Well-Posedness of Nonlinear Schrödinger Equations

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Notation

D^α $= \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$. for a multi-index $\alpha \in \mathbb{N}^N$.

∇u $= (\partial_1 u, \ldots, \partial_N u)$.

Δ $= \Sigma_{j=1}^N \partial_j^2$.

$\mathcal{S}(\mathbb{R}^n)$ Schwartz space; i.e., the set of all real- or complex -valued C^∞ functions on $\mathbb{R}$ such that for every nonegative interger m and every multi-index α,

$$p_{m,\alpha}(u) = \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{\frac{m}{2}} |D^\alpha u(x)| < \infty.$$

$\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space when equipped with the seminorms $p_{m,\alpha}$.

$\mathcal{S}'(\mathbb{R}^n)$ space of tempered distributions on $\mathbb{R}^n$; i.e.,the topological dual of $\mathcal{S}(\mathbb{R}^n)$. $\mathcal{S}'(\mathbb{R}^n)$ is a subspace of $C_0^\infty(\mathbb{R}^n)$.

$L^p(\mathbb{R})$ Banach space of measurable functions $u : \mathbb{R} \to \mathbb{R}$ or $(u : \mathbb{R} \to \mathbb{C})$ such that $||u||_{L^p} < \infty$, with

$$||u||_{L^p} = \begin{cases} \left(\int_{\mathbb{R}} |u(x)|^p dx\right)^{1/p} & \text{if } (p < \infty) \\ \text{esssup}_{\mathbb{R}} |u(x)| & \text{if } (p = \infty). \end{cases}$$

$H^{s,p}(\mathbb{R}^n)$ $(s \in \mathbb{R}, 1 \le p \le \infty)$ Banach space of elements $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}} \widehat{u}] \in L^p(\mathbb{R}^n)$. $= H^{s,p}(\mathbb{R}^n)$ is equipped with the norm

$$\|u\|_{H^{s,p}} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}} \widehat{u}\|_{L^p}.$$

$H^s(\mathbb{R}^n)$ $= H^{s,2}(\mathbb{R}^n)$

$\dot{H}^{s,p}(\mathbb{R}^n)$ $(s \in \mathbb{R}, 1 \le p \le \infty)$ homogeneous version of the Sobolev space $H^{s,p}(\mathbb{R}^n)$

$B^s_{p,q}(\mathbb{R}^n)$ $(s \in \mathbb{R}, 1 \le p \le \infty)$ Banach space of elements $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|u\|_{B^s_{p,q}} < \infty$ with

$$||u||_{B^s_{p,q}} = \|\mathcal{F}^{-1}(\eta\widehat{u})\|_{L^p} + \begin{cases} \left(\Sigma_{j=1}^\infty (2^{sj} \|\mathcal{F}^{-1}(\phi_j \widehat{u})\|_{L^p})^q\right)^{1/q} & \text{if } (q < \infty) \\ \sup_{j\ge1} (2^{sj} \|\mathcal{F}^{-1}(\phi_j \widehat{u})\|_{L^p}) & \text{if } (q = \infty). \end{cases}$$

where $\mathcal{F}^{-1}(\phi_j \widehat{u})$ is the j^{th} dyadic block of the Littlewood-Paley decomposition of u.

$\dot{B}^s_{p,q}(\mathbb{R}^n)$ $(s \in \mathbb{R}, 1 \le p, q \le \infty)$ homogeneous version of the Sobolev space $B^s_{p,q}(\mathbb{R}^n)$.

$X_{s,b}$ $(s, b \in \mathbb{R})$ Fourier restriction space corresponding to the Schrödinger equation of elements $u \in \mathcal{S}'(\mathbb{R}^2)$ such that

$$\|u\|_{X_{s,b}} = \|\langle\xi\rangle^s \langle\tau + \xi^2\rangle \widehat{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$.

$A \lesssim B$	for any positive A and B means $A \leq CB$ with constant C.
$A \sim B$	$A \lesssim B$ and $B \lesssim A$.
$A \ll B$	$A \leq cB$ with small constant c.

Chapter 1

Introduction

In this paper we study the Cauchy problem of nonlinear Schrödinger equations with initial data in Sobolev space

$$iu_t + \Delta u + \lambda |u|^{p-1} u = 0, \tag{1.1}$$

$$u(0,t) = \phi(x) \in H^s(\mathbb{R}^n), \tag{1.2}$$

where u is a complex valued function with $\lambda \in \mathbb{C}$ and $p > 1$. There are many problems concerning the short time and the long time behaviour of solutions of nonlinear Schrödinger equations. We are particularly interested in two problems such as the unconditional well-posedness and the global well-posedness under H^1 norm.

First, we briefly recall the definition of well-posedness in H^s that we use here:

Definition 1.0.1. *For any initial data $\phi \in H^s$, there exists a positive time $T = T(\|u_0\|_{H^s})$ depending on the norm of initial data such that* (1.1)-(1.2) *has a strong unique solution u in $X \subset C([-T,T], H^s)$, where X is a Banach spaces and the solution map from H^s to $C([-T,T], H^s)$ continuously depends on the initial data. If T can be chosen arbitrarily large, we say that* (1.1)-(1.2) *is globally well-posed. If we can take $X = C([-T,T], H^s)$, then we say that* (1.1)-(1.2) *is unconditionally well-posed.*

One may show that existence and uniqueness result by fixed point method via the following equation

$$u(t) = U(t)u_0 - i\int_0^t U(t-s)|u|^{p-1}u(s)ds,$$

where $U(t)$ is a free Schrödinger evolution operator. It is known that the solution of (1.1)-(1.2) satisfies the conservation of mass

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}$$

and the conservation of energy

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{C}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx.$$

Now we discuss the unconditional well-posedness of nonlinear Schrödinger equation with power nonlinearity.

Unconditionally well-posed

Kato [28] introduces the concept of unconditional well-posedness of nonlinear Schrödinger equation. He explains that the well-posedness problem needs the auxiliary condition which is related to Strichartz estimates to ensure uniqueness with two examples. First we start with the definition of admissible pair.

Let (q, r) be an admissible pair such that

$$\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r}),$$

when $2 \leq r \leq \frac{2n}{n-2}$ ($2 \leq r \leq \infty$ if $n = 1, 2 \leq r < \infty$ if $n = 2$).

Let us consider the equations (1.1)-(1.2) in the following two cases.

(i) Problem (A)

Assume $p \leq 1+\frac{4}{N-2}$. There exists a unique solution $u \in C([0,T), H^1(\mathbb{R}^N)) \cap L^q((0,T), L^r(\mathbb{R}^n))$, for some $T \geq 0$, where (q, r) is an admissible pair.

(ii) Problem (B)

Assume $p < 1 + \frac{4}{N}$. There exists a unique solution $u \in C([0,T], L^2(\mathbb{R}^N)) \cap L^q((0,T), L^r(\mathbb{R}^n))$, for some $T \geq 0$, where (q, r) is an admissible pair.

In problem (A), auxiliary space $L^q((0,T), L^r(\mathbb{R}^n))$ can be removed, it is bonus as Kato points out, which may or may not appear in theorem. Hence we say that problem (A) is unconditional well-posedness. But in problem (B), auxiliary space $L^q((0,T), L^r(\mathbb{R}^n))$ is essential part of the well-posedness because we might not prove the uniqueness without auxiliary conditions. Hence we say that, problem (B) is conditional well-posedness in $L^2(\mathbb{R}^N)$ with $L^q((0,T), L^r(\mathbb{R}^n))$. In this case there are infinitely many auxiliary spaces but they are consistent. The unconditional uniqueness is a concept of uniqueness which does not depend on how to construct the solution.

Our another interesting is the global well-posedness of nonlinear Schrödinger equations when the regularity is below H^1.

Globally well-posed below H^1

It is known that the global well-posedness in $H^s(\mathbb{R}^N)$ for $s \geq 1$, is obtained via the laws of conservation of mass and energy. These two laws lead to L^2 and H^1 regularities, respectively and the both norms stay bounded for all time. The question then arises whether (1.1)-(1.2) is globally well-posed or not when regularity is below H^1. In that direction, two methods have been developed our understanding recent years, namely *Fourier truncation method* of Bourgain and *method of almost conserved quantities* or *I-method* of J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao. They both proceed by choosing a large frequency N which depend on time and dividing a solution into two parts of lower and higher frequencies than N.

Let us consider for the particular case

$$iv_t + \frac{1}{2}\Delta v \;=\; |v|^{p-1}v, \ (t,x) \in \mathbb{R} \times \mathbb{R}^2, \tag{1.3}$$

with the conservation of energy

$$E(v) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla v|^2 + \frac{1}{p+1}|v|^{p+1}dx \tag{1.4}$$

where $p = 3$ or 5. It is also known that this nonlinear Schrödinger equation enjoys the scaling symmetry

$$v(t,x) \mapsto \lambda^{-2/(p-1)}v(\frac{t}{\lambda^2}, \frac{x}{\lambda}); \ v_0(x) \mapsto \lambda^{-2/(p-1)}v_0(\frac{x}{\lambda}).$$

To explain the I-method, we start with the definition of Fourier multiplier operator

$$\widehat{Iv}(\xi) = m(\xi)\hat{v}(\xi)$$

where $m(\xi)$ is an arbitrary real valued multiplier which is the identity on the low frequencies and like as a fractional integral operator of order $1-s$ on the high frequencies. Then we see that

$$\|v(t)\|_{H^s} \lesssim \|Iv(t)\|_{H^1} \lesssim N^{1-s}\|v(t)\|_{H^s}.$$

We set $u = Iv$ by applying I to (1.3), then u solves the equation

$$iu_t + \frac{1}{2}\Delta u \;=\; I(|v|^{p-1}v), \ x \in \mathbb{R}^2, t \geq 0. \tag{1.5}$$

Hence we have

$$\partial_t E(u) = -2\mathrm{Re}\int_{\mathbb{R}^2} \overline{I\partial_t v}(|Iv|^{p-1}Iv - I(|v|^{p-1}v))dx.$$

Then it can be shown that there exists $\alpha > 0$ such that

$$E(Iv(t)) - E(Iv(0)) \lesssim N^{-\alpha},$$

for all $t \in [0,1]$ which is controlled by bilinear or multi-linear estimates and usual Strichartz estimates under $X_{s,b}$ spaces setting.

Here we divide the proof into three main parts such as low-low, high-low and high-high frequency interaction. In low-low case, $E(Iu)$ is conserved because the operator I is the same as the identity. In high-high case, we get the small error in IH^1_x. The most difficult case is high-low frequencies interaction. Hence we have to improve some estimates (for instance, the refinement of bilinear Strichartz estimate, see Chapter 6) to get a good decay which needs to match the error of high-high case.

For instance, when $p = 3$, $n = 2$, J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao [11] show

$$E(Iv(t)) \lesssim E(Iv(0)) + N^{-\frac{3}{2}+}.$$

By using time iteration and scaling, global well-posedness for H^s with $s > \frac{4}{7}$ is shown.

In this subject, we shall mainly concentrate on the derivative nonlinear Schrödinger equation (DNLS) in one dimension

$$\begin{aligned} i\partial_t u + \partial_{xx} u &= \delta \partial_x(|u|^2 u), \\ u(0,x) &= u_0(x), \end{aligned}$$

where $u(t,x)$ is a complex valued function and $\delta = +1$ or -1. It is not necessary to distinguish the defocusing and the focusing because we consider the suffuciently small initial data.

In Chapter 2, we give the function spaces, several applications such as a gauge transformation on both periodic and non-periodic cases, paraproduct method, the functions with $2\pi\lambda$-periodic and several properties for rescaled functions. In Chapter 3, we are concerned with the Fourier restriction spaces. In Chapter 4, we discuss the Hartree type nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + \lambda(|x|^{-\gamma} * |u|^2)u = 0, \ (t,x) \in [0,T) \times \mathbb{R}^n, \tag{1.6}$$

$$u(0,x) = u_0, \tag{1.7}$$

where $\lambda \in \mathbb{R}$ and $T > 0$. Let $0 < \gamma < \min(4,n)$. We show that (1.6)-(1.7) is unconditionally well-posed in $C([0,T]; \dot{H}^s(\mathbb{R}^n))$ when $n \geq 3$, $0 < s < \frac{n}{2}$ and $\gamma < 2s + 2$. In Chapter 5, we study the derivative nonlinear Schrödiger equation as an unconditional well-posedness in energy space. In Chapter 6 we examine the global well-posedness of the derivative nonlinear Schrödinger equation for H^s with $\frac{1}{2} < s < 1$.

Chapter 2

Preliminary

2.1 Basic function spaces and fundamental properties

In this section we present some well-known spaces and their properties.

Let $C^\infty(= C^\infty(\mathbb{R}^n, \mathbb{C}))$ be the linear space of infinitely differentiable functions equipped with the topology induced by the semi-norms;

$$p_{m,\alpha}(f) = \sup_{x\in\mathbb{R}^n}(1+|x|^2)^{\frac{m}{2}}|D_x^\alpha f(x)|$$

is finite for all m and $\alpha = (\alpha_1, \ldots, \alpha_n)$ and D_x^α is the differential operator

$$D_x^\alpha = \frac{1}{i^{|\alpha|}}\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n \geq 0$ and $m \in \mathcal{N} \cup \{0\}$.

A C^∞ function is of class $\mathcal{S}(\mathbb{R}^n)$ if f and all its partial derivatives are rapidly decreasing. Any functions in $C_0^\infty(\mathbb{R}^n)$ belong to $\mathcal{S}(\mathbb{R}^n)$, it is a larger class of functions. We also know that $\mathcal{S}(\mathbb{R}^n)$ is a Fŕechet space and closed under differentiation. Moreover, $\mathcal{S}(\mathbb{R}^n)$ is closed under translations and multiplication by complex potentials $e^{ix\xi}$.

Let $\mathcal{S}'(\mathbb{R}^n)$ denote the set of all tempered distributions, the topological dual of $\mathcal{S}(\mathbb{R}^n)$. We set $f(g) = \langle f, g\rangle$ for $f \in \mathcal{S}'(\mathbb{R}^n),\ g \in \mathcal{S}(\mathbb{R}^n)$.

Let T and T' be the linear operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ such that the adjoint identity

$$\int (T\psi(x))g(x)dx = \int \psi(x)T'g(x)dx$$

for $\psi, g \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\langle Tf, g\rangle = \langle f, T'g\rangle \text{ for any } f \in \mathcal{S}'.$$

Here we note that in Hilbert space theory the above identity requires complex conjugates while in distribution theory we do not take complex conjugates even if the function is a complex valued function.

Let $f \in \mathcal{S}(\mathbb{R}^n)$, we define

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx, \ \xi \in \mathbb{R}^n$$

then $\mathcal{F}\cdot : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism with inverse

$$\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} e^{ix\xi} f(\xi)d\xi$$

and the identity

$$\int \mathcal{F}f \ g \ dx = \int f \ \mathcal{F}g \ dx$$

holds.

The Plancherel formula is a simple consequence of this identity. We take $f(x) = \overline{\mathcal{F}g(x)}$. We have $f(x) = \int \overline{g(\xi)} e^{ix\xi} d\xi = \mathcal{F}^{-1}\overline{g}(x)$ then $\mathcal{F}f(x) = \mathcal{F}\mathcal{F}^{-1}\overline{g} = \overline{g}(x)$. Hence the identity becomes

$$\int |g(x)|^2 dx = \int |\mathcal{F}g(x)|^2 dx.$$

Now it is turn to define the Fourier transformation

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \ \ \langle \mathcal{F}f, g\rangle = \langle f, \mathcal{F}g\rangle$$

If f is any integrable function, we could define the Fourier transform of f directly.

2.1.1 Sobolev and Besov spaces on $\mathbb{R}^n$

To introduce Sobolev spaces and Besov spaces we need some special systems of functions contained in $\mathcal{S}(\mathbb{R}^n)$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a real bump function with

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2, \end{cases}$$

where $0 \le \phi(x) \le 1$ and $\sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = 1, \ \xi \ne 0$.

Let $\phi_k, \ \psi \in \mathcal{S}(\mathbb{R}^n)$ be define by

$$\begin{aligned} \mathcal{F}\phi_k(\xi) &= \phi(2^{-k}\xi), \\ \mathcal{F}\psi(\xi) &= (1 - \sum_{k=1}^{\infty} \phi(2^{-k}\xi))\hat{f}(\xi). \end{aligned}$$

with $k \in \mathbb{Z}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $s \in \mathbb{R}$. We shall denote two operators J^s and I^s, both from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ are defined by

$$\begin{aligned} J^s f &= \mathcal{F}^{-1}\{\langle\xi\rangle^s \mathcal{F}f\}, \ \langle \mathcal{F}J^s f, \phi\rangle = \langle \mathcal{F}f, \langle\xi\rangle^s \phi\rangle \\ I^s f &= \mathcal{F}^{-1}\{|\xi|^s \mathcal{F}f\}, \ \langle \mathcal{F}I^s f, \phi\rangle = \langle \mathcal{F}f, |\xi|^s \phi\rangle, \ 0 \notin \operatorname{supp} \mathcal{F}f \end{aligned}$$

where $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$. The operators J^{-s} and I^{-s} are called the Bessel and Riesz potential of order s respectively. Then we give some simple properties of operators J^s and I^s.

Lemma 2.1.1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $1 \le p \le \infty$, $s \in \mathbb{R}$. Assume that $\phi_k * f \in L^p(\mathbb{R}^n)$, then there exist a consant $C > 0$ such that*

$$\begin{aligned} \|J^s \phi_k * f\|_{L^p} &\le C(n,s)2^{sk}\|\phi_k * f\|_{L^p(\mathbb{R}^n)} \ (k \ge 1), \\ \|I^s \phi_k * f\|_{L^p} &\le C(n,s)2^{sk}\|\phi_k * f\|_{L^p(\mathbb{R}^n)} \ (all \ \ k) \end{aligned}$$

and if $\psi * f \in L^p(\mathbb{R}^n)$

$$\|J^s \psi * f\|_{L^p} \le C(n,s)\|\psi * f\|_{L^p(\mathbb{R}^n)}.$$

Definition 2.1.2. *Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$. We define the (generalized) Sobolev space $H^s_p(\mathbb{R}^n)$ and the Besov space $B^s_{p,q}(\mathbb{R}^n)$ by*

$$\begin{aligned} H^s_p &= \{f : f \in \mathcal{S}'(\mathbb{R}^n), \|f\|_{H^s_p(\mathbb{R}^n)} < \infty\}, \\ B^s_{p,q} &= \{f : f \in \mathcal{S}'(\mathbb{R}^n), \|f\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty\} \end{aligned}$$

endowed with the norms

$$\begin{aligned} \|f\|_{H^s_p} &= \|J^s f\|_{L^p(\mathbb{R}^n)}, \\ \|f\|_{B^s_{p,q}} &= \|\psi * f\|_{L^p(\mathbb{R}^n)} + (\sum_{k=1}^{\infty}(2^{sk}\|\phi_k * f\|_{L^p(\mathbb{R}^n)})^q)^{\frac{1}{q}}. \end{aligned}$$

$H^s_p(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ are norm linear spaces with norm $\|\cdot\|_{H^s_p}$ and $\|\cdot\|_{B^s_{p,q}}$ respectively. Moreover, they are complete and therefore Banach spaces.

We denote that $A_1 \hookrightarrow A_2$ is continuous embedding , i.e there is a constant C such that

$$\|f\|_{A_2} \le C\|f\|_{A_1}, \forall f \in A_1.$$

Remark 2.1.3 (Dilation of $H^s_p(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$)**.** *Let $s > n\max(0, \frac{1}{p} - 1)$. Let $f(\cdot) \to f(\lambda\cdot)$, $\lambda > 0$. Then there exist a constant $C > 0$ which does not depend on λ and f such that*

$$\begin{aligned} \|f(\lambda\cdot)\|_{H^s_p} &\le C\lambda^{-\frac{n}{p}}\max(1,\lambda)^s\|f\|_{H^s_p(\mathbb{R}^n)}, \\ \|f(\lambda\cdot)\|_{B^s_{p,q}} &\le C\lambda^{-\frac{n}{p}}\max(1,\lambda)^s\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \end{aligned}$$

for all $f \in H^s_p(\mathbb{R}^n)$ and $f \in B^s_{p,q}(\mathbb{R}^n)$ respectively.

Theorem 2.1.4. *If $s_1 \le s_2$ and $1 \le p, q \le \infty$ we have*

$$\begin{aligned} H^{s_2}_p(\mathbb{R}^n) &\hookrightarrow H^{s_1}_p(\mathbb{R}^n) \\ B^{s_2}_{p,q}(\mathbb{R}^n) &\hookrightarrow B^{s_1}_{p,q}(\mathbb{R}^n). \end{aligned}$$

Moreover, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s_p(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ respectively.

Theorem 2.1.5 (Sobolev's embedding for $H_p^s(\mathbb{R}^n)$)**.** *Let $1 \leq p \leq p_1 < \infty$, and $s_1, s_2 \in \mathbb{R}$.*

$$H_p^s(\mathbb{R}^n) \hookrightarrow H_{p_1}^{s_1}(\mathbb{R}^n)$$

if $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$. In particular, if $1 \leq p < \infty$, and $0 < s < \frac{n}{p}$ then we have

$$H_p^s(\mathbb{R}^n) \hookrightarrow L^{\frac{pn}{n-sp}}(\mathbb{R}^n).$$

If $p > 1$ and $s > \frac{n}{p}$ then $H_p^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

Theorem 2.1.6 (Sobolev's embedding for $B_{p,q}^s$)**.** *Let $1 \leq p \leq p_1 < \infty, 1 \leq q \leq q_1 < \infty$, and $s, s_1 \in \mathbb{R}$.*

$$B_{p,q}^s(\mathbb{R}^n) \hookrightarrow B_{p_1,q_1}^{s_1}(\mathbb{R}^n)$$

if $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$.

Remark 2.1.7. *Next we give some relation between $H_p^s(\mathbb{R}^n)$ and $B_{p,q}^s(\mathbb{R}^n)$.*

(i) If $1 < p \leq 2$, then $B_{p,p}^s(\mathbb{R}^n) \hookrightarrow H_p^s(\mathbb{R}^n) \hookrightarrow B_{p,2}^s(\mathbb{R}^n)$.

(ii) If $2 < p < \infty$, then $B_{p,2}^s(\mathbb{R}^n) \hookrightarrow H_p^s(\mathbb{R}^n) \hookrightarrow B_{p,p}^s(\mathbb{R}^n)$.

(iii) In particular $B_{2,2}^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

Remark 2.1.8. *Let $s_0 < s < s_1$. There exists a constant $C > 0$ such that*

$$\begin{aligned} \|f\|_{H_p^s} &\leq C(\|f\|_{H_p^{s_1}(\mathbb{R}^n)} + \|f\|_{H_p^{s_0}(\mathbb{R}^n)}), \\ \|f\|_{B_{p,q}^s} &\leq C(\|f\|_{B_{p,q}^{s_1}(\mathbb{R}^n)} + \|f\|_{B_{p,q}^{s_0}(\mathbb{R}^n)}) \end{aligned}$$

for $f \in H_p^{s_1}(\mathbb{R}^n)$ and $f \in B_{p,q}^{s_1}(\mathbb{R}^n)$ respectively.

Theorem 2.1.9. *J^σ is an isomorphism between $H_p^s(\mathbb{R}^n)$ and $H_p^{s-\sigma}(\mathbb{R}^n)$, $B_{p,,q}^s(\mathbb{R}^n)$ and $B_{p,q}^{s-\sigma}(\mathbb{R}^n)$ respectively.*

Corollary 2.1.10 (Duality)**.** *Let $s \in \mathbb{R}$. If $1 \leq p, q < \infty$, we have*

$$(H_p^s)' = H_{p'}^{-s},$$

$$(B_{p,q}^s)' = B_{p',q'}^{-s},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

We have a several interpolations corresponding to the indices s, p, q.

Theorem 2.1.11 (Interpolation). *Let $0<\theta<1$. Assume $s_0, s, s_1 \in \mathbb{R}$, $0 \le p_0, p, p_1, q_0, q, q_1 \le \infty$ are satisfy the following formulas*

$$\begin{aligned} s &= (1-\theta)s_0 + \theta s_1, \\ \frac{1}{p} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \\ \frac{1}{q} &= \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \end{aligned}$$

Then, we have

$$(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1})_{[\theta]} = B^s_{p,q},$$

$$(H^{s_0}_{p_0}, H^{s_1}_{p_1})_{[\theta]} = H^s_p.$$

2.1.2 The Homogeneous Sobolev and Besov spaces

We now introduce the homogeneous Sobolev spaces $\dot{H}^s_p(\mathbb{R}^n)$ and the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$. The function ϕ_k is defined as a previous subsection.

Let $s \in \mathbb{R}$, $1 \le p \le \infty$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$, we define

$$\|f\|_{\dot{H}^s_p(\mathbb{R}^n)} = \left\| \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}(|\xi|^s \mathcal{F}\phi_k * f) \right\|_{L^p(\mathbb{R}^n)}$$

if the series $\sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}(|\xi|^s \mathcal{F}\phi_k * f)$ converges in $\mathcal{S}'$ to an L^p function. We note that $\dot{H}^s_p(\mathbb{R}^n)$ is a semi-normed space and $\|f\|_{\dot{H}^s_p(\mathbb{R}^n)} = 0$ if and only if f is a polynomial.

Next, $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$, We write

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} \left(2^{ks} \|\phi_k * f\|_{L^p(\mathbb{R}^n)}\right)^q \right)^{\frac{1}{q}}.$$

When the norm $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ is finite, we call $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is a homogeneous Besov space. We again note that $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is a semi-normed space and $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = 0$ if and only if $\operatorname{supp} \widehat{f} = \{0\}$, i.e. if and only if f is a polynomial.

Furthermore, for $0<s<1$ and $q<\infty$, one defines

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \sim \left\{ \int_0^\infty \left(t^{-s} \sup_{|y|\le t} \|f(\cdot - y) - f(\cdot)\|_{L^p(\mathbb{R}^n)} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}}.$$

and when $q = \infty$

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \sim \sup_{t>0} t^{-s} \sup_{|y|\le t} \|f(\cdot - y) - f(\cdot)\|_{L^p(\mathbb{R}^n)}.$$

Several results of $H^s_p(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ carry to the homogeneous spaces $\dot{H}^s_p(\mathbb{R}^n)$, $\dot{B}^s_{p,q}(\mathbb{R}^n)$. For instance,

Theorem 2.1.12. *If $s_1 < s_2$ and $1 \leq p, q \leq \infty$ we have*

$$\begin{aligned} \dot{H}_p^{s_2}(\mathbb{R}^n) &\hookrightarrow \dot{H}_p^{s_1}(\mathbb{R}^n), \\ \dot{B}_{p,q}^{s_2}(\mathbb{R}^n) &\hookrightarrow \dot{B}_{p,q}^{s_1}(\mathbb{R}^n). \end{aligned}$$

Moreover $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{H}_p^s(\mathbb{R}^n)$ and $\dot{B}_{p,q}^s(\mathbb{R}^n)$ respectively.

Theorem 2.1.13. *J^σ is an isomorphism between $\dot{H}_p^s(\mathbb{R}^n)$ and $\dot{H}_p^{s-\sigma}(\mathbb{R}^n)$, $\dot{B}_{p,,q}^s(\mathbb{R}^n)$ and $\dot{B}_{p,q}^{s-\sigma}(\mathbb{R}^n)$.*

Corollary 2.1.14 (Duality)**.** *Let $s \in \mathbb{R}$. If $1 \leq p, q < \infty$, we have*

$$\begin{aligned} (\dot{H}_p^s(\mathbb{R}^n))' &= \dot{H}_{p'}^{-s}(\mathbb{R}^n), \\ (\dot{B}_{p,q}^s(\mathbb{R}^n))' &= \dot{B}_{p',q'}^{-s}(\mathbb{R}^n), \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

2.1.3 Strichartz's estimates

In this section we present the Strichartz's estimates which are useful to study the nonlinear Schrödinger equation in the fractional order Sobolev spaces $H^s(\mathbb{R}^n)$. The estimates are introduced by Strichartz [36] as a Fourier restriction theorem.

Let consider the nonlinear Schrödinger equation

$$\begin{aligned} iu_t + \Delta u &= F(u) \\ u(0, x) &= u_0(x) \end{aligned}$$

satisfy the solution $u(t) = U(t)u_0$ when $F(u) = 0$.

It is known that the free Schrödinger evolution operator $U(t) = e^{it\Delta}$ is a unitary over L^2 but not stable on L^p spaces. We now start with dispersive inequality.

Proposition 2.1.15. *If $p \in [2, \infty]$ and $t \neq 0$, then $U(t)$ maps from $L^{p'}(\mathbb{R}^n)$ continuous to $L^p(\mathbb{R}^n)$;*

$$\|U(t)u_0\|_{L^p(\mathbb{R}^n)} \leq (4\pi|t|)^{-n(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{L^{p'}(\mathbb{R}^n)} \tag{2.1}$$

for all $u_0 \in L^{p'}(\mathbb{R}^n)$.

This decay estimate is useful for the long time theory of nonlinear Schrödinger equations, when the dimension n is large and the initial data u_0 has good integrability properties. But in many situations, the initial data is assumed in L^2 based Sobolev spaces such as $H_x^s(\mathbb{R}^n)$. For that direction, Strichartz estimates are introduced by combining the dispersive estimates with some duality arguments.

We continue with the notion of admissible pair.

Definition 2.1.16. *We say that the exponent pair (q,r) is admissible if*

$$\frac{2}{q} = N(\frac{1}{2} - \frac{1}{r})$$

where

$$2 \leq r \leq \frac{2N}{N-2}(2 \leq r \leq \infty \; if N = 1, \; 2 \leq r < \infty \; if N = 2).$$

Proposition 2.1.17 (Strichartz's estimates). *If $n \geq 1$, $s \in \mathbb{R}$, (p,q) and $(\bar{p},\bar{q})$ are admissible and $\frac{1}{r} + \frac{1}{q} = 1$, $\frac{1}{r'} + \frac{1}{q'} = 1$, then we have the homogeneous Strichartz estimate*

$$\|U(t)u_0\|_{L^q_t(L^r_x)(\mathbb{R}^{n+1})} \leq C(n,q,r)\|u_0\|_{L^2_x(\mathbb{R}^n)}, \tag{2.2}$$

the dual homogeneous Strichartz estimate

$$\left\| \int_0^t U(s)F(s)ds \right\|_{L^2_x(\mathbb{R}^n)} \leq C(n,\bar{q}',\bar{r}')\|F\|_{L^{\bar{q}'}_t L^{\bar{r}'}_x(\mathbb{R}^{n+1})}, \tag{2.3}$$

and the inhomogeneous Strichartz estimate

$$\left\| \int_0^t U(t-s)F(s)ds \right\|_{L^q_t L^r_x(\mathbb{R}^{n+1})} \leq C(n,q,r,\bar{q}',\bar{r}')\|F\|_{L^{\bar{q}'}_t L^{\bar{r}'}_x(\mathbb{R}^{n+1})}, \tag{2.4}$$

for all test function u_0, F on $\mathbb{R}^n, \mathbb{R}^{n+1}$ respectively, for all $t > 0$.

We notice that Strichartz's estimates fail in a bounded domain $\Omega \subset \mathbb{R}^n$.

2.2 Paraproduct method

In this section, we study the product of two functions with paraproduct method. This method is easier to estimate when the derivatives are involved because they identify which of the factor is high frequency and which is low frequency.

We begin with the definition of Fourier multipliers. It is an important concept of paraproduct method. For $k, k_1 \in \mathbb{Z}$, we define the Fourier multipliers

$$\begin{aligned} P_{\leq k}f(\xi) &= \mathcal{F}^{-1}\{\phi(2^{-k}\xi)\widehat{f}(\xi)\}, \\ P_{>k}f(\xi) &= \mathcal{F}^{-1}\{1 - \phi(2^{-k}\xi)\widehat{f}(\xi)\}, \\ P_k f(\xi) &= \mathcal{F}^{-1}\{(\phi(2^{-k}\xi) - \phi(2^{-k+1}\xi))\widehat{f}(\xi)\}, \end{aligned}$$

where the smooth function ϕ_k is defined in Subsection 2.1.1. Hence $P_{\leq k}$, $P_{>k}$, P_k are smoothed out projections to the region $|\xi| \leq 2^k$, $|\xi| > 2^k$, $|\xi| \sim 2^k$ respectively. By the Littlewood-Paley decomposition, we have

$$f = \sum_k P_k f.$$

In addition we shall use

$$P_{\le k}f = \sum_{k_1 \le k} P_{k_1}f;\ P_{>k}f = \sum_{k_1>k} P_{k_1}f;\ f = \sum_{k_1} P_{k_1}f$$

for all $f \in \mathcal{S}'$ with dyadic numbers 2^{k_1}.

We write the L^p norm by using the Littlewood-Paley inequality

$$\|f\|_{L^p(\mathbb{R}^n)} \sim \left\| \left(\sum_{k\ge 0} |P_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)},$$

where $1 < p < \infty$. Similarly, by Plancherel theorem, we have

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \sim \left(\sum_{k\ge 0} 2^{2ks} \|P_k f\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}}$$

and

$$\|f\|_{H^s(\mathbb{R}^n)} \sim \|P_{\le 0}f\|_{L^2(\mathbb{R}^n)} + \left(\sum_{k>0} 2^{2ks} \|P_k f\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}}.$$

We often use the Bernstein inequalities for $\mathbb{R}^n$ with $s \ge 0$. It is useful to estimate when the frequency is localized. Applying the Bernstein inequality on the low and median frequency to improve low Lebesgue integrability to high Lebesgue integrability is the best . It can be seen in Chapter 4 (Lemma 4.2.1 and Lemma 4.2.3).

Lemma 2.2.1. *Assume $s \ge 0$, $1 \le p \le q \le \infty$. Then the following inequalities hold:*

(i) $\|P_{\ge k}f\|_{L^p_x(\mathbb{R}^n)} \lesssim C(p,s,n)2^{-sk}\||\nabla|^s P_{\ge k}f\|_{L^p_x(\mathbb{R}^n)}$,

(ii) $\|P_{\le k}|\nabla|^s f\|_{L^p_x(\mathbb{R}^n)} \lesssim C(p,s,n)2^{sk}\||P_{\le k}f\|_{L^p_x(\mathbb{R}^n)}$,

(iii) $\|P_k|\nabla|^{\pm s} f\|_{L^p_x(\mathbb{R}^n)} \lesssim C(p,s,n)2^{\pm sk}\|P_k f\|_{L^p_x(\mathbb{R}^n)}$,

(iv) $\|P_{\le k}f\|_{L^q_x(\mathbb{R}^n)} \lesssim C(p,q,n)2^{nk(\frac{1}{p}-\frac{1}{q})}\|P_{\le k}f\|_{L^p_x(\mathbb{R}^n)}$,

(v) $\|P_k f\|_{L^q_x(\mathbb{R}^n)} \lesssim C(p,q,n)2^{nk(\frac{1}{p}-\frac{1}{q})}\|P_k f\|_{L^p_x(\mathbb{R}^n)}$.

Lemma 2.2.2. *For all $1 < p < \infty$, the following inequality holds*

$$\left\| \left(\sum_{k>0} |P_k f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{k>0} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}$$

This lemma is useful when we want the projection $P_{\leq k}$ and P_k to throw away. See [42] (Lecture notes 3).

Proposition 2.2.3. *If $s \geq 0$, then we have*

$$\|fg\|_{H^s_p(\mathbb{R}^n)} \leq C(s,n)\|f\|_{H^s_p(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{H^s_p(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} \tag{2.5}$$

for all $f, g \in H^s_p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. In particular, if $s \geq \frac{n}{p}$, we see that H^s_p space is closed under multiplication, i.e.

$$\|fg\|_{H^s_p(\mathbb{R}^n)} \leq C(s,n)\|f\|_{H^s_p(\mathbb{R}^n)}\|g\|_{H^s_p(\mathbb{R}^n)} \tag{2.6}$$

Proof. Fix $p = 2$. The case where $s = 0$ is trivial. For $s > 0$. By Littlewood-Paley decomposition, we write

$$\|fg\|_{H^s(\mathbb{R}^n)} \sim \|P_{\leq 0}(fg)\|_{L^2(\mathbb{R}^n)} + \left(\sum_{k>0} 2^{2ks}\|P_k(fg)\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}}. \tag{2.7}$$

We consider

$$P_k(fg) = \sum_{k',k''\in\mathbb{Z}} P_k\left((P_{k'}f)(P_{k''}g)\right) \tag{2.8}$$

It is clear that $P_{k'}f$ has Fourier support in $2^{k'-1} \leq |\xi| \leq 2^{k'+1}$ and $P_{k''}g$ has Fourier support in $2^{k''-1} \leq |\xi| \leq 2^{k''+1}$ so that $P_k((P_{k'}f)(P_{k''}g)$ has Fourier support in the sum of these two annuli. This sum needs to intersect the annulus $2^{k-1} \leq |\xi| \leq 2^{k+1}$. If $k' < k-5$ then the sum vanishes unless $k-3 < k'' < k+3$. Similarly if $k'' < k-5$ then the sum vanishes unless $k-3 < k' < k+3$. If $k' > k-5$ then the sum vanishes unless $k'-3 < k'' < k'+3$, then

$$\begin{aligned} P_k(fg) &= P_k[(P_{\leq k-5}f)\,(P_{k-3\leq\cdot\leq k+3}g) + ((P_{k-3\leq\cdot\leq k+3}f)(P_{\leq k-5}g)) \\ &\quad + ((P_{k-3\leq\cdot\leq k+3}g)(P_{k-3\leq\cdot\leq k+3}f)) + \left(\sum_{\substack{k',k''>k+5 \\ |k'-k''|\leq 3}} (P_{k'}f)(P_{k''}g)\right)]. \end{aligned}$$

The first term of (2.7) is clear for $s \geq \frac{n}{p}$ because

$$\|P_{\leq 0}(fg)\|_{L^2(\mathbb{R}^n)} \lesssim \|fg\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{H^s(\mathbb{R}^n)}\|g\|_{H^s(\mathbb{R}^n)}, \tag{2.9}$$

by Hölder and Sobolev embedding. For the second term of (2.7), we first consider the low-high frequency interaction

$$\begin{aligned} &\left\|\left(\sum_{k>0}\left|2^{sk}P_k[(P_{\leq k-5}f)(P_{k-3\leq\cdot\leq k+3}\,g)]\right|^2\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)} \qquad (2.10) \\ \lesssim &\left\|\left(\sum_{k>0}\left|2^{sk}(P_{\leq k-5}f)(P_{k-3\leq\cdot\leq k+3}\,g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

by Lemma 2.2.2.

If $|P_{\le k-5}f(x)| \lesssim M(f)$, (2.10) is bounded by

$$\|M(f)\|_{L^\infty(\mathbb{R}^n)} \left\| \left(\sum_{k>0} |2^{sk}(P_{k-3\le\cdot\le k+3}\ g)|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{H^s}$$

where $M(f)(x)$ is the Hardy-Littlewood maximal operator with the ball B centered at x and the radius r, that is

$$M(f)(x) = \sup_{r>0} \frac{1}{B(x,r)} \int_{B(x,r)} |f(y)|dy.$$

Hence we have to show $|P_{\le k-5}f(x)| \lesssim M(f)$. We start with

$$\begin{aligned}
|P_{\le k-5}f(x)| &= \frac{1}{(2\pi)^n}\left|\int \widehat{\phi}(y) f(x+\frac{y}{2^{k-5}})dy\right| \\
&= \left|\int f(y) 2^{n(k-5)}\widehat{\phi}(2^{k-5}(x-y))dy\right| \\
&\lesssim \int |f(y)| 2^{nk}(1+2^k|x-y|)^{-100n}dy \\
&\lesssim 2^{nk}\int_{B(x,2^{-k})} |f(y)|dy + \sum_{j>0} 2^{nk}2^{-100nj}\int_{B(x,2^{-k+j})}|f(y)|dy \\
&\lesssim Mf(x) + \sum_{j>0} 2^{nj}2^{-100nj}Mf(x) \lesssim Mf(x).
\end{aligned}$$

The proof of high-low and low-low frequencies interaction are same as above. We remain to consider the high-high interaction;

$$\left\| \left(\sum_{k>0} |2^{sk}P_k(\sum_{\substack{k',k''>k+5 \\ |k'-k''|\le 3}} (P_{k'}f)(P_{k''}g))|^2 \right) \right\|_{L^2(\mathbb{R}^n)}. \tag{2.11}$$

Let $k' = k+a$, $k'' = k+b$. (2.11) becomes

$$\left\| \left(\sum_{k>0} |2^{sk}P_k(\sum_{\substack{a,b>5 \\ |a-b|\le 3}} (P_{k+a}f)(P_{k+b}g))|^2 \right) \right\|_{L^2(\mathbb{R}^n)}. \tag{2.12}$$

It is difficult to handle both high-high frequencies interaction simultaneously, hence we use the triangle inequality to isolate the contribution of individual a

and b, then

$$\sum_{\substack{a,b>5\\|a-b|\le 3}} \left\| \left(\sum_{k>0} |2^{sk} P_k((P_{k+a}f)(P_{k+b}g))|^2 \right) \right\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim \sum_{\substack{a,b>5\\|a-b|\le 3}} \|Mf(\sum_{k>0} |2^{sk}(P_{k+b}\ g)|^2)^{\frac{1}{2}}\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim \sum_{\substack{a,b>5\\|a-b|\le 3}} \|Mf\|_{L^\infty(\mathbb{R}^n)} \|(\sum_{k>0} |2^{sk}(P_{k+b}\ g)|^2)^{\frac{1}{2}}\|_{L^2(\mathbb{R}^n)}$$

if $|P_{k+a}f(x)| \lesssim M(f)$.

We can estimate that

$$\begin{aligned}(\sum_{k>0} |2^{sk}(P_{k+b}\ g)|^2)^{\frac{1}{2}} &= (\sum_{k>0} 2^{-skb}|2^{s(k+b)}(P_{k+b}\ g)|^2)^{\frac{1}{2}} \\ &\lesssim 2^{-sb} \left(\sum_{k>0} |2^{sk}(P_k\ g)|^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Then (2.12) becomes

$$\left\| \left(\sum_{k>0} |2^{sk} P_k(\sum_{\substack{a,b>5\\|a-b|\le 3}} (P_{k+a}f)(P_{k+b}g))|^2 \right) \right\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{\substack{a,b>5\\|a-b|\le 3}} 2^{-s(a+b)} \|f\|_{H^s} \|g\|_{H^s}.$$

The proof is completed. $\square$

Proposition 2.2.4. *[35] Let $s \ge 0$, $1 \le p < \infty$, $s < \frac{n}{p}$ and $\frac{1}{p} - \frac{s}{n} \le \frac{1}{2}$. Let $t = \frac{n}{s+2(\frac{n}{p}-s)}$. Then there exists a constant $C > 0$ such that for all $f, g \in \dot{H}^s_p(\mathbb{R}^n)$*

$$\|fg\|_{\dot{H}^s_t(\mathbb{R}^n)} \le C\|f\|_{\dot{H}^s_p(\mathbb{R}^n)} \|g\|_{\dot{H}^s_p(\mathbb{R}^n)}. \tag{2.13}$$

2.3 Gauge transformation

In this section we discuss the gauge transformation for non periodic and periodic cases. It is known that for the Schrödinger equation with power nonlinearity, the nonlinearity is controlled by $L^p - L^q$ (Strichartz's estimate) estimate but the derivative nonlinear Schrodinger equations (DNLS) has loss of derivative in nonlinearity. It seems unlikely that $L^p - L^q$ estimate is helpful as in the power nonlinearity case. To overcome this difficulty N. Hayashi and T. Ozawa [24], [25] introduce for non-periodic and S. Herr [26] adjusts for periodic case to derive from the derivative nonlinearity to some new nonlinearity. It is one way to avoid the loss of derivative in the nonlinearity among the several ways.

2.3.1 Non-periodic case

Definition 2.3.1. *Let $\mathcal{G}$ be the nonlinear map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ by*

$$\mathcal{G}f(x) = e^{-i\int_{-\infty}^{x}|f(y)|^2dy}f(x).$$

The inverse transform of f is given by

$$\mathcal{G}^{-1}f(x) = e^{i\int_{-\infty}^{x}|f(y)|^2dy}f(x).$$

Proposition 2.3.2. *The gauge transformation is a bi-continuous operator from H^1 to H^1.*

Proof. For any $f, g \in H^1$ and $0 < \theta < 1$. By mean value theorem

$$\begin{aligned}
\mathcal{G}f(x) - \mathcal{G}g(x) &= e^{-i\int_{-\infty}^{x}|f(y)|^2dy}(f(x) - g(x)) \\
&\quad + i\left(\int_{-\infty}^{x}|g(y)|^2dy - \int_{-\infty}^{x}|f(y)|^2dy\right)g(x) \\
&\quad \partial_\theta \int_0^1 \exp\left(\theta i\int_{-\infty}^{x}|f(y)|^2dy + (1-\theta)i\int_{-\infty}^{x}|g(y)|^2dy\right)d\theta.
\end{aligned}$$

Taking L^2 norm on both sides and applying the Hölder inequality, we have

$$\begin{aligned}
\|\mathcal{G}f - \mathcal{G}g\|_{L^2} &\leq C\|f-g\|_{L^2} + \||f|^2 - |g|^2\|_{L^1}\|g\|_{L^2} \\
&\leq C\|f-g\|_{L^2}(1 + \|f\|_{L^2}^2 + \|g\|_{L^2}^2).
\end{aligned}$$

Again we consider the derivative of gauge transform,

$$\begin{aligned}
\|\partial_x(\mathcal{G}f - \mathcal{G}g)\|_{L^2} &\leq C\|\partial_x(f-g)\|_{L^2} + \|f-g\|_{L^2} \\
&\quad \times\{\|\partial_x g\|_{L^2} + \|f\|_{L^2} + \|g\|_{L^2} + (\|f\|_{L^2} + \|g\|_{L^2})^4\}.
\end{aligned}$$

Then we get

$$\|\mathcal{G}f - \mathcal{G}g\|_{H^1} \leq C\|f-g\|_{H^1}.$$

Similarly, we can prove the inverse of gauge transformation. □

2.3.2 Periodic case

We now discuss the gauge transformation for the periodic setting which is adjusted by S. Herr [26]. Before we give definition, we start with some proposition.

Proposition 2.3.3. *Let $T > 0$, $s \geq 0$. Then the following translations are continuous.*

$$\tau : \mathbb{R} \times C([-T,T], H^s(\mathbb{T})) \to C([-T,T], H^s(\mathbb{T})), \tau(h,u)(t,x) = u(t, x+ht).$$

Proof. See [26]. □

Proposition 2.3.4. *Let $u \in C([-T,T], L^2(\mathbb{T}))$, define $\mu(u) = \frac{1}{2\pi}\|u(0)\|^2_{L^2(\mathbb{T})}$. Then, for any $s \geq 0$, the following translations are homeomorphisms.*

$$\tau^{\mp} : \mathbb{R} \times C([-T,T], H^s(\mathbb{T})) \to C([-T,T], H^s(\mathbb{T})), u \mapsto \tau(\mp 2\mu(u), u).$$

Definition 2.3.5. *Let $f(x)$ be a 2π- periodic function. We define the gauge transformation such that $\mathcal{G}(f) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by*

$$\mathcal{G}(f)(x) = e^{-iJ(f)} f(x) \tag{2.14}$$

where $J(u)(x) := \frac{1}{2\pi} \int_0^{2\pi} \int_\theta^x |u(y)|^2 - \frac{1}{2\pi}\|u\|^2_{L^2} dy d\theta$.

We have seen that $Jf(x)$ is a 2π periodic function since $|f(y)|^2 - \frac{1}{2\pi}\|f(t)\|^2_{L^2}$ is zero mean value function. Hence $\mathcal{G}f(x)$ is also 2π periodic function.

Proposition 2.3.6. *[26] For $s \geq 0$, $r > 0$ and there exists $c > 0$ such that*

$$u, v \in B_r = \{u \in C([-T,T], H^s(\mathbb{T})) | \sup_{|t| \leq T} \|u(t)\|_{H^s(\mathbb{T})} \leq r\}$$

the map $\mathcal{G}$ is locally Lipschitz from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$, i.e.,

$$\|\mathcal{G}(u)(t) - \mathcal{G}(v)(t)\|_{H^s(\mathbb{T})} \leq \|u(t) - v(t)\|_{H^s(\mathbb{T})}, \ t \in [-T,T] \tag{2.15}$$

The inverse map

$$\mathcal{G}^{-1}(f) = e^{iJ(f)} f(x)$$

and $\mathcal{G}^{-1}$ also satisfies (2.15) *on B_r, hence $\mathcal{G}$ is bi-Lipschitz on bounded subsets.*

Proof. First, we consider the plus sign

$$(e^{iJ(f)} - e^{iJ(g)})h = ih(J(f) - J(g)) \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} (iJ(f))^j (iJ(g))^{k-1-j}. \tag{2.16}$$

We fix for $s > \frac{1}{2}$. Then we take the H^s norm on both sides, left hand side becomes

$$\|h\|_{H^s} \|J(f) - J(g)\|_{H^s} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} \left(c\|J(f)\|_{H^s}\right)^j \left(c\|J(g)\|_{H^s}\right)^{k-1-j}. \tag{2.17}$$

We note that

$$\Sigma_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} \left(c\|J(f)\|_{H^s}\right)^j \left(c\|J(g)\|_{H^s}\right)^{k-1-j} \leq c e^{c\|J(f)\|_{H^s} + c\|J(g)\|_{H^s}} \tag{2.18}$$

We note that $\||f|^2\|_{H^{s-1}} \leq c\||f|^2\|_{L^{\frac{1}{1-\epsilon}}} \leq c\|f\|^2_{L^{\frac{2}{1-\epsilon}}} \leq c\|f\|^2_{H^{\frac{\epsilon}{2}}}$ by Sobolev embedding. Then

$$\|J(f)\|_{H^s} \ \leq \ \||f|^2\|_{H^{s-1}} + \|f\|^2_{L^2} \leq \|f\|^2_{H^s} \tag{2.19}$$

choosing $\epsilon \leq 2s$. Similarly

$$\|J(f) - J(g)\|_{H^s} \leq c(\|f\|_{H^s} + \|g\|_{H^s})\|f - g\|_{H^s} \tag{2.20}$$

Then combining (2.16)-(2.20),

$$\|(e^{iJ(f)} - e^{iJ(g)})h\|_{H^s} \leq ce^{c\|f\|^2_{H^s}+c\|g\|^2_{H^s}}(\|f\|_{H^s}) + \|g\|_{H^s}\|f - g\|_{H^s}\|h\|_{H^s}. \tag{2.21}$$

For $s = 0$, it can be proved that

$$\begin{aligned}
\|(e^{iJ(f)} - e^{iJ(g)})h\|_{L^2} &\leq \|e^{iJ(f)} - e^{iJ(g)}\|_{L^\infty}\|h\|_{L^2} \\
&\leq \|J(f) - J(g)\|_{L^\infty}\|h\|_{L^2} \\
&\leq 2(\|f\|_{L^2} + \|g\|_{L^2})\|f - g\|_{L^2}\|h\|_{L^2}.
\end{aligned}$$

Similarly, we can prove minus sign. □

2.4 Scaling and $2\pi\lambda$-periodic functions

In this section, we prepare for Chapter 6, for the proof of global well-posedness in periodic setting. Let $u(t, x)$ be a large spacial periodic on $\mathbb{R} \times \mathbb{R}/2\pi\lambda\mathbb{Z}$. We recall that the Cauchy problem (1.1)-(1.2) is L^2 invariant under scaling $(t, x) \to \lambda^{-1/2}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$. Hence $u(t, x)$ solve (1.1)-(1.2) on $\mathbb{R} \times \mathbb{R}/2\pi\lambda\mathbb{Z}$ then

$$u^\lambda(t, x) = \lambda^{-\frac{1}{2}}u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$$

is a solution of (1.1)-(1.2).

We recall the Fourier transform of spacial periodic with a large period λ. Define $(d\xi)_\lambda$ to be the renormalized measure on $\mathbb{Z}/\lambda$:

$$\int a(\xi)(d\xi)_\lambda := \frac{1}{\lambda}\sum_{\xi\in\mathbb{Z}/\lambda} a(\xi)$$

$(d\xi)_\lambda$ is the counting measure when $\lambda = 1$ and weakly converge to the Lebesgue measure when $\lambda \to \infty$.

For a $2\pi\lambda$-periodic function f, we define the Lebesgue measure of its function

$$\|f\|_{L^p_\lambda} := \left(\int_{\mathbb{R}/2\pi\lambda\mathbb{Z}} |f|^p dx\right)^{1/p}$$

for $1 \leq p \leq \infty$ (usual modification, if $p = \infty$). The spatial Fourier transform is defined as

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}/2\pi\lambda\mathbb{Z}} e^{-ix\xi}f(x)dx, \ \forall\xi \in \mathbb{Z}/\lambda$$

and the transformation is inverted by

$$f(x) = \int e^{ix\xi}\hat{f}(\xi)(d\xi)_\lambda.$$

When we consider ∂_x^m, $m \in \mathbb{Z}$, we may doing Fourier inverse transform

$$\partial_x^m f(x) = \int e^{ix\xi}(i\xi)^m \hat{f}(\xi)(d\xi)_\lambda.$$

Moreover, the following properties hold for large period:

(i) $\|f\|_{L^2_\lambda} = \|\widehat{f}\|_{L^2((d\xi)_\lambda)}$,

(ii) $\int_{\mathbb{R}/2\pi\lambda\mathbb{Z}} f(x)\overline{g}(x)dx = \int \widehat{f}(x)\overline{\widehat{g}(x)}dx$,

(iii) $\widehat{fg}(\xi) = \widehat{f} *_\lambda \widehat{g}(\xi) = \int \widehat{f}(\xi_1)\overline{\widehat{g}(\xi - \xi_1)}(d\xi_1)_\lambda$.

2.5 Notes and references

In this chapter we present the well-known properties of function spaces and some notion which can be found in several text books.

For Section 2.1, we refer to the text books of T. Cazenave [6] (Chapter 1-2), J. Bergh -J. L$\ddot{o}$fstr$\ddot{o}$m [1] (Chapter 6) , T. Runst - W. Sickel [35] (Chapter 2-3) and R. Strichartz [37]. In Subsection 2.1.3, Strichartz estimate is introduced by R. Strichartz [36] as a Fourier restriction theorem. The end point Strichartz estimates are created by M. Keel and T. Tao [29]. The application of Strichartz estimates can be seen in several papers, for instance T. Kato [28], J. Ginibre and G. Velo [17], G. Furioli and E. Terraneo [14] etc. See also T. Tao [41] (Chapter 2, Section 2.3).

In Section 2.2, we present the paraproduct method which is based on T. Tao [41] (Chapter A, Lemma A.8) and his Lecture notes (Math 254A (Time-frequency analysis)), T. Runst and W. Sickel [35] (Chapter 4, Section 4.4.4, Theorem 1). In Sectin 2.3 we discuss the gauge transformation for both periodic and non-periodic cases and we refer to some papers such as N. Hayashi and T. Ozawa [24], [25], [34] for non-periodic and S. Herr [26] for periodic. For Section 2.4 we consult the papers of J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao [10], [12], D. De Silva, Nata$\check{s}$a Pavlovi$\acute{c}$, G. Staffilani and N. Tzirakis [13] and L. Molinet [33].

Chapter 3

The Fourier restriction space

In this chapter, we define the two parameter family spaces which come from the linear dispersive equations associated to the space time Fourier transform. We study the some fundamental properties of Fourier restriction space with respect to the Schrödinger equation of both periodic and non-periodic cases and it will be applied in Chapter 5 and 6.

3.1 Definitions and fundamental properties

Let $h(\xi) : \mathbb{R}^n \to \mathbb{R}$ be a phase function corresponding to the dispersive linear equation

$$\partial_t u = \text{Ł} u,\ u_0(0,x) = u_0$$

where $\text{Ł} = ih(\Delta/i)$. $U(t)$ denote the group of unitary opertor

$$\mathcal{F}_x U(t) u_0(\xi) = e^{-ith(\xi)} \mathcal{F}_x u_0(\xi).$$

The space-time Fourier transform of solution u will be supported in the region $\{(\tau,\xi); \tau + h(\xi) = 0\}$. We can localize the solution in time, assume that $u(t,x) = \eta(t)U(t)u_0(x)$ where $\eta \in C_0^\infty$ with $0 \le \eta \le 1$. $\eta(t) = 1$ on $|t| \le 1$, $\eta(t) = 0$ on $|t| \ge 2$. Then

$$\mathcal{F}u(\tau,\xi) = \widehat{\eta}(\tau - h(\xi))\mathcal{F}_x u_0(\xi).$$

We see that the Fourier transform of $\widehat{\eta u}$ will be focus in the region of $\{(\tau,\xi); \tau + h(\xi) = O(1)\}$.

Example 3.1.1 (The free Schrödinger equation). *Consider*

$$\partial_t u - i\Delta u = 0$$

with $h(\xi) = -\xi^2, n = 1,\ h(\xi) = -|\xi|^2, n \in \mathcal{N}$.

Example 3.1.2 (The Airy equation). *Consider*

$$\partial_t u + \partial_x^3 u = 0$$

with $h(\xi) = \xi^3, n = 1$.

Definition 3.1.3 (For $\mathbb{R}$). *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. For $s, b \in \mathbb{R}$. Let $X_{s,b}(\mathbb{R} \times \mathbb{R})$ be the closure of the Schwartz functions $\mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{R}^n)$ under norm*

$$\|u\|_{X_{s,b}^{\tau=h(\xi)}} = \|\langle \xi \rangle^s \langle \tau - h(\xi) \rangle^b \hat{u}(\tau, \xi)\|_{L^2_\tau L^2_\xi(\mathbb{R}\times\mathbb{R})}.$$

Definition 3.1.4 (For $\mathbb{T}$). *Let $s, b \in \mathbb{R}$. Let $X_{s,b}(\mathbb{R} \times \mathbb{T})$ be the closure of the Schwartz functions $\mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{T})$ with*

$$X_{s,b} := \{u \in \mathcal{S}'(\mathbb{R} \times \mathbb{T}) : u(t, x + 2\pi) = u(t, x), \|u\|_{X_{s,b}} < +\infty\}$$

under norm

$$\|u\|_{X_{s,b}^{\tau=h(k)}} = \|\langle k \rangle^s \langle \tau - h(k) \rangle^b \widehat{u}(\tau, k)\|_{L^2_\tau L^2_k(\mathbb{R}\times\mathbb{T})}.$$

We call $X_{s,b}^{\tau=h(\xi)}$ is Fourier restriction spaces, Bourgain spaces or dispersive Sobolev spaces. For any time interval I, we define the restricted space $X_{s,b}(I \times \mathbb{T})$ ($X_{s,b}(I \times \mathbb{R})$) by

$$\|u\|_{X_{s,b}(I\times\mathbb{T})} := \inf\{\|\tilde{u}\|_{X_{s,b}(\mathbb{R}\times\mathbb{T})} : \tilde{u}|_{I\times\mathbb{T}} = u\}.$$

Remark 3.1.5. *When $b = 0$, the dispersion relation $\tau = h(\xi)$ is irrelevant, $X_{s,0}$ space represent $L^2_t H^s_x$. When $h(\xi) = 0$ ($h(k) = 0$), $X_{s,b}^{\tau=h(\xi)}$ space is simply represented the product space $H^b_t \otimes H^s_x$.*

Remark 3.1.6. *There are some fundamental properties of the space $X_{s,b}^{\tau=h(\xi)}$.*

(i) $X_{s,b}^{\tau=h(\xi)} = X_{s,b}$, $X_{0,0} = L^2_\tau L^2_\xi$ *(same norms),*

(ii) $\|\bar{u}\|_{X_{s,b}^{\tau=-h(-\xi)}} = \|u\|_{X_{s,b}^{\tau=h(\xi)}}$ *(complex conjugation),*

(iii) $(X_{s,b}^{\tau=h(\xi)})' = X_{-s,-b}^{\tau=-h(-\xi)}$ *(duality),*

(iv) $X_{s_1,b_1} \hookrightarrow X_{s_2,b_2}$ *if* $s_1 \geq s_2$, $b_1 \geq b_2$ *(trivial embedding),*

(v) $(X_{s_1,b_1}, X_{s_2,b_2})_{[\theta]} = X_{s,b}$ *if* $0 < \theta < 1$, $s, s_1, s_2, b, b_1, b_2 \in \mathbb{R}$ *(interpolation).*

Remark 3.1.7 (Sobolev embedding). *We have the following well-known embedding,*

(i) if $2 \leq p < \infty, \quad b \geq \frac{1}{2} - \frac{1}{p}$; $\|u\|_{L^p_t H^s_x} \leq C\|u\|_{X_{s,b}}$,

(ii) if $2 \leq p, q < \infty, \quad b \geq \frac{1}{2} - \frac{1}{p}, \quad s \geq \frac{1}{2} - \frac{1}{q}$; $\|u\|_{L^p_t L^q_x} \leq C\|u\|_{X_{s,b}}$,

(iii) if $1 < p \leq 2, \quad b \leq \frac{1}{2} - \frac{1}{p}$; $\|u\|_{X_{s,b}} \leq C\|u\|_{L^p_t H^s_x}$,

for some constant $C > 0$.

We consider the relation between $X_{s,b}$ norms and the Schrödinger evolution operator $U(t)$, $t \in \mathbb{R}$. Define $H_{s,b}$ space under norm

$$\|f\|_{H_{s,b}} = \|\langle\xi\rangle^s\langle\tau\rangle^b \hat{f}(\xi,\tau)\|_{L^2_\tau L^2_\xi}.$$

Since

$$\begin{aligned}\hat{U}(\cdot)f(\xi,\tau) &= \int e^{-ix\xi}e^{it\tau}e^{-it\xi^2}f(\xi,\tau)dxdt \\ &= \hat{f}(\xi,\tau+\xi^2),\end{aligned}$$

we see that

$$\begin{aligned}\|U(-\cdot)f\|_{H_{s,b}} &= \|\langle\xi\rangle^s\langle\tau\rangle^b U\hat{(\cdot)}f(\xi,\tau)\|_{L^2_\tau L^2_\xi} \\ &= \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle^b \hat{f}(\xi,\tau)\|_{L^2_\tau L^2_\xi} \\ &= \|f\|_{X_{s,b}}.\end{aligned}$$

Lemma 3.1.8. *Let $s \in \mathbb{R}$. Let Y be a Banach space of on $\mathbb{R}\times\mathbb{R}^n$ with*

$$\|e^{it\tau}e^{it\partial_{xx}}f\|_Y \lesssim \|f\|_{H^s(\mathbb{R}^n)} \tag{3.1}$$

for all $f \in H^s(\mathbb{R}^n)$, $\tau \in \mathbb{R}$. Then we have

$$\|u\|_Y \lesssim \|u\|_{X_{s,b}}$$

for $b > \frac{1}{2}$.

Proof. Set $f = \mathcal{F}_t(e^{-it\partial_{xx}}u)$. We can write

$$\begin{aligned}u(t) &= e^{it\partial_{xx}}e^{-it\partial_{xx}}u(t) \\ &= ce^{it\partial_{xx}}\int e^{it\tau}\mathcal{F}_t(e^{-it\partial_{xx}}u)(\tau)d\tau \\ &= c\int e^{it\partial_{xx}}e^{it\tau}f(\tau)d\tau.\end{aligned}$$

Take Y norm, using (3.1) and the Minkowski inequality

$$\begin{aligned}\|u\|_Y &\lesssim \int \|f(\tau)\|_{H^s_x}d\tau \\ &\lesssim \left(\int \langle\tau\rangle^{2b}\|f\|^2_{H^s_x}d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

□

Consequently, we apply to $Y = C_t(\mathbb{R}, H^s_x)$, then we get the embedding $X_{s,b} \hookrightarrow C(\mathbb{R}, H^s_x)$ for all $b > \frac{1}{2}$. We use only $b = \frac{1}{2}$ on both periodic and non-periodic case, the other space is needed to ensure the embedding of $C(\mathbb{R}, H^s_x)$.

Definition 3.1.9. *We define the function space $Y_{s,b}$ which is slightly smaller than $X_{s,\frac{1}{2}}$ and it is the closure of the Schwartz function space $\mathcal{S}_{t,x}(\mathbb{R}\times\mathbb{R})$ under the norm*

$$\|u\|_{Y_{s,b}} := \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle^b\hat{u}\|_{L^1_\tau L^2_\xi}. \tag{3.2}$$

For any time interval I, we define the restricted space $Y_{s,b}(I\times\mathbb{T})$ ($Y_{s,b}(I\times\mathbb{R})$) by

$$\|u\|_{Y_{s,b}(I\times\mathbb{T})} := \inf\{\|\tilde{u}\|_{Y_{s,b}(\mathbb{R}\times\mathbb{T})} : \tilde{u}|_{I\times\mathbb{T}} = u\}.$$

Let $Z_{s,b}$ be the space with $Z_s := X_{s,\frac{1}{2}}\cap Y_{s,0}$ under norm

$$\|u\|_{Z_s} := \|u\|_{X_{s,\frac{1}{2}}} + \|u\|_{Y_{s,0}}. \tag{3.3}$$

Similarly, we define the restricted space of Z_s for any time interval I such that

$$\|u\|_{Z_s(I\times\mathbb{T})} := \inf\{\|\tilde{u}\|_{Z_s(\mathbb{R}\times\mathbb{T})} : \tilde{u}|_{I\times\mathbb{T}} = u\}.$$

Moreover, we have the embedding for any $s\in\mathbb{R}$

$$\|u\|_{C(\mathbb{R},H^s(\mathbb{T}))} \le c\|u\|_{Z^s}.$$

We note that $X_{s,b}$ space is an invariant under time localization.

Lemma 3.1.10. *Let $s,b\in\mathbb{R}$, $u\in\mathcal{S}(\mathbb{R}\times\mathbb{R})$. Let $\eta(t)$ be a Schwartz function in time. Then*

$$\|\eta(t)u\|_{X_{s,b}} \le C(\eta,b)\|u\|_{X_{s,b}}.$$

Furthermore, $-\frac{1}{2}<b'\le b<\frac{1}{2}$, then

$$\|\eta(t/T)u\|_{X_{s,b'}} \le C(\eta,b,b')T^{b-b'}\|u\|_{X_{s,b}}.$$

Proof. It is clear that

$$\begin{aligned}\|\eta(t/T)u\|_{X_{s,b'}} &= \|U(-t)\eta(t/T)u\|_{H_{s,b'}}\\ &\lesssim \|\eta(t/T)\|_{H^{b'}_t}\|U(-t)u\|_{H^s_x}\\ &\lesssim T^{b-b'}\|u\|_{X_{s,b}}.\end{aligned}$$

□

We will often use the following well-known lemma; see [15], Lemma 4.2.

Lemma 3.1.11. *Let $0\le a\le b$ such that $a+b>\frac{1}{2}$. Then*

$$\int_{\mathbb{R}}\langle y-s\rangle^{-2a}\langle y-t\rangle^{-2b}dy \lesssim \langle s-t\rangle^{1-2(a+b)} \tag{3.4}$$

either a or b is less then $\frac{1}{2}$.

3.2 Linear estimates

Let $u(t) = U(t)u_0$ be the solution of free Schrodinger equation

$$\partial_t u = i\partial_{xx} u,$$

for all $t \in [0, T]$. We have the usual Strichartz estimate

$$\|U(t)u_0\|_{L_t^q L_x^r} \le c\|u_0\|_{L_x^2}, \tag{3.5}$$

where (q, r) is the admissible pair. It can be adopted by Bourgain space setting as follows

$$\|u\|_{L_t^q L_x^r} \le c\|u\|_{X_{0,\frac{1}{2}+\epsilon}} \tag{3.6}$$

for any $\epsilon > 0$. More precisely, we have $\|u\|_{L^\infty} \le \|\hat{u}\|_{L^1}$ and the right hand side of (3.5) becomes

$$\begin{aligned} \|U(t)U(-t)u_0\|_{L_t^\infty L_x^2} &\le \|\mathcal{F}(U(-t)u_0)\|_{L_\tau^1 L_\xi^2} \\ &= \|\langle\tau\rangle^{-\frac{1}{2}-\epsilon}\langle\tau\rangle^{\frac{1}{2}+\epsilon}\hat{u}(\tau-\xi^2,\xi)\|_{L_\tau^1 L_\xi^2} \\ &\lesssim \|u\|_{X_{0,\frac{1}{2}+\epsilon}}. \end{aligned}$$

Here we shall discuss the bilinear estimates in large periodic setting. The first lemma is for L^4 Strichartz estimate which is improved by J. Bourgain [2]. See also A. Grünrock [18] for alternative method.

Lemma 3.2.1 (Large periodic case). *[18] If $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ are $2\pi\lambda$ periodic which frequencies are supported on $\{\xi_1 : |\xi_1| \sim N_1\}$ and $\{\xi_2 : |\xi_2| \sim N_2\}$ with $N_1 \sim N_2$ respectively, then*

$$\|u_1u_2\|_{L_t^2 L_x^2} \lesssim \|u_1\|_{X_{0,b}}\|u_2\|_{X_{0,b}}, \tag{3.7}$$

where $b > \frac{3}{8}$.

Proof. We have

$$u_{N_1} = \langle\xi_1\rangle^s\langle\tau_1+\xi_1^2\rangle^b\widehat{u_1}(\tau_1,\xi_1)$$

and $\|u_{N_1}\|_{L_t^2L_x^2} = \|u_1\|_{X_{s,b}}$. By the Plancherel theorem, for $\lambda > 1$, it is sufficient to show that

$$\begin{aligned} &\left\| \sum_{\xi_1\in\mathbb{Z}/\lambda} \int \frac{u_{N_1}u_{N_2}d\tau_1}{\langle\tau_1+\xi_1^2\rangle^b\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^b} \right\|_{L_\tau^2 L_\xi^2} \\ \lesssim &\sup_{(\tau,\xi)\in\mathbb{R}\times\mathbb{Z}/\lambda} \left(\sum_{\xi_1\in\mathbb{Z}/\lambda} \int \frac{d\tau_1}{\langle\tau_1+\xi_1^2\rangle^{2b}\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b}} \right)^{\frac{1}{2}} \|u_{N_1}\|_{L_t^2L_x^2}\|u_{N_2}\|_{L_t^2L_x^2}. \end{aligned}$$

We denote

$$S(\tau,\xi) = \sup_{(\tau,\xi)\in\mathbb{R}\times\mathbb{Z}/\lambda} \left(\sum_{\xi_1\in\mathbb{Z}/\lambda} \int \frac{d\tau_1}{\langle \tau_1+\xi_1^2\rangle^{2b} \langle \tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b}} \right)^{\frac{1}{2}}$$

and we have to show that $S(\tau,\xi) \lesssim 1$.

By Lemma 4.2 of [15], we see that

$$S(\tau,\xi) \lesssim \sup_{(\tau,\xi)\in\mathbb{R}\times\mathbb{Z}/\lambda} \sum_{\xi_1\in\mathbb{Z}/\lambda} \langle \tau-\xi_1^2-(\xi-\xi_1)^2\rangle^{1-4b}.$$

By [18], let $k=\xi_1-\xi_2$, $\xi=\xi_1+\xi_2$, hence $k+\xi=2\xi_1$, $k-\xi=-2\xi_2$. Then $S(\tau,\xi)$ becomes

$$\begin{aligned} S(\tau,\xi) &\lesssim \sup_{(\tau,\xi)\in\mathbb{R}\times\mathbb{Z}/\lambda} \sum_{k\in\mathbb{Z}/\lambda} \langle 4\tau+2(\xi^2+k^2)\rangle^{1-4b} \\ &\lesssim \sup_{(\tau,\xi)\in\mathbb{R}\times\mathbb{Z}/\lambda} \sum_{k\in\mathbb{Z}/\lambda} \langle k^2-|2\tau+\xi^2|\rangle^{1-4b}. \end{aligned}$$

We take $2\tau+\xi^2=x_0^2$, if $|k-x_0|<1$ or $|k+x_0|<1$, the number of k is at most 4. On the other cases, by Cauchy-Schwarz inequality,

$$\begin{aligned} S(\tau,\xi) &\lesssim 1+\left(\sum_{k\in\mathbb{Z}/\lambda} \langle k-x_0\rangle^{2(1-4b)}\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}/\lambda} \langle k+x_0\rangle^{2(1-4b)}\right)^{\frac{1}{2}} \\ &\lesssim 1+\frac{1}{\lambda} \end{aligned}$$

with $b>\frac{3}{8}$. □

Now it is turn to discuss the refinement of Strichartz estimate. It is very useful to get a good decay in improvement of global well-posedness.

Proposition 3.2.2 (The refinement of Strichartz estimate)**.** *[13] Let φ_1 and φ_2 be $2\pi\lambda$-periodic functions which frequnecies are supported on $\{\xi_1 : |\xi_1| \sim N_1\}$ and $\{\xi_2 : |\xi_2| \sim N_2\}$ with $N_1 \gg N_2$ respectively. Then*

$$\|\eta(t)(U_\lambda(t)\varphi_1)\eta(t)(U_\lambda(t)\varphi_2)\|_{L^2_{t,x}} \lesssim C(\lambda,N_1)\|\varphi_1\|_{L^2_x}\|\varphi_2\|_{L^2_x} \tag{3.8}$$

$$C(\lambda.N_1) = \begin{cases} 1 & \text{if } N_1 \le 1, \\ (\frac{1}{N_1}+\frac{1}{\lambda})^{\frac{1}{2}} & \text{if } N_1 \ge 1. \end{cases}$$

Moreover, even if $N_1 \sim N_2$ the estimate (3.8) holds when ξ_1 and ξ_2 have same sign.

It is see that

$$\|\varphi_1\varphi_2\|_{L^2_{t,x}} \lesssim C(\lambda,N_1)\|\varphi_1\|_{X_{0,\frac{1}{2}+}}\|\varphi_2\|_{X_{0,\frac{1}{2}+}}. \tag{3.9}$$

Proof. Let $\psi = \hat{\eta}$ be a positive even Schwartz function. We rewrite the left hand side of (3.8) by Plancherel,

$$\| \int_{\xi=\xi_1+\xi_2} \int_{\tau=\tau_1+\tau_2} \hat{\varphi}_1(\xi_1)\hat{\varphi}_2(\xi_2)\psi(\tau_1-\xi_1^2)\psi(\tau_1-\xi_1^2)d\psi_1(d\xi_1)_\lambda \|_{L^2_{\xi,\tau}}. \tag{3.10}$$

We estimate the τ_1 integration by

$$\int \psi(\tau_1-\xi_1^2)\psi(\tau_1-\xi_1^2)d\psi_1 = \tilde{\psi}(\tau-\xi_1^2-\xi_2^2)$$

with $\tilde{\psi}$ also Schwartz function. Substituting this into (3.10) and using Cauchy-Schwarz, we get

$$\begin{aligned} & \|(\int \tilde{\psi}(\tau-\xi_1^2-\xi_2^2))^{\frac{1}{2}}(\int \tilde{\psi}(\tau-\xi_1^2-\xi_2^2)|\hat{\varphi}(\xi_1)|^2|\hat{\varphi}(\xi_2)|^2(d\xi_1)_\lambda)^{\frac{1}{2}}\|_{L^2_{\xi,\tau}} \\ \lesssim\ & M\|\varphi_1\|_{L^2_x}\|\varphi_2\|_{L^2_x} \end{aligned}$$

by integrating in τ on $\tilde{\psi}$, using Fubini's theorem over ξ_1, ξ_2 and then Plancherel with

$$M = \| \int \psi(\tau-\xi_1^2-\xi_2^2)(d\xi_1)\|^{\frac{1}{2}}_{L^\infty_{\xi,\tau}}.$$

We estimate M as follows:

$$M \lesssim (\frac{1}{\lambda} sup_{\tau,\xi}\#S)^{\frac{1}{2}}$$

where $S = \{\xi_1 \in \frac{1}{\lambda}\mathbb{Z} : |\xi_1| \sim N_1, |\xi-\xi_1| \sim N_2, \xi^2 - 2\xi_1(\xi-\xi_1) = \tau + O(1)\}$ and $\#S$ denote the number of elements of S. When $N \lesssim 1$, then $\#S \lesssim O(\lambda)$ which implies $C(\lambda, N_1) \lesssim 1$. When $N > 1$ rename $\xi_1 = z$. Then

$$S = \{z \in \frac{1}{\lambda}\mathbb{Z} : |z| \sim N_1, |\xi - z| \sim N_2, \xi^2 - 2z(\xi - z) = \tau + O(1))\}.$$

Let z_0 be an element of S, i.e

$$|z_0| \sim N_1, |\xi - z_0| \sim N_2;\ \xi^2 - 2z_0(\xi - z_0) = \tau + O(1). \tag{3.11}$$

We shall count the number of $\bar{z}'s \in \frac{1}{\lambda}\mathbb{Z}$ such that $z_0 + \bar{z} \in S$ to obtain the upper bound of $\#S$. Then

$$|z_0+\bar{z}| \sim N_1, |z_0+\bar{z}-\xi| \sim N_2;\ \xi^2 - 2(z_0+\bar{z})^2 - 2(z_0+\bar{z})\xi = \tau + O(1). \tag{3.12}$$

We rewrite the left hand side of (3.12) as

$$\xi^2 + 2z_0^2 + 2\bar{z}^2 + 4z_0\bar{z} - 2z_0\xi - 2\bar{z}\xi = \tau + O(1) + 2\bar{z}^2 + 4z_0\bar{z} - 2\bar{z}\xi$$

by using (3.11). Since (3.10)-(3.11),

$$|\bar{z}| = |(\bar{z} + z_0 + \xi) - (z_0 - \xi)| \lesssim N_2 \ll N_1. \tag{3.13}$$

On the other hand, by (3.8), we have $|z_0-\frac{\xi}{2}|\sim N_1$. Hence it is sufficient to count $\bar{z}'s\in\frac{1}{\lambda}\mathbb{Z}$ satisfying (3.11) and such that $\bar{z}^2+2\bar{z}(z_0-\frac{\xi}{2})=)(1)$ where z_0 satisfies (3.10). Then

$$\bar{z}(\bar{z}+2(z_0-\frac{\xi}{2}))=O(1)$$
$$|\bar{z}|=\frac{O(1)}{N_1},\ \bar{z}\in\frac{\mathbb{Z}}{\lambda}$$

this implies

$$\#S\lesssim 1+\frac{\lambda}{N_1}.$$

Hence $M\lesssim(\frac{1}{N_1}+\frac{1}{\lambda})^{\frac{1}{2}}$. □

Remark 3.2.3. *We interpolate between* (3.7) *and* (3.8), *then*

$$\|u_1u_2\|_{L_t^2L_x^2}\lesssim C(\lambda,N_1)^{1-}\|u_1\|_{X_{0,\frac{1}{2}}}\|u_2\|_{X_{0,\frac{1}{2}}}. \tag{3.14}$$

Let us consider the inhomogeneous Schrödinger equation

$$i\partial_t u+\partial_{xx}u+F(u)=0, \tag{3.15}$$
$$u(0,x)=u_0(x). \tag{3.16}$$

Then the solution $u(t)$ of (3.15)-(3.16) satisfy the Duhamel formula

$$u(t)=U(t)u_0-i\int_0^t U(t-t')F(t')dt'. \tag{3.17}$$

Lemma 3.2.4. *Let $u_0\in H^s(\mathbb{R})$ for some $s\in\mathbb{R}$. Then for any Schwartz time cut off function $\eta(t)\in\mathcal{S}(\mathbb{R})$ we have*

$$\|\eta(t)U(t)u_0\|_{X_{s,b}}\lesssim C(\eta,b)\|u_0\|_{H^s(\mathbb{R})}.$$

Proof. We have

$$\|u\|_{X_{s,b}}=\|U(-t)u\|_{H^{s,b}}, \tag{3.18}$$

and $X_{s,b}$ is stable with respect to time localisation then

$$\|\eta(t)U(t)u_0\|_{X_{s,b}}\leq C(\eta,b)\|U(t)u_0\|_{X_{s,b}}=C(\eta,b)\|u_0\|_{H^s}. \tag{3.19}$$

□

Next we consider that the convolution part of (3.17) is estimated by $X_{s,b}$ norm

$$\|(U*F)\|_{X_{s,b}}\leq C\|F\|_{X_{s',b'}},$$

where $*$ is convolution in time and it is equivalent to

$$\|LF\|_{H^{s,b}}\leq C\|F\|_{H^{s',b'}}$$

with the same constant C, and $X_{s',b'}$ and $H^{s',b'}$ is the dual of $X_{s,b}$ and $H^{s,b}$ respectively. The operator L is defined by

$$(LF)=\int_0^t F(t')dt'.$$

Lemma 3.2.5. *Let $s \in \mathbb{R}$ and the following estimates are hold.*

(i) $\|LF\|_{H_t^{\frac{1}{2}}} \lesssim \|F\|_{H_t^{-\frac{1}{2}}} + \int_{|\tau|T\geq 1} |\tau|^{-1}\widehat{F}(\tau)d\tau$,

(ii) $\|(U * F)\|_{X_{s,\frac{1}{2}}} \lesssim \|F\|_{X_{s,-\frac{1}{2}}} + \left\{\int \langle\xi\rangle^{2s}(\int \langle\tau-\xi^2\rangle^{-1}\widehat{F}_+(\tau,\xi)d\tau)^2 d\xi\right\}^{\frac{1}{2}}$,

where $\widehat{F}_+(\tau,\xi) = \chi\{|\tau-\xi^2|T \geq 1\}\widehat{F}(\tau,\xi)$.

Proof. We first prove (i). There exists a positive constant C, we define

$$\int_0^t F(t')dt' = C\int_{-\infty}^{\infty} \frac{e^{it\tau}-1}{i\tau}\widehat{F}(\tau)d\tau \tag{3.20}$$

and we split $F = F_+ + F_-$ where

$$\begin{aligned} \widehat{F}_+(\tau) &= \widehat{F}(\tau)\chi(|\tau|T \geq 1), \\ \widehat{F}_-(\tau) &= \widehat{F}(\tau)\chi(|\tau|T \leq 1). \end{aligned}$$

Then (3.20) becomes

$$\psi_T \int_0^t F(t')dt' = I + II + III$$

where

$$\begin{aligned} I &= \psi_T \sum_{k=1}^{\infty} \frac{t^k}{k!} \int \widehat{F}_-(\tau)(i\tau)^{(k-1)}d\tau, \\ II &= \psi_T \mathcal{F}^{-1}(\widehat{F}_+(\tau)(i\tau)^{-1}), \\ III &= -\psi_T \int \frac{\widehat{F}_+(\tau)}{i\tau}d\tau. \end{aligned}$$

The first contribution is bounded by

$$\|I\|_{H^{\frac{1}{2}}} \lesssim \sum_{k=1}^{\infty} \left\| \langle\tau\rangle^{\frac{1}{2}} \mathcal{F}_t\left(\psi_T t^k\right) * \mathcal{F}_t\left(\int \widehat{F}_-(\tau)|\tau|^{k-1}d\tau\right)\right\|_{L^2_\tau}.$$

We apply the Young inequality,

$$\|I\|_{H^{\frac{1}{2}}} \lesssim \sum_{k=1}^{\infty} \left\|\langle\tau\rangle^{\frac{1}{2}}\widehat{\psi}_T^{(k)}(\tau)\right\|_{L^2_\tau} \left\|\mathcal{F}_t\left(\int \widehat{F}_-(\tau)|\tau|^{k-1}d\tau\right)\right\|_{L^1_\tau},$$

since the support of ψ, hence the first norm is estimated by T^k. The second norm is bounded by

$$\begin{aligned} \int |\widehat{F}_-(\tau)||\tau|^{k-1}d\tau &\lesssim T^{1-k}\|F\|_{H^{-\frac{1}{2}}} \left(\int_{|\tau|T\leq 1} \langle\tau\rangle d\tau\right)^{\frac{1}{2}} \\ &\lesssim T^{-k}\|F\|_{H^{-\frac{1}{2}}}. \end{aligned}$$

Combining them, the first contribution is bounded by

$$\|I\|_{H^{\frac{1}{2}}} \lesssim \|F\|_{H^{-\frac{1}{2}}}. \tag{3.21}$$

When we apply the $H^{\frac{1}{2}}$ norm on II, we see that

$$\begin{aligned} \|II\|_{H^{\frac{1}{2}}} &= \left\| \langle\tau\rangle^{\frac{1}{2}} \left(\mathcal{F}_t\psi_T * \mathcal{F}_t F_+(\tau)(i\tau)^{-1}\right) \right\|_{L^2_\tau} \\ &\lesssim \left\| |\tau|^{\frac{1}{2}} \widehat{\psi}_T * \widehat{F}_+(\tau)(i\tau)^{-1} \right\|_{L^2_\tau} + \left\| \widehat{\psi}_T * \langle\tau\rangle^{\frac{1}{2}} \widehat{F}_+(\tau)(i\tau)^{-1} \right\|_{L^2_\tau}. \end{aligned}$$

By using the Young inequality,

$$\begin{aligned} \|II\|_{H^{\frac{1}{2}}} &\lesssim \|\widehat{\psi}_T |\tau|^{\frac{1}{2}}\|_{L^1_\tau} \|\widehat{F}_+(\tau)|\tau|^{-1}\|_{L^2_\tau} + \|\widehat{\psi}_T\|_{L^1_\tau} \|\langle\tau\rangle^{\frac{1}{2}} \widehat{F}_+(\tau)|\tau|^{-1}\|_{L^2_\tau} \\ &\lesssim \big(\sup_{|\tau|\geq T^{-1}} (|\tau|^{-1+\frac{1}{2}}) T^{-\frac{1}{2}} + \sup_{|\tau|\geq T^{-1}} |\tau|^{\frac{1}{2}-1+\frac{1}{2}} \big) \|F\|_{H^{-\frac{1}{2}}} \\ &\lesssim \|F\|_{H^{-\frac{1}{2}}}. \end{aligned} \tag{3.22}$$

We apply the $H^{\frac{1}{2}}$ norm on III and by using the Young inequality, we get

$$\begin{aligned} \|III\|_{H^{\frac{1}{2}}} &\lesssim \|\psi_T\|_{H^{\frac{1}{2}}} \int |\tau|^{-1} \widehat{F}_+(\tau) d\tau \\ &\lesssim \int_{|\tau|T\geq 1} |\tau|^{-1} \widehat{F}(\tau) d\tau. \end{aligned} \tag{3.23}$$

Combining $H^{\frac{1}{2}}$ norm of I, II and III, we get

$$\|LF\|_{H^{\frac{1}{2}}} \lesssim \|F\|_{H^{-\frac{1}{2}}} + \int_{|\tau|T\geq 1} |\tau|^{-1} \widehat{F}(\tau) d\tau.$$

Fix ξ and multiplying each by $\langle\xi\rangle^{2s}$ and taking the L^2 norm over ξ. We obtain

$$\|\psi_T U * F\|_{H^{s,\frac{1}{2}}} \lesssim \|F\|_{H^{s,-\frac{1}{2}}} + \left\{ \int \langle\xi\rangle^{2s} \Big(\int_{|\tau|T\geq 1} |\tau|^{-1} \widehat{F}(\tau) d\tau \Big)^2 d\xi \right\}^{\frac{1}{2}}. \tag{3.24}$$

Substitute $U(-t)f$ for f , then we get (ii). $\square$

3.3 Notes and references

The Fourier restriction norm space of Schrodinger equation is introduced by J. Bourgain [2], he also concern for KdV equation. T. Tao [41] (Chapter 2, Section 2.3) is mostly support for this chapter. The refinement of Strichartz estmate for periodic Schrodinger equation is from the paper of D. De Silva, N. Pavlović, G. Staffilani and N. Tzirakis [13], and for the case of periodic KdV equation can be found in the paper of J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao [10]. We also refer to J. Ginibre, Y. Tsutsumi and G. Velo [15], A. Grünrock [21] for this chapter.

Chapter 4

Unconditional well-posedness of Hartree NLS

4.1 Introduction

In this chapter we study the Hartree nonlinear Schrödinger equation

$$\begin{aligned} i\partial_t u + \Delta u + \lambda(|x|^{-\gamma} * |u|^2)u &= 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, & (4.1) \\ u(0,x) &= u_0, & (4.2) \end{aligned}$$

where $\lambda \in \mathbb{R}$ and $T > 0$. Let $0 < \gamma < \min(4, n)$ and $s \geq 0$.

Our aim is to prove the unconditional well-posedness in $C([0,T]; \dot{H}^s(\mathbb{R}^n))$ for $0 < s < \frac{n}{2}$. The proof is based on the paraproduct method and the end point Strichartz estimates.

The concept of unconditional well-posedness of Schrödinger equation is introduced by T. Kato [28] and he proves for power nonlinearity by using Strichartz estimates and Sobolev embedding. More precisely, we consider the following equation,

$$\begin{aligned} i\partial_t u + \Delta u + |u|^\alpha u &= 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, & (4.3) \\ u(0,x) &= u_0(x) \in H^s(\mathbb{R}^n), & (4.4) \end{aligned}$$

where $\alpha > 0$. We first recall Kato's result. Let $s \geq 0$. If the following cases satisfy

(i) $s \geq \frac{n}{2}$,

(ii) $n \geq 2$, $0 \leq s < \frac{n}{2}$ and $\alpha < \min(\frac{4}{n-2s}, \frac{2s+2}{n-2s})$,

(iii) $n = 1$, $0 \leq s < \frac{1}{2}$ and $\alpha \leq \frac{1+2s}{1-2s}$,

then (4.3) - (4.4) is unconditional well-posedness in $C([0,T); H^s(\mathbb{R}^n))$. He proves the solution of (4.3) - (4.4) is unique in $L^\infty((0,T); L^2(\mathbb{R}^n)) \cap L^\infty((0,T); L^p(\mathbb{R}^n))$. Then the result is followed by Sobolev embedding with $H^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. This result was improved by G. Furioli and E. Terraneo [14] for low dimensional case. They show that if the following cases satisfy

(i) $n = 3$, $\frac{2s}{n-2s} < \alpha \leq \frac{n+2-2s}{n-2s}$ and

(ii) $3 \leq n \leq 5$, $1 < \alpha < \min\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}, \frac{2+4s}{n-2s}\}$,

(4.3) - (4.4) is unconditional well-posedness in $C([0,T); \dot{H}^s(\mathbb{R}^n))$. They prove the solution is unique in $L^\infty((0,T); \dot{B}^\sigma_{p,2}(\mathbb{R}^n))$ for $\sigma < 0$. Their proof is based on the end point Strichartz estimate, the paraproduct technique and Sobolev embedding $\dot{H}^s(\mathbb{R}^n) \subset \dot{B}^\sigma_{p,2}(\mathbb{R}^n)$. We use the same idea as in [14] on Hartree nonlinearity.

Our main theorem is as follows:

Theorem 4.1.1. *Let $n \geq 3$, $0 < s < \frac{n}{2}$, $\gamma < 2s + 2$ and $u, v \in C([0,T]; \dot{H}^s(\mathbb{R}^n))$ be two solutions of (4.1)-(4.2) with the same initial data $u_0 \in \dot{H}^s(\mathbb{R}^n)$, Then the solution is unique for all $t \in [0,T)$.*

This chapter is organized as follows. Section 4.2 contain the proof of several estimates, some are based on the paraproduct method with Besov spaces. In Section 4.3 we give the uniqueness result of our main theorem.

4.2 Lemmas

In this section, we prepare for the argument of uniqueness theorem. We start with the following lemma.

Lemma 4.2.1. *Let $u \in \dot{H}^s$ with $0 < s < \frac{n}{2}$. There exist σ, p such that $s - \frac{n}{2} = \sigma - \frac{n}{p}$ and we assume that $s + \sigma > 0$, $\sigma < 0$ and $s \geq \sigma > s - 1$. If $w \in \dot{B}^\sigma_{p,2}$ then the following estimates hold.*

(i) $\|u^2\|_{\dot{H}^s_t} \lesssim \|u\|^2_{\dot{H}^s}$ *where* $t = \frac{n}{s+2(\frac{n}{2}-s)}$,

(ii) $\|uw\|_{\dot{B}^\sigma_{q,2}} \lesssim \|u\|_{\dot{H}^s}\|w\|_{\dot{B}^\sigma_{p,2}}$ *where* $\frac{1}{q} = \frac{1}{p} + \frac{n-2s}{2n}$.

Proof. By Proposition 2.2.4, we see that (i) is clear. Now we prove (ii) by paraproduct method. We may write the product of uw by Littlewood-Paley decomposition,

$$P_k(uw) = \Sigma_{k',k''\in\mathbb{Z}} P_k((P_{k'}u)(P_{k''}w)).$$

It says that $P_{k'}u$ has Fourier support in $2^{k'-1} \leq |\xi| \leq 2^{k'+1}$ and $P_{k''}w$ has Fourier support in $2^{k''-1} \leq |\xi| \leq 2^{k''+1}$ so that $P_k((P_{k'}u)(P_{k''}w)$ has Fourier support in the sum of these two annuli. This sum needs to intersect the annulus $2^{k-1} \leq |\xi| \leq 2^{k+1}$. If $k' < k - 2$ then the sum vanishes unless $k - 1 < k'' < k + 1$. Similarly if $k'' < k - 2$ then the sum vanishes unless $k - 1 < k' < k + 1$. If $k' > k - 2$ then the sum vanishes unless $k' - 1 < k'' < k' + 1$, then

$$\begin{aligned} P_k(uw) &= P_k(P_{\leq k-2}uP_kw) + P_k(P_kuP_{\leq k-2}w) \\ &\quad + P_k\left(\sum_{\substack{k',k''>k+1 \\ |k'-k''|\leq 2}} P_{k'}uP_{k''}w\right). \end{aligned} \tag{4.5}$$

We consider the first term of (4.5). We take the L^q norm on both sides and apply Hölder's inequality,

$$\|P_k(P_{\leq k-2}uP_kw)\|_{L^q} \lesssim \|P_{\leq k-2}u\|_{L^{\frac{2n}{n-2s}}}\|P_kw\|_{L^p}.$$

where $\frac{1}{q} = \frac{1}{p} + \frac{n-2s}{2n}$. Product by $2^{k\sigma}$ and then we take l^2 norm on both sides, then

$$\begin{aligned} \{2^{k\sigma}\|P_k(P_{\leq k-2}uP_kw)\|_{L^q}\}_{l^2} &\lesssim \|P_{\leq k-2}u\|_{L^{\frac{2n}{n-2s}}}\{2^{k\sigma}\|P_kw\|_{L^p}\}_{l^2} \\ &\lesssim \|u\|_{\dot{H}^s}\|w\|_{\dot{B}^\sigma_{p,2}} \end{aligned} \tag{4.6}$$

by Sobolev embedding. Similarly, we can prove the second term of (4.5), we have

$$\|P_k(P_{\leq k-2}wP_ku)\|_{L^q} \lesssim \|P_{\leq k-2}w\|_{L^p}\|P_ku\|_{L^{\frac{2n}{n-2s}}},$$

product by $2^{k\sigma}$ and then we take the l^2 norm on both sides, then

$$\{2^{k\sigma}\|P_k(P_{\leq k-2}w.P_ku)\|_{L^q}\}_{l^2} \lesssim \{2^{(k-j)\sigma}2^{j\sigma}\|P_{\leq k-2}w\|_{L^p}\}_{l^2}\|P_ku\|_{L^{\frac{2n}{n-2s}}} \tag{4.7}$$

Here, $2^{|k-j|\sigma} \in l^1$, since $\sigma < 0$

$$\{2^{|k-j|\sigma}2^{j\sigma}\|P_{\leq k-2}w\|_{L^p}\}_{l^2} \lesssim \sum_{j\leq k-2}\{2^{j\sigma}\|P_jw\|_{L^p}\}_{l^2} \lesssim \|g\|_{\dot{B}^\sigma_{p,2}}.$$

It follows that we get the desired estimate.

Finally, we consider the third term of (4.5) using the triangle inequality and Bernstein inequality with $\frac{1}{q} \leq \frac{1}{p} + \frac{1}{2}$ with $p > 2$, then

$$\begin{aligned} \left\|P_k\left(\sum_{\substack{k',k''>k+2\\|k'-k''|\leq 1}} P_{k'}uP_{k''}w\right)\right\|_{L^q} &= \left\|P_k\left(\sum_{\substack{a,b>2\\|a-b|\leq 1}} P_{k+a}uP_{k+b}w\right)\right\|_{L^q} \\ &\leq \sum_{\substack{a,b>2\\|a-b|\leq 1}} 2^{nk\{\frac{1}{2}-\frac{n-2s}{2n}\}}\|P_{k+a}uP_{k+b}w\|_{L^{\frac{2p}{p+2}}} \\ &\leq \sum_{\substack{a,b>2\\|a-b|\leq 1}} 2^{ks}\|P_{k+a}u\|_{L^2}\|P_{k+b}w\|_{L^p}. \end{aligned}$$

Product by $2^{k\sigma}$ and then we apply the l^2 norm on both sides, then

$$\begin{aligned} \left\{2^{k\sigma}\left\|P_k\left(\sum_{\substack{k',k''>k+2\\|k'-k''|\leq 1}} P_{k'}uP_{k''}w\right)\right\|_{L^q}\right\}_{l^2} &\lesssim 2^{-sa}2^{-\sigma b}\|u\|_{\dot{H}^s}\|w\|_{\dot{B}^\sigma_{p,2}} \\ &\sim 2^{-(s+\sigma)a}\|f\|_{\dot{H}^s}\|w\|_{\dot{B}^\sigma_{p,2}} \end{aligned}$$

for $a \sim b$. Then we get the desired estimate since $s + \sigma > 0$.

□

Next we have to prove the following lemma.

Lemma 4.2.2. *We put $0 < \gamma < \min(4, n)$. Let $\sigma < 0 < s$. Let $2s < \gamma$ for (i) and $2s - \sigma < \gamma$ for (ii). If $u \in \dot{H}^s(\mathbb{R}^n)$ and $w \in \dot{B}^{\sigma}_{p,2}$, then the following estimates hold.*

(i) $\||x|^{-\gamma} * |u|^2\|_{\dot{H}^s_{\frac{n}{\gamma - s}}} \leq \|u\|^2_{\dot{H}^s}$,

(ii) $\||x|^{-\gamma} * wu\|_{\dot{B}^{\sigma}_{\frac{n}{\gamma - 2s + \sigma}}} \leq \|w\|_{\dot{B}^{\sigma}_{p,2}} \|u\|_{\dot{H}^s}$.

Proof. We first prove (i). Let $u \in \dot{H}^s$, by Lemma 4.2.1's (i) we have that $u^2 \in \dot{H}^s_{\frac{n}{n-s}}$. We put $f = |u|^2$ and $\psi_k = \phi(2^{-k}\xi) - \phi(2^{-k+1}\xi)$, then

$$\begin{aligned}
\||x|^{-\gamma} * f\|_{\dot{H}^s_{\frac{n}{\gamma - s}}} &= \|\mathcal{F}^{-1}(|\xi|^s \psi_k \mathcal{F}(|x|^{-\gamma} * f))\|_{L^{\frac{n}{\gamma - s}}} \\
&= \||x|^{-\gamma} * \mathcal{F}^{-1}(|\xi|^s \psi_k \mathcal{F} f)\|_{L^{\frac{n}{\gamma - s}}} \\
&\lesssim \|\mathcal{F}^{-1}(|\xi|^s \psi_k \mathcal{F} f)\|_{L^{\frac{n}{n-s}}} \\
&= \|f\|_{\dot{H}^s_{\frac{n}{n-s}}},
\end{aligned}$$

by using Hardy-Littlewood inequality, we get the desired estimate.

Now we prove (ii). By Lemma 4.2.1's (ii) we know that $uw \in \dot{B}^{\sigma}_{q,2}$ with $\frac{1}{q} = \frac{1}{p} + \frac{n-2s}{2n}$. We put $g = uw$, then

$$\begin{aligned}
\||x|^{-\gamma} * g\|^2_{\dot{B}^{\sigma}_{\frac{n}{\gamma - 2s + \sigma}}} &= \sum_{k=-\infty}^{+\infty} 2^{2\sigma k} \|\mathcal{F}^{-1}\{\psi_k \mathcal{F}(|x|^{-\gamma} * g)\}\|^2_{L^{\frac{n}{\gamma - 2s + \sigma}}} \\
&= \sum_{k=-\infty}^{+\infty} 2^{2\sigma k} \||x|^{-\gamma} * \mathcal{F}^{-1}\{\psi_k \mathcal{F} g\}\|^2_{L^{\frac{n}{\gamma - 2s + \sigma}}} \\
&\lesssim \sum_{k=-\infty}^{+\infty} 2^{2\sigma k} \|\mathcal{F}^{-1}\{\psi_k \mathcal{F} g\}\|^2_{L^q} \\
&= \|g\|^2_{\dot{B}^{\sigma}_{q,2}},
\end{aligned}$$

where $\frac{1}{q} = \frac{1}{p} + \frac{n-2s}{2n}$ and by using Hardy-Littlewood inequality.

□

Finally, we are ready to prove the following main estimates.

Lemma 4.2.3. *Let $u, v \in \dot{H}^s$ with $0 < s < \frac{n}{2}$ and $0 < \gamma < \min(4, n)$. There exist σ, p justifying $s - \frac{n}{2} = \sigma - \frac{n}{p}$ and the following two conditions (a) $s + \sigma > 0, \sigma < 0$, $s \geq \sigma > s - 1$, (b) $3s - \gamma \leq \sigma \leq 3s - \gamma + 1$, $\sigma \leq \frac{n}{2} + 2s - \gamma$ such that if $w \in \dot{B}^{\sigma}_{p,2}$, then the following estimates hold.*

(i) $\|(|x|^{-\gamma} * |u|^2)w\|_{\dot{B}^{\sigma}_{r',2}} \lesssim \|w\|_{\dot{B}^{\sigma}_{p,2}} \|u\|^2_{\dot{H}^s}$ *where* $\frac{1}{r'} = \frac{1}{p} + \frac{\gamma - 2s}{n}$ *and*

(ii) $\|(|x|^{-\gamma} * wu)\overline{v}\|_{\dot{B}^{\sigma}_{r',2}} \lesssim \|w\|_{\dot{B}^{\sigma}_{p,2}}\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^s}$ *where* $\frac{1}{r'} = \frac{\gamma-2s+\sigma}{n} + \frac{n-2s}{2n}$,

for $\frac{2n}{n+2} \leq r' \leq 2$.

Proof. It is enough to prove (i). We put $f = |x|^{-\gamma} * |u|^2$. By Lemma 4.2.2, $f \in \dot{H}^s_{\frac{n}{\gamma-s}}$. We may write the product of fw by Littlewood-Paley decomposition,

$$P_k(fw) = \Sigma_{k',k''\in\mathbb{Z}} P_k((P_{k'}f)(P_{k''}w)).$$

We consider that the Fourier support of f and w are same as u and w in Lemma 4.2.1 respectively, then

$$\begin{aligned} P_k(fw) &= P_k(P_{\leq k-2}fP_kw) + P_k(P_kfP_{\leq k-2}w) \\ &\quad + P_k\left(\sum_{\substack{k',k''>k+1 \\ |k'-k''|\leq 2}} P_{k'}fP_{k''}w\right). \end{aligned} \tag{4.8}$$

We first consider the first term of (4.8). We take the $L^{r'}$ norm on both sides and apply Hölder's inequality,

$$\|P_k(P_{\leq k-2}fP_kw)\|_{L^{r'}} \lesssim \|P_{\leq k-2}f\|_{L^{\frac{n}{\gamma-2s}}}\|P_kw\|_{L^p}.$$

where $\frac{1}{r'} = \frac{1}{p} + \frac{\gamma-2s}{n}$ and $3s-\gamma \leq \sigma \leq 3s-\gamma+1$ ensure that $\frac{2n}{n+2} \leq r' \leq 2$. Again, product by $2^{k\sigma}$ and then we take l^2 norm on both sides, then

$$\begin{aligned} \{2^{k\sigma}\|P_k(P_{\leq k-2}fP_kw)\|_{L^{r'}}\}_{l^2} &\lesssim \|P_{\leq k-2}f\|_{L^{\frac{n}{\gamma-2s}}}\{2^{k\sigma}\|P_kw\|_{L^p}\}_{l^2} \\ &\lesssim \|f\|_{\dot{H}^s_{\frac{n}{\gamma-s}}}\|w\|_{\dot{B}^{\sigma}_{p,2}} \end{aligned} \tag{4.9}$$

by Sobolev embedding. Similarly, we can consider the second term of (4.8), we have

$$\|P_k(P_{\leq k-2}wP_kf)\|_{L^{r'}} \lesssim \|P_{\leq k-2}w\|_{L^p}\|P_kf\|_{L^{\frac{n}{\gamma-2s}}},$$

product by $2^{k\sigma}$ and then we take the l^2 norm on both sides, then

$$\{2^{k\sigma}\|P_k(P_{\leq k-2}w.P_kf)\|_{L^{r'}}\}_{l^2} \lesssim \{2^{(k-j)\sigma}2^{j\sigma}\|P_{\leq k-2}w\|_{L^p}\}_{l^2}\|P_kf\|_{L^{\frac{n}{\gamma-2s}}} \tag{4.10}$$

Here, $2^{|k-j|\sigma} \in l^1$, since $\sigma < 0$

$$\{2^{|k-j|\sigma}2^{j\sigma}\|P_{\leq k-2}w\|_{L^p}\}_{l^2} \lesssim \sum_{j\leq k-2}\{2^{j\sigma}\|P_jw\|_{L^p}\}_{l^2} \lesssim \|g\|_{\dot{B}^{\sigma}_{p,2}}.$$

It follows that we get the desired estimate.

Finally, we consider the third term of (4.8) using the triangle inequality and Bernstein inequality with $\frac{1}{r'} \leq \frac{1}{p} + \frac{\gamma-s}{n}$ since by assumption $\sigma \leq \frac{n}{2} + 2s - \gamma$, then

$$\begin{aligned}
\left\| P_k \left(\sum_{\substack{k',k''>k+2 \\ |k'-k''|\leq 1}} P_{k'} f P_{k''} w \right) \right\|_{L^{r'}} &= \left\| P_k \left(\sum_{\substack{a,b>2 \\ |a-b|\leq 1}} P_{k+a} f P_{k+b} w \right) \right\|_{L^{r'}} \\
&\leq \sum_{\substack{a,b>2 \\ |a-b|\leq 1}} 2^{nk\{(\frac{\gamma-s}{n})+\frac{1}{p}-\frac{1}{r'}\}} \|P_{k+a} f P_{k+b} w\|_{L^{\frac{pn}{n+p(\gamma-s)}}} \\
&\leq \sum_{\substack{a,b>2 \\ |a-b|\leq 1}} 2^{ks} \|P_{k+a} f\|_{L^{\frac{n}{\gamma-s}}} \|P_{k+b} w\|_{L^p}.
\end{aligned}$$

Product by $2^{k\sigma}$ and then we apply the l^2 norm on both sides, then

$$\begin{aligned}
\left\{ 2^{k\sigma} \left\| P_k \left(\sum_{\substack{k',k''>k+2 \\ |k'-k''|\leq 1}} P_{k'} f P_{k''} w \right) \right\|_{L^{r'}} \right\}_{l^2} &\lesssim 2^{-sa} 2^{-\sigma b} \|f\|_{\dot{H}^s_{\frac{n}{\gamma-s}}} \|w\|_{\dot{B}^\sigma_{p,2}} \\
&\sim 2^{-(s+\sigma)a} \|f\|_{\dot{H}^s_{\frac{n}{\gamma-s}}} \|w\|_{\dot{B}^\sigma_{p,2}}
\end{aligned}$$

for $a \sim b$. Then we get the desired estimate since $s+\sigma>0$. □

4.3 Uniqueness Theorem

In this section, we consider the uniqueness of solution only, since the uniqueness is a main part of unconditional well-posedness problem.

Proof of Theorem 4.1.1. Let u and v be the solutions of (4.1)-(4.2) with the same initial data. By Duhamel formula, we have that

$$(u-v)(t) = \int_0^t U(t-s)[(|x|^{-\gamma} * |u|^2)u - (|x|^{-\gamma} * |v|^2)v](s)ds.$$

We put $w = u - v$. By using the Strichartz estimate, we see that

$$\begin{aligned}
\|(|x|^{-\gamma} * |u|^2)u - (|x|^{-\gamma} * |v|^2)v\|_{L^{q'}(\dot{B}^\sigma_{r',2})} &\lesssim \|(|x|^{-\gamma} * |u|^2)w\|_{L^{q'}(0,T;\dot{B}^\sigma_{r',2})} \\
&\quad + \|(|x|^{-\gamma} * \bar{w}u)v\|_{L^{q'}(0,T;\dot{B}^\sigma_{r',2})} \\
&\quad + \|(|x|^{-\gamma} * wu)\bar{v}\|_{L^{q'}(0,T;\dot{B}^\sigma_{r',2})}
\end{aligned}$$

where the pair (q,r) is admissible, and $(q,q'), (r,r')$ are dual pairs. By Lemma 4.2.3,

$$\begin{aligned}
&\|(|x|^{-\gamma} * |u|^2)u - (|x|^{-\gamma} * |v|^2)v\|_{L^{q'}(0,T;\dot{B}^\sigma_{r',2})} \\
&\lesssim T^\alpha \|w\|_{L^q(\dot{B}^\sigma_{r,2})} \left(\|u\|^2_{L^\infty(0,T;\dot{H}^s)} + 2\|u\|_{L^\infty(0,T;\dot{H}^s)} \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)
\end{aligned}$$

where $\alpha = \frac{4}{2s-\gamma+2} < +\infty$ for $\gamma < 2s+2$. Then, we conclude that

$$\|u-v\|_{L^q(0,T;\dot{B}^{\sigma}_{r,2})} \lesssim T^{\alpha}\|u-v\|_{L^q(0,T;\dot{B}^{\sigma}_{r,2})}.$$

Hence, we choose T small enough so that $u(t) = v(t)$ on $[0,T)$. $\square$

Remark 4.3.1. *If $s < \frac{n}{2}$, $n \geq 3$, the inequality*

$$\|F(u)-F(v)\|_{L^{\rho'}} \lesssim (\|u\|_{L^p}^2 + \|u\|_{L^p}^2)\|u-v\|_{L^r}$$

holds if $\delta(\rho)+\delta(r)+2\delta(p) = \gamma$, where $\delta(\rho) = n(\frac{1}{2}-\frac{1}{\rho})$ and $\frac{1}{\rho}+\frac{1}{\rho'} = 1$. Since the admissible values of ρ and r are $2 \leq r, \rho \leq \frac{2n}{n-2}$, the admissibe value of p are $\frac{2n}{n+2-\gamma} \leq p \leq \infty$. We want $\dot{H}^s \hookrightarrow L^p$, which requires $\gamma < 2s+2$.

Chapter 5

Unconditional well-posedness of DNLS

5.1 Introduction

In this chapter we consider the Cauchy problem of derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_{xx} u = \delta\partial_x(|u|^2 u) \quad \text{on } (0,T)\times\mathbb{R},\ T>0, \tag{5.1}$$
$$u(0,x) = u_0(x) \in H^1(\mathbb{R}), \tag{5.2}$$

where $u(t,x)$ is a complex valued function and $\delta \in \mathbb{R}$.

Our purpose is to prove the unconditional well-posedness in energy space. The proof is based on gauge transformation and Fourier restriction method.

We prove the unconditional well-posedness in the following sense.

Theorem 5.1.1. *Let $u_0 \in H^1(\mathbb{R})$ and $T > 0$, assume that u and v are two solutions of* (5.1)-(5.2) *in $L^\infty(0,T;H^1(\mathbb{R}))$ with the same initial data. Then $u(t) = v(t)$, $t \in [0,T]$.*

One may show the existence and uniqueness result by fix pointed method via the following equation

$$u(t) = e^{it\partial_{xx}}u_0 - i\delta\int_0^t e^{i(t-s)\partial_{xx}}(|u|^2u)_x ds.$$

It is known that in usual Schrödinger equation, nonlinearity is controlled by $L^p - L^q$ type inequality but (5.1)-(5.2) has loss of derivative in nonlinearity. It is unlikely to be used directly in the same way as usual Schrödinger equation. In [24], [25], Hayashi and Ozawa proposed the gauge transformation to overcome this difficulty. They use two gauge transformations to convert (5.1)-(5.2) to a system equation without derivative in nonlinearity and then the local well-posedness of (5.1)-(5.2) in $H^1(\mathbb{R})$ is shown.

Suppose u is a smooth solution to (5.1)-(5.2). Let

$$v(x,t) = e^{-i\int_{-\infty}^x |u(y)|^2 dy}u(x).$$

A strighfoward calculation yields, (5.1)-(5.2) becomes

$$^{i}\partial_t v + \partial_{xx} v = i\delta|v|^2\partial_x v + \delta^2|v|^4 v \quad \text{on } (0,T)\times\mathbb{R},\ T>0, \delta\in\mathbb{R} \tag{5.3}$$

$$v(0,x) = v_0(x) \in H^1. \tag{5.4}$$

In the periodic case, M. Tsutsumi and I. Fukuda [43] proved that the solution of (5.1) - (5.2) is local existence in $L^\infty(0,T;H^s)\cap C(0,T;H^{s-1})$, $s>3/2$ and it is also hold for non-periodic case. Their results was improved by N. Hayashi, T. Ozawa [25] and they proved the uniqueness in $C([0,T],H^1(\mathbb{R}))\cap L^4_{\text{loc}}([0,T],H^{1,\infty}(\mathbb{R}))$. Moreover, N. Hayashi [24] proved the uniqueness in $C([0,T], H^1(\mathbb{R}))\cap L^{12}([0,T],H^{1,3}(\mathbb{R}))$.

In the case of real line, H. Takaoka [39] uses the Fourier restriction method to handle the transformed equation containing derivative nonlinearity and shows the resulting problem is locally well-posed in $H^s(\mathbb{R})$ for $s\geq\frac{1}{2}$ and ill-posedness for $s<\frac{1}{2}$ in the sense that solution map is no longer uniformally continuous. More precisely he showed the solution of (5.1)-(5.2) is unique in $C([0,T),H^s(\mathbb{R}))\cap L^4_T(W^{s,\infty}_x)$ for $s\geq\frac{1}{2}$.

we start the following observation. let

$$\tilde{u}(t) = \begin{cases} u(t) & \text{if } t\in(0,T), \\ 0 & \text{otherwise.} \end{cases}$$

We first see that if $\tilde{u}\in L^\infty(\mathbb{R};H^1(\mathbb{R}))$ then we have $\tilde{u}\in L^2(\mathbb{R};H^1(\mathbb{R}))$, which means $\tilde{u}\in X^T_{1,0}$. On the other hand, by taking the Fourier transform on both sides of (5.1), we can see

$$(\tau+\xi^2)\mathcal{F}v(\tau,\xi) = \mathcal{F}[(|\tilde{u}|^2\tilde{u})_x]\in L^2(\mathbb{R}^2)$$

where $v(t)=\tilde{u}(t), t\in(0,T)$, i.e., $v\in X^T_{0,1}$.

By using the interpolation between above two cases, we get the solution $\tilde{u}\in X^T_{\frac{1}{2},\frac{1}{2}}$. Hence, if the solution is unique in $X_{\frac{1}{2},\frac{1}{2}}$, the problem (5.1)-(5.2) is unconditionally well-posed in H^1.

The unconditional uniqueness is a concept of uniqueness which does not depend on how to construct the solution. For example, we consider (5.1)-(5.2) with zero Drichlet boundary condition on the ball B_R centered at the origin with radius $R>0$. Let u_0 be a compactly supported function in $H^1(\mathbb{R})$. If $\|u_0\|_{L^2(B_R)}$ is small, we have a global solution $u_R\in L^\infty(\mathbb{R};H^1_0(B_R))$ such that

$$\|u(t)\|_{L^2(B_R)} = \|u_0\|_{L^2(B_R)}$$

and

$$E(u(t))\leq E(u_0)$$

where $E(u(t)) = \|\partial_x u\|^2_{L^2(B_R)} - \frac{1}{2}\text{Im}\int_{B_R} u\bar{u}u\partial_x\bar{u}dx$. Indeed, because we have by the Gagliardo-Nirenberg inequality,

$$\left|\text{Im}\int_{B_R} u\bar{u}u\partial_x\bar{u}dx\right| \leq C\|u\|^2_{L^2(B_R)}\|u\|^2_{H^1(B_R)},$$

there exists a solution $u \in L^\infty(\mathbb{R}, H_0^1(B_R))$ when $\|u_0\|_{L^2(B_R)}$ is sufficiently small. Then the passage to the limit as $R \to \infty$ leads to a solution in $L^\infty(\mathbb{R}; H^1(\mathbb{R}))$. In this case, the proof of existence does not imply that a such solution is in the auxiliary spaces.

The plan of this chapter is composed as follows. In Section 5.2, we derived the estimates of nonlinear terms on Bourgain space. In Section 5.3 we prove that the solution which comes from gauge transfotmation is unique in $X^T_{\frac{1}{2},\frac{1}{2}}$ and the solution of (5.1)-(5.2) is unique in $L^\infty(0,T;H^1)$.

5.2 Nonlinear Estimates

In this section we derive the nonlinear estimates in the framework of L^2 based Bourgain space. Let f be a nonnegative function such that

$$\begin{aligned} f_i(\tau_i,\xi_i) &= \langle \tau_i + \xi_i^2\rangle^{\frac{1}{2}}\langle \xi_i\rangle^{\frac{1}{2}}|\widehat{u}(\tau_i,\xi_i)|, \\ f_3(\tau_3,\xi_3) &= \langle \tau_3 - \xi_3^2\rangle^{\frac{1}{2}}\langle \xi_3\rangle^{\frac{1}{2}}|\widehat{u}(\tau_3,\xi_3)| \end{aligned}$$

where i=1,2 and $\tau, \xi \in \mathbb{R}$. Here the scales of time and space are different for the Schrödinger equation. Let $\tau = \Sigma_{i=1}^3 \tau_i$, $\xi = \Sigma_{i=1}^3 \xi_i$.

We consider [39],

$$\sigma - \sigma_1 - \sigma_2 - \sigma_3 = 2(\xi - \xi_1)(\xi - \xi_2), \tag{5.5}$$

where $\sigma_i = \tau_i + \xi_i^2, i = 1,2$ and $\sigma_3 = \tau_3 - \xi_3^2$. So either of the following two cases happens

$$\begin{aligned} |\xi - \xi_1| \le 1 \ &or\ |\xi - \xi_2| \le 1, \qquad (5.6)\\ |\xi - \xi_1|\ > 1 \ &and\ |\xi - \xi_2|\ > 1. \qquad (5.7) \end{aligned}$$

Lemma 5.2.1. *When $|\xi - \xi_1| \le 1$ or $|\xi - \xi_2| \le 1$ then*

$$\frac{\langle \xi\rangle^{\frac{1}{2}}\langle \xi_3\rangle}{\langle \xi_1\rangle^{\frac{1}{2}}\langle \xi_2\rangle^{\frac{1}{2}}\langle \xi_3\rangle^{\frac{1}{2}}} \le 1.$$

When $|\xi - \xi_1|\ > 1$ and $|\xi - \xi_2|\ > 1$

(i) $|\xi_1| \le |\xi|/2$ and $|\xi_2| \le |\xi|/2$,

(ii) $|\xi| << |\xi_1|$ and $|\xi| << |\xi_2|$,

then

$$\xi_3| \le |\xi| + |\xi_1| + |\xi_2| \le c(|\xi_1| + |\xi_2|)$$

If either $|\xi_1| \le |\xi_2|$ or $|\xi_2| \le |\xi_1|$ we then conclude

$$\frac{\langle \xi\rangle^{\frac{1}{2}}\langle \xi_3\rangle}{\langle \xi_1\rangle^{\frac{1}{2}}\langle \xi_2\rangle^{\frac{1}{2}}\langle \xi_3\rangle^{\frac{1}{2}}\langle \xi-\xi_1\rangle^{\frac{1}{2}}\langle \xi-\xi_2\rangle^{\frac{1}{2}}} \le \frac{c}{\langle \xi_1\rangle^{\frac{1}{2}}\langle \xi_2\rangle^{\frac{1}{2}}}.$$

Proof. In the region $|\xi - \xi_1| > 1$ and $|\xi - \xi_2| > 1$.

We first assume $|\xi_1| \leq |\xi|/2$ and $|\xi_2| \leq |\xi|/2$. Since $|\xi| \leq |\xi - \xi_1| + |\xi_1|$, $|\xi| \leq |\xi - \xi_2| + |\xi_2|$, $|\xi| \leq c|\xi - \xi_1|^{\frac{1}{2}}|\xi - \xi_2|^{\frac{1}{2}}$. In addition, if $|\xi| \leq |\xi_3|/2$, since $|\xi_3| \leq |\xi - \xi_1| + |\xi - \xi_2| + |\xi|$, we get $|\xi_3| \leq c|\xi - \xi_1|^{\frac{1}{2}}|\xi - \xi_2|^{\frac{1}{2}}$. We conclude that

$$\frac{\langle\xi\rangle^{\frac{1}{2}}\langle\xi_3\rangle}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi_3\rangle^{\frac{1}{2}}\langle\xi-\xi_1\rangle^{\frac{1}{2}}\langle\xi-\xi_2\rangle^{\frac{1}{2}}} \leq \frac{c}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}}. \tag{5.8}$$

On the other hand $|\xi| << |\xi_1|$ and $|\xi| << |\xi_2|$, we have the identity

$$\frac{1}{\xi - \xi_1} = \frac{\xi}{(\xi - \xi_1)\xi_1} - \frac{1}{\xi_1}.$$

We can show that

$$\begin{aligned} \frac{1}{|\xi - \xi_1|} &\leq \frac{|\xi|}{|\xi - \xi_1||\xi_1|} + \frac{1}{|\xi_1|} \\ &\leq \frac{|\xi|^{1/2}}{|\xi - \xi_1|^{1/2}|\xi_1|^{1/2}}\frac{1}{|\xi_1|^{1/2}} \\ \frac{1}{|\xi - \xi_1|^{1/2}} &\leq \frac{c}{|\xi_1|^{1/2}}. \end{aligned}$$

Similarly, we can show that

$$\frac{1}{|\xi - \xi_2|^{1/2}} \leq \frac{c}{|\xi_2|^{1/2}}.$$

Since

$$|\xi_3| \leq |\xi| + |\xi_1| + |\xi_2| \leq c(|\xi_1| + |\xi_2|)$$

and if either $|\xi_1| \leq |\xi_2|$ or $|\xi_2| \leq |\xi_1|$ we then conclude

$$\frac{\langle\xi\rangle^{\frac{1}{2}}\langle\xi_3\rangle}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi_3\rangle^{\frac{1}{2}}\langle\xi-\xi_1\rangle^{\frac{1}{2}}\langle\xi-\xi_2\rangle^{\frac{1}{2}}} \leq \frac{c}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}}. \tag{5.9}$$

□

Lemma 5.2.2. *Let $u(t)$ be the support on $\{t; |t| \leq T\}$ with $0 < T \leq 1$ and there exists $c, \epsilon \geq 0$. Then the following estimate holds.*

$$\|u_1\ u_2\ \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} \leq cT^{\epsilon}\|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}}\|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}}\|u_3\|_{\bar{X}_{\frac{1}{2},\frac{1}{2}}}. \tag{5.10}$$

Proof. Consider

$$\begin{aligned} &\|u_1 u_2 \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} \\ &\leq \left\| \int \cdots \int_{\mathbb{R}^4} \left(\frac{\langle\xi\rangle^{\frac{1}{2}}\langle\xi_3\rangle}{\Pi_{i=1}^3\langle\xi_i\rangle^{\frac{1}{2}}} \right) \frac{f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)}{\langle\sigma\rangle^{\frac{1}{2}}\Pi_{i=1}^3\langle\sigma_i\rangle^{\frac{1}{2}}} d\mu_1 d\mu_2 \right\|_{L^2_{\tau,\xi}} \end{aligned}$$

where $d\mu = d\tau d\xi$.

Case 1: In the region $|\xi - \xi_1| \leq 1$ or $|\xi - \xi_2| \leq 1$, we have,

$$\frac{\langle\xi\rangle^{\frac{1}{2}}\langle\xi_3\rangle}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi_3\rangle^{\frac{1}{2}}} \leq 1. \tag{5.11}$$

Then

$$\begin{aligned}\|u_1u_2\partial_x\bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq \left\|\int\cdots\int_{\mathbb{R}^4}\frac{f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)}{\langle\sigma\rangle^{\frac{1}{2}}\Pi_{i=1}^3\langle\sigma_i\rangle^{\frac{1}{2}}}d\mu_1d\mu_2\right\|_{L^2_{\tau,\xi}} \\ &= c\left\|\langle\sigma\rangle^{-\frac{1}{2}}\left(\langle\xi_1\rangle^{\frac{1}{2}}|\hat{u}_1| * \langle\xi_2\rangle^{\frac{1}{2}}|\hat{u}_2| * \langle\xi_3\rangle^{\frac{1}{2}}|\hat{\bar{u}}_3|\right)\right\|_{L^2_{\tau,\xi}}. \end{aligned}\tag{5.12}$$

By Plancherel theorem,

$$\begin{aligned}\|u_1u_2\partial_x\bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq c\left\|\langle P\rangle^{-\frac{1}{2}}\Pi_{i=1}^3D_x^{\frac{1}{2}}u_i\right\|_{L^2_{t,x}} \\ &= \left\|\Pi_{i=1}^3D_x^{\frac{1}{2}}u_i\right\|_{X_{0,-\frac{1}{2}}}\end{aligned}\tag{5.13}$$

where $P = i\partial_t + \partial_{xx}$. Then we apply the $L^4_{t,x}$ dual Strichartz estimate and Hölder inequality,

$$\begin{aligned}\left\|\Pi_{i=1}^3D_x^{\frac{1}{2}}u_i\right\|_{X_{0,-\frac{1}{2}}} &\leq c\left\|\Pi_{i=1}^3D_x^{\frac{1}{2}}u_i\right\|_{L^{\frac{4}{3}}_{t,x}} \\ &\leq c\Pi_{i=1}^3\left\|D_x^{\frac{1}{2}}u_i\right\|_{L^4_{t,x}} \\ &\leq c\|D_x^{\frac{1}{2}}u_i\|_{X_{0,\frac{3}{8}+\epsilon}}.\end{aligned}\tag{5.14}$$

We conclude that

$$\|u_1\ u_2\ \partial_x\bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} \leq cT^{\epsilon}\|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}}\|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}}\|u_3\|_{X_{\frac{1}{2},\frac{1}{2}}}. \tag{5.15}$$

Next, we consider the region $|\xi - \xi_1| \gg 1$ and $|\xi - \xi_2| \gg 1$ and we divided four subcases according to the one of $\sigma' s$ is the largest.

Subcase 1. $(\langle\sigma\rangle \geq \max\{\langle\sigma_1\rangle, \langle\sigma_2\rangle, \langle\sigma_3\rangle\})$

By Plancherel theorem, lemma 5.2.1 and Hölder inequality,

$$\begin{aligned}\|u_1u_2\partial_x\bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq c\left\|\int\cdots\int_{\mathbb{R}^4}\frac{f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\Pi_{i=1}^3\langle\sigma_i\rangle^{\frac{1}{2}}}d\mu_1d\mu_2\right\|_{L^2_{\tau,\xi}} \\ &= c\left\|u_1u_2D_x^{\frac{1}{2}}\bar{u}_3\right\|_{L^2_{t,x}} \\ &\leq c\|u_1\|_{L^8_{t,x}}\|u_2\|_{L^8_{t,x}}\|D_x^{\frac{1}{2}}\bar{u}_3\|_{L^4_{t,x}}\end{aligned}\tag{5.16}$$

This estimate follows after applying the Sobolev embedding at both space and time for u_1, u_2 as well as L^4 Strichartz estimate for u_3. We use again Sobolev embedding, we get

$$\|u_1 u_2 \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} \leq c\Pi_{i=1}^3 T^{\epsilon}\|u_i\|_{X_{\frac{1}{2},\frac{1}{2}}}. \tag{5.17}$$

Subcase 2. ($\langle\sigma_1\rangle \geq \max\{\langle\sigma\rangle, \langle\sigma_2\rangle, \langle\sigma_3\rangle\}$) By lemma 5.2.1, we have

$$\begin{aligned}
\|u_1 u_2 \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq c\left\|\int \cdots \int_{\mathbb{R}^4} \frac{f_1(\tau_1,\xi_1) f_2(\tau_2,\xi_2) f_3(\tau_3,\xi_3)}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\sigma\rangle^{\frac{1}{2}}\langle\sigma_2\rangle^{\frac{1}{2}}\langle\sigma_3\rangle^{\frac{1}{2}}} d\mu_1\, d\mu_2\right\|_{L^2_{\tau,\xi}} \\
&= c\left\|\langle\sigma\rangle^{-\frac{1}{2}}\left(\langle\sigma_1\rangle^{\frac{1}{2}}|\hat{u}_1| * |\hat{u}_2| * \langle\xi_3\rangle^{\frac{1}{2}}|\hat{\bar{u}}_3|\right)\right\|_{L^2_{\tau,\xi}}. \qquad (5.18)
\end{aligned}$$

By Plancherel theorem and Sobolev embedding in time,

$$\begin{aligned}
\|\langle P\rangle^{-\frac{1}{2}}(\langle P\rangle^{\frac{1}{2}} u_1 u_2 D_x^{\frac{1}{2}}\bar{u}_3)\|_{L^2_{t,x}} &\leq c\|\langle P\rangle^{\frac{1}{2}} u_1 u_2 D_x^{\frac{1}{2}}\bar{u}_3\|_{X_{0,-\frac{3}{8}}} \\
&\leq c\|\langle P\rangle^{\frac{1}{2}} u_1 u_2 D_x^{\frac{1}{2}}\bar{u}_3\|_{L^2_x, L^{\frac{8}{7}}_t} \\
&\leq c\|\langle P\rangle^{\frac{1}{2}} u_1\|_{L^2_t, L^8_x}\|u_2\|_{L^8_{t,x}}\|D_x^{\frac{1}{2}}\bar{u}_3\|_{L^4_{t,x}}. \qquad (5.19)
\end{aligned}$$

Applying Sobolev embedding again,

$$\begin{aligned}
\|\langle P\rangle^{\frac{1}{2}} u_1\|_{L^2_T L^8_x} &\leq c\|\langle P\rangle^{\frac{1}{2}} u_1\|_{X_{\frac{3}{8},0}} \\
&\leq c\|u_1\|_{X_{\frac{3}{8},\frac{1}{2}}}, \qquad (5.20)
\end{aligned}$$

We obtain the required estimate.

Subcase 3. ($\langle\sigma_2\rangle \geq \max\{\langle\sigma\rangle, \langle\sigma_1\rangle, \langle\sigma_3\rangle\}$). The proof is same as subcase 2.

Finally, Subcase 4. ($\langle\sigma_3\rangle \geq \max\{\langle\sigma\rangle, \langle\sigma_1\rangle, \langle\sigma_2\rangle\}$)

By lemma 5.2.1, we have

$$\begin{aligned}
\|u_1 u_2 \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq c\left\|\int \cdots \int_{\mathbb{R}^4} \frac{f_1(\tau_1,\xi_1) f_2(\tau_2,\xi_2) f_3(\tau_3,\xi_3)}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\sigma\rangle^{\frac{1}{2}}\langle\sigma_1\rangle^{\frac{1}{2}}\langle\sigma_2\rangle^{\frac{1}{2}}} d\mu_1\, d\mu_2\right\|_{L^2_{\tau,\xi}} \\
&= \|u_1 u_2 \mathcal{F}^{-1} f_3\|_{X_{0,-\frac{1}{2}}}. \qquad (5.21)
\end{aligned}$$

Applying the dual of Strichartz estimate and Hölder inequality, we get

$$\begin{aligned}
\|u_1 u_2 \mathcal{F}^{-1} f_3\|_{L^{\frac{4}{3}}_{t,x}} &\leq c\|u_1\|_{L^8_{t,x}}\|u_2\|_{L^8_{t,x}}\|\mathcal{F}^{-1} f_3\|_{L^2_{t,x}} \\
&\leq cT^{\epsilon}\Pi_{i=1}^3\|u_i\|_{X_{\frac{1}{2},\frac{1}{2}}}. \qquad (5.22)
\end{aligned}$$

□

Lemma 5.2.3. *There exists $c, \epsilon > 0$ such that for $T \in (0,1)$, then the following estimate holds.*

$$\left\{\int \langle\xi\rangle \left(\int \langle\sigma\rangle^{-1}|\widehat{F}_+(\tau,\xi)|d\tau\right)^2 d\xi\right\}^{\frac{1}{2}} \leq c\, T^{\epsilon}\, \Pi_{i=1}^3\|u_i\|_{X_{\frac{1}{2},\frac{1}{2}}}, \tag{5.23}$$

where each $u_j(t), j = 1,2,3$ has support in $(0,T)$.

Proof. We consider in the following two cases.

Case (1). In the region $|\xi-\xi_1|\le 1$ or $|\xi-\xi_2|\le 1$.

$$\left\{\int\langle\xi\rangle\left(\int\langle\sigma\rangle^{-1}|\widehat{F}_+(\tau,\xi)|d\tau\right)^2 d\xi\right\}^{\frac12}=c\left\|\int\dots\int_{\mathbb{R}^4}\frac{\langle\xi\rangle^{\frac12}|\xi_3|\Pi_{i=1}^3 f_i(\tau,\xi)}{\langle\sigma\rangle\Pi_{i=1}^3\langle\xi_i\rangle\Pi_{i=1}^3\langle\sigma_i\rangle^{\frac12}}\right\|_{L^2_\xi,L^1_\tau} \tag{5.24}$$

Applying Schwartz estimate in time, (5.24) is bounded by

$$c\|\langle\sigma\rangle^{-1+\rho}\|_{L^2_\tau}\|\langle\sigma\rangle^{-\rho}\left(\langle\xi_1\rangle^{1/2}|\hat{u}_1|*\langle\xi_2\rangle^{1/2}|\hat{u}_2|*\langle\xi_3\rangle^{1/2}|\hat{\bar{u}}_3|\right)\|_{L^2_{\xi,\tau}}. \tag{5.25}$$

where $\frac13\le\rho\le\frac12$, by case 1 of lemma 3.1, we obtain the required estimate.

Case(2). In the region of $|\xi-\xi_1|>1$ or $|\xi-\xi_2|>1$. We note that [26],

$$\langle\tau-\xi^2\rangle^{\frac12}\ge c\langle\tau_1-\xi_1^2\rangle^{\delta}\langle\tau_2-\xi_2^2\rangle^{\delta}\langle\tau_3-\xi_3^2\rangle^{\delta}\langle\xi-\xi_1\rangle^{\frac12-3\delta}\langle\xi-\xi_2\rangle^{\frac12-3\delta} \tag{5.26}$$

We can estimate multipliers as the previous lemma,

$$\frac{\langle\xi\rangle^{\frac12}|\xi_3|}{\Pi_{i=1}^3\langle\xi_i\rangle^{\frac12}\langle\xi-\xi_1\rangle^{1-3\delta}\langle\xi-\xi_2\rangle^{1-3\delta}}\le\frac{c}{\langle\xi_1\rangle^{\frac12}\langle\xi_2\rangle^{\frac12}\langle\xi\rangle^{\frac12-3\delta}\langle\xi_3\rangle^{\frac12-3\delta}}$$

Next, we consider the four subcases according to the one of $\sigma's$ is the largest.

Subcase 1. ($\langle\sigma\rangle\ge\max\{\langle\sigma_1\rangle,\langle\sigma_2\rangle,\langle\sigma_3\rangle\}$)

Hence (5.24) can be estimated by

$$\left\|\int\dots\int_{\mathbb{R}^4}\frac{f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)}{\langle\xi_1\rangle^{\frac12}\langle\xi_2\rangle^{\frac12}\langle\xi\rangle^{\frac12-3\delta}\langle\xi_3\rangle^{\frac12-3\delta}\langle\sigma_1\rangle^{\frac12+\delta}\langle\sigma_2\rangle^{\frac12+\delta}\langle\sigma_3\rangle^{\frac12+\delta}}d\mu_1\,d\mu_2\right\|_{L^2_\xi,L^1_\tau} \tag{5.27}$$

where $\delta\le\frac16$. Fix ξ and applying Young inequality for variable τ, (5.27) is bounded by

$$\left\|\int_{\mathbb{R}^2}\langle\xi\rangle^{-\frac12+3\delta}\Pi_{i=1}^2\left\|\frac{f_i}{\langle\xi_i\rangle^{\frac12}\langle\sigma_i\rangle^{\frac12+\delta}}\right\|_{L^1_\tau}\left\|\frac{f_3}{\langle\xi_3\rangle^{\frac12-3\delta}\langle\sigma_3\rangle^{\frac12+\delta}}\right\|_{L^1_\tau}d\xi_1 d\xi_2\right\|_{L^2_\xi}. \tag{5.28}$$

Since Cauchy Schwarz inequality

$$\left\|\frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\frac12+\delta}}\right\|_{L^1_\tau}\le\left\|\langle\sigma_i\rangle^{-\frac12-\epsilon}\right\|_{L^2}\left\|\frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\delta-\epsilon}}\right\|_{L^2}$$

(5.28) is bounded by

$$\left\|\langle\xi\rangle^{-\frac12+3\delta}\right\|_{L^4_\xi}\left\|\Pi_{i=1}^2\langle\xi_i\rangle^{-\frac12}\left\|\frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\delta-\epsilon}}\right\|_{L^2_\tau}\langle\xi_3\rangle^{-\frac12+3\delta}\left\|\frac{f_3(\xi_3)}{\langle\sigma_3\rangle^{\delta-\epsilon}}\right\|_{L^2_\tau}\right\|_{L^4_\xi}. \tag{5.29}$$

We use again Young inequality for ξ, (5.29) is bounded by

$$\Pi_{i=1}^{2}\left\|\langle\xi_i\rangle^{-\frac{1}{2}}\left\|\frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\delta-\epsilon}}\right\|_{L^2_\tau}\right\|_{L_\xi^{\frac{4}{3}}}\left\|\langle\xi_3\rangle^{-\frac{1}{2}+3\delta}\left\|\frac{f_3(\xi_3)}{\langle\sigma_3\rangle^{\delta-\epsilon}}\right\|_{L^2_\tau}\right\|_{L_\xi^{\frac{4}{3}}} \tag{5.30}$$

Applying Hölder inequality and δ is chosen sufficiently small, (5.30) is bounded by

$$\begin{aligned}\Pi_{i=1}^{3}\|\langle\xi_i\rangle^{-\frac{1}{2}-\epsilon}\|_{L^4_\xi}\left\|\frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\delta-\epsilon}}\right\|_{L^2_{\tau,\xi}} &\leq c\Pi_{i=1}^{3}\|\langle\xi_i\rangle^{\frac{1}{2}}\langle\sigma_i\rangle^{\frac{1}{2}-(\delta-\epsilon)}\hat{u}_i\|_{L^2_{\tau,\xi}}\\ &\leq c\Pi_{i=1}^{3}\|u_i\|_{X_{\frac{1}{2},\frac{1}{2}-\epsilon}}.\end{aligned}$$

Similarly we can prove for $\sigma_i's$ which is the largest of σ_j, $i\neq j$, where $1\leq i,j\leq 3$ by using Cauchy-Schwarz inequality for τ. The desired estimates are followed as (5.17), (5.19), (5.20) and (5.22).

□

Remark 5.2.4. *The prove of nonlinearity estimates are same as [26] and [39].*

Lemma 5.2.5. *Let $s\in\mathbb{R}$, there exists $c,\acute{\epsilon}>0$ such that for $T\in(0,1)$ then the following estimate holds.*

$$\left\|\Pi_{i=1}^{5}u_i\right\|_{X_{s,-\frac{1}{2}}}\leq c\,T^{\acute{\epsilon}}\Pi_{i=1}^{5}\|u_i\|_{X_{s,\frac{1}{2}}}, \tag{5.31}$$

where each $u_i(t), i=1,...,5$ has support in $(0,T)$.

Proof. Let $\xi=\sum_{i=1}^{5}\xi_i$ then $\langle\xi\rangle\leq\sum_{i=1}^{5}\langle\xi_i\rangle$. We can see simply

$$\left\|\Pi_{i=1}^{5}u_i\right\|_{X_{s,-\frac{1}{2}}}=\left\|\langle\xi\rangle^s\,\Pi_{i=1}^{5}u_i\right\|_{X_{0,-\frac{1}{2}}}.$$

Then

$$\begin{aligned}\left\|\langle D_x\rangle^s\,\Pi_{i=1}^{5}u_i\right\|_{X_{0,-\frac{1}{2}}} &\leq C\sum_{i=1}^{5}\|\langle D_x\rangle^s\,u_i\Pi_{k=1,i\neq k}u_k\|_{X_{0,-\frac{1}{2}}}\\ &\leq c\sum_{i=1}^{5}\left\|D_x^s u_i\Pi_{k=1,i\neq k}^{5}\,u_k\right\|_{L_{t,x}^{\frac{4}{3}}}\\ &\leq c\sum_{i=1}^{5}\|D_x^s\,u_i\|_{L^4_{t,x}}\Pi_{k=1,i\neq k}^{5}\|u_k\|_{L^8_{t,x}}\\ &\leq c\,T^{\acute{\epsilon}}\,\Pi_{i=1}^{5}\|u_i\|_{X_{s,\frac{1}{2}}}\end{aligned}$$

where we used the duality of Strichartz's estimate and the Hölder inequality. □

5.3 Uniqueness Theorem

Since the uniqueness of solution is a main part of the unconditional well-posedness, we consider the uniqueness only.

Theorem 5.3.1. *We assume that $v_0 \in H^1(\mathbb{R})$, there exists T such that $0 < T < 1$. Set $v = G(u)$. Let $v, \tilde{v} \in X^T_{\frac{1}{2},\frac{1}{2}}$ be two solutions of (5.3)-(5.4) with the same initial data in H^1. Then there exists a unique solution in $X^T_{\frac{1}{2},\frac{1}{2}}$ for all $t \in (0,T)$.*

Proof. For any $v_0 \in H^1$ and let $M > 0$ with $\|v_0\|_{H^1} \leq M$ and there exists T such that $0 < T < 1$, we want to prove that the transformation

$$v(t) = \psi_1(t)U(t)v_0(t) + i\psi_T(t)\int_0^t U(t-s)(i\delta|v|^2\partial_x v + \delta^2|v|^4 v)ds$$

is a contraction. Let $v, \tilde{v} \in X^T_{\frac{1}{2},\frac{1}{2}}$ be two solutions of (5.3)-(22)with the same initial data. Let

$$\begin{aligned}\psi_T\tilde{v}(t) &= \psi_1(t)U(t)\tilde{v}_0(t)\\ &\quad + i\psi_T(t)\int_0^t U(t-s)(i\delta\psi_T^3(s)|\tilde{v}|^2\partial_x\tilde{v} + \delta^2\psi_T^5(s)|\tilde{v}|^4\tilde{v})ds.\end{aligned}$$

Assume $\|v\|_{X^T_{\frac{1}{2},\frac{1}{2}}} = \|\psi_T\tilde{v}\|_{X^T_{\frac{1}{2},\frac{1}{2}}} \leq M$. Then for all $t \in (0,T)$ with $0 < T < 1$, applying lemma (2.1), (3.1), (3.2) and (3.3), we get

$$\|v(t) - \psi_T\tilde{v}(t)\|_{X^T_{\frac{1}{2},\frac{1}{2}}} \leq CT^{\epsilon}(M^2 + M^4)\|v(t) - \psi_T\tilde{v}(t)\|_{X^T_{\frac{1}{2},\frac{1}{2}}},$$

choose $T < \frac{1}{2C(M^2+M^4)^{\frac{1}{\epsilon}}}$ is sufficiently small, then

$$\|v(t) - \psi_T\tilde{v}(t)\|_{X^T_{\frac{1}{2},\frac{1}{2}}} \leq \frac{1}{2}\|v(t) - \psi_T\tilde{v}(t)\|_{X^T_{\frac{1}{2},\frac{1}{2}}}.$$

Hence $v(t) = \psi_T\tilde{v}(t)$, we conclude that $v(t) = \tilde{v}(t)$ for all $t \in [0,T)$. □

Proof of Theorem 5.1.1. Let $u_0 \in H^1$. We define $v_0 = G(u_0) \in H^1$, since the gauge transformation is continuous. Let $v_0^{(n)} \in C^\infty$ with $v_0^{(n)} \longrightarrow v_0$ in H^1.

We define

$$\begin{aligned}u_n(t,x) &= G^{-1}(v_n)(t,x),\\ u_0^{(n)}(t,x) &= G^{-1}(v_0^{(n)})(t,x).\end{aligned}$$

We have

$$\|u_n\|_{L_t^\infty(L_x^2)} = \|v_n\|_{L_t^\infty(L_x^2)}. \tag{5.32}$$

We can see that

$$\partial_x u_n = \left(\partial_x v_n(x) + |v_n|^2 v_n(x)\right) e^{-i\int_{-\infty}^x |v_n|^2 dy}.$$

Then

$$\|\partial_x u_n\|_{L_t^\infty(L_x^2)} \le c\|\partial_x v_n\|_{L_t^\infty(L_x^2)} + c\|v_n\|^3_{L_t^\infty(L_x^2)}.$$

By Gagliardo -Nirenberg's inequality,

$$\|\partial_x u_n\|_{L_t^\infty(L_x^2)} \le c\|\partial_x v_n\|_{L_t^\infty(L_x^2)} + c\|\partial_x v_n\|_{L_t^\infty(L_x^2)}\|v_n\|^2_{L_t^\infty(L_x^2)},$$

then

$$\|\partial_x u_n\|_{L_t^\infty(L_x^2)} \le c(1 + \|v_n\|^2_{L_t^\infty(L_x^2)})\|v_n\|_{L_t^\infty(H_x^1)}, \tag{5.33}$$

for all $v_n \in H^1$. Similarly,

$$\begin{aligned} \|u_n - u_m\|_{L_t^\infty(H_x^1)} &= \|G^{-1}(v_n) - G^{-1}(v_m)\|_{L_t^\infty(H_x^1)} \\ &\le C(M)\|v_n - v_m\|_{L_t^\infty(H_x^1)}. \end{aligned}$$

We conclude that there exists a unique solution in $L^\infty(0, T; H^1(\mathbb{R}))$ such that $u_n \to u$ in $L^\infty(0, T; H^1(\mathbb{R}))$, then we obtain the unique solution of (5.1)-(5.2). $\square$

Chapter 6

Global well-posedness of DNLS

6.1 Intorduction

In this chapter, we discuss the derivative nonlinear Schrödinger equation with periodic boundary condition:

$$
\begin{aligned}
i\partial_t u + \partial_{xx} u &= \delta \partial_x(|u|^2 u), && (6.1)\\
u(0,x) &= u_0(x), && (6.2)
\end{aligned}
$$

where $u(t,x)$ is a complex valued function of $(t,x) \in \mathbb{R} \times \mathbb{T}$, and δ=+1 or -1. For simplicity, throughout this chapter, we assume $\delta = 1$, because it makes no difference between the cases $\delta = 1$ and $\delta = -1$ in the results of this paper.

Our aim is to prove the global well-posedness of (6.1)-(6.2) in $H^s(\mathbb{T})$ for $s > \frac{1}{2}$ with small data in L^2. We use the method of almost conserved energy or I-method which was introduced by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao and the refinement of bilinear estimate.

We improve global well-posedness result in the following sense.

Theorem 6.1.1. *The initial value problem* (6.1)-(6.2) *is globally well-posed for initial data* $u_0 \in H^s(\mathbb{T})$, $s > \frac{1}{2}$ *if* $\|u_0\|_{L^2(\mathbb{T})} < \sqrt{2\pi}$.

We use the gauge transformation on (6.1) - (6.2), the new nonlinearities still contain derivative like $u^2\overline{u}_x$. We then apply the Fourier restriction method to handle the transformed equation containing derivative nonlinearity as H. Takaoka [39].

Suppose u is a smooth solution to (6.1) - (6.2). Let

$$w(t) = \mathcal{G}(u(t,x))$$

and $\mathcal{G}(u)(x) = e^{-iJ(u)}u(x)$ where

$$J(u)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x |u(y)|^2 - \frac{1}{2\pi}\|u\|_{L^2}^2 dy d\theta$$

for all time t. The strightforward calculation yields, (6.1)-(6.2) becomes

$$i\partial_t w - \partial_{xx} w - 2\mu(w)\partial_x w = -w^2\partial_x \bar{w} + \frac{i}{2}|w|^4 w - i\mu(w)|w|^2 w + i\psi(w)w \quad (6.3)$$

where

$$\psi(w)(t) = \frac{1}{2\pi}\int_0^{2\pi} 2\mathrm{Im}\,(\partial_x \bar{w} w)(t,\theta) - \frac{1}{2}|w|^4(t,\theta)d\theta + \frac{1}{4\pi^2}\|w(0)\|^4_{L^2(\mathbb{T})}$$

and $\mu(w) = \|w(t)\|_{L^2}$.

The linear term $2\mu(w)\partial_x w$ is cancelled by transformation

$$v(t,x) := w(t, x - 2\mu t) \ \ with\ \mu = \|w(0)\|_{L^2}.$$

Then

$$\begin{aligned} i\partial_t v - \partial_{xx} v &= -v^2\partial_x \bar{v} + \frac{i}{2}|v|^4 v - i\mu(v)|v|^2 v + i\psi(v)v && (6.4)\\ v(0,x) &= v_0(x) && (6.5)\end{aligned}$$

We then rewrite the equation (6.4) as follows:

$$\begin{aligned} v_t &= iv_{xx} - \left(v\bar{v}_x - \frac{i}{2\pi}\int_0^{2\pi} 2\mathrm{Im}\ v\bar{v}_x d\theta\right)\ v(x,t) && (6.6)\\ &\quad + i\left(|v|^4 - \frac{1}{4\pi^2}\int_0^{2\pi}|v|^4 d\theta\right) v(x,t) \\ &\quad - i(\int_0^{2\pi}|v|^2 d\theta)\left(|v|^2 - \frac{1}{2\pi}\int_0^{2\pi}|v|^2 d\theta\right) v(x,t) \\ v(0,x) &= v_0(x). && (6.7)\end{aligned}$$

We note that the global well-posedness of (6.1)-(6.2) in H^s is equivalent to that of (6.6)-(6.7) according to the properties of gauge transformation.

We shall write notation as $\xi_{ij} = \xi_i - \xi_j$ for i and j are even and odd integers but in some cases $\xi_{ij} = \xi_i + \xi_j$ for both i and j are even or odd. For instance, $\xi_{123} = \xi_1 - \xi_2 + \xi_3$, $\xi_{234} = \xi_2 - \xi_3 + \xi_4$, sometime we may write $\xi_{i-j} = \xi_i - \xi_j$. We also use if $m(\xi)$ is a function defined on frequency space, we write $m(\xi_i) = m_i$, $m(\xi_{ij}) = m(\xi_i - \xi_j) = m_{ij}$ for i and j are even and odd respectively.

We organized the sections as follows. In Seciton 6.2 and 6.3, we define gauge transformation, conservation of energy, operator I and construct differential equations associated with the I-method, which is called the I-system. In Section 6.4 we prove local well-posedness by I-syatem for $s > \frac{1}{2}$. In Section 6.5 we present a main proof of global well-posedness for $s > \frac{1}{2}$.

6.2 Almost conserved energy

In this section we begin with the conservation of energy but we do not here observe the potential term which is not going to make any trouble.

Definition 6.2.1. *[8] - [9] If $v \in H^1(\mathbb{T})$, the energy $E(v)$ is defined by*

$$E(v) := \int_0^{2\pi} \partial_x v \partial_x \bar{v} d\theta - \frac{1}{2} Im \int_0^{2\pi} |v|^2 v \partial_x \bar{v} d\theta. \tag{6.8}$$

Now we give the lemma that the energy $E(v)$ is strong enough to control by H^1.

Lemma 6.2.2. *Let f be a smooth and 2π- periodic function. Assume $f(t) \in H^1(\mathbb{T})$ such that $\|f(t)\|_{L^2(\mathbb{T})} < \delta$, δ is sufficiently small. Then we have*

$$\|\partial_x f(t)\|_{L^2(\mathbb{T})} \leq C(\delta) E(f)^{1/2}. \tag{6.9}$$

Proof. Let us define

$$g(x) = e^{\frac{i}{2}J(f)} f(x)$$

where $J(f)(x) = \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x \left(|f(y)|^2 - \frac{1}{2\pi}\|f\|_{L^2}^2\right) dy d\theta$. It is easily seen that $\|g\|_{L^2} = \|f\|_{L^2} < \delta$.

Since $\partial_x f(x) = e^{-\frac{i}{2}J(f)}(\partial_x g - \frac{i}{2}(|g|^2 - \mu)g)$ where $\mu = \frac{1}{2\pi}\int_0^{2\pi} |g|^2 d\theta$, the energy (6.8) becomes

$$\begin{aligned} E(f) &\geq \|\partial_x g\|_{L^2(\mathbb{T})}^2 - i\int_{\mathbb{T}} \partial_x \bar{g}(|g|^2 - \mu)g - \frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} |g|^2 g(\partial_x \bar{g} + \frac{i}{2}(|g|^2 - \mu)\bar{g}) \quad (6.10) \\ &\geq \|\partial_x g\|_{L^2}^2 - \|(|g|^2 - \mu)\bar{g}\|_{L^2}\|\partial_x g\|_{L^2} - \frac{1}{2}\|g\|_{L^6}^3(\|\partial_x g\|_{L^2} + \frac{1}{2}\|(|g|^2 - \mu)\bar{g}\|_{L^2}) \end{aligned}$$

Let $h = |g|^2 - \mu$ be a zero mean value function. By translation $x \to x + \xi$ such that $g(\xi) = 0$ for some $\xi \in [0, 2\pi]$, we have $h(x) = \int_0^{x+\xi} h'(y)dy$ and $h(x) = \int_{x+\xi}^{2\pi} h'(y)dy$. Applying Hölder inequality, we obtain

$$\|h\|_{L^\infty(\mathbb{T})} \leq \|h'\|_{L^1} \leq \|gg'\|_{L^1} \leq \|g\|_{L^2}\|g'\|_{L^2}.$$

Similarly,

$$\|(|g|^2 - \mu)\bar{g}\|_{L^2(\mathbb{T})} \leq \|g\|_{L^2}\ \||g|^2 - \mu\|_{L^\infty} \leq \|g\|_{L^2}^2\|g'\|_{L^2}.$$

By Gaglirado-Nirenberg inequality and (6.10) becomes

$$E(f) \geq \|\partial_x g\|_{L^2}^2\ (1 - \frac{3}{2}\|g\|_{L^2}^2 - \frac{1}{4}\|g\|_{L^2}^4).$$

Thank to the small data $\|g\|_{L^2} < \delta$, we get

$$\|\partial_x g(t)\|_{L^2(\mathbb{T})} \leq C(\|g\|_{L^2}) E(f)^{1/2}. \tag{6.11}$$

On the other hand, we may show that

$$\begin{aligned} \|\partial_x f\|_{L^2(\mathbb{T})} &\le \|J(f)_x g\|_{L^2(\mathbb{T})} + \|\partial_x g\|_{L^2(\mathbb{T})} \\ &\le \|(|g|^2 - \mu) g\|_{L^2(\mathbb{T})} + \|\partial_x g\|_{L^2(\mathbb{T})} \\ &\le (1 + \|g\|_{L^2}^2)\|\partial_x g\|_{L^2(\mathbb{T})} \end{aligned} \tag{6.12}$$

Combining (6.11) and (6.12), we obtain (6.9). □

We give the definition of multiplier operator $\mathcal{I}$ which is the identity on low frequencies and like as a fractional integral operator of order $1-s$ on high frequencies. We then construct differential equation via (6.6) - (6.7). We also study modefied energies introduced in [8] - [9] giving the name as generation of modefied energies.

Let $m(\xi)$ be an arbitrary real valued 1- multiplier and defined by

$$m(\xi) = \begin{cases} 1 & \text{if } |\xi| < N, \\ (\frac{|\xi|}{N})^{s-1} & \text{if } |\xi| > 2N. \end{cases} \tag{6.13}$$

Define the multiplier operator $\mathcal{I} : H^s \to H^1$ such that $\widehat{\mathcal{I}v}(\xi) := m(\xi)\hat{v}(\xi)$. Then we see that

$$\|v(t)\|_{H^s} \lesssim \|Iv(t)\|_{H^1} \lesssim N^{1-s}\|v(t)\|_{H^s}.$$

By Plancherel, we may rewrite the equation (6.8) by Λ notation

$$E(v) := -\Lambda_2(\xi_1\xi_2; v) + \frac{1}{4}\Lambda_4(\xi_{13}; v).$$

Here we note that $E(v)$ could not be directly controlled the H^1 norm like as Lemma 6.2.2. Hence we define the substitute energy which have a very slow increment in time (in term of N) such that

$$E_N(v) := E(Iv).$$

This energy make sense even if v is only in H^s. Let $m(\xi)$ be a symmetric multiplier. Let I be the multiplier operator associated with $m(\xi)^2$. We define

$$E^1(v) := E(Iv).$$

If $m(\xi)$ is the multiplier in (6.13), then

$$E^1(v) = E_N(v).$$

We then define the $E^1(v)$ is a first generation of family of modified energy such that

$$E^1(v) := -\Lambda_2(m_1\xi_1 m_2\xi_2, v(t)) + \frac{1}{4}\Lambda_4((\xi_1+\xi_3)m_1m_2m_3m_4, v(t)). \tag{6.14}$$

Proposition 6.2.3. *Let v be a solution of (6.6) - (6.7) and let M_n be a n-multiplier of order $n \geq 2$. Then*

$$\begin{aligned}
\frac{d}{dt}\Lambda_n(M_n; v(t)) &= -i\Lambda_n(\alpha_n M_n; v(t)) \qquad (6.15)\\
&\quad +i\Lambda_{n+2}\left(\sum_{j=1}^{n}(-1)^j X_j^2(M_n)\xi_{j+1}\phi(\xi_j - \xi_{j+1}); v(t)\right)\\
&\quad -ic\Lambda_{n+2}\left(\sum_{j=1}^{n}(-1)^{j-1} X_j^2(M_n); v(t)\right)\\
&\quad +i\Lambda_{n+4}\left(\sum_{j=1}^{n}(-1)^{j-1} X_j^4(M_n); v(t)\right)
\end{aligned}$$

where $\alpha_n = \Sigma_{j=1}^{n}(-1)^{j+1}\xi_j^2$ and the characteristic function

$$\varphi(\xi) = \begin{cases} 1 & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases} \qquad (6.16)$$

Proof. Fix $n = 2$. We consider only nonlinear term containing derivative which play main role.

$$\begin{aligned}
\frac{d}{dt}|v|^2 &= v\overline{v_t} + \bar{v}v_t\\
&= -i(v\bar{v}_{xx} - \bar{v}v_{xx}) - (v_x\bar{v} - \frac{i}{2\pi}\int_{\mathbb{T}} \operatorname{Im} v_x\bar{v}dx)|v|^2\\
&\quad -(\bar{v_x}v - \frac{i}{2\pi}\int_{\mathbb{T}} \operatorname{Im} \bar{v_x}vdx)|v|^2 \qquad (6.17)\\
&\quad +\frac{i}{2\pi}\int_{\mathbb{T}} 4\operatorname{Im} \bar{v_x}vdx|v|^2
\end{aligned}$$

We take Fourier coefficient in spatial variable of (6.17), the first term is cancelled. We may write by symmetry

$$\begin{aligned}
v_x\bar{v}v - \frac{i}{2\pi}\int_{\mathbb{T}} \operatorname{Im} v_x\bar{v}d\theta v &= \int_{\xi_1-\xi_2+\xi_3-\xi_4=0} i\varphi(\xi_{12})\varphi(\xi_{14})\xi_1\hat{v}(\xi_1)\overline{\hat{v}(\xi_2)}\hat{v}(\xi_3)\\
&\quad +i\xi_1\hat{v}(\xi_1)\bar{\hat{v}}(\xi_1)\hat{v}(\xi_1)
\end{aligned}$$

and we consider the last term of right hand side of (6.17) as a constant.

Hence we get

$$\frac{d}{dt}\int_{\xi}|\hat{v}(\xi)|^2 = -i\int_{\xi_1-\xi_2+\xi_3-\xi_4=0}(\xi_1 - \xi_2)\varphi(\xi_{12})\varphi(\xi_{14})\hat{v}(\xi_1)\overline{\hat{v}(\xi_2)}\hat{v}(\xi_3)\overline{\hat{v}(\xi_4)},$$

implies

$$\frac{d}{dt}\Lambda_2(1; v) = -i\varphi(\xi_{12})\varphi(\xi_{14})\Lambda_4((\xi_1 - \xi_2); v).$$

On the other hand,

$$\frac{d}{dt}\int_{\xi_1-\xi_2=0} m(\xi_1)m(\xi_2)\hat{v}(\xi_1)\overline{\hat{v}(\xi_2)}$$
$$= -i\int_{\substack{\xi_1-\xi_2+\xi_3-\xi_4=0\\ \xi_1-\xi_2\neq 0,\xi_1-\xi_4\neq 0}} (\xi_1-\xi_2)m(\xi_1)\dots m(\xi_4)\hat{v}(\xi_1)\overline{\hat{v}(\xi_2)}\hat{v}(\xi_3)\overline{\hat{v}(\xi_4)}.$$

Using Λ notation with I-system and appropriate summetry,

$$\frac{d}{dt}\Lambda_2(1;Iv) = i\varphi(\xi_{12})\varphi(\xi_{14})\Lambda_4(m_{123}m_4\xi_2 - m_1m_{234}\xi_3;v) \tag{6.18}$$

We also omit $\int_{\mathbb{T}}|v|^4dx$ and $\int_{\mathbb{T}}|v|^2dx$ as constants for the other nonlinear terms. In the rest of this paper we shall drop $v(t)$ from Λ notation. $\square$

Applying proposition 6.2.3, differentating (6.14) with respect to time, using the identity $\xi_1-\xi_2+\xi_3-\ldots+(-1)^{n-1}\xi_n=0$, we have

$$\begin{aligned}
\frac{d}{dt}\Lambda_2(m_1\xi_1m_2\xi_2) &= -i\Lambda_2(m_1\xi_1m_2\xi_2(\xi_1^2-\xi_2^2))\\
&\quad -i\varphi(\xi_{12})\varphi(\xi_{14})\Lambda_4(m_1\xi_1m_{234}\xi_{234}\xi_3 - m_4\xi_4m_{123}\xi_{123}\xi_2)_{\mathrm{sym}}\\
&\quad -ic\Lambda_4(m_{123}\xi_{123}m_4\xi_4 - m_1\xi_1m_{234}\xi_{234})_{\mathrm{sym}}\\
&\quad +i\Lambda_6(m_{12345}\xi_{12345}m_6\xi_6 - m_1\xi_1m_{23456}\xi_{23456})_{\mathrm{sym}}\\
&= -i\varphi(\xi_{12})\varphi(\xi_{14})\Lambda_4(m_1^2\xi_1^2\xi_3 - m_4^2\xi_4^2\xi_2)_{\mathrm{sym}}\\
&\quad +ic\Lambda_4(m_1^2\xi_1^2 - m_4^2\xi_4^2)_{\mathrm{sym}} - i\Lambda_6(m_1^2\xi_1^2 - m_6^2\xi_6^2)_{\mathrm{sym}}
\end{aligned}$$

Note that in this derivative, Λ_2 is zero because the factor $\xi_1^2-\xi_2^2=0$ over the set $\xi_1-\xi_2=0$.

Next, we differentiate the contribution of Λ_4, by using proposition (6.2.3)

$$\begin{aligned}
\frac{d}{dt}\Lambda_4(\xi_{13-24}m_1m_2m_3m_4) &= -\frac{1}{8}i\Lambda_4(\alpha_4m_1m_2m_3m_4\xi_{13-24})_{\mathrm{sym}}\\
&\quad -i\Lambda_6(\varphi(\xi_{12})\varphi(\xi_{23})m_1m_2m_3m_{456}\xi_{13-2456}\xi_5\\
&\quad -\varphi(\xi_{12})\varphi(\xi_{16})m_1m_2m_{345}m_6\xi_{1345-26}\xi_4)_{\mathrm{sym}}\\
&\quad +ic\Lambda_6(m_{123}m_4m_5m_6\xi_{1235-46} - m_1m_2m_3m_{456}\xi_{13-2456})_{\mathrm{sym}}\\
&\quad -i\Lambda_8(m_{12345}m_6m_7m_8\xi_{12357-68}\\
&\quad -m_1m_2m_3m_{45678}\xi_{13-245678})_{\mathrm{sym}},
\end{aligned}$$

where $\alpha_4=\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2$. To cancel the first Λ_4 terms of $\frac{d}{dt}\Lambda_2$ and $\frac{d}{dt}\Lambda_4$, we choose

$$m_1m_2m_3m_4\xi_{13-24} = -\frac{(m_1^2\xi_1^2\xi_3 - m_4^2\xi_4^2\xi_2)_{sym}}{\alpha_4}.$$

with $\alpha_4 \neq 0$. We may define the second generation of modified energy

$$E^2(v(t)) = -\Lambda_2(m_1\xi_1 m_2\xi_2) + \frac{1}{8}\Lambda_4(M_4(\xi_1,\xi_2,\xi_3,\xi_4)) \tag{6.19}$$

where

$$M_4(\xi_1,\xi_2,\xi_3,\xi_4) = -\frac{m_1^2\xi_1^2\xi_3 - m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 - m_4^2\xi_4^2\xi_2}{\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2}.$$

In low frequency case, $|\xi_1|, |\xi_2|, |\xi_3|, |\xi_4| \ll 1$, each of m_i is 1. we have

$$\begin{aligned} M_4(\xi_1,\xi_2,\xi_3,\xi_4) &= -\frac{(\xi_1+\xi_3)(\xi_1\xi_3 - \xi_2\xi_4)}{2(\xi_1-\xi_2)(\xi_1-\xi_4)} \\ &= -\frac{(\xi_1+\xi_3)(\xi_1-\xi_2)(\xi_3-\xi_2)}{2(\xi_1-\xi_2)(\xi_1-\xi_4)} \\ &= \frac{1}{2}(\xi_1+\xi_3) \end{aligned}$$

over the set $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0$ and the second modified energy still satisfies (6.14). Now we differentiate the second modified energy with respect to time by using proposition 6.2.3 and (6.19), we get

$$\begin{aligned} \frac{d}{dt}E^2(v(t)) &= ic\Lambda_4(\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4)) \\ &\quad + i\Lambda_6[M_6'(\xi_1,\xi_2,...,\xi_6) + M_6''(\xi_1,\xi_2,...,\xi_6) - \sigma_6(\xi_1,\xi_2,...,\xi_6)] \\ &\quad - i\Lambda_8(M_8(\xi_1,\xi_2,...,\xi_8)). \end{aligned}$$

where

$$\sigma_n(\xi_1,\xi_2,\ldots,\xi_n) = (m_1^2\xi_1^2 - m_2^2\xi_2^2 + \ldots - (-1)^{n-1}m_n^2\xi_n^2), n = 4, 6,$$

$$\begin{aligned} M_6'(\xi_1,\xi_2,...,\xi_6) &= \sum_* \Lambda_6(\varphi(\xi_e-\xi_f)\varphi(\xi_e-\xi_d)M_4(\xi_{abc},\xi_d,\xi_e,\xi_f)\xi_b \\ &\quad + \varphi(\xi_e-\xi_f)\varphi(\xi_f-\xi_a)M_4(\xi_a,\xi_{bcd},\xi_e,\xi_f)\xi_c \\ &\quad + \varphi(\xi_a-\xi_b)\varphi(\xi_a-\xi_f)M_4(\xi_a,\xi_b,\xi_{cde},\xi_f)\xi_d \\ &\quad + \varphi(\xi_a-\xi_b)\varphi(\xi_b-\xi_c)M_4(\xi_a,\xi_b,\xi_c,\xi_{def})\xi_e), \end{aligned}$$

$$\begin{aligned} M_6''(\xi_1,\xi_2,...,\xi_6) &= \sum_* \Lambda_6(M_4(\xi_{abc},\xi_d,\xi_e,\xi_f) - M_4(\xi_a,\xi_{bcd},\xi_e,\xi_f) + M_4(\xi_a,\xi_b,\xi_{cde},\xi_f) \\ &\quad - M_4(\xi_a,\xi_b,\xi_c,\xi_{def})) \end{aligned}$$

and

$$\begin{aligned} M_8(\xi_1,\xi_2,...,\xi_8) &= \sum_{**} \Lambda_8(M_4(\xi_{abcde},\xi_f,\xi_g,\xi_h) - M_4(\xi_a,\xi_{bcdef},\xi_g,\xi_h)) \\ &\quad - M_4(\xi_a,\xi_b,\xi_{cdefg},\xi_h) + M_4(\xi_a,\xi_b,\xi_c,\xi_{defgh}) \end{aligned}$$

where $\sum_* = \sum_{\substack{\{a,c,e\}=\{1,3,5\} \\ \{b,d,f\}=\{2,4,6\}}}$, $\sum_{**} = \sum_{\substack{\{a,c,e,g\}=\{1,3,5,7\} \\ \{b,d,f,h\}=\{2,4,6,8\}}}$.

We make a summary of above consideration as follows:

Proposition 6.2.4. *Let v be a H^1 global solution to* (6.6). *Then*

$$E_N^2(v(T+\delta) - E_N^2(v(T))) = \int_T^{T+\delta} [\Lambda_4(\sigma_4; v(t)) + \Lambda_6(M_6; v(t)) + \Lambda_8(M_8; v(t))]dt$$

for any $T \in \mathbb{R}$, $\delta > 0$ and $M_6 = M_6' + M_6'' - \sigma_6$. Furthermore, if $|\xi_i| \ll N$, for all j, then the multiplier M_4, M_6 and M_8 vanish.

Then we show the lemma that the second modified energy E_N^2 is still strong enough to control by H^1.

Lemma 6.2.5. *Assume that v satisfies $\|v(t)\|_{L^2} \leq \sqrt{2\pi}$ and $\|Iv(t)\|_{H^1} \leq \epsilon$. Then*

$$\|\partial_x Iv\|^2_{L^2(\mathbb{T})} \lesssim E_N^2(v).$$

Proof. We have

$$\begin{aligned} E_N^2(v) &= -\Lambda_2(m_1\xi_1 m_2\xi_2) + \frac{1}{8}\Lambda_4(\xi_{13-42}m_1m_2m_3m_4) \\ &+\frac{1}{8}\Lambda_4(M_4(\xi_1,\xi_2,\xi_3,\xi_4) - \xi_{13-42}m_1m_2m_3m_4). \end{aligned}$$

In Lemma 6.2.2, we proved that

$$\|\partial_x Iv\|^2_{L^2(\mathbb{T})} \lesssim E^1(v)$$

for $\|Iv\|_{L^2(\mathbb{T})} < \sqrt{2\pi}$. Hence it is enough to show that

$$|\Lambda_4(M_4(\xi_1,\xi_2,\xi_3,\xi_4) - \xi_{13-42}m_1m_2m_3m_4)| \lesssim O(N^{-\alpha})\|Iv\|^4_{H^1(\mathbb{T})}$$

for some $\alpha > 0$.

By a Littlewood-Paley decomposition, we may restrict the frequency of $\widehat{u_i}$ is as $|\xi_i| \sim N_i$ with dyadically and assume that $N_1 \geq N_2 \geq N_3 \geq N_4$. We also assume that each of u_i is nonnegative.

When $N_4 \gg N$, by equation (6.21) (see next section) we may write

$$\begin{aligned} |\Lambda_4(M_4(\xi_1,\xi_2,\xi_3,\xi_4))| &\lesssim \left|\int_{\xi_1-\xi_2+\xi_3-\xi_4=0} m(N_{max})^2 N_{\max}\widehat{u_1}\overline{\widehat{u_2}}\widehat{u_3}\overline{\widehat{u_4}}\right| \\ &\lesssim \left|\int_{\xi_1-\xi_2+\xi_3-\xi_4=0} \frac{\langle N_1\rangle Iu_1\langle N_2\rangle \overline{Iu_2}\langle N_3\rangle^{\frac{1}{2}-}Iu_3\langle N_4\rangle^{\frac{1}{2}-}\overline{Iu_4}}{\langle N_2\rangle\langle N_4\rangle^{1-}m(N_4)^2}\right| \end{aligned}$$

Since $m(N_4)\langle N_4\rangle^{1/2}$ is nondecreasing and L^2 for u_1, u_2, L^∞ for u_3, u_4 by Hölder and Sobolev embedding, we obtain

$$|\Lambda_4(M_4(\xi_1,\xi_2,\xi_3,\xi_4))| \lesssim \frac{1}{N^{1-}}\|Iu(t)\|^4_{H^1} \tag{6.20}$$

Next, we consider the second contribution

$$\begin{aligned}
|\Lambda_4(\xi_{13-24}m_1m_2m_3m_4))| &= 2\left|\int_{\substack{\xi_1-\xi_2+\xi_3-\xi_4=0\\|\xi_i|\sim N_i}} \xi_{13}m_1\widehat{u}_1m_2\overline{\widehat{u}_2}m_3\widehat{u}_3m_4\overline{\widehat{u}_4}\right| \\
&\lesssim \left|\int_{\xi_1-\xi_2+\xi_3-\xi_4=0} \frac{N_1N_2Iu_1\overline{Iu_2}Iu_3\overline{Iu_4}}{N_2}\right| \\
&\lesssim \frac{1}{N}\|Iu(t)\|_{H^1}^2\|Iu(t)\|_{L^\infty}^2 \\
&\lesssim \frac{1}{N}\|Iu(t)\|_{H^1}^4
\end{aligned}$$

where we use again Hölder and Sobolev embedding.

$\square$

6.3 Multiplier estimate.

In this section, we prove M_4 multilinear estimate. Our destination is to get the same estimate as [8] and it is a main point in our proof. we often use the following elementary tool.

Lemma 6.3.1. *(Double mean-value theorem) Let a be a smooth function of real variable ξ. Then*

$$|a(\xi+\eta+\lambda)-a(\xi+\eta)-a(\xi+\lambda)+a(\xi)| \lesssim |a''(\theta)||\eta||\lambda|$$

where $|\theta|\sim|\xi|$ and $\max(|\eta|,|\lambda|)\ll|\xi|$.

Lemma 6.3.2. *Let*

$$M_4(\xi_1,\xi_2,\xi_3,\xi_4) = \frac{m_1^2\xi_1^2\xi_3 - m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 - m_4^2\xi_4^2\xi_2}{(\xi_1-\xi_2)(\xi_1-\xi_4)} \tag{6.21}$$

where $m_i = m(\xi_i)$ is even and $\mathbb{R}$ valued and is defined as (6.13). Then

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| \le m(N_{\max})^2N_{\max}. \tag{6.22}$$

Proof. Let $M_4'(\xi_1,\xi_2,\xi_3,\xi_4) = m_1^2\xi_1^2\xi_3 - m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 - m_4^2\xi_4^2\xi_2$.

We define $f(\xi)=m(\xi)^2\xi^2$. Consider in the set $\xi_1-\xi_2+\xi_3-\xi_4=0$, we have

$$\begin{aligned}
M_4'(\xi_1,\xi_2,\xi_3,\xi_4) &= \xi_3[f(\xi_1)-f(\xi_2)-f(\xi_4)+f(\xi_3)] + (\xi_3-\xi_2)(f(\xi_4)-f(\xi_3)) \\
&\quad +(\xi_1-\xi_2)(f(\xi_3)-f(\xi_2)) \\
&= \xi_3[f(\xi_1)-f(\xi_1-\xi_{12})-f(\xi_1-\xi_{14})+f(\xi_{1-12-14})] \\
&\quad +(\xi_3-\xi_2)(f(\xi_4)-f(\xi_4-\xi_{12}) \\
&\quad +(\xi_1-\xi_2)(f(\xi_3)-f(\xi_3-\xi_{41})).
\end{aligned}$$

Applying DMVT on the first term and mean value theorem (MVT) on remaining terms. Then

$$|M_4'(\xi_1,\xi_2,\xi_3,\xi_4)| \le |\xi_3||f''(\xi_1)||\xi_{12}||\xi_{14}| + |f'(\xi_4)||\xi_{12}||\xi_{14}| + |f'(\xi_3)||\xi_{12}||\xi_{14}| \tag{6.23}$$

Since we have $(m(\xi)^2\xi^2)'' \sim m(\xi)^2$, then

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| \le m(N_{\max})^2 N_{\max}.$$

□

Lemma 6.3.3. *We also consider in the following cases*

Case (i). Assume that $\frac{1}{2}|\xi_1| \ge |\xi_2|, |\xi_4|$*; then*

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| \le m(N_{max})^2 N_3,. \tag{6.24}$$

Case (ii). Assume that $|\xi_1| \sim |\xi_2| \ge N \gg |\xi_3|, |\xi_4|$*; then*

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| + |m_2^2\xi_2| \le \frac{|(m_1^2\xi_1^2)'|\,|\xi_3|}{|\xi_1|} + |\xi_3|, \tag{6.25}$$

where $(\xi_1-\xi_2)(\xi_1-\xi_4) \neq 0$.

Proof. Case (i). In the region $\frac{1}{2}|\xi_1| \ge |\xi_2|, |\xi_4|$ we then see that $|\xi_1| \le |\xi_1 - \xi_2|, |\xi_1-\xi_4|$ and $|\xi_3| \sim |\xi_1|$. We often use $m(\xi)$ is an even function.

$$\begin{aligned}
|M_4'(\xi_1,\xi_2,\xi_3,\xi_4)| &= |\xi_3(m_1^2\xi_1^2 - m_3^2\xi_3^2) + \xi_4(m_3^2\xi_3^2 - m_2^2\xi_2^2) + \xi_2(m_3^2\xi_3^2 - m_4^2\xi_4^2)| \\
&\le |\xi_3[m(\xi_1)^2\xi_1^2 - m(\xi_1-\xi_{24})^2(\xi_1-\xi_{24})^2]| \\
&\quad + |\xi_4[m(\xi_3)^2\xi_3^2 - m(\xi_3-\xi_{41})^2(\xi_3-\xi_{14})^2]| \\
&\quad + |\xi_2[m(\xi_3)^2\xi_3^2 - m(\xi_3-\xi_{21})^2(\xi_3-\xi_{12})^2]| \\
&\le |\xi_3||(m_1^2\xi_1^2)'||\xi_2+\xi_4| + |\xi_4||(m_3^2\xi_3^2)'||\xi_1-\xi_4| \\
&\quad + |\xi_2||(m_3^2\xi_3^2)'||\xi_1-\xi_2| \\
&\le |\xi_3|\left[m_1^2|\xi_1|^2 + m_3^2|\xi_1||\xi_1-\xi_4| + m_3^2|\xi_1||\xi_1-\xi_2|\right]
\end{aligned}$$

since $(m(\xi)^2\xi^2)' \sim m(\xi)^2\xi$, we obtain

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4))| \le |\xi_3|(m_1^2 + m_3^2) \le |\xi_3|\; m_1^2$$

Case(ii). In the region $|\xi_1| \sim |\xi_2| \ge N \gg |\xi_3|, |\xi_4|$, then we see that

$$m(\xi_3)^2 = m(\xi_4)^2 = 1.$$

Since, we have $|\xi_1| \le |\xi_1 - \xi_4|$, we may rewrite

$$\begin{aligned}
M_4(\xi_1,\xi_2,\xi_3,\xi_4) &= \frac{(m_1^2\xi_1^2 - m_2^2\xi_2^2)\xi_3 - m_2^2\xi_2^2(\xi_1-\xi_2)}{(\xi_1-\xi_2)\xi_1} \\
&\quad + \frac{\xi_3^2(\xi_1-\xi_2) + \xi_2(\xi_3^2-\xi_4^2)}{(\xi_1-\xi_2)\xi_1},
\end{aligned}$$

using MVT on the first term and the last term is very small, then

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| + |m_2^2\xi_2| \le \frac{|(m_1^2\xi_1^2)'|\,|\xi_3|}{|\xi_1|} + |\xi_3|$$

we obtain case (ii). □

6.4 Local estimates

In this section, we study the local estimates of (6.1) - (6.2) which have already known from [26] for all $s \geq \frac{1}{2}$.

Let f_i be a nonnegative function such that

$$f_i(\tau_i, \xi_i) = \langle \tau_i + (-1)^i \xi_i^2 \rangle^{\frac{1}{2}} \langle \xi_i \rangle^{\frac{1}{2}} |\widehat{u}_i(\tau, \xi)|.$$

Let $\tau_4 = \Sigma_{i=1}^3 \tau_i$, $\xi_4 = \Sigma_{i=1}^3 (-1)^{i-1} \xi_i$. We consider ([2], [30] and [39])

$$\Sigma_{i=1}^4 (\tau_i + (-1)^i \xi_i^2) = 2(\xi_1 - \xi_2)(\xi_1 - \xi_4).$$

We use the operator $J^s = J_x^s$ in space variable, which is defined by $\widehat{J^s f}(\xi) = \langle \xi \rangle^s \hat{f}(\xi)$ and we will use the same notation in time as $J^s = J_t^s$.

We recall the transformed equation of (6.6) - (6.7) that

$$\begin{aligned} v_t - i v_{xx} &= \mathcal{N}(v), && (6.26) \\ v(0,x) &= v_0(x), && (6.27) \end{aligned}$$

where

$$\begin{aligned} \mathcal{N}(v) &= -\left(v\bar{v}_x - \frac{i}{2\pi} \int_0^{2\pi} 2\mathrm{Im}\ v\bar{v}_x d\theta \right) v(t,x) \\ &\quad + i \left(|v|^4 - \frac{1}{4\pi^2} \int_0^{2\pi} |v|^4 d\theta \right) v(t,x) \\ &\quad - i (\int_0^{2\pi} |v|^2 d\theta) \left(|v|^2 - \frac{1}{2\pi} \int_0^{2\pi} |v|^2 d\theta \right) v(t,x). \end{aligned}$$

In Lemma 7.1, we shall estimate the expression of Proposition 6.2.4. In that case, it is not enough to use the spatial norms such as $\|Iv\|_{H^1}$, hence we shall use the space-time norms such as $\|Iv\|_{Z_1}$. For that purpose we prove the local existence of solution in time. Before we start local existence theorem, we set $u := Iv$. Then (6.26) - (6.27) become

$$\begin{aligned} u_t - i u_{xx} &= \mathcal{I}(N(v)), && (6.28) \\ u(0,x) &= u_0(x), && (6.29) \end{aligned}$$

Theorem 6.4.1. *Let v be a global solution of* (6.26) - (6.27)*. Assume that for some $\theta > 0$, $\|Iv_0\|_{H^1(\mathbb{T})} \leq \theta$. Then, there exist two positive constants T and C such that*

$$\|Iv\|_{Z_1([-T,T]\times\mathbb{T})} \leq C\theta,$$

where C is indepentent of N appearing in the definition of I.

Theorem 6.4.1 is a consequence of the following two lemmas. The first lemma is for the estimates of homogeneous and inhomogeneous problem. See [2], [15], [26].

Lemma 6.4.2. *Let $s \in \mathbb{R}$. Let $u_0 \in H^s(\mathbb{T})$. Then*

$$\|\chi(t)U(t)u_0\|_{Z_1} \lesssim \|u_0\|_{H^1(\mathbb{T})},$$

$$\|\chi(t)\int_0^t U(t-s)I(\mathcal{N}(v))ds\|_{Z_1} \lesssim \|I(\mathcal{N}(v))\|_{X_{1,-\frac{1}{2}}} + \|I(\mathcal{N}(v))\|_{Y_{1,-1}},$$

where $\chi \in C_0^\infty((-2,2))$ with $\chi \equiv 1$ in $[-1,1]$ and supp $f \subset \{(t,x)||t| \le 2\}$.

In next lemma, we prove the multilinear estimates as follows.

Lemma 6.4.3. *Let v be the Schwartz function with spatial periodic. There exists $\epsilon' > 0$ such that for $T \in (0,1]$, then*

$$\left\| I\left(v_1\overline{\partial_x v_2}v_3 - \frac{i}{2\pi}(Im \int_{\mathbb{T}} v_1\overline{\partial_x v_2}d\theta)v_3(x) \right)\right\|_{X_{1,-\frac{1}{2}}} \lesssim T^{\epsilon'}\prod_{i=1}^3 \|Iv_i\|_{X_{1,\frac{1}{2}}}, \tag{6.30}$$

Proof. We take the Fourier transform on the left hand side of (6.30) to have

$$\begin{aligned} v_1\overline{\partial_x v_2}v_3 \quad - \quad & \frac{i}{2\pi}(\text{Im}\int_{\mathbb{T}} v_1\overline{\partial_x v_2}d\theta)v_3(x) \\ &= \int_{\substack{\xi_4=\xi_1-\xi_2+\xi_3 \\ \xi_1\neq\xi_4,\xi_4\neq\xi_3}} \xi_2\hat{v}(\xi_1)\hat{\bar{v}}(\xi_2)\hat{v}(\xi_3) + i\xi_1\hat{\bar{v}}(\xi_1)\hat{v}(\xi_1)\hat{v}(\xi_1). \end{aligned}$$

Assume that each of $\hat{v}_i$ is nonnegative. We have to show two main parts such that

$$\left\| I\left(\mathcal{F}_x^{-1}\int_{\substack{\xi_4=\xi_1-\xi_2+\xi_3 \\ \xi_1\neq\xi_4,\xi_4\neq\xi_3}} |\xi_2|\hat{v}(\xi_1)\hat{\bar{v}}(\xi_2)\hat{v}(\xi_3)\right)\right\|_{X_{1,-\frac{1}{2}}} \lesssim T^{\varepsilon'}\prod_{i=1}^3 \|Iv_i\|_{X_{1,\frac{1}{2}}} \tag{6.31}$$

and

$$\left\| I\left(\mathcal{F}_x^{-1}\{|\xi|\hat{v}(\xi)\hat{\bar{v}}(\xi)\hat{v}(\xi)\}\right)\right\|_{X_{1,-\frac{1}{2}}} \lesssim T^{\varepsilon'}\prod_{i=1}^3 \|Iv_i\|_{X_{1,\frac{1}{2}}}. \tag{6.32}$$

We first prove (6.31). We can write the left hand side of (6.31) as

$$\begin{aligned} &\left\| \langle\xi_4\rangle\langle\tau_4+\xi_4^2\rangle^{-\frac{1}{2}}\mathcal{F}_{t,x}I\left(\mathcal{F}_x^{-1}\int_{\substack{\xi_4=\xi_1-\xi_2+\xi_3 \\ \xi_1\neq\xi_4,\xi_4\neq\xi_3}} \xi_2\hat{v}(\xi_1)\hat{\bar{v}}(\xi_2)\hat{v}(\xi_3)\right)\right\|_{L_t^2L_x^2} \\ &= \left\| \int_* \varphi(\xi_{12})\varphi(\xi_{14})\langle\xi_4\rangle\langle\tau_4+\xi_4^2\rangle^{-\frac{1}{2}}m(\xi_4)|\xi_2|\hat{v}(\xi_1)\hat{\bar{v}}(\xi_2)\hat{v}(\xi_3)\right\|_{L_t^2L_x^2} \end{aligned}$$

by definition of operator I and $\int_*$ denote the integration over the measure $\delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)\delta(\xi_1 - \xi_2 + \xi_3 - \xi_4)$. Applying the Plancherel and duality, we have to show that

$$\left| \int_* \frac{\varphi(\xi_{12})\varphi(\xi_{14})m(\xi_4)\langle\xi_4\rangle|\xi_2|\Pi_{i=1}^4 f_i}{\langle\tau_4+\xi_4^2\rangle^{\frac{1}{2}}\prod_{i=1}^3 m(\xi_i)\langle\xi_i\rangle\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}}}\right| \lesssim \prod_{i=1}^4 \|f_i\|_{L_{\tau_i}^2L_{\xi_i}^2}. \tag{6.33}$$

where all of f_i are real and nonnegative functions .

Case(1). If

$$\frac{m(\xi_4)\langle\xi_4\rangle|\xi_2|}{\prod_{i=1}^3 m(\xi_i)\langle\xi_i\rangle} \lesssim 1,$$

we need to show that

$$\left|\int_* \frac{\prod_{i=1}^4 f_i(\tau_i,\xi_i)}{\langle\tau_4+\xi_4^2\rangle^{\frac12}\Pi_{i=1}^3\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac12}}\right| \lesssim \prod_{i=1}^4 \|f_i\|_{L^2_{\tau_i}L^2_{\xi_i}}. \tag{6.34}$$

We apply the Cauchy Schwarz inequality for left hand side of (6.34), then

$$\begin{aligned}
&\left|\int_* \frac{\prod_{i=1}^4 f_i(\tau_i,\xi_i)}{\langle\tau_4+\xi_4^2\rangle^{\frac12}\prod_{i=1}^3\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac12}}\right| \\
&\le \|f_4\|_{L^2_{\tau_4}L^2_{\xi_4}}\left\|\frac{\prod_{i=1}^3 f_i(\tau_i,\xi_i)}{\langle\tau_4+\xi_4^2\rangle^{\frac12}\Pi_{i=1}^3\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac12}}\right\|_{L^2_{\tau_4}L^2_{\xi_4}} \\
&= \|f_4\|_{L^2_{\tau_4}L^2_{\xi_4}}\|\langle\tau_4+\xi_4^2\rangle^{-\frac12}\prod_{i=1}^3\langle\xi_i\rangle m(\xi_i)\widehat{v}(\xi_i)\|_{L^2_{\tau_4}L^2_{\xi_4}}.
\end{aligned}$$

We use the L^4 dual Strichartz estimate to obtain

$$\begin{aligned}
\|\langle\tau_4+\xi_4^2\rangle^{-\frac12}\prod_{i=1}^3\langle\xi_i\rangle m(\xi_i)\widehat{v}(\xi_i)\|_{L^2_{\tau_4}L^2_{\xi_4}} &= \|(J_xIv_1)(J_xIv_2)(J_xIv_3)\|_{X_{0,-\frac12}} \\
&\le C\|(J_xIv_1)(J_xIv_2)(J_xIv_3)\|_{L_t^{\frac43}L_x^{\frac43}} \\
&\le C\Pi_{i=1}^3\|J_xIv_i\|_{L_t^4L_x^4}.
\end{aligned}$$

We finally use the L^4 Strichartz estimate and Sobolev embedding in time to get the desired estimate,

$$\|J_xIv\|_{L_t^4L_x^4} \le C\|Iv\|_{X_{1,\frac38+}} \le CT^{\varepsilon'}\|Iv\|_{X_{1,\frac12}}.$$

Case (2). We may assume that

$$\frac{m(\xi_4)\langle\xi_4\rangle|\xi_2|}{\prod_{i=1}^3 m(\xi_i)\langle\xi_i\rangle} \gg 1$$

for $|\xi_1-\xi_2|\gg 1$, $|\xi_1-\xi_4|\gg 1$. We note that [2], [30] and [39]

$$\max_{i=1,2,3,4}|\tau_i+(-1)^i\xi_i^2| \ge |\xi_1-\xi_2||\xi_1-\xi_4| \tag{6.35}$$

on the set $\tau_1+\tau_2+\tau_3+\tau_4=0$. We consider the following four subcases according to the one of $\langle\tau_i+(-1)^i\xi_i^2\rangle's$ is the largest. Case(a) if $\max_{i=1,2,3,4}\langle\tau_i+(-1)^i\xi_i^2\rangle = \langle\tau_4+\xi_4^2\rangle$ and case (b) if $\max_{i=1,2,3,,4}\langle\tau_i+(-1)^i\xi_i^2\rangle = \langle\tau_j+(-1)^j\xi_j^2\rangle$ for some $j=1,2,3$.

We divide two regions such that $|\xi_1| \leq \frac{1}{2}|\xi_4| \leq |\xi_3|$ and $|\xi_3| \leq \frac{1}{2}|\xi_4| \leq |\xi_1|$. In both cases we see that the inequality (See [8]-[9]),

$$\frac{m(\xi_4)\langle\xi_4\rangle^{1-s}}{\prod_{i=1}^{3} m(\xi_i)\langle\xi_i\rangle^{1-s}} \lesssim 1, \tag{6.36}$$

for all $\xi, \ldots, \xi_4$ such that $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0$, since $m(\xi)\langle\xi\rangle$ is increasing in $|\xi|$ and $m(\xi)\langle\xi\rangle \gtrsim 1$. Hence it is enough to show that

$$\left| \int_* \frac{\langle\xi_4\rangle^s \langle\xi_2\rangle \varphi(\xi_{12})\varphi(\xi_{12})}{\langle\tau_4 + \xi_4^2\rangle^{\frac{1}{2}} \prod_{i=1}^{3} \langle\xi_i\rangle^s \langle\tau_i + (-1)^i \xi_i^2\rangle^{\frac{1}{2}}} \prod_{i=1}^{4} f_i(\tau_i, \xi_i) \right| \lesssim \prod_{i=1}^{4} \|f_i\|_{L_t^2 L_x^2}. \tag{6.37}$$

Subcase(i). In the region $|\xi_1| \leq \frac{1}{2}|\xi_4| \leq |\xi_3|$.

We may assume that $|\xi_1 - \xi_4| \leq |\xi_1 - \xi_2|$ on the set $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0$. Since $|\xi_4| \leq |\xi_1 - \xi_4|$, we have $|\xi_1| \leq \frac{1}{2}|\xi_1 - \xi_4| \leq \frac{1}{2}|\xi_1 - \xi_2|$. We observe that with $s = \frac{1}{2} + \epsilon$,

$$\begin{aligned} \frac{\langle\xi_4\rangle^s |\xi_2|}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s \langle\xi_3\rangle^s} &\leq \frac{\langle\xi_1 - \xi_4\rangle^s (\langle\xi_1 - \xi_2\rangle^{1-s} + \langle\xi_1\rangle^{1-s})}{\langle\xi_1\rangle^s \langle\xi_3\rangle^s} \\ &\leq \frac{2\langle\xi_1 - \xi_2\rangle^{\frac{1}{2}} \langle\xi_1 - \xi_4\rangle^{\frac{1}{2}}}{\langle\xi_1\rangle^s \langle\xi_3\rangle^s} = A. \end{aligned}$$

On the other hand, we assume $|\xi_1 - \xi_4| \geq |\xi_1 - \xi_2|$, then

$$\begin{aligned} \frac{\langle\xi_4\rangle^s |\xi_2|}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s \langle\xi_3\rangle^s} &\lesssim \frac{\langle\xi_4\rangle^s (\langle\xi_1 - \xi_2\rangle + \langle\xi_1\rangle)}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s \langle\xi_3\rangle^s} \\ &\lesssim \frac{\langle\xi_4\rangle^s \langle\xi_1 - \xi_2\rangle}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s \langle\xi_3\rangle^s} + \frac{\langle\xi_4\rangle^s \langle\xi_1\rangle}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s \langle\xi_3\rangle^s} \\ &= B + C. \end{aligned}$$

In case B, it is clear that

$$B \leq \frac{\langle\xi_1 - \xi_2\rangle}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s} \leq \frac{\langle\xi_1 - \xi_2\rangle^{\frac{1}{2}} \langle\xi_1 - \xi_4\rangle^{\frac{1}{2}}}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s}.$$

Now we consider the region C, we divide in two subregion such that $C = C_1 + C_2$ where C_1 for $|\xi_1 - \xi_2| \geq \frac{1}{10}|\xi_1|$ and C_2 for $|\xi_1 - \xi_2| \leq \frac{1}{10}|\xi_1|$. When $|\xi_1 - \xi_2| \geq \frac{1}{10}|\xi_1|$,

$$C_1 \leq \frac{\langle\xi_1 - \xi_2\rangle}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s} \leq \frac{\langle\xi_1 - \xi_2\rangle^{\frac{1}{2}} \langle\xi_1 - \xi_4\rangle^{\frac{1}{2}}}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s}.$$

We next consider the region $|\xi_1 - \xi_2| \leq \frac{1}{10}|\xi_1|$. In this case we see $|\xi_1| \sim |\xi_2| \leq |\xi_3|$ and so that

$$C_2 = \frac{\langle\xi_4\rangle^s \langle\xi_1\rangle}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s \langle\xi_3\rangle^s} \lesssim \frac{1}{\langle\xi_1\rangle^{2s-1}} \lesssim 1,$$

since $s > \frac{1}{2}$.

Subcase(ii). $|\xi_3| \le \frac{1}{2}|\xi_4| \le |\xi_1|$. We used the same idea as subcase (i).

Now it is turn to prove the following cases.

Case (a). We first consider in the case of $\max_{i=1,2,3,4}\langle \tau_i + (-1)^i \xi_i^2\rangle = \langle \tau_4 + \xi_4^2\rangle$. We apply the multiplier estimates of case (i) except C_2, hence we have to show that

$$\left| \int_* \frac{\prod_{i=1}^4 f_i(\tau_i, \xi_i)}{\langle \xi_1\rangle^s \langle \xi_3\rangle^s \Pi_{i=1}^3 \langle \tau_i + (-1)^i \xi_i^2\rangle^{\frac{1}{2}}} \right| \lesssim T^{\varepsilon'} \prod_{i=1}^4 \|f_i\|_{L_t^2 L_x^2}. \tag{6.38}$$

We apply Hölder inequality, the left hand side of (6.38) becomes

$$\begin{aligned}
&\left| \int_* \frac{\prod_{i=1}^4 f_i(\tau_i, \xi_i)}{\langle \xi_1\rangle^s \langle \xi_3\rangle^s \Pi_{i=1}^3 \langle \tau_i + (-1)^i \xi_i^2\rangle^{\frac{1}{2}}} \right| \\
&\lesssim \left\| \mathcal{F}_{t,x}^{-1}\left(\frac{f_1(\tau_1, \xi_1)}{\langle \xi_1\rangle^s \langle \tau_1 - \xi_1^2\rangle^{\frac{1}{2}}} \right) \right\|_{L_t^8 L_x^8} \left\| \mathcal{F}_{t,x}^{-1}\left(\frac{f_2(\tau_2, \xi_2)}{\langle \tau_2 + \xi_2^2\rangle^{\frac{1}{2}}} \right) \right\|_{L_t^4 L_x^4} \\
&\times \left\| \mathcal{F}_{t,x}^{-1}\left(\frac{f_3(\tau_3, \xi_3)}{\langle \xi_3\rangle^s \langle \tau_3 - \xi_3^2\rangle^{\frac{1}{2}}} \right) \right\|_{L_t^8 L_x^8} \|\mathcal{F}_{t,x}^{-1} f_4\|_{L_t^2 L_x^2} \\
&\lesssim T^{\varepsilon'} \prod_{i=1}^4 \|f_i\|_{L_t^2 L_x^2}.
\end{aligned}$$

by Sobolev embedding for f_1 and f_2, L^4 Strichartz estimate for f_2 and finally we use again the Sobolev embedding in time, we get the desired estimate.

Case (b). Now we consider the case of $\max_{i=1,2,3,4}\langle \tau_i + (-1)^i \xi_i^2\rangle = \langle \tau_j + (-1)^i \xi_j^2\rangle$ for some $j = 1, 2, 3$. We use case (i) except the region C_2. For $j = 1$, hence we need to show that

$$\left| \int_* \frac{\prod_{i=1}^4 f_i(\tau_i, \xi_i)}{\langle \xi_1\rangle^s \langle \xi_3\rangle^s \langle \tau_4 + \xi_4^2\rangle^{\frac{1}{2}} \langle \tau_2 - \xi_2^2\rangle^{\frac{1}{2}} \langle \tau_3 + \xi_3^2\rangle^{\frac{1}{2}}} \right| \le T^{\varepsilon'} \prod_{i=1}^4 \|f_i\|_{L_t^2 L_x^2} \tag{6.39}$$

By Cauchy Schwarz inequality, we have that

$$\begin{aligned}
&\left| \int_* \frac{\prod_{i=1}^4 f_i(\tau_i, \xi_i)}{\langle \xi_1\rangle^s \langle \xi_3\rangle^s \langle \tau_4 + \xi_4^2\rangle^{\frac{1}{2}} \langle \tau_2 - \xi_2^2\rangle^{\frac{1}{2}} \langle \tau_3 + \xi_3^2\rangle^{\frac{1}{2}}} \right| \\
&\le \|f_4\|_{L_t^2 L_x^2} \left\| \frac{\prod_{i=1}^3 f_i(\tau_i, \xi_i)}{\langle \xi_1\rangle^s \langle \xi_3\rangle^s \langle \tau_4 + \xi_4^2\rangle^{\frac{1}{2}} \langle \tau_2 - \xi_2^2\rangle^{\frac{1}{2}} \langle \tau_3 + \xi_3^2\rangle^{\frac{1}{2}}} \right\|_{L_{\tau_4}^2 L_{\xi_4}^2}.
\end{aligned}$$

Now we need to prove that

$$\left\| \frac{\prod_{i=1}^3 f_i(\tau_i, \xi_i)}{\langle \xi_1\rangle^s \langle \xi_3\rangle^s \langle \tau_4 + \xi_4^2\rangle^{\frac{1}{2}} \langle \tau_2 - \xi_2^2\rangle^{\frac{1}{2}} \langle \tau_3 + \xi_3^2\rangle^{\frac{1}{2}}} \right\|_{L_t^2 L_x^2} \le T^{\varepsilon'} \prod_{i=1}^3 \|I v_i\|_{X_{1,\frac{1}{2}}}. \tag{6.40}$$

We can rewrite the left hand side of (6.40) as

$$\begin{aligned}\|(J_x^{1-s}J_t^{\frac{1}{2}}Iv_1)(J_x\overline{Iv_2})(J_x^{1-s}Iv_3)\|_{X_{0,-\frac{1}{2}}} &\leq \|(J_x^{1-s}J_t^{\frac{1}{2}}Iv_1)(J_x\overline{Iv_2})(J_x^{1-s}Iv_3)\|_{L_t^{1-}L_x^2}\\ &\lesssim \|J_x^{1-s}J_t^{\frac{1}{2}}Iv_1\|_{L_t^pL_x^8}\|J_xIv_2\|_{L_t^4L_x^4}\\ &\quad\times\|J_x^{1-s}Iv_3\|_{L_t^8L_x^8}\\ &\lesssim T^{\epsilon'}\|Iv_1\|_{X_{1,\frac{1}{2}}}\|Iv_2\|_{X_{1,\frac{1}{2}}}\|Iv_3\|_{X_{1,\frac{1}{2}}},\end{aligned}$$

by Sobolev embedding in time for the first inequality and the second is used by Hölder's inequality with $p > \frac{8}{5}$. We fix $p = 2$ to get the best estimate what we desired. We note that the proof of multilinear estimate (6.37) for C_2 region is the same as case (1). Now it is turn to prove (6.32), it is enough to prove that

$$\left\|I\left(\mathcal{F}_x^{-1}\{|\xi|\hat{v}(\xi)\hat{\bar{v}}(\xi)\hat{v}(\xi)\}\right)\right\|_{X_{1,0}} \lesssim T^{\varepsilon'}\prod_{i=1}^{3}\|Iv_i\|_{X_{1,\frac{1}{2}}}. \tag{6.41}$$

We can rewrite the left hand side of (6.41) as

$$\begin{aligned}&\left\|\int_{\sum_{i=1}^4\tau_i=0}\frac{\langle\xi\rangle^2m(\xi)f_1(\tau_1,\xi)f_2(\tau_2,\xi)f_3(\tau_3,\xi)}{m(\xi)^3\langle\xi\rangle^3\prod_{i=1}^3\langle\tau_i+(-1)^i\xi^2\rangle^{\frac{1}{3}+}}\right\|_{L^2_{\tau,\xi}}\\ &\leq\left\|\int_{\sum_{i=1}^4\tau_i=0}\frac{f_1(\tau_1,\xi)f_2(\tau_2,\xi)f_3(\tau_3,\xi)}{\prod_{i=1}^3\langle\tau_i+(-1)^i\xi^2\rangle^{\frac{1}{3}+}}\right\|_{L^2_{\tau,\xi}}\end{aligned} \tag{6.42}$$

since $m(\xi)\langle\xi\rangle^{\frac{1}{2}} \gtrsim 1$. By Young's inequality in time, (6.42) is bounded by

$$\left\|\left\|\frac{f_1(\tau_1,\xi)}{\langle\tau_1-\xi^2\rangle^{\frac{1}{3}+}}\right\|_{L^{\frac{6}{5}}_{\tau_1}}\left\|\frac{f_2(\tau_2,\xi)}{\langle\tau_2+\xi^2\rangle^{\frac{1}{3}+}}\right\|_{L^{\frac{6}{5}}_{\tau_2}}\left\|\frac{f_3(\tau_3,\xi)}{\langle\tau_3+\xi^2\rangle^{\frac{1}{3}+}}\right\|_{L^{\frac{6}{5}}_{\tau_3}}\right\|_{L^2_\xi}. \tag{6.43}$$

Then we use the Hölder's inequality for two of $v_i's$ in L^∞_ξ and other in L^2_ξ. Then (6.43) is bounded by

$$\left\|\left\|\frac{f_1(\tau_1,\xi)}{\langle\tau_1-\xi^2\rangle^{\frac{1}{3}+}}\right\|_{L^{\frac{6}{5}}_{\tau_1}}\right\|_{L^\infty_\xi}\left\|\left\|\frac{f_2(\tau_2,\xi)}{\langle\tau_2-\xi^2\rangle^{\frac{1}{3}+}}\right\|_{L^{\frac{6}{5}}_{\tau_2}}\right\|_{L^\infty_\xi}\left\|\left\|\frac{f_3(\tau_3,\xi)}{\langle\tau_3-\xi^2\rangle^{\frac{1}{3}+}}\right\|_{L^{\frac{6}{5}}_{\tau_3}}\right\|_{L^2_\xi}.$$

We get the desired estimate by using the space time Sobolev embedding.

Lemma 6.4.4. *Let v be the Schwartz function with spatial periodic. There exists $\epsilon' > 0$ such that for $T \in (0,1]$, then*

$$\left\|I\left(v_1\overline{\partial_xv_2}v_3-\frac{i}{2\pi}(Im\int_{\mathbb{T}}v_1\overline{\partial_xv_2}d\theta)v_3(x)\right)\right\|_{Y_{1,-1}} \lesssim T^{\epsilon'}\prod_{i=1}^{3}\|Iv_i\|_{X_{1,\frac{1}{2}}}, \tag{6.44}$$

Proof. We apply the same idea of Lemma 6.4.3, hence we need to prove two main parts as follows.

$$\left\| \int_* \frac{\langle \xi_4 \rangle m(\xi_4) \varphi(\xi_{12}) \varphi(\xi_{14}) |\xi_2| \prod_{i=1}^3 f_i(\tau_i, \xi_i)}{\langle \tau_4 + \xi_4^2 \rangle \prod_{i=1}^3 m(\xi_i) \langle \xi_i \rangle \langle \tau_i + (-1)^i \xi_i^2 \rangle^{\frac{1}{2}}} \right\|_{L^1_{\tau_4} L^2_{\xi_4}} \leq T^{\varepsilon'} \prod_{i=1}^3 \|Iv_i\|_{X_{1,\frac{1}{2}}} \tag{6.45}$$

where all of f_i are real and nonnegative functions and $\int_*$ denote the integration over the measure $\delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)\delta(\xi_1 - \xi_2 + \xi_3 - \xi_4)$ and

$$\|I(\mathcal{F}_x^{-1}\{|\xi| \hat{v}_1(\xi) \widehat{\overline{v}_2}(\xi) \hat{v}_3(\xi)\})\|_{Y_{1,-1}} \leq T^{\varepsilon'} \prod_{i=1}^3 \|Iv_i\|_{X_{1,\frac{1}{2}}}. \tag{6.46}$$

Now we prove (6.45). Case(1). If

$$\frac{m(\xi_4)\langle \xi_4 \rangle |\xi_2|}{\prod_{i=1}^3 m(\xi_i) \langle \xi_i \rangle} \lesssim 1,$$

by using Cauchy Schwarz inequality in the left hand side of (6.45), then

$$\|\langle \tau_4 + \xi_4^2 \rangle^{-\frac{1}{2}-}\|_{L^2_{\tau_4}} \left\| \int_* \frac{\prod_{i=1}^3 f_i(\tau_i, \xi_i)}{\langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}-} \prod_{i=1}^3 \langle \tau_i + (-1)^i \xi_i^2 \rangle^{\frac{1}{2}}} \right\|_{L^2_{\xi_4} L^2_{\tau_4}}.$$

The claim is the same as case (1) of Lemma 6.4.3, we obtain desired estimate.

Case (2) We may assume that

$$\frac{m(\xi_4)\langle \xi_4 \rangle |\xi_2|}{\prod_{i=1}^3 m(\xi_i) \langle \xi_i \rangle} \gg 1$$

for $|\xi_1 - \xi_2| \gg 1$, $|\xi_1 - \xi_4| \gg 1$. We divide two regions such that $|\xi_1| \leq \frac{1}{2}|\xi_4| \leq |\xi_3|$ and $|\xi_3| \leq \frac{1}{2}|\xi_4| \leq |\xi_1|$. In both case we see that,

$$\frac{m(\xi_4)\langle \xi_4 \rangle^{1-s}}{\prod_{i=1}^3 m(\xi_i) \langle \xi_i \rangle^{1-s}} \leq 1,$$

for all $\xi, \ldots, \xi_4$ such that $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0$, since $m(\xi)\langle \xi \rangle$ is increasing in $|\xi|$ and $m(\xi)\langle \xi \rangle \gtrsim 1$. Hence it is enough to show that

$$\left\| \int_* \frac{\langle \xi_4 \rangle^s \langle \xi_2 \rangle \varphi(\xi_{12}) \varphi(\xi_{14})}{\langle \tau_4 + \xi_4^2 \rangle \prod_{i=1}^3 \langle \xi_i \rangle^s \langle \tau_i + (-1)^i \xi_i^2 \rangle^{\frac{1}{2}}} \prod_{i=1}^3 f_i(\tau_i, \xi_i) \right\|_{L^1_{\tau_4} L^2_{\xi_4}} \lesssim T^{\varepsilon'} \prod_{i=1}^3 \|Iv_i\|_{X_{1,\frac{1}{2}}}. \tag{6.47}$$

We note by [26] that,

$$\langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}} \geq c \langle \tau_1 - \xi_1^2 \rangle^{\delta} \langle \tau_2 + \xi_2^2 \rangle^{\delta} \langle \tau_3 - \xi_3^2 \rangle^{\delta} \langle \xi_1 - \xi_2 \rangle^{\frac{1}{2}-3\delta} \langle \xi_1 - \xi_4 \rangle^{\frac{1}{2}-3\delta}, \tag{6.48}$$

where δ is sufficiently small. We may estimate the Fourier multipliers as Lemma 6.4.3. We divide the following two regions such that (i) $|\xi_3| \leq \frac{1}{2}|\xi_4| \leq |\xi_1|$ and (ii) $|\xi_1| \leq \frac{1}{2}|\xi_4| \leq |\xi_3|$.

We first recall the the multiplier estimate of Lemma 6.4.3

$$\begin{aligned}\frac{\langle\xi_4\rangle^s\langle\xi_2\rangle}{\langle\tau_4+\xi_4^2\rangle\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s} &\lesssim \frac{2\langle\xi_1-\xi_2\rangle^{\frac{1}{2}}\langle\xi_1-\xi_4\rangle^{\frac{1}{2}}}{\langle\tau_4+\xi_4^2\rangle\langle\xi_1\rangle^s\langle\xi_3\rangle^s}+\frac{1}{\langle\tau_4+\xi_4^2\rangle} \\ &= I_1+I_2. \end{aligned}\tag{6.49}$$

Here it is enough to consider only case $\max_{i=1,2,3,4}\langle\tau_i+(-1)^i\xi_i^2\rangle=\langle\tau_4+\xi_4^2\rangle$ and the other cases can be controlled as (6.39).

Now we consider the region $|\xi_3|\le\frac{1}{2}|\xi_4|\le|\xi_1|$ on the set $\xi_1-\xi_2+\xi_3-\xi_4=0$. We have $|\xi_4|\le 2|\xi_1-\xi_2|$, then

$$|\xi_4||\xi_1-\xi_4|\le 2|\xi_1-\xi_2||\xi_1-\xi_4|.$$

If $|\xi_3|\le|\xi_1-\xi_4|$, then

$$|\xi_2|\le|\xi_1-\xi_4|+|\xi_3|\le 2|\xi_1-\xi_4|.$$

We conclude that $|\xi_2||\xi_4|\le 4|\xi_1-\xi_2||\xi_1-\xi_4|$. We see that (6.48) becomes

$$\langle\tau_4+\xi_4^2\rangle^{\frac{1}{2}}\ge C\prod_{i=1}^{3}\langle\tau_i+(-1)^i\xi_i^2\rangle^{\delta}\langle\xi_2\rangle^{\frac{1}{2}-3\delta}\langle\xi_4\rangle^{\frac{1}{2}-3\delta}.$$

Then

$$I_1\lesssim\frac{1}{\prod_{i=1}^{3}\langle\tau_i+(-1)^i\xi_i^2\rangle^{\delta}\langle\xi_1\rangle^s\langle\xi_3\rangle^s\langle\xi_2\rangle^{\frac{1}{2}-3\delta}\langle\xi_4\rangle^{\frac{1}{2}-3\delta}}.$$

If $|\xi_3|\ge|\xi_1-\xi_4|$, then $|\xi_2|\le 2|\xi_3|$. Thus we obtain

$$\frac{\langle\xi_4\rangle^s\langle\xi_2\rangle}{\langle\tau_4+\xi_4^2\rangle\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}\lesssim\frac{1}{\langle\tau_4+\xi_4^2\rangle\langle\xi_2\rangle^{2s-1}}\lesssim\frac{1}{\langle\tau_4+\xi_4^2\rangle}.$$

This case and I_2 can be controlled as case (1). We need to show (6.47) for I_1 region,

$$\left\|\int_*\frac{\prod_{i=1}^{3}f_i(\tau_i,\xi_i)}{\langle\xi_1\rangle^s\langle\xi_3\rangle^s\langle\xi_2\rangle^{\frac{1}{2}-3\delta}\langle\xi_4\rangle^{\frac{1}{2}-3\delta}\prod_{i=1}^{3}\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}+\delta}}\right\|_{L^1_{\tau_4}L^2_{\xi_4}}\lesssim T^{\varepsilon'}\prod_{i=1}^{3}\|Iv_i\|_{X_{1,\frac{1}{2}}}.\tag{6.50}$$

Fix ξ and we apply Young's inequality for time, then

$$\left\|\int_{\sum_{i=1}^4\tau_i=0}\prod_{i=1}^{3}\frac{f_i(\tau_i,\xi_i)}{\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}+\delta}}\right\|_{L^1_{\tau_4}}\le\prod_{i=1}^{3}\left\|\frac{f_i(\tau_i,\xi_i)}{\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}+\delta}}\right\|_{L^1_{\tau_i}}.\tag{6.51}$$

By Cauchy Schwarz inequality

$$\left\|\frac{f(\tau,\xi)}{\langle\tau+\xi^2\rangle^{\frac{1}{2}+\delta}}\right\|_{L^1_\tau}\le\|\langle\tau+\xi^2\rangle^{-\frac{1}{2}-\frac{\delta}{2}}\|_{L^2_\tau}\left\|\frac{f(\tau,\xi)}{\langle\tau+\xi^2\rangle^{\frac{\delta}{2}}}\right\|_{L^2_\tau}.\tag{6.52}$$

By (6.51) and (6.52), left hand side of (6.50) becomes

$$\begin{aligned}
&\left\|\langle\xi_1\rangle^{-s}\langle\xi_3\rangle^{-s}\langle\xi_2\rangle^{-\frac{1}{2}+3\delta}\langle\xi_4\rangle^{-\frac{1}{2}+3\delta}\prod_{i=1}^{3}\left\|\frac{f_i(\tau_i,\xi_i)}{\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{\delta}{2}}}\right\|_{L^2_{\tau_i}}\right\|_{L^2_{\xi_4}}\\
&\lesssim \|\langle\xi_4\rangle^{-\frac{1}{2}+3\delta}\|_{L^4_{\xi_4}}\left\|\langle\xi_1\rangle^{-s}\langle\xi_3\rangle^{-s}\langle\xi_2\rangle^{-\frac{1}{2}+3\delta}\prod_{i=1}^{3}\left\|\frac{f_i(\tau_i,\xi_i)}{\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{\delta}{2}}}\right\|_{L^2_{\tau_i}}\right\|_{L^4_{\xi_4}}\\
&\lesssim \prod_{i=1,3}\left\|\frac{f_i(\tau_i,\xi_i)}{\langle\tau_i-\xi_i^2\rangle^{\frac{\delta}{2}}\langle\xi_i\rangle^{s}}\right\|_{L^2_{\tau_i}L^{\frac{4}{3}}_{\xi_i}}\left\|\frac{f_2(\tau_2,\xi_2)}{\langle\tau_2+\xi_2^2\rangle^{\frac{\delta}{2}}\langle\xi_2\rangle^{\frac{1}{2}-3\delta}}\right\|_{L^2_{\tau_2}L^{\frac{4}{3}}_{\xi_2}},
\end{aligned}$$

by using Hölder and Young inequality. Next we prove that

$$\begin{aligned}
\left\|\frac{f_i(\tau_i,\xi_i)}{\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{\delta}{2}}\langle\xi_i\rangle^{s}}\right\|_{L^2_{\tau_i}L^{\frac{4}{3}}_{\xi_i}} &\le \|\langle\xi_i\rangle^{-s}\|_{L^4_{\xi_4}}\|\langle\tau_i+(-1)^i\xi_i^2\rangle^{-\frac{\delta}{2}}f_i(\tau_i,\xi_i)\|_{L^2_{\tau_i}L^2_{\xi_i}}\\
&\lesssim \|\langle\xi_i\rangle\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}-\frac{\delta}{2}}\widehat{Iv_i}(\tau_i,\xi_i)\|_{L^2_{\tau_i}L^2_{\xi_i}}\\
&\lesssim T^{\varepsilon'}\|Iv_i\|_{X_{1,\frac{1}{2}}}, \qquad (6.53)
\end{aligned}$$

we get the desired estimate. Similary, one can show that

$$\|I\left(\mathcal{F}_x^{-1}\{|\xi|\hat{\bar{v}}_1(\xi)\hat{v}_2(\xi)\hat{v}_3(\xi)\}\right)\|_{Y_{1,-1}} \lesssim T^{\varepsilon'}\Pi_{i=1}^3\|Iv_i\|_{X_{1,\frac{1}{2}}}.$$

□

Lemma 6.4.5. *Let v be the Schwartz function with spatial periodic. There exists $\epsilon'>0$ such that for $T\in(0,1]$, then*

$$\left\|I\{(|v|^n-\int_{\mathbb{T}}|v|^n)v\}\right\|_{X_{1,-\frac{1}{2}}\cap Y_{1,-1}} \lesssim T^{\epsilon'}\|Iv\|^n_{X_{1,\frac{1}{2}}}, \qquad (6.54)$$

where $n=2,4$.

Fix $n=2$, taking Fourier transform, we have

$$\mathcal{F}_x[(v_1\bar{v}_2-\int_{\mathbb{T}}v_1\bar{v}_2)v_3]=\int_{\xi=\xi_1-\xi_2+\xi_3}\varphi(\xi_1-\xi_2)\hat{v}_1(\xi_1,\tau_1)\overline{\hat{v}_2(\xi_2,\tau_2)}\hat{v}_3(\xi_3,\tau_3).$$

By Plancherel and duality, it is enough to show that

$$\left|\int_*\frac{\varphi(\xi_{12})\langle\xi_4\rangle m(\xi_4)\prod_{i=1}^4 f_i(\tau_i,\xi_i)}{\langle\tau_4+\xi_4^2\rangle^{\frac{1}{2}}\prod_{i=1}^3 m(\xi_i)\langle\xi_i\rangle\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}}}\right| \lesssim T^{\varepsilon'}\prod_{i=1}^4\|f_i\|_{L^2_{\tau_i}L^2_{\xi_i}}, \qquad (6.55)$$

where all of f_i are real and nonnegative functions and $\int_*$ denote the integration over the measure $\delta(\tau_1+\tau_2+\tau_3+\tau_4)\delta(\xi_1-\xi_2+\xi_3-\xi_4)$. We first assume that

$$\frac{m(\xi_4)\langle\xi_4\rangle|\xi_2|}{\prod_{i=1}^3 m(\xi_i)\langle\xi_i\rangle}\lesssim 1.$$

In this case we use Cauchy Schwarz inequality on the left hand side of (6.55), then

$$\left|\int_* \frac{\varphi(\xi_{12})\langle\xi_4\rangle m(\xi_4)\prod_{i=1}^4 f_i(\tau_i,\xi_i)}{\langle\tau_4+\xi_4^2\rangle^{\frac{1}{2}}\prod_{i=1}^3 m(\xi_i)\langle\xi_i\rangle\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}}}\right|$$
$$\le \|f_4\|_{L^2_{\tau_4}L^2_{\xi_4}}\left\|\prod_{i=1}^3 \frac{\langle\tau_4+\xi_4^2\rangle^{-\frac{1}{2}} f_i(\tau_i,\xi_i)}{\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}}}\right\|_{L^2_{\tau_4}L^2_{\xi_4}}.$$

Thus we can show that

$$\left\|\prod_{i=1}^3 \frac{f_i(\tau_i,\xi_i)}{\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}}}\right\|_{X_{0,-\frac{1}{2}}} \lesssim \left\|\prod_{i=1}^3 J_x I v_i(\tau_i,\xi_i)\right\|_{L_t^{\frac{4}{3}}L_x^{\frac{4}{3}}}.$$

The desired estimate follow by Hölder inequality, L^4 Strichartz estimate and Sobolev embedding in time.

Next we may assume that $\frac{m(\xi_4)\langle\xi_4\rangle|\xi_2|}{\prod_{i=1}^3 m(\xi_i)\langle\xi_i\rangle} \ge 1$. Here we know that

$$\frac{m(\xi_4)\langle\xi_4\rangle^{1-s}}{\prod_{i=1}^3 m(\xi_i)\langle\xi_i\rangle^{1-s}} \lesssim 1$$

for two region such that $|\xi_1| \le \frac{1}{2}|\xi_4| \le |\xi_3|$ and $|\xi_3| \le \frac{1}{2}|\xi_4| \le |\xi_1|$. Hence it is enough to show that

$$\left|\int_* \frac{\varphi(\xi_{12})\langle\xi_4\rangle^s\prod_{i=1}^4 f_i(\tau_i,\xi_i)}{\langle\tau_4+\xi_4^2\rangle^{\frac{1}{2}}\prod_{i=1}^3\langle\xi_i\rangle^s\langle\tau_i+(-1)^i\xi_i^2\rangle^{\frac{1}{2}}}\right| \le \prod_{i=1}^4 \|f_i\|_{L^2_{\tau_i}L^2_{\xi_i}}.$$

By undoing the duality we can write

$$\left\|(v_1\overline{v}_2 - \int_{\mathbb{T}} v_1\overline{v}_2)v_3\right\|_{X_{s,-\frac{1}{2}}} \lesssim T^{\varepsilon'}\|v_1\|_{X_{\frac{3}{8},\frac{1}{2}}}\|v_2\|_{X_{\frac{3}{8},\frac{1}{2}}}\|v_3\|_{X_{s,\frac{1}{2}}}.$$

We consider two cases. We first assume $|\xi_3| = \max_{i=1,2,3}|\xi_i|$, then

$$\left\|(v_1\overline{v}_2 - \int_{\mathbb{T}} v_1\overline{v}_2)v_3\right\|_{X_{s,-\frac{1}{2}}} \le \|v_1\|_{X_{\frac{3}{8},\frac{3}{8}}}\|v_2\|_{X_{\frac{3}{8},\frac{3}{8}}}\|v_3\|_{X_{s,\frac{3}{8}}}.$$

On the other hand, if $|\xi_3| \le \max_{i=1,2,3}|\xi_i|$, then

$$\left\|(v_1\overline{v}_2 - \int_{\mathbb{T}} v_1\overline{v}_2)v_3\right\|_{X_{s,-\frac{1}{2}}} \le \|v_1\|_{X_{\frac{3}{8},\frac{3}{8}}}\|v_2\|_{X_{s,\frac{3}{8}}}\|v_3\|_{X_{\frac{3}{8},\frac{3}{8}}},$$

by using the fractional Leibnitz rule, Hölder and L^4 dual Strichartz estimate. Similarly, we can prove that

$$\left\|I\{(v_1\overline{v}_2 - \int_{\mathbb{T}} v_1\overline{v}_2)v_3\}\right\|_{Y_{1,-1}} \lesssim T^{\epsilon'}\|Iv\|^3_{X_{1,\frac{1}{2}}}.$$

We may prove for $n=4$, if we use more Sobolev estimates. □

It is now turn to prove Theorem 6.4.1.

Proof of Theorem 6.4.1. Fix $T > 0$, to be chosen later. Let $Iv_0 \in H^1$, there exists a constant $\theta > 0$ such that $\|Iv_0\|_{H^1} \leq \theta$. Consider the set $B = \{Iv \in Z_1 : \|Iv\|_{Z_1} \leq 2C\theta\}$ for some constant $C > 0$. We have

$$\mathcal{H}(Iv)(t) = \chi U(t)Iv_0 + i\chi\int_0^t U(t-s)I(\mathcal{N}(v))(s)ds.$$

We assume that $\|\chi Iv\|_{Z_1} \simeq \|Iv\|_{Z_1([-T,T]\times\mathbb{T})}$. We then apply Lemmas 6.4.2, 6.4.3, 6.4.4 and 6.4.5 to have

$$\|\mathcal{H}(Iv)\|_{Z_1([-T,T]\times\mathbb{T})} \leq C\|Iv_0\|_{H^1(\mathbb{T})} + CT^{\epsilon'}(1+\|Iv\|_{Z_1([-T,T]\times\mathbb{T})})^3\|Iv\|^2_{Z_1([-T,T]\times\mathbb{T})}.$$

We choose T small enough so that $T < (8C^3\theta(1+2C\theta)^3)^{-1/\epsilon'}$. We note that T depend only on $\|Iv_0\|_{H^1}$. This completes the proof of Theorem 6.4.1. $\square$

6.5 Estimates of $\Lambda_n, n = 4, 6, 8$.

In this section, we compute the following main estimates.

Lemma 6.5.1. *Let $v_i = v_i(x,t)$ be the $2\pi\lambda$-periodic function with $\lambda < N$ for any δ come from local theory, then*

$$\left|\int_0^\delta \Lambda_n(M_n(\xi_1, \xi_2, ..., \xi_n))\right| \lesssim N^{-1-}\lambda^{-1+}\|Iv\|^n_{Z_1}, \tag{6.56}$$

where $n = 4, 6, 8$.

Lemma 6.5.2. *[8] We consider the following multiplier terms*

$$M_6'(\xi_a, \cdots, \xi_6) = \{\varphi(\xi_{c-d})M_4(\xi_a, \xi_b, \xi_{cde}, \xi_f)\xi_d - \varphi(\xi_{d-e})M_4(\xi_a, \xi_b, \xi_c, \xi_{def})\xi_e\}_{sym},$$

$$M_6''(\xi_a, \cdots, \xi_f) = \{M_4(\xi_{abc}, \xi_d, \xi_e, \xi_f) - M_4(\xi_a, \xi_b, \xi_c, \xi_{def})\}_{sym}$$

and

$$\sigma_6(\xi_1, \cdots, \xi_6) = m_1^2\xi_1^2 - m_2^2\xi_2^2 + \ldots - m_6^2\xi_6^2$$

then the following inequalities are hold:

(i) If $|\xi_3| \geq N$, then $|M_6(\xi_1, \xi_2, ..., \xi_6)| \leq m(N_{max})^2N_{max}^2$,

(ii) If $|\xi_3| \ll N$, then $|M_6(\xi_1, \xi_2, ..., \xi_6)| \leq N_{max}N_3$,

(iii) $|M_6''(\xi_1, \cdots, \xi_6)| \leq N_{max}m^2(N_{max})$.

where $M_6 = M_6' + \sigma_6$ and $\{1, \cdots, 6\} = \{a, \cdots, f\}$.

Proof. The proof of (i) follows from (6.22). For the proof of case (ii), we shall consider two cases and assume that $N_1 \geq \cdots \geq N_6$. .

Case (1). $N_1 \sim N_2$. Then

$$\begin{aligned} |\sigma_6(\xi_1, \cdots, \xi_6)| &\lesssim m(N_1)^2 N_1 N_{12} + m(N_3)^2 N_3^2 \\ &\lesssim m(N_1)^2 N_1 N_3. \end{aligned}$$

Apply (6.22) to estimate M_6', then

$$|M_4(\xi_{abc}, \xi_d, \xi_e, \xi_f)\xi_g| \lesssim m(N_1)^2 N_1 N_3 \tag{6.57}$$

for every $a, \ldots, g \in 1, \cdots, 6$ and $g \neq 1, 2$. Thus,

$$\begin{aligned} &\left| \sum_{\substack{(a,e)=\{3,5\} \\ (d,f)=\{4,6\}}} M_4(\xi_{a21}, \xi_d, \xi_e, \xi_f)\xi_2 + M_4(\xi_a, \xi_{21d}, \xi_e, \xi_f)\xi_1 \right| \\ &+ \left| \sum_{\substack{(a,c)=\{3,5\} \\ (d,f)=\{4,6\}}} M_4(\xi_a, \xi_{12b}, \xi_e, \xi_f)\xi_1 + M_4(\xi_a, \xi_b, \xi_{12e}, \xi_f)\xi_2 \right| \\ &+ \left| \sum_{\substack{(a,c)=\{3,5\} \\ (d,f)=\{4,6\}}} M_4(\xi_a, \xi_b, \xi_{12c}, \xi_f)\xi_2 + M_4(\xi_a, \xi_b, \xi_c, \xi_{12f})\xi_1 \right| \\ &+ \left| \sum_{\substack{(a,e)=\{3,5\} \\ (d,f)=\{4,6\}}} M_4(\xi_{a2c}, \xi_d, \xi_1, \xi_f)\xi_2 + M_4(\xi_a, \xi_2, \xi_c, \xi_{d1f})\xi_1 \right| = \sum_{i=1}^{4} I_i. \end{aligned}$$

The function M_4 in $\Sigma_{i=1}^3$ are strictly smaller than $\frac{N}{2}$ and by (6.22) then

$$\Sigma_{i=1}^3 I_i \lesssim N_1 N_3.$$

We use (6.24) and the symmetry of M_4, then

$$I_4 \lesssim N_1 N_3.$$

Case(2). $N_1 \sim N_3$. We need some cancellation between the large terms coming from $\sigma_6(\xi_1, \ldots, \xi_6)$ and the large terms of the sum of the M_4. From (6.57), then

$$\begin{aligned} \tilde{M}_6(\xi_1, \cdots, \xi_6) &= -\frac{1}{6}(m_1^2 \xi_1^2 + m_3^2 \xi_3^2) \\ &\quad -\frac{\xi_1}{36}\left(\sum_{(b,d,f)=\{2,4,6\}} M_4(\xi_a, \xi_{b1d}, \xi_3, \xi_f) + M_4(\xi_a, \xi_b, \xi_3, \xi_{d1f}) \right) \\ &\quad -\frac{\xi_3}{36}\left(\sum_{(b,d,f)=\{2,4,6\}} M_4(\xi_a, \xi_b, \xi_1, \xi_{d3f}) + M_4(\xi_a, \xi_{b3d}, \xi_1, \xi_f)) \right). \end{aligned}$$

We use (6.25) and the symmetry of M_4,

$$\begin{aligned}
\tilde{M}_6(\xi_1,\cdots,\xi_6) &= -\frac{1}{6}(m_1^2\xi_1^2+m_3^2\xi_3^2)\\
&\quad -\frac{\xi_1}{72}\left(\sum_{(b,d,f)=\{2,4,6\}}\frac{m_3^2(\xi_{b1d}^2+\xi_{b1f}^2)}{\xi_3}\right)+O(N_1N_3)\\
&\quad -\frac{\xi_3}{72}\left(\sum_{(b,d,f)=\{2,4,6\}}\frac{m_3^2(\xi_{d3f}^2+\xi_{b3d}^2)}{\xi_1}\right)+O(N_1N_3)\\
&= -\frac{1}{6}(m_1^2\xi_1^2+m_3^2\xi_3^2)\\
&\quad +\frac{1}{72}\left(\sum_{(b,d,f)=\{2,4,6\}} m_3^2(\xi_{b1d}^2+\xi_{b1f}^2)\right)+O(N_1N_3)\\
&\quad +\frac{1}{72}\left(\sum_{(b,d,f)=\{2,4,6\}} m_3^2(\xi_{d3f}^2+\xi_{b3d}^2)\right)+O(N_1N_3)\\
&= -\frac{1}{72}m_3^2\sum_{(b,d,f)=\{2,4,6\}}(\xi_{b1d}^2+\xi_{b1f}^2)(\xi_3^2-\xi_{1fb}^2)\\
&\quad -\frac{1}{72}m_1^2\sum_{(b,d,f)=\{2,4,6\}}(\xi_{d3f}^2+\xi_{b3d}^2)(\xi_1^2-\xi_{b3d}^2)+O(N_1N_3)
\end{aligned}$$

We obtain

$$\tilde{M}_6(\xi_1,\cdots,\xi_6)\lesssim N_1N_3.$$

Finally, $M_6^{''}$ is consequence of (6.22). □

In our proof, we work with the Littlewood-Paley decomposition. We may restrict the frequency of $\widehat{v_i}$ is as $|\xi_i|\sim N_i$ according to the dyadic decomposition and assume that $N_1\geq N_2\cdots\geq N_i$, i= 4, 6, 8. When all frequencies are smaller than N, the multiplier $M_j, j=4,6,8$ vanish over the integral of Λ_j.

Lemma 6.5.3. *Let $v_i=v_i(t,x)$ be the $2\pi\lambda$-periodic function with $\lambda<N$ for any δ come from the local theory, then*

$$\left|\int_0^\delta \Lambda_6(M_6'(\xi_1,\cdots,\xi_n))\right|\lesssim N^{-1-}\lambda^{-1+}\|Iv\|_{Z_1}^6. \tag{6.58}$$

Proof. It is enough to consider the following three cases.

Case (1). $N_4\geq N\gg N_5$. By Lemma 6.5.2 we can rewrite the left hand side of (6.58) as

$$\left|\int_0^\delta\int m(N_1)N_1m(N_2)N_2\Pi_{i=1}^6\widehat{v_i}\right|\sim\int_0^\delta\int\left|\frac{N_1Iv_1N_2\overline{Iv_2}Iv_3\Pi_{i=4}^6Iv_i}{m(N_3)m(N_4)}\right|.$$

Since $m(N_j) \sim \langle N_j \rangle^{s-1} N^{1-s}$ with $j = 3, 4$, the above estimate is bounded by

$$N^{2(s-1)} \int_0^\delta \int \left| \Pi_{i=1}^2 \langle N_i \rangle I v_i \langle N_3 \rangle^{1-s} \langle N_4 \rangle^{1-s} \overline{I v_4} v_5 \overline{v_6} \right|$$

Using the Hölder's inequality, L^5 for Iv_1, Iv_2, Iv_3, Iv_4 and L^{10} for v_5 and v_6. Then

$$\begin{aligned} \left| \int_0^\delta \Lambda_6(M_6'(\xi_1, \cdots, \xi_6)) \right| &\lesssim N^{2(s-1)} \Pi_{i=1,2} \| J I v_i \|_{L_t^5 L_x^5} \Pi_{j=3,4} \| J^{1-s} I v_j \|_{L_t^5 L_x^5} \\ &\quad \times \Pi_{k=5,6} \| v_k \|_{L_t^{10} L_x^{10}}. \end{aligned}$$

We then apply the Strichartz estimate for L^5 and Sobolev embedding for L^{10}. We get the estimate

$$\left| \int_0^\delta \Lambda_6(M_6'(\xi_1, \cdots, \xi_6)) \right| \lesssim N^{-2+} \Pi_{i=1}^6 \| I v \|_{X_{1, \frac{1}{2}-}}.$$

Case(2). $N_3 \geq N \gg N_4$. By Lemma (6.5.2), the left hand side of (6.58) becomes

$$\left| \int \eta(t) \int m(N_1) N_1 m(N_2) N_2 \Pi_{i=1}^6 \widehat{v_i} \right| \leq N^{s-1} \left| \int_0^\delta \int \eta(t) N_1 I v_1 N_2 \overline{I v_2} N_3^{1-s} I v_3 \Pi_{i=4}^6 v_i \right|$$

By the fact that $m(N_3) \sim \langle N_3 \rangle^{s-1} N^{1-s}$ and Hölder's inequality, we have

$$N^{-1} \| J I v_1 J I v_3 \|_{L_t^2 L_x^2} \| \eta(t) J I v_2 v_4 \|_{L_t^2 L_x^2} \Pi_{i=5,6} \| v_i \|_{L_t^\infty L_x^\infty}$$

Since $N_1 + N_3 \sim N_2$, applying (3.14) for $v_1 v_3$ and $v_2 v_4$, we get

$$N^{-1} \left(\frac{1}{\lambda} + \frac{1}{N_2} \right)^{1-} \| J I v_1 \|_{X_{0,\frac{1}{2}}} \| J I v_3 \|_{X_{0,\frac{1}{2}}} \| J I v_2 \|_{X_{0,\frac{1}{2}}} \| v_4 \|_{X_{0,\frac{1}{2}}} \Pi_{i=5,6} \| v_i \|_{L_t^\infty L_x^\infty}.$$

Since $\lambda < N_2$ we see that

$$\begin{aligned} \left| \int \Lambda_6(M_6'(\xi_1, \cdots, \xi_6)) \right| &\lesssim N^{-1} \lambda^{-1+} \| J I v_1 \|_{X_{0,\frac{1}{2}}} \| J I v_3 \|_{X_{0,\frac{1}{2}}} \| J I v_2 \|_{X_{0,\frac{1}{2}}} \\ &\quad \times \| v_4 \|_{X_{0,\frac{1}{2}}} \Pi_{i=5,6} \| v_i \|_{X_{\frac{1}{2}+, \frac{1}{2}+}}. \end{aligned}$$

Furthermore, since v_5 and v_6 are of low frequency and $\langle \tau - \xi^2 \rangle \ll N$, we obtain

$$\left| \int_0^\delta \Lambda_6(M_6'(\xi_1, \cdots, \xi_6)) \right| \lesssim N^{-1-} \lambda^{-1+} \lambda^{0+} \Pi_{i=1}^6 \| I v_i \|_{X_{1,\frac{1}{2}}}.$$

Case (3). $N_1 \sim N_2 \geq N \gg N_3$. By Lemma 6.5.2, the left hand side of (6.58) becomes

$$\left| \int \int \eta(t) \Lambda_6(M_6'(\xi_1, \cdots, \xi_6)) \right| \leq \int \int \eta(t) \left| N_1 I v_1 \overline{I} v_2 N_3 v_3 \overline{v}_4 v_5 \overline{v}_6 \right|.$$

As in the proof of case (1), we prove that

$$\left| \int_0^\delta \int \Lambda_6(M_6'(\xi_1, \cdots, \xi_6)) \right| \leq N^{-1-} \lambda^{-1+} \lambda^{0+} \Pi_{i=1}^6 \| I v_i \|_{X_{1,\frac{1}{2}}}.$$

$\square$

Now we are ready to show the estimate of $\Lambda_6(M_6^{''})$.

Lemma 6.5.4. *Let $v_i = v_i(t,x)$ be the $2\pi\lambda$-periodic function with $\lambda < N$ for any δ come from the local theory, then*

$$\left|\int_0^\delta \Lambda_6(M_6^{''}(\xi_1,\cdots,\xi_6))\right| \lesssim N^{-1-}\lambda^{-1+}\|Iv\|_{Z_1}^6. \tag{6.59}$$

Proof. We can rewrite the left hand side of (6.59) as

$$\left|\int_0^\delta \Lambda_6(M_6^{''}(\xi_1,\cdots,\xi_6))\right| \lesssim \int_0^\delta\int\left|\frac{N_1Iv_1\overline{Iv_2}Iv_3\Pi_{i=4}^6 v_i}{m(N_3)}\right|.$$

In case(1) $N_3 \gg N \geq N_4$.

Since $m(N_3) \sim \langle N_3\rangle^{s-1}N^{1-s}$, then

$$\begin{aligned}\left|\int_0^\delta \Lambda_6(M_6^{''}(\xi_1,\cdots,\xi_6))\right| &\lesssim \lambda^{0+}N^{-2}\Pi_{i=1}^3\|JIv_i\|_{L_t^4L_x^4}\Pi_{j=4}^6\|v_j\|_{L_t^{12}L_x^{12}}\\ &\lesssim \lambda^{0+}N^{-2}\|Iv\|_{X_{1,\frac{3}{8}}}^3\|Iv\|_{X_{\frac{5}{12},\frac{5}{12}}}^3.\end{aligned}$$

In case(2) $N_2 \gg N \geq N_3$.

By Hölder's inequality,

$$\left|\int_0^\delta \Lambda_6(M_6^{''}(\xi_1,\cdots,\xi_6))\right| \lesssim N^{-1}\|Iv_1v_3\|_{L_{t,x}^2}\|\eta(t)Iv_2v_4\|_{L_t^2L_x^2}\Pi_{i=5,6}\|v_i\|_{L_t^\infty L_x^\infty}.$$

Then desired estimate follows by (3.14) and Sobolev embedding,

$$\left|\int_0^\delta \Lambda_6(M_6^{''}(\xi_1,\cdots,\xi_6))\right| \lesssim \lambda^{0+}N^{-1-}\lambda^{-1+}\|Iv\|_{X_{1,\frac{1}{2}}}^6.$$

□

Lemma 6.5.5. *We consider the M_8 multiplier*

$$M_8(\xi_1,\cdots,\xi_8) = \{M_4(\xi_{12345},\xi_6,\xi_7,\xi_8) - M_4(\xi_1,\xi_2,\xi_3,\xi_{45678})\}_{sym}$$

then

$$|M_8(\xi_1,\cdots,\xi_8)| \leq N_{max}m^2(N_{max}).$$

Proof. This lemma is a consequence of Lemma 6.21. □

Lemma 6.5.6. *Let $v_i = v_i(t,x)$ be the $2\pi\lambda$-periodic function with $\lambda < N$ for any δ come from the local theory, then*

$$\left|\int_0^\delta \Lambda_8(M_n(\xi_1,\cdots,\xi_8))\right| \lesssim N^{-1-}\lambda^{-1+}\|Iv\|_{Z_1}^8. \tag{6.60}$$

Proof. It is enough to consider two cases, case (1) $N_3 \geq N$; case (2) $N_3 \leq N$. When all frequencies are smaller than N, we observe that M_8 vanishes over the Λ_8 integral.

In case(1) $N_3 \gg N \geq N_4$.

By Lemma 6.5.5, we can rewrite the left hand side of (6.60) as

$$\left|\int_0^\delta \Lambda_8(\xi_1, \cdots, \xi_8)\right| \lesssim \int_0^\delta \int \left|\frac{N_1 Iv_1 \overline{Iv_2} Iv_3 \Pi_{i=4}^8 v_i}{m(N_3)}\right|.$$

since $m(N_3) \sim \langle N_3\rangle^{s-1} N^{1-s}$, then

$$\begin{aligned}\left|\int_0^\delta \Lambda_8(\xi_1, \cdots, \xi_8)\right| &\lesssim \lambda^{0+} N^{-2} \Pi_{i=1}^3 \|JIv_i\|_{L_t^4 L_x^4} \Pi_{j=4}^8 \|v_j\|_{L_t^{20} L_x^{20}} \\ &\lesssim \lambda^{0+} N^{-2+} \|Iv\|_{X_{1,\frac{3}{8}}}^3 \|Iv\|_{X_{\frac{9}{20},\frac{9}{20}}}^5.\end{aligned}$$

In case(2). $N_2 \gg N \geq N_3$.

By Lemma 6.5.5, and the Hölder's inequality,

$$\left|\int_0^\delta \Lambda_8(\xi_1, \cdots, \xi_8)\right| \lesssim N^{-1} \|Iv_1 v_3\|_{L_{t,x}^2} \|Iv_2 v_4\|_{L_t^2 L_x^2} \Pi_{i=5}^8 \|v_i\|_{L_t^\infty L_x^\infty}.$$

Then desired estimate follows by (3.14) and Sobolev embedding,

$$\left|\int_0^\delta \Lambda_8(\xi_1, \cdots, \xi_8)\right| \lesssim \lambda^{0+} N^{-1-} \lambda^{-1+} \|Iv\|_{X_{1,\frac{1}{2}}}^8.$$

□

Lemma 6.5.7. *Assume that*

$$\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4) = m_1^2 \xi_1^2 - m_2^2 \xi_2 + m_3^2 \xi_3^2 - m_4^2 \xi_4^2.$$

We prove the following cases

(i) $|\xi_1|, |\xi_2|, |\xi_3| \geq 1 \gg |\xi_4|$,

(ii) $|\xi_1|, |\xi_2| \geq 1 \gg |\xi_3|, |\xi_4|$

then

$$|\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim m_{\max}^2 |\xi_1||\xi_3|.$$

Proof. If $|\xi_i| \ll 1$ for (i=1,2,3,4), we may easily seen that σ_4 is vanish.

(i). By DMVT, we obtain

$$\begin{aligned}|\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4)| &\leq m_1^2 |\xi_1 - \xi_2||\xi_1 - \xi_4| \\ &\lesssim m_{\max}^2 |\xi_{\max}|^2.\end{aligned}$$

(iii). We may assume $|\xi_1 - \xi_2| \leq 2|\xi_3|$ by symmetry and MVT

$$\begin{aligned}|\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4)| &\leq m_1^2 |\xi_1 - \xi_2||\xi_1| + (\xi_3 - \xi_4)(\xi_3 + \xi_4) \\ &\lesssim m_{\max}^2\ |\xi_1||\xi_3| + |\xi_1||\xi_3|.\end{aligned}$$

□

Lemma 6.5.8. *Let $v_i = v_i(t,x)$ be the $2\pi\lambda$-periodic function with $\lambda < N$ for any δ come from the local theory, then*

$$\left|\int_0^\delta \Lambda_4(\sigma_4(\xi_1,\cdots,\xi_4))\right| \lesssim N^{-1-}\lambda^{-1+}\|Iv\|_{Z_1}^4. \tag{6.61}$$

Proof. It is enough to consider two cases, case(1) $N_3 \geq N$; case (2) $N_3 \leq N$. When all frequencies are smaller than N, we observe that σ_4 vanishes over the Λ_4 integral.

Case (1). $N_3 \geq N \geq N_4$. By Lemma 6.5.7, we may rewrite the left hand side of (6.61) as

$$\begin{aligned}\left|\int_0^\delta \Lambda_4(\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4))\right| &\lesssim \left|\int_0^\delta\int \eta(t)N_1Iv_1N_2\overline{Iv_2}v_3\overline{v_4}\right| \\ &\lesssim N^{-1}\|\eta(t)N_1Iv_1N_3Iv_3\|_{L_t^2L_x^2}\|N_2Iv_2v_4\|_{L_t^2L_x^2} \\ &\lesssim N^{-1+}(\frac{1}{\lambda}+\frac{1}{N_2})^{1-}\Pi_{i=1,2}\|Iv_i\|_{X_{1,\frac{1}{2}}}\|Iv_3\|_{X_{1,\frac{1}{2}}}\|v_4\|_{X_{0,\frac{1}{2}}} \\ &\lesssim \lambda^{0+}N^{-1}\lambda^{-1+}\|Iv\|_{X_{1,\frac{1}{2}}}^4,\end{aligned}$$

by Hölder's inequality and the L^4 Strichartz estimate, we get the desired estimate.

Case(2). $N_1 \sim N_2 \geq N \gg N_3$. By Lemma 6.5.7, we may rewrite the left hand side of (6.61) as

$$\left|\int_0^\delta \Lambda_4(\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4))\right| \lesssim \left|\int_0^\delta\int N_1Iv_1\overline{Iv_2}N_3v_3\overline{v_4}\right|.$$

We apply (3.14), Hölder and Strichartz estimate, then

$$\left|\int_0^\delta \Lambda_4(\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4))\right| \lesssim N^{-1}\lambda^{-1+}\|Iv\|_{X_{1,\frac{1}{2}}}^4.$$

□

Now we are in a position to prove Theorem 6.1.1.

Proof of Theorem 6.1.1. Let $u_0 \in H^s$ with $\frac{1}{2} < s < 1$. Let $u_\lambda(t,x) = \lambda^{-1/2}u(t/\lambda^2, x/\lambda)$, where $\lambda > 1$. By Plancherel,

$$\|\partial_x Iu_\lambda(0)\|_{L^2} \leq \frac{N^{1-s}}{\lambda^s}\|u(0)\|_{H^s}.$$

We choose $\lambda \sim N^{\frac{1-s}{s}}$ for $\frac{1}{2} < s < 1$ which implies $\lambda \lesssim N$. Then we see that

$$\|Iu_\lambda(0)\|_{H^1} \lesssim 1,$$

since the initial data is in H^s.

We use Proposition 6.2.4 to have

$$E_N^2(t) \lesssim E_N^2(0) + \int_0^\delta [\Lambda_4(M_4) + \Lambda_6(M_6) + \Lambda_8(M_8)]dt.$$

By Lemma 6.5.1, we have that

$$E_N^2(t) \lesssim E_N^2(0) + N^{-1-}\lambda^{-1+}(\|Iv\|_{Z_1}^4 + \|Iv\|_{Z_1}^6 + \|Iv\|_{Z_1}^8).$$

Now we apply Theorem 6.4.1 to obtain

$$E_N^2(t) \lesssim E_N^2(0) + N^{-1-}\lambda^{-1+}(\theta^4 + \theta^6 + \theta^8),$$

for any $t \in (0,1)$. We may assume $\theta \geq 1$ without loss of generality. By iteration, we can prove for all $t \in [0,T]$,

$$E_N^2(t) \lesssim E_N^2(0) + N^{-1-}\lambda^{-1+}T\theta^8.$$

Here we can choose the size of time T such that $TN^{-1-}\lambda^{-1+} \lesssim 1$ and take $N = N(T)$ that for $t \in [0, N^{1+}\lambda^{1-}]$,

$$N^{1+}\lambda^{1-} > \lambda^2 T$$

$$T < N^{1+}\lambda^{-1+} \sim N^{1-(\frac{1-s}{s})+} \to \infty$$

as $N \to \infty$, if $s > \frac{1}{2}$.

Hence the proof is completed for global well-posedness with $s > \frac{1}{2}$. $\square$

6.6 Notes and references

Equation (6.1)-(6.2) describe the long wavelength dynamics of dispersive Alfvén waves propagating along an ambient magnetic field and satisfies the infinitely many conservation laws [38]. It is known that the global well-posedness in H^1 regularity is obtained via the laws of conservation of mass and energy. In the real line case, N. Hayashi and T. Ozawa [24], [25], [34] prove the global well-posedness in H^1 assuming the smallness condition $\|u_0\|^2_{L^2(\mathbb{R})} \leq 2\pi$. This result was improved by H. Takaoka [40]. He proves the global well-posedness in H^s for $s > \frac{32}{33}$ and the argument is based on the Fourier truncation method of J. Bourgain [2], estimating separately high and low frequencies of initial data.

J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao [8] improve the global result for H^s, $(s > \frac{2}{3})$ which satisfies the small initial data in L^2 with I-method. In [9], the same authors modify the energies in almost conservation laws and extend the order of dispersive derivative operator $(\partial_t - i\partial_{xx})$, then the global result for H^s, $(s > \frac{1}{2})$ except the end point is obtained. In their proof, a bilinear refinement of Strichartz estimate is a central tool and they use the space $X_{s,b}$ with $b > \frac{3}{4}$.

In the case of periodic, S. Herr [26] adjusts the gauge transformation for periodic setting and proves the same local well-posedness result as H. Takaoka [39]. He uses only the L^4 Strichartz estimate because of the absence of local smoothing and he also shows a global well-posedness in $H^s(s \geq 1)$ via conservation laws.

The crucial differences between the periodic and non-periodic cases are the condition of gauge transformation and the dispersive properties of solution. Due to gauge transform with periodic setting the transformed equation has new nonlinear terms which do not appear in the real line case. In the periodic case, one of difficulties is how to handle these new nonlinear terms of transformed equations. We use the structure of the equation to cancel some additional terms out, when we consider the differential equations associated with the almost conservation laws. See section 6.3.

As the absent of local smoothing (a sense of weak dispersive properties) in periodic setting, when we consider the local well-posedness of DNLS, the L^4 Strichartz estimate with the loss of $\epsilon > 0$ derivative is only available to contral derivative in nonlinear terms. In the case of global well-posedness, we found again the difficulty that the order of dispersive derivative $(\partial_t - i\partial_{xx})$ could not extended to get a good decay like a real line case. To get over this we consider how to handle the interaction between two functions with frequency of the same order. See section 3.3.

We use the same idea of the proof of local estimate as in [23], [26] and [39]. The proof of Lemma 6.2.2 is used the same idea as Lemma 3.6 in [9]. In the statement of Proposition 6.2.3, we use the same notation as in [8]-[9]. Lemma 6.2.4 is Lemma 3.8 in [8]. The M_4 multilinear estimates of Lemma 6.3.2 and 6.3.3 are Lemma 4.1 and 4.2, M_6 multilinear estimates of Lemma 6.5.2 is Lemma 6.4 in [8].

It is known that when u solves the initial value problem (6.1)-(6.2), $u_\lambda(t,x) =$

$\lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda x)$, for all $\lambda > 0$, is also a solution. We have $\|D_x^s u_\lambda(0)\|_{L_x^2} = \lambda^s \|D_x^s u_0\|_{L_x^2}$, hence the critial regularity for the scaling argument is L^2 but it is still open. We see that low regularity problem is easy to extend from local to global in time by using conservation laws. When one work the Cauchy problem under scaling invariant regularities, example below L^2 and H^1, the almost conserved quantities is essential to extend a local to global theory. Here we notice that conservation laws are based on L^2 space, hence they do not yield in general L^p based norms for global.

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