Classical and quantum behavior of the integrated density of states for a randomly perturbed lattice

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Abstract

The asymptotic behavior of the integrated density of states for a randomly perturbed lattice at the infimum of the spectrum is investigated. The leading term is determined when the decay of the single site potential is slow. The leading term depends only on the classical effect from the scalar potential. Contrarily the quantum effect appears when the decay of the single site potential is fast. The corresponding leading term is estimated and the leading order is determined. In the multidimensional cases, the leading order varies in different ways from the known results in the Poisson case. The same problem is considered for the negative potential. These estimates are applied to investigate the long time asymptotics of Wiener integrals associated with the random potentials.

Keywords: perturbed lattice; Random Schrödinger operators; Lifshitz tail; Brownian motion; Wiener integrals

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Running head: IDS for perturbed lattice

1 Introduction

In this paper, we are concerned with the self-adjoint operator in the form of

$$H_\xi = -h\Delta + \sum_{q\in\mathbb{Z}^d} u(\,\cdot - q - \xi_q) \tag{1}$$

defined on the L^2-space on $\mathbb{R}^d \setminus \bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K)$ with the Dirichlet boundary condition, where h is a positive constant and K is a compact set in $\mathbb{R}^d$. Our assumptions on the potential term are the following: (i) $\xi = (\xi_q)_{q\in\mathbb{Z}^d}$ is a collection of independently and identically distributed $\mathbb{R}^d$-valued random variables with

$$\mathbb{P}_\theta(\xi_q \in dx) = \exp(-|x|^\theta)dx/Z(d,\theta) \tag{2}$$

for some $\theta > 0$ and the normalizing constant $Z(d,\theta)$; (ii) u is a nonnegative function belonging to the Kato class K_d (cf. [3] p-53) and satisfying

$$u(x) = C_0|x|^{-\alpha}(1+o(1)) \tag{3}$$

as $|x| \to \infty$ for some $\alpha > d$ and $C_0 > 0$.

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We will consider the integrated density of states $N(\lambda)$ ($\lambda \in \mathbb{R}$) of H_ξ, defined by the thermodynamic limit

$$\frac{1}{|\Lambda_R|} N_{\xi,\Lambda_R}(\lambda) \longrightarrow N(\lambda) \qquad \text{as } R \to \infty. \tag{4}$$

In (4) we denote by Λ_R a box $(-R/2, R/2)^d$ and by $N_{\xi,\Lambda_R}(\lambda)$ the number of eigenvalues not exceeding λ of the self-adjoint operator $H^D_{\xi,R}$ on the L^2-space on $\Lambda_R \setminus \bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K)$ with the Dirichlet boundary condition. It is well known that the above limit exists for almost every ξ and define a deterministic increasing function $N(\lambda)$ (cf. [3], [10]). We here note that the potential term in (1) belongs to the local Kato class $K_{d,loc}$ (cf. [3] p-53) as we will show in Section 9 below.

In this paper we prove the following:

Theorem 1. *If $d < \alpha \le d+2$ and $\operatorname{ess\,inf}_{|x|\le R} u(x)$ is positive for any $R \ge 1$, then we have*

$$\log N(\lambda) \asymp -\lambda^{-(d+\theta)/(\alpha-d)}, \tag{5}$$

where $f(\lambda) \asymp g(\lambda)$ means $0 < \underline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) \le \overline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) < \infty$. Moreover if $\alpha < d+2$, then we have

$$\lim_{\lambda\downarrow 0} \lambda^\kappa \log N(\lambda) = \frac{-\kappa^\kappa}{(\kappa+1)^{\kappa+1}} \left\{ \int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \left(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \right) \right\}^{\kappa+1}, \tag{6}$$

where $\kappa = (d+\theta)/(\alpha-d)$.

Theorem 2. *If $d = 1$ and $\alpha > 3$, then we have*

$$\lim_{\lambda\downarrow 0} \lambda^{(1+\theta)/2} \log N(\lambda) = -\frac{\pi^{1+\theta} h^{(1+\theta)/2}}{(1+\theta)2^\theta}. \tag{7}$$

If $d = 2$ and $\alpha > 4$, then we have

$$\log N(\lambda) \asymp -\lambda^{-1-\theta/2} \left(\log \frac{1}{\lambda} \right)^{-\theta/2}. \tag{8}$$

If $d \ge 3$ and $\alpha > d+2$, then we have

$$\log N(\lambda) \asymp -\lambda^{-(d+\mu\theta)/2}, \tag{9}$$

where $\mu = 2(\alpha-2)/(d(\alpha-d))$.

These results are generalizations of Corollary 3.1 in [5] to the case that $\mathrm{supp}(u)$ is not compact (cf. Theorem 11 below). The results in Theorem 1 are independent of the constant h. This means that only the classical effect from the scalar potential affects the leading term for $\alpha < d+2$ and the leading order for $\alpha \le d+2$. Contrarily the quantum effect appears in Theorem 2. In fact the right hand side of (7) depends on h and the right hand sides of (8) and (9) are strictly less than that of (5). We here note that the right hand side of (5) gives an upper bound not only for $\alpha \le d+2$ but also for $\alpha > d+2$ (see Proposition 4 below). For the critical case $\alpha = d+2$, the quantum effect appears at least in some cases. We shall elaborate this aspect in Section 4 below.

In our model, the single site potentials are randomly displaced from the lattice. As is mentioned in [5], such a model describes the Frenkel disorder in solid state physics and is called a random displacement model in the theory of random Schrödinger operator. Though it is quite natural model in physics, there are only a few mathematical studies and in particular the displacements have been assumed to be bounded in almost all works. For that case, Kirsch and Martinelli [11] discussed the existence of band gaps and Klopp [12] proved spectral localization in a semi-classical limit. More recently, Baker, Loss and Stolz [1], [2] studied which configuration minimizes the spectrum of (1). On the other hand, the displacements are unbounded in our model.

In a slightly broader class of models where the potentials are randomly located, the most studied model is the Poisson model, where the random points $(q+\xi_q)_{q\in\mathbb{Z}^d}$ are replaced by the sample points of the Poisson random measure (cf. [3], [16]). The Poisson model is usually regarded as a model of completely disordered materials, whereas the unperturbed lattice is regarded as completely ordered crystals. As is mentioned in [5], our model describes an intermediate situation between these two extremal situations (see also Remark 1 (i) below). This is the character of our model. In the case of the unperturbed lattice, the infimum of the spectrum becomes positive. Thus it is natural that the decay rates of $N(\lambda)$ explode in the limit $\theta \to \infty$. On the other hand, in the limit of $\theta \to 0$, the above results coincide with the corresponding results for the Poisson model obtained by Pastur [17], Lifshitz [13], Donsker and Varadhan [4], Nakao [14], and Ôkura [15]. As in the Poisson model, the critical value is always $\alpha = d+2$ and, in the one-dimensional case, the leading order increases continuously as α increases to $d+2$ and does not vary for $\alpha \geq d+2$. However contrarily to the Poisson case, the leading order jumps at $\alpha = d+2$ for $d=2$, and that varies also on $\alpha \geq d+2$ for $d \geq 3$. These phenomena are due to the fact that the supports of the states with low energies for the multidimensional case have many holes and some of the potentials are located there, as observed in [5]. This is a characteristic difference with the Poisson case.

For the proof of Theorem 2, we use a method based on a functional analytic approach (cf. [3], [10]). This is different from the method in [5], where a coarse graining method following Sznitman [20] is applied. The method employed here can also be used to give a simpler proof of the results in the compact case in [5]. For this aspect, we will discuss in Section 3 below. On the other hand, the method used in [5] gives finer results in some special cases. This aspect will be discussed in Section 7 (see Theorem 19 for the results). Our proof of Theorem 1 is an extension of that of the corresponding result for the Poisson case (cf. [17], [16]).

Remarks 1. **(i)** In the definition of our model, only the tail of the distribution

$$\mathbb{P}_\theta(\xi_q \in x + [0,1]^d) \asymp \exp(-|x|^\theta)$$

and the leading term $C_0|x|^{-\alpha}$ of the decay of the potential $u(x)$ as $|x| \to \infty$ are essential for our theory. In particular, we may replace $|x|^\theta$ by $(1+|x|)^\theta$ in (2). Then our model tends to that of a completely ordered lattice as $\theta \to \infty$.
(ii) In the definition of the operator (1), the presence of "*hard obstacles*" K has no meanings for the above results. We introduce the hard obstacle for applications in the case that the potential u has a local singularity (see our proof of Theorem 18 in Section 6).

We also consider the operator

$$H_\xi^- = -h\Delta - \sum_{q\in\mathbb{Z}^d} u(\,\cdot - q - \xi_q) \tag{10}$$

obtained by replacing the potential u in H_ξ by $-u$. For this operator, we assume $K = \emptyset$ since we are interested only in the effect of the negative potential. The spectrum of this operator extends to $-\infty$. For the asymptotic distribution, we show the following:

Theorem 3. *If $K = \emptyset$, $\sup u = u(0) < \infty$ and, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $u(x) \geq u(0) - \varepsilon$ for $|x| < R_\varepsilon$, then the integrated density of states $N^-(\lambda)$ of H_ξ^- satisfies*

$$\lim_{\lambda\downarrow-\infty} \frac{\log N^-(\lambda)}{(-\lambda)^{1+\theta/d}} = \frac{-C_1}{u(0)^{1+d/\theta}}, \tag{11}$$

where $C_1 = d^{1+\theta/d}/\{(d+\theta)|S^{d-1}|^{d/\theta}\}$ and $|S^{d-1}|$ is the volume of the $(d-1)$-dimensional surface S^{d-1}.

For the Poisson model, Pastur [17] showed that the corresponding integrated density of states $N^-_{\mathrm{Poi}}(\lambda)$ satisfies

$$\lim_{\lambda\downarrow-\infty} \frac{\log N^-_{\mathrm{Poi}}(\lambda)}{(-\lambda)\log(-\lambda)} = \frac{-1}{u(0)}.$$

The power of λ in (11) tends to that of Poisson model. However, the logarithmic term is not recovered and thus the approximation in rather implicit. Both for the Poisson and our cases, only the classical effect from the scalar potential determines the leading term.

We prove Theorems 1, 2, and 3 in Sections 2, 3, and 5, respectively. In Section 3 we also give a simple proof of the corresponding results for the case that $\mathrm{supp}(u)$ is compact. In Section 4, we discuss the critical case $\alpha = d+2$. We next recall that the main motivation in [5] was to study the survival probability of the Brownian motion in a random environment, which are of interest in their own rights. In Section 6 we recall the connection and extend the theory to the present settings. Finally, we discuss the extension of the method in [5] to our case in Section 7 and asymptotics of higher moments in Section 8.

2 Proof of Theorem 1

2.1 Upper estimate

Let $\widetilde{N}(t)$ be the Laplace-Stieltjes transform of the integrated density of states $N(\lambda)$:

$$\widetilde{N}(t) = \int_0^\infty e^{-t\lambda} dN(\lambda).$$

To prove the upper estimate, we have only to show the following:

Proposition 4. *If $K = \emptyset$ and $\mathrm{ess\,inf}_{|x|\leq R} u(x)$ is positive for any $R \geq 1$, then we have*

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)/(\alpha+\theta)}} \leq -\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) \tag{12}$$

for any $\alpha > d$.

Proof. We use the bound

$$\widetilde{N}(t) \leq \widetilde{N}_1(t)(4\pi t h)^{-d/2}, \tag{13}$$

where

$$\widetilde{N}_1(t) = \int_{\Lambda_1} dx \mathbb{E}_\theta \Big[\exp\Big(-t \sum_{q\in\mathbb{Z}^d} u(\, x - q - \xi_q) \Big) \Big].$$

This is a simple modification of the bound in Theorem (9.6) in [16] for the $\mathbb{Z}^d$-stationary random field. By replacing the summation by the integration, we have

$$\log \widetilde{N}_1(t) \leq \int_{\mathbb{R}^d} dq \log \mathbb{E}_\theta \Big[\exp\Big(-t \inf_{x\in\Lambda_2} u(x - q - \xi_0) \Big) \Big].$$

We restrict the integration to $|q| \le \mathcal{L}$ for some finite $\mathcal{L}$. For any $\varepsilon_1 > 0$, there exists R_1 such that $u(x) \ge C_0(1-\varepsilon_1)|x|^{-\alpha}$ for any $|x|_\infty \ge R_1$, where $|x|_\infty = \max_{1\le i\le d}|x_i|$. Thus the right hand side is dominated by

$$\int_{|q|\le \mathcal{L}} dq \log\left\{ \int_{|q+y|_\infty \ge R_1+1} \frac{dy}{Z(d,\theta)} \exp\Big(-t \inf_{x\in\Lambda_2} \frac{C_0(1-\varepsilon_1)}{|x-q-y|^\alpha} - |y|^\theta \Big) + \exp\Big(-t \inf_{\Lambda_{2R_1+4}} u \Big) \right\}.$$

By changing the variables, this equals

$$t^{d\eta} \int_{|q|\le L} dq \log\Big\{ \widetilde{N}_2(t,q) + \exp\Big(-t \inf_{\Lambda_{2R_1+4}} u \Big) \Big\},$$

where

$$\widetilde{N}_2(t,q) = t^{d\eta} \int_{|q+y|_\infty \ge (R_1+1)t^{-\eta}} \frac{dy}{Z(d,\theta)} \exp\Big(-t^{\theta\eta} \inf_{x\in\Lambda_{2t^{-\eta}}} \frac{C_0(1-\varepsilon_1)}{|x-q-y|^\alpha} - t^{\theta\eta}|y|^\theta \Big),$$

$\eta = 1/(\alpha+\theta)$ and $L = \mathcal{L}t^{-\eta}$. We take L as an arbitrary constant independent of t. Then, taking $\varepsilon_2, \varepsilon_3 > 0$ sufficiently small and using the positivity assumption, we can dominate $\widetilde{N}_2(t,q)$ by $\exp(-t^{\theta\eta}\widetilde{N}_3(q))\varepsilon_2^{-d/\theta}$ for large enough t, where

$$\widetilde{N}_3(q) = \inf\Big\{ \frac{C_0(1-\varepsilon_1)}{|x-q-y|^\alpha} + (1-\varepsilon_2)|y|^\theta : x \in \Lambda_{\varepsilon_3}, y \in \mathbb{R}^d \Big\}.$$

Therefore we obtain

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)\eta}} \le -\int_{|q|\le L} \widetilde{N}_3(q)dq.$$

Since $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and L are arbitrary, we can complete the proof. □

2.2 Lower estimate

To prove the lower estimate, we have only to show the following:

Proposition 5. *If $\alpha < d+2$, then we have*

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)/(\alpha+\theta)}} \ge -\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big). \tag{14}$$

Moreover, this bound remains valid for $\alpha = d+2$ with a smaller constant in the right hand side.

For the case of $\alpha = d+2$, we discuss in more detail in Section 4 below.

Proof of Proposition 5. We use the bound

$$\widetilde{N}(t) \ge R^{-d} \exp(-th\|\nabla\psi_R\|_2^2)\widetilde{N}_1(t), \tag{15}$$

for any $R \in \mathbb{N}$ and $\psi_R \in C_0^\infty(\Lambda_R)$ such that $\|\psi_R\|_2 = 1$, where $\|\cdot\|_2$ is the L^2-norm, and

$$\widetilde{N}_1(t) = \mathbb{E}_\theta\left[\exp\Big(-t \sum_{q\in\mathbb{Z}^d} \int dx \psi_R(x)^2 u(\, x - q - \xi_q) \Big) : \bigcup_{q\in\mathbb{Z}^d} (q + \xi_q + K) \cap \Lambda_R = \emptyset \right].$$

This is proven by the same method as for the corresponding bound in Theorem (9.6) in [16] for the $\mathbb{R}^d$-stationary random field. By replacing the summation by the integration, we have

$$\log \widetilde{N}_1(t) \geq \int_{\mathbb{R}^d} \widetilde{N}_2(t,q)dq,$$

where

$$\widetilde{N}_2(t,q) = \log \mathbb{E}_\theta \Big[\exp\Big(-t \int dx \psi_R(x)^2 \sup_{z\in\Lambda_1} u(x-q-z-\xi_0) \Big) : (q+\xi_0+K)\cap\Lambda_R = \emptyset \Big].$$

For any $\varepsilon_1 > 0$, there exists R_1 such that $K \subset B(R_1)$ and $u(x) \leq C_0(1+\varepsilon_1)|x|^{-\alpha}$ for any $|x| \geq R_1$ by the assumption (3). To use this bound in the above right hand side, we need $\inf\{|x-q-z-y| : x\in\Lambda_R, z\in\Lambda_1\} \geq R_1$. However we shall deal with a simpler sufficient condition $|y| \leq |q|/2$ and $|q| \geq 2(R_1+\sqrt{d}R)$ instead. Now let $\beta > 0$ be fixed and take t large enough so that $t^\beta > 2(R_1+\sqrt{d}R)$. Then we obtain

$$\int_{|q|\geq t^\beta} \widetilde{N}_2(t,q)dq \geq \int_{|q|\geq t^\beta} dq \Big(-\frac{tC_0(1+\varepsilon_1)2^\alpha}{(|q|-2\sqrt{d}R)^\alpha} + \log \mathbb{P}_\theta(|\xi_0| \leq |q|/2) \Big). \tag{16}$$

By a simple estimate using $\log(1-X) \geq -2X$ for $0 \leq X \leq 1/2$, we can dominate the right hand side from below by $-c_1 t^{1-\beta(\alpha-d)} - c_2\exp(-c_3 t^{\beta\theta})$. The other part is dominated as

$$\begin{aligned} &\int_{|q|\leq t^\beta} \widetilde{N}_2(t,q)dq \\ &\geq \int_{|q|\leq t^\beta} dq \log \int_{|q+y|\geq R_1+\sqrt{d}R} \frac{dy}{Z(d,\theta)} \exp\Big(-\frac{tC_0(1+\varepsilon_1)}{\inf\{|x-q-z-y|^\alpha : x\in\Lambda_R, z\in\Lambda_1\}} - |y|^\theta \Big). \end{aligned} \tag{17}$$

By changing the variables, this equals

$$t^{d\eta} \int_{|q|\leq t^{\beta-\eta}} dq \log \int_{|q+y|\geq (R_1+\sqrt{d}R)t^{-\eta}} \frac{dy t^{d\eta}}{Z(d,\theta)} \exp(-t^{\theta\eta}\widetilde{N}_3(y,q)),$$

where $\eta = 1/(\alpha+\theta)$ and

$$\widetilde{N}_3(y,q) = \frac{C_0(1+\varepsilon_1)}{\inf\{|x-q-z-y|^\alpha : x\in\Lambda_{Rt^{-\eta}}, z\in\Lambda_{t^{-\eta}}\}} + |y|^\theta. \tag{18}$$

Taking $\gamma > 0$, we restrict the integration with respect to y to the ball $B(y_0, t^{-\gamma})$ with the center y_0 and the radius $t^{-\gamma}$. Then we can dominate the integrand with respect to q from below by

$$\log \frac{|B(0,1)|t^{d(\eta-\gamma)}}{Z(d,\theta)} - t^{\theta\eta}\widetilde{N}_4(q,t), \tag{19}$$

where

$$\widetilde{N}_4(q,t) = \inf\Big\{ \sup_{y\in B(y_0,t^{-\gamma})} \widetilde{N}_3(y,q) : y_0\in\mathbb{R}^d, d(B(y_0,t^{-\gamma}),-q) \geq (R_1+\sqrt{d}R)t^{-\eta} \Big\}. \tag{20}$$

We now specify R as the integer part of $\varepsilon_2 t^\eta$, where ε_2 is an arbitrarily fixed positive number. We take ψ_R as a normalized ground state of the Dirichlet Laplacian on the cube Λ_R and take β between η and $\eta(1+\theta/d)$. Then, for $\alpha < d+2$, we obtain

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)\eta}} \geq -\varlimsup_{t\uparrow\infty} \int_{|q|\leq t^{\beta-\eta}} dq \widetilde{N}_4(q,t), \tag{21}$$

since $th\|\nabla\psi_R\|_2 \asymp tR^{-2}$ and (16) is negligible compared with $t^{(d+\theta)\eta}$. When $|q| \leq t^{\beta-\eta}$, we can dominate $1/t$ by a power of q. Thus, for large $|q|$, by taking y_0 as 0, we can dominate $\widetilde{N}_4(q,t)$ by $|q|^{-\alpha} + |q|^{-\gamma\theta/(\beta-\eta)}$. This is integrable if we take γ large enough so that $\gamma\theta/(\beta-\eta) > d$. Thus, by the Lebesgue convergence theorem, we have

$$\lim_{t\uparrow\infty} \int_{|q|\leq t^{\beta-\eta}} dq \widetilde{N}_4(q,t) = \int_{\mathbb{R}^d} dq \inf\Big\{ \frac{C_0(1+\varepsilon_1)}{\inf\limits_{x\in\Lambda_{\varepsilon_2}} |x-q-y|^\alpha} + |y|^\theta : y \in \mathbb{R}^d, d(y,q) \geq \varepsilon_2\sqrt{d} \Big\}.$$

Since ε_1 and ε_2 are arbitrary, we can complete the proof of the former part of Proposition 5. For the case $\alpha = d+2$, we take $\varepsilon_2 = 1$. Then we have $th\|\nabla\psi_R\|_2 \asymp t^{(d+\theta)\eta}$ and the latter part of Proposition 5 follows from the same argument as above. □

3 Proof of Theorem 2 and the compact case

In this section, we use some additional notations to simplify the presentation. For any self-adjoint operator A, let $\lambda_1(A)$ be the infimum of its spectrum and, for any locally integrable function V and $R > 0$, let $(-h\Delta + V)_R^D$ and $(-h\Delta + V)_R^N$ be the self-adjoint operators $-h\Delta + V$ on the L^2-space on the cube Λ_R with the Dirichlet and the Neumann boundary conditions, respectively.

3.1 Proof of Theorem 2 (I): One-dimensional case

To obtain the upper estimate, we have only to show the following:

Proposition 6. *If* $d = 1$, $K = \emptyset$, $\operatorname{supp}(u)$ *is compact,*

$$\liminf_{x\downarrow 0} \int_0^x u(y)dy/x > 0, \text{ and } \liminf_{x\downarrow 0} \int_{-x}^0 u(y)dy/x > 0, \tag{22}$$

then we have

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(1+\theta)/(3+\theta)}} \leq -\frac{3+\theta}{1+\theta}\Big(\frac{h\pi^2}{4}\Big)^{(1+\theta)/(3+\theta)}. \tag{23}$$

Proof. We assume $h = 1$ for simplicity. In the well known expression

$$\widetilde{N}(t) = \int_{\Lambda_1} \mathbb{E}_\theta[\exp(-tH_\xi)(x,x)]dx,$$

we apply the Feynman-Kac formula and an estimate on the exit time of the Brownian motion (cf. [8]) to obtain

$$\widetilde{N}(t) \leq \int_{\Lambda_1} \mathbb{E}_\theta[\exp(-tH_{\xi,t}^D)(x,x)]dx + c_1 e^{-c_2 t},$$

where $\exp(-tH_\xi)(x,y)$ and $\exp(-tH^D_{\xi,t})(x,y)$, $t>0$, $x,y\in\mathbb{R}$, are the integral kernels of the heat semigroups generated by H_ξ and $H^D_{\xi,t}$, respectively. By the eigenfunction expansion of the integral kernel, we have

$$\widetilde{N}(t)\le c_3t\widetilde{N}_1(t)+c_4e^{-c_5t},$$

where $\widetilde{N}_1(t)=\mathbb{E}_\theta[\exp(-t\lambda_1(H^D_{\xi,t}))]$. Thus we have only to prove (23) with $\widetilde{N}(t)$ replaced by $\widetilde{N}_1(t)$. Now we use Theorem 3.1 in the page 123 in [20], which states

$$\lambda_1(H^D_{\xi,t})\ge \pi^2/(\sup_k|I_k|+c_6)^2$$

for large enough t under the assumption (22), where $\{I_k\}_k$ are the random open intervals such that $\sum_k I_k=\Lambda_t-\{q+\xi_q:q\in\mathbb{Z}\}$, and $|I_k|$ is the length of I_k. If $\sup_k|I_k|\ge s$ for some $0\le s\le t$, then there exists $p\in\mathbb{Z}\cap\Lambda_t$ such that $\{q+\xi_q:q\in\mathbb{Z}\}\cap[p,p+s-2]=\emptyset$. The probability of this event is estimated as

$$\begin{aligned}
&\mathbb{P}_\theta(\sup_k|I_k|\ge s)\le \sum_{p\in\mathbb{Z}\cap\Lambda_t}\prod_{q\in\mathbb{Z}\cap[p,p+s-2]}\mathbb{P}_\theta(q+\xi_q\notin[p,p+s-2])\\
&\le t\prod_{q\in\mathbb{Z}\cap[p,p+s-2]}\exp(-(1-\varepsilon)d(q,[p,p+s-2]^c)^\theta)/\varepsilon^{1/\theta}\\
&\le t\exp\Big(-(1-\varepsilon)\int_0^{s-3}d(q,[0,s-3]^c)^\theta dq+\frac{s}{\theta}\log\frac{1}{\varepsilon}\Big)\\
&\le t\exp\Big(-\frac{2(1-\varepsilon)}{\theta+1}\Big(\frac{s-3}{2}\Big)^{\theta+1}+\frac{s}{\theta}\log\frac{1}{\varepsilon}\Big)
\end{aligned}$$

if $s\ge 3$, where $0<\varepsilon<1$ is arbitrary. Therefore we have

$$\widetilde{N}_1(t)\le c_7t^2\exp\Big(-\inf_{R>3}\Big(t\frac{\pi^2}{(R+c_6)^2}+\frac{(1-\varepsilon)}{2^\theta(\theta+1)}(R-3)^{\theta+1}-\frac{R}{\theta}\log\frac{1}{\varepsilon}\Big)\Big)+c_8e^{-c_9t}$$

for large t. Now it is easy to see that the infimum in the right hand side is attained by $R\sim 2(\pi^2t/4)^{1/(3+\theta)}$ and we obtain (23). □

Remark 1. We put the additional assumption (22) only to use Theorem 3.1 in the page 123 in [20]. These assumptions are not restrictive at all since we can always find a $z\in\mathbb{R}$ such that $u(\,\cdot\,+z)$ satisfies them by the fundamental theorem of calculus and such a finite translation of u does not affect the above argument.

Proposition 7. *If $d=1$ and $\alpha>3$, then we have*

$$\varliminf_{t\uparrow\infty}\frac{\log\widetilde{N}(t)}{t^{(1+\theta)/(3+\theta)}}\ge -\frac{3+\theta}{1+\theta}\Big(\frac{h\pi^2}{4}\Big)^{(1+\theta)/(3+\theta)}. \tag{24}$$

Proof. This is proven by modifying our proof of Proposition 5. We take ψ_R as the normalized ground state of $(-\Delta)^D_R$. In (17), we restrict the integral with respect to y to $|q+y|\ge R_1+(R+1)/2$. In (19), we take $\eta=1/(3+\theta)$ and R as the integer part of $\mathcal{R}t^\eta$ for a positive number $\mathcal{R}>0$. Then since $t\|\nabla\psi_R\|_2^2\sim t^{(1+\theta)\eta}(\pi/\mathcal{R})^2$ is not negligible, (21) is modified as

$$\varliminf_{t\uparrow\infty}\frac{\log\widetilde{N}(t)}{t^{(1+\theta)\eta}}\ge -h\Big(\frac{\pi}{\mathcal{R}}\Big)^2-\varlimsup_{t\uparrow\infty}\int_{|q|\le t^{\beta-\eta}}dq\widetilde{N}_4(q,t),$$

where $\widetilde{N}_4(q,t)$ is defined by replacing $\widetilde{N}_3(y,q)$ and $R_1+\sqrt{d}$ by

$$\frac{C_0(1+\varepsilon_1)}{t^{(\alpha-3)\eta}\inf\{|x-q-z-y|^\alpha : x\in\Lambda_{Rt^{-\eta}}, z\in\Lambda_{t^{-\eta}}\}}+|y|^\theta$$

and $R_1+(R+1)/2$, respectively, in (20). Since

$$\overline{\lim_{t\uparrow\infty}}\,\widetilde{N}_4(q,t)\le \inf_{y\notin\Lambda_{\mathcal{R}}(-q)}|y|^\theta = d(q,\Lambda_{\mathcal{R}}^c)^\theta,$$

we obtain

$$\underline{\lim_{t\uparrow\infty}}\frac{\log\widetilde{N}(t)}{t^{(1+\theta)\eta}}\ge -h\Big(\frac{\pi}{\mathcal{R}}\Big)^2-\frac{\mathcal{R}^{\theta+1}}{2^\theta(\theta+1)},$$

by the Lebesgue convergence theorem. By taking the supremum over $\mathcal{R}>0$, we obtain the result. $\square$

3.2 Proof of Theorem 2 (II) : Upper estimate for the multidimensional case

In the two-dimensional case, we only use Corollary 3.1 in [5]:

$$N(\lambda)\le c_1\exp(-c_2\lambda^{-1-\theta/2}(\log(1/\lambda))^{-\theta/2}), \tag{25}$$

for $0\le\lambda\le c_3$, where c_1, c_2 and c_3 are finite constants depending on h and C_0. We give another proof in subsection 3.4 below.

In the rest of this subsection we assume $d\ge 3$. Then our goal is the following:

Proposition 8. *Let $\alpha\ge d+2$ and $K=\emptyset$. There exist finite positive function $k_1(h)$ and $k_2(h)$ of h and a finite constant c such that*

$$N(\lambda)\le k_1(h)\exp(-c((h\wedge h^{(\alpha-d)/(\alpha-2)})/\lambda)^{(d+\mu\theta)/2}) \tag{26}$$

for $0\le\lambda\le k_2(h)$.

We first see that Proposition 8 follows from the following:

Proposition 9. *For small enough $\varepsilon_1,\varepsilon_2>0$, there exist a finite constant c independent of (h,R), and finite constants c' and c'' independent of (c_0,h,R) such that $\#\{q\in\mathbb{Z}^d\cap\Lambda_R : |\xi_q|\ge\varepsilon_1R^\mu\}\le\varepsilon_2R^d$, $R^{\mu d}\ge c'h/c_0$ and $R^{\mu(\alpha-2-d)}\ge c''c_0/h$ imply*

$$\lambda_1\Big(\Big(-h\Delta+\sum_{q\in\mathbb{Z}^d\cap\Lambda_R}\frac{c_0 1_{B(q+\xi_q,R_0)^c}(x)}{|x-q-\xi_q|^\alpha}\Big)_R^N\Big)\ge c(h\wedge h^{(\alpha-d)/(\alpha-2)})/R^2, \tag{27}$$

where c_0 and R_0 are arbitrarily fixed positive constants, and 1_D is the characteristic function of D for any subset D in $\mathbb{R}^d$.

Proof of Proposition 8. It is well known that

$$N(\lambda)\le\frac{c_1}{(R\wedge\sqrt{h})^d}\mathbb{P}_\theta(\lambda_1(H_R^N)\le\lambda)$$

(cf. (10.10) in [16]). We can take c_0 and R_0 so that

$$u(x)\ge c_0 1_{B(R_0)^c}(x)|x|^{-\alpha}.$$

Thus by Proposition 9, there exists a constant c_2 such that

$$N(c_2(h \wedge h^{(\alpha-d)/(\alpha-2)})/R^2) \leq \frac{c_1}{(R \wedge \sqrt{h})^d} \mathbb{P}_\theta(\#\{q \in \mathbb{Z}^d \cap \Lambda_R : |\xi_q| \geq \varepsilon_1 R^\mu\} \geq \varepsilon_2 R^d).$$

We here should take c_0 sufficiently small so that the conditions of Proposition 9 are satisfied if $\alpha = d+2$. When the event in the right hand side occurs, we have

$$\sum_{q \in \mathbb{Z}^d \cap \Lambda_R} |\xi_q|^\theta \geq \varepsilon_1^\theta \varepsilon_2 R^{d+\mu\theta}.$$

Thus it is easy to show

$$N(c_2(h \wedge h^{(\alpha-d)/(\alpha-2)})/R^2) \leq \frac{c_3}{(R \wedge \sqrt{h})^d} \exp(-c_4 R^{d+\mu\theta}),$$

and (26) follows immediately. □

We next proceed to the proof of Proposition 9. To this end, we prepare the following:

Lemma 1. $\inf\{\lambda_1((-\Delta + 1_{B(b,1)})_R^N) : b \in \Lambda_R\} \geq cR^{-d}$.

This lemma follows immediately from the Proposition 2.3 of Taylor [21] using the scaling with the factor R^{-1}. That proposition is stated in terms of the scattering length. We here give an elementary proof following a lemma in the page 378 in Rauch [18] for the reader's convenience.

Proof. We rewrite as $\lambda_1((-\Delta+1_{B(b,1)})_R^N) = \lambda_1((-\Delta+1_{B(1)})_{R,b}^N)$, where, for any locally integrable function V and $R > 0$, $(-\Delta + V)_{R,b}^N$ is the self-adjoint operator $-\Delta + V$ on the L^2 space on the cube $\Lambda_R(b) = b + \Lambda_R$ with the the Neumann boundary condition, and $B(1) = B(0,1)$. For any smooth function φ on the closure of $\Lambda_R(b)$, we have

$$\begin{aligned}
&\int_{\Lambda_R(b)} \varphi^2(x)dx \\
&= \int_1^{R(b)} dr r^{d-1} \int_{\theta \in S^{d-1}:(r,\theta) \in \Lambda_R(b)} dS\Big(\varphi(g(r),\theta) + \int_{g(r)}^r \partial_s \varphi(s,\theta)ds\Big)^2 + \int_{B(1)\cap\Lambda_R(b)} \varphi^2(x)dx,
\end{aligned}$$

where (r,θ) is the polar coordinate, $R(b) = \sup\{|x| : x \in \Lambda_R(b)\}$, dS is the volume element of the $(d-1)$-dimensional surface S^{d-1} and $g(r) = \{(r-1)/(R(b)-1)+1\}/2$. By the Schwarz inequality and a simple estimate, we can show

$$\int_1^{R(b)} dr r^{d-1} \int_{\theta \in S^{d-1}:(r,\theta) \in \Lambda_R(b)} dS\Big(\int_{g(r)}^r \partial_s \varphi(s,\theta)ds\Big)^2 \leq cR(b)^d \int_{\Lambda_R(b)} |\nabla\varphi|^2(x)dx,$$

where c is a constant depending only on d. By changing the variable, we can also show

$$\int_1^{R(b)} dr r^{d-1} \int_{\theta \in S^{d-1}:(r,\theta) \in \Lambda_R(b)} dS \varphi(g(r),\theta)^2 \leq c' R(b)^d \int_{B(1)\cap\Lambda_R(b)} \varphi^2(x)dx,$$

where c' is also a constant depending only on d. Since $\sup_{b \in \Lambda_R} R(b) \leq \sqrt{d}R$, we can complete the proof. □

Lemma 2. *There exist finite constants c, c' and c'' such that*

$$\inf\Big\{\lambda_1\Big(\Big(-h\Delta+\sum_{j=1}^{n}\frac{c_0 1_{B(b_j,R_0)^c}(x)}{|x-b_j|^\alpha}\Big)_R^N\Big):b_1,\ldots,b_n\in\Lambda_R\Big\}\geq c(c_0n)^{(d-2)/(\alpha-2)}h^{(\alpha-d)/(\alpha-2)}/R^d$$

for $n\geq c'h/c_0$ and $R\geq c''(c_0n/h)^{1/(\alpha-2)}$.

Proof. Since $\lambda_1(A+B)\geq\lambda_1(A)+\lambda_1(B)$ for any self-adjoint operators A and B, the left hand side is bounded from below by

$$\inf\{\lambda_1((-h\Delta+c_0n1_{B(b,R_0)^c}(x)|x-b|^{-\alpha})_R^N):b\in\Lambda_R\}.$$

By changing the variable, this equals

$$hk^{-2}\inf\{\lambda_1((-\Delta+c_0nk^{2-\alpha}h^{-1}1_{B(b,R_0/k)^c}(x)|x-b|^{-\alpha})_{R/k}^N):b\in\Lambda_{R/k}\}$$

for any $k>0$. We can dominate this from below by

$$hk^{-2}\inf\{\lambda_1((-\Delta+c_0nk^{2-\alpha}h^{-1}3^{-\alpha}1_{B(b',1)}(x))_{R/k}^N):b'\in\Lambda_{R/k}\}$$

for $k\geq R_0$ and $R>4\sqrt{d}k$, and we can use Lemma 1 to complete the proof by taking k as $(c_0n3^{-\alpha}h^{-1})^{1/(\alpha-2)}$. In fact, for each $b\in\Lambda_{R/k}$, we set $b':=b-(1+R_0/k)b/|b|$ if b is not the zero vector. If b is the zero vector, we set b' as an arbitrarily chosen vector with the norm $1+R_0/k$. Since $R_0/k\leq|x-b|\leq 2+R_0/k$ on $B(b',1)$, we have

$$1_{B(b,R_0/k)^c}(x)|x-b|^{-\alpha}\geq(2+R_0/k)^{-\alpha}1_{B(b',1)}(x).$$

We dominate this from below by $3^{-\alpha}1_{B(b',1)}(x)$ by assuming $k\geq R_0$. Moreover we claim $b'\in\Lambda_{R/k}$ for all $b\in\Lambda_{R/k}$. A sufficient condition for this is $R\geq 2\sqrt{d}(R_0+k)$, since b' for b with $|b|\geq 1+R_0/k$ is a contraction of b and $\sup\{|b'|_\infty:|b|\leq 1+R_0/k\}=\sqrt{d}(1+R_0/k)$. □

Lemma 3. *Let V be any locally integrable nonnegative function on $\mathbb{R}^d$. Then any eigenfunction ϕ of $(-h\Delta+V)_R^N$ satisfies*

$$\|\phi\|_\infty\leq c(1/R+\sqrt{\lambda/h})^{d/2}\|\phi\|_2,$$

where c is a finite constant depending only on d, λ is the corresponding eigenvalue, and $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are L^∞ and L^2 norms, respectively.

The proof of this lemma is same with that of (3.1.55) in [20]. Now we prove Proposition 9:

Proof of Proposition 9. We use the following classification:

$$\mathcal{F}=\{a\in\Lambda_R\cap R^\mu\mathbb{Z}^d:\#(\Lambda_{R^\mu}(a)\cap\{q+\xi_q:q\in\mathbb{Z}^d\cap\Lambda_R\})<R^{\mu d}/2\}$$

and

$$\mathcal{N}=\{a\in\Lambda_R\cap R^\mu\mathbb{Z}^d:\#(\Lambda_{R^\mu}(a)\cap\{q+\xi_q:q\in\mathbb{Z}^d\cap\Lambda_R\})\geq R^{\mu d}/2\}.$$

By Lemma 2, $\lambda_1((-h\Delta+\sum_q c_0 1_{B(q+\xi_q,R_0)^c}(x)|x-q-\xi_q|^{-\alpha})_{R^\mu,a}^N)\geq ch^{(\alpha-d)/(\alpha-2)}/R^2$ for any $a\in\mathcal{N}$. Then the normalized ground state φ of the operator $(-h\Delta+\sum_q c_0 1_{B(q+\xi_q,R_0)^c}(x)|x-q-\xi_q|^{-\alpha})_R^N$ satisfies

$$\lambda_1\Big(\Big(-h\Delta+\sum_q\frac{c_0 1_{B(q+\xi_q,R_0)^c}(x)}{|x-q-\xi_q|^\alpha}\Big)_R^N\Big)\geq\frac{ch^{(\alpha-d)/(\alpha-2)}}{R^2}\sum_{a\in\mathcal{N}}\int_{\Lambda_{R^\mu}(a)}\varphi^2dx.$$

If we assume $\lambda_1((-h\Delta + \sum_q c_0 1_{B(q+\xi_q,R_0)^c}(x)|x-q-\xi_q|^{-\alpha})^N_{R^\mu,a}) \le Mh/R^2$, then Lemma 3 implies that the right hand side is bounded from below by

$$cR^{-2}h^{(\alpha-d)/(\alpha-2)}(1-c' M^{d/2}R^{(\mu-1)d}\#\mathcal{F}). \tag{28}$$

Since $\#(\Lambda_{R^\mu}(a) \cap \{q+\xi_q : q \in \mathbb{Z}^d \cap \Lambda_R\}) \ge \#\{q \in \Lambda_{(1-2\varepsilon_1)R^\mu}(a) \cap \mathbb{Z}^d : |\xi_q| \le \varepsilon_1 R^\mu\}$, we have $\#\{q \in \Lambda_{(1-2\varepsilon_1)R^\mu}(a) \cap \mathbb{Z}^d : |\xi_q| \le \varepsilon_1 R^\mu\} < R^{\mu d}/2$ and $\#\{q \in \Lambda_{(1-2\varepsilon_1)R^\mu}(a) \cap \mathbb{Z}^d : |\xi_q| \ge \varepsilon_1 R^\mu\} > \{(1-2\varepsilon_1)^d - 1/2\}R^{\mu d}$ for $a \in \mathcal{F}$. Thus, by the assumption of this proposition, we have $\varepsilon_2 R^d \ge (\#\mathcal{F})\{(1-2\varepsilon_1)^d - 1/2\}R^{\mu d}$ and $\#\mathcal{F} \le R^{d(1-\mu)}\varepsilon_2/\{(1-2\varepsilon_1)^2 - 1/2\}$. By substituting this to (28), we can complete the proof. $\square$

3.3 Proof of Theorem 2 (III) : Lower estimate for the multidimensional case

In this subsection, we prove the lower estimate. We shall work with $h = C_0 = 1$ for simplicity.

Proposition 10. *Suppose $d=2$ and $\alpha > 4$ or $d \ge 3$ and $\alpha \ge d+2$. There exist finite constants c_1, c_2 and c_3 such that*

$$N(\lambda) \ge \begin{cases} c_1 \exp\Big(-c_2\lambda^{-1-\theta/2}\left(\log\frac{1}{\lambda}\right)^{-\theta/2}\Big) & (d=2), \\ c_1\exp(-c_2\lambda^{-(d+\mu\theta)/2}) & (d\ge 3), \end{cases} \tag{29}$$

for $0 \le \lambda \le c_3$.

Proof. We consider the event

$$\begin{aligned}&\{\text{For any } p \in R_1\mathbb{Z}^d \cap \Lambda_{3R} \text{ and } q \in \mathbb{Z}^d \cap \Lambda_{R_1}(p) \cap \Lambda_{2R}, q+\xi_q \in \Lambda_1(p).\\ &\text{For any } q \in \mathbb{Z}^d \setminus \Lambda_{2R}, |\xi_q| \le |q|/4\}\end{aligned} \tag{30}$$

where $R_1 = R^\mu$ for $d \ge 3$ and $R_1 = R/\sqrt{\log R}$ for $d=2$. Then we have

$$N(\lambda) \ge R^{-d}\mathbb{P}_\theta\Big(\|\nabla\Phi_R\|_2^2 + \Big(\Phi_R, \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)\Phi_R\Big) \le \lambda \text{ and the event (30) holds}\Big), \tag{31}$$

where Φ_R is an element of the domain of the Dirichlet Laplacian on the cube $\Lambda_R \setminus \bigcup_{p\in R_1\mathbb{Z}^d\cap\Lambda_{3R}}(p+K)$ such that $\|\Phi_R\|_2 = 1$ (cf. Theorem (5.25) in [16]). We take Φ_R as $\phi_R\psi_R/\|\phi_R\psi_R\|_2$, where ψ_R is the normalized ground state of the Dirichlet Laplacian on Λ_R and

$$\phi_R(x) = \begin{cases} \Big(2d_\infty\Big(x, \sum_{p\in R^\mu\mathbb{Z}^d\cap\Lambda_R}\Lambda_{R^\nu}(p)\Big)R^{-\nu}\Big) \wedge 1 & (d\ge 3), \\ \Big(\log d_\infty(x, \Lambda_R \cap \frac{R\mathbb{Z}^2}{\sqrt{\log R}}) - \frac{4}{\alpha}\log R\Big)_+ \Big/ \Big(\log\frac{R}{2\sqrt{\log R}} - \frac{4}{\alpha}\log R\Big) & (d=2). \end{cases} \tag{32}$$

In (32), $d_\infty(\cdot,\cdot)$ is the distance function with respect to the maximal norm, $\nu = 2/(\alpha-d)$, and $(\cdot)_+$ is the positive part. Then it is not difficult to see $\|\nabla\Phi_R\|_2^2 \le c_4R^{-2}$. On the event (30), we have

$$\sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q) \le \frac{c_5R_1^d}{d(x, \sum_{p\in R_1\mathbb{Z}^d\cap\Lambda_{2R}}\Lambda_1(p))^\alpha} + c_6R_1^{-(\alpha-d)} \tag{33}$$

in Λ_R. Then we have

$$\Big(\Phi_R, \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)\Phi_R\Big) \le c_7R^{-2}.$$

On the other hand, the probability of the event (30) can be estimated as

$$\begin{aligned}
&\log \mathbb{P}_\theta(\text{ the event in (30) occurs })\\
\geq& -\#(R_1\mathbb{Z}^d \cap \Lambda_{3R}) \sum_{q\in\mathbb{Z}^d\cap\Lambda_{R_1}} \log \mathbb{P}_\theta(\xi_0 \in \Lambda_1(q)) + \sum_{q\in\mathbb{Z}^d\setminus\Lambda_{2R}} \log(1-\mathbb{P}_\theta(|\xi_0| \geq |q|/4))\\
\geq& -c_8 R^d R_1^\theta
\end{aligned}$$

by using $\log(1-X) \geq -2X$ for $0 \leq X \leq 1/2$ in the last line. Therefore, we have

$$N(c_9R^{-2}) \geq R^{-d}\exp\Big(-c_{10}R^dR_1^\theta\Big)$$

and the proof is finished. □

Remark 2. For the manner of taking the function ϕ_R in (32) and the event in (30), we refer the reader to the notion of the "constant capacity regime" (cf. Section 3.2.B of [20]). The same technique is used in Appendix B of [5].

3.4 Compact case

In this subsection, we modify the methods in the preceding sections to give a simple proof of the following results in [5]:

Theorem 11. *We assume* $\Lambda_{r_1} \subset \mathrm{supp}(u) \cup K \subset \Lambda_{r_2}$ *for some* $0 < r_1 \leq r_2 < \infty$ *instead of* (3). *Then we have*

$$\log N(\lambda) \begin{cases} \sim -\Big(\frac{\pi^2 h}{\lambda}\Big)^{(1+\theta)/2} \frac{1}{(1+\theta)2^\theta} & (d=1),\\ \asymp -\lambda^{-1-\theta/2}\Big(\log\frac{1}{\lambda}\Big)^{-\theta/2} & (d=2),\\ \asymp -\lambda^{-(d/2+\theta/d)} & (d\geq 3) \end{cases}$$

as $\lambda \downarrow 0$, *where* $f(\lambda) \sim g(\lambda)$ *means* $\lim_{\lambda\downarrow 0} f(\lambda)/g(\lambda) = 1$ *and* $f(\lambda) \asymp g(\lambda)$ *means* $0 < \underline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) \leq \overline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) < \infty$.

Remark 3. The assumption on u in this theorem is only for giving a simple proof in the multidimensional case. If $d = 1$, then the assumption in Proposition 6 is enough. If $d \geq 3$, then this theorem is extended to the case that the scattering length of u is positive.

The proof for $d = 1$ is given in Subsection 3.1. The lower estimate for $d = 2$ is given in Subsection 3.3. To prove the lower estimate for $d \geq 3$, we replace R^ν by $2r_2 + 1$ in the proof of Proposition 10. Then the rest of the proof is simpler than that of the proposition since

$$\Big(\Phi_R, \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)\Phi_R\Big) = 0$$

under the event in (30) with $R_1 = R^{2/d}$. To prove the upper estimate for $d \geq 3$, we have only to apply the following instead of Proposition 9 in the proof of Proposition 8:

Proposition 12. *For small enough* $\varepsilon_1, \varepsilon_2 > 0$, *there exists a finite constant c such that* $\#\{q \in \mathbb{Z}^d \cap \Lambda_R : |\xi_q| \geq \varepsilon_1 R^{2/d}\} \leq \varepsilon_2 R^d$ *implies*

$$\lambda_1\Big(\Big(-\Delta + c_0 \sum_{q\in\mathbb{Z}^d\cap\Lambda_R} 1_{B(q+\xi_q, r_0)}\Big)_R^N\Big) \geq c/R^2, \tag{34}$$

where c_0 *and* r_0 *are arbitrarily fixed positive constants.*

Proof. In the proof of Proposition 9, we use the classification

$$\mathcal{F}_0 = \{a \in \Lambda_R \cap R^{2/d}\mathbb{Z}^d : \Lambda_{R^{2/d}}(a) \cap \{q + \xi_q : q \in \mathbb{Z}^d \cap \Lambda_R\} = \emptyset\}$$

and

$$\mathcal{N}_0 = \{a \in \Lambda_R \cap R^{2/d}\mathbb{Z}^d : \Lambda_{R^{2/d}}(a) \cap \{q + \xi_q : q \in \mathbb{Z}^d \cap \Lambda_R\} \neq \emptyset\},$$

instead of $\mathcal{F}$ and $\mathcal{N}$. Then we can complete the proof by Lemmas 1 and 3 without using Lemma 2. □

To prove the upper estimate for $d = 2$, we have only to apply the following instead of Proposition 9 in the proof of Proposition 8:

Proposition 13. *For small enough $\varepsilon_1, \varepsilon_2 > 0$, there exists a finite constant c such that $\#\{q \in \mathbb{Z}^2 \cap \Lambda_R : |\xi_q| \geq \varepsilon_1 R/\sqrt{\log R}\} \leq \varepsilon_2 R^2$ implies*

$$\lambda_1\Big(\Big(-\Delta + c_0 \sum_{q \in \mathbb{Z}^2 \cap \Lambda_R} 1_{B(q+\xi_q, r_0)}\Big)_R^N\Big) \geq c/R^2. \tag{35}$$

To prove this, we replace $R^{2/d}$ by $R/\sqrt{\log R}$ in the proof of Proposition 12 and we further need to extend Lemma 1 to the 2-dimensional case. By a simple modification of the proof of Lemma 1, we have the following, which is enough for our purpose:

Lemma 4. *If $d = 2$, then we have $\inf\{\lambda_1((-\Delta + c_0 1_{B(b,r_0)})_R^N) : b \in \Lambda_R\} \geq c/(R^2 \log R)$.*

4 Critical case

In this section we discuss the case of $\alpha = d + 2$. By modifying our proof of Proposition 5, we can prove the following:

Proposition 14. *If $\alpha = d + 2$, then we have*

$$\varliminf_{t \uparrow \infty} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)/(d+2+\theta)}} \geq -K_0(h, C_0), \tag{36}$$

where

$$\begin{aligned} &K_0(h, C_0) \\ &= \inf\Big\{h\|\nabla\psi\|_2^2 + \int_{\mathbb{R}^d} dq \inf_{y \notin \mathrm{supp}(\psi) - q} \Big(\int_{\mathbb{R}^d} \frac{dx C_0 \psi(x)^2}{|x - q - y|^{d+2}} + |y|^\theta\Big) : \psi \in W_2^1(\mathbb{R}^d), \|\psi\|_2 = 1\Big\} \end{aligned} \tag{37}$$

and $W_2^1(\mathbb{R}^d) = \{\psi \in L^2(\mathbb{R}^d) : \nabla\psi \in L^2(\mathbb{R}^d)\}$.

Proof. In (15), we replace ψ_R by an arbitrary function $\varphi \in H_0^1(\Lambda_R)$ with $\|\varphi\|_2 = 1$, where $H_0^1(\Lambda_R)$ is the completion of $C_0^\infty(\Lambda_R)$ in $W_2^1(\mathbb{R}^d)$. Then (17) is modified as

$$\begin{aligned} &\int_{|q| \leq t^\beta} \widetilde{N}_2(t, q) dq \\ &\geq \int_{|q| \leq t^\beta} dq \log \int_{y \in [\mathrm{supp}(\varphi) : R_1 + \sqrt{d}/2]^c - q} \frac{dy}{Z(d, \theta)} \exp\Big(-\int \frac{dx \varphi(x)^2 t C_0 (1 + \varepsilon_1)}{\inf\{|x - q - z - y|^{d+2} : z \in \Lambda_1\}} - |y|^\theta\Big), \end{aligned}$$

where $[A:r]=\{x\in\mathbb{R}^d : d(x,A)<r\}$ for any $A\subset\mathbb{R}^d$ and $r>0$. We take η as $1/(d+2+\theta)$. Then, by changing the variables, this equals

$$t^{d\eta}\int_{|q|\le t^{\beta-\eta}} dq \log \int_{y\in[\mathrm{supp}(\varphi_\eta):(R_1+\sqrt{d}/2)/t^\eta]^c-q} \frac{dy t^{d\eta}}{Z(d,\theta)}\exp(-t^{\theta\eta}\widetilde{N}_3(y,q;\varphi_\eta)),$$

where

$$\widetilde{N}_3(y,q;\varphi_\eta)=\int\frac{dx\varphi_\eta(x)^2C_0(1+\varepsilon_1)}{\inf\{|x-q-z-y|^{d+2}: z\in\Lambda_{t-\eta}\}}+|y|^\theta$$

and $\varphi_\eta(x)=t^{d\eta/2}\varphi(t^\eta x)$. We take R as the integer part of $\mathcal{R}t^\eta$ for a positive number $\mathcal{R}$, and take φ so that $\varphi_\eta=\psi$ is a t-independent element of $H_0^1(\Lambda_{\mathcal{R}})$. Since $t\|\nabla\varphi\|_2^2=t^{(d+\theta)\eta}\|\nabla\psi\|_2^2$ is not negligible, (21) is modified as

$$\varliminf_{t\uparrow\infty}\frac{\log\widetilde{N}(t)}{t^{(d+\theta)\eta}}\ge -h\|\nabla\psi\|_2^2-\varlimsup_{t\uparrow\infty}\int_{|q|\le t^{\beta-\eta}}dq\widetilde{N}_4(q,t),$$

where

$$\widetilde{N}_4(q,t)=\inf\Big\{\sup_{y\in B(y_0,t^{-\gamma})}\widetilde{N}_3(y,q;\psi): y_0\in\Big[\mathrm{supp}(\psi):\frac{R_1+\sqrt{d}/2}{t^\eta}+\frac{1}{t^\gamma}\Big]^c-q\Big\}.$$

Since

$$\varlimsup_{t\uparrow\infty}\widetilde{N}_4(q,t)\le\inf_{y\in(\mathrm{supp}(\psi))^c-q}\Big(\int\frac{dx\psi(x)^2C_0(1+\varepsilon_1)}{|x-q-y|^{d+2}}+|y|^\theta\Big),$$

we obtain

$$\varliminf_{t\uparrow\infty}\frac{\log\widetilde{N}(t)}{t^{(d+\theta)\eta}}\ge -h\|\nabla\psi\|_2^2-\int_{\mathbb{R}^d}dy\inf_{y\in(\mathrm{supp}(\psi))^c-q}\Big(\int\frac{dx\psi(x)^2C_0(1+\varepsilon_1)}{|x-q-y|^{d+2}}+|y|^\theta\Big)$$

by the Lebesgue convergence theorem. By taking the supremum with respect to ε_1, ψ and $\mathcal{R}$, we obtain the result. $\square$

If we apply Donsker and Varadhan's large deviation theory without caring the topological problems, then the formal upper estimate

$$\varlimsup_{t\uparrow\infty}\frac{\log\widetilde{N}(t)}{t^{(d+\theta)/(d+2+\theta)}}\le -K(h,C_0) \tag{38}$$

is expected, where $K(h,C_0)$ is the quantity obtained by removing the restriction $y\notin\mathrm{supp}(\psi)-q$ in the definition (37) of $K_0(h,C_0)$. For the corresponding Poisson case, this is rigorously established in Ôkura [15]. In that case, the space $\mathbb{R}^d$ can be replaced by a d-dimensional torus and the Feynman-Kac functional becomes a lower semicontinuous functional, so that Donsker and Varadhan's theory applies. However, both verification of the replacement of the space and the continuity of the functional seem to be difficult in our case.

From the conjecture (38), we expect that the quantum effect appears in the leading term. By Proposition 8 in Section 3, we can justify this if $d\ge 3$ and h is large:

Proposition 15. *If $d\ge 3$ and $\alpha=d+2$, then we have*

$$\varlimsup_{h\to\infty}\varlimsup_{\lambda\to 0}\lambda^{(d+\theta)/2}\log N(\lambda)=-\infty. \tag{39}$$

In the one-dimensional case we can show the same statement with a more explicit bound

$$\varlimsup_{\lambda\to 0} \lambda^{(1+\theta)/2}\log N(\lambda) \leq -\frac{\pi^{1+\theta}h^{(1+\theta)/2}}{(1+\theta)2^\theta}$$

by Theorem 2, since the leading order does not vary for $\alpha \geq 3$. In the two-dimensional case we have no such result.

5 Proof of Theorem 3

5.1 Upper estimate

Let $\widetilde{N}^-(t)$ be the Laplace-Stieltjes transform of the integrated density of states $N^-(\lambda)$:

$$\widetilde{N}^-(t) = \int_{-\infty}^{\infty} e^{-t\lambda} dN^-(\lambda).$$

To prove the upper estimate, we have only to show the following:

Proposition 16. *Under the condition that* $u \geq 0, \sup u = u(0) < \infty$ *and* $\sup |x|^\alpha u(x) < \infty$ *for some* $\alpha > d$*, we have*

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}^-(t)}{t^{1+d/\theta}} \leq u(0)^{1+d/\theta}\int_{|q|\leq 1} dq(1-|q|^\theta). \tag{40}$$

Proof. We use the bound

$$\widetilde{N}^-(t) \leq \widetilde{N}_1^-(t)(4\pi th)^{-d/2}$$

as in (13), where

$$\widetilde{N}_1^-(t) = \int_{\Lambda_1} dx \mathbb{E}_\theta\Bigg[\exp\Big(t\sum_{q\in\mathbb{Z}^d} u(\,x-q-\xi_q)\Big)\Bigg].$$

Here we have used the path integral expression of $\widetilde{N}^-(t)$ in Theorem VI.1.1 of [3]. The assumption required in that theorem will be checked in Lemma 11 in Section 9. By replacing the summation by the integration, we have

$$\log \widetilde{N}_1^-(t) \leq \int_{\mathbb{R}^d} dq \log \widetilde{N}_2^-(t,q),$$

where

$$\widetilde{N}_2^-(t,q) = \mathbb{E}_\theta\Big[\exp\Big(t\sup_{x\in\Lambda_2} u(x-q-\xi_0)\Big)\Big].$$

Now we fix an arbitrary small number $\varepsilon > 0$ and let $C = \sup|x|^\alpha u(x)$. When $|q| > (1+\varepsilon)(u(0)t)^{1/\theta}$, we estimate as

$$\widetilde{N}_2^-(t,q) \leq \exp(t\sup\{u(x-y) : x\in\Lambda_2, |y|\geq \delta|q|\}) + \exp(tu(0))\mathbb{P}_\theta(|\xi_0| \geq (1-\delta)|q|), \tag{41}$$

where $\delta > 0$ is taken to satisfy $(1-\delta)^{\theta+2}(1+\varepsilon)^\theta = 1$. For the first term in the right hand side, we use an obvious bound

$$\sup\{u(x-y) : x\in\Lambda_2, |y|\geq\delta|q|\} \leq C(\delta|q|-\sqrt{d})^{-\alpha}.$$

For the second term, it is easy to see

$$\mathbb{P}_\theta(|\xi_q| \geq (1-\delta)|q|) \leq M(\delta,\theta)\exp(-(1-\delta)^{\theta+1}|q|^\theta)$$

for some large $M(\delta,\theta) > 0$. Moreover, we have

$$\begin{aligned}(1-\delta)^{\theta+1}|q|^\theta &= (1-\delta)^{\theta+2}|q|^\theta + \delta(1-\delta)^{\theta+1}|q|^\theta \\ &\geq u(0)t + \delta(1-\delta)^{\theta+1}|q|^\theta\end{aligned}$$

thanks to $|q| > (1+\varepsilon)(u(0)t)^{1/\theta}$ and our choice δ. Combining above three estimates, we get

$$\widetilde{N}_2^-(t,q) \leq \exp(tC(\delta|q|-\sqrt{d})^{-\alpha})(1+M(\delta,\theta)\exp(-\delta(1-\delta)^{\theta+1}|q|^\theta)) \tag{42}$$

and thus

$$\log \widetilde{N}_2^-(t,q) \leq tC(\delta|q|-\sqrt{d})^{-\alpha} + M(\delta,\theta)\exp(-\delta(1-\delta)^{\theta+1}|q|^\theta), \tag{43}$$

using $\log(1+X) \leq X$. Since the integral of the right hand side over $\{|q| > (1+\varepsilon)(u(0)t)^{1/\theta}\}$ is easily seen to be $o(t^{1+d/\theta})$, we can neglect this region.

For q with $|q| \leq (1+\varepsilon)(u(0)t)^{1/\theta}$, we estimate as

$$\widetilde{N}_2^-(t,q) \leq \exp(t\sup\{u(x-y) : x \in \Lambda_2, |y| \geq L\}) + \exp(tu(0))\mathbb{P}_\theta(|q+\xi_0| \leq L), \tag{44}$$

where $L = 2\varepsilon(u(0)t)^{1/\theta}$. We use obvious bounds

$$\sup\{u(x-y) : x \in \Lambda_2, |y| \geq L\} \leq C(L-\sqrt{d})_+^{-\alpha}$$

for the first term and

$$\mathbb{P}_\theta(|q+\xi_0| \leq L) \leq \exp(-(|q|-L)_+^\theta)|B(0,L)|/Z(d,\theta)$$

for the second term. Note also that we have

$$tc(L-\sqrt{d})_+^{-\alpha} \leq tu(0) - (|q|-L)_+^\theta$$

for large t, from $|q| \leq (1+\varepsilon)(u(0)t)^{1/\theta}$ and our choice of L. Using these estimates, we obtain

$$\begin{aligned}&\int_{|q|\leq(1+\varepsilon)(u(0)t)^{1/\theta}} dq \log \widetilde{N}_2^-(t,q) \\ &\leq \int_{|q|\leq(1+\varepsilon)(u(0)t)^{1/\theta}} dq \Big\{ \log\Big(\frac{|B(0,L)|}{Z(d,\theta)} + 1\Big) + tu(0) - (|q|-L)_+^\theta \Big\}.\end{aligned}$$

By changing the variable and taking the limit, it follows

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{1+d/\theta}} \leq u(0)^{1+d/\theta} \int_{|q|\leq 1+\varepsilon} dq\{1-(|q|-2\varepsilon)_+^\theta\}.$$

This completes the proof of Proposition 16 since $\varepsilon > 0$ is arbitrary. □

5.2 Lower estimate

To prove the lower estimate, we have only to show the following:

Proposition 17. *If $u \geq 0, \sup u = u(0) < \infty$ and, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $u(x) \geq u(0) - \varepsilon$ for $|x| < R_\varepsilon$, then we have*

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}^-(t)}{t^{1+d/\theta}} \geq u(0)^{1+d/\theta} \int_{|q|\leq 1} dq(1-|q|^\theta). \tag{45}$$

Proof. We use the bound

$$\widetilde{N}^-(t) \geq \exp(-th\|\nabla\psi_\varepsilon\|_2)\widetilde{N}_1^-(t),$$

for any $\psi_\varepsilon \in C_0^\infty(\Lambda_\varepsilon)$ such that the L^2-norm of ψ_ε is 1, where

$$\widetilde{N}_1^-(t) = \mathbb{E}_\theta\left[\exp\left(t\sum_{q\in\mathbb{Z}^d}\inf_{x\in\Lambda_\varepsilon} u(\,x-q-\xi_q)\right)\right]. \tag{46}$$

This is proven by the same estimate used in (15). We take ψ_ε as a normalized ground state of the Dirichlet Laplacian on the cube Λ_ε. Since a sufficient condition for $\sup_{x\in\Lambda_\varepsilon}|x-q-\xi_q| \leq R_\varepsilon$ is $|q+\xi_q| \leq R_\varepsilon - \varepsilon\sqrt{d}/2$, we restrict as

$$\log \widetilde{N}_1^-(t) \geq \sum_{q\in\mathbb{Z}^d} \log \int_{|q+y|\leq R_\varepsilon-\varepsilon\sqrt{d}/2} \frac{dy}{Z(d,\theta)} \exp(t(u(0)-\varepsilon)-|y|^\theta).$$

Since a sufficient condition for $\inf\{u(0)-\varepsilon)-|y|^\theta \leq R_\varepsilon : |q+y| \leq R_\varepsilon - \varepsilon\sqrt{d}/2\} \geq 0$ is $|q| \leq \{t(u(0)-\varepsilon)\}^{1/\theta} - R_\varepsilon + \varepsilon\sqrt{d}/2$, we restrict as

$$\begin{aligned}
\log \widetilde{N}_1^-(t) &\geq \int_{|q|\leq h(t)} \{c'\log(|B(0,R_\varepsilon-\varepsilon\sqrt{d}/2)|/Z(d,\theta)) + t(u(0)-\varepsilon) - (|q|+R_\varepsilon-c))^\theta\} \\
&= h(t)^d \int_{|q|\leq 1} \{c'\log(|B(0,R_\varepsilon-\varepsilon\sqrt{d}/2)|/Z(d,\theta)) + t(u(0)-\varepsilon) - (h(t)|q|+R_\varepsilon+c))^\theta\}
\end{aligned}$$

for large t and small ε, where $h(t) = \{t(u(0)-\varepsilon)\}^{1/\theta} - R_\varepsilon - c$ and c and c' are positive constants. Then we obtain

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}^-(t)}{t^{1+d/\theta}} \geq (u(0)-\varepsilon)^{1+d/\theta} \int_{|q|\leq 1} dq(1-|q|^\theta).$$

Since ε is arbitrary, we can complete the proof of Proposition 17. □

6 Asymptotics for associated Wiener integrals

In the previous work [5], the asymptotic behaviors of the integrated density of states were derived from those of certain Wiener integrals. In this section, we recall the connection and derive estimates of the asymptotic behaviors of the associated Wiener integrals in our settings. Let $h = 1/2$ for simplicity and E_x denote the expectation with respect to the standard Brownian motion $(B_s)_{0\leq s\leq\infty}$ starting at x. Then the Laplace-Stieltjes transform of the integrated density of states can be expressed as follows:

$$\begin{aligned}
\widetilde{N}(t) = (2\pi t)^{-d/2} \int_{\Lambda_1} dx \mathbb{E}_\theta \otimes E_x &\left[\exp\left\{-\int_0^t \sum_{q\in\mathbb{Z}^d} u(B_s-q-\xi_q)ds\right\}\right. \\
&\left. : B_s \notin \bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K) \text{ for } 0\leq s\leq t \,\middle|\, B_t = x\right].
\end{aligned} \tag{47}$$

This expression is also valid for $\widetilde{N}^-(t)$ by changing the sign of u and setting $K=\emptyset$ in the right hand side. In view of (47), $\widetilde{N}(t)$ is essentially same with the Wiener integral

$$\begin{aligned}S_{t,\,x}=\mathbb{E}_\theta\otimes E_x\bigg[&\exp\bigg\{-\int_0^t\sum_{q\in\mathbb{Z}^d}u(B_s-q-\xi_q)ds\bigg\}\\&:B_s\notin\bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K)\text{ for }0\le s\le t\bigg],\end{aligned}\tag{48}$$

which was the main object in [5]. This quantity is of interest itself since not only it gives the average of the solution of a heat equation with random sinks but also can be interpreted as the annealed survival probability of the Brownian motion among killing potentials. Similarly, $\widetilde{N}^-(t)$ is essentially the same with the average of the solution of a heat equation with random sources

$$S^-_{t,\,x}=\mathbb{E}_\theta\otimes E_x\bigg[\exp\bigg\{\int_0^t\sum_{q\in\mathbb{Z}^d}u(B_s-q-\xi_q)ds\bigg\}\bigg],\tag{49}$$

which can also be interpreted as the average number of the branching Brownian motions in random media. We refer the readers to [7, 6, 20] about the interpretations of $S_{t,\,x}$ and $S^-_{t,\,x}$. The connection between the asymptotics of $\widetilde{N}(t)$ and $S_{t,\,x}$ are discussed in many reference for the case that $\{q+\xi_q\}_q$ is replaced by an $\mathbb{R}^d$-stationary random field (see e.g. [14], [19]). However our case is only $\mathbb{Z}^d$-stationary.

We first prepare a lemma which gives upper bounds on $\log S_{t,\,x}$ and $\log S^-_{t,\,x}$ in terms of $\log\widetilde{N}(t)$ and $\log\widetilde{N}^-(t)$, respectively. We shall state the results only for $x\in\Lambda_1$ since they automatically extend to the whole space by the $\mathbb{Z}^d$-stationarity.

Lemma 5. *For any $x\in\Lambda_1$ and $\varepsilon>0$, we have*

$$\log S_{t,\,x}\le\log\widetilde{N}(t-\varepsilon)(1+o(1))\tag{50}$$

and

$$\log S^-_{t,\,x}\le\log\widetilde{N}^-(t-t^{-2d/\theta})(1+o(1))\tag{51}$$

as $t\to\infty$.

Proof. We give the proof of (51) first. Let $V_\xi(x)$ denotes the potential $\sum_{q\in\mathbb{Z}^d}u(x-q-\xi_q)$ for simplicity. We divide the expectation as

$$\begin{aligned}S^-_{t,\,x}=&\mathbb{E}_\theta\otimes E_x\bigg[\exp\bigg\{\int_0^tV_\xi(B_s)ds\bigg\}:\sup_{0\le s\le t}|B_s|_\infty<[t^{1+d/\theta}]\bigg]\\&+\sum_{n>[t^{1+d/\theta}]}\mathbb{E}_\theta\otimes E_x\bigg[\exp\bigg\{\int_0^tV_\xi(B_s)ds\bigg\}:n-1\le\sup_{0\le s\le t}|B_s|_\infty<n\bigg].\end{aligned}\tag{52}$$

The summands in the second term can be bounded above by

$$\begin{aligned}&\mathbb{E}_\theta\bigg[\exp\bigg\{t\sup_{y\in\Lambda_{2n}}V_\xi(y)\bigg\}\bigg]P_x\left(n-1\le\sup_{0\le s\le t}|B_s|_\infty\right)\\\le&c_1n^d\mathbb{E}_\theta\bigg[\exp\bigg\{t\sup_{y\in\Lambda_1}V_\xi(y)\bigg\}\bigg]\exp\{-c_2n^2/t\}\\\le&c_1n^d\exp\{c_3t^{1+d/\theta}-c_2n^2/t\},\end{aligned}\tag{53}$$

where we have used a standard Brownian estimate (cf. [8] Section 1.7) and the $\mathbb{Z}^d$-stationarity in the second line, and Lemma 11 below in the third line. Then, it is easy to see that the second term in (52) is bounded above by a constant and hence it is negligible compared with $\widetilde{N}^-(t)$.

Now let us turn to the estimate on the first term in (52). Note first that we can derive an upper large deviation bound

$$\mathbb{P}_\theta\Big(\sup_{y\in\Lambda_{[t^{1+d/\theta}]}} V_\xi(y)\geq v\Big)\leq [t^{1+d/\theta}]^d\mathbb{P}_\theta\Big(\sup_{y\in\Lambda_1} V_\xi(y)\geq v\Big)\leq \exp(-c_4v^{1+\theta/d}) \tag{54}$$

which is valid for all sufficiently large t and $v\geq t$, from the exponential moment estimate in Lemma 11 below. Using this estimate, we get

$$\begin{aligned}
&\mathbb{E}_\theta\otimes E_x\Big[\exp\Big\{\int_0^t V_\xi(B_s)ds\Big\}:\sup_{0\leq s\leq t}|B_s|_\infty<[t^{1+d/\theta}],\\
&\qquad\qquad\qquad\qquad\sup_{y\in\Lambda_{2[t^{1+d/\theta}]}} V_\xi(y)\geq t^{2d/\theta}\Big]\\
\leq&\mathbb{E}_\theta\Big[\exp\Big\{t\sup_{y\in\Lambda_{2[t^{1+d/\theta}]}} V_\xi(y)\Big\}:\sup_{y\in\Lambda_{2[t^{1+d/\theta}]}} V_\xi(y)\geq t^{2d/\theta}\Big]\\
\leq&\sum_{n\geq t^{2d/\theta}}\exp\{tn\}\mathbb{P}_\theta\Big(n-1\leq\sup_{y\in\Lambda_{2[t^{1+d/\theta}]}} V_\xi(y)<n\Big)\\
\leq&\sum_{n\geq t^{2d/\theta}}\exp\Big\{tn-c_4(n-1)^{1+\theta/d}\Big\}.
\end{aligned}\tag{55}$$

Since the last expression converges to 0 as $t\to\infty$, we can restrict ourselves on the event $\{\sup V_\xi(x)\leq t^{2d/\theta}\}$. Hereafter, we let $T=[t^{1+d/\theta}]$ since its exact form will be irrelevant in the sequel. Then, the Markov property at time $\varepsilon=t^{-2d/\theta}$ yields

$$\begin{aligned}
&\mathbb{E}_\theta\otimes E_x\Big[\exp\Big\{\int_0^t V_\xi(B_s)ds\Big\}:\sup_{0\leq s\leq t}|B_s|_\infty<T,\sup_{y\in\Lambda_{2T}} V_\xi(y)<t^{2d/\theta}\Big]\\
\leq&e\int_{\Lambda_{2T}}\frac{dy}{(2\pi\varepsilon)^{d/2}}\exp\Big(-\frac{|x-y|^2}{2\varepsilon}\Big)\mathbb{E}_\theta\otimes E_y\Big[\exp\Big\{\int_0^{t-\varepsilon} V_\xi(B_s)ds\Big\}:\sup_{0\leq s\leq t-\varepsilon}|B_s|_\infty<T\Big]\\
\leq&\frac{e}{(2\pi\varepsilon)^{d/2}}\int_{\Lambda_{2T}}dy\int_{\Lambda_{2T}}dz\mathbb{E}_\theta[\exp(-(t-\varepsilon)H^{-,D}_{\xi,2T})(y,z)],
\end{aligned}\tag{56}$$

where $\exp(-tH^{-,D}_{\xi,2T})(x,y)$, $t>0$, $x,y\in\Lambda_{2T}$, is the integral kernels of the heat semigroup generated by the self-adjoint operator H^-_ξ on the L^2-space on the cube Λ_{2T} with the Dirichlet boundary condition.

Finally, we use the estimate

$$\exp(-tH^D_{\xi,2T})(y,z)\leq\big\{\exp(-tH^D_{\xi,2T})(y,y)\exp(-tH^D_{\xi,2T})(z,z)\big\}^{1/2}$$

for the kernel of self-adjoint semigroup and the Schwarz inequality to dominate the right hand side in (56) by $T^{2d}\widetilde{N}^-(t-\varepsilon)$ multiplied by some constant.

Combining all the estimates above, we finish the proof of (51). We can also prove (50) in the same way as (56). However it is much simpler since we do not have to care about $\sup V_\xi(\,\cdot\,)$ and thus we omit the details. $\square$

The next lemma gives the converse relation between $\log S_{t,x}$ and $\log \widetilde{N}(t)$, while the lower estimate on $\log S^{-}_{t,x}$ will be derived directly. (See the proof of Theorem 18.)

Lemma 6. *For any $x \in \Lambda_1$ and $\varepsilon > 0$, we have*

$$\log \widetilde{N}(t) \le \log S^{v,K'}_{t-\varepsilon,x}(1+o(1)) \tag{57}$$

as $t \to \infty$, where $S^{v,K'}_{t,x}$ is the expectation defined by replacing K and u by $K' = \{x \in K : d(x,K^c) \ge \sqrt{d}\}$ and $v(y) = \inf\{u(y-x+z) : z \in \Lambda_1\}$, respectively, in the definitions (48), respectively. If u is a function satisfying the conditions in Theorem 1 or 2, then v is also a function satisfying the same conditions.

Proof. Let $\varepsilon > 0$ be an arbitrarily small number. By the Chapman-Kolmogorov identity, we have

$$\begin{aligned}\widetilde{N}(t) \le (2\pi\varepsilon)^{-d/2} \int_{\Lambda_1} dz \mathbb{E}_\theta \otimes E_z \Bigg[\exp\Bigg\{ -\int_0^{t-\varepsilon} \sum_{q\in\mathbb{Z}^d} u(B_s - q - \xi_q) ds \Bigg\} \\ : B_s \notin \bigcup_{q\in\mathbb{Z}^d} (q + \xi_q + K) \text{ for } 0 \le s \le t-\varepsilon \Bigg].\end{aligned}$$

The right hand side is dominated by $(2\pi\varepsilon)^{-d/2} S^{v,K'}_{t-\varepsilon,x}$ and the proof of (57) is completed. □

We now state our results on the asymptotics of $S_{t,x}$ and $S^{-}_{t,x}$:

Theorem 18. (i) *We assume $d = 1$ and $\operatorname{ess\,inf}_{B(R)} u > 0$ for any $R \ge 1$ if $\alpha \le 3$. Then we have*

$$\log S_{t,x} \begin{cases} \sim -t^{(1+\theta)/(\alpha+\theta)} \displaystyle\int_{\mathbb{R}} dq \inf_{y\in\mathbb{R}} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) & (1 < \alpha < 3), \\ \asymp -t^{(1+\theta)/(3+\theta)} & (\alpha = 3), \\ \sim -t^{(1+\theta)/(3+\theta)} \dfrac{3+\theta}{1+\theta} \Big(\dfrac{\pi^2}{8}\Big)^{(1+\theta)/(3+\theta)} & (\alpha > 3) \end{cases} \tag{58}$$

as $t \to \infty$, where $f(t) \sim g(t)$ means $\lim_{t\to\infty} f(t)/g(t) = 1$ and $f(t) \asymp g(t)$ means $0 < \underline{\lim}_{t\to\infty} f(t)/g(t) \le \overline{\lim}_{t\to\infty} f(t)/g(t) < \infty$.

(ii) *We assume $d = 2$ and $\operatorname{ess\,inf}_{B(R)} u > 0$ for any $R \ge 1$ if $\alpha \le 4$. Then we have*

$$\log S_{t,x} \begin{cases} \sim -t^{(2+\theta)/(\alpha+\theta)} \displaystyle\int_{\mathbb{R}^2} dq \inf_{y\in\mathbb{R}^2} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) & (2 < \alpha < 4), \\ \asymp -t^{(2+\theta)/(4+\theta)} & (\alpha = 4), \\ \asymp -t^{(2+\theta)/(4+\theta)} (\log t)^{-\theta/(4+\theta)} & (\alpha > 4) \end{cases} \tag{59}$$

as $t \to \infty$.

(iii) *We assume $d \ge 3$ and $\operatorname{ess\,inf}_{B(R)} u > 0$ for any $R \ge 1$ if $\alpha \le d+2$. Then we have*

$$\log S_{t,x} \begin{cases} \sim -t^{(d+\theta)/(\alpha+\theta)} \displaystyle\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) & (d < \alpha < d+2), \\ \asymp -t^{(d+\theta\mu)/(d+2+\theta\mu)} & (\alpha \ge d+2) \end{cases} \tag{60}$$

as $t \to \infty$, where $\mu = 2(\alpha-2)/(d(\alpha-d))$ as in Theorem 2.

(iv) *We assume* $\sup u = u(0) < \infty$ *and the existence of* $R_\varepsilon > 0$ *for any* $\varepsilon > 0$ *such that* $\operatorname{ess\,inf}_{B(R_\varepsilon)} u \geq u(0) - \varepsilon$*. Then we have*

$$\log S_{t,x}^- \sim t^{1+d/\theta} u(0)^{1+d/\theta} \int_{|q| \leq 1} dq (1 - |q|^\theta) \tag{61}$$

as $t \to \infty$*.*

Proof. We first consider the corresponding results for $\widetilde{N}(t)$ and $\widetilde{N}^-(t)$: the estimates (58)–(61) with $S_{t,x}$ and $S_{t,x}^-$ replaced by $\widetilde{N}(t)$ and $\widetilde{N}^-(t)$, respectively. Those are already proven in earlier sections except for the case of $\alpha > d+2$ and $d \geq 2$. The results for the remaining case follow from Propositions 8 and 10 using the exponential Abelian theorem due to Kasahara [9]. Then by Lemma 5, we obtain the upper estimates for $S_{t,x}$ and $S_{t,x}^-$. For the lower estimate of $S_{t,x}$, we set $u^\#(y) = \sup\{u(y+x+z) : z \in \Lambda_1\} 1_{B(R_1)^c}(y) + 1_{B(R_1)}(y)$ with $R_1 \geq 0$. If u satisfies the conditions in Theorems 1 and 2, and R_1 is sufficiently large, then $u^\#$ satisfies also the same conditions. Therefore we obtain the corresponding lower estimate of $\widetilde{N}(t)$ where K is replaced by $B(R_2)$ with any $R_2 \geq R_1$ and u is replaced by $u^\#$. Then by Lemma 6, we obtain the corresponding lower estimates of $S_{t,x}^{v^\#, B(R_2+\sqrt{d})}$, where $v^\#(y) = \inf\{u^\#(y - x + z) : z \in \Lambda_1\}$. Since $K \subset B(R_2 + \sqrt{d})$ and $v^\# \geq u$ on $B(R_2)^c$ for some $R_2 \geq R_1$, we obtain the corresponding lower estimates of $S_{t,x}$. For the lower estimate of $S_{t,x}^-$, we restrict the expectation to the event $B_s \in \Lambda_\varepsilon$ for any $s \in [1, t]$ to obtain

$$S_{t,x}^- \geq \int_{\Lambda_\varepsilon} dy e^{\Delta/2}(x,y) \int_{\Lambda_\varepsilon} dz e^{(t-1)\Delta_\varepsilon^D/2}(y,z) \widetilde{N}_1^-(t-1) \geq c_1 e^{-c_2 t} \widetilde{N}_1^-(t-1),$$

where $\widetilde{N}_1^-(t)$ is the function defined in (46), and $\exp(t\Delta/2)(x,y)$, $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and $\exp(t\Delta_\varepsilon^D/2)(x,y)$, $(t,x,y) \in (0,\infty) \times \Lambda_\varepsilon \times \Lambda_\varepsilon$ are the integral kernels of the heat semigroups generated by the Laplacian and the Dirichlet Laplacian on Λ_ε, respectively, multiplied by $-1/2$. Therefore the lower estimate of $S_{t,x}^-$ is given by our proof of Proposition 17. □

7 Leading term for the light tail cases

In Theorem 2.9 of [5], the leading term for $\log S_{t,x}$ was investigated in the case of discrete trap configuration using Sznitman's "method of enlargement of obstacles". We shall apply the same method to improve our previous estimate for $\alpha \geq d+2$ when the distribution of ξ_q is discretized as

$$\mathbb{P}_\theta(\xi_q \in dx) = \frac{1}{Z_{\mathrm{disc}}(d,\theta)} \sum_{p \in \mathbb{Z}^d} \exp(-|p|^\theta) \delta_p(dx). \tag{62}$$

7.1 Multidimensional cases

In this subsection, we consider the cases $d \geq 2$ and $\alpha \geq d+2$. We shall set $h = 1/2$ and the hard trap $K = \emptyset$ for simplicity and take $x \in \Lambda_1$ as in the previous section. We call a domain R a lattice animal if it is represented as

$$R = \overline{\bigcup_{q \in S(R)} \Lambda_1(q)}^{\circ},$$

where $S(R) \subset \mathbb{Z}^d$ consists of adjacent sites. This means that R is a combination of unit cubes connected via faces. We introduce a scaling with the factor $r = t^{1/(d+2+\mu\theta)}$ for the cases $d \geq 3$ and $(d, \alpha) = (2, 4)$. For the case $d = 2$ with $\alpha > 4$, we take $r = t^{1/(4+\theta)}(\log t)^{\theta/(8+2\theta)}$. We set

$$\begin{aligned}\mathcal{S}_r = \big\{ &(R_r, \zeta = (\zeta_q)_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d}) : R_r \text{ is a lattice animal included in } \Lambda_{t/r}, \\ &|R_r| < r^{\chi}, q + \zeta_q \in [\Lambda_t : t^{1/(\mu\theta)}] \cap \mathbb{Z}^d \text{ for all } q \in (r[R_r \,:\, l]) \cap \mathbb{Z}^d \big\},\end{aligned} \tag{63}$$

where χ is an arbitrarily fixed number in $((\mu - 2/d)\theta, \mu\theta)$, l is a positive number specified later, and $[A : l] = \{x \in \mathbb{R}^d : d(x, A) < l\}$ for any $A \subset \mathbb{R}^d$. For any union U of lattice animals and $\xi = (\xi_q)_{q\in\mathbb{Z}^d} \in (\mathbb{R}^d)^{\mathbb{Z}^d}$, we denote by $\lambda^r_\xi(U)$ the bottom of the spectrum of

$$-\frac{1}{2}\Delta + V^r_\xi = -\frac{1}{2}\Delta + \sum_{q\in\mathbb{Z}^d} r^2 u(rx - q - \xi_q)$$

in U with the Dirichlet boundary condition. Similarly, for any $(R_r, \zeta) \in \mathcal{S}_r$, we write

$$V^r_\zeta(x) = \sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^2 u(rx - q - \zeta_q)$$

with a slight abuse of the notation and define $\lambda^r_\zeta(R_r)$ accordingly. In this section we show the following:

Theorem 19. *Under the above setting, we have the following:*
(i) For any $\varepsilon > 0$ and $l > 0$, there exists $t_{\varepsilon,l} > 0$ such that

$$t^{-1}r^2 \log S_{t,x} \leq -(1-\varepsilon) \inf_{(R_r,\zeta)\in\mathcal{S}_r} \left\{ \lambda^r_\zeta(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta \right\} \tag{64}$$

for any $t \geq t_{\varepsilon,l}$, where

$$\gamma(r) = \begin{cases} \sqrt{(4+\theta)\log r} & (d = 2 \text{ and } \alpha > 4), \\ r^{1-\mu} & (d \geq 3 \text{ or } (d,\alpha) = (2,4)). \end{cases} \tag{65}$$

(ii) If $\alpha > d + 2$, then for any $\varepsilon > 0$ and $l > 0$, there exists $t_{\varepsilon,l} > 0$ such that

$$t^{-1}r^2 \log S_{t,x} \geq -(1+\varepsilon) \inf_{(R_r,\zeta)\in\mathcal{S}_r} \left\{ \lambda^r_\zeta(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta \right\} \tag{66}$$

for any $t \geq t_{\varepsilon,l}$.
(iii) If $\alpha = d + 2$, then for any $\varepsilon > 0$, there exist $t_\varepsilon > 0$ and $l_\varepsilon > 0$ such that

$$t^{-1}r^2 \log S_{t,x} \geq -(1+\varepsilon) \inf_{(R_r,\zeta)\in\mathcal{S}_r} \left\{ \lambda^r_\zeta(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta \right\} \tag{67}$$

for any $t \geq t_\varepsilon$ and $l \geq l_\varepsilon$.

Remark 4. The above proposition shows

$$-t^{-1}r^2 \log S_{t,x} \sim \inf_{(R_r,\zeta)\in\mathcal{S}_r} \left\{ \lambda^r_\zeta(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta \right\} \tag{68}$$

for the case of $\alpha > d+2$. For the case of $\alpha = d+2$, we have to take l large to make upper and lower bounds close. By the variational formula for the bottom of the spectrum, we can rewrite the infimum in the right hand side as

$$\begin{aligned}&\inf_{(R_r,\zeta)\in\mathcal{S}_r}\inf_{\phi\in C_0^\infty(R_r),\,\|\phi\|_2=1}\Bigg\{\frac{1}{2}\int_{R_r}|\nabla\phi|^2(x)\,dx\\&+r^2\int_{R_r}\sum_{q\in(r[R_r:l])\cap\mathbb{Z}^d}u(rx-q-\zeta_q)\phi^2(x)\,dx+\gamma(r)^\theta\sum_{q\in(r[R_r:l])\cap\mathbb{Z}^d}r^{-d}\Big|\frac{\zeta_q}{r}\Big|^\theta\Bigg\}.\end{aligned} \tag{69}$$

If we formally replace $u(x)$ by $C_0|x|^{-\alpha}$ and scaled sums by integrals and also interchange the infimum over ζ and that over ϕ, we obtain the expression

$$\begin{aligned}&\inf_{R_r}\inf_{\phi\in C_0^\infty(R_r),\,\|\phi\|_2=1}\Bigg\{\frac{1}{2}\int_{R_r}|\nabla\phi|^2(x)\,dx\\&+\int_{[R_r:l]}dq\inf_{y\in[\Lambda_{t/r}:t^{1/(\mu\theta)}/r]}\left(r^{d+2-\alpha}\int_{R_r}\frac{C_0\phi^2(x)}{|x-q-y|^\alpha}\,dx+\gamma(r)^\theta|y|^\theta\right)\Bigg\},\end{aligned} \tag{70}$$

which is quite similar to K_0 appearing in Proposition 14. However there still exists difference between them and also we have no idea about how to justify the above formal argument.

The rest of this section is devoted to the proof of Theorem 19. To prove the upper bound (i), we recall the elements of the methods developed in [5]. We take $\eta \in (0,1)$ so small that

$$\chi > (\mu - \frac{2}{d})\theta + 2\eta^2 + \left(d - 2 + \frac{2\theta}{d}\right)\eta$$

and

$$\gamma := \frac{d-2}{d} + \frac{2\eta}{d} < 1.$$

We further introduce a notation concerning a diadic decomposition of $\mathbb{R}^d$. For each $k \in \mathbb{Z}_+$, let $\mathcal{I}_k$ be the collection of indices $\dot{\imath} = (i_0, i_1, \ldots, i_k)$ with $i_0 \in \mathbb{Z}^d$ and $i_1, \ldots, i_k \in \{0,1\}^d$. For each $\dot{\imath} \in \mathcal{I}_k$, we associate the box

$$C_{\dot{\imath}} = q_{\dot{\imath}} + 2^{-k}[0,1]^d,$$

where

$$q_{\dot{\imath}} = i_0 + 2^{-1}i_1 + \cdots + 2^{-k}i_k.$$

For $\dot{\imath} \in \mathcal{I}_k$ and $\dot{\imath}' \in \mathcal{I}_{k'}$ $(k' \le k)$, $\dot{\imath} \preceq \dot{\imath}'$ means that the first k' coordinates coincide. Finally, we introduce

$$n_\beta = \left[\beta\frac{\log r}{\log 2}\right]$$

for $\beta > 0$ so that $2^{-n_\beta - 1} < r^{-\beta} \le 2^{-n_\beta}$.

We can now define the density set, which we can discard from the consideration.

Definition 1. *We call a unit cube C_q with $q \in \mathbb{Z}^d$ a density box if all $q \preceq \dot{\imath} \in \mathcal{I}_{n_{\eta\gamma}}$ satisfy the following: for at least half of $\dot{\imath} \preceq \dot{\imath}' \in \mathcal{I}_{n_\gamma}$,*

$$(q_{\dot{\imath}'} + 2^{-n_\gamma - 1}[0,1]^d) \cap \{(q+\xi_q)/r : q \in \mathbb{Z}^d\} \neq \emptyset. \tag{71}$$

The union of all density boxes is denoted by $\mathcal{D}_r(\xi)$.

We can replace $\underline{\mathcal{D}}_r(\xi)$ by a hard trap by the following theorem.

Spectral control. *There exists $\rho > 0$ such that for all $M > 0$ and sufficiently large r,*

$$\sup_{\xi \in (\mathbb{R}^d)^{\mathbb{Z}^d}} \left(\lambda_\xi^r\left(\underline{\mathcal{R}}_r(\xi)\right) \wedge M - \lambda_\xi^r\left(\Lambda_{t/r}\right) \wedge M\right) \le r^{-\rho}, \tag{72}$$

where $\underline{\mathcal{R}}_r(\xi) = \Lambda_{t/r} \setminus \underline{\mathcal{D}}_r(\xi)$.

By Proposition 2.7 in [5], the proof of this theorem is reduced to the extension of Theorem 4.2.3 in [20] from the compactly supported single site potentials to the Kato class single site potentials, which is straightforward.

For $\underline{\mathcal{R}}_r(\xi)$, we can gives the following quantitative estimate on its volume:

Lemma 7. *There exists a positive constant c independent of r such that*

$$\mathbb{P}_\theta(|\underline{\mathcal{R}}_r(\xi)| \ge r^\chi) \le \exp\left\{-cr^{d(1-\eta\gamma)+(1-\gamma)\theta+\chi}\right\}. \tag{73}$$

In particular, $\mathbb{P}_\theta(|\underline{\mathcal{R}}_r(\xi)| \ge r^\chi) = o(S_{t,\,x})$.

Proof. The first inequality is Proposition 2.8 in [5]. The second claim follows from Theorem 18 and our choice of χ. $\square$

We now see that the *relevant* configurations of $(\underline{\mathcal{R}}_r(\xi), \xi)$ are only the pairs in $\mathcal{S}_r$. In fact removing the points $\{q + \xi_q : q \in \mathbb{Z}^d \setminus (r[R_r : l])\}$, which should be cared in proving the lower bound, is permitted as we will show in Lemma 9 below. We also have

$$\lambda_\xi^r\left(\underline{\mathcal{R}}_r(\xi)\right) = \lambda_\xi^r\left(R_r\right)$$

for some lattice animal R_r included in $\underline{\mathcal{R}}_r(\xi)$ and

$$\mathbb{P}_\theta(q + \xi_q \notin [\Lambda_t : t^{1/(\mu\theta)}] \text{ for some } q \in (r[R_r : l]) \cap \mathbb{Z}^d)$$

decays exponentially in t. The latter easily follows by observing that

$$d(r[R_r : l], [\Lambda_t : t^{1/(\mu\theta)}]^c) > t^{1/\theta},$$

which is due to $lr + t^{1/\theta} < t^{1/(\mu\theta)}$, for large t.

The key point in our coarse graining method is that the number of relevant configurations is estimated as

$$\#\mathcal{S}_r \le t^{dr^\chi}(t + 2t^{1/(\mu\theta)})^{dr^{d+\chi}c(1+l)} = o(S_{t,\,x}^{-1}) \tag{74}$$

by an elementary counting argument, where c is a finite constant depending only on d. The second relation comes from our choice of χ.

We now prove the upper bound in (i). By a standard Brownian estimate and scaling, we have

$$\begin{aligned} S_{t,\,x} &\le \mathbb{E}_\theta \otimes E_x\left[\exp\left\{-\int_0^t V_\xi(B_s)ds\right\} : \sup_{0\le s\le t} |B_s|_\infty < \frac{t}{2}\right] + e^{-ct} \\ &\le \mathbb{E}_\theta \otimes E_{x/r}\left[\exp\left\{-\int_0^{tr^{-2}} V_\xi^r(B_s)ds\right\} : \sup_{0\le s\le tr^{-2}} |B_s|_\infty < \frac{t}{2r}\right] + e^{-ct}. \end{aligned} \tag{75}$$

For any $\varepsilon \in (0,1)$, there exists a finite constant c_ε depending only on d and ε such that the first term of the right hand side is less than

$$c_\varepsilon \mathbb{E}_\theta \left[\exp\left\{-(1-\varepsilon)\lambda^r_\xi(\Lambda_{t/r})tr^{-2}\right\}\right]$$

by (3.1.9) of [20]. By the spectral control (72), Lemma 7, and (74), this quantity is less than

$$\begin{aligned}&o(S_{t,\,x}^{-1}) \sup_{(R_r,\,\zeta)\in\mathcal{S}_r} \mathbb{P}_\theta(\xi_q = \zeta_q \text{ for all } q \in (r[R_r:l]) \cap \mathbb{Z}^d)\\ &\times \exp\left\{-(1-\varepsilon)(\lambda^r_\zeta(R_r)\wedge M - r^{-\rho})tr^{-2}\right\} + o(S_{t,\,x}).\end{aligned}$$

Thus, we have

$$\begin{aligned}t^{-1}r^2 \log S_{t,\,x} \leq &-(1-2\varepsilon) \inf_{(R_r,\,\zeta)\in\mathcal{S}_r} \Bigg\{\lambda^r_\zeta(R_r)\wedge M\\ &+ t^{-1}r^2 \sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} \Big(|\zeta_q|^\theta + \log Z_{\mathrm{disc}}(d,\theta)\Big)\Bigg\}\end{aligned} \tag{76}$$

for sufficiently large t. We can drop M and $r^{-\rho}$ from the right hand side since Theorem 18 tells us that the left hand side is bounded from below. Moreover, we can also neglect $\log Z_{\mathrm{disc}}(d,\theta)$ since

$$\#((r[R^*_r:l])\cap\mathbb{Z}^d) \leq cr^{d+\chi} = o(tr^{-2}). \tag{77}$$

We next proceed to the lower bound. We pick a pair (R^*_r, ζ^*) which attains the infimum in the right hand side of (66). Then we have the following estimate for the L^2-normalized nonnegative eigenfunction ϕ^* corresponding to $\lambda^r_{\zeta^*}(R^*_r)$.

Lemma 8. *There exist $p^* \in (rR^*_r)\cap\mathbb{Z}^d$ and $c_0 > 0$ such that $\sup_{x\in\Lambda_{2/r}(p^*/r)} V^r_{\zeta^*}(x) \leq c_0 r^{d+\chi+2}$ and*

$$\int_{\Lambda_{1/r}(p^*/r)} \phi^*(x)\,dx \geq \frac{1}{2\|\phi^*\|_\infty} r^{-d-\chi}. \tag{78}$$

Proof. We fix $1 < r_0 < \infty$ so that $C_0/2|x|^{-\alpha} \leq u(x) \leq 2C_0|x|^{-\alpha}$ for all $|x| > r_0$ and take $k \in \mathbb{N}$ satisfying $2^{-k-3} \leq r_0/r < 2^{-k-2}$. We divide R^*_r into subboxes of sidelength 2^{-k} as

$$R^*_r = \bigcup_{\dot{\imath}\in\mathcal{I}^*} C_{\dot{\imath}} \quad \text{for some } \mathcal{I}^* \subset \mathcal{I}_k.$$

We take a covering $\mathcal{C}$ of the centers of the obstacles defined by the union of all boxes $C_{\dot{\imath}}$ in R^*_r whose enlarged boxes $q_{\dot{\imath}} + 2^{-k}[-1,2]^d$ intersect with $\{r^{-1}(q+\zeta^*_q) : q \in (r[R^*_r:l])\cap\mathbb{Z}^d\}$. Then it is easy to see that if $C_{\dot{\imath}} \subset \mathcal{C}$, there exists $a \in C_{\dot{\imath}}$ and $c_1 > 0$ for which $V^r_{\zeta^*} \geq c_1 r^2 1_{B(a,1/r)}$. Thus, by using Lemma 1 and the scaling with the factor r, we have

$$\inf_{\phi\in C^\infty(C_{\dot{\imath}})} \left\{\frac{1}{\|\phi\|_2^2}\int_{C_{\dot{\imath}}} \Big(\frac{1}{2}|\nabla\phi(x)|^2 + V^r_{\zeta^*}(x)\phi(x)^2\Big)dx\right\} \geq c_2 r^2$$

for all $C_{\dot{\imath}} \subset \mathcal{C}$ and consequently

$$c_2 r^2 \int_{\mathcal{C}} \phi^*(x)^2 dx \leq \int_{\mathcal{C}} \Big(\frac{1}{2}|\nabla\phi^*|^2(x) + V^r_{\zeta^*}(x)\phi^*(x)^2\Big)dx.$$

Since the right hand side is bounded from above by $\lambda^r_{\zeta^*}(R^*_r)$, it follows that $\int_{\mathcal{C}} \phi^*(x)^2 dx \le c_3 r^{-2}$. This implies $\int_{R^*_r \setminus \mathcal{C}} \phi^*(x)^2 dx \ge 1/2$ for large r and we can find a $\Lambda_{1/r}(p^*/r)$ in $R^*_r \setminus \mathcal{C}$ such that

$$\|\phi^*\|_\infty \int_{\Lambda_{1/r}(p^*/r)} \phi^*(x)\, dx \ge \int_{\Lambda_{1/r}(p^*/r)} \phi^*(x)^2\, dx \ge \frac{1}{2} r^{-d-\chi}.$$

Finally, we show the bound $\sup_{x \in \Lambda_{2/r}(p^*/r)} V^r_{\zeta^*}(x) \le c_0 r^{d+\chi+2}$. Note first that we have $\sup_{x \in \Lambda_{2/r}(p^*/r)} r^2 u(rx - q - \zeta^*_q) \le c_4 r^2$ for each q since $R^*_r \setminus \mathcal{C}$ keeps the distance larger than $(r_0+1)/r$ from $\{r^{-1}(q+\zeta^*_q) : q \in (r[R^*_r : l]) \cap \mathbb{Z}^d\}$. Multiplying the total number of points $\#\{r^{-1}(q+\zeta^*_q : q \in (r[R^*_r : l]) \cap \mathbb{Z}^d\} \le (2l+1)^d r^{d+\chi}$, we obtain the result. $\square$

We bound $S_{t,x}$ as follows:

$$\begin{aligned}
S_{t,x} \ge & \mathbb{P}_\theta \left(\xi_q = \zeta^*_{p^*+q} \text{ for } q \in (r[R^*_r : l]) \cap \mathbb{Z}^d - p^* \right) \\
& \times \mathbb{P}_\theta \left(\sup_{x \in (rR^*_r - p^*) \cup \Lambda_2} \sum_{q \in \mathbb{Z}^d \setminus \{(r[R^*_r : l]) \cap \mathbb{Z}^d - p^*\}} u(x - q - \xi_q) < c_1 (rl)^{-\alpha+d} \right) \\
& \times E_x \Bigg[\exp\left\{ - \int_0^t \sum_{q \in (r[R^*_r : l]) \cap \mathbb{Z}^d - p^*} u(B_s - q - \zeta^*_{p^*+q}) ds \right\} : \\
& \qquad B_s \in \Lambda_2 \text{ for } 0 \le s \le 1, B_1 \in \Lambda_1, B_s \in rR^*_r - p^* \text{ for } 1 \le s \le t \Bigg] \\
& \times \exp\left(- \frac{c_1 t}{(rl)^{\alpha-d}} \right).
\end{aligned} \tag{79}$$

The first factor is greater than or equal to

$$\exp\left(- \sum_{q \in (r[R_r : l]) \cap \mathbb{Z}^d} |\zeta_q|^\theta - c r^{d+\chi} \right)$$

by the same argument using (77) for the upper bound. The last factor is greater than $\exp(-\varepsilon t r^{-2})$ for sufficiently large r if $\alpha > d+2$, and for sufficiently large r and l if $\alpha = d+2$. To bound the second factor we use the following:

Lemma 9. *Let $\{R_r : r \ge 1\}$ be a family of lattice animals satisfying $R_r \subset \Lambda_{t/r}$ and $|R_r| < r^\chi$. Let $k, l > 0$. Then there exists $c_1, c_2, c_3 > 0$ independent of $\{R_r\}$ such that*

$$\mathbb{P}_\theta \left(\sup_{x \in [rR_r : k]} \sum_{q \in \mathbb{Z}^d \setminus (r[R_r : l])} u(x - q - \xi_q) < c_1 (rl)^{-\alpha+d} \right) \ge c_2 \tag{80}$$

for any $r \ge c_3$.

Proof. We consider the event

$$d(q + \xi_q, [rR_r : k]) \ge \frac{1}{2} d(q, [rR_r : k]) \text{ for all } q \in \mathbb{Z}^d \setminus (r[R_r : l]). \tag{81}$$

On this event, we have

$$\sum_{q\in\mathbb{Z}^d\setminus(r[R_r:l])}|x-q-\xi_q|^{-\alpha}\le\sum_{q\in\mathbb{Z}^d\setminus(r[R_r:l])}\Big(\frac{2}{d(q,[rR_r:k])}\Big)^{\alpha}$$
$$\le c_4\sum_{q\in\mathbb{Z}^d:\,d(q,rR_r)\ge rl}d(q,[rR_r:k])^{-\alpha}\le c_5(rl)^{-\alpha+d}$$

for any $x\in[rR_r:k]$ and large r. This estimate implies the event in the result since $u(x)=C_0|x|^{-\alpha}(1+o(1))$. Since the event (81) occurs if

$$|\xi_q|\le d(q,[rR_r:k])/2 \text{ for all } q\in\mathbb{Z}^d\setminus(r[R_r:l]),$$

the probability of the event (81) is greater than or equal to

$$\prod_{q\in\mathbb{Z}^d\setminus(r[R_r:l])}\left(1-\frac{1}{Z_{\rm disc}(d,\theta)}\sum_{y\in\mathbb{Z}^d:|y|\ge d(q,\,[rR_r:k])/2}\exp(-|y|^{\theta})\right). \tag{82}$$

It is easy to see that

$$\frac{1}{Z_{\rm disc}(d,\theta)}\sum_{y\in\mathbb{Z}^d:|y|\ge d(q,\,[rR_r:k])/2}\exp(-|y|^{\theta})\le\exp(-c_6d(q,[rR_r:k])^{\theta})$$

and

$$\#\{q\in\mathbb{Z}^d:n\le d(q,[rR_r:k])<n+1\}\le c_7r^{\chi+d}n^{d-1}.$$

By using also an elementary inequality $(1-x)^p\ge 1-px$ for any $p\ge1$ and $0<x<1$, the quantity in (82) is greater than or equal to

$$\prod_{rl-k\le n\in\mathbb{N}}\Big(1-\exp(-c_6n^{\theta})\Big)^{c_7r^{\chi+d}n^{d-1}}\ge\prod_{rl-k\le n\in\mathbb{N}}\Big(1-c_8r^{\chi+d}\exp(-c_9n^{\theta})\Big).$$

Since the right hand side is a convergent infinite product, we conclude that the event in (81) has a uniformly positive probability. □

It remains to bound the third factor in (79). We use the bound $\sup_{x\in\Lambda_{2/r}(p^*/r)}V^r_{\zeta^*}(x)\le c_0r^{d+\chi+2}$ in Lemma 8 for $0\le s\le1$ and the positivity of

$$\inf_{x,y\in\Lambda_1}\exp(\Delta_2^D/2)(x,y),$$

where $\exp(t\Delta_2^D/2)(x,y)$, $(t,x,y)\in(0,\infty)\times\Lambda_2\times\Lambda_2)$ is the integral kernel of the heat semigroup generated by the Dirichlet Laplacian on Λ_2 multiplied by $-1/2$. Then, we can show that the third factor is greater than

$$r^d\exp(-c_0r^{d+\chi})\int_{\Lambda_{1/r}}dy\int_{R_r^*-p^*/r}dz\exp(-(t-1)r^{-2}H^*)(y,z) \tag{83}$$

for large r by using a scaling, where $\exp(-tH^*)(x,y)$, $(t,x,y)\in(0,\infty)\times(R_r^*-p^*/r)\times(R_r^*-p^*/r))$ is the integral kernel of the heat semigroup generated by the Schrödinger operator

$$H^*=-\Delta/2+\sum_{q\in(r[R_r^*:l])\cap\mathbb{Z}^d-p^*}r^2u(rx-q-\zeta^*_{p^*+q})$$

in $R_r^* - p^*/r$ with the Dirichlet boundary condition. By (78), the integral in (83) is greater than or equal to

$$\begin{aligned}&\int_{\Lambda_{1/r}} dy \int_{R_r^*-p^*/r} dz \exp(-(t-1)r^{-2}H^*)(y,z)\frac{\phi^*(z+p^*/r)}{\|\phi^*\|_\infty}\\ &\geq \exp(-(t-1)r^{-2}\lambda^r_{\zeta^*}(R_r^*))/(2\|\phi^*\|_\infty^2 r^{d+\chi}).\end{aligned}$$

Finally $\|\phi^*\|_\infty$ is bounded since

$$\phi^*(y) = \exp(\lambda^r_{\zeta^*}(R_r^*)) \int \exp(-H^0)(y,z)\phi^*(z)\,dz,$$

$\|\exp(-H^0)(y,\cdot)\|_2 \leq 1$, and $\lambda^r_{\zeta^*}(R_r^*)$ is bounded by Theorem 18 and the upper bound in (i), where $\exp(-tH^0)(x,y)$, $(t,x,y) \in (0,\infty) \times R_r^* \times R_r^*$, is the integral kernel of the heat semigroup generated by the Schrödinger operator $H^0 = -\Delta/2 + V^r_{\zeta^*}$ in R_r^* with the Dirichlet boundary condition. By all these the lower bounds (ii) and (iii) are proven.

7.2 One-dimensional critical case

For the one-dimensional case, we only consider $\alpha = 3$ since we have already known the leading term in other cases. We first fix a constant $M > 0$ such that

$$\begin{aligned}\mathbb{P}_\theta\left(\{q+\xi_q : q \in \mathbb{Z}\} \cap (0, Mt^{1/(3+\theta)}) = \emptyset\right) &\leq \exp\left\{-cM^{1+\theta}t^{(1+\theta)/(3+\theta)}\right\}\\ &= o(S_{t,x}),\end{aligned}$$

which is possible in view of Theorem 18. We define $r = t^{1/(3+\theta)}$ and the set of *relevant* configurations as

$$\begin{aligned}\mathcal{S}_r = \big\{&((m,n), \zeta = (\zeta_q)_{q\in(m-lr,n+lr)\cap\mathbb{Z}}) : m,n \in \mathbb{Z}, -t \leq m < n \leq t,\\ &n-m \leq Mr, |\zeta_q| \leq t^{1/\theta}, \{q+\zeta_q : q \in (m-lr,n+lr)\cap\mathbb{Z}\} \cap (m,n) = \emptyset\big\}.\end{aligned}$$

Then, we have the following:

Theorem 20. *Let $(d,\alpha) = (1,3)$ and assume (22). Then for any $\varepsilon > 0$, there exist $t_\varepsilon > 0$ and $l_\varepsilon > 0$ such that*

$$t^{-(1+\theta)/(3+\theta)} \log S_{t,x} \begin{cases} \geq -(1+\varepsilon) \inf_{((m,n),\zeta)\in\mathcal{S}_r} \left\{\lambda^r_\zeta((m/r,n/r)) + \sum_{q\in(m-lr,n+lr)\cap\mathbb{Z}} r^{-1}\left|\frac{\zeta_q}{r}\right|^\theta\right\},\\ \leq -(1-\varepsilon) \inf_{((m,n),\zeta)\in\mathcal{S}_r} \left\{\lambda^r_\zeta((m/r,n/r)) + \sum_{q\in(m-lr,n+lr)\cap\mathbb{Z}} r^{-1}\left|\frac{\zeta_q}{r}\right|^\theta\right\},\end{cases}$$

for all $t > t_\varepsilon$ and $l > l_\varepsilon$.

Proof. We only prove the upper bound. After having it, the lower bound follows exactly in the same way as in the previous subsection.

Let $\varepsilon > 0$ be an arbitrary constant. We use a simple version of the method of enlargement of obstacles where $\gamma = 1$ and any 2^{-n_1}-box containing a point of $\{r^{-1}(q+\xi_q) : q \in \mathbb{Z}\}$ is a density box. Such a box indeed satisfies the quantitative Wiener criterion (2.12) in page 152 of [20] since even a point has positive capacity when $d = 1$ (cf. page 153 of [20]). Then, the spectral control (72) implies that we can impose the Dirichlet boundary condition on each point in $\{r^{-1}(q+\xi_q)\}_{q\in\mathbb{Z}}$. Combining this observation with a standard Brownian estimate and (3.1.9) in [20], we find

$$S_{t,x} \le \mathbb{E}_\theta\left[c\left(1+\left(\lambda_\xi^1((-t,t))t\right)^{1/2}\right)\exp\left\{-\lambda_\xi^1((-t,t))t\right\}\right]+e^{-ct}$$
$$\le c_\varepsilon \mathbb{E}_\theta\left[\sup_k \exp\left\{-(1-\varepsilon)\lambda_\xi^r\left(r^{-1}I_k\right)tr^{-2}\right\}\right]+e^{-ct},$$

where $\{I_k\}_k$ are the random open intervals such that $\sum_k I_k = \Lambda_t \setminus \{q+\xi_q : q\in\mathbb{Z}\}$. By considering all possibilities of I_k, we can bound the $\mathbb{E}_\theta$-expectation in the right hand side by

$$\sum_{m,n\in\mathbb{Z}:-t\le m<n\le t} \mathbb{E}_\theta\left[\exp\left\{-(1-\varepsilon)\lambda_\xi^r\left((m/r,n/r)\right)tr^{-2}\right\} : \{q+\xi_q : q\in\mathbb{Z}\}\cap(m,n)=\emptyset\right].$$

Note that we can discard (m,n) whose interval $n-m>Mr$ thanks to our choice of M. Hence, we can restrict our consideration on $\mathcal{S}_r$ and we can also show $\#\mathcal{S}_r = \exp\{o(t^{(1+\theta)/(3+\theta)})\}$ by an elementary counting argument. Now, we have

$$S_{t,x} \le \sum_{((m,n),\zeta)\in\mathcal{S}_r} \exp\left\{-(1-\varepsilon)\lambda_\zeta^r\left((m/r,n/r)\right)tr^{-2}\right\}\mathbb{P}_\theta\left(\xi_q=\zeta_q \text{ for all } q\right)+o(S_{t,x})$$
$$\le \exp\left\{-(1-2\varepsilon)t^{(1+\theta)/(3+\theta)}\inf_{((m,n),\zeta)\in\mathcal{S}_r}\left\{\lambda_\zeta^r((m/r,n/r))+\sum_{q\in(m-lr,n+lr)\cap\mathbb{Z}} r^{-1}\left|\frac{\zeta_q}{r}\right|^\theta\right\}\right\},$$

which is the desired estimate. □

8 Asymptotics of higher moments

In [5], a result on the asymptotics for higher moments of the survival probability is shown as an application of the precise form of the leading term. We shall extend the result to our cases in this section. Our objects are the p-th moments defined by

$$S_{t,x}^{(p)} := \mathbb{E}_\theta\left[E_{x/\sqrt{2h}}\left[\exp\left\{-\int_0^t \sum_{q\in\mathbb{Z}^d} u(\sqrt{2h}B_s-q-\xi_q)ds\right\} : \sqrt{2h}B_s \notin \bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K) \text{ for } 0\le s\le t\right]^p\right]$$

and

$$S_{t,x}^{(p),-} := \mathbb{E}_\theta\left[E_{x/\sqrt{2h}}\left[\exp\left\{\int_0^t \sum_{q\in\mathbb{Z}^d} u(\sqrt{2h}B_s-q-\xi_q)ds\right\}\right]^p\right].$$

We consider their asymptotics in Subsection 8.1. In Subsection 8.2, we discuss a related quantitative estimate on intermittency for the parabolic Anderson problem.

8.1 Asymptotics for each case

Proposition 21. *Under the settings in Section 7, there exist $c_1, c_2 \in (0,\infty)$ depending on d,θ and u such that for any $p\ge 1$,*

$$-c_1 p^{(d+\mu\theta)/(d+2+\mu\theta)} \le t^{-1}r^2\log S_{t,x}^{(p)} \le -c_2 p^{(d+\mu\theta)/(d+2+\mu\theta)}$$

holds for sufficiently large t, uniformly in $x\in\Lambda_1$, where we take $\mu=1$ in the case $d=1$.

Proof. We first assume $d \geq 3$ and $\alpha > d+2$. The same argument as in Section 7, using the scaling with factor $s = (pt)^{1/(d+2+\mu\theta)}$ instead of $r = t^{1/(d+2+\mu\theta)}$ in (75) and (83), yields

$$\log S_{t,x}^{(p)} \sim -(pt)^{(d+\mu\theta)/(d+2+\mu\theta)} \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda_\zeta^s(R_s) + s^{(1-\mu)\theta} \sum_{q\in(s[R_s:\,l])\cap\mathbb{Z}^d} s^{-d} \left| \frac{\zeta_q}{s} \right|^\theta \right\} \tag{84}$$

as $t \to \infty$ for any l. Since we know

$$\begin{aligned} 0 &< \varliminf_{s\to\infty} \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda_\zeta^s(R_s) + s^{(1-\mu)\theta} \sum_{q\in(s[R_s:\,l])\cap\mathbb{Z}^d} s^{-d} \left| \frac{\zeta_q}{s} \right|^\theta \right\} \\ &\leq \varlimsup_{s\to\infty} \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda_\zeta^s(R_s) + s^{(1-\mu)\theta} \sum_{q\in(s[R_s:\,l])\cap\mathbb{Z}^d} s^{-d} \left| \frac{\zeta_q}{s} \right|^\theta \right\} < \infty \end{aligned} \tag{85}$$

from Theorems 18 and 19, the proof is completed. The other cases can be treated exactly in the same way. $\square$

Remark 5. If $-\lim_{t\to\infty} t^{-1}r^2 \log S_{t,x}$ exists under the setting of the last proposition, denoting it by L, we have

$$t^{-1}r^2 \log S_{t,x}^{(p)} \sim -Lp^{(d+\mu\theta)/(d+2+\mu\theta)}. \tag{86}$$

In fact, when $d \geq 2$ and $\alpha > d+2$, the existence of the above limit implies

$$\lim_{t\to\infty} \inf_{(R_r,\,\zeta)\in\mathcal{S}_r} \left\{ \lambda_\zeta^r(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^{-d} \left| \frac{\zeta_q}{r} \right|^\theta \right\} = L$$

by Theorem 19. Then (86) is obvious from the proof of the last proposition. When $\alpha = d+2$, we know only that the superior limit and the inferior limit in (85) tend to L as $l \to \infty$. This is still enough to show (86).

The above remark actually applies for the case $d = 1$ and $\alpha > 3$:

Proposition 22. *Under the conditions of Theorem 2 with $d = 1$, we have*

$$\lim_{t\uparrow\infty} t^{-(1+\theta)/(3+\theta)} \log S_{t,x}^{(p)} = -\frac{3+\theta}{1+\theta} \left(\frac{ph\pi^2}{4} \right)^{(1+\theta)/(3+\theta)} \tag{87}$$

for any $p \geq 1$, uniformly in $x \in \Lambda_1$.

Proof. As in the proof of the last proposition we have

$$\log S_{t,x}^{(p)} \sim -(pt)^{(1+\theta)/(3+\theta)} \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda_\zeta^s((m/s, n/s)) + \sum_{q\in(m-ls,n+ls)\cap\mathbb{Z}} s^{-1} \left| \frac{\zeta_q}{s} \right|^\theta \right\} \tag{88}$$

as $t \to \infty$ for any l in the notations of Subsection 7.2, where $s = (pt)^{1/(3+\theta)}$. We here note that this asymptotics is common for the continuous distribution (2) and the discrete distribution (62). When $\alpha > 3$, we know the limit

$$\lim_{s\to\infty} \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda_\zeta^s((m/s, n/s)) + \sum_{q\in(m-ls,n+ls)\cap\mathbb{Z}} s^{-1} \left| \frac{\zeta_q}{s} \right|^\theta \right\} = \frac{3+\theta}{1+\theta} \left(\frac{h\pi^2}{4} \right)^{(1+\theta)/(3+\theta)}.$$

$\square$

Proposition 23. *Under the conditions of Theorem 1 with $\alpha < d+2$, we have*

$$\lim_{t\uparrow\infty} t^{-(d+\theta)/(\alpha+\theta)} \log S_{t,x}^{(p)} = -p^{(d+\theta)/(\alpha+\theta)} \int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) \tag{89}$$

for any $p \geq 1$, uniformly in $x \in \Lambda_1$.

Proof. We have only to show

$$\lim_{t\uparrow\infty} t^{-(d+\theta)/(\alpha+\theta)} \log S_{t,x}^{(p)} = -\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \Big(\frac{pC_0}{|q+y|^\alpha} + |y|^\theta \Big).$$

The upper estimate is easy since we have

$$S_{t,x}^{(p)} \leq \mathbb{E}_\theta \bigg[E_{x/\sqrt{2h}} \bigg[\exp\bigg\{ -\int_0^t V_\xi(\sqrt{2h}B_s)ds \bigg\} \bigg]^p \bigg] \leq \mathbb{E}_\theta \otimes E_{x/\sqrt{2h}} \bigg[\exp\bigg\{ -p\int_0^t V_\xi(\sqrt{2h}B_s)ds \bigg\} \bigg]$$

by removing the Dirichlet condition and using the Hölder inequality. For the lower estimate, we take R, R_1 and β as in the proof of Proposition 5 and restrict the integral as

$$S_{t,x}^{(p)} \geq \mathbb{E}_\theta \bigg[E_{x/\sqrt{2h}} \bigg[\exp\bigg\{ -\int_0^t V_\xi(\sqrt{2h}B_s)ds \bigg\} : \sqrt{2h}B_s \in \Lambda_R \text{ for } 0 \leq s \leq t \bigg]^p : \Xi_t \bigg]$$

for $t^\beta \geq 2(R_1 + R\sqrt{d})$, where Ξ_t is the event

$$\{|\xi_q| \leq |q|/2 \text{ for } |q| \geq t^\beta, \text{ and } |q+\xi_q| \geq R_1 + R\sqrt{d} \text{ for } |q| < t^\beta\}.$$

The right hand side is bounded from below by

$$\mathbb{E}_\theta \bigg[\exp\bigg\{ -pt \sup_{y\in\Lambda_R} V_\xi(y) \bigg\} : \Xi_t \bigg] \exp(-chptR^{-2}).$$

This is estimated by the same method as in our proof of Proposition 5. □

Proposition 24. *Under the conditions of Theorem 3, we have*

$$\lim_{t\uparrow\infty} t^{-(1+d/\theta)} \log S_{t,x}^{(p),-} = (pu(0))^{1+d/\theta} \int_{|q|\leq 1} dq (1 - |q|^\theta)$$

for any $p \geq 1$, uniformly in $x \in \Lambda_1$.

Proof. The upper estimate is obtained similarly as in the proof of Proposition 23 and the lower estimate is obtained similarly as in the proof of (61). □

8.2 Intermittency

The Brownian expectations appearing in $S_{t,x}^{(p)}$ and $S_{t,x}^{(p),-}$ solve the initial value problems

$$\frac{\partial}{\partial t} u_\xi(t,x) = -H_\xi u_\xi(t,x) \text{ with } u_\xi(0,x) \equiv 1$$

and

$$\frac{\partial}{\partial t} u_\xi^-(t,x) = -H_\xi^- u_\xi^-(t,x) \text{ with } u_\xi^-(0,x) \equiv 1,$$

respectively. This kind of initial value problems with random potential are usually referred as the "parabolic Anderson problem". For a wide class of random potentials, solutions of parabolic Anderson problems are known to reveal so-called "intermittency" (cf. [7]):

$$\frac{\langle u_\xi(t,x)^{p_2}\rangle^{1/p_2}}{\langle u_\xi(t,x)^{p_1}\rangle^{1/p_1}} \overset{t\to\infty}{\longrightarrow} \infty \quad \text{for } p_1 < p_2, \tag{90}$$

where $\langle\cdot\rangle$ denotes the expectation with respect to the law of random potentials. In particular, if we consider a slightly different moment

$$\overline{S}_t^{(p)} := \mathbb{E}_\theta\left[\int_{\Lambda_1} u_\xi(t,x)^p dx\right]$$

in our model, then intermittency follows from the same argument as for Theorem 3.2 of [7]. Note that this quantity naturally arises when we relate statistical average to spatial one as

$$\mathbb{E}_\theta\left[\int_{\Lambda_1} u_\xi(t,x)^p dx\right] = \lim_{n\to\infty}\frac{1}{|\Lambda_{2n+1}|}\int_{\Lambda_{2n+1}} u_\xi(t,x)^p dx$$

by ergodic theorem. (Recall that our model is *not* $\mathbb{R}^d$-stationary.)

In our model, we can derive the rates of the divergence in (90) from the results in the previous subsections as follows:

1. Under the settings in Section 7, we have

$$\begin{aligned}&\exp\left\{tr^{-2}\left(c_2p_1^{-2/(d+2+\mu\theta)} - c_1p_2^{-2/(d+2+\mu\theta)}\right)\right\}\\ &\le \frac{\left(S_{t,x}^{(p_2)}\right)^{1/p_2}}{\left(S_{t,x}^{(p_1)}\right)^{1/p_1}} \le \exp\left\{tr^{-2}\left(c_1p_1^{-2/(d+2+\mu\theta)} - c_2p_2^{-2/(d+2+\mu\theta)}\right)\right\},\end{aligned}$$

   for sufficiently large t, where $\infty > c_1 \ge c_2 > 0$ are the constants in Proposition 21.

2. Under the conditions of Theorem 2 with $d = 1$, for any $\varepsilon > 0$, it holds that

$$\begin{aligned}&\exp\left\{\frac{3+\theta}{1+\theta}\Big(\frac{h\pi^2t}{4}\Big)^{(1+\theta)/(3+\theta)}\left(p_1^{-2/(3+\theta)} - p_2^{-2/(3+\theta)} - \varepsilon\right)\right\}\\ &\le \frac{\left(S_{t,x}^{(p_2)}\right)^{1/p_2}}{\left(S_{t,x}^{(p_1)}\right)^{1/p_1}} \le \exp\left\{\frac{3+\theta}{1+\theta}\Big(\frac{h\pi^2t}{4}\Big)^{(1+\theta)/(3+\theta)}\left(p_1^{-2/(3+\theta)} - p_2^{-2/(3+\theta)} + \varepsilon\right)\right\}\end{aligned}$$

   for sufficiently large t.

3. Under the conditions of Theorem 1 with $\alpha < d+2$, for any $\varepsilon > 0$, it holds that

$$\begin{aligned}&\exp\left\{c_3t^{(d+\theta)/(\alpha+\theta)}\left(p_1^{(d-\alpha)/(\alpha+\theta)} - p_2^{(d-\alpha)/(\alpha+\theta)} - \varepsilon\right)\right\}\\ &\le \frac{\left(S_{t,x}^{(p_2)}\right)^{1/p_2}}{\left(S_{t,x}^{(p_1)}\right)^{1/p_1}} \le \exp\left\{c_3t^{(d+\theta)/(\alpha+\theta)}\left(p_1^{(d-\alpha)/(\alpha+\theta)} - p_2^{(d-\alpha)/(\alpha+\theta)} + \varepsilon\right)\right\}\end{aligned}$$

   for sufficiently large t, where $c_3 = \int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d}\big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta\big)$.

4. Under the conditions of Theorem 3, for any $\varepsilon > 0$, it holds that

$$\exp\left\{c_4 t^{1+d/\theta}\left(p_2^{d/\theta} - p_1^{d/\theta} - \varepsilon\right)\right\} \le \frac{\left(S_{t,x}^{(p_2),-}\right)^{1/p_2}}{\left(S_{t,x}^{(p_1),-}\right)^{1/p_1}} \le \exp\left\{c_4 t^{1+d/\theta}\left(p_2^{d/\theta} - p_1^{d/\theta} + \varepsilon\right)\right\}$$

for sufficiently large t, where $c_4 = u(0)^{1+d/\theta}\int_{|q|\le 1} dq(1-|q|^\theta)$.

Note that in the first case, the left hand side goes to infinity only when p_2/p_1 is sufficiently large. On the other hand, the left hand sides go to infinity for any $p_2/p_1 > 1$ in other cases. This is slightly better than Theorem 3.2 of [7] where $p_2 \ge 2$ is required. Note also that all these estimates hold uniformly in $x \in \Lambda_1$ and therefore, the same estimates hold for $\overline{S}_t^{(p)}$ and

$$\overline{S}_t^{(p),-} := \mathbb{E}_\theta\left[\int_{\Lambda_1} u_\xi^-(t,x)^p dx\right]$$

as well.

9 Appendix

We here show the following lemma, which is used to define the integrated density of states $N(\lambda)$ and to represent it by the Feynman-Kac formula:

Lemma 10. *Let u be a nonnegative function belonging to the class K_d and satisfying* (3)*. Let $\xi = (\xi_q)_{q\in\mathbb{Z}^d}$ be a collection of independently and identically distributed $\mathbb{R}^d$-valued random variables satisfying* (2)*. Then the almost all sample functions of the random field defined by $V_\xi(x) = \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)$ belong to the class $K_{d,loc}$.*

Proof. For any $\varepsilon, \delta > 0$, by the Chebyshev inequality, we have

$$\mathbb{P}_\theta(|\xi_q| \ge |q|^\varepsilon) \le \mathbb{E}_\theta[(|\xi_q|/|q|^\varepsilon)^\delta] \le c_1/|q|^{\varepsilon\delta}.$$

For any ε, there exists δ such that

$$\sum_{q\in\mathbb{Z}^d} \mathbb{P}_\theta(|\xi_q| \ge |q|^\varepsilon) < \infty.$$

By the Borel-Cantelli lemma, for almost all ξ, we have $N_\xi \in \mathbb{N}$ such that $|\xi_q| < |q|^\varepsilon < |q|/3$ for any $q \in \mathbb{Z} - B(N_\xi)$. By the condition (3) we also have R_ε such that $u(x) \le (C_0+\varepsilon)/|x|^\alpha$ for any $x \in B(R_\varepsilon)^c$. We now take $R > 0$ arbitrarily. If $x \in B(R)$ and $q \in \mathbb{Z}^d - B(3(R\vee R_\varepsilon)\vee N_\xi)$, then

$$|x-q-\xi_q| \ge |q| - |\xi_q| - |x| \ge |q|/3 \ge R_\varepsilon$$

and

$$V_\xi(x) \le \sum_{q\in\mathbb{Z}^d\cap B(3(R\vee R_\varepsilon)\vee N_\xi)} u(x-q-\xi_q) + c_2.$$

Since the right hand side is a finite sum, we have $1_{B(R)}V_\xi \in K_d$. Since R is arbitrary, we can complete the proof. $\square$

To treat the integrated density of states $N^-(\lambda)$ and relate it with the integral $S_{t,x}^-$, we further use the following:

Lemma 11. *Let u be a bounded nonnegative function satisfying* (3). *Then there exist finite constants c_1 and c_2 such that*

$$\mathbb{E}_\theta\Big[\exp\Big(r\sup_{x\in\Lambda_1} V_\xi(x)\Big)\Big] \le c_1\exp(c_2 r^{1+d/\theta})$$

for any $r \ge 0$, where ξ and V_ξ are same as in the last lemma.

Proof. We first dominate as

$$\log\mathbb{E}_\theta\Big[\exp\Big(r\sup_{x\in\Lambda_1} V_\xi(x)\Big)\Big] \le \int_{\mathbb{R}^d}\log I(q)dq,$$

where

$$I(q) = \mathbb{E}_\theta\Big[\exp\Big(r\sup_{x\in\Lambda_2} u(x-q-\xi_0)\Big)\Big].$$

For sufficiently large $R>0$, we have $u(x) \le 2C_0|x|^{-\alpha}$ for $|x| \ge R_0$. A sufficient condition for $\inf_{x\in\Lambda_2}|x-q-\xi_0| \ge R$ is $|q+\xi_0| \ge R+\sqrt{d}$. Then, for $q \in B(2(R+\sqrt{d}))^c$, we dominate as

$$\begin{aligned} I(q) \le& \mathbb{E}_\theta\Big[\exp\Big(\sup_{x\in\Lambda_2}\frac{2rC_0}{|x-q-\xi_0|^\alpha}\Big) : |q+\xi_0| \ge \frac{|q|}{2}\Big] \\ &+ \exp(r\sup u)\mathbb{P}_\theta\Big(|q+\xi_0| < \frac{|q|}{2}\Big) \\ \le& \exp\Big(\frac{2rC_0}{(|q|/2-\sqrt{d})^\alpha}\Big)(1+c_1\exp(r\sup u - c_2|q|^\theta)) \end{aligned}$$

Since $\log(1+X) \le X$ for any $X \ge 0$, we have

$$\begin{aligned} \int_{B(2(R+\sqrt{d}))^c}\log I(q)dq \le& \int_{B(2(R+\sqrt{d}))^c}\frac{2rC_0}{(|q|/2-\sqrt{d})^\alpha}dq + \int_{B(2(R+\sqrt{d}))^c} c_1\exp(r\sup u - c_2|q|^\theta))dq \\ \le& \frac{c_3 r}{R^{\alpha-d}} + c_4\exp(r\sup u - c_5 R^\theta). \end{aligned}$$

By a simple uniform estimate, we have

$$\int_{B(2(R+\sqrt{d}))}\log I(q)dq \le c_6 r\sup u R^d.$$

We set $R = (r\sup u/c_5)^{1/\theta}$. Then we have

$$\int \log I(q)dq \le c_7 r^{1+d/\theta}$$

for sufficiently large $r>0$. □

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