q > 2$. Moreover, we deduce from Lemma 2.4.1 that

$$\begin{aligned}\left\|\nabla(|x|^{-\gamma} * |a^\varepsilon|^2)\right\|_{L^p} &\leqslant C \left\| |\nabla|^{1+n(\frac{1}{2}-\frac{1}{p})} (|x|^{-\gamma} * |a^\varepsilon|^2) \right\|_{L^2} \leqslant C \|a^\varepsilon\|^2_{H^{s+1}}, \\ \left\|\nabla^2(|x|^{-\gamma} * |a^\varepsilon|^2)\right\|_{L^q} &\leqslant C \left\| |\nabla|^{2+n(\frac{1}{2}-\frac{1}{q})} (|x|^{-\gamma} * |a^\varepsilon|^2) \right\|_{L^2} \leqslant C \|a^\varepsilon\|^2_{H^{s+1}},\end{aligned}$$

provided $n/p-1<\gamma \leqslant n-2$ and $n/q-2<\gamma \leqslant n-2$, respectively. By the assumptions $p>2^*$ and $q>2$, we see that

$$\max\left(\frac{n}{p}-1,\frac{n}{q}-2\right)<\frac{n}{2}-2.$$

Hence, for so small T that $TC(E_{\mathrm{part}}(0))<1/3$, we see from (4.4.16) that

$$\|v^\varepsilon\|_{L^\infty([0,T];L^p)}+\|\nabla v^\varepsilon\|_{L^\infty([0,T];L^q)} \leqslant 3\|\nabla\Phi_0\|_{L^p}+3\left\|\nabla^2\Phi_0\right\|_{L^q}+C(E_{\mathrm{part}}(0)). \tag{4.4.17}$$

Plugging (4.4.17) to (4.4.16), we obtain desired energy estimate: There exist T and C such that

$$\sup_{t\in[0,T]} E(t) \leqslant C(E(0)).$$

Hence, we obtain the solution $(a^\varepsilon,v^\varepsilon)\in C([0,T];H^{s+1}\times Y^{s+2}_{p,q})$.

Tail estimate of v^ε and the uniqueness

We next investigate the decay property of v^ε: By the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we have

$$v^\varepsilon-\nabla\Phi_0=-\int_0^t\left((v^\varepsilon\cdot\nabla)v^\varepsilon+\lambda\nabla(|x|^{-\gamma}*|a^\varepsilon|^2)\right)ds\in L^{\max\left(\frac{pq}{p+q},\frac{n}{\gamma+1}+\right)}. \tag{4.4.18}$$

Let us proceed to the proof of the uniqueness of (4.4.3). Let $(a_1^\varepsilon,v_1^\varepsilon)$ and $(a_2^\varepsilon,v_2^\varepsilon)$ be two solutions of (4.4.3) in $C([0,T];H^{s+1}\times Y^{s+2}_{p,q})$ with $(a_i^\varepsilon,v_i^\varepsilon)(0)=(A_0^\varepsilon,\nabla\Phi_0)$. Put $d_a^\varepsilon=a_1^\varepsilon-a_2^\varepsilon$ and $d_v^\varepsilon=v_1^\varepsilon-v_2^\varepsilon$. We remark that $d_a^\varepsilon(0)\equiv 0$ and $d_v^\varepsilon(0)\equiv 0$. Moreover, we see from the above estimate (4.4.18) that $d_v^\varepsilon=(v_1^\varepsilon-\nabla\Phi_0)-(v_2^\varepsilon-\nabla\Phi_0)$ and so $d_v^\varepsilon\to 0$ as $|x|\to\infty$. Now, we estimate

$$E_d(t):=\|d_a^\varepsilon\|_{L^2}^2+\|\nabla d_v^\varepsilon\|_{L^2}^2.$$

It is important to note that ∇v_1^ε and ∇v_2^ε do not necessarily belong to L^2 by definition of $Y^s_{p,q}$. Nevertheless, their difference d_v^ε does so because it is identically zero and so belongs to L^2 at the initial time. We shall follow this part precisely. The system for $(d_a^\varepsilon,d_v^\varepsilon)$ is rewritten as

$$\begin{cases}\partial_t d_a^\varepsilon+d_v^\varepsilon\cdot\nabla a_1^\varepsilon+v_2^\varepsilon\cdot\nabla d_a^\varepsilon+\dfrac{1}{2}d_a^\varepsilon\cdot\nabla v_1^\varepsilon+\dfrac{1}{2}a_2^\varepsilon\cdot\nabla d_v^\varepsilon=i\dfrac{\varepsilon}{2}\Delta d_a^\varepsilon,\\ \partial_t d_v^\varepsilon+d_v^\varepsilon\cdot\nabla v_1^\varepsilon+v_2^\varepsilon\cdot\nabla d_v^\varepsilon+\lambda\nabla(|x|^{-\gamma}*(d_a^\varepsilon\overline{a_1^\varepsilon}+a_2^\varepsilon\overline{d_a^\varepsilon}))=0.\end{cases} \tag{4.4.19}$$

Now estimate the L^2 norm of d_a^ε. From the first equation in (4.4.19), it holds that

$$\begin{aligned}\frac{d}{dt}\|d_a^\varepsilon\|_{L^2}^2&=2\operatorname{Re}\langle\partial_t d_a^\varepsilon,d_a^\varepsilon\rangle\\ &\leqslant C|\operatorname{Re}\langle d_v^\varepsilon\cdot\nabla a_1^\varepsilon,d_a^\varepsilon\rangle|+C|\operatorname{Re}\langle v_2^\varepsilon\cdot\nabla d_a^\varepsilon,d_a^\varepsilon\rangle|+C|\operatorname{Re}\langle d_a^\varepsilon\cdot\nabla v_1^\varepsilon,d_a^\varepsilon\rangle|\\ &\qquad+C|\operatorname{Re}\langle a_2^\varepsilon\cdot\nabla d_v^\varepsilon,d_a^\varepsilon\rangle|+|\operatorname{Re}\langle i\Delta d_a^\varepsilon,d_a^\varepsilon\rangle|.\end{aligned}$$

Now, Hölder's inequality and integration by parts show that

$$
\begin{aligned}
|\operatorname{Re}\langle a_2^\varepsilon \cdot \nabla d_v^\varepsilon, d_a^\varepsilon\rangle| &\leqslant \|a_2^\varepsilon\|_{L^\infty} \|\nabla d_v^\varepsilon\|_{L^2} \|d_a^\varepsilon\|_{L^2}, \\
|\operatorname{Re}\langle v_2^\varepsilon \cdot \nabla d_a^\varepsilon, d_a^\varepsilon\rangle| + |\operatorname{Re}\langle d_a^\varepsilon \cdot \nabla v_1^\varepsilon, d_a^\varepsilon\rangle| &\leqslant (\|\nabla v_1^\varepsilon\|_{L^\infty} + \|\nabla v_2^\varepsilon\|_{L^\infty}) \|d_a\|_{L^2}^2, \\
\operatorname{Re}\langle i\Delta d_a^\varepsilon, d_a^\varepsilon\rangle &= 0.
\end{aligned}
$$

Another use of Hölder's and Sobolev inequalities shows

$$
\begin{aligned}
|\langle d_v^\varepsilon \cdot \nabla a_1^\varepsilon, d_a^\varepsilon\rangle| &\leqslant \|d_v^\varepsilon\|_{L^{\frac{2n}{n-2}}} \|\nabla a_1^\varepsilon\|_{L^n} \|d_a^\varepsilon\|_{L^2} \\
&\leqslant C \|\nabla a_1^\varepsilon\|_{H^{\frac{n}{2}-1}} \|\nabla d_v^\varepsilon\|_{L^2} \|d_a^\varepsilon\|_{L^2}
\end{aligned}
$$

Thus, we end up with the estimate

$$
\frac{d}{dt} \|d_a^\varepsilon\|_{L^2}^2 \leqslant C \left(\|d_a^\varepsilon\|_{L^2}^2 + \|\nabla d_v^\varepsilon\|_{L^2}^2 \right),
$$

where the constant C depends only on $\|a_1^\varepsilon\|_{H^{n/2}}$, $\|a_2^\varepsilon\|_{L^\infty}$, and $\|\nabla v_i^\varepsilon\|_{L^\infty}$ $(i = 1, 2)$. Therefore, this can be written as

$$
\frac{d}{dt} \|d_a^\varepsilon\|_{L^2}^2 \leqslant C(\|a_i^\varepsilon\|_{H^{s+1}}, \|v_i^\varepsilon\|_{Y_{p,q}^{s+2}}) E_d(t). \tag{4.4.20}
$$

Similarly, for all $1 \leqslant i, j \leqslant n$, we have the estimates for $\partial_i d_{v,j}^\varepsilon$:

$$
\begin{aligned}
\left|\left\langle ((\partial_i d_v^\varepsilon) \cdot \nabla) v_{1,j}^\varepsilon, \partial_i d_{v,j}^\varepsilon \right\rangle\right| &\leqslant C \left\|\nabla v_{1,j}^\varepsilon\right\|_{L^\infty} \|\partial_i d_v^\varepsilon\|_{L^2}^2, \\
\left|\left\langle (d_v^\varepsilon \cdot \nabla) \partial_i v_{1,j}^\varepsilon, \partial_i d_{v,j}^\varepsilon \right\rangle\right| &\leqslant C \left\|\nabla \partial_i v_{2,j}^\varepsilon\right\|_{L^n} \|d_v^\varepsilon\|_{L^{\frac{2n}{n-2}}} \left\|\partial_i d_{v,j}^\varepsilon\right\|_{L^2}, \\
\left|\left\langle ((\partial_i v_2^\varepsilon) \cdot \nabla) d_{v,j}^\varepsilon, \partial_i d_{v,j}^\varepsilon \right\rangle\right| &\leqslant C \|\partial_i v_2^\varepsilon\|_{L^\infty} \left\|\nabla d_{v,j}^\varepsilon\right\|_{L^2}^2, \\
\left|\left\langle (v_2^\varepsilon \cdot \nabla) \partial_i d_{v,j}^\varepsilon, \partial_i d_{v,j}^\varepsilon \right\rangle\right| &\leqslant C \|\nabla v_1^\varepsilon\|_{L^\infty} \left\|\partial_i d_{v,j}^\varepsilon\right\|_{L^2}^2, \\
\left|\left\langle \partial_i \partial_j (|x|^{-\gamma} * (d_a^\varepsilon \overline{a_1^\varepsilon})), \partial_i d_{v,j}^\varepsilon \right\rangle\right| &\leqslant C(\|a_1^\varepsilon\|_{L^\infty} + \|a_1^\varepsilon\|_{L^2}) \|d_a^\varepsilon\|_{L^2} \left\|\partial_i d_{v,j}^\varepsilon\right\|_{L^2}, \\
\left|\left\langle \partial_i \partial_j (|x|^{-\gamma} * (a_2^\varepsilon \overline{d_a^\varepsilon})), \partial_i d_{v,j}^\varepsilon \right\rangle\right| &\leqslant C(\|a_2^\varepsilon\|_{L^\infty} + \|a_2^\varepsilon\|_{L^2}) \|d_a^\varepsilon\|_{L^2} \left\|\partial_i d_{v,j}^\varepsilon\right\|_{L^2},
\end{aligned}
$$

where $v_{1,j}^\varepsilon$ and $d_{v,j}^\varepsilon$ denote the j-th components of v_1^ε and d_v^ε, respectively. Summing up over i and j, we obtain

$$
\frac{d}{dt} \|\nabla d_v^\varepsilon\|_{L^2}^2 \leqslant C(\|a_i^\varepsilon\|_{H^{s+1}}, \|v_i^\varepsilon\|_{Y_{p,q}^{s+2}}) E_d(t). \tag{4.4.21}
$$

Plugging (4.4.20) and (4.4.21), we obtain

$$
\frac{d}{dt} E_d(t) \leqslant C(\|a_i^\varepsilon\|_{H^{s+1}}, \|v_i^\varepsilon\|_{Y_{p,q}^{s+2}}) E_d(t).
$$

Hence, we conclude from Gronwall's lemma that

$$
E_d(t) \leqslant C(\|a_i^\varepsilon\|_{H^{s+1}}, \|v_i^\varepsilon\|_{Y_{p,q}^{s+2}}) E_d(0) = 0
$$

as long as the solutions $(a_i^\varepsilon, v_i^\varepsilon)$ exist. This implies that $d_a^\varepsilon \equiv 0$ and $\nabla d_v^\varepsilon \equiv 0$. In particular, there exists a function $d = d(t)$ of time such that $d_v^\varepsilon(t, x) = d(t)$. Recall that $d_v^\varepsilon(t, x) \to 0$ as $|x| \to \infty$. As a result, $d(t) \equiv 0$ follows and we finally obtain $(a_1^\varepsilon, v_1^\varepsilon) = (a_2^\varepsilon, v_2^\varepsilon)$.

Construction of ϕ^ε.

Since we have obtained the uniqueness of the solution to (4.4.3), we can use the direct definition introduced in Section 2.2.2: We define ϕ^ε by

$$\phi^\varepsilon(t) = \Phi_0 - \int_0^t \left(\frac{1}{2}|v^\varepsilon|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2) \right) ds$$

then $(a^\varepsilon, \phi^\varepsilon)$ is a unique solution to (4.4.2). Though ϕ^ε and Φ_0 themselves do not necessarily belong to any Lebesgue space, it follows from the Hölder inequality and the Hardy-Littlewood-Sobolev inequality that

$$\phi^\varepsilon(t) - \Phi_0 = - \int_0^t \left(\frac{1}{2}|v^\varepsilon|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2) \right) ds \in L^{\max\left(\frac{p}{2}, \frac{n}{\gamma}+\right)}.$$

Moreover, it is bounded uniformly in $\varepsilon \in [0,1]$. If $n \geqslant 5$ then, applying Lemma 2.2.1 twice, we see that there exist constants $c_0 \in \mathbb{R}$ and $c_1 \in \mathbb{R}^n$ such that

$$\|\Phi_0 - c_0 - c_1 \cdot \|_{L^{\frac{2n}{n-4}}} \leqslant C \|\nabla\Phi_0 - c_1\|_{L^{\frac{2n}{n-2}}} \leqslant C \left\|\nabla^2 \Phi_0\right\|_{L^2}.$$

Since $\nabla\Phi_0 \in L^q$ $(q < \infty)$, we see that $\nabla\Phi_0 \to 0$ as $|x| \to \infty$, and so that $c_1 = 0$. By the Sobolev embedding, we also have

$$\|\Phi_0 - c_0\|_{L^\infty} \leqslant C \left\|\nabla^2 \Phi_0\right\|_{H^s},$$

which shows $\Phi_0 \in L^\infty$ and so $\phi^\varepsilon \in L^\infty$. □

Remark 4.4.4. There is another way to construct ϕ^ε from v^ε which depends on the characteristic curve method. As long as v^ε exists with $\nabla v^\varepsilon \in L^\infty$, we can define corresponding characteristic curve (classical trajecotry) uniquely. Then, by an argument in [17] shows that irrotational property propagates along the characteristic curve. Then, we can apply the first method in Section 2.2.2 based on the Poincaré lemma. However, this method does not give the uniqueness (of v^ε).

4.5 Large time WKB analysis

4.5.1 Main result

We now in a position to prove Theorem 4.1.1. Using all results in the previous sections of this chapter, we shall justify the WKB approximation (4.1.1) of the solution u^ε of (SP). The strategy of the proof is the same as the general strategy observed in Section 2.2.3. Let us recall briefly: We consider the system

$$
\begin{cases}
\partial_t a^\varepsilon + (\nabla\phi^\varepsilon\cdot\nabla)a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, \\
\partial_t\phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda V_{\mathrm{P}}^\varepsilon = 0, \\
-\Delta V_{\mathrm{P}}^\varepsilon = |a^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x|\to\infty, \\
(a^\varepsilon(0,x),\phi^\varepsilon(0,x)) = (A_0^\varepsilon,\Phi_0).
\end{cases}
\tag{4.1.2}
$$

Our goal is to show an ε-power expansion of this solution

$$
a^\varepsilon = a_0 + \sum_{j=1}^{k}\varepsilon^j a_j + o(\varepsilon^k), \qquad \phi^\varepsilon = \phi_0 + \sum_{j=1}^{k}\varepsilon^j\phi_j + o(\varepsilon^k)
$$

for large time. The difference is that we have already known that the zeroth order term (a_0,ϕ_0) can be defined globally in time. Recall that (a_0,ϕ_0) solves the system

$$
\begin{cases}
\partial_t a_0 + (\nabla\phi_0\cdot\nabla)a_0 + \dfrac{1}{2}a_0\Delta\phi_0 = 0, \\
\partial_t\phi_0 + \dfrac{1}{2}|\nabla\phi_0|^2 + \lambda V_{\mathrm{P}} = 0, \\
-\Delta V_{\mathrm{P}} = |a_0|^2, \quad V_{\mathrm{P}} \to 0 \text{ as } |x|\to\infty, \\
(a_0(0,x),\phi_0(0,x)) = (A_0(x),\Phi_0(x)).
\end{cases}
\tag{4.1.3}
$$

We have given a global solution to this system in Theorem 4.2.1 and observed its regularity in Theorem 4.3.2. It will turn out that, for any fixed $T>0$, $a^\varepsilon - a_0$ and $\phi^\varepsilon - \phi_0$ are finite for sufficiently small $\varepsilon>0$ and tend to zero as $\varepsilon\to 0$. Thanks to this fact, we infer that the existence time of $(a^\varepsilon,\phi^\varepsilon)$ can be chosen arbitrarily large as long as ε is sufficiently small. We will also verify that if (a_0,ϕ_0) is global in time, then it is true for all (a_j,ϕ_j). Then, the ε-power expansion of $(a^\varepsilon,\phi^\varepsilon)$ is valid on an arbitrarily large time interval if ε is enough small. For making the notation simpler, we write

$$
\lambda V_{\mathrm{P}}^\varepsilon = \lambda\Delta^{-1}|a^\varepsilon|^2 = \lambda c_n(|x|^{-(n-2)} * |a^\varepsilon|^2)
$$

and denote λc_n again by λ, where c_n is a positive constant. This changes the second and the third lines of equations (4.1.2) into

$$
\partial_t\phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda(|x|^{-(n-2)} * |a^\varepsilon|^2) = 0.
$$

Assumptions

We now clarify our assumptions. First of all, the initial data (A_0^ε,Φ_0) of (4.1.2) should lie in the framework of Theorem 4.4.2:

Assumption 4.5.1 (Assumption for local existence)**.** *Suppose* $n \geqslant 3$ *and* $\lambda \in \mathbb{R}$*. Let* $s > n/2 + 1$*. Assume that*

- $\Phi_0 \in C^4$ *with* $\nabla\Phi_0 \in Y^{s+2}_{p,q}$ *for* $p \in (2*, \infty]$ *and* $q \in (2, \infty)$ *with* $p \geqslant q$*;*
- A_0^ε *is uniformly bounded in* H^{s+1} *for* $\varepsilon \in [0, 1]$*,*

where $Y^{s+2}_{p,q}$ *is defined by* (4.3.3) *and* (4.3.4)*.*

If this Assumption 4.5.1 is met, we can apply Theorem 4.4.2 and give a unique local solution $(a^\varepsilon, \phi^\varepsilon) \in C([0, T]; H^{s+1} \times C^4)$ of (4.1.2) with $\nabla\phi^\varepsilon \in C([0, T]; Y^{s+2}_{p,q})$. In order to apply Theorems 4.2.1 and 4.3.2, we further make the following assumption on $A_0 := \lim_{\varepsilon\to 0} A_0^\varepsilon$ and Φ_0, which corresponds to the initial data of the limit equation (4.1.3).

Assumption 4.5.2 (Assumption for global existence of the limit solution)**.** *In addition to Assumption 4.5.1, we assume* $\lambda < 0$ *and that* $A_0 := \lim_{\varepsilon\to 0} A_0^\varepsilon$ *exists in the* H^{s+1} *sense. Assume that there exist functions* $\mathbf{A}_0 \colon \mathbb{R}_+ \to \mathbb{C}$ *and* $\mathbf{\Phi}_0 \colon \mathbb{R}_+ \to \mathbb{R}$ *such that* $A_0(x) = \mathbf{A}_0(|x|)$ *and* $\Phi_0(x) = \mathbf{\Phi}_0(|x|)$*, respectively, and satisfy following properties:*

- $\mathbf{A}_0 \in C^{\lceil s+3 \rceil}([0, \infty))$ *is nontrivial function satisfying*

$$\begin{aligned} &r^{j-\lceil s \rceil}\partial_r^j \mathbf{A}_0 \in L^2((0, \infty), r^{n-1}dr) && 1 \leqslant j \leqslant \lceil s+3 \rceil, \\ &\partial_r^j \mathbf{A}_0 \in L^2((0, \infty), r^{n-1}dr) && 0 \leqslant j \leqslant \lceil s+3 \rceil, \\ &\partial_r^j \mathbf{A}_0 = O(r^{-n/2}) \quad \text{as } r \to \infty && 0 \leqslant j \leqslant \lceil s+3 \rceil, \end{aligned}$$

  and that there exits $k_0 \geqslant \lceil s+3 \rceil - \lceil (n-1)/2 \rceil$ *such that*

$$(\partial_r^j \mathbf{A}_0)(0) = 0 \quad \text{for } j \in [0, k_0 - 1], \quad (\partial_r^{k_0} \mathbf{A}_0)(0) \neq 0,$$

  where $\lceil s \rceil$ *denotes the minimum integer larger than or equal to* s*.*

- $\mathbf{\Phi}_0$ *is given from* $\mathbf{A}_0$ *by the formula*

$$\mathbf{\Phi}_0(r) = \int_0^r \sqrt{\frac{2|\lambda|}{(n-2)s^{n-2}} \int_0^s |\mathbf{A}_0(\sigma)|^2 \sigma^{n-1} d\sigma} ds + \text{const.} \qquad (4.2.2)$$

One verifies that if this Assumption 4.5.2 is satisfied then Theorem 4.3.2 gives the global (radial) solution $(a_0, \phi_0) \in C([0, \infty); H^{s+3} \times C^6)$ of (4.1.3) with $\nabla\phi_0 \in C([0, T]; Y^{s+4}_{p,q})$. Here we remark that the limit solution (a_0, ϕ_0) is assumed to have more regularity than $(a^\varepsilon, \phi^\varepsilon)$. This is because we rely on the regularity of (a_0, ϕ_0) when we close an energy estimate of error term. Furthermore, in order to justify the ε-power expansion of $(a^\varepsilon, \phi^\varepsilon)$, we assume that this expansion is already known at the initial time $t = 0$:

Assumption 4.5.3 (Assumption for expansion). *In addition to Assumptions 4.5.1 and 4.5.2, we assume that there exists a positive integer k such that s satisfies $s > n/2 + 2k + 1$ and that A_0^ε is expanded as*

$$A_0^\varepsilon = A_0 + \sum_{j=1}^{k} \varepsilon^j A_j + O(\varepsilon^{k+1}) \quad \textit{in } H^{s+1}.$$

Namely, we assume $\varepsilon^{-(k+1)}(A_0^\varepsilon - \sum_{j=0}^{k} \varepsilon^j A_j)$ is bounded in H^{s+1} uniformly in $\varepsilon \in (0, 1]$.

Remark 4.5.4. The assumption on Φ_0 in Assumption 4.5.1 is automatically satisfied if Φ_0 is given as in Assumption 4.5.2 (see Proposition 4.3.4).

Remark 4.5.5. In the above assumptions, $A_0 = \lim_{\varepsilon\to 0} A_0^\varepsilon$ and Φ_0 are assumed to be radial. However, A_0^ε itself is not necessarily radial function.

Main theorem.

Before stating the result, we define

$$\eta(T) := \|a_0\|_{L^\infty([0,T];H^{s+3})} + \|\nabla\phi_0\|_{L^\infty([0,T];Y^{s+4}_{(2^*,\infty],(2,\infty]})}. \tag{4.5.1}$$

It follows from Theorem 4.3.2 that $\eta(T) < \infty$ for all $T > 0$. Under the assumption 4.5.3, we have the following theorem:

Theorem 4.5.6. *Let Assumption 4.5.3 be satisfied. Let $(a^\varepsilon, \phi^\varepsilon)$ be the solution to* (4.1.2) *given by Theorem 4.4.2 and (a_0, ϕ_0) be the global solution to* (4.1.3) *given by Theorem 4.2.1. Then, there exist*

$$(a_j, \phi_j) \in C([0,\infty); H^{s-2j+3} \times Y^{s-2j+5}_{(\frac{n}{n-2},\infty],(\frac{n}{n-1},\infty]})$$

($1 \leqslant j \leqslant k$) and constant C_s depending only on n and s such that, for any $T > 0$, it holds that

$$\begin{cases} a^\varepsilon = a_0 + \displaystyle\sum_{j=1}^{k} \varepsilon^j a_j + O(\varepsilon^{k+1}) \quad \textit{in } L^\infty([0,T], H^{s-2k+1}(\mathbb{R}^n)), \\ \phi^\varepsilon = \phi_0 + \displaystyle\sum_{j=1}^{k} \varepsilon^j \phi_j + O(\varepsilon^{m+1}) \quad \textit{in } L^\infty([0,T], Y^{s-2k+3}_{(\frac{n}{n-2},\infty],(\frac{n}{n-1},\infty]}(\mathbb{R}^n)) \end{cases}$$

for $\varepsilon \leqslant C\eta(T)e^{-3C_s\eta(T)T}$, and so (4.1.5) *holds.*

Remark 4.5.7. We note that ϕ^ε and ϕ_0 themselves do not belong to the space $Y^{s+3}_{(\frac{n}{n-2},\infty],(\frac{n}{n-1},\infty]}(\mathbb{R}^n)$.

Theorem 4.1.1 immediately follows from this theorem by the argument in Section 2.2.4. In order to avoid the complexity, we separate the proof of Theorem 4.5.6 into three steps, and prove them individually in the following Sections 4.5.2–4.5.4.

4.5.2 Proof the theorem – part 1: the zeroth order

In this section, we shall estimate the distance $a^\varepsilon - a_0$ and $\phi^\varepsilon - \phi_0$ for large time, where $(a^\varepsilon, \phi^\varepsilon)$ is the solution to (4.1.2) given by Theorem 4.4.2 and (a_0, ϕ_0) is the global solution to (4.1.3) given by Theorem 4.2.1.

Proposition 4.5.8. *Let Assumption 4.5.3 be satisfied. Let $(a^\varepsilon, \phi^\varepsilon)$ be the solution to* (4.1.2) *given by Theorem 4.4.2 and (a_0, ϕ_0) be the global solution to* (4.1.3) *given by Theorem 4.2.1. Let η be as in* (4.5.1). *Then, there exists a constant C_s depending on n and s and Γ_1 on A_0^ε such that*

$$\|a^\varepsilon - a_0\|_{L^\infty([0,T],H^{s+1})} + \|\nabla\phi^\varepsilon - \nabla\phi_0\|_{L^\infty([0,T],Y^{s+2}_{(\frac{n}{n-1},\infty],[2,\infty])})} \leqslant \varepsilon\Gamma_1 e^{C_s\eta(T)T} \tag{4.5.2}$$

holds for all $0 < \varepsilon \leqslant \varepsilon_0(T) \leqslant \eta(T)Ce^{-C_s\eta(T)T}$. In particular, the existence time T of $(a^\varepsilon, \phi^\varepsilon)$ can be chosen so that $\varepsilon \sim \eta(T)e^{-C_s\eta(T)T}$.

Proof. Denote $v^\varepsilon = \nabla\phi^\varepsilon$ and $v_0 = \nabla\phi_0$. We set $(\widetilde{a}_0^\varepsilon, \widetilde{v}_0^\varepsilon) = (a^\varepsilon - a_0, v^\varepsilon - v_0)$. Then, we deduce from (4.1.2) and (4.1.3) that $(\widetilde{a}_0^\varepsilon, \widetilde{v}_0^\varepsilon)$ solves the equation

$$\begin{cases} \partial_t\widetilde{a}_0^\varepsilon + \widetilde{v}_0^\varepsilon\cdot\nabla\widetilde{a}_0^\varepsilon + \dfrac{1}{2}\widetilde{a}_0^\varepsilon\cdot\nabla\widetilde{v}_0^\varepsilon + \widetilde{v}_0^\varepsilon\cdot\nabla a_0 + v_0\cdot\nabla\widetilde{a}_0^\varepsilon \\ \qquad\qquad + \dfrac{1}{2}\widetilde{a}_0^\varepsilon\cdot\nabla v_0 + \dfrac{1}{2}a_0\cdot\nabla\widetilde{v}_0^\varepsilon = i\dfrac{\varepsilon}{2}\Delta\widetilde{a}_0^\varepsilon + i\dfrac{\varepsilon}{2}\Delta a_0, \\ \partial_t\widetilde{v}_0^\varepsilon + \widetilde{v}_0^\varepsilon\cdot\nabla\widetilde{v}_0^\varepsilon + \lambda\nabla(|x|^{-(n-2)} * |\widetilde{a}_0^\varepsilon|^2) + \widetilde{v}_0^\varepsilon\cdot\nabla v_0 \\ \qquad + v_0\cdot\nabla\widetilde{v}_0^\varepsilon + 2\lambda\nabla(|x|^{-(n-2)} * (\mathrm{Re}(\widetilde{a}_0^\varepsilon\overline{a_0}))) = 0. \end{cases} \tag{4.5.3}$$

The point is that we exclude all a^ε and v^ε by using $a^\varepsilon = a_0 + \widetilde{a}_0^\varepsilon$ and $v^\varepsilon = v_0 + \widetilde{v}_0^\varepsilon$, respectively. We set

$$\widetilde{E}_0(t) := \|\widetilde{a}_0^\varepsilon\|_{H^{s+1}} + \|\widetilde{v}_0^\varepsilon\|_{Y^{s+2}_{(\frac{n}{n-1},\infty],[2,\infty]}}.$$

Since $\widetilde{v}_0^\varepsilon(0) = 0$ and so we can repeat the energy estimate which we made in the proof of 2.4.8 and obtain

$$\frac{d}{dt}\widetilde{E}_0(t) \leqslant C_s(\widetilde{E}_0(t)^2 + C_0(t)\widetilde{E}_0(t) + \varepsilon C_0(t)), \tag{4.5.4}$$

where C_s depends on n and s, and C_0 on $\|a(t)\|_{H^{s+3}} + \|v(t)\|_{Y^{s+4}_{(2^*,\infty],(2,\infty]}}$. We recall that $\eta(T) := \sup_{t\in[0,T]} C_0(t)$ and $\eta(T) < \infty$ for $T < \infty$. Therefore, to prove the theorem, it suffices to show the estimate

$$\sup_{t\in[0,T]} \widetilde{E}_0(t) \leqslant \varepsilon\Gamma_1 e^{C_s\eta(T)T} \tag{4.5.2}$$

holds for $\varepsilon \leqslant \varepsilon_0(T) \leqslant C\eta(T)e^{-C_s\eta(T)T}$. Once this is proven, then we have

$$\sup_{t\in[0,T]}\left(\|a^\varepsilon(t)\|_{H^{s+1}} + \|v^\varepsilon(t)\|_{Y^{s+2}_{(2^*,\infty],(2,\infty]}}\right) \leqslant \eta(T) + \sup_{t\in[0,T]}\widetilde{E}_0(t) < \infty$$

for $\varepsilon \leqslant \varepsilon_0(T)$, which implies the solution $(a^\varepsilon, v^\varepsilon)$ exists until $t = T$. The following Lemma 4.5.9 completes the proof. $\square$

Lemma 4.5.9. *Let $\widetilde{E}_0(t)$ be a nonnegative function depending on a parameter ε and satisfying the inequality* (4.5.4)*. Assume that $\limsup_{\varepsilon\to 0} \widetilde{E}_0(0)/\varepsilon < \infty$. Let η be a function such that $C_0(t) \leqslant \eta(T) < \infty$ for all $0 \leqslant t \leqslant T < \infty$. Then, for any $T > 0$ there exist $\varepsilon_0 = \varepsilon_0(T) \leqslant C\eta(T)e^{-C_s\eta(T)T}$ and a constant $\Gamma_1 = \Gamma_1(\limsup_{\varepsilon\to 0} \widetilde{E}_0(0)/\varepsilon) > 0$ such that $\sup_{t\in[0,T]} \widetilde{E}_0(t) \leqslant \varepsilon\Gamma_1 e^{C_s\eta(T)T}$ for $\varepsilon \leqslant \varepsilon_0$.*

Proof. We fix some $T > 0$ and analyze (4.5.4) for $t \in [0, T]$;

$$\frac{d}{dt}\widetilde{E}_0(t) \leqslant C_s(\varepsilon\eta(T) + \eta(T)\widetilde{E}_0(t) + \widetilde{E}_0(t)^2),$$

where we denote $\eta(T)$ by η for short. This gives the inequality for $Z(t) = \widetilde{E}_0(t)e^{-C_s t\eta}$,

$$\frac{d}{dt}Z(t) \leqslant C_s\eta\varepsilon e^{-C_s t\eta} + C_s e^{C_s t\eta} Z(t)^2, \quad Z(0) = \widetilde{E}_0(0).$$

By assumption, there exist $\varepsilon_{0,1} > 0$ such that $\widetilde{E}_0(0) < \beta\varepsilon$ holds for some $\beta_0 > 0$ all $\varepsilon \leqslant \varepsilon_{0,1}$. We set

$$\delta_0 := \frac{\sqrt{1+\beta_0} - 1}{\beta_0}, \quad \theta_0 := \frac{\delta_0 e^{C_s T\eta}}{2\varepsilon(e^{C_s T\eta} - 1)}.$$

Multiplying the above inequality by $\frac{\theta_0}{(1+\theta_0 Z)^2}$, we obtain

$$\frac{\theta_0 Z'(t)}{(1+\theta_0 Z(t))^2} \leqslant C_s\eta\theta_0\varepsilon e^{-C_s t\eta} + C_s\theta_0^{-1}e^{C_s t\eta}.$$

Integration over $[0, t]$ gives

$$\frac{1}{1+\theta_0 Z(t)} \geqslant \frac{1}{1+\theta_0\widetilde{E}_0(0)} - \varepsilon\theta_0(1 - e^{-C_s t\eta}) - \eta^{-1}\theta_0^{-1}(e^{C_s t\eta} - 1). \quad (4.5.5)$$

We now show that for small ε the right hand side is bounded by $\delta_0/2$ from below. If $\varepsilon \leqslant \varepsilon_{0,1}$ and $T > T_0 = T_0(\beta)$, then it holds that

$$\begin{aligned}\frac{1}{1+\theta_0\widetilde{E}_0(0)} - \delta_0 &\geqslant \frac{1}{1+\theta_0\varepsilon\beta_0} - \delta_0 = \frac{e^{C_s T\eta} - 2 + 2\delta_0}{(\sqrt{1+\beta_0}+1)e^{C_s T\eta} - 2} \\ &\geqslant \frac{1}{2(1+\sqrt{1+\beta_0})} = \frac{\delta_0}{2}. \end{aligned} \quad (4.5.6)$$

Moreover, the right hand side of (4.5.5) is monotone decreasing in t, and $\delta_0 - \varepsilon\theta_0(1 - e^{-C_s T\eta}) - \eta^{-1}\theta_0^{-1}(e^{C_s T\eta} - 1) \geqslant 0$ is equivalent to

$$\varepsilon(1 - e^{-C_s T\eta})\theta_0^2 - \delta_0\theta_0 + \eta^{-1}(e^{C_s T\eta} - 1) \leqslant 0. \quad (4.5.7)$$

Since θ_0 is the minimizer of the left hand side, we see that if $\varepsilon \leqslant \varepsilon_{0,2} := \delta_0^2 \eta e^{C_s T\eta}/4(e^{C_s T\eta}-1)^2$ then (4.5.7) holds. Plugging (4.5.6) and (4.5.7) to (4.5.5), we obtain $(1+\theta_0 Z(t))^{-1} \geqslant \delta_0/2$, which implies

$$Z(t) \leqslant (1+2\sqrt{1+\beta_0})\theta_0^{-1} \leqslant 3\sqrt{1+\beta_0}\theta_0^{-1}.$$

We set $\varepsilon_0 = \min(\varepsilon_{0,1}, \varepsilon_{0,2})$. Notice that $\varepsilon_{0,1}$ is independent of T, and $\varepsilon_{0,2} \sim (\delta_0^2/4)\eta e^{-C_s T\eta}$ if T is large. Then, for all $T > T_0 = T_0(\beta_0)$ and $\varepsilon \leqslant \varepsilon_0 = \varepsilon_0(\beta, T)$, we conclude that

$$\sup_{t\in[0,T]} \widetilde{E}_0(t) \leqslant \left(\frac{6(1+\sqrt{1+\beta_0})e^{C_s\eta(T)T}}{\delta_0}\right)\varepsilon =: \varepsilon\Gamma_1 e^{C_s\eta(T)T},$$

where Γ_1 depends on β_0, that is, on $\limsup_{\varepsilon\to 0}\widetilde{E}_0(0)/\varepsilon$. $\square$

4.5.3 Proof the theorem – part 2: the first order

In this section, we show the following two point: First is that (a_1, ϕ_1) is defined globally in time as a limit $\varepsilon \to 0$ of $(\widetilde{a}_0^\varepsilon, \widetilde{\phi}_0^\varepsilon)$ (Proposition 4.5.10). Second is the asymptotics

$$a^\varepsilon = a_0 + \varepsilon a_1 + O(\varepsilon^2), \quad v^\varepsilon = v_0 + \varepsilon v_1 + O(\varepsilon^2)$$

for large time (Proposition 4.5.11). If the number k in Assumption 4.5.3 is one, then Proposition 4.5.11 completes the proof of Theorem 4.5.6.

Proposition 4.5.10. *Let Assumption 4.5.3 be satisfied. Then, there exists*

$$(a_1, \phi_1) \in C([0,\infty), H^{s+1} \times Y^{s+3}_{(\frac{n}{n-2},\infty],(\frac{n}{n-1},\infty]}).$$

Let $E_1(t) := \|a_1(t)\|_{H^{s+1}} + \|\nabla\phi_1(t)\|_{Y^{s+2}_{(\frac{n}{n-1},\infty],[2,\infty]}}$. *Then, for any* $T>0$, *we have the following bound*

$$\sup_{t\in[0,T]} E_1(t) \leqslant \Gamma_1 e^{C_s\eta(T)T} =: \eta_1(T), \tag{4.5.8}$$

where Γ_1, C_s, *and* η *are the same one as in Proposition 4.5.8. In particular,* (a_1, ϕ_1) *is defined globally in time.*

Proposition 4.5.11. *Let Assumption 4.5.3 be satisfied. Let* $(a^\varepsilon, \phi^\varepsilon)$ *be the solution to* (4.1.2) *given by Theorem 4.4.2 and* (a_0, ϕ_0) *be the global solution to* (4.1.3) *given by Theorem 4.2.1. Let* (a_1, ϕ_1) *be the limit defined in Proposition 4.5.10. Let* C_s *be the same one as in Proposition 4.5.8. Let* η *be as in* (4.5.1). *Then, there exists a constant* Γ_2 *depending on* A_0^ε *such that*

$$\begin{aligned}\|a^\varepsilon - a_0 - \varepsilon a_1\|_{L^\infty([0,T],H^{s-1})} + \|\nabla(\phi^\varepsilon - \phi_0 - \varepsilon\phi_1)\|_{L^\infty([0,T],Y^s_{(\frac{n}{n-1},\infty],[2,\infty]})}&\\ \leqslant \varepsilon^2\Gamma_2\eta(T)^{-1}e^{3C_s\eta(T)T}e^{\varepsilon C_s\eta_1(T)T}&\end{aligned} \tag{4.5.9}$$

holds for all $0<\varepsilon\leqslant\varepsilon_1(T)\leqslant Ce^{-2C_s\eta(T)T}$. *In particular, the existence time* T *of* $(a^\varepsilon, \phi^\varepsilon)$ *can be chosen so that* $\varepsilon \sim e^{-2C_s\eta(T)T}$.

Proofs.

Proof of Proposition 4.5.10. Fix $T > 0$. By (4.5.2), we infer that $(a^\varepsilon - a_0)/\varepsilon$ and $(\nabla\phi^\varepsilon - \nabla\phi_0)/\varepsilon$ are uniformly bounded in the limit $\varepsilon \to 0$. Therefore, there exists a weak limit (a_1, v_1). This limit satisfies (4.5.8) by lower semi-continuity of the weak limit. Since $(\widetilde{a}_0^\varepsilon, \widetilde{v}_0^\varepsilon)$ solves (4.5.3), one sees that (a_1, v_1) solves

$$\begin{cases} \partial_t a_1 + (v_1 \cdot \nabla) a_0 + (v_0 \cdot \nabla) a_1 + \dfrac{1}{2} a_1 \nabla \cdot v_0 + \dfrac{1}{2} a_0 \nabla \cdot v_1 = \dfrac{i}{2} \Delta a_0, \\ \partial_t v_1 + (v_1 \cdot \nabla) v_0 + (v_0 \cdot \nabla) v_1 + \lambda \nabla (|x|^{-(n-2)} * (2 \operatorname{Re}(a_1 \overline{a_0})) = 0, \\ a_1(0) = A_1, \quad v_1(0) = 0. \end{cases} \tag{4.5.10}$$

We verify that the solution is unique by a standard energy method. We now define ϕ_1 by

$$\phi_1(t) = -\int_0^t \Big(v_0 \cdot v_1 + 2\lambda (|x|^{-(n-2)} * (\operatorname{Re}(a_1 \overline{a_0}))) \Big)\, ds.$$

It is easy to see that $(a_1, \nabla\phi_1)$ solves (4.5.10). Hence, $\nabla\phi_1 = v_1$ by uniqueness. The first term of the integrand belongs to $L^{\frac{n}{2(n-1)}+} \cap L^\infty$ and the second term belongs to L^q for $q \in (n/(n-2), \infty)$ by the Hardy-Littlewood-Sobolev inequality. We also deduce from Lemma 2.4.1 that

$$\begin{aligned} \left\| (|x|^{-(n-2)} * (\operatorname{Re}(a_1 \overline{a_0})) \right\|_{L^\infty} &\leqslant C \left\| \nabla (|x|^{-(n-2)} * (\operatorname{Re}(a_1 \overline{a_0})) \right\|_{H^{\frac{n}{2}-1+}} \\ &\leqslant C (\|a_1 a_0\|_{H^{s+1}} + \|a_1 a_0\|_{L^1}) < \infty. \end{aligned}$$

Therefore, $\phi_1 \in C([0,\infty); L^{\frac{n}{n-2}+} \cap L^\infty)$. □

Proof of Proposition 4.5.11. We first put $b_1^\varepsilon = (\widetilde{a}_0^\varepsilon/\varepsilon - a_1)/\varepsilon$ and $w_1^\varepsilon = (\widetilde{v}_0^\varepsilon/\varepsilon - v_1)/\varepsilon$, where $v_1 = \nabla\phi_1$. One verifies from (4.5.3) and (4.5.10) that $(b_1^\varepsilon, w_1^\varepsilon)$ solves

$$\begin{cases} \partial_t b_1^\varepsilon + \varepsilon^2 \left(w_1^\varepsilon \cdot \nabla b_1^\varepsilon + \dfrac{1}{2} b_1^\varepsilon \nabla \cdot w_1^\varepsilon \right) + w_1^\varepsilon \cdot \nabla (a_0 + \varepsilon a_1) \\ \quad + (v_0 + \varepsilon v_1) \cdot \nabla b_1^\varepsilon + \dfrac{1}{2} b_1^\varepsilon \nabla \cdot (v_0 + \varepsilon v_1) + \dfrac{1}{2} (a_0 + \varepsilon a_1) \nabla \cdot w_1^\varepsilon \\ \quad + v_1 \cdot \nabla a_1 + \dfrac{1}{2} a_1 \nabla \cdot v_1 = i \dfrac{\varepsilon}{2} \Delta b_1^\varepsilon + i \dfrac{1}{2} \Delta a_1, \\ \partial_t w_1^\varepsilon + \varepsilon^2 \left(w_1^\varepsilon \cdot \nabla w_1^\varepsilon + \lambda \nabla (|x|^{-\gamma} * |b_1^\varepsilon|^2) \right) + w_1^\varepsilon \cdot \nabla (v + \varepsilon v_1) \\ \quad + (v + \varepsilon v_1) \cdot \nabla w_1^\varepsilon + 2\lambda \nabla (|x|^{-\gamma} * (\operatorname{Re}(b_1^\varepsilon \overline{(a + \varepsilon a_1)})) \\ \quad + v_1 \cdot \nabla v_1 + \lambda \nabla (|x|^{-\gamma} * |a_1|^2) = 0, \\ b_1^\varepsilon(0) = \dfrac{A_0^\varepsilon - A_0 - \varepsilon A_1}{\varepsilon^2}, \quad w_1^\varepsilon(0) = 0. \end{cases} \tag{4.5.11}$$

We now put $\widetilde{E}_1(t) := \|b_1^\varepsilon\|_{H^{s-1}} + \|w_1^\varepsilon\|_{Y^s_{(\frac{n}{n-1},\infty],[2,\infty]}}$. Mimicking the previous arguments, we obtain

$$\frac{d}{dt}\widetilde{E}_1(t) \leqslant C_s\left(\varepsilon^2\widetilde{E}_1(t)^2 + (\eta(t)+\varepsilon\eta_1(t))\widetilde{E}_1(t) + c_1\eta_1(t)^2\right), \qquad (4.5.12)$$

The constant C_s can be exactly the same as in (4.5.4) since the quadratic and linear parts in the left hand sides of the first and the second equation of (4.5.11) is the same as (4.5.3) up to multiplications by ε. If necessary, we denote $\sup_{s'\in(2/n+1,s]} C_{s'}$ again by C_s. c_1 is an adjusting constant. Then, the following Lemma completes the proof. This lemma is a modification of Lemma 4.5.9. □

Lemma 4.5.12. *Let $\widetilde{E}_1(t)$ be a function $\mathbb{R}_+ \to \mathbb{R}_+$ depending on a parameter ε and satisfying* (4.5.12) *and* $\limsup_{\varepsilon\to 0}\widetilde{E}_1(0) < \infty$. *Let $\eta(t)$ and $\eta_1(t)$ be as in* (4.5.1) *and* (4.5.8)*, respectively. Then, for large $T > 0$ there exists $\varepsilon_1 = \varepsilon_1(T) \sim \eta e^{-2C_s\eta(T)T}$ such that if $\varepsilon \leqslant \varepsilon_1$ then*

$$\sup_{t\in[0,T]}\widetilde{E}_1(t) \leqslant \frac{\Gamma_2 e^{3C_s\eta(T)T}}{\eta(T)} e^{\varepsilon C_s\eta_1(T)T}.$$

Proof. We fix $T > 0$ and analyze (4.5.12) for $t \in [0,T]$. We write $\eta = \eta(T)$ and $\eta_1 = \eta_1(T)$, for short. Take $\varepsilon_{1,1} > 0$ and $\beta_1 > 0$ so that $\widetilde{E}_1(0) \leqslant \beta_1$ for $\varepsilon \leqslant \varepsilon_{1,1}$. Put $Z_1(t) = \widetilde{E}_1(t)e^{-C_s t(\eta+\varepsilon\eta_1)}$. Then, for $t \leqslant T$,

$$\frac{d}{dt}Z_1(t) \leqslant C_s\eta_1^2 e^{-C_s t(\eta+\varepsilon\eta_1)} + c_1 C_s\varepsilon^2 e^{C_s t(\eta+\varepsilon\eta_1)} Z_1(t)^2$$

with $Z_1(0) = \widetilde{E}_1(0)$. We set

$$\delta_1 := \frac{\sqrt{1+\beta_1}-1}{\beta_1}, \quad \theta_1 := \frac{\delta_1(\eta+\varepsilon\eta_1)e^{C_s T(\eta+\varepsilon\eta_1)}}{2\eta_1^2(e^{C_s T(\eta+\varepsilon\eta_1)}-1)}.$$

Multiplying the above inequality by $\frac{\theta_1}{(1+\theta_1 Z_1)^2}$, we obtain

$$\frac{\theta_1 Z_1'}{(1+\theta_1 Z_1)^2} \leqslant C_s\theta_1\eta_1^2 e^{-C_s t(\eta+\varepsilon\eta_1)} + c_1 C_s\theta_1^{-1}\varepsilon^2 e^{C_s t(\eta+\varepsilon\eta_1)}.$$

Integration over $[0,t]$ gives

$$\frac{1}{1+\theta_1 Z_1} \geqslant \frac{1}{1+\theta_1\widetilde{E}_1(0)} - \theta_1\frac{\eta_1^2}{\eta+\varepsilon\eta_1}(1-e^{-C_s t(\eta+\varepsilon\eta_1)}) \\ - \theta_1^{-1}\frac{c_1\varepsilon^2}{\eta+\varepsilon\eta_1}(e^{C_s t(\eta+\varepsilon\eta_1)}-1). \quad (4.5.13)$$

We now show that for small ε and large T the right hand side is bounded by $\delta_1/2$ from below. We first prove that

$$\frac{1}{1+\theta \widetilde{E}_1(0)} - \delta_1 \geqslant \frac{\delta_1}{2}. \tag{4.5.14}$$

Indeed, if $\varepsilon \leqslant \varepsilon_{1,1}$ then we have

$$\frac{1}{1+\theta_1 \widetilde{E}_1(0)} - \delta_1 = \frac{(2-2\delta_1-\beta_1\delta_1^2 P_{\varepsilon,T})e^{C_sT(\eta+\varepsilon\eta_1)} - 2(1-\delta_1)}{(2+\beta_1\delta_1 P_{\varepsilon,T})e^{C_sT(\eta+\varepsilon\eta_1)} - 2},$$

where $P_{\varepsilon,T} := (\eta+\varepsilon\eta_1)/\eta_1^2$. Recall that $\eta_1 = \Gamma_1 e^{C_sT\eta}$. There exists T_0 depending on Γ_1 and C_s such that if $T \geqslant T_0$ then $P_{\varepsilon,T} \leqslant 1$ for $\varepsilon \leqslant 1$. We suppose $T \geqslant T_0$. Then,

$$\frac{1}{1+\theta_1 \widetilde{E}_1(0)} - \delta_1 \geqslant \frac{e^{C_sT(\eta+\varepsilon\eta_1)} - 2(1-\delta_1)}{(1+\sqrt{1+\beta_1})e^{C_sT(\eta+\varepsilon\eta_1)} - 2} \geqslant \frac{1}{2(1+\sqrt{1+\beta_1})} = \frac{\delta_1}{2}$$

for $T \geqslant T_1 = \exists T_1(C_s,\beta_1)$, where we have used the relation $1-2\delta_1-\beta_1\delta_1^2 = 0$. We next consider the inequality

$$\frac{\eta_1^2}{\eta+\varepsilon\eta_1}(1-e^{-C_sT(\eta+\varepsilon\eta_1)})\theta_1^2 - \delta_1\theta_1 + \frac{c_1\varepsilon^2}{\eta+\varepsilon\eta_1}(e^{C_sT(\eta+\varepsilon\eta_1)}-1) \leqslant 0. \tag{4.5.15}$$

Since θ_1 is the minimizer of the left hand side, we see that if

$$\varepsilon \leqslant \frac{\delta_1(\eta+\varepsilon\eta_1)}{\sqrt{c_1}\eta_1}\sqrt{\frac{e^{C_sT(\eta+\varepsilon\eta_1)}}{(e^{C_sT(\eta+\varepsilon\eta_1)}-1)^2}}$$

then (4.5.15) holds. We assume $\varepsilon \leqslant \eta/\eta_1$. Then, $\eta+\varepsilon\eta_1 \leqslant 2\eta$ and so

$$\frac{\delta_1(\eta+\varepsilon\eta_1)}{\sqrt{c_1}\eta_1}\sqrt{\frac{e^{C_sT(\eta+\varepsilon\eta_1)}}{(e^{C_sT(\eta+\varepsilon\eta_1)}-1)^2}} \geqslant \frac{\delta_1\eta}{\sqrt{c_1}\eta_1}\sqrt{\frac{e^{C_sT(\eta+\varepsilon\eta_1)}}{(e^{C_sT(\eta+\varepsilon\eta_1)})^2}} \geqslant \frac{\delta_1\eta}{\sqrt{c_1}\eta_1 e^{C_sT\eta}}$$

We denote the most right hand side by $\varepsilon_{1,2}$. Then, (4.5.15) holds if $\varepsilon \leqslant \varepsilon_{1,2}$. Note that, in this case, the $\varepsilon \leqslant eta/\eta_1$ is automatically satisfied. The right hand side of (4.5.13) is monotone decreasing in t. Hence, plugging (4.5.14) and (4.5.15) to (4.5.13), we obtain $(1+\theta_1 Z_1(t))^{-1} \geqslant \delta_1/2$ for $t \leqslant T$, which implies

$$Z_1(t) \leqslant 3\sqrt{1+\beta_1}\theta_1^{-1}$$

for $t \leqslant T$. We set $\varepsilon_1 = \min(\varepsilon_{1,1}, \varepsilon_{1,2})$. Notice that $\varepsilon_{1,1}$ is independent of T, and that if T is large then $\varepsilon_{1,2} \sim \eta e^{-2C_sT\eta}$ by (4.5.8). Hence, for all $T > \max(T_0, T_1)$ and $\varepsilon \leqslant \varepsilon_1 = \varepsilon_1(\beta, T)$, we conclude that

$$\sup_{t\in[0,T]} \widetilde{E}_1(t) \leqslant \frac{3\sqrt{1+\beta_1}\Gamma_1^2}{\delta_1\eta} e^{3C_sT\eta} e^{\varepsilon C_sT\eta_1}.$$

We set $\Gamma_2 := 3\sqrt{1+\beta_1}\delta_1^{-1}\Gamma_1^2$ to obtain the desired estimate. $\square$

4.5.4 Proof the theorem – part 3: higher order

We finally consider the higher order expansion. Assume that the constant k in Assumption 4.5.3 is bigger than one. It is because if $k=1$ then the proof is already finished with Proposition 4.5.11. The proof is based on the induction argument. We make following notations and definitions: Our goal is to show that the asymptotics

$$\begin{cases} a^\varepsilon = a_0 + \displaystyle\sum_{j=1}^{m} \varepsilon^j a_j + O(\varepsilon^{m+1}) \quad \text{in } L^\infty([0,T], H^{s-2m+1}(\mathbb{R}^n)), \\ \phi^\varepsilon = \phi_0 + \displaystyle\sum_{j=1}^{m} \varepsilon^j \phi_j + O(\varepsilon^{m+1}) \quad \text{in } L^\infty([0,T], Y^{s-2m+3}_{(\frac{n}{n-2},\infty],(\frac{n}{n-1},\infty]}(\mathbb{R}^n)) \end{cases} \tag{4.5.16}$$

for $m=k$. We introduce the system

$$\begin{cases} \partial_t a_j + \displaystyle\sum_{i_1+i_2=j} \nabla\phi_{i_1}\cdot\nabla a_{i_2} + \sum_{i_1+i_2=j} \frac{1}{2} a_{i_1}\Delta\phi_{i_2} - i\frac{1}{2}\Delta a_{j-1} = 0, \qquad a_j(0) = A_j \\ \partial_t \phi_j + \displaystyle\sum_{i_1+i_2=j} \frac{1}{2}\nabla\phi_i\cdot\nabla\phi_j + \lambda \sum_{i_1+i_2=j} (|x|^{-(n-2)} * \operatorname{Re}(a_{i_1}\overline{a_{i_2}})) = 0 \quad \phi_j(0) = 0. \end{cases} \tag{4.5.17}$$

We define the following function:

$$\eta_j(T) := \frac{\Gamma_j}{\eta(T)^{j-1}} e^{(2j-1)C_s\eta(T)T} \tag{4.5.18}$$

with $\eta(T)$ is an increasing function defined in (4.5.1), C_s is the same constant as in (4.5.4) (and in (4.5.12)) depending on s and n, Γ_1 and Γ_2 are as in Propositions (4.5.8) and (4.5.11), respectively, and Γ_j ($j \geqslant 3$) is a constant depending only on A_0^ε to be chosen later. Note that

$$\eta_m(T) \gg \eta_{m-1}(T) \gg \cdots \gg \eta_1(T) \gg \eta(T) > 0$$

for large T. The following two propositions complete the proof by induction.

Proposition 4.5.13. *Let Assumption 4.5.3 be satisfied for some $k \geqslant 2$. Let (a_0,ϕ_0) be the global solution to (4.1.3) given in Theorem 4.3.2. Fix $k_0 \in [1, k-1]$. Assume that $(a_j,\phi_j) \in C([0,\infty); H^{s-2j+3} \times Y^{s-2j+5}_{(\frac{n}{n-2},\infty],(\frac{n}{n-1},\infty]})$ ($1 \leqslant j \leqslant k_0$) exist and all of them solve (4.5.17). We further assume that there exists Γ_{k_0+1} such that*

$$\varlimsup_{\varepsilon\to 0} \sup_{t\in[0,T]} \left(\left\| \frac{a^\varepsilon - \sum_{j=0}^{k_0} \varepsilon^j a_j}{\varepsilon^{k_0+1}} \right\|_{H^{s-2k_0+1}} + \left\| \frac{\nabla(\phi^\varepsilon - \sum_{j=0}^{k_0}\varepsilon^j\phi_j)}{\varepsilon^{k_0+1}} \right\|_{Y^{s-2k_0+2}_{(\frac{n}{n-1},\infty],[2,\infty])}} \right)$$

is bounded by $\eta_{k_0+1}(T)$ *defined in* (4.5.18) *for any fixed* $T > 0$. *Then, there exists* $(a_{k_0+1}, \phi_{k_0+1}) \in C([0,\infty); H^{s-2k_0+1} \times Y^{s-2k_0+3}_{(\frac{n}{n-2},\infty],(\frac{n}{n-1},\infty]})$ *which solves* (4.5.17) *and satisfies*

$$\sup_{t\in[0,T]} \left(\|a_{k_0+1}\|_{H^{s-2k_0+1}} + \|\nabla\phi_{k_0+1}\|_{Y^{s-2k_0+2}_{(\frac{n}{n-1},\infty],[2,\infty])}} \right) \leqslant \eta_{k_0+1}(T).$$

Proposition 4.5.14. *Let Assumption 4.5.3 be satisfied for some* $k \geqslant 2$. *Let* (a_0, ϕ_0) *be a global solution to* (4.1.3) *given in Theorem 4.3.2. Fix* $k_0 \in [1, k-1]$. *Assume that, for all* $1 \leqslant j \leqslant k_0 + 1$, *the solution* $(a_j, \phi_j) \in C([0,\infty); H^{s-2j+3} \times (L^{\frac{n}{n-2}+} \cap L^\infty))$ *of* (4.5.17) *exists and satisfies*

$$\sup_{t\in[0,T]} \|a_j\|_{H^{s-2j+3}} + \|\nabla\phi_j\|_{Y^{s-2j+2}_{(\frac{n}{n-1},\infty],[2,\infty])}} \leqslant \eta_j(T).$$

Then, for any fixed $T > 0$,

$$\sup_{t\in[0,T]} \left(\left\| \frac{a^\varepsilon - \sum_{j=0}^{k_0+1} \varepsilon^j a_j}{\varepsilon^{k_0+2}} \right\|_{H^{s-2k_0-1}} + \left\| \frac{\nabla(\phi^\varepsilon - \sum_{j=0}^{k_0+1} \varepsilon^j \phi_j)}{\varepsilon^{k_0+2}} \right\|_{Y^{s-2k_0}_{(\frac{n}{n-1},\infty],[2,\infty])}} \right)$$

is bounded uniformly in $\varepsilon \in (0, \varepsilon_{k_0+2}]$. *In particular, the asymptotics* (4.5.16) *holds with* $m = k_0+1$ *for* $\varepsilon \in (0, \varepsilon_{k_0+2}]$. ε_{k_0+2} *can be chosen so that* $\varepsilon_{k_0+2} \leqslant C\eta(T)e^{-3C_s\eta(T)T}$. *Moreover, there exists a constant* Γ_{k_0+2} *depending only on* A_0^ε *such that* $\eta_{k_0+2}(T)$ *defined in* (4.5.18) *bounds*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \left(\left\| \frac{a^\varepsilon - \sum_{j=0}^{k_0+1} \varepsilon^j a_j}{\varepsilon^{k_0+2}} \right\|_{H^{s-2k_0-1}} + \left\| \frac{\nabla(\phi^\varepsilon - \sum_{j=0}^{k_0+1} \varepsilon^j \phi_j)}{\varepsilon^{k_0+2}} \right\|_{Y^{s-2k_0}_{(\frac{n}{n-1},\infty],[2,\infty])}} \right)$$

for any fixed large $T > 0$.

Indeed, once these two propositions are shown, we immediately obtain the theorem: Proposition 4.5.11 implies that the assumption of Proposition 4.5.13 is satisfied for $k_0 = 1$. Then, by Proposition 4.5.13, the assumption of Proposition 4.5.14 is met for $k_0 = 1$, which ensures the assumption of Proposition 4.5.13 for $k_0 = 2$. After repeating this argument $k-1$ times, we see that Proposition 4.5.14 holds for $k_0 = k-1$. Then, this gives (4.5.16) with $m = k$.

Proofs

Before the proof, we introduce the notation. We write

$$b_m^\varepsilon = \frac{a^\varepsilon - \sum_{j=0}^m \varepsilon^j a_j}{\varepsilon^{m+1}}, \qquad w_m^\varepsilon = \frac{\nabla\phi^\varepsilon - \sum_{j=0}^m \varepsilon^j \nabla\phi_j}{\varepsilon^{m+1}}.$$

An elementary computation shows that $(b^\varepsilon_m, w^\varepsilon_m)$ satisfies

$$\begin{aligned}
&\partial_t b^\varepsilon_m + \varepsilon^{m+1}\left(w^\varepsilon_m\cdot\nabla b^\varepsilon_m + \frac{1}{2}b^\varepsilon_m\nabla\cdot w^\varepsilon_m\right)\\
&\qquad+\sum_{\ell=0}^{m}\varepsilon^\ell\left(w^\varepsilon_m\cdot\nabla a_\ell + v_\ell\cdot\nabla b^\varepsilon_m + \frac{1}{2}b^\varepsilon_m\nabla\cdot v_\ell + \frac{1}{2}a_\ell\nabla\cdot v^\varepsilon_m\right)\\
&\quad+\sum_{\ell=0}^{m-1}\varepsilon^\ell\sum_{i,j\leqslant m, i+j=m+1+\ell}\left(v_i\cdot\nabla a_j + \frac{1}{2}a_i\nabla\cdot v_j\right) - i\frac{1}{2}\Delta a_m = i\frac{\varepsilon}{2}\Delta b^\varepsilon_m,
\end{aligned} \tag{4.5.19}$$

$$\begin{aligned}
&\partial_t w^\varepsilon_m + \varepsilon^{m+1}\left(w^\varepsilon_m\cdot\nabla w^\varepsilon_m + \lambda\nabla(|x|^{-(n-2)} * |b^\varepsilon_m|^2)\right)\\
&\qquad+\sum_{\ell=0}^{m}\varepsilon^\ell((w^\varepsilon_m\cdot\nabla v_\ell + v_\ell\cdot\nabla w^\varepsilon_m) + \lambda\nabla(|x|^{-(n-2)} * \mathrm{Re}(a_\ell\overline{b^\varepsilon_m}))\\
&\quad+\sum_{\ell=0}^{m-1}\varepsilon^\ell\sum_{i,j\leqslant m, i+j=m+1+\ell}\left(v_i\cdot\nabla v_j + \lambda\nabla(|x|^{-(n-2)} * \mathrm{Re}(a_i\overline{a_j}))\right) = 0,
\end{aligned} \tag{4.5.20}$$

and

$$b^\varepsilon_m(0) = \sum_{j=0}^{k-1-m}\varepsilon^j A_{j+m+1} + \varepsilon^{k-m}r^\varepsilon_{k+1}, \qquad w^\varepsilon_m(0) = 0 \tag{4.5.21}$$

as long as $(a_0, v_0) := (a_0, \nabla\phi_0)$ and $(a_j, v_j) := (a_j, \nabla\phi_j)$ $(1 \leqslant j \leqslant m)$ solve (4.1.3) and (4.5.17), respectively, where r^ε_{k+1} is $\varepsilon^{-k+1}(A^\varepsilon_0 - \sum_{j=0}^k \varepsilon^j A_j)$. If Assumption 4.5.3 is satisfied then r^ε_{k+1} is bounded in H^{s+1} as $\varepsilon \to 0$.

Proof of Proposition 4.5.13. By assumption, $(b^\varepsilon_{k_0}, w^\varepsilon_{k_0})$ is uniformly bounded in

$$L^\infty([0,T), H^{s-2k_0+1} \times (H^{s-2k_0+2} \cap L^{\frac{n}{n-1}+}))$$

in the limit $\varepsilon \to 0$. Note that $Y^{s-2k_0+2}_{(\frac{n}{n-1},\infty],[2,\infty]} = H^{s-2k_0+2} \cap L^{\frac{n}{n-1}+}$ since $n/(n-1) < 2$ for $n \geqslant 3$. Therefore, extracting a subsequence, there exists a weak limit, denoted by (a_{k_0+1}, v_{k_0+1}), in the same class. Moreover, we obtain the bound

$$\sup_{t\in[0,T]}\left(\|a_{k_0+1}\|_{H^{s-2k_0+1}} + \|\nabla\phi_{k_0+1}\|_{Y^{s-2k_0+2}_{(\frac{n}{n-1},\infty],[2,\infty]}}\right) \leqslant \eta_{k_0+1}(T).$$

by the lower semi-continuity of the weak limit. Since $(b^\varepsilon_{k_0}, w^\varepsilon_{k_0})$ solves (4.5.19)–(4.5.21), we see that (a_{k_0+1}, v_{k_0+1}) solves

$$\begin{cases} \partial_t a_j + \sum_{i_1+i_2=j} v_{i_1}\cdot\nabla a_{i_2} + \sum_{i_1+i_2=j} \frac{1}{2} a_{i_1}\nabla\cdot v_{i_2} - i\frac{1}{2}\Delta a_{j-1} = 0, & a_j(0) = A_j \\ \partial_t v_j + \nabla \sum_{i_1+i_2=j} \frac{1}{2} v_i\cdot v_j + \lambda\nabla \sum_{i_1+i_2=j} (|x|^{-(n-2)} * \mathrm{Re}(a_{i_1}\overline{a_{i_2}})) = 0 & v_j(0) = 0. \end{cases} \tag{4.5.22}$$

for $j = k_0+1$. By the way, once we know (a_j, v_j) $(j = [0, k_0])$, we can solve this system directly by a standard argument and obtain unique solution (a_{k_0+1}, v_{k_0+1}) in the same space. Therefore, the above weak limit is the unique solution to (4.5.22). We now define ϕ_{k_0+1} by

$$\phi_{k_0+1}(t) = -\int_0^t \left(\sum_{i_1+i_2=j} \frac{1}{2} v_i\cdot v_j + \lambda \sum_{i_1+i_2=j} (|x|^{-(n-2)} * \mathrm{Re}(a_{i_1}\overline{a_{i_2}})) \right) ds.$$

Then, $\nabla\phi_{k_0+1} = v_{k_0+1}$ holds by the uniqueness of (4.5.22). Hence, $\nabla\phi_{k_0+1}$ is the unique solution to (4.5.17) for $j = k_0+1$. By definition of ϕ_{k_0+1}, we see it decays at spatial infinity. Thus, Lemma 2.2.1 provides

$$\|\phi_{k_0+1}\|_{Y^{s-2k_0+3}_{(\frac{n}{n-2},\infty],(\frac{n}{n-1},\infty]}} \leqslant C\,\|\nabla\phi_{k_0+1}\|_{Y^{s-2k_0+2}_{(\frac{n}{n-1},\infty],[2,\infty]}}.$$

T is arbitrary, and so we obtain the proposition. □

Proof. By assumption, we can define $(b^\varepsilon_{k_0+1}, w^\varepsilon_{k_0+1})$ solving (4.5.19)–(4.5.21). We will bound

$$\widetilde{E}_{k_0+1}(t) := \left\|b^\varepsilon_{k_0+1}(t)\right\|_{H^{s-2k_0-1}} + \left\|w^\varepsilon_{k_0+1}(t)\right\|_{Y^{s-2k_0}_{(\frac{n}{n-1},\infty],[2,\infty]}}.$$

Recall that the quadratic part and the linear part of (4.5.19)–(4.5.20) are the same as (4.5.11). Hence, we deduce by the standard energy estimate that, for any fixed $T > 0$,

$$\frac{d}{dt}\widetilde{E}_{k_0+1}(t) \leqslant C_s(\varepsilon^{k_0+1}\widetilde{E}_{k_0+1}(t)^2 + \mu^\varepsilon_{k_0+1}\widetilde{E}_{k_0+1}(t) + c_{k_0+1}\nu^\varepsilon_{k_0+1}) \tag{4.5.23}$$

holds for all $t \in [0, T]$. Here, we define

$$\mu^\varepsilon_{k_0+1} = \mu^\varepsilon_{k_0+1}(T) := \eta(T) + \sum_{j=1}^{k_0+1} \varepsilon^j \eta_j(T)$$

which bounds the constant part

$$\sup_{t\in[0,T]} \left(\left\| \sum_{\ell=0}^{k_0+1} \varepsilon^\ell a_\ell \right\|_{H^{s-2k_0+1}} + \|v_0\|_{Y^{s-2k_0+2}_{(2^*,\infty],(2,\infty]}} + \left\| \sum_{\ell=1}^{k_0+1} \varepsilon^\ell v_\ell \right\|_{Y^{s-2k_0+2}_{(\frac{n}{n-1},\infty],[2,\infty]}} \right)$$

and

$$\nu^\varepsilon_{k_0+1} = \nu^\varepsilon_{k_0+1}(T) := \eta_{k_0+1}(T) + \sum_{\ell=0}^{k_0} \varepsilon^\ell \sum_{i=\ell+1}^{k_0+1} \eta_i(T)\eta_{k_0+2+\ell-i}(T)$$

which is an upper bound of the linear terms

$$\sup_{t\in[0,T]} \left(\frac{1}{2} \|\Delta a_{k_0}\|_{H^{s-2k_0-1}} + C \sum_{\ell=0}^{k_0} \varepsilon^\ell \sum_{\substack{i,j\leqslant k_0+1,\\ i+j=k_0+2+\ell}} \left(\|v_i\|_{Y^{s-2k_0+2}_{(\frac{n}{n-1},\infty],[2,\infty]}} \|v_j\|_{Y^{s-2k_0+2}_{(\frac{n}{n-1},\infty],[2,\infty]}} + \|a_i\|_{H^{s-2k_0+1}} \|a_j\|_{H^{s-2k_0+1}} \right) \right)$$

up to an adjusting constant c_{k_0+1}.

Uniform bound of $\widetilde{E}_{k_0+1}$

We now show that $\sup_{t\in[0,T]} \widetilde{E}_{k_0+1}(t)$ is uniformly bounded for small ε. We keep fixing $T > 0$. By Assumption (4.5.3), we see that there exists a positive constant β_{k_0+1} depending only on A^ε_0 such that $\widetilde{E}_{k_0+1}(0) \leqslant \beta_{k_0+1}$ holds for $\varepsilon \in (0, 1]$. Set a function

$$Z_{k_0+1}(t) := \widetilde{E}_{k_0+1}(t) \exp(-C_s \mu^\varepsilon_{k_0+1}(T)t)$$

and two constants

$$\begin{aligned} \delta_{k_0+1} &:= (1 + \sqrt{1 + \beta_{k_0+1}})^{-1}, \\ \theta_{m+1} &:= \frac{\delta\mu^\varepsilon_{k_0+1}(T)}{2c_{k_0+1}\nu^\varepsilon_{k_0+1}(T)(1 - e^{-C_s\mu^\varepsilon_{k_0+1}(T)T})}. \end{aligned}$$

Then, multiplying the both sides of (4.5.23) by $\frac{\theta_{k_0+1}\exp(-C_s t\mu^\varepsilon_{k_0+1})}{(1+\theta_{k_0+1}Z_{k_0+1}(t))^2}$, we obtain

$$\frac{\theta_{k_0+1}Z'_{k_0+1}(t)}{(1 + \theta_{k_0+1}Z_{k_0+1}(t))^2} \leqslant C_s\varepsilon^{k_0+2}e^{C_s t\mu^\varepsilon_{k_0+1}}\theta^{-1}_{k_0+1} + C_s c_{k_0+1}\nu^\varepsilon_{k_0+1}e^{-C_s t\mu^\varepsilon_{k_0+1}}\theta_{k_0+1},$$

where we denote $\mu^\varepsilon_{k_0+1}(T)$ and $\nu^\varepsilon_{k_0+1}(T)$ by $\mu^\varepsilon_{k_0+1}$ and $\mu^\varepsilon_{k_0+1}$, respectively, for short. Integration over $[0, t]$ gives

$$\begin{aligned} \frac{1}{1 + \theta_{k_0+1}Z_{k_0+1}(t)} &\geqslant \frac{1}{1 + \theta_{k_0+1}\widetilde{E}_{k_0+1}(0)} \\ &- \frac{c_{k_0+1}\nu^\varepsilon_{k_0+1}}{\mu^\varepsilon_{k_0+1}}(1 - e^{-C_s t\mu^\varepsilon_{k_0+1}})\theta_{k_0+1} - \frac{\varepsilon^{k_0+2}}{\mu^\varepsilon_{k_0+1}}(e^{C_s t\mu^\varepsilon_{k_0+1}} - 1)\theta^{-1}_{k_0+1}. \end{aligned} \tag{4.5.24}$$

Let us show that the right hand side of (4.5.24) is bounded by $\delta_{k_0+1}/2$ from below. For simplicity, in the followings, we omit the index $k_0 + 1$ and denote β_{k_0+1}, c_{k_0+1}, δ_{k_0+1}, $\mu^\varepsilon_{k_0+1}$, $\nu^\varepsilon_{k_0+1}$, and θ_{k_0+1} by β, c, δ, μ^ε, ν^ε, and θ, respectively. We also omit T variable in $\eta(T)$ and $\eta_j(T)$. By the fact that $\eta_{j+1} \gg \eta_j$ for each j and large T and by definitions of μ^ε and ν^ε, if T is large then $c\nu^\varepsilon \geqslant \mu^\varepsilon$ holds for all $\varepsilon \in [0,1]$. Then, replacing T with larger one if necessary, we obtain

$$\begin{aligned}
\frac{1}{1+\theta\widetilde{E}_{k_0+1}(0)} - \delta &\geqslant \frac{1}{1+\theta\beta} - \delta = \frac{e^{C_s\mu^\varepsilon T}-1}{e^{C_s\mu^\varepsilon T}(1+\frac{\mu^\varepsilon}{c\nu^\varepsilon}\frac{\delta\beta}{2})-1} - \delta \\
&\geqslant \frac{e^{C_s\mu^\varepsilon T}-1}{e^{C_s\mu^\varepsilon T}(1+\frac{\delta\beta}{2})-1} - \delta = \frac{e^{C_s\mu^\varepsilon T}(2-2\delta-\delta^2\beta)-2+2\delta}{e^{C_s\mu^\varepsilon T}(2+\delta\beta)-2} \\
&\geqslant \frac{2-2\delta-\delta^2\beta}{2(2+\delta\beta)} = \frac{1}{2(2+\delta\beta)} = \frac{1+\sqrt{1+\beta}}{2(1+\sqrt{1+\beta})^2} = \frac{\delta}{2}, \quad (4.5.25)
\end{aligned}$$

where we have used the relation $1-2\delta-\delta^2\beta = 0$. Moreover, θ is the minimizer of the quantity

$$\frac{c\nu^\varepsilon}{\mu^\varepsilon}(1-e^{-C_s\mu^\varepsilon T})\theta^2 - \delta\theta + \frac{\varepsilon^{k_0+2}}{\mu^\varepsilon}(e^{C_s\mu^\varepsilon T}-1)$$

and so this quantity becomes less than or equal to zero if

$$\varepsilon \leqslant \left(\frac{\delta^2(\mu^\varepsilon)^2 e^{C_s\mu^\varepsilon T}}{c\nu^\varepsilon(e^{C_s\mu^\varepsilon T}-1)^2}\right)^{\frac{1}{k_0+2}}. \quad (4.5.26)$$

We now replace this condition with stronger but clearer one. We first let ε be so small that

$$\varepsilon \leqslant \min_{j\in[1,k_0+1]} \left(\frac{\eta}{\eta_j}\right)^{\frac{1}{j}} = \min_{j\in[1,k_0+1]} \frac{\eta}{\Gamma_j^{1/j} e^{(2-1/j)C_s\eta(T)T}}. \quad (4.5.27)$$

For such ε, we have $\mu^\varepsilon \leqslant (k_0+2)\eta$ and, by definition of η_j (4.5.18),

$$\begin{aligned}
\nu^\varepsilon &= \eta_{k_0+1} + \sum_{\ell=0}^{k_0} \varepsilon^\ell \sum_{i=\ell+1}^{k_0+1} \eta_i\eta_{k_0+2+\ell-i} \\
&\leqslant \eta_{k_0+1} + \widetilde{\Gamma}_1 \frac{e^{(2k_0+2)C_s\eta T}}{\eta^{k_0}} + \frac{e^{(2k_0+3)C_s\eta T}}{\eta^{k_0+1}} \sum_{\ell=1}^{k_0} \widetilde{\Gamma}_2\eta\left(\frac{\varepsilon}{\eta}e^{(2-1/\ell)C_s\eta T}\right)^\ell \\
&\leqslant \eta_{k_0+1} + \widetilde{\Gamma}_1 \frac{e^{(2k_0+2)C_s\eta T}}{\eta^{k_0}} + \widetilde{\Gamma}_3 \frac{e^{(2k_0+3)C_s\eta T}}{\eta^{k_0}} \\
&\leqslant \widetilde{\Gamma}_4 \frac{e^{(2k_0+3)C_s\eta T}}{\eta^{k_0}},
\end{aligned}$$

where $\widetilde{\Gamma}_i$ is a constant depending on k_0 and Γ_j $(1 \leqslant j \leqslant k_0+1)$. Therefore, the right hand side of (4.5.26) is bounded below by

$$\begin{aligned}
\left(\frac{\delta^2(\mu^\varepsilon)^2 e^{C_s\mu^\varepsilon T}}{c\nu^\varepsilon(e^{C_s\mu^\varepsilon T}-1)^2}\right)^{\frac{1}{k_0+2}} \geqslant & \left(\frac{\delta^2\eta^2}{c\nu^\varepsilon e^{C_s\mu^\varepsilon T}}\right)^{\frac{1}{k_0+2}} \\
&\geqslant \widetilde{\Gamma}_5\left(\frac{\eta^2}{(\eta^{-k_0}e^{(2k_0+3)C_s\eta T})e^{(k_0+2)C_s\eta T}}\right)^{\frac{1}{k_0+2}} \\
&=\widetilde{\Gamma}_5\frac{\eta}{e^{(3-\frac{1}{k_0+2})C_s\eta T}} \geqslant \widetilde{\Gamma}_5\frac{\eta}{e^{3C_s\eta T}} =: \varepsilon^{k_0+2},
\end{aligned}$$

where $\widetilde{\Gamma}_5$ depends on $\widetilde{\Gamma}_4$, β, and c. Then, the condition $\varepsilon \leqslant \varepsilon_{k_0+2}$ ensures (4.5.26) and so

$$\delta - \frac{c\nu^\varepsilon}{\mu^\varepsilon}(1-e^{-C_s\mu^\varepsilon T})\theta - \frac{\varepsilon^{k_0+2}}{\mu^\varepsilon}(e^{C_sT\mu^\varepsilon}-1)\theta^{-1} \geqslant 0. \tag{4.5.28}$$

Note that ε_{k_0+2} is smaller than the right hand side of (4.5.27) and so that $\varepsilon \leqslant \varepsilon_{k_0+2}$ is stronger than (4.5.27). Furthermore, plugging (4.5.25) and (4.5.28) to (4.5.24), we obtain

$$\sup_{t\in[0,T]} \widetilde{E}_{k_0+1}(t) \leqslant 3\sqrt{1+\beta}\,\theta^{-1}e^{C_s\mu^\varepsilon T} \leqslant \frac{6c\sqrt{1+\beta}\,\nu^\varepsilon}{\delta\mu^\varepsilon}e^{C_s\mu^\varepsilon T}, \tag{4.5.29}$$

which is the desired uniform bound. Indeed, the right hand side is bounded by

$$\frac{6c\sqrt{1+\beta}\,\widetilde{\Gamma}_4}{\delta\eta^{k_0+1}}e^{C_s(3k_0+5)\eta T}$$

as long as $\varepsilon \leqslant \varepsilon_{k_0+2}$. We finally confirm that the right hand side of (4.5.29) tends to $\eta_{k_0+2}(T)$ with a suitable constant. By definition, it holds that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mu^\varepsilon \lim_{\varepsilon\to 0}\mu^\varepsilon_{k_0+1}(T) = \eta(T), \\
&\lim_{\varepsilon\to 0}\nu^\varepsilon = \lim_{\varepsilon\to 0}\nu^\varepsilon_{k_0+1}(T) = \eta_{k_0+1} + \sum_{i=1}^{k_0+1}\eta_i\eta_{k_0+2-i} \leqslant \frac{\hat{\Gamma}_{k_0+2}}{\eta(T)^{k_0}}e^{(2k_0+2)C_s\eta(T)T},
\end{aligned}$$

where $\hat{\Gamma}_{k_0+2}$ depends on k_0 and Γ_j $(1 \leqslant j \leqslant k_0+1)$. Therefore, we end up with the estimate

$$\begin{aligned}
\limsup_{\varepsilon\to 0}\sup_{t\in[0,T]} e_{m+1}(t) &\leqslant \frac{6\sqrt{1+\beta}(\hat{\Gamma}_{k_0+2}\eta(T)^{-k_0}e^{(2k_0+2)C_s g(T)T})}{\delta\eta(T)}e^{C_s\eta(T)T} \\
&=: \frac{\Gamma_{k_0+2}}{\eta(T)^{k_0+1}}e^{(2k_0+3)C_s\eta(T)T} = \eta_{k_0+2}(T),
\end{aligned}$$

which completes the proof. $\square$

Appendix A

Tool box

A.1 Basic inequalities

A.1.1 The Hölder inequality

Lemma A.1.1 (The Young inequality)**.** *Let* $1 < p, q < \infty$ *and* $p^{-1}+q^{-1} = 1$*. Then*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

for all $a, b > 0$*.*

Proof. By convexity of e^x,

$$ab = e^{\frac{\log a^p}{p}+\frac{\log b^q}{q}} \leqslant \frac{e^{\log a^p}}{p} + \frac{e^{\log b^q}}{q} = \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Lemma A.1.2 (The Hölder inequality)**.** *Let* $1 \leqslant p, q, r \leqslant \infty$ *satisfy* $p^{-1} = q^{-1} + r^{-1}$*. Then it holds for all* $f \in L^q(\mathbb{R}^n)$ *and* $g \in L^r(\mathbb{R}^n)$ *that*

$$\|fg\|_{L^p(\mathbb{R})} \leqslant \|f\|_{L^q(\mathbb{R})} \|g\|_{L^r(\mathbb{R})} .$$

Proof. If one of p, q, r is infinity then the result is trivial. So, we assume $p, q, r, < \infty$. In the case $p = 1$, it follows from the Young inequality that

$$\begin{aligned}\frac{\|fg\|_{L^1(\mathbb{R})}}{\|f\|_{L^q(\mathbb{R})} \|g\|_{L^r(\mathbb{R})}} &= \int_{\mathbb{R}^n} \left|\frac{f}{\|f\|_{L^q(\mathbb{R})}}\right| \left|\frac{g}{\|g\|_{L^r(\mathbb{R})}}\right| dx \\ &\leqslant \frac{1}{q\,\|f\|^q_{L^q(\mathbb{R})}} \int_{\mathbb{R}^n} |f|^p dx + \frac{1}{r\,\|g\|^r_{L^r(\mathbb{R})}} \int_{\mathbb{R}^n} |g|^r dx = 1.\end{aligned}$$

If $p > 1$ then $\|fg\|_{L^p} = \||fg|^p\|^{1/p}_{L^1} \leqslant \||f|^p\|^{1/p}_{L^{q/p}} \||g|^p\|^{1/p}_{L^{r/p}} = \|f\|_{L^q} \|g\|_{L^r}$. □

Corollary A.1.3. *Let f be a complex-valued function on $\mathbb{R}^n$. If f belongs to both $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ ($1 \leqslant p < q \leqslant \infty$), then f belongs to $L^r(\mathbb{R}^n)$ for all $r \in [p, q]$. In particular,*

$$\|f\|_{L^r(\mathbb{R})} \leqslant \|f\|^{\theta}_{L^p(\mathbb{R})} \|f\|^{1-\theta}_{L^q(\mathbb{R})},$$

where $\theta \in [0,1]$ is given by $\theta/p + (1-\theta)/q = 1/r$.

Proof. It is an immediately consequence of the Hölder inequality. □

A.1.2 The Sobolev inequality

Lemma A.1.4 (The Sobolev inequality)**.** *Let $1 \leqslant q \leqslant p < \infty$ and $\alpha = n(q^{-1} - p^{-1}) \geqslant 0$. Then, there exists a constant $C > 0$ such that*

$$\|f\|_{L^p} \leqslant C \, \||\nabla|^{\alpha} f\|_{L^q}$$

holds, provided that the right hand side is finite. Moreover, if $p > 1$ and $s > n/p$ then there exists a constant $C > 0$ such that

$$\|f\|_{L^\infty} \leqslant C \|f\|_{W^{s,p}(\mathbb{R}^n)}$$

holds, provided the right hand side is finite.

For some more properties of Sobolev spaces we refer to [1, 7, 70].

A.1.3 The Hardy-Littlewood-Sobolev inequality

Lemma A.1.5 (The Hardy-Littlewood-Sobolev inequality)**.** *Let $\gamma \in (0, n)$ and $1 < p < q < \infty$ satisfies*

$$\frac{1}{p} = \frac{1}{q} - \frac{n-\gamma}{n}.$$

Then, there exists a constant $C > 0$ such that

$$\left\| |x|^{-\gamma} * f \right\|_{L^p} \leqslant C \|f\|_{L^q}$$

holds, provided the right hand side is finite.

A.2 Tools for energy estimates

A.2.1 Gronwall's lemma

Lemma A.2.1 (Gronwall's lemma)**.** *Let g and h be continuous functions on $\mathbb{R}$. If f satisfies the inequality*

$$f'(t) \leqslant g(t) f(t) + h(t),$$

then it holds that

$$f(t) \leqslant e^{\int_0^t g(s) ds} \left(f(0) + \int_0^t h(s) e^{-\int_0^s g(\sigma) d\sigma} \right).$$

Proof. Multiply the both side by $e^{-\int_0^t g(s)ds}$ to obtain

$$\frac{d}{dt}(fe^{-\int_0^t g(s)ds})(t) \leqslant h(t)e^{-\int_0^t g(s)ds}$$

Integration over $(0,t)$ gives

$$f(t)e^{-\int_0^t g(s)ds} - f(0) \leqslant \int_0^t h(s)e^{-\int_0^s g(\sigma)d\sigma}.$$

Hence the Lemma. □

The point is that f is bounded by $f(0)$ and the coefficient functions g and h. We mainly use this lemma to estimates the energy. By this lemma, we can give the upper bound of the energy from its initial value.

A.2.2 Commutator estimate

We denote $1-\Delta$ by Λ. The following lemma can be found [6, 40].

Lemma A.2.2 (Commutator estimate)**.** *Let $s \geqslant 0$ be a real number and and $k \geqslant 0$ be an integer. There exists $C > 0$ such that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \leqslant C(\|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|\nabla^k f\|_{H^{s-k}} \|g\|_{L^\infty}).$$

Lemma A.2.3. *Let $s \geqslant 0$ be a real number and and $k \geqslant 0$ be an integer. There exists $C > 0$ such that*

$$\|\Lambda^s(fg)\|_{L^2} \leqslant C(\|f\|_{H^s} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|\nabla^k g\|_{H^{s-k}}), \tag{A.2.1}$$

for all $f \in H^s \cap L^\infty$ and $g \in \dot{H}^k \cap \dot{H}^s \cap L^\infty$, and that

$$\|\Lambda^s \nabla(fg)\|_{L^2} \leqslant C(\|\nabla f\|_{H^s} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|\nabla g\|_{H^s}), \tag{A.2.2}$$

for all $f, g \in \dot{H}^1 \cap \dot{H}^s \cap L^\infty$.

The next lemma is the estimate of a composite function.

Lemma A.2.4. *Let I be closed interval of $\mathbb{R}$. Take a nonnegative integer m. Let $s > -m$ be a real number and let σ be the smallest integer such that $\sigma \leqslant s$. Take a complex-valued function $F \in W^{\sigma+m+1,\infty}(I)$ and let v be a valued in I and such that $|\nabla|^m v \in H^s$. Then, there exists a constant $C = C_{s,I}$ such that*

$$\||\nabla|^m(F(v))\|_{H^s} \leqslant C(1 + \|v\|_{L^\infty})^{\sigma+m} \|F'\|_{W^{\sigma+m,\infty}(I)} \||\nabla|^m v\|_{H^s}$$

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