Ampleness of two-sided tilting complexes

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Abstract

In this paper we define the notion of ampleness for two-sided tilting complexes over finite dimensional algebras and prove its basic properties.

We call a finite dimensional k-algebra A of finite global dimension Fano if $(A^*[-d])^{-1}$ is ample for some $d \geq 0$. For example geometric algebras in the sense of Bondal-Polishchuk are Fano. We give a characterization of representation type of a quiver from a noncommutative algebro-geometric view point, that is, a finite acyclic quiver has finite representation type if and only if its path algebra is fractional Calabi-Yau, and a finite acyclic quiver has infinite representation type if and only if its path algebra is Fano.

0 Introduction

Let X be a nonsingular projective variety over a field k and let ω_X be its canonical bundle. Then the functor $S_X := - \otimes_X^{\mathbf{L}} \omega_X[\dim X] : D^b(\operatorname{coh} X) \longrightarrow D^b(\operatorname{coh} X)$ is the Serre functor ,i.e., $\operatorname{Hom}_X(\mathcal{G}^\cdot, \mathcal{F}^\cdot)^*$ is functorially isomorphic to $\operatorname{Hom}_X(\mathcal{F}^\cdot, S_X(\mathcal{G}^\cdot))$ for $\mathcal{F}^\cdot, \mathcal{G}^\cdot \in D^b(\operatorname{coh} X)$. By this fact, from a noncommutative (or categorical) algebro-geometric view point, one thinks of a triangulated category $\mathcal{T}$ as the derived category of coherent sheaves of some "space" X and of the Serre functor $S_\mathcal{T}$ of $\mathcal{T}$ (if exists) as the derived tensor product of "$\dim X$"-shifted "canonical bundle" ω_X. From this view point, the notion of Calabi-Yau algebra (and Calabi-Yau category) is defined and studied extensively by many researchers.

In this paper we introduce the notion of ampleness for two-sided tilting complexes over finite dimensional k-algebras. Let A be a finite dimensional k-algebra of finite global dimension.

Definition 0.1 (Definition 2.6). *A two-sided tilting complex σ over A is called very ample if* $\mathrm{H}^i(\sigma) = 0$ *for $i \geq 1$ and σ^n is pure for $n \gg 0$. σ is called ample if σ^n is pure for $n \gg 0$.*

In Section 2, we justify this definition by using the theory of noncommutative projective schemes due to Artin-Zhang [AZ] and Polishchuk [Po]. In the theory of noncommutative projective schemes , for a graded coherent ring R over k we attach an imaginary geometric object $\operatorname{proj} R = \big(\operatorname{cohproj} R, \overline{R}, (1)\big)$. An abelian category $\operatorname{cohproj} R$ is considered as the category of coherent sheaves on $\operatorname{proj} R$. (See Section 1.) In Section 2 we show that the following facts hold. If σ is a very ample tilting complex over A, then the tensor algebra $T := T_A(\mathrm{H}^0(\sigma))$ of $\mathrm{H}^0(\sigma)$ over A is a graded connected coherent ring over A and there is a t-structure D^σ defined by σ in $\operatorname{Perf} A$ and its heart $\mathcal{H}^\sigma$ is equivalent to $\operatorname{cohproj} T$. Moreover the following Theorem holds.

Theorem 0.2 (Theorem 2.8). *Let A be a finite dimensional k-algebra of finite global dimension and let σ be a very ample two-sided tilting complex. Then there is a natural equivalence of triangulated categories*

$$D^b\,(\text{mod-}A) \xrightarrow{\sim} D^b\,(\operatorname{cohproj} T)\,.$$

where $T := T_A(\mathrm{H}^0(\sigma))$ is the tensor algebra of $\mathrm{H}^0(\sigma)$ over A.

In [Be] Beilinson showed that $\mathbb{P}^n$ is derived equivalent to a finite dimensional k-algebra. This result has been generalized to other varieties. The above Theorem gives a partial converse.

A finite dimensional k-algebra A of finite global dimension is called Fano if $(A^*[-d])^{-1}$ is ample for some $d \geq 0$.

We give a characterization of representation type of a quiver from noncommutative algebro-geometric view point, that is, a finite acyclic quiver has finite representation type if and only if its path algebra is fractional Calabi-Yau, and a finite acyclic quiver has infinite representation type if and only if its path algebra is Fano.

In [Le] and [GL], geometric notions are introduced to study certain class of algebras. This paper develop a formal aspect of these works.

This paper is generalization of [Mi].

We organize the present paper as follows: in Section 1 we introduce some definitions and results on noncommutative projective schemes :in Section 2 we give the definition of ampleness of two-sided tilting complexes and prove its basic property : in Section 3 we show that some finite dimensional algebra studied before has ample or anti-ample "canonical bundle".

Notation and convention. Throughout this paper k denotes a field. If there would be no confusion, we denote by the same symbol T a two-sided tilting complex T and the functor $- \otimes^{\mathbf{L}}_A T$ induced by T. For a ring A we denote by Mod-A (resp. mod-A) the abelian category of right A-modules (resp. the abelian category of finite right A-modules). For a k-vector space M, we denote by M^* its k-dual vector space.

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1 Preliminaries on Noncommutative Projective Schemes

This section is a summary of the paper [Po] by A. Polishchuk. Although connected $\mathbb{Z}$-algebras are treated in [Po], we treat connected $\mathbb{N}$-graded algebras over some finite dimensional k-algebra A. One can see that the argument in [Po] is applied to our case.

Let k be a field and let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a graded coherent ring over k. We assume that the degree zero part R_0 of R is a finite dimensional algebra over k. $\operatorname{Gr} R$ (resp. $\operatorname{coh} R$) denotes the category of graded right R-modules (resp. finitely presented graded right R-modules). $\operatorname{Tor} R$ (resp. $\operatorname{tor} R$) denote the full subcategory of torsion modules (resp. modules finite dimensional over k). Note that $\operatorname{Tor} R$ and $\operatorname{tor} R$ are dense subcategories of $\operatorname{Gr} R$ and $\operatorname{coh} R$ respectively, hence the quotient categories $\operatorname{QGr} R = \operatorname{Gr} R / \operatorname{Tor} R$ and $\operatorname{cohproj} R = \operatorname{coh} R / \operatorname{tor} R$ are abelian categories.

For a graded right R-module $M = \oplus_{n \in \mathbb{Z}} M_n$, we denote by $M(1)$ the 1-degree shift of M. i.e., $M(1)_n = M_{n+1}$. The degree shift operator $(1) : \operatorname{coh} R \longrightarrow \operatorname{coh} R$ induces the autoequivalence (1) on $\operatorname{cohproj} R$. We denote by $\overline{R}$ the image in $\operatorname{cohproj} R$ of the regular module R_R. The (coherent) *noncommutative projective scheme* $\operatorname{proj} R$ associated to R is the triple $(\operatorname{cohproj} R, \overline{R}, (1))$. The autoequivalence (1) is called the *canonical polarization* on $\operatorname{proj} R$.

In noncommutative projective geometry, one thinks of $\operatorname{cohproj} R$ as the category of coherent sheaves on a noncommutative projective scheme $\operatorname{proj} R$ associated to a graded ring R ([AZ, Po]).

Let us consider a triple $(\mathcal{C}, \mathcal{O}, s)$ consisting of a k-linear abelian category $\mathcal{C}$ such that $\dim_k(\mathcal{F}, \mathcal{G}) < \infty$ for any $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, an object $\mathcal{O} \in \mathcal{C}$ and an autoequivalence s on $\mathcal{C}$. For $\mathcal{F} \in \mathcal{C}$, we define

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \geq 0} \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}(n)),$$

where $\mathcal{F}(n) := s^n\mathcal{F}$, and we set

$$R = \Gamma_*(\mathcal{C}, \mathcal{O}, s) = \Gamma_*(\mathcal{O}).$$

Multiplication is defined as follows: If $x \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}(l))$, $b \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{O}(m))$ and $a \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{O}(n))$, then

$$x \cdot a = s^n(x) \circ a \quad \text{and} \quad a \cdot b = s^m(a) \circ b.$$

With this law of composition, $\Gamma_*(\mathcal{F})$ becomes a graded right module over the graded algebra R over k.

Definition 1.1 ([AZ, Section 4.2],[Po, Section 2])**.** *Let $(\mathcal{C}, \mathcal{O}, s)$ be a triple as above. Then the pair $(\mathcal{O}, s)$ is called ample if the following conditions hold:*

(1) For every object $\mathcal{F} \in \mathcal{C}$, there are positive integers $l_1, \ldots, l_p$ and an epimorphism $\oplus_{i=1}^{p}\mathcal{O}(-l_i) \longrightarrow \mathcal{F}$.

(2) For every epimorphism $f : \mathcal{F} \longrightarrow \mathcal{G}$, there exists an integer n_0 such that for every $n \geq n_0$ the induced map $\operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}(n)) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{G}(n))$ is surjective.

Let $\pi : \operatorname{Gr} R \longrightarrow \operatorname{QGr} R$ be the quotient functor. Set $\overline{\Gamma}_* = \pi \circ \Gamma_*$.

Theorem 1.2 ([Po, Theorem 2.4])**.** *Let $(\mathcal{C}, \mathcal{O}, s)$ be a triple as above. If $(\mathcal{O}, s)$ is ample, then the graded ring $R = \Gamma_*(\mathcal{C}, \mathcal{O}, s)$ is coherent, $\Gamma_*(\mathcal{F})$ is finitely presented R-module for $\mathcal{F} \in \mathcal{C}$ and the functor $\overline{\Gamma}_* : \mathcal{C} \longrightarrow \operatorname{cohproj} R$ induces an equivalence of triples between $(\mathcal{C}, \mathcal{O}, s)$ and $\operatorname{proj} R = (\operatorname{cohproj} R, \overline{R}, (1))$, i.e.,*

$\overline{\Gamma}_* : \mathcal{C} \xrightarrow{\sim} \operatorname{cohproj} R$ *is an equivalence of categories,*

$\overline{\Gamma}_*(\mathcal{O}) \cong \overline{R}$, *and* $\overline{\Gamma}_* \circ s = (1) \circ \overline{\Gamma}_*$.

2 t-structures defined by two-sided tilting complexes

Let X be a projective variety over k and $\mathcal{L}$ be an ample line bundle on X. Let $(D^{\geq 0}, D^{\leq 0})$ be the standard t-structure in $D^b(\operatorname{coh} X)$,i.e., $D^{\geq 0}$ (resp. $D^{\leq 0}$) is the full subcategory of $D^b(\operatorname{coh} X)$ with objects $\mathcal{F}^{\cdot}$ such that $\mathrm{H}^i(\mathcal{F}^{\cdot}) = 0$ for $i < 0$ (resp. $i > 0$) . By Serre's vanishing theorem [Har, Propsition III.5.3], a complex $\mathcal{F} \in D^b(\operatorname{coh} X)$ belongs to $D^{\geq 0}$ (resp. $D^{\leq 0}$) if and only if $\mathcal{F}$ satisfies the following condition:

$$\mathbb{R}\operatorname{Hom}^{\cdot}(\mathcal{O}_X, \mathcal{F}^{\cdot} \otimes^{\mathbf{L}} \mathcal{L}^n) \in D^{\geq 0}(k\text{-}vect) \quad \text{for } n \gg 0$$
$$(\text{resp.} \quad \mathbb{R}\operatorname{Hom}^{\cdot}(\mathcal{O}_X, \mathcal{F}^{\cdot} \otimes^{\mathbf{L}} \mathcal{L}^n) \in D^{\leq 0}(k\text{-}vect). \quad \text{for } n \gg 0)$$

Reversing this observation, to formulate ampleness in the study of derived categories, we define the following.

Definition 2.1. *Let A be a k-algebra and let σ be a two-sided tilting complex over A. The full subcategory $D^{\sigma,\geq 0}$ (resp. $D^{\sigma,\leq 0}$) of $D^b(\text{mod-}A)$ consists of objects $M^{\cdot}$ which satisfy*

$$\sigma^n M \in D^{\geq 0}(\text{Mod-}A) \quad \textit{for } n \gg 0$$
$$(\textit{resp.} \quad \sigma^n M \in D^{\leq 0}(\text{Mod-}A) \quad \textit{for } n \gg 0).$$

We define $D^{\sigma} := (D^{\sigma,\geq 0}, D^{\sigma,\leq 0})$.

Since $\sigma^n M \simeq \mathbb{R}\operatorname{Hom}(A, \sigma^n M)$, we think of A as the "structure sheaf" in Definition 2.1.

Theorem 2.2. *Let A be a right Noetherian k-algebra of finite global dimension and let σ be a two-sided tilting complex over A. If $\mathrm{H}^i(\sigma) = 0$ for $i \geq 1$, then D^σ is a t-structure in $D^b(\text{mod-}A)$.*

To prove Theorem 2.2 we need the following Lemma.

Lemma 2.3. *Let $f : L \longrightarrow M$ be a morphism in $D^-(\text{Mod-}A \otimes_k B^{\mathrm{op}})$ and $N \in D^-(\text{Mod-}C \otimes_k A^{\mathrm{op}})$ for k-algebras A, B, C. If $\mathrm{H}^i(f) = 0$ for any $i \in \mathbb{Z}$, then $\mathrm{H}^i(f \otimes^{\mathbf{L}}_A 1_N) = 0$ for any $i \in \mathbb{Z}$.*

The above Lemma is easily proved by the following Lemma taken from [Y, Lemma 2.1]. (See also [ML, Theorem XII. 12.2].)

Lemma 2.4. *Let $M \in D^-(\text{Mod-}A \otimes_k B^{\mathrm{op}})$ and $N \in D^-(\text{Mod-}C \otimes_k A^{\mathrm{op}})$ for k-algebras A, B, C. Then there is a convergent Künneth spectral sequence*

$$\mathrm{E}_2^{p,q} = \bigoplus_{i+j=q} \mathrm{H}^p\left(\mathrm{H}^i(M) \otimes^{\mathbf{L}}_A \mathrm{H}^j(N)\right) \Longrightarrow \mathrm{H}^{p+q}(M \otimes^{\mathbf{L}}_A N)$$

in $\text{Mod-}C \otimes_k B^{\mathrm{op}}$ which is functorial in $M^{\cdot}$ and $N^{\cdot}$. If $i_0 \geq \sup\{i \mid \mathrm{H}^i(M) \neq 0\}$ and $j_0 \geq \sup\{j \mid \mathrm{H}^j(N) \neq 0\}$, then $\mathrm{H}^{i_0}(M) \otimes_A \mathrm{H}^{j_0}(N) \simeq \mathrm{H}^{i_0+j_0}\left(M \otimes^{\mathbf{L}}_A N\right)$.

The latter part of this Lemma will be used in the sequel.

Proof of Theorem 2.2. The only nontrivial part is the following statement:
For any $M \in D^b(\text{mod-}A)$ there is an exact triangle

$$M' \longrightarrow M \longrightarrow M'' \xrightarrow{[1]}$$

in $D^b(\text{mod-}A)$ such that $M' \in D^{\sigma,\leq 0}$ and $M'' \in D^{\sigma,\geq 1}$.

Let $(D^{\leq 0}, D^{\geq 0})$ be a standard t-structure in $D(\text{Mod-}A)$ and let $\tau_{\leq 0}$ and $\tau_{\geq 1}$ be standard truncation functors. Let $N \in D^b(\text{mod-}A)$. Applying σ to the canonical morphism $\tau_{\leq 0}N \longrightarrow N$, we get the morphism $\sigma(\tau_{\leq 0}N) \longrightarrow \sigma(N)$. Since σ is the derived tensor $- \otimes^{\mathbf{L}}_A \sigma$ of the complex σ such that $\mathrm{H}^i(\sigma) = 0$ for $i \geq 1$, $\sigma(\tau_{\leq 0}(N)) \in D^{\leq 0}$. Therefore we get a morphism $\sigma(\tau_{\leq 0}(N)) \longrightarrow \tau_{\leq 0}(\sigma(N))$. Setting $N = \sigma^n(M)$ for $n \geq 0$, we get a morphism $\sigma(\tau_{\leq 0}(\sigma^n M)) \longrightarrow \tau_{\leq 0}(\sigma^{n+1}M)$. Applying $\sigma^{-(n+1)}$ to this morphism, we get a morphism $\phi_n : \sigma^{-n}(\tau_{\leq 0}(\sigma^n M)) \longrightarrow \sigma^{-(n+1)}(\tau_{\leq 0}(\sigma^{n+1}M))$. Set $\tau^{\sigma,n}_{\leq 0} := \sigma^{-n}\tau_{\leq 0}\sigma^n$ and $\tau^{\sigma,n}_{\geq 1} := \sigma^{-n}\tau_{\geq 1}\sigma^n$. Applying σ^{-n} to the exact triangle

$$\tau_{\leq 0}(\sigma^n M) \longrightarrow \sigma^n M \longrightarrow \tau_{\geq 1}(\sigma^n M) \xrightarrow{[1]}, \tag{1}$$

we get the following exact triangle

$$\tau^{\sigma,n}_{\leq 0} M \xrightarrow{\alpha_n} M \xrightarrow{\beta_n} \tau^{\sigma,n}_{\geq 1}(M) \xrightarrow{[1]} . \tag{2}$$

We have the following commutative diagram:

$$\begin{tikzcd}
\tau^{\sigma,n}_{\leq 0} M \arrow[r, "\alpha_n"] \arrow[d, "\phi_n"] & M \arrow[r, "\beta_n"] \arrow[d, "="] & \tau^{\sigma,n}_{\geq 1} M \arrow[r, "{[1]}"] \arrow[d, "\psi_n"] & {} \\
\tau^{\sigma,n+1}_{\leq 0} M \arrow[r, "\alpha_{n+1}"] & M \arrow[r, "\beta_{n+1}"] & \tau^{\sigma,n+1}_{\geq 1} M \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

where ψ_n is induced morphism.

Let us consider the following cohomology long exact sequence of (2):

$$\cdots \longrightarrow \mathrm{H}^{i-1}(\tau^{\sigma,n}_{\geq 1} M) \xrightarrow{\partial^{i-1}} \mathrm{H}^i(\tau^{\sigma,n}_{\leq 0} M) \xrightarrow{\mathrm{H}^i(\alpha_n)} \mathrm{H}^i(M) \rightarrow \cdots$$

where ∂^{i-1} is the connecting morphism. Let $\delta : \tau_{\geq 1}\sigma^n M \longrightarrow \tau_{\leq 0}\sigma^n M[1]$ be the morphism obtained by rotating the exact triangle (1). Then $\mathrm{H}^i(\delta) = 0$ for any $i \in \mathbb{Z}$. Since $\partial^{i-1} \simeq \mathrm{H}^{i-1}(\delta \otimes^{\mathbf{L}}_A 1_{\sigma^{-n}})$, $\partial^{i-1} = 0$ by Lemma 2.3. Therefore $\mathrm{H}^i(\alpha_n)$ is injective. Hence $\mathrm{H}^i(\phi_n)$ is injective. We have the following system of injections

$$\cdots \hookrightarrow \mathrm{H}^i(\tau^{\sigma,n}_{\leq 0} M) \hookrightarrow \mathrm{H}^i(\tau^{\sigma,n+1}_{\leq 0} M) \hookrightarrow \cdots \quad \mathrm{H}^i(M).$$

Since $\mathrm{H}^i(M) = 0$ except for finitely many i and $\mathrm{H}^i(M)$ is Noethrian for each i, there is a positive integer n_0 such that $\mathrm{H}^i(\phi_n)$ is isomorphism for $n \geq n_0$ and $i \in \mathbb{Z}$. Therefore ϕ_n and ψ_n is quasi-isomorphism for $n \geq n_0$. Thus if we set $M' := \tau^{\sigma,n_0}_{\leq 0} M$ and $M'' := \tau^{\sigma,n_0}_{\geq 1} M$, then $M' \in D^{\rho,\leq 0}$ and $M'' \in D^{\rho,\geq 1}$. This complete the proof of the Theorem. □

Let A be a finite dimensional k-algebra of finite global dimension and σ be a two-sided tilting complex over A such that $\mathrm{H}^i(\sigma) = 0$ for $i \geq 1$. Then by Theorem 2.2, D^σ is t-structure in $D^b(\text{mod-}A)$. Let $\mathcal{H}^\sigma$ be the heart of the t-structure D^σ. Then σ acts on $\mathcal{H}^\sigma$. Furthermore assume that σ^n is pure in standard t-structure for each $n \gg 0$. Then $A \in \mathcal{H}^\sigma$ and the triple $(\mathcal{H}^\sigma, A, \sigma)$ satisfies the conditions in Section 1.

Lemma 2.5. *With the assumptions above the pair (A, σ) is ample in the sense of Definition 1.1 on the triple $(\mathcal{H}^\sigma, A, \sigma)$.*

Proof. We check the conditions (1) and (2) of Definition 1.1.

First note that the cokernel of the morphism $f : M \longrightarrow N$ in the abelian category $\mathcal{H}^\sigma$ is $\tau^{\sigma}_{\geq 0}(\mathrm{Cone}(f))$, where $\tau^{\sigma}_{\geq 0} : D^b(\text{mod-}A) \longrightarrow D^{\sigma,\geq 0}$ is the truncation functor of the t-structure D^σ (See [GM, IV.4]). So f is an epimorphism in $\mathcal{H}^\sigma$ if and only if $\mathrm{Cone}(f) \in D^{\sigma,\leq -1}$.

(1) Let $M \in \mathcal{H}^\sigma$ and let $n \geq 0$ be an integer such that $\sigma^n M$ is pure. Let P be a bounded complex of finite projective right A-modules which represents $\sigma^n M$. We may assume that $P^i = 0$ for $i \geq 1$ and $P^0 \cong A^{\oplus p}$ for some $p \in \mathbb{N}$. An isomorphism $A^{\oplus p} \xrightarrow{\sim} P^0$ induces a morphism $\phi : A^{\oplus p} \longrightarrow \sigma^n M$. Let C be a cone of ϕ. Then $C \in D^{\leq -1}$ and we have the following exact triangle

$$\sigma^{-n} A^{\oplus p} \xrightarrow{\sigma^{-n}\phi} M \longrightarrow \sigma^{-n} C \xrightarrow{[1]} .$$

Since $\sigma^{-n} C \in D^{\sigma,\leq -1}$, $\sigma^{-n}\phi$ is an epimorphism in $\mathcal{H}^\sigma$.

(2) Let

$$M \xrightarrow{f} N \longrightarrow L \xrightarrow{[1]}$$

be an exact triangle such that $M, N \in \mathcal{H}^\sigma$ and $L \in D^{\sigma,\leq -1}$. Take an integer n_0 such that $\mathrm{Hom}(A, \sigma^n L^{\cdot}) \cong \mathrm{H}^0(\sigma^n L^{\cdot}) = 0$ for each $n \geq n_0$. Then the induced morphism $\mathrm{Hom}(A, \sigma^n M) \longrightarrow \mathrm{Hom}(A, \sigma^n N)$ is surjective in Mod-A for each $n \geq n_0$. □

Definition 2.6. *Let A be a finite dimensional k-algebra and let σ be a two-sided tilting complex over A. σ is called extremely ample if σ^n is pure for $n \geq 0$. σ is called very ample if $\mathrm{H}^i(\sigma) = 0$ for $i \geq 1$ and σ^n is pure for $n \gg 0$. σ is called ample if σ^n is pure for $n \gg 0$.*

Let σ be a very ample two-sided tilting complex over a finite dimensional k-algebra A. Then $\mathrm{H}^0(\sigma)^{\otimes_A n} \simeq \mathrm{H}^0(\sigma^n)$ for $n \geq 0$ by Lemma 2.4. Therefore the tensor algebra

$$T_A(\mathrm{H}^0(\sigma)) = \bigoplus_{n \geq 0} \left(\mathrm{H}^0(\sigma)\right)^{\otimes_A n}$$

of $\mathrm{H}^0(\sigma)$ over A is naturally isomorphic to the homogeneous coordinate ring

$$\Gamma_*(\mathcal{H}^\sigma, A, \sigma) = \bigoplus_{n\geq 0} \mathrm{Hom}(A, \sigma^n A) \cong \bigoplus_{n\geq 0} \mathrm{H}^0(\sigma^n)$$

of the triple $(\mathcal{H}^\sigma, A, \sigma)$. By Theorem 1.2 we obtain the following Corollary.

Corollary 2.7. *Let A be a finite dimensional k-algebra of finite global dimension and let σ be a very ample two-sided tilting complex over A. Then the tensor algebra $T := T_A(\mathrm{H}^0(\sigma))$ of $\mathrm{H}^0(\sigma)$ over A is a graded coherent ring and the triple $(\mathcal{H}^\sigma, A, \sigma)$ is equivalent to the triple $\big(\mathrm{cohproj}\, T, \overline{T}, (1)\big)$ as triple. In particular the abelian category $\mathcal{H}^\sigma$ is equivalent to the abelian category* $\mathrm{cohproj}\, T$.

In [Be] Beilinson showed that $\mathbb{P}^n$ is derived equivalent to a finite dimensional k-algebra. This result has been generalized to other varieties. The next Theorem gives a partial converse.

Theorem 2.8. *Let A be a finite dimensional k-algebra of finite global dimension and let σ be a very ample two-sided tilting complex. Then there is a natural equivalence of triangulated categories*

$$D^b\,(\text{mod-}A) \xrightarrow{\sim} D^b\left(\mathrm{cohproj}\, T_A(\mathrm{H}^0(\sigma))\right).$$

Proof. We set $T := T_A(\mathrm{H}^0(\sigma))$. Let $\mathcal{P}_A$ be the full subcategory of mod-A consisting all finite projective A modules. We can extend the functor

$$\mathcal{P}_A \longrightarrow \mathrm{cohproj}\, T, \quad P \mapsto \bigoplus_{i\geq 0} \mathrm{Hom}\left(A, P\otimes_A \mathrm{H}^0(\sigma^i)\right)$$

to the functor $\gamma : K^b(\mathcal{P}_A) \longrightarrow K^b(\mathrm{cohproj}\, T)$ between the homotopy category of complexes. Let $\Phi : K^b(\mathcal{P}_A) \xrightarrow{\sim} D^b\,(\text{mod-}A)$ be the natural equivalence and let $\pi : K^b(\mathrm{cohproj}\, T) \longrightarrow D^b(\mathrm{cohproj}\, T)$ be the natural quotient functor. Define $\mathbf{L}\overline{\Gamma}_* := \pi\circ\gamma\circ\Phi^{-1}$. Then we obtain the following commutative diagram.

$$\begin{tikzcd} D^b\,(\text{mod-}A) \arrow[r, "\mathbf{L}\overline{\Gamma}_*"] & D^b(\mathrm{cohproj}\, T) \\ \mathcal{H}^\rho \arrow[u, hook, "i_A"] \arrow[r, "\sim", "\overline{\Gamma}_*"'] & \mathrm{cohproj}\, T \arrow[u, hook, "i_T"'] \end{tikzcd} \tag{3}$$

where i_A, i_Π are inclusions. We prove that $\mathbf{L}\overline{\Gamma}_*$ is an equivalence.

By Lemma 2.5 and Theorem 1.2 the functor $\overline{\Gamma}_* : \mathcal{H}^\sigma \longrightarrow \mathrm{cohproj}\, T$ is an equivalence. Therefore $\mathbf{L}\overline{\Gamma}_*$ is essentially surjective. To complete the proof it suffices to show that $\mathbf{L}\overline{\Gamma}_*$ is fully faithful. Since every complex $M^{\cdot} \in D^b\,(\text{mod-}A)$ is obtained from A by taking finite number of cones, shifts and direct summand, the problem is reduced to the following lemma. $\square$

Lemma 2.9. *The map*

$$\mathrm{Hom}_{D^b(\text{mod-}A)}(A, A[i]) \xrightarrow{\mathbf{L}\overline{\Gamma}_{*A,A[i]}} \mathrm{Hom}_{D^b(\mathrm{cohproj}\, T)}(\mathbf{L}\overline{\Gamma}_*(A), \mathbf{L}\overline{\Gamma}_*(A)[i])$$

is an isomorphism for every $i \in \mathbb{Z}$.

Proof. For the case $i = 0$, the map $\mathbf{L}\overline{\Gamma}_{*A,A}$ is equal to

$$\begin{aligned}\mathrm{Hom}_{D^b(\text{mod-}A)}(A, A) \cong \mathrm{Hom}_{\mathcal{H}^\sigma}(A, A) &\cong \mathrm{Hom}_{\mathrm{cohproj}\, T}(\overline{\Gamma}_*(A), \overline{\Gamma}_*(A)) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{cohproj}\, T)}(\mathbf{L}\overline{\Gamma}_*(A), \mathbf{L}\overline{\Gamma}_*(A))\end{aligned}$$

where the second isomorphism is induced by the equivalence $\overline{\Gamma}_*$. Hence $\mathbf{L}\overline{\Gamma}_{*A,A}$ is an isomorphism.

For the case $i \neq 0$, since $\mathbf{L}\overline{\Gamma}_*(A) \cong \overline{T}$, we have only to show that $\mathrm{Ext}^i_{qcohT}\left(\overline{T}, \overline{T}\right) = 0$ for $i \geq 1$. First note that $\mathrm{Ext}^i_{\mathrm{cohproj}\,T}\left(\overline{T}, \overline{T}\right) \cong \lim_{n\to\infty} \mathrm{Ext}^i_{\mathrm{coh}\,T}\left(T_{\geq n}(-n), T\right)$ where $T_{\geq n} := \oplus_{m\geq n}\mathrm{H}^0\left(\sigma\right)^{\otimes_A m}$ and $(-n)$ is the $-n$-graded degree shift operator (See [AZ]). Let $n_0 \geq 0$ be a positive integer such that σ^m is pure for $m \geq n_0$. Then by Lemma 2.4 $\mathrm{H}^0\left(\sigma\right)^{\otimes_A m+n} \simeq \mathrm{H}^0\left(\sigma\right)^{\otimes_A m} \otimes^{\mathbf{L}}_A \sigma^n$ for $n \geq n_0$ and $m \geq 0$. Thus $T_{\geq n} \simeq T \otimes^{\mathbf{L}}_A \sigma^n$ for $n \geq n_0$. Therefore for $n \geq n_0$

$$\begin{aligned}\mathbb{R}\,\mathrm{Hom}_{\mathrm{coh}\,T}\left(T_{\geq n}(-n), T\right) \simeq \mathbb{R}\,\mathrm{Hom}_{\mathrm{coh}\,T}\left(T \otimes^{\mathbf{L}}_A \sigma^n(-n), T\right) &\simeq \mathbb{R}\,\mathrm{Hom}_{\mathrm{coh}\,T}\left(T, \sigma^{-n} \otimes^{\mathbf{L}}_A T(n)\right)\\ &\simeq \sigma^{-n} \otimes^{\mathbf{L}}_A \sigma^n \simeq A.\end{aligned}$$

This complete the proof of the Lemma , which also complete the proof of Theorem 2.8. $\square$

Lemma 2.10. *Let A be a finite dimensional k-algebra of finite global dimension and let σ be a very ample two-sided tilting complex over A. Then $D^\sigma = D^{\sigma^n}$ for $n \in \mathbb{N}$.*

Definition 2.11. *Let A be a finite dimensional k-algebra of finite global dimension and let σ be an ample two-sided tilting complex over A. We define $D^\sigma := D^{\sigma^n}$ and $\mathcal{H}^\sigma := \mathcal{H}^{\sigma^n}$ where n is a natural number such that σ^n is very ample. By the above Lemma, this is well-defined.*

Proposition 2.12. *Let A be a finite dimensional k-algebra of finite global dimension and let σ be an ample two-sided tilting complex. Then the following conditions are equivalent.*

(1) the inverse σ^{-1} is ample.

(2) the t-structure D^σ is equal to the standard t-structure D^A.

(3) $\sigma^n \in \mathrm{Pic}\,A$ for $n \gg 0$.

Lemma 2.13. *If $M \in \mathcal{H}^\sigma$, then $\mathrm{H}^i(M) = 0$ for $i < 0$ and $i > \max\{\mathrm{pd}\left({}_A\sigma^n\right) \mid n \geq 0\}$ where $\mathrm{pd}\left({}_A\sigma^n\right)$ is the projective dimension of σ^n as a left A-module.*

Proof. We prove that $\mathrm{H}^i(M) = 0$ for $i > \max\{\mathrm{pd}\,\sigma^n \mid n \geq 0\}$. The case when $i < 0$ can be proved in the same way. Set $d := \max\{\mathrm{pd}\left({}_A\sigma^n\right) \mid n \geq 0\}$. Let $\tau_{\leq d}$ and $\tau_{\geq d+1}$ be the standard truncation functors of $D^b(\text{mod-}A)$. We have the following exact triangle:

$$\sigma^n\left(\tau_{\leq d}M\right) \longrightarrow \sigma^n M \longrightarrow \sigma^n\left(\tau_{\geq d+1}M\right) \xrightarrow{[1]} \tag{4}$$

for each $n \geq 0$. Let n be a positive integer such that $\sigma^n M$ and σ^n are pure. Since $d = \max\{\mathrm{pd}\left({}_A\sigma^n\right) \mid n \geq 0\}$, $\mathrm{H}^i\left(\sigma^n\left(\tau_{\geq d+1}M\right)\right) = 0$ for $i \leq 0$. For $i \geq 1$, we consider the following part of the cohomology long exact sequence

$$\mathrm{H}^i\left(\sigma^n M\right) = 0 \longrightarrow \mathrm{H}^i\left(\sigma^n\left(\tau_{\geq d+1}M\right)\right) \xrightarrow{\partial^i} \mathrm{H}^{i+1}\left(\sigma^n\left(\tau_{\leq d}M\right)\right)$$

of the exact triangle (4). Applying the same argument in the proof of Theorem 2.2, we conclude that the connecting morphism $\partial^i = 0$ and $\mathrm{H}^i\left(\sigma^n\left(\tau_{\geq d+1}M\right)\right) = 0$ for $i \geq 1$. Therefore $\sigma^n\left(\tau_{\geq d+1}M\right) = 0$ and hence $\tau_{\geq d+1}M = 0$. This completes the proof. $\square$

Let $\mathcal{P} = (\mathcal{C}, \mathcal{O})$ be a pair consisting of a k-linear abelian category $\mathcal{C}$ and an object $\mathcal{O} \in \mathcal{C}$. In [AZ, Sction 7] the cohomology group of $\mathcal{F} \in \mathcal{C}$ is defined to be $\mathrm{H}^i\left(\mathcal{P}, \mathcal{F}\right) := \mathrm{Ext}^i_{\mathcal{C}}\left(\mathcal{O}, \mathcal{F}\right)$. for $i \geq 0$. The cohomological dimension of the pair $\mathcal{P}$ is defined to be

$$\mathrm{cd}\left(\mathcal{P}\right) := \max\{i \mid \mathrm{H}^i(\mathcal{P}, \mathcal{F}) \neq 0 \ for\ \mathcal{F} \in \mathcal{C}\}.$$

By definition cohomology groups and cohomological dimension of a triple $(\mathcal{C}, \mathcal{O}, s)$ in Section 1 are that of the pair $(\mathcal{C}, \mathcal{O})$.

Let A be a finite dimensional k-algebra of finite global dimension and let σ be a very ample tilting complex over A. Set $T = T_A\left(\mathrm{H}^0(\sigma)\right)$. We consider the triple $\mathrm{proj}\,T = \left(\mathrm{cohproj}\,T, \overline{T}, (1)\right)$.

Corollary 2.14. $\mathrm{cd}\,(\mathrm{proj}\,T) = \max\{\mathrm{pd}\,({}_A\sigma^n) \mid n \geq 0\}$.

Proof. Let $\mathcal{F} \in \mathrm{cohproj}\,T$ and let $M \in \mathcal{H}^\sigma$ be an object of $D^b(\text{mod-}A)$ which corresponds to $\mathcal{F}$ under the equivalence of Theorem 2.8. Then $\mathrm{H}^i\,(\mathrm{proj}\,T, \mathcal{F}) \cong \mathrm{Hom}(A, M[i]) \cong \mathrm{H}^i(M)$ where the right hand side is the i-th cohomology group of the complex M of right A-modules. Therefore by Lemma 2.13 $\mathrm{H}^i\,(\mathrm{proj}\,T, \mathcal{F}) = 0$ for $i < 0, \max\{\mathrm{pd}\,\sigma^n \mid n \geq 0\} < i$. Hence $\mathrm{cd}\,(\mathrm{proj}\,T) \leq \max\{\mathrm{pd}\,\sigma^n \mid n \geq 0\}$. It is clear that $\sigma^{-n}A \in \mathcal{H}^\sigma$. If $\mathrm{pd}\,({}_A\sigma^n) = d$, then $\mathrm{H}^d\,(\sigma^{-n}A) \neq 0$. Therefore $\mathrm{cd}\,(\mathrm{proj}\,T) = \max\{\mathrm{pd}\,\sigma^n \mid n \geq 0\}$. $\square$

The global dimension of $\mathrm{cohproj}\,T$ is bounded by $\mathrm{gl.\,dim}\,A$ from above.

Proposition 2.15. $\mathrm{gl.\,dim}\,(\mathrm{cohproj}\,T) \leq \mathrm{gl.\,dim}\,A$.

Proof. Let $\mathcal{F}, \mathcal{G} \in \mathrm{cohproj}\,T$ and let $M, N \in \mathcal{H}^\sigma$ be an object of $D^b(\text{mod-}A)$ which corresponds to $\mathcal{F}, \mathcal{G}$ under the equivalence of Theorem 2.8. Let $n \gg 0$ be a positive integer such that $\sigma^n M$ and $\sigma^n N$ are pure. Then $\mathrm{Ext}^i_{\mathrm{cohproj}\,T}\,(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^i_{\text{mod-}A}\,(\sigma^n M, \sigma^n N) = 0$ for $i > \mathrm{gl.\,dim}\,A$. $\square$

Remark 2.16. *In general* $\mathrm{gl.\,dim}\,(\mathrm{cohproj}\,T) < \mathrm{gl.\,dim}\,A$. *See Section 3.3.*

3 Fano algebras and algebras with ample canonical bundle

3.1 definition and basic properties

Let A be a finite dimensional k-algebra of finite global dimension. The k-dual A^* has the natural A-bimodule structure. It is known that $- \otimes^{\mathbf{L}}_A A^* : D^b(\text{mod-}A) \longrightarrow D^b(\text{mod-}A)$ is the Serre functor ([Hap, I.4.6]). For a nonsingular projective variety X over k, the $[\dim X]$-shifted derived tensor $- \otimes^{\mathbf{L}}_X \omega_X[\dim X]$ of the canonical bundle ω_X is the Serre functor of $D^b\,(\mathrm{coh}\,X)$. From a view point of noncommutative algebraic geometry A^* is thought as "shifted canonical bundle". For example, if $(A^*)^m \simeq [n]$ for some positive integers m, n, then A is called *fractional Calabi-Yau of CY dimension* $\dfrac{n}{m}$, which is apparently named after analogy to the property of the derived category of a Calabi-Yau variety.

Definition 3.1. *Let A be a finite dimensional k-algebra of finite global dimension, let d be a non-negative integer, and set $\omega := (A^*[-d])$. A is said to be a Fano algebra of Fano dimension d if the two-sided tilting complex ω^{-1} is ample.*

Remark 3.2. *It is not known that in general finite dimensional k-algebras of finite global dimension have ample $A^*[-d]$ for some d. Therefore we don't use the term "algebra of general type".*

Remark 3.3. *(1) Let A be a finite dimensional k-algebra of finite global dimension. If for some positive integer d,* $\mathrm{Ext}^i(A^*, A) = 0$, $i \neq d$, *then* $\mathrm{gl.\,dim}\,A = d$. *Therefore if $\omega_A^{-1} = (A^*[-d])^{-1}$ is extremely ample, then the global dimension of A is equal to d. In general global dimension of a Fano algebra is not equal to its Fano dimension. (See Section3.3.)*

(2) By the standard argument we can prove that if $\omega := A^*[-d]$ is ample (resp. anti-ample) then $\mathrm{gl.\,dim}\,\mathcal{H}^\omega = d$ *(resp.* $\mathrm{gl.\,dim}\,\mathcal{H}^{\omega^{-1}} = d$*).*

Lemma 3.4. *Let A be a finite dimensional k-algebra of finite global dimension. If A is fractional Calabi-Yau then A is not Fano. Conversely if A is Fano then A is not fractional Calabi-Yau.*

Proof. We prove that a fractionally Calabi-Yau algebra is not Fano. We set $\omega := A^*[-d]$ for some $d \geq 0$. Let m, n be integers such that $(A^*)^m \simeq [n]$. Then $\omega^{-m} \simeq [dm - n]$. If $dm - n \neq 0$, then ω^{-lm} is not pure for $l > 0$. If $dm - n = 0$ then $\omega^{-(lm-1)} \simeq \omega = A^*[-d]$ is not pure for $l > 0$. In any case ω^{-1} is not ample. $\square$

Example 3.5 (Geometric algebras)**.** *Let $\mathcal{T}$ be an algebraic k-linear triangulated category such that* $\dim_k \mathrm{Hom}(E, F) < \infty$ *for $E, F \in \mathcal{T}$ and $E_\bullet := (E_0, E_1, \dots, E_d)$ be a full geometric collection in $\mathcal{T}$ (See [BP, ELO] for the definition and the properties below of a geometric collection). The endomorphism algebra $A := \mathrm{End}\left(\oplus_{i=0}^d E_i\right)$ is called a geometric algebra in [BP]. Then A is a finite dimensional k-algebra of global dimension d and $(A^*[-d])^{-n}$ is pure for $n \geq 0$. Therefore the geometric algebra A is a Fano algebra. By Corollary 2.7, The tensor algebra $T_A(\rho)$ is coherent. Therefore the $\mathbb{Z}$-algebra $\mathcal{A}$ associated to geometric collection $E_\bullet$ is coherent. In particular, the homogeneous coordinate ring $\mathcal{A}^{m,V}$ of noncommutative Grassmanian* $\mathrm{NGr}(m, V)$ *([ELO]) is coherent.*

3.2 A noncommutative algebro-geometric characterization of representation type of a quiver

Let Q be a finite acyclic quiver, i.e., a quiver with finitely many vertexes and finitely many arrows without loops and oriented cycles. Then the path algebra $A = kQ$ of Q is a finite dimensional k-algebra of global dimension 1. Note that $\omega_Q^{-1} = (A^*[-1])^{-1}$ is the inverse of the Auslander-Reiten translation. Therefor if the quiver Q has infinite representation type, then ω_Q^{-n} is pure for any $n \geq 0$ by [Hap, II.4.7]. Therefore the anti-canonical bundle ω_Q^{-1} is extremely ample.

Theorem 3.6. *Let Q be a finite acyclic quiver of infinite representation type. Then the path algebra kQ of Q is a Fano algebra of Fano dimension* 1*.*

If a finite acyclic quiver Q has finite representation type, then its path algebra kQ is fractional Calabi-Yau. (This fact has been known by specialists. See [MY] for the precise CY dimension of these algebras.) By Lemma 3.4 and Theorem 3.6 we obtain the following characterization of representation type of a quiver from a noncommutative algebro-geometric view point.

Corollary 3.7. *A finite acyclic quiver has finite representation type if and only if its path algebra is fractional Calabi-Yau, and a finite acyclic quiver has infinite representation type if and only if its path algebra is Fano.*

By Theorem 2.8 and Theorem 3.6 we obtain the following corollary.

Corollary 3.8. *Let Q be a finite acyclic quiver of infinite representation type. Then there is a natural equivalence of triangulated categories*

$$D^b(\text{mod-}kQ) \xrightarrow{\sim} D^b(\mathrm{cohproj}\,\Pi(Q))$$

where $\Pi(Q)$ is the preprojective algebra of Q.

Remark 3.9. *The above equivalence is essentially proved in* [Le].

Remark 3.10. *Set* $\mathcal{T}' = \{N \in \text{mod-}A \mid \mathrm{Hom}(A, \omega_Q^{-n} N) = 0 \, \textit{for } n \gg 0\}$ *and* $\mathcal{F}' = \{N \in \text{mod-}A \mid \mathrm{Ext}^{-1}(A, \omega_Q^{-n} N) = 0 \, \textit{for } n \gg 0\}$*. Then we can prove that $(\mathcal{T}', \mathcal{F}')$ is a torsion pair on* mod-A*. From this torsion pair we can define a t-structure in $D^b(\mathsf{mod}\text{-}A)$ by setting*

$$\begin{aligned} D'^{\geq 0} :=& \{M^\cdot \in D^{\geq 0}(\text{mod-}A) \mid H^0(M^\cdot) \in \mathcal{F}'\} \\ D'^{\leq 0} :=& \{M^\cdot \in D^{\leq 1}(\text{mod-}A) \mid H^1(M^\cdot) \in \mathcal{T}'\}. \end{aligned}$$

(*See* [HRS, Proposition I.2.1]). *However, this is not a new t-structure. It can be proved that* $(D'^{\geq 0}, D'^{\leq 0}) = D^{\omega_Q^{-1}}$.

Remark 3.11. *Let Q be a finite acyclic quiver. By Happel's theorem ([Hap, Theorem.II.4.9]), there is a natural equivalence of triangulated categories*

$$D^b(\text{mod-}kQ) \xrightarrow{\sim} \underline{\text{grmod}}\text{-}T(Q)$$

where $T(Q) := kQ \oplus (kQ)^*$ is a trivial extension algebra and $\underline{\text{grmod}}\text{-}T(Q)$ is the stable category of finite graded $T(Q)$ modules. In the case Q has infinite representation type, compositing above equivalence and the equivalence of corollary 3.8, we obtain the equivalence of triangulated categories

$$D^b(\text{cohproj}\,\Pi(Q)) \simeq \underline{\text{grmod}}\text{-}T(Q).$$

It seems that this equivalence asserts that $\Pi(Q)$ and $T(Q)$ are Koszul dual to each other over kQ. In the classical theory of Koszul algebras, graded algebras over a semi-simple algebra are treated. But path algebras are not semi-simple in general. The related theory will be developed in [MT].

3.3 canonical algebras

The concept of a weighted projective line was given by Geigle and Lewnzing [GL] to treat geometrically canonical algebras.

Let $p = (p_0, \dots, p_n)$ be the $n+1$-tuple of positive integers, called a *weight sequence*. Denote by $\mathbf{L}(\mathbf{p})$ the rank one abelian group on generators $\vec{x}_0, \dots, \vec{x}_n$ with relations $p_0\vec{x}_0 = \cdots = p_n\vec{x}_n$. The element $\vec{c} = p_0\vec{x}_0 = \cdots = p_n\vec{x}_n$ is called the *canonical element* of $\mathbf{L}(p)$ and the element $\vec{\omega} = (n-1)\vec{c} - \sum_{i=0}^{n} \vec{x}_i$ is called the *dualizing element* of $\mathbf{L}(p)$. $\mathbf{L}(p)$ is an ordered group with $\mathbf{L}(p)^+ = \sum_{i=0}^{n} \mathbb{N}\vec{x}_i$ as its set of positive elements.

Let $\mathbb{X} = \mathbb{X}(p, \lambda)$ be a weighted projective line of type $p = (p_0, \dots, p_n)$ and $\lambda = (\lambda_2, \dots, \lambda_n)$ where λ is a sequence of pairwise distinct elements of $k^\times$, normalized such that $\lambda_2 = 1$.

The abelian category $\text{coh}\,\mathbb{X}$ of coherent sheaves on $\mathbb{X}$ has global dimension 1.

For each $\vec{x} \in \mathbf{L}(p)$ we can attach a line bundle $\mathcal{O}_{\mathbb{X}}(\vec{x})$. This correspondence is additive ,i.e., there are a natural isomorphisms $\mathcal{O}_{\mathbb{X}}(\vec{x} + \vec{y}) \cong \mathcal{O}_{\mathbb{X}}(\vec{x}) \otimes_{\mathbb{X}} \mathcal{O}_{\mathbb{X}}(\vec{y})$ and $\mathcal{O}_{\mathbb{X}}(0) \cong \mathcal{O}_{\mathbb{X}}$.

- (Serre duality) The functor $- \otimes_{\mathbb{X}}^{\mathbf{L}} \mathcal{O}_{\mathbb{X}}(\vec{\omega})[1]$ is the Serre functor of $D^b(\text{coh}\,\mathbb{X})$.
- (Serre vanishing) Let $\vec{x} \in \mathbf{L}(p)^+$. For $\mathcal{F} \in \text{coh}\,\mathbb{X}$,

$$\mathrm{H}^i\left(\mathbb{X}, \mathcal{F} \otimes_{\mathbb{X}} \mathcal{O}_{\mathbb{X}}(n\vec{x})\right) = \mathrm{Ext}^i_{\text{coh}\,\mathbb{X}}\left(\mathcal{O}_{\mathbb{X}}, \mathcal{F} \otimes_{\mathbb{X}} \mathcal{O}_{\mathbb{X}}(n\vec{x})\right) = 0$$

for $i > 0$ and $n \gg 0$.

The endomorphism algebra $\Lambda = \text{End}(T)$ of $T := \bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}(\vec{x})$ is isomorphic to a *canonical algebra* in the sense of Ringel [R]. It is given by the quiver

$$\begin{tikzcd}
& \vec{x}_0 \arrow[r, "x_0"] & 2\vec{x}_0 \arrow[r] & \cdots \arrow[r] & (p_0-1)\vec{x}_0 \arrow[ddr] & \\
& \vec{x}_1 \arrow[r, "x_1"] & 2\vec{x}_1 \arrow[r] & \cdots \arrow[r] & (p_1-1)\vec{x}_1 \arrow[dr] & \\
0 \arrow[uur] \arrow[ur] \arrow[dr] & \vdots & \vdots & & \vdots & \vec{c} \\
& \vec{x}_n \arrow[r, "x_n"] & 2\vec{x}_n \arrow[r] & \cdots \arrow[r] & (p_n-1)\vec{x}_n \arrow[ur] &
\end{tikzcd}$$

with relations $x_i^{p_i} - x_1^{p_1} + \lambda x_0^{p_0}$, $i = 2, \dots, n$. The global dimension of the canonical algebra Λ is bounded by 2 from above. Moreover T is a tilting sheaf on $\mathbb{X}$, i.e., T induces a natural equivalence of triangulated categories

$$D^b(\text{coh}\,\mathbb{X}) \simeq D^b(\text{mod-}\Lambda) \tag{5}$$

The *genus* $g_{\mathbb{X}}$ of a weighted projective line $\mathbb{X}$ is by definition $g_{\mathbb{X}} = 1 + \frac{1}{2}\left((n-1) - \sum_{i=0}^{n} \frac{p}{p_i}\right)$. If $g_{\mathbb{X}} < 1$ ($g_{\mathbb{X}} = 1$ resp. $g_{\mathbb{X}} > 1$), then $\mathbb{X}$ is called of domestic (tubular resp. wild) type. Note that if $g_{\mathbb{X}} < 1$ ($g_{\mathbb{X}} = 1$ resp. $g_{\mathbb{X}} > 1$), then $\vec{\omega} < 0$ ($\vec{\omega} = 0$ resp. $\vec{\omega} > 0$).

Set $\omega_\Lambda := \Lambda^*[-1]$. Let $\mathcal{F}, \mathcal{G} \in D^b(\operatorname{coh}\mathbb{X})$ and let $M, N \in \mathcal{H}^\sigma$ be an object of $D^b(\text{mod-}\Lambda)$ which corresponds to $\mathcal{F}, \mathcal{G}$ under the equivalence (5). Then by the uniqueness of Serre functor there is a natural isomorphism

$$\operatorname{Hom}_{D^b(\operatorname{coh}\mathbb{X})}\left(\mathcal{F}, \mathcal{G} \otimes^{\mathbf{L}}_{\mathbb{X}} \mathcal{O}_{\mathbb{X}}(n\vec{\omega})\right) \cong \operatorname{Hom}_{D^b(\text{mod-}\Lambda)}\left(M, N \otimes^{\mathbf{L}}_{\Lambda} \omega_\Lambda^n\right)$$

for $n \in \mathbb{Z}$.

In the domestic case, by Serre vanishing theorem we can prove that the canonical algebra Λ is a Fano algebra of Fano dimension 1. The triple $(\mathcal{H}^{\omega_\Lambda^{-1}}, \Lambda, \omega_\Lambda^{-1})$ is equivalent to $(\operatorname{coh}\mathbb{X}, T, - \otimes^{\mathbf{L}}_{\mathbb{X}} \mathcal{O}(-\vec{\omega}))$ under the equivalence (5) as a triple.

In the wild case, the canonical bundle $\omega_\Lambda = \Lambda^*[-1]$ is ample. The triple $(\mathcal{H}^{\omega_\Lambda}, \Lambda, \omega_\Lambda)$ is equivalent to $(\operatorname{coh}\mathbb{X}, T, - \otimes^{\mathbf{L}}_{\mathbb{X}} \mathcal{O}(\vec{\omega}))$ under the equivalence (5) as a triple. In general the global dimension $\operatorname{gl.dim}\Lambda$ of a canonical algebra Λ is equal to 2. In both case, equal sign is not true in the inequality of Proposition 2.15.

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