GAUGE THEORY ON INFINITE CONNECTED SUM AND MEAN DIMENSION

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Abstract. We study the geometry of infinite dimensional moduli spaces coming from the Yang-Mills gauge theory over infinite connected sum spaces. We develop the technique of gluing infinitely many instantons, and apply it to the evaluation of the mean dimension of the moduli spaces.

1. Introduction

Nonlinear analysis on non-compact manifolds is a challenging research field. We study the infinite energy Yang-Mills gauge theory on certain non-compact 4-manifolds (infinite connected sums of S^4).

Let Γ be a finitely generated infinite group with a finite generating set S. We suppose that S does not contain the identity element. We consider the infinite connected sum space $(S^4)^{\sharp(\Gamma,S)}$ by gluing the copies of S^4 "along the Cayley graph of (Γ, S)". (Its precise definition will be given in Section 2.) $(S^4)^{\sharp(\Gamma,S)}$ is a non-compact 4-manifold. Γ naturally acts on $(S^4)^{\sharp(\Gamma,S)}$. For example, if $(\Gamma, S) = (\mathbb{Z}, \{1\})$, then $(S^4)^{\sharp(\Gamma,S)}$ is conformally equivalent to $S^3 \times \mathbb{R}$.

Fix $c > 0$. We want to study $SU(2)$-ASD connections A on $(S^4)^{\sharp(\Gamma,S)}$ satisfying $\|F_A\|_{L^\infty} \leq c$. (Here we consider "$L^\infty$-norm condition" for simplicity of the explanation. We will consider more general conditions later.) Since the base space $(S^4)^{\sharp(\Gamma,S)}$ is non-compact, such ASD connections can have infinite L^2-energy, and their moduli space $\mathcal{M}$ can be an infinite dimensional space. The moduli space $\mathcal{M}$ admits a natural Γ-action. The main subject of this paper is the evaluation of the "mean dimension" $\dim(\mathcal{M} : \Gamma)$. Mean dimension is a notion introduced by Gromov [8]. (See also Lindenstrauss-Weiss [10] and Lindenstrauss [9].) Intuitively, the mean dimension $\dim(\mathcal{M} : \Gamma)$ is given by

$$\dim(\mathcal{M} : \Gamma) = \dim \mathcal{M}/|\Gamma|.$$

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(We give the precise definition of mean dimension in Appendix B.) In particular, if $\mathcal{M}$ is a finite dimensional space (in the usual sense), then $\dim(\mathcal{M} : \Gamma) = 0$. Hence the value of $\dim(\mathcal{M} : \Gamma)$ has an information about "infinite dimensional geometry" of $\mathcal{M}$.

We study $\mathcal{M}$ by using the technique of "gluing an infinite number of instantons". Gluing instantons is a famous technique in the gauge theory. (Taubes [12], Donaldson [3], etc.) In Tsukamoto [13], we studied the technique of gluing infinitely many instantons. In the present paper we will develop this gluing technique more thoroughly and apply it to the evaluation of the mean dimension $\dim(\mathcal{M} : \Gamma)$. The main body of the paper is devoted to the detailed (rather technical) study of this infinite gluing construction.

The application of the gluing technique to the theory of mean dimension is suggested by Gromov [8, p. 403, 3.6.6] in the context of (pseudo-)holomorphic curves. Gournay [7] studies the application of the gluing technique of pseudo-holomorphic curves to the problem of mean dimension.

2. Gauge theory on infinite connected sum of S^4

2.1. **Main results.** Let Γ be a finitely generated infinite group. Let $S = \{s_1, \cdots, s_{|S|}\} \subset \Gamma$ be a finite generating set which does not contain the identity element. Here we don't suppose that $S = S^{-1}$. Easy examples are $(\Gamma, S) = (\mathbb{Z}, \{1\}), (\mathbb{Z}^2, \{(1,0),(0,1)\})$.

Let S^4 be the 4-sphere and x_s and y_s ($s \in S$) be $2|S|$ distinct points in S^4. We will construct an infinite connected sum $(S^4)^{\sharp(\Gamma,S)}$ by patching the copies of S^4 "along the Cayley graph of (Γ, S)". The following construction is based on the "conformal connected sum" described in Donaldson-Kronheimer [5, Section 7.2].

Since the standard metric on S^4 is conformally flat, S^4 has a Riemannian metric h satisfying the following:
(i) h is conformally equivalent to the standard metric.
(ii) h is flat in some neighborhood of each x_s and y_s ($s \in S$).

Of course, h is not uniquely determined by these conditions. The important condition is the first one. The second condition is just for simplicity. Let λ be a positive (very small) parameters. For $x \in S^4$ and $r > 0$, we denote $B(x, r)$ (resp. $\bar{B}(x, r)$) as the open (resp. closed) ball of radius r centered at x (with respect to the metric h). Set

$$U := S^4 \setminus \left(\bigsqcup_{s \in S} \bar{B}(x_s, \sqrt{\lambda}/2) \sqcup \bar{B}(y_s, \sqrt{\lambda}/2) \right).$$

For each $s \in S$ we choose an orientation-*reversing* linear isometry $\sigma_s : T_{x_s} S^4 \to T_{y_s} S^4$.

For each $\gamma \in \Gamma$, let (S^4_γ, h_γ) be the isometric copy of (S^4, h). Let $x_{\gamma,s}$ and $y_{\gamma,s}$ ($s \in S$) be the points on S^4_γ corresponding to x_s and y_s on S^4. S^4_γ has the open set U_γ corresponding to U in S^4. We define the connected sum $(S^4)^{\sharp(\Gamma,S)}$ by

$$(S^4)^{\sharp(\Gamma,S)} := \left(\bigsqcup_{\gamma \in \Gamma} U_\gamma \right) / \sim,$$

where the identification $\sim$ is given as follows; We identify the annulus region $B(x_{\gamma,s}, 2\sqrt{\lambda}) \setminus \bar{B}(x_{\gamma,s}, \sqrt{\lambda}/2)$ in S^4_γ with $B(y_{\gamma s,s}, 2\sqrt{\lambda}) \setminus \bar{B}(y_{\gamma s,s}, \sqrt{\lambda}/2)$ in $S^4_{\gamma s}$ by

$$
\begin{aligned}
(1) \qquad & B(x_{\gamma,s}, 2\sqrt{\lambda}) \setminus \bar{B}(x_{\gamma,s}, \sqrt{\lambda}/2) \ni \xi \sim \eta \in B(y_{\gamma s,s}, 2\sqrt{\lambda}) \setminus \bar{B}(y_{\gamma s,s}, \sqrt{\lambda}/2), \\
& \overset{\text{def}}{\Longleftrightarrow} \eta = \lambda\sigma_s(\xi)/|\xi|^2.
\end{aligned}
$$

Here ξ and η are the normal coordinates centered at $x_{\gamma,s}$ and $y_{\gamma s,s}$, and we consider σ_s as a map from $T_{x_{\gamma,s}}S^4_\gamma$ to $T_{y_{\gamma s,s}}S^4_{\gamma s}$ by identifying $T_{x_{\gamma,s}}S^4_\gamma$ (resp. $T_{y_{\gamma s,s}}S^4_{\gamma s}$) with $T_{x_s}S^4$ (resp. $T_{y_s}S^4$). $(S^4)^{\sharp(\Gamma,S)}$ admits a natural left Γ-action as follows. For $\delta \in \Gamma$ we define $\delta : U_\gamma \to U_{\delta\gamma}$ by sending $p \in U_\gamma$ to $q \in U_{\delta\gamma}$ corresponding to the same point in S^4. This is compatible with the above identification (1). This action is fixed point free, i.e., every $\delta \neq 1$ in Γ has no fixed point.

We choose a Γ-invariant Riemannian metric g on $(S^4)^{\sharp(\Gamma,S)}$ as follows; Let w be a smooth function in S^4 such that $0 \leq w \leq 1$ all over S^4, $w = 1$ in the complement of the balls $B(x_s, \sqrt{\lambda})$ and $B(y_s, \sqrt{\lambda})$ $(s \in S)$, and $w = 0$ on each $\bar{B}(x_s, \sqrt{\lambda}/2)$ and $\bar{B}(y_s, \sqrt{\lambda}/2)$. Let w_γ $(\gamma \in \Gamma)$ be the copy of w defined in S^4_γ. We set

$$g := \sum_{\gamma \in \Gamma} w_\gamma h_\gamma,$$

where h_γ is the Riemannian metric given before. Since the map $\eta = \lambda\sigma_s(\xi)/|\xi|^2$ in (1) is conformal, g is conformally equivalent to each h_γ over U_γ.

We want to study $SU(2)$-ASD connections over $(S^4)^{\sharp(\Gamma,S)}$. Let $c > 0$ be a positive real number and $d \in (2, +\infty]$ (d may be $+\infty$). Let E be a principal $SU(2)$-bundle over $(S^4)^{\sharp(\Gamma,S)}$ and A be an ASD connection on E. We want to study such a pair (E, A) satisfying

$$(2) \qquad \|F_A\|_{L^d(U_\gamma, g)} \leq c \quad \text{for all } \gamma \in \Gamma,$$

where the norm denotes the L^d-norm over the region U_γ defined by the metric g. Let (E, A) and (E', A') be two pairs of a principal $SU(2)$-bundle over $(S^4)^{\sharp(\Gamma,S)}$ and an ASD connection on it. They are called gauge equivalent if there exists a bundle isomorphism $u : E \to E'$ satisfying $u(A) = A'$. We define $\mathcal{M}(c, d)$ as the space of the gauge equivalence classes $[E, A]$ of a principal $SU(2)$-bundle E over $(S^4)^{\sharp(\Gamma,S)}$ and an ASD connection A on E satisfying (2). This space admits a natural right Γ-action: For $[E, A] \in \mathcal{M}(c, d)$ and $\gamma \in \Gamma$, we set

$$(3) \qquad [E, A].\gamma := [\gamma^* E, \gamma^* A],$$

where $\gamma^* E$ and $\gamma^* A$ are the pull-backs of E and A by the map $\gamma : (S^4)^{\sharp(\Gamma,S)} \to (S^4)^{\sharp(\Gamma,S)}$.

Remark 2.1. Since $(S^4)^{\sharp(\Gamma,S)}$ is a non-compact 4-manifold, all principal $SU(2)$-bundle on it is isomorphic to the product bundle $(S^4)^{\sharp(\Gamma,S)} \times SU(2)$. Therefore we can define $\mathcal{M}(c, d)$ as the space of gauge equivalence classes of ASD connections on $(S^4)^{\sharp(\Gamma,S)} \times SU(2)$ satisfying (2). But the above formulation is more flexible.

Remark 2.2. An ASD connection A satisfying the condition (2) is a Yang-Mills analogue of "Brody curve" in the theory of entire holomorphic curves. (cf. Brody [2], Tsukamoto [14, 15].) A holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$ is called a Brody curve if it satisfies $|df|(z) \leq 1$ (or $|df|(z) \leq C$ for some positive constant C) for all $z \in \mathbb{C}$.

$\mathcal{M}(c,d)$ is equipped with the topology of $\mathcal{C}^\infty$-convergence on compact subsets. That is, a sequence $\{[E_n, A_n]\}_{n\geq 1} \subset \mathcal{M}(c,d)$ converges to $[E,A] \in \mathcal{M}(c,d)$ if for any compact set $K \subset (S^4)^{\sharp(\Gamma,S)}$ there exist $n_0(K) > 0$ and bundle maps $u_n : E_n|_K \to E|_K$ (for $n \geq n_0(K)$) such that $u_n(A_n|_K)$ converge to $A|_K$ in the $\mathcal{C}^\infty$-topology. This topology is metrizable.

From $d > 2$ and Uhlenbeck's compactness result ([16, Theorem 1.5 (3.6)], [17]), the moduli space $\mathcal{M}(c,d)$ becomes compact. But I think that this compactness is not so obvious. In some cases (e.g. $(\Gamma, S) = (\mathbb{Z}, \{1\})$) it directly follows from [17, Theorem E']. But the general case needs some clarification. So we will give its proof in Appendix A.

The group action $\mathcal{M}(c,d) \times \Gamma \to \mathcal{M}(c,d)$ defined in (3) is continuous. If Γ is amenable, then we can define the mean dimension $\dim(\mathcal{M}(c,d) : \Gamma)$. (See Gromov [8], Lindenstrauss-Weiss [10], Lindenstrauss [9] and Appendix B.)

For $c > 0$ and $d > 2$ we define $M_{S^4}(c,d)$ as the space of the gauge equivalence classes of $SU(2)$-ASD connections $[A]$ on S^4 satisfying

$$\|F_A\|_{L^d(S^4,h)} \leq c.$$

We denote $\dim M_{S^4}(c,d)$ as the topological (covering) dimension of $M_{S^4}(c,d)$. For $d \in (2, +\infty]$ (d may be $+\infty$), we set

$$\begin{aligned} c_0(d) &:= \sup\{c \geq 0 | \dim M_{S^4}(c,d) = 0\}, \\ &= \sup\{c \geq 0 | M_{S^4}(c,d) = \{[\text{the product connection}]\}\}. \end{aligned}$$

We have $c_0(d) > 0$, and $\dim M_{S^4}(c,d) > 0$ for any $c > c_0(d)$. Our main results on the gauge theory over $(S^4)^{\sharp(\Gamma,S)}$ are the following. The first result concerns with the upper bound on the mean dimension:

Theorem 2.3. *(i) For any $d \in (2, +\infty]$ and $0 \leq c < c_0(d)$ there exists $\lambda_0(c,d) > 0$ such that if $\lambda \leq \lambda_0(c,d)$ then $\mathcal{M}(c,d)$ is equal to the space of the gauge equivalence classes of flat $SU(2)$-connections on $(S^4)^{\sharp(\Gamma,S)}$. Hence if $(S^4)^{\sharp(\Gamma,S)}$ is simply connected, then $\mathcal{M}(c,d)$ is the one-point space.*
(ii) Suppose Γ is amenable. Then for any $0 \leq c < \bar{c} < +\infty$ and $d \in (2, +\infty]$, there exists $\lambda_1(c, \bar{c}, d) > 0$ such that if $\lambda \leq \lambda_1(c, \bar{c}, d)$ then

$$\dim(\mathcal{M}(c,d) : \Gamma) \leq 3|S| + \dim M_{S^4}(\bar{c}, d).$$

The next result is the lower bound on the mean dimension.

Theorem 2.4. *Suppose Γ is amenable. Let $0 < \underline{c} < c < +\infty$ and $2 < d < +\infty$ (d must be finite). If $\dim M_{S^4}(\underline{c}, d) > 0$, then there exists $\lambda_2(\underline{c}, c, d) > 0$ such that for any*

$\lambda \leq \lambda_2(\underline{c}, c, d)$

$$\dim(\mathcal{M}(c,d) : \Gamma) \geq 3|S| + \dim M_{S^4}(\underline{c}, d).$$

Theorem 2.5. *Suppose* Γ *is amenable. For each* $d \in (2, +\infty)$ *there exists a countable set* $\Delta(d) \subset (c_0(d), +\infty)$ *satisfying the following: For any* $c \in (c_0(d), +\infty) \setminus \Delta(d)$ *there exists* $\lambda_3(c,d) > 0$ *such that if* $\lambda \leq \lambda_3(c,d)$ *then we have*

$$\dim(\mathcal{M}(c,d) : \Gamma) = 3|S| + \dim M_{S^4}(c,d).$$

We will prove Theorem 2.3 and 2.4 in Section 8 and 9. Here we prove Theorem 2.5, assuming Theorem 2.3 and 2.4.

Proof of Theorem 2.5. Consider the following non-decreasing function

$$(c_0(d), +\infty) \to \mathbb{Z}_{>0}, \quad c \mapsto \dim M_{S^4}(c,d). \tag{4}$$

Let $\Delta(d) \subset (c_0(d), +\infty)$ be the set of points where (4) is not continuous. $\Delta(d)$ is a countable set. For any $c \in (c_0(d), +\infty) \setminus \Delta(d)$, if we choose $\underline{c}$ and $\bar{c}$ ($\underline{c} < c < \bar{c}$) sufficiently close to c, then we have

$$\dim M_{S^4}(\underline{c}, d) = \dim M_{S^4}(c,d) = \dim M_{S^4}(\bar{c}, d) > 0.$$

Using Theorem 2.3 and 2.4, we get

$$\dim(\mathcal{M}(c,d) : \Gamma) = 3|S| + \dim M_{S^4}(c,d),$$

for $\lambda \ll 1$. $\square$

2.2. **Outline of the proofs of the main theorems.** The proofs of the main theorems (Theorem 2.3, 2.4) need lengthy technical arguments. So we want to describe the brief outline of the proofs in this subsection.

For $c \geq 0$ and $d \in (2, +\infty]$, we call $\theta = (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma \in \Gamma, s \in S}$ a (c,d)-gluing data if the following conditions are satisfied: Each E_γ ($\gamma \in \Gamma$) is a principal $SU(2)$-bundle over S^4_γ, and A_γ is an ASD connection on E_γ satisfying $\|F(A_\gamma)\|_{L^d(S^4_\gamma, h_\gamma)} \leq c$. $\rho_{\gamma,s}$ is an $SU(2)$-isomorphism from $(E_\gamma)_{x_{\gamma,s}}$ to $(E_{\gamma s})_{y_{\gamma s,s}}$. We can consider a natural equivalence relation in the set of (c,d)-gluing data. For each (c,d)-gluing data θ, we will construct a principal $SU(2)$-bundle $\boldsymbol{E}(\theta)$ over $(S^4)^{\sharp(\Gamma,S)}$ and an ASD connection $\boldsymbol{A}(\theta)$ on it by using a "gluing construction".

The proof of Theorem 2.3 proceeds as follows; $0 \leq c < \bar{c} < +\infty$ and $d \in (2, +\infty]$. For each $[E, A] \in \mathcal{M}(c,d)$ we can find a $(\bar{c}, d)$-gluing data θ satisfying $[E, A] = [\boldsymbol{E}(\theta), \boldsymbol{A}(\theta)]$ ($\lambda \ll 1$). (This is the most difficult part of the proof.) If $\bar{c} < c_0(d)$, then we can (easily) prove that $\boldsymbol{A}(\theta)$ is flat. This shows Theorem 2.3 (i). In general, we have

$$\dim(\mathcal{M}(c,d) : \Gamma) \leq \dim(\text{space of } (\bar{c}, d)\text{-gluing data} : \Gamma),$$

and the right-hand-side can be estimated as follows. (The following argument is not rigorous.) Let $\theta = (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma \in \Gamma, s \in S}$ be a $(\bar{c}, d)$-gluing data. For each $\gamma \in \Gamma$, (E_γ, A_γ)

has $\dim M_{S^4}(\bar{c}, d)$ parameters of deformation, and $\rho_{\gamma,s}$ has three parameters (for each $s \in S$). Therefore the number of "deformation parameters" of θ is $(\dim M_{S^4}(\bar{c}, d) + 3|S|)|\Gamma|$. Hence we have

$$\begin{aligned} \dim(\text{space of } (\bar{c}, d)\text{-gluing data} : \Gamma) &= \dim(\text{space of } (\bar{c}, d)\text{-gluing data})/|\Gamma|, \\ &\approx \dim M_{S^4}(\bar{c}, d) + 3|S|. \end{aligned} \tag{5}$$

From this we get Theorem 2.3 (ii).

On the other hand, if $0 < \underline{c} < c < \infty$ and $d \in (2, +\infty)$ then we can prove that, for each $(\underline{c}, d)$-gluing data θ, $[\boldsymbol{E}(\theta), \boldsymbol{A}(\theta)]$ belongs to $\mathcal{M}(c, d)$. (The proof of this facts needs $d < \infty$.) Therefore we have

$$\dim(\mathcal{M}(c, d) : \Gamma) \geq \dim(\text{space of } (\underline{c}, d)\text{-gluing data} : \Gamma).$$

Using (5), we get Theorem 2.4. The above argument does not explain the meaning of the condition "$\dim M_{S^4}(\underline{c}, d) > 0$" in the assumption in Theorem 2.4. This condition concerns with the validity of the equation (5).

3. Infinite gluing construction: preparations

From this section we will develop a theory of "gluing infinitely many instantons" for general closed 4-manifolds. Let X be a compact, oriented Riemannian 4-manifold with prescribed $2|S|$-points x_s and y_s ($s \in S$). We suppose that the metric is flat in some neighborhood of each x_s and y_s. Fix a real number p with $2 < p < 4$ and define $q \in (4, +\infty)$ by $1 - 4/p = -4/q$, i.e., $L^p_1 \hookrightarrow L^q$. (These p and q are fixed throughout the paper.)

3.1. **Infinite connected sum.** First we briefly describe a construction of an infinite connected sum of X. This is essentially the same as in Section 2. But we need to introduce one more extra parameter $N > 0$ for several technical reasons. Let λ and N be positive parameters. We choose them so that $\lambda \ll 1$, $N \gg 1$ and $N\sqrt{\lambda} \ll 1$. We set (we follow the notation of Donaldson-Kronheimer [5, Section 7.2])

$$X' := X \setminus \left(\bigsqcup_{s \in S} \bar{B}(x_s, \sqrt{\lambda}/N) \sqcup \bar{B}(y_s, \sqrt{\lambda}/N) \right),$$

$$X'' := X \setminus \left(\bigsqcup_{s \in S} \bar{B}(x_s, \sqrt{\lambda}/2) \sqcup \bar{B}(y_s, \sqrt{\lambda}/2) \right).$$

X'' corresponds to the region U in Section 2. We define annulus regions $\Omega(x_s)$ and $\Omega(y_s)$ in X by $\Omega(x_s) := B(x_s, N\sqrt{\lambda}) \setminus \bar{B}(x_s, \sqrt{\lambda}/N)$ and $\Omega(y_s) := B(y_s, N\sqrt{\lambda}) \setminus \bar{B}(y_s, \sqrt{\lambda}/N)$. For each $s \in S$ we choose an orientation-reversing isometry $\sigma_s : T_{x_s}X \to T_{y_s}X$.

Let X_γ ($\gamma \in \Gamma$) be the copy of X with the points $x_{\gamma,s}$ and $y_{\gamma,s}$ ($s \in S$) corresponding to x_s and y_s. X_γ has the open sets X'_γ, X''_γ, $\Omega(x_{\gamma,s})$, $\Omega(y_{\gamma,s})$ corresponding to X', X'', $\Omega(x_s)$,

$\Omega(y_s)$ in X, respectively. We define $X^{\sharp(\Gamma,S)}$ by

$$X^{\sharp(\Gamma,S)} := \left(\bigsqcup_{\gamma\in\Gamma} X'_\gamma\right) / \sim,$$

where we identify $\Omega(x_{\gamma,s})$ in X'_γ with $\Omega(y_{\gamma s,s})$ in $X'_{\gamma s}$ by

$$\Omega(x_{\gamma,s}) \ni \xi \sim \eta \in \Omega(y_{\gamma s,s}) \overset{\text{def}}{\Longleftrightarrow} \eta = \lambda\sigma_s(\xi)/|\xi|^2. \tag{6}$$

Here ξ and η are the normal coordinates centered at $x_{\gamma,s}$ and $y_{\gamma s,s}$, and we consider σ_s as a map from $T_{x_{\gamma,s}}X_\gamma$ to $T_{y_{\gamma s,s}}X_{\gamma s}$ as in Section 2. Γ freely acts on $X^{\sharp(\Gamma,S)}$; for $g\in\Gamma$, we define $g: X'_\gamma \to X'_{g\gamma}$ by sending $p\in X'_\gamma$ to $q\in X'_{g\gamma}$ corresponding to the same point in X.

Let g_γ ($\gamma\in\Gamma$) be the Riemannian metric on X_γ which is the copy of the metric on X. Let w be a smooth function in X satisfying $0\le w\le 1$, $w=1$ in the complement of the balls $B(x_s,\sqrt{\lambda})$ and $B(y_s,\sqrt{\lambda})$ ($s\in S$), and $w=0$ on each $B(x_s,\sqrt{\lambda}/2)$ and $B(y_s,\sqrt{\lambda}/2)$. We define a metric on $X^{\sharp(\Gamma,S)}$ by

$$g := \sum_{\gamma\in\Gamma} w_\gamma g_\gamma,$$

where the weight function w_γ is the copy of w. We have

$$X^{\sharp(\Gamma,S)} = \bigcup_{\gamma\in\Gamma} X'_\gamma = \bigcup_{\gamma\in\Gamma} X''_\gamma,$$

and the Riemannian structure on $X^{\sharp(\Gamma,S)}$ is independent of N. Hence the above connected sum construction is compatible with that in Section 2.

The Riemannian metric g is conformally equivalent to g_γ over X'_γ ($g = m_\gamma^2 g_\gamma$) and satisfies

$$1\le m_\gamma \le N^2 \text{ on } X'_\gamma, \quad 1\le m_\gamma\le 2^3 \text{ on } X''_\gamma. \tag{7}$$

Moreover, on each neighborhood of $x_{\gamma,s}$ and $y_{\gamma,s}$,

$$N^{5/3}\le m_\gamma\le N^2 \quad (\sqrt{\lambda}/N \le |\xi| \le \sqrt{\lambda}/N^{5/6}), \tag{8}$$

where ξ is the Euclidean coordinate (in X_γ) around $x_{\gamma,s}$ or $y_{\gamma,s}$.

The important point for the later argument is the following (essentially the same things are discussed in [5, pp. 293-294]): For a 1-form α and a 2-form ξ on X'_γ we have

$$|\alpha|_g^q d\mathrm{vol}_g = m_r^{4-q}|\alpha|_{g_\gamma}^q d\mathrm{vol}_{g_\gamma}, \quad |\xi|_g^p d\mathrm{vol}_g = m_r^{4-2p}|\xi|_{g_\gamma}^p d\mathrm{vol}_{g_\gamma}, \tag{9}$$

where $d\mathrm{vol}_g$ denotes the volume form defined by g. Since $2<p<4$ and $q>4$, (7) implies

$$\begin{aligned} &\|\alpha\|_{L^q(X'_\gamma,g)} \le \|\alpha\|_{L^q(X'_\gamma,g_\gamma)} \le N^{2-8/q}\|\alpha\|_{L^p(X'_\gamma,g)}, \quad \|\alpha\|_{L^q(X''_\gamma,g_\gamma)} \le 8^{1-4/q}\|\alpha\|_{L^q(X''_\gamma,g)}, \\ &\|\xi\|_{L^p(X'_\gamma,g)} \le \|\xi\|_{L^p(X'_\gamma,g_\gamma)} \le N^{4-8/p}\|\xi\|_{L^p(X'_\gamma,g)}, \quad \|\xi\|_{L^p(X''_\gamma,g_\gamma)} \le 8^{2-4/p}\|\xi\|_{L^p(X''_\gamma,g)}. \end{aligned} \tag{10}$$

3.2. **Gluing principal $SU(2)$-bundles.** Let M be a set of (not necessarily all) gauge equivalence classes of (E, A), where E is a principal $SU(2)$-bundle over X and A is an ASD connection on E. We suppose that M can be decomposed as $M = M_0 \sqcup M_1$ (disjoint union) satisfying the following conditions:
(a) M_0 and M_1 are compact with respect to the topology of $\mathcal{C}^\infty$-convergence. (In particular, the number of the possible topological types of E is finite.)
(b) For all $[E, A] \in M_0$, A is a regular connection. That is, the following two conditions are satisfied: A is irreducible (i.e., if a gauge transformation $g : E \to E$ satisfies $g(A) = A$, then $g = \pm 1$) and the operator $d_A^+ : \Omega^1(\mathrm{ad}E) \to \Omega^+(\mathrm{ad}E)$ is surjective. Here Ω^1 is the space of 1-forms and Ω^+ is the space of self-dual forms. $\mathrm{ad}E$ is the Lie algebra bundle associated with E.
(c) If X satisfies $b_1(X) = b^+(X) = 0$ (e.g., $X = S^4, \overline{\mathbb{C}P^2}$), then

$$M_1 \subset \{[X \times SU(2), \text{the product connection}]\}.$$

Otherwise we set $M_1 = \emptyset$. Therefore M_1 is the one-point space or empty.

Remark 3.1. Let $E := X \times SU(2)$ and A be the product connection. If $b^+(X) = 0$, then $d_A^+ : \Omega^1(\mathrm{ad}E) \to \Omega^+(\mathrm{ad}E)$ is surjective. But A is not irreducible. All constant gauge transformations fix A. The condition $b^+(X) = b_1(X) = 0$ implies that $[A]$ has no local deformation as an ASD connection.

In our application to Theorem 2.3, we need to consider the product connection. The condition (c) is added for this purpose. But if the reader does not want to consider reducible connections, you should consider only the case $M = M_0$.

Definition 3.2. A sequence $(E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma, s\in S}$ is called a gluing data (or M-gluing data) if it satisfies the following:
(i) For all $\gamma \in \Gamma$, E_γ is a principal $SU(2)$-bundle over X_γ and A_γ is an ASD connection on it which satisfies $[E_\gamma, A_\gamma] \in M$. (Here we naturally identify X_γ with X.)
(ii) $\rho_{\gamma,s} : (E_\gamma)_{x_{\gamma,s}} \to (E_{\gamma s})_{y_{\gamma s,s}}$ $(\gamma \in \Gamma, s \in S)$ is an $SU(2)$-isomorphism between the fibers $(E_\gamma)_{x_{\gamma,s}}$ and $(E_{\gamma s})_{y_{\gamma s,s}}$. We call $\rho = (\rho_{\gamma,s})_{\gamma\in\Gamma,s\in S}$ "gluing parameter".

We consider that two gluing data $(E_{1\gamma}, A_{1\gamma}, \rho_{1\gamma,s})_{\gamma\in\Gamma,s\in S}$ and $(E_{2\gamma}, A_{2\gamma}, \rho_{2\gamma,s})_{\gamma\in\Gamma,s\in S}$ are equivalent if there exist bundle isomorphisms $g_\gamma : E_{1\gamma} \to E_{2\gamma}$ $(\gamma \in \Gamma)$ satisfying $g_\gamma(A_{1\gamma}) = A_{2\gamma}$ and $g_{\gamma s}\rho_{1\gamma,s} = \rho_{2\gamma,s}g_\gamma$ (in particular, $E_{1\gamma}$ and $E_{2\gamma}$ are isomorphic). We define GlD as the set of the equivalence classes of gluing data. (We sometimes use the notation "GlD_M" when we need to make the dependence on M explicit.) There exists a natural projection $\mathrm{GlD} \to M^\Gamma$ defined by $(E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma,s\in S} \mapsto (E_\gamma, A_\gamma)_{\gamma\in\Gamma}$.

Let $\theta = (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma,s\in S}$ be a gluing data. For $x \in X_\gamma$ in a small neighborhood of $x_{\gamma,s}$, the fiber of E_γ over the point x can be identified with the fiber over $x_{\gamma,s}$ by using the parallel transport (defined by A_γ) along the radial line from $x_{\gamma,s}$ to x. This trivialization is usually called "exponential gauge" (or sometimes "radial gauge"); see [6, Chapter 9] or

[5, Section 2.3.1]. Using these exponential gauges centered at $x_{\gamma,s}$ or $y_{\gamma,s}$, we trivialize the bundle E_γ over $\bigsqcup_{s\in S}\Omega(x_{\gamma,s})\sqcup\Omega(y_{\gamma,s})$. ($\Omega(x_{\gamma,s})$ and $\Omega(y_{\gamma,s})$ are the annulus regions over X_γ defined in Section 3.1: $\Omega(x_{\gamma,s}) = B(x_{\gamma,s}, N\sqrt{\lambda})\setminus \bar{B}(x_{\gamma,s}, \sqrt{\lambda}/N)$.)

Each $\rho_{\gamma,s}$ is an isomorphism between $(E_\gamma)_{x_{\gamma,s}}$ and $(E_{\gamma s})_{y_{\gamma s,s}}$. We have the identification $\Omega(x_{\gamma,s})\cong\Omega(y_{\gamma s,s})$ defined by (6), and the above exponential gauges give the bundle trivializations $E_\gamma|_{\Omega(x_{\gamma,s})}\cong\Omega(x_{\gamma,s})\times(E_\gamma)_{x_{\gamma,s}}$ and $E_{\gamma s}|_{\Omega(y_{\gamma s,s})}\cong\Omega(y_{\gamma s,s})\times(E_{\gamma s})_{y_{\gamma s,s}}$. Therefore $\rho_{\gamma,s}$ gives an identification map between $E_\gamma|_{\Omega(x_{\gamma,s})}$ and $E_{\gamma s}|_{\Omega(y_{\gamma s,s})}$ covering the base space identification $\Omega(x_{\gamma,s})\cong\Omega(y_{\gamma s,s})$ (see the diagram (11)).

$$\begin{tikzcd} E_\gamma|_{\Omega(x_{\gamma,s})} \arrow[r, "\text{exp. gauge}"] \arrow[d] & \Omega(x_{\gamma,s})\times(E_\gamma)_{x_{\gamma,s}} \arrow[d, "\rho_{\gamma,s}"] \\ E_{\gamma s}|_{\Omega(y_{\gamma s,s})} \arrow[r, "\text{exp. gauge}"] & \Omega(y_{\gamma s,s})\times(E_{\gamma s})_{y_{\gamma s,s}} \end{tikzcd} \tag{11}$$

We define a principal $SU(2)$ bundle $\boldsymbol{E}(\theta)$ over $X^{\sharp(\Gamma,S)}$ by setting

$$\boldsymbol{E}(\theta) := \left(\bigsqcup_{\gamma\in\Gamma} E_\gamma|_{X'_\gamma}\right)/\sim,$$

where we identify $E_\gamma|_{\Omega(x_{\gamma,s})}$ with $E_{\gamma s}|_{\Omega(y_{\gamma s,s})}$ by (11).

3.3. **Cut-off functions.** We need to introduce several cut-off functions. We basically follow the description of Donaldson-Kronheimer [5, Section 7.2]. First note that the following fact. Since M is compact, there exists an uniform upper bound of $|F_A|$ for all $[E,A]\in M$:

$$|F_A|\leq \mathrm{const}_M, \tag{12}$$

where const_M denotes a positive constant depending only on M.

Set $b := 4N\sqrt{\lambda}$ ($\ll 1$). Let ψ be the cut-off function on X such that $\psi = 0$ over $\bigsqcup_{s\in S} B(x_s, b/2)\sqcup B(y_s, b/2)$ and $\psi = 1$ over the complement of $\bigsqcup_{s\in S} B(x_s,b)\sqcup B(y_s,b)$ and $|d\psi|\leq 4/b$. Let ψ_γ be the copy of ψ defined on X_γ.

Let $\theta = (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma, s\in S}$ be a gluing data. As in Section 3.2, we trivialize the bundle E_γ around $x_{\gamma,s}$ and $y_{\gamma,s}$ by using the exponential gauges. Then we can define a connection A'_γ on E_γ by setting

$$A'_\gamma := \psi_\gamma A_\gamma. \tag{13}$$

Here we consider A_γ as a connection matrix over each neighborhood of $x_{\gamma,s}$ and $y_{\gamma,s}$ by using the above trivialization. In the exponential gauge we have $|A_\gamma(x)|\leq |x|\sup|F_{A_\gamma}|\leq \mathrm{const}_M|x|$ (see Donaldson-Kronheimer [5, p. 54]). Therefore we have

$$|A'_\gamma - A_\gamma|\leq \mathrm{const}_M\cdot b,\quad |F^+(A'_\gamma)|\leq\mathrm{const}_M,\quad |F(A'_\gamma)-F(A_\gamma)|\leq \mathrm{const}_M, \tag{14}$$

where const_M is a positive constant which only depends on M (and is independent of γ, b, λ, N). Then

$$\begin{aligned} &\left\|A'_\gamma - A_\gamma\right\|_{L^4(X_\gamma,g_\gamma)} \leq \mathrm{const}_M \cdot b^2, \quad \left\|F^+(A'_\gamma)\right\|_{L^p(X_\gamma,g_\gamma)} \leq \mathrm{const}_M \cdot b^{4/p}, \\ &\left\|F(A'_\gamma) - F(A_\gamma)\right\|_{L^p(X_\gamma,g_\gamma)} \leq \mathrm{const}_M \cdot b^{4/p}. \end{aligned} \tag{15}$$

A'_γ and $A'_{\gamma s}$ ($s \in S$) coincide with each other over $X'_\gamma \cap X'_{\gamma s}$ under the identification (11). Hence there exists an unique (not necessarily ASD) connection $\boldsymbol{A}'(\theta)$ on $\boldsymbol{E}(\theta)$ compatible with each A'_γ over X'_γ.

Remark 3.3. If $[E_\gamma, A_\gamma] \in M_1$ (i.e., A_γ is gauge equivalent to the product connection), then $A'_\gamma = A_\gamma$. Hence if $[E_\gamma, A_\gamma] \in M_1$ for all $\gamma \in \Gamma$, then $\boldsymbol{A}'(\theta)$ is a flat connection on $\boldsymbol{E}(\theta)$ (which might have a non-trivial holonomy).

Later (in Section 5.1) we will need the following $\{\psi'_\gamma\}$ also; Let ψ' be the cut-off function on X such that $\psi' = 0$ over $\bigsqcup_{s\in S} B(x_s, b/4) \sqcup B(y_s, b/4)$ and $\psi' = 1$ over the complement of $\bigsqcup_{s\in S} B(x_s, b/2) \sqcup B(y_s, b/2)$ and $|d\psi'| \leq 8/b$. Let ψ'_γ be the copy of ψ' defined on X_γ.

The following lemma is essentially the copy of [5, Lemma (7.2.10)]:

Lemma 3.4. *There exists a positive number K satisfying the following: For any λ and N there exists a smooth function $\beta = \beta_{\lambda,N}$ defined in $\mathbb{R}^4$ such that $\beta(x) = 0$ for $|x| \leq \sqrt{\lambda}/N$, $\beta(x) = 1$ for $|x| \geq \sqrt{\lambda}/N^{5/6}$ and*

$$\|d\beta\|_{L^4} \leq K(\log N)^{-3/4}.$$

Proof. Note that the L^4-norm of a 1-form is conformally invariant. So we can change the description from the Euclidean $\mathbb{R}^4$ to the cylinder $S^3 \times \mathbb{R}$ by the coordinate transform $t = \log|x| - \log\sqrt{\lambda}$. $\sqrt{\lambda}/N \leq |x| \leq \sqrt{\lambda}/N^{5/6}$ becomes $-\log N \leq t \leq -(5/6)\log N$. Then the proof is easy. $\square$

The condition $\mathrm{supp}(d\beta) \subset \{\sqrt{\lambda}/N \leq |x| \leq \sqrt{\lambda}/N^{5/6}\}$ will be used in Section 6.1 (cf. (8)). Using the above Lemma 3.4, we define a cut-off function β on X by putting the above $\beta_{\lambda,N}$ around each x_γ and y_γ. That is, β is a function with $0 \leq \beta \leq 1$ such that $\beta = 0$ on $\bigsqcup_{s\in S} B(x_s, \sqrt{\lambda}/N) \sqcup B(y_s, \sqrt{\lambda}/N)$, $\beta = 1$ on the complement of $\bigsqcup_{s\in S} B(x_s, \sqrt{\lambda}/N^{5/6}) \sqcup B(y_s, \sqrt{\lambda}/N^{5/6})$ and

$$\|d\beta\|_{L^4} \leq K(\log N)^{-3/4}. \tag{16}$$

(Strictly speaking, the above constant K should be $(2|S|)^{1/4}K$. But for simplicity we use the abuse of notation.) Let β_γ be the copy of β defined on X_γ.

We need to introduce one more cut-off. Let β' be a smooth function on X such that $0 \leq \beta' \leq 1$, $\beta' = 0$ on $\bigsqcup_{s\in S} B(x_s, \sqrt{\lambda}/2) \sqcup B(y_s, \sqrt{\lambda}/2)$, $\beta' = 1$ on the complement of $\bigsqcup_{s\in S} B(x_s, 2\sqrt{\lambda}) \sqcup B(y_s, 2\sqrt{\lambda})$ (hence $\mathrm{supp}\,\beta' \subset X''$). We can choose β' so that the L^4-norm $\|d\beta'\|_{L^4}$ is independent of λ (and N). Since $N \gg 1$, we have $\sqrt{\lambda}/2 \geq \sqrt{\lambda}/N^{5/6}$ and hence

$$\beta \cdot \beta' = \beta'. \tag{17}$$

Let β'_γ be the copy of β defined on X_γ ($\beta_\gamma \cdot \beta'_\gamma = \beta'_\gamma$). Moreover we choose β' so that these β'_γ becomes a partition of unity on $X^{\sharp(\Gamma,S)}$:

$$\sum_{\gamma\in\Gamma} \beta'_\gamma = 1. \tag{18}$$

In particular we have $\beta'_\gamma + \beta'_{\gamma s} = 1$ over $\Omega(x_{\gamma,s}) = \Omega(y_{\gamma s,s})$.

3.4. **Preliminary estimates.** In this subsection we prepare several estimates. I think that they are essentially well-known. Therefore we omit most of the proofs. If some readers feel this subsection cumbersome, you should skip it and return to this subsection when it is used.

3.4.1. *Right inverse of* d_A^+. Let $[E, A] \in M$, and set $\Delta_A := (d_A^+)(d_A^+)^* : \Omega^+(\mathrm{ad}E) \to \Omega^+(\mathrm{ad}E)$, where $(d_A^+)^* : \Omega^+(\mathrm{ad}E) \to \Omega^1(\mathrm{ad}E)$ is the formal adjoint of d_A^+. From the conditions (b) and (c) in the beginning of Section 3.2, there exists the inverse Δ_A^{-1} (see also Remark 3.1). Set $P_A := (d_A^+)^* \cdot \Delta_A^{-1} : \Omega^+(\mathrm{ad}E) \to \Omega^1(\mathrm{ad}E)$. P_A becomes a right inverse of d_A^+: $d_A^+ P_A = 1$.

Remember that $2 < p < 4$, $q > 4$ and $1-4/p = -4/q$. We have the Sobolev embedding: $L^p_1(X) \hookrightarrow L^q(X)$. Since M is compact, there exists a positive constant const_M depending only on M (and independent of A) such that

$$\|P_A\xi\|_{L^q} \leq \mathrm{const}_M \|\xi\|_{L^p}, \quad \|d_A P_A(\xi)\|_{L^p} \leq \mathrm{const}_M \|\xi\|_{L^p}. \tag{19}$$

for any $[E, A] \in M$ and any $\xi \in \Omega^+(\mathrm{ad}E)$.

3.4.2. *The cohomology* H^1_A. Let $[E, A] \in M$ and set

$$H^1_A := \ker(d_A^* + d_A^+ : \Omega^1(\mathrm{ad}E) \to (\Omega^0 \oplus \Omega^+)(\mathrm{ad}E)).$$

(If $[E, A] \in M_1$, then $H^1_A = 0$.) There exists $\delta_M > 0$ such that for any $\alpha \in H^1_A$ with $\|\alpha\|_{L^q} \leq \delta_M$ we have $\tilde{\alpha} = \tilde{\alpha}(A, \alpha) \in \Omega^1(\mathrm{ad}E)$ satisfying the following:

$$d_A^*\tilde{\alpha} = 0, \quad F^+(A + \tilde{\alpha}) = 0,$$

$$\|\tilde{\alpha} - \alpha\|_{L^q} \leq \mathrm{const}_M \|\alpha\|^2_{L^q}.$$

(We have $\tilde{\alpha}(A, 0) = 0$.) Moreover

$$\|\alpha\|_{L^q} \leq \mathrm{const}_M \cdot d_{L^q}([A], [A + \tilde{\alpha}]). \tag{20}$$

Here, for connections A_1 and A_2 on E, we define L^q-distance $d_{L^q}([A_1], [A_2])$ by

$$d_{L^q}([A_1], [A_2]) := \inf_{g:E\to E} \|A_2 - g(A_1)\|_{L^q(X)}.$$

Lemma 3.5. *There is $\delta'_M > 0$ such that if an ASD connection B on E with $[E,B] \in M$ satisfies $d_{L^q}([A],[B]) \le \delta'_M$ then there exists $\alpha \in H^1_A$ with $\|\alpha\|_{L^q} \le \delta_M$ satisfying $[B] = [A + \tilde{\alpha}]$.*

Lemma 3.6. *If we choose δ_M sufficiently small, then for any $\xi \in \Omega^+(\mathrm{ad}E)$ and $\alpha \in H^1_A$ with $\|\alpha\|_{L^q} \le \delta_M$,*

$$\|P_A(\xi) - P_{A+\tilde{\alpha}}(\xi)\|_{L^q} \le \mathrm{const}_M \|\alpha\|_{L^q} \|\xi\|_{L^p}.$$

3.4.3. *Auxiliary estimates.* For $\varepsilon > 0$ let $X_\varepsilon \subset X$ be the complement of the union of the balls $\bar{B}(x_s,\varepsilon)$ and $\bar{B}(y_s,\varepsilon)$ $(s \in S)$:

$$X_\varepsilon := X \setminus \left(\bigcup_{s\in S} \bar{B}(x_s,\varepsilon) \cup \bar{B}(y_s,\varepsilon) \right).$$

Lemma 3.7. *There is $\varepsilon_M > 0$ such that if $\varepsilon \le \varepsilon_M$ then for any two $[E_i, A_i] \in M$ $(i = 1,2)$ we have the following:*
(1) If E_1 is isomorphic to E_2, then

$$d_{L^q}([A_1],[A_2]) \le \mathrm{const}_M \cdot d_{L^q}([A_1|_{X_\varepsilon}],[A_2|_{X_\varepsilon}]),$$

where $d_{L^q}([A_1|_{X_\varepsilon}],[A_2|_{X_\varepsilon}])$ is given by

$$d_{L^q}([A_1|_{X_\varepsilon}],[A_2|_{X_\varepsilon}]) := \inf_{g: E_1|_{X_\varepsilon} \to E_2|_{X_\varepsilon}} \|A_2 - g(A_1)\|_{L^q(X_\varepsilon)}.$$

(2) If E_1 is not isomorphic to E_2, then

$$d_{L^q}([A_1|_{X_\varepsilon}],[A_2|_{X_\varepsilon}]) \ge \mathrm{const}_M > 0.$$

For $[E,A] \in M$ we denote I_A as the set of gauge transformations $g : E \to E$ satisfying $g(A) = A$. If $[E,A] \in M_0$, then $I_A = \{\pm 1\}$, and if $[E,A] \in M_1$, then $I_A \cong SU(2)$ (the set of constant gauge transformations).

Lemma 3.8. *There is $\varepsilon'_M > 0$ such that if $\varepsilon \le \varepsilon'_M$ then we have the following: Let $[E,A] \in M$ and $g : E|_{X_\varepsilon} \to E|_{X_\varepsilon}$ be a bundle map over X_ε. Then*

$$\min_{h \in I_A} \|g - h\|_{\mathcal{C}^0(X_\varepsilon)} \le \mathrm{const}_M \|d_A g\|_{L^q(X_\varepsilon)}.$$

3.4.4. *Estimates about the exponential gauge.* Let $D \subset \mathbb{R}^4$ be a ball centered at the origin in the Euclidean space $\mathbb{R}^4$, and $E = D \times SU(2)$ be a principal $SU(2)$ bundle over D with smooth (not necessarily ASD) connections A_1 and A_2. Let $u_i : E \to D \times E_0$ $(i = 1,2)$ be the exponential gauges associated with A_i centered at the origin. (E_0 is the fiber of E at the origin.) We have $\partial u_i/\partial r = u_i A_{i,r}$ $(r = |x|)$ and hence $\partial(u_1 u_2^{-1})/\partial r = u_1(A_{1,r} - A_{2,r})u_2^{-1}$. Therefore

$$|u_1(x) - u_2(x)| \le |x| \cdot \|A_1 - A_2\|_{\mathcal{C}^0(B)}. \tag{21}$$

Let $B_i := u_i(A_i)$ be the connection matrices in the exponential gauge $(i = 1,2)$.

Lemma 3.9.

$$|B_1 - B_2| \leq r \|F(A_1) - F(A_2)\|_{\mathcal{C}^0} + \frac{r^2}{2} \|A_1 - A_2\|_{\mathcal{C}^0} (\|F(A_1)\|_{\mathcal{C}^0} + \|F(A_2)\|_{\mathcal{C}^0}).$$

Proof. We have

$$B_{i,\theta} = \int_0^r F(B_i)_{r\theta} dr = \int_0^r u_i F(A_i)_{r\theta} u_i^{-1} dr,$$

where r, θ denote the polar coordinate. (Of course, θ has three components.) Hence

$$B_{1,\theta} - B_{2,\theta} =$$
$$\int_0^r \{(u_1 - u_2)F(A_1)_{r\theta} u_1^{-1} + u_2(F(A_1)_{r\theta} - F(A_2)_{r\theta})u_1^{-1} + u_2 F(A_2)_{r\theta}(u_1^{-1} - u_2^{-1})\} dr.$$

Then

$$\begin{aligned} |B_1 - B_2| &\leq \int_0^r |u_1 - u_2|(|F(A_1)| + |F(A_2)|) + |F(A_1) - F(A_2)| dr, \\ &\leq \frac{r^2}{2} \|A_1 - A_2\|_{\mathcal{C}^0} (\|F(A_1)\|_{\mathcal{C}^0} + \|F(A_2)\|_{\mathcal{C}^0}) + r \|F(A_1) - F(A_2)\|_{\mathcal{C}^0}. \end{aligned}$$

$\square$

Let $A = A(t)$ be a family of connections on E depending smoothly on the parameter $t \in (-1, 1)$. Let $u = u(t) : E \to D \times E_0$ be the exponential gauge of A about the origin, and set $B = B(t) = u(A)$. Suppose that there exists a family of sections $w = w(t)$ of $\mathrm{ad}E$ such that $u(t) = u(0)e^{w(t)}$ and $w(0) \equiv 0$. Let ψ and ϕ be smooth functions on D satisfying $0 \leq \phi, \psi \leq 1$. Set $A_1 = A_1(t) := u^{-1}(\psi B)$ (we consider B as a connection matrix), and $A_2 = A_2(t) := e^{\phi w}(A_1(t))$.

Lemma 3.10.

$$\left| \left. \frac{\partial A_2}{\partial t} \right|_{t=0} \right| \leq (1 + r|d\phi| + 3r^2 \|F(A(0))\|_{\mathcal{C}^0}) \left\| \left. \frac{\partial A}{\partial t} \right|_{t=0} \right\|_{\mathcal{C}^0} + r \left\| d_A \left(\frac{\partial A}{\partial t} \right) \right|_{t=0} \right\|_{\mathcal{C}^0}.$$

Proof. We can assume that $A(0)$ is already a connection matrix in the exponential gauge. Then $u(0) \equiv 1$, $u(t) = e^{w(t)}$ and $B(0) \equiv A(0)$. Let (r, θ) be the polar coordinate. Set $A_r = A_r(t) := \langle A(t), \partial/\partial r \rangle$. We have $A_r(0) \equiv 0$ and $\partial u/\partial r = uA_r$. Hence

$$\frac{\partial}{\partial r} \left(\frac{\partial u}{\partial t} \right) \bigg|_{t=0} = u \frac{\partial A_r}{\partial t} \bigg|_{t=0}.$$

Since $u = 1$ at the origin for all t, we have $\partial u/\partial t = 0$ at the origin. Hence

$$\left| \left. \frac{\partial u}{\partial t} \right|_{t=0} \right| \leq r \left\| \left. \frac{\partial A}{\partial t} \right|_{t=0} \right\|_{\mathcal{C}^0}. \tag{22}$$

We have

$$B_\theta = \int_0^r F(B)_{r\theta} dr = \int_0^r u F(A)_{r\theta} u^{-1} dr.$$

Differentiating this equation and using the above (22), we get

$$\left|\frac{\partial B}{\partial t}\Big|_{t=0}\right| \leq r^2 \left\|F(A(0))\right\|_{\mathcal{C}^0} \left\|\frac{\partial A}{\partial t}\Big|_{t=0}\right\|_{\mathcal{C}^0} + r\left\|d_A\left(\frac{\partial A}{\partial t}\right)\Big|_{t=0}\right\|_{\mathcal{C}^0}. \tag{23}$$

We have $d_A u = (A-B)u$. Differentiating this (and using $u(0)\equiv 1$), we get

$$d_A\left(\frac{\partial u}{\partial t}\right)\Big|_{t=0} = \left(\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t}\right)\Big|_{t=0}.$$

We have $A_1 = u^{-1}(\psi B) = u^{-1}du + \psi u^{-1}Bu$ and $A(0) = B(0)$. Then

$$\begin{aligned}\frac{\partial A_1}{\partial t}\Big|_{t=0} &= d_A\left(\frac{\partial u}{\partial t}\right)\Big|_{t=0} + (\psi-1)[A, \partial u/\partial t]|_{t=0} + \psi\frac{\partial B}{\partial t}\Big|_{t=0}, \\ &= \frac{\partial A}{\partial t}\Big|_{t=0} + (\psi-1)\frac{\partial B}{\partial t}\Big|_{t=0} + (\psi-1)[A,\partial u/\partial t]|_{t=0}.\end{aligned}$$

We have $w(0)\equiv 0$, $\partial w/\partial t|_{t=0} = \partial u/\partial t|_{t=0}$ and $A_1(0) = \psi B(0) = \psi A(0)$. Some calculation shows

$$\begin{aligned}\frac{\partial A_2}{\partial t}\Big|_{t=0} =& - d\phi\otimes\frac{\partial u}{\partial t}\Big|_{t=0} + (1-\phi)\frac{\partial A}{\partial t}\Big|_{t=0} \\ &+ (\psi+\phi-1)\frac{\partial B}{\partial t}\Big|_{t=0} + (\psi-1)(1-\phi)[A,\partial u/\partial t]|_{t=0}.\end{aligned}$$

We have $|A(0)| \leq r\left\|F(A(0))\right\|_{\mathcal{C}^0}$. Therefore

$$\left|\frac{\partial A_2}{\partial t}\Big|_{t=0}\right| \leq \left(1 + r|d\phi| + 3r^2\left\|F(A(0))\right\|_{\mathcal{C}^0}\right)\left\|\frac{\partial A}{\partial t}\Big|_{t=0}\right\|_{\mathcal{C}^0} + r\left\|d_A\left(\frac{\partial A}{\partial t}\right)\Big|_{t=0}\right\|_{\mathcal{C}^0}.$$

□

Moreover suppose that $A = A(t)$ is a family of ASD connections. We have $F^+(A_1) = u^{-1}F^+(\psi B)u$ and $F^+(\psi B) = (d\psi\wedge B)^+ + \psi(\psi-1)(B\wedge B)^+$. Using the inequalities (22) and (23), we get (at $t=0$)

$$\begin{aligned}&\left|\frac{\partial}{\partial t}F^+(A_1)\right| \leq 2r|F^+(\psi B)|\left\|\frac{\partial A}{\partial t}\right\|_{\mathcal{C}^0} + \left|\frac{\partial}{\partial t}F^+(\psi B)\right|, \\ &|F^+(\psi B)| \leq \left(r|d\psi| + r^2\left\|F_A\right\|_{\mathcal{C}^0}\right)\left\|F_A\right\|_{\mathcal{C}^0}, \\ &\left|\frac{\partial}{\partial t}F^+(\psi B)\right| \leq \left(r|d\psi| + 2\sqrt{2}r^2\left\|F_A\right\|_{\mathcal{C}^0}\right)\left(r\left\|F_A\right\|_{\mathcal{C}^0}\left\|\frac{\partial A}{\partial t}\right\|_{\mathcal{C}^0} + \left\|d_A\left(\frac{\partial A}{\partial t}\right)\right\|_{\mathcal{C}^0}\right).\end{aligned} \tag{24}$$

4. Infinite gluing: basic construction

In the following three sections we will develop the technique of gluing an infinite number of ASD connections. Our approach is based on the method of Donaldson-Kronheimer [5, Section 7.2]. We also use the ideas of Angenent [1] and Macrì-Nolasco-Ricciardi [11] in the construction of the right inverse of $d^+_{\boldsymbol{A}'}$. A different approach using "alternating method" in Donaldson [4] is developed in Tsukamoto [13]. Recall that $2 < p < 4$, $q > 4$ and $1 - 4/p = -4/q$.

4.1. **Construction.** Let $\theta = (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma, s\in S}$ be a gluing data. That is, E_γ ($\gamma \in \Gamma$) is a principal $SU(2)$ bundle over X_γ, and A_γ is an ASD connection on E_γ satisfying $[E_\gamma, A_\gamma] \in M$. $\rho := (\rho_{\gamma,s})_{\gamma\in\Gamma,s\in S}$ is a gluing parameter. We have constructed the principal $SU(2)$ bundle $\boldsymbol{E} = \boldsymbol{E}(\theta)$ on $X^{\sharp(\Gamma,S)}$. We want to construct an ASD connection on $\boldsymbol{E}$ by gluing the given ASD connections A_γ.

Let α and ξ be ad$\boldsymbol{E}$-valued 1-form and self-dual 2-form on $X^{\sharp(\Gamma,S)}$ respectively. We define BL^q-norm (bounded L^q-norm) of α and BL^p-norm of ξ by

$$\|\alpha\|_{BL^q} := \sup_{\gamma\in\Gamma} \|\alpha\|_{L^q(X''_\gamma, g)}, \quad \|\xi\|_{BL^p} := \sup_{\gamma\in\Gamma} \|\xi\|_{L^p(X''_\gamma, g)}. \tag{25}$$

Let BL^q be the Banach space of all locally-L^q, ad$\boldsymbol{E}$-valued 1-forms whose BL^q-norms are finite, and BL^p be the Banach space of all locally-L^p, ad$\boldsymbol{E}$-valued self-dual 2-forms whose BL^p-norms are finite. This type of function space is used in Macrí-Nolasco-Ricciardi [11] for the study of self-dual vortices. It is also used in Gournay [7] for the study of gluing infinitely many pseudo-holomorphic curves. For $\xi \in BL^p$ we define an ad$\boldsymbol{E}$-valued 1-form $Q(\xi) = Q_\theta(\xi)$ by

$$Q(\xi) := \sum_{\gamma\in\Gamma} \beta_\gamma P_{A_\gamma}(\beta'_\gamma \xi), \tag{26}$$

where P_{A_γ} is the right inverse of $d^+_{A_\gamma}$ defined in Section 3.4.1. The above infinite sum is a locally finite sum, and $Q(\xi)$ becomes locally-L^p_1. Using (19) and (10), we have (note that $\mathrm{supp}\beta'_\gamma \subset X''_\gamma$)

$$\begin{aligned} \left\|P_{A_\gamma}(\beta'_\gamma\xi)\right\|_{L^q(X'_\gamma, g)} &\le \left\|P_{A_\gamma}(\beta'_\gamma\xi)\right\|_{L^q(X'_\gamma, g_\gamma)} \le \mathrm{const}_M \left\|\beta'_\gamma\xi\right\|_{L^p(X_\gamma, g_\gamma)}, \\ &\le \mathrm{const}_M \|\xi\|_{L^p(X''_\gamma, g_\gamma)} \le \mathrm{const}'_M \|\xi\|_{L^p(X''_\gamma, g)}. \end{aligned} \tag{27}$$

Therefore

$$\|Q(\xi)\|_{BL^q} \le \mathrm{const}_M \cdot \|\xi\|_{BL^p}. \tag{28}$$

Let $\boldsymbol{A}' = \boldsymbol{A}'(\theta)$ be the connection on $X^{\sharp(\Gamma,S)}$ defined in Section 3.3. We have

$$d^+_{\boldsymbol{A}'} Q(\xi) = \sum_{\gamma\in\Gamma} d^+_{A'_\gamma}(\beta_\gamma P_{A_\gamma}(\beta'_\gamma\xi)).$$

Since $d^+_{A_\gamma} P_{A_\gamma} = 1$ and $\beta'_\gamma\beta_\gamma = \beta'_\gamma$ (see (17)),

$$\begin{aligned} d^+_{A'_\gamma}(\beta_\gamma P_{A_\gamma}(\beta'_\gamma\xi)) &= (d\beta_\gamma \wedge P_{A_\gamma}(\beta'_\gamma\xi))^+ + \beta_\gamma d^+_{A'_\gamma} P_{A_\gamma}(\beta'_\gamma\xi), \\ &= \beta'_\gamma\xi + (d\beta_\gamma \wedge P_{A_\gamma}(\beta'_\gamma\xi))^+ + \beta_\gamma[(A'_\gamma - A_\gamma)\wedge P_{A_\gamma}(\beta'_\gamma\xi)]^+. \end{aligned}$$

$\{\beta'_\gamma\}$ is a partition of unity (see (18)). So

$$d^+_{\boldsymbol{A}'} Q(\xi) = \xi + \sum_{\gamma\in\Gamma} (d\beta_\gamma \wedge P_{A_\gamma}(\beta'_\gamma\xi))^+ + \sum_{\gamma\in\Gamma} \beta_\gamma[(A'_\gamma - A_\gamma)\wedge P_{A_\gamma}(\beta'_\gamma\xi)]^+ \tag{29}$$

From Hölder's inequality ($L^4 \times L^q \to L^p$) and (27)

$$\left\|(d\beta_\gamma \wedge P_{A_\gamma}(\beta'_\gamma \xi))^+\right\|_{L^p(X'_\gamma, g)} + \left\|[(A'_\gamma - A_\gamma) \wedge P_{A_\gamma}(\beta'_\gamma \xi)]^+\right\|_{L^p(X'_\gamma, g)}$$
$$\leq \mathrm{const}_M \cdot (\|d\beta_\gamma\|_{L^4(X_\gamma, g)} + \left\|A_\gamma - A'_\gamma\right\|_{L^4(X_\gamma, g)}) \|\xi\|_{L^p(X''_\gamma, g)}.$$

Note that the L^4-norm of a 1-form is conformally invariant. So $\|d\beta_\gamma\|_{L^4(X_\gamma, g)}$ and $\left\|A_\gamma - A'_\gamma\right\|_{L^4(X_\gamma, g)}$ are equal to $\|d\beta_\gamma\|_{L^4(X_\gamma, g_\gamma)}$ and $\left\|A_\gamma - A'_\gamma\right\|_{L^4(X_\gamma, g_\gamma)}$, and these are very small (see (15) and (16)) for $N \gg 1$ and $b = 4N\sqrt{\lambda} \ll 1$. Then we get

$$\left\|d^+_{\boldsymbol{A}'} Q(\xi) - \xi\right\|_{BL^p} \leq \mathrm{const}_M((\log N)^{-3/4} + b^2) \|\xi\|_{BL^p}.$$

Thus

Lemma 4.1. *Set $R := R_\theta := d^+_{\boldsymbol{A}'}Q - 1 : BL^p \to BL^p$. For any $\xi \in BL^p$*

$$\|R(\xi)\|_{BL^p} \leq \mathrm{const}_M((\log N)^{-3/4} + b^2) \|\xi\|_{BL^p}.$$

Hence there exist positive constants $N_0 = N_0(M)$ and $b_0 = b_0(M)$ such that if $N \geq N_0$ and $b = 4N\sqrt{\lambda} \leq b_0$ then $\|R\| \leq 1/2$ and there exists $(1+R)^{-1} = \sum_{n\geq 0}(-R)^n : BL^p \to BL^p$. $P := P_\theta := Q(1+R)^{-1}$ gives a right inverse of $d^+_{\boldsymbol{A}'}$:

$$d^+_{\boldsymbol{A}'} P(\xi) = \xi,$$

for any $\xi \in BL^p$. From (28), for any $\xi \in BL^p$

$$\|P(\xi)\|_{BL^q} \leq \mathrm{const}_M \cdot \|\xi\|_{BL^p}. \tag{30}$$

We want to find an ASD connection of the form $\boldsymbol{A}' + P(\xi)$ ($\xi \in BL^p$). Since P is a right inverse of $d^+_{\boldsymbol{A}'}$, this is equivalent to solving the following equation for $\xi \in BL^p$:

$$\xi + (P(\xi) \wedge P(\xi))^+ = -F^+(\boldsymbol{A}'). \tag{31}$$

From (7) we have $\mathrm{vol}(X''_\gamma, g) \leq 8^4 \mathrm{vol}(X''_\gamma, g_\gamma) \leq 8^4 \mathrm{vol}(X)$. From this and $q > 4$, we have

$$\sup_{\gamma\in\Gamma} \|P(\xi)\|_{L^4(X''_\gamma, g)} \leq \mathrm{const} \cdot \|P(\xi)\|_{BL^q},$$

where this "const" is a positive constant depending only on $\mathrm{vol}(X)$. Then, using Hölder's inequality ($L^4 \times L^q \to L^p$) and (27), we get

$$\left\|(P(\xi_1) \wedge P(\xi_1))^+ - (P(\xi_2) \wedge P(\xi_2))^+\right\|_{BL^p} \leq \mathrm{const}_M(\|\xi_1\|_{BL^p} + \|\xi_2\|_{BL^p}) \|\xi_1 - \xi_2\|_{BL^p}.$$

Then we use the following lemma (this is [5, p. 289, Lemma (7.2.23)])

Lemma 4.2. *Let B be a Banach space and k be a positive constant. Let $S : B \to B$ be a (not necessarily linear) map satisfying $S(0) = 0$ and $\|S(x) - S(y)\| \leq k(\|x\| + \|y\|) \|x - y\|$. Then for any $y \in B$ with $\|y\| \leq 1/(10k)$ there uniquely exists $x \in B$ with $\|x\| \leq 1/(5k)$ satisfying*

$$x + S(x) = y.$$

Moreover x satisfies $\|x\| \leq \|y\| + 2k\|y\|^2$.

Proof. Set $T(x) := y - S(x)$. It is easy to check that for $\|x\| \leq 1/(5k)$ we have $\|T(x)\| \leq 1/(5k)$ and for $\|x_i\| \leq 1/(5k)$ $(i = 1, 2)$ we have $\|T(x_1) - T(x_2)\| \leq (2/5)\|x_1 - x_2\|$. Then the contraction mapping principle implies that there uniquely exists x with $\|x\| \leq 1/(5k)$ satisfying $T(x) = x$. If $x + S(x) = y$ and $\|x\| \leq 1/(5k)$, then $\|x\| \leq \|y\| + k\|x\|^2 \leq \|y\| + (1/5)\|x\|$. Hence $\|x\| \leq (5/4)\|y\|$. Therefore

$$\|x - y\| \leq k\|x\|^2 \leq (25k/16)\|y\|^2 \leq 2k\|y\|^2.$$

□

From (15) we have

$$\left\|F^+(\boldsymbol{A}')\right\|_{BL^p} \leq \mathrm{const}_M \cdot b^{4/p}.$$

Hence we can solve the equation (31) if $b \ll 1$.

Proposition 4.3. *There are positive constants $N_0 = N_0(M)$, $b_0 = b_0(M)$* [1]*, $C_1 = C_1(M)$ such that if $N \geq N_0$ and $b = 4N\sqrt{\lambda} \leq b_0$ then there exists $\boldsymbol{\xi} = \boldsymbol{\xi}(\theta) \in BL^p$ with $\|\boldsymbol{\xi}\|_{BL^p} \leq C_1$ satisfying*

$$F^+(\boldsymbol{A}' + P(\boldsymbol{\xi})) = 0 \quad \textit{and} \quad \|\boldsymbol{\xi}\|_{BL^p} \leq \mathrm{const}_M \cdot b^{4/p}.$$

Moreover this $\boldsymbol{\xi}$ is unique, i.e., if $\eta \in BL^p$ with $\|\eta\|_{BL^p} \leq C_1$ satisfies $F^+(\boldsymbol{A}' + P(\eta)) = 0$ then $\eta = \boldsymbol{\xi}$. From (30),

$$\|P(\boldsymbol{\xi})\|_{BL^q} \leq \mathrm{const}_M \cdot b^{4/p}.$$

We will denote $\boldsymbol{A}(\theta) := \boldsymbol{A}' + P(\boldsymbol{\xi})$.

Remark 4.4. We assume $2 < p < 4$. But the above construction argument is still true for $p = 2$. In particular, we have

$$\|\boldsymbol{\xi}\|_{BL^2} := \sup_{\gamma\in\Gamma} \|\boldsymbol{\xi}\|_{L^2(X''_\gamma, g)} \leq \mathrm{const}_M \cdot b^2,$$

$$\|P(\boldsymbol{\xi})\|_{BL^4} := \sup_{\gamma\in\Gamma} \|P(\boldsymbol{\xi})\|_{L^4(X''_\gamma, g)} \leq \mathrm{const}_M \cdot b^2.$$

Remark 4.5. If $[E_\gamma, A_\gamma] \in M_1$ for all $\gamma \in \Gamma$, then $\boldsymbol{A}'(\theta)$ is a flat connection (cf. Remark 3.3). In particular, $F^+(\boldsymbol{A}'(\theta)) = 0$. Hence we have $\boldsymbol{\xi}(\theta) = 0$ and $\boldsymbol{A}(\theta) = \boldsymbol{A}'(\theta)$.

4.2. **Estimate on the curvature.** We want to estimate the curvature of $\boldsymbol{A} = \boldsymbol{A}(\theta)$. From (15),

$$\|F(\boldsymbol{A}')\|_{BL^p} \leq \sup_{\gamma\in\Gamma} \|F(A_\gamma)\|_{L^p(X_\gamma, g_\gamma)} + \mathrm{const}_M \cdot b^{4/p}. \tag{32}$$

For any $\xi \in BL^p$ we have

$$d_{\boldsymbol{A}'} Q(\xi) = \sum_\gamma d\beta_\gamma \wedge P_{A_\gamma}(\beta'_\gamma \xi) + \beta_\gamma d_{A_\gamma} P_{A_\gamma}(\beta'_\gamma \xi) + \beta_\gamma [(A'_\gamma - A_\gamma) \wedge P_{A_\gamma}(\beta'_\gamma \xi)].$$

[1]This is an abuse of notation; these N_0 and b_0 are not necessarily equal to the constants in Lemma 4.1.

We can estimate this as in the previous subsection by using (19) and get:

$$\|d_{\boldsymbol{A}'}Q(\xi)\|_{BL^p} \leq \mathrm{const}_M \|\xi\|_{BL^p}.$$

Since $P = Q(1+R)^{-1}$, we have

$$\|d_{\boldsymbol{A}'}P(\xi)\|_{BL^p} \leq \mathrm{const}_M \left\|(1+R)^{-1}\xi\right\|_{BL^p} \leq \mathrm{const}'_M \|\xi\|.$$

We have $\boldsymbol{A} = \boldsymbol{A}' + P(\boldsymbol{\xi})$ and $F(\boldsymbol{A}) = F(\boldsymbol{A}') + d_{\boldsymbol{A}'}P(\boldsymbol{\xi}) + P(\boldsymbol{\xi}) \wedge P(\boldsymbol{\xi})$. Using $\|\boldsymbol{\xi}\|_{BL^p} \leq \mathrm{const}_M b^{4/p}$, we get

$$\|F(\boldsymbol{A})\|_{BL^p} \leq \|F(\boldsymbol{A}')\|_{BL^p} + \|d_{\boldsymbol{A}'}P(\boldsymbol{\xi})\|_{BL^p} + \mathrm{const}\, \|P(\boldsymbol{\xi})\|^2_{BL^q} \leq \|F(\boldsymbol{A}')\|_{BL^p} + \mathrm{const}_M b^{4/p}.$$

Using (32) we get the conclusion:

Proposition 4.6. *The ASD connection $\boldsymbol{A}(\theta)$ satisfies*

$$\|F(\boldsymbol{A}(\theta))\|_{BL^p} \leq \sup_{\gamma\in\Gamma} \|F(A_\gamma)\|_{L^p(X_\gamma, g_\gamma)} + \mathrm{const}_M \cdot b^{4/p}.$$

5. Infinite gluing: injectivity problem

Section 5 and Section 6 are technical. Some readers should skip these sections and go to Section 7, and return to them when the results in these sections are used. The main result in Section 5 is Proposition 5.5, and the main result in Section 6 is Theorem 6.11. They will be used later. Some arguments in Section 5.1, 6.2 and 6.3 (in particular, Corollary 6.8 and Lemma 6.10) will be also used later.

5.1. **Variation.** For each $\gamma \in \Gamma$, let $\theta := (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma, s\in S}$ be a gluing data. Let $\alpha_\gamma \in H^1_{A_\gamma}$ with $\|\alpha_\gamma\|_{L^q} \leq \delta_M$ (see Section 3.4.2), and $\tilde{A}_\gamma := A_\gamma + \tilde{\alpha}_\gamma$ be the ASD connection on E_γ given in Section 3.4.2. Set $\boldsymbol{\alpha} := (\alpha_\gamma)_{\gamma\in\Gamma}$. Let $v_{\gamma,s} \in (\mathrm{ad}E_\gamma)_{x_{\gamma,s}}$ ($\gamma\in\Gamma, s\in S$) with $|v_{\gamma,s}| \leq \mathrm{Diam}(SU(2))$. Set $\rho'_{\gamma,s} := \rho_{\gamma,s} e^{v_{\gamma,s}}$ and $\boldsymbol{v} := (v_{\gamma,s})_{\gamma\in\Gamma, s\in S}$. We define

$$\|\boldsymbol{\alpha}\| := \sup_{\gamma\in\Gamma} \|\alpha_\gamma\|_{L^q(X_\gamma, g_\gamma)}, \quad \|\boldsymbol{v}\| := \sup_{\gamma\in\Gamma, s\in S} |v_{\gamma,s}|.$$

Suppose $[E_\gamma, \tilde{A}_\gamma] \in M$ and set $\tilde{\theta} := (E_\gamma, \tilde{A}_\gamma, \rho'_{\gamma,s})_{\gamma\in\Gamma, s\in S}$. We want to compare $\tilde{\boldsymbol{A}} := \boldsymbol{A}(\tilde{\theta})$ with $\boldsymbol{A} := \boldsymbol{A}(\theta)$. First we will construct a gauge transformation h from $\tilde{\boldsymbol{E}} = \boldsymbol{E}(\tilde{\theta})$ to $\boldsymbol{E} = \boldsymbol{E}(\theta)$.

Let $u_\gamma : E_\gamma|_{B(x_{\gamma,s},b)} \to B(x_{\gamma,s}, b) \times (E_\gamma)_{x_{\gamma,s}}$ and $u_\gamma : E_\gamma|_{B(y_{\gamma,s},b)} \to B(y_{\gamma,s}, b) \times (E_\gamma)_{y_{\gamma,s}}$ be the exponential gauge of A_γ around $x_{\gamma,s}$ and $y_{\gamma,s}$ ($\gamma\in\Gamma, s\in S$). We also denote $\tilde{u}_\gamma$ as the exponential gauge of $\tilde{A}_\gamma$ around $x_{\gamma,s}$ and $y_{\gamma,s}$ ($\gamma\in\Gamma, s\in S$). From (21) we have $|u_\gamma - \tilde{u}_\gamma| \leq \mathrm{const}_M \cdot b\,\|\boldsymbol{\alpha}\| \ll 1$. Hence there uniquely exists a section w_γ of $\mathrm{ad}E_\gamma$ with $|w_\gamma| \leq \mathrm{const}'_M \cdot b\,\|\boldsymbol{\alpha}\| \ll 1$ over $\bigsqcup_{s\in S} B(x_{\gamma,s}, b) \sqcup B(y_{\gamma,s}, b)$ satisfying $\tilde{u}_\gamma = u_\gamma e^{w_\gamma}$. We define a section $\hat{v}_\gamma$ of $\mathrm{ad}E_\gamma$ over $\bigsqcup_{s\in S} B(x_{\gamma,s}, b) \sqcup B(y_{\gamma,s}, b)$ by setting

$$\hat{v}_\gamma := \begin{cases} u_\gamma^{-1} \circ v_{\gamma,s} \circ u_\gamma & \text{on } B(x_{\gamma,s}, b)\ (s\in S), \\ -u_\gamma^{-1} \circ (\rho_{\gamma s^{-1},s} \circ v_{\gamma s^{-1},s} \circ \rho^{-1}_{\gamma s^{-1},s}) \circ u_\gamma & \text{on } B(y_{\gamma,s}, b)\ (s\in S). \end{cases} \tag{33}$$

We define the gauge transformation $h_\gamma : E_\gamma \to E_\gamma$ by

$$h_\gamma := \begin{cases} e^{(1-\beta'_\gamma)\hat{v}_\gamma} e^{(1-\psi'_\gamma)w_\gamma} & \text{on } B(x_{\gamma,s}, b) \text{ and } B(y_{\gamma,s}, b) \ (s \in S), \\ 1 & \text{otherwise}, \end{cases} \tag{34}$$

where β'_γ and ψ'_γ are the cut-off functions introduced in Section 3.3. (ψ'_γ satisfies $|d\psi'_\gamma| \leq 8/b$, $\psi'_\gamma = 0$ over $\bigsqcup_{s\in S} B(x_s, b/4) \sqcup B(y_s, b/4)$ and $\psi'_\gamma = 1$ over the complement of $\bigsqcup_{s\in S} B(x_s, b/2) \sqcup B(y_s, b/2)$.) Since $\beta'_\gamma + \beta'_{\gamma s} = 1$ and $\psi'_\gamma = 0$ over $\Omega(x_{\gamma,s}) = \Omega(y_{\gamma s,s})$ (remember: $\Omega(x_{\gamma,s}) = B(x_{\gamma,s}, N\sqrt{\lambda}) \setminus \bar{B}(x_{\gamma,s}, \sqrt{\lambda}/N)$), the diagram (35) becomes commutative.

$$\begin{array}{ccc} E_\gamma|_{\Omega(x_{\gamma,s})} & \xrightarrow{h_\gamma} & E_\gamma|_{\Omega(x_{\gamma,s})} \\ {\scriptstyle \tilde{u}_{\gamma s}^{-1}\circ\rho'_{\gamma,s}\circ\tilde{u}_\gamma}\big\downarrow & & \big\downarrow{\scriptstyle u_{\gamma s}^{-1}\circ\rho_{\gamma,s}\circ u_\gamma} \\ E_{\gamma s}|_{\Omega(y_{\gamma s,s})} & \xrightarrow{h_{\gamma s}} & E_{\gamma s}|_{\Omega(y_{\gamma s,s})} \end{array} \tag{35}$$

Therefore $\{h_\gamma\}$ compatibly define the gauge transformation $h = h_{\boldsymbol{v},\boldsymbol{\alpha}} : \tilde{\boldsymbol{E}} \to \boldsymbol{E}$.

Set $P := P_\theta$ and $P_{\boldsymbol{v},\boldsymbol{\alpha}} := h \circ P_{\tilde{\theta}} \circ h^{-1} : BL^p \to BL^q$. We set $\tilde{\boldsymbol{A}}' := \boldsymbol{A}'(\tilde{\theta})$ (see Section 3.3) and

$$\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}} := h(\tilde{\boldsymbol{A}}') \quad \text{(this is a connection on } \boldsymbol{E}). \tag{36}$$

Lemma 5.1. *For any* $\xi \in BL^p$,

$$\|P_{\boldsymbol{v},\boldsymbol{\alpha}}(\xi) - P(\xi)\|_{BL^q} \leq \mathrm{const}_M(\|\boldsymbol{\alpha}\| + \|\boldsymbol{v}\|)\, \|\xi\|_{BL^p}.$$

Proof. The proof is just a confirmation of the definitions. We have

$$P_{\boldsymbol{v},\boldsymbol{\alpha}} = Q_{\boldsymbol{v},\boldsymbol{\alpha}}(d^+_{\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}}} Q_{\boldsymbol{v},\boldsymbol{\alpha}})^{-1},$$

where

$$\begin{aligned} Q_{\boldsymbol{v},\boldsymbol{\alpha}}(\xi) &= \sum_{\gamma\in\Gamma} h_\gamma \cdot \beta_\gamma P_{\tilde{A}_\gamma}(\beta'_\gamma h_\gamma^{-1}\xi), \\ &= \sum_{\gamma\in\Gamma} \Big((h_\gamma - 1)\beta_\gamma P_{\tilde{A}_\gamma}(\beta'_\gamma h_\gamma^{-1}(\xi)) + \beta_\gamma P_{\tilde{A}_\gamma}(\beta'_\gamma(h_\gamma^{-1} - 1)\xi) + \beta_\gamma P_{\tilde{A}_\gamma}(\beta'_\gamma \xi) \Big). \end{aligned} \tag{37}$$

By using the definition (34) and Lemma 3.6, we get

$$\|Q_{\boldsymbol{v},\boldsymbol{\alpha}}(\xi) - Q(\xi)\|_{BL^q} \leq \mathrm{const}_M(\|\boldsymbol{\alpha}\| + \|\boldsymbol{v}\|)\, \|\xi\|_{BL^p}.$$

In a similar way (cf. (29)),

$$\left\| (d^+_{\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}}} Q_{\boldsymbol{v},\boldsymbol{\alpha}})^{-1}(\xi) - (d^+_{\boldsymbol{A}'} Q)^{-1}(\xi) \right\|_{BL^p} \leq \mathrm{const}_M(\|\boldsymbol{\alpha}\| + \|\boldsymbol{v}\|)\, \|\xi\|_{BL^p}.$$

Then we get the above conclusion. □

Set $\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}} := h(\boldsymbol{\xi}(\tilde{\theta})) \in BL^p(\Omega^+(\mathrm{ad}\boldsymbol{E}))$. We have $\|\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}}\|_{BL^p} \leq \mathrm{const}_M \cdot b^{4/p}$ (see Proposition 4.3).

Lemma 5.2.

$$\|\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}} - \boldsymbol{\xi}\|_{BL^p} \leq \mathrm{const}_M \cdot b^{4/p} \cdot (\|\boldsymbol{\alpha}\| + b^2 \|\boldsymbol{v}\|).$$

Proof. We have (see (31) and Proposition 4.3)

$$\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}} + (P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}}) \wedge P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}}))^+ = -F^+(\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}}), \quad \boldsymbol{\xi} + (P(\boldsymbol{\xi}) \wedge P(\boldsymbol{\xi}))^+ = -F^+(\boldsymbol{A}').$$

By using (21), Lemma 3.9 and $h_\gamma = 1$ in the support of $F^+(\tilde{A}'_\gamma)$ (cf. Section 3.3), we have

$$\left\|F^+(\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}}) - F^+(\boldsymbol{A}')\right\|_{BL^p} \leq \mathrm{const}_M \cdot b^{4/p} \|\boldsymbol{\alpha}\|.$$

From Proposition 4.3, Remark 4.4, Lemma 5.1 and Hölder's inequality $BL^4 \times BL^q \to BL^p$,

$$\begin{aligned} &\left\|(P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}}) \wedge P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}}))^+ - (P(\boldsymbol{\xi}) \wedge P(\boldsymbol{\xi}))^+\right\|_{BL^p}, \\ &\leq \mathrm{const}_M \cdot b^2 \|\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}} - \boldsymbol{\xi}\|_{BL^p} + \mathrm{const}_M \cdot b^{2+4/p}(\|\boldsymbol{\alpha}\| + \|\boldsymbol{v}\|). \end{aligned}$$

Hence

$$\|\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}} - \boldsymbol{\xi}\|_{BL^p} \leq \mathrm{const}_M \cdot b^2 \|\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}} - \boldsymbol{\xi}\|_{BL^p} + \mathrm{const}_M \cdot b^{4/p}(\|\boldsymbol{\alpha}\| + b^2 \|\boldsymbol{v}\|)$$

Since $b \ll 1$, we get the desired estimate. □

Corollary 5.3.

$$\|P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}}) - P(\boldsymbol{\xi})\|_{BL^q} \leq \mathrm{const}_M \cdot b^{4/p}(\|\boldsymbol{\alpha}\| + \|\boldsymbol{v}\|).$$

Set $a_\gamma := P(\boldsymbol{\xi})|_{X''_\gamma} \in \Omega^1_{X''_\gamma}(\mathrm{ad}E_\gamma)$ and $\tilde{a}_\gamma := P_{\tilde{\theta}}(\boldsymbol{\xi}_{\tilde{\theta}})|_{X''_\gamma} = h_\gamma^{-1}(P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}}))|_{X''_\gamma} \in \Omega^1_{X''_\gamma}(\mathrm{ad}E_\gamma)$ for each $\gamma \in \Gamma$. We have $\boldsymbol{A}|_{X''_\gamma} = A'_\gamma + a_\gamma$ and $\tilde{\boldsymbol{A}}|_{X''_\gamma} = \tilde{A}'_\gamma + \tilde{a}_\gamma$.

Lemma 5.4.

$$\sup_{\gamma \in \Gamma} \|a_\gamma - \tilde{a}_\gamma\|_{L^q(X''_\gamma, g_\gamma)} \leq \mathrm{const}_M \cdot b^{4/p}(\|\boldsymbol{\alpha}\| + \|\boldsymbol{v}\|).$$

Proof.

$$\tilde{a}_\gamma - a_\gamma = (h_\gamma^{-1} - 1)P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}})h_\gamma + P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}})(h_\gamma - 1) + (P_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\xi}_{\boldsymbol{v},\boldsymbol{\alpha}}) - P(\boldsymbol{\xi})).$$

From Proposition 4.3 and Corollary 5.3, we get the above estimate. □

5.2. **Injectivity problem.** The purpose of this subsection is to prove the following.

Proposition 5.5. *There exists $N_0 = N_0(M)$ and $b_0 = b_0(M)$ such that if $N \geq N_0$ and $b = 4N\sqrt{\lambda} \leq b_0$ then the following holds: Let $\theta = (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma, s\in S}$ and $\theta' = (F_\gamma, B_\gamma, \rho'_{\gamma,s})_{\gamma\in\Gamma,s\in S}$ be two gluing data. Then $\boldsymbol{A}(\theta)$ is gauge equivalent to $\boldsymbol{A}(\theta')$ if and only if $[\theta] = [\theta']$ in* GlD.

Proof. The "if" part is a direct consequence of the definitions. So we will give the proof of the "only if" part. We set $\boldsymbol{A}_1 := \boldsymbol{A}(\theta)$ and $\boldsymbol{A}_2 := \boldsymbol{A}(\theta')$. Suppose there is a gauge transformation $g : \boldsymbol{E}(\theta) \to \boldsymbol{E}(\theta')$ satisfying $g(\boldsymbol{A}_1) = \boldsymbol{A}_2$. We define $X_{\gamma,b}$ ($\gamma \in \Gamma$) as the complement of the b-balls $\bar{B}(x_{\gamma,s}, b)$ and $\bar{B}(y_{\gamma,s}, b)$ in X_γ:

$$X_{\gamma,b} := X_\gamma \setminus \left(\bigcup_{s\in S} \bar{B}(x_{\gamma,s}, b) \cup \bar{B}(y_{\gamma,s}, b) \right).$$

By the definitions of the cut-off functions in Section 3.3, we have $A'_\gamma = A_\gamma$ and $B'_\gamma = B_\gamma$ over $X_{\gamma,b}$. From Proposition 4.3, we have

$$\|A_\gamma - \boldsymbol{A}_1\|_{L^q(X_{\gamma,b},g_\gamma)}, \|B_\gamma - \boldsymbol{A}_2\|_{L^q(X_{\gamma,b},g_\gamma)} \le \mathrm{const}_M \cdot b^{4/p}.$$

Since $\boldsymbol{A}_1$ is gauge equivalent to $\boldsymbol{A}_2$,

$$d_{L^q}([A_\gamma|_{X_{\gamma,b}}], [B_\gamma|_{X_{\gamma,b}}]) \le \mathrm{const}_M \cdot b^{4/p} \ll 1.$$

From Lemma 3.7 (2), this implies (for $b \ll 1$) $E_\gamma \cong F_\gamma$ for all $\gamma \in \Gamma$. Moreover, from Lemma 3.7 (1), Lemma 3.5 and the inequality (20), there exists $\alpha_\gamma \in H^1_{A_\gamma}$ with $\|\alpha_\gamma\|_{L^q} \le \mathrm{const}_M \cdot b^{4/p}$ for each $\gamma \in \Gamma$ such that B_γ is gauge equivalent to $\tilde{A}_\gamma := A_\gamma + \tilde{\alpha}_\gamma$. We can suppose $B_\gamma = \tilde{A}_\gamma$ without loss of generality.

$\rho_{\gamma,s}$ and $\rho'_{\gamma,s}$ are $SU(2)$-isomorphisms between $(E_\gamma)_{x_{\gamma,s}}$ and $(E_{\gamma s})_{y_{\gamma s,s}}$ ($\gamma \in \Gamma, s \in S$). Take $v_{\gamma,s} \in (\mathrm{ad}E)_{x_{\gamma,s}}$ such that $\rho'_{\gamma,s} = \rho_{\gamma,s} e^{v_{\gamma,s}}$ and $|v_{\gamma,s}| = d(\rho_{\gamma,s}, \rho'_{\gamma,s})$ ($\le \mathrm{Diam}(SU(2))$). Set $\boldsymbol{\alpha} := (\alpha_\gamma)_{\gamma\in\Gamma}$ and $\boldsymbol{v} := (v_{\gamma,s})_{\gamma\in\Gamma,s\in S}$ as in Section 5.1. From the assumption, there are gauge transformations g_γ of E_γ over X''_γ such that

$$g_\gamma(A'_\gamma + a_\gamma) = \tilde{A}'_\gamma + \tilde{a}_\gamma \quad \text{over } X''_\gamma, \tag{38}$$

where a_γ and $\tilde{a}_\gamma$ are the element of $\Omega^1_{X''_\gamma}(\mathrm{ad}E_\gamma)$ satisfying $\boldsymbol{A}(\theta) = A'_\gamma + a_\gamma$ and $\boldsymbol{A}(\theta') = \tilde{A}'_\gamma + \tilde{a}_\gamma$ over X''_γ as in Section 5.1. Moreover g_γ satisfy the following compatibility condition:

$$\rho'_{\gamma,s} \circ g_\gamma = g_{\gamma s} \circ \rho_{\gamma,s} \quad \text{over } \Omega(x_{\gamma,s}) = \Omega(y_{\gamma s,s}). \tag{39}$$

Let I_{A_γ} be the isotropy group of A_γ. From Lemma 3.8, we have

$$\min_{h\in I_{A_\gamma}} \|g_\gamma - h\|_{\mathcal{C}^0(X''_\gamma)} \le \mathrm{const}_M \left\| d_{A'_\gamma} g_\gamma \right\|_{L^q(X''_\gamma, g_\gamma)},$$

Using the action of $\prod_{\gamma\in\Gamma} I_{A_\gamma}$ on the gluing data, we can assume that

$$\|g_\gamma - 1\|_{\mathcal{C}^0(X''_\gamma)} \le \mathrm{const}_M \left\| d_{A'_\gamma} g_\gamma \right\|_{L^q(X''_\gamma, g_\gamma)}. \tag{40}$$

From (38),

$$\left\| d_{A'_\gamma} g_\gamma \right\|_{L^q(X''_\gamma)} \le \left\| A'_\gamma - \tilde{A}'_\gamma \right\|_{L^q(X''_\gamma)} + 2\|g_\gamma - 1\|_{\mathcal{C}^0(X''_\gamma)} \|a_\gamma\|_{L^q(X''_\gamma)} + \|\tilde{a}_\gamma - a_\gamma\|_{L^q(X''_\gamma)}.$$

Using Lemma 3.9, we get $\left\| A'_\gamma - \tilde{A}'_\gamma \right\|_{L^q(X''_\gamma)} \le \mathrm{const}_M \cdot \|\alpha_\gamma\|_{L^q(X_\gamma)}$. From Proposition 4.3 and Lemma 5.4, we get $\|a_\gamma\|_{L^q(X''_\gamma)} \le \mathrm{const}_M \cdot b^{4/p} \ll 1$ and $\|a_\gamma - \tilde{a}_\gamma\|_{L^q(X''_\gamma)} \le \mathrm{const}_M \cdot b^{4/p}(\|\boldsymbol{\alpha}\| + \|\boldsymbol{v}\|)$. Therefore (using (40))

$$\|g_\gamma - 1\|_{\mathcal{C}^0(X''_\gamma)} \le \mathrm{const}_M \|\boldsymbol{\alpha}\| + \mathrm{const}_M \cdot b^{4/p} \|\boldsymbol{v}\|. \tag{41}$$

On the other hand, from (38), $g_\gamma(A_\gamma) - \tilde{A}_\gamma = (\tilde{a}_\gamma - a_\gamma) + (1-g_\gamma)a_\gamma + g_\gamma a_\gamma(1-g_\gamma^{-1})$ over $X_{\gamma,b}$ where $A'_\gamma = A_\gamma$ and $\tilde{A}'_\gamma = \tilde{A}_\gamma$. Hence (using (20) and Lemma 3.7 (1))

$$\|\alpha_\gamma\| \leq \mathrm{const}_M \cdot d_{L^q}([A_\gamma|_{X_{\gamma,b}}], [\tilde{A}_\gamma|_{X_{\gamma,b}}]) \leq \mathrm{const}_M \left(\|a_\gamma - \tilde{a}_\gamma\|_{L^q(X''_\gamma)} + \|g_\gamma - 1\|_{\mathcal{C}^0(X''_\gamma)} \|a_\gamma\|_{L^q(X_\gamma)}\right)$$

Using (41), Proposition 4.3 and Lemma 5.4,

$$\|\boldsymbol{\alpha}\| \leq \mathrm{const}_M \cdot b^{4/p}(\|\boldsymbol{\alpha}\| + \|\boldsymbol{v}\|).$$

Since $b \ll 1$, we get

$$\|\boldsymbol{\alpha}\| \leq \mathrm{const}_M \cdot b^{4/p} \|\boldsymbol{v}\|. \tag{42}$$

Substituting this into (41), we get $\|g_\gamma - 1\|_{\mathcal{C}^0(X''_\gamma)} \leq \mathrm{const}_M \cdot b^{4/p}\|\boldsymbol{v}\|$.

From the compatibility condition (39), $\rho'_{\gamma,s} - \rho_{\gamma,s} = (g_{\gamma s}-1)\rho_{\gamma,s}g_\gamma + \rho_{\gamma,s}(g_\gamma^{-1}-1)$. Hence

$$|\rho'_{\gamma,s} - \rho_{\gamma,s}| \leq \|g_{\gamma s} - 1\|_{\mathcal{C}^0(X''_\gamma)} + \|g_\gamma - 1\|_{\mathcal{C}^0(X''_\gamma)} \leq \mathrm{const}_M \cdot b^{4/p} \|\boldsymbol{v}\|.$$

Then

$$\|\boldsymbol{v}\| \leq \mathrm{const} \cdot \sup_{\gamma,s} |\rho'_{\gamma,s} - \rho_{\gamma,s}| \leq \mathrm{const}_M \cdot b^{4/p} \|\boldsymbol{v}\|.$$

Since $b \ll 1$, we get $\|\boldsymbol{v}\| = 0$. This implies $\rho' = \rho$ and (using (42)) $\|\boldsymbol{\alpha}\| = 0$. Therefore $B_\gamma = \tilde{A}_\gamma = A_\gamma$. □

6. Infinite gluing: surjectivity problem

In this section we will study a "surjectivity problem". We basically follow the argument of Donaldson-Kronheimer [5, Section 7.2.4, 7.2.5]. But our case is more involved because we cannot use the usual index-theorem argument. (This difficulty is suggested in [5, p. 298].)

6.1. **Linearized problem.** Let $\theta = (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma, s\in S}$ be a gluing data. We denote I as the set of $\gamma \in \Gamma$ satisfying $[E_\gamma, A_\gamma] \in M_1$. If $M_1 = \emptyset$, then $I = \emptyset$. Set $\boldsymbol{E} := \boldsymbol{E}(\theta)$ and $\boldsymbol{A}' := \boldsymbol{A}'(\theta)$. We define $\boldsymbol{V}$ and $\boldsymbol{H}$ by

$$\begin{aligned} &\boldsymbol{V} := \{\boldsymbol{v} = (v_{\gamma,s})_{\gamma\in\Gamma,s\in S} \in \prod_{\gamma\in\Gamma,s\in S} (\mathrm{ad}E_\gamma)_{x_{\gamma,s}} |\ \|\boldsymbol{v}\| := \sup_{\gamma,s} |v_{\gamma,s}| < \infty\}, \\ &\boldsymbol{H} := \{\boldsymbol{\alpha} = (\alpha_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} H^1_{A_\gamma} |\ \|\boldsymbol{\alpha}\| := \sup_\gamma \|\alpha_\gamma\|_{L^q(X_\gamma, g_\gamma)} < \infty\}. \end{aligned} \tag{43}$$

Note that $H^1_{A_\gamma} = 0$ for $[E_\gamma, A_\gamma] \in M_1$. For $\boldsymbol{v} \in \boldsymbol{V}$ and $\boldsymbol{\alpha} \in \boldsymbol{H}$, we define $j_1(\boldsymbol{v})$ and $j_2(\boldsymbol{\alpha})$ in $\Omega^1(\mathrm{ad}\boldsymbol{E})$ by

$$j_1(\boldsymbol{v}) := \left.\frac{\partial}{\partial t}\right|_{t=0} \boldsymbol{A}'_{t\boldsymbol{v},0}, \quad j_2(\boldsymbol{\alpha}) := \left.\frac{\partial}{\partial t}\right|_{t=0} \boldsymbol{A}'_{0,t\boldsymbol{\alpha}}, \tag{44}$$

where $\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}}$ denotes a connection on $\boldsymbol{E}$ defined as in (36).

$\rho_{\gamma,s}$ is a $SU(2)$-isomorphism between the fibers $(E_\gamma)_{x_{\gamma,s}}$ and $(E_{\gamma s})_{y_{\gamma s,s}}$. In this subsection, we identify the fibers $(E_\gamma)_{x_{\gamma,s}}$ and $(E_{\gamma s})_{y_{\gamma,s}}$ by the given $\rho_{\gamma,s}$. Let $\boldsymbol{v} = (v_{\gamma,s})_{\gamma,s}$

where $v_{\gamma,s} \in (\mathrm{ad}E_\gamma)_{x_{\gamma,s}} \cong (\mathrm{ad}E_{\gamma s})_{y_{\gamma s,s}}$. We often consider $v_{\gamma,s}$ as a section of $\mathrm{ad}E_\gamma$ (or $\mathrm{ad}E_{\gamma s}$) over $\Omega(x_{\gamma,s})$ (or $\Omega(y_{\gamma s,s})$) by using the exponential gauge of A_γ (or $A_{\gamma s}$): $E_\gamma|_{\Omega(x_{\gamma,s})} \cong \Omega(x_{\gamma,s}) \times (E_\gamma)_{x_{\gamma,s}}$ (or $E_{\gamma s}|_{\Omega(y_{\gamma s,s})} \cong \Omega(y_{\gamma s,s}) \times (E_{\gamma s})_{y_{\gamma s,s}}$). Then the above $j_1(\boldsymbol{v})$ is expressed by

$$j_1(\boldsymbol{v}) = \begin{cases} d_{A'_\gamma}(\beta'_\gamma v_{\gamma s}) = d\beta'_\gamma \otimes v_{\gamma,s} & \text{over } \Omega(x_{\gamma,s}) \\ -d_{A'_\gamma}(\beta'_\gamma v_{\gamma s^{-1},s}) = -d\beta'_\gamma \otimes v_{\gamma s^{-1},s} & \text{over } \Omega(y_{\gamma,s}), \end{cases} \tag{45}$$

and we have $\mathrm{supp}(j_1(\boldsymbol{v})) \subset \bigcup_\gamma \mathrm{supp}(d\beta'_\gamma) \subset \bigcup_{\gamma,s}(\Omega(x_{\gamma,s}) \cup \Omega(y_{\gamma,s}))$. From this we easily deduce that $d^+_{\boldsymbol{A}'} j_1(\boldsymbol{v}) = 0$.

Lemma 6.1.

$$\left\| j_2(\boldsymbol{\alpha}) - \sum_{\gamma\in\Gamma} \beta'_\gamma \alpha_\gamma \right\|_{BL^q} \leq \mathrm{const}_M \cdot b^{4/q} \left\|\boldsymbol{\alpha}\right\|.$$

In particular, $\left\|j_2(\boldsymbol{\alpha})\right\|_{BL^q} \leq \mathrm{const}_M \left\|\boldsymbol{\alpha}\right\|$. *Moreover*

$$\left\|d^+_{\boldsymbol{A}'} j_2(\boldsymbol{\alpha})\right\|_{BL^p} \leq \mathrm{const}_M \cdot b^{4/p} \left\|\boldsymbol{\alpha}\right\|.$$

Proof. We have $j_2(\boldsymbol{\alpha}) = \alpha_\gamma$ over $X_\gamma \setminus \bigcup_s (B(x_{\gamma,s}, b) \cup B(y_{\gamma,s}, b))$. Using Lemma 3.10, we have

$$|j_2(\alpha)|_{g_\gamma} \leq \mathrm{const}_M \left\|\boldsymbol{\alpha}\right\| \quad \text{over } \bigcup_s (B(x_{\gamma,s}, b) \cup B(y_{\gamma,s}, b)).$$

Therefore we get the first inequality. We have $d^+_{\boldsymbol{A}'} j_2(\boldsymbol{\alpha}) = 0$ over $X_\gamma \setminus \bigcup_s (B(x_{\gamma,s}, b) \cup B(y_{\gamma,s}, b))$, where $F^+(\boldsymbol{A}_{0,t\boldsymbol{\alpha}}) = 0$. Since $d^+_{\boldsymbol{A}'} j_2(\boldsymbol{\alpha}) = \partial F^+(\boldsymbol{A}'_{0,t\boldsymbol{\alpha}})/\partial t\big|_{t=0}$ and $h_\gamma = 1$ in the support of $F^+(\boldsymbol{A}'_{0,t\boldsymbol{\alpha}})$, the estimate (24) gives

$$|d^+_{\boldsymbol{A}'} j_2(\boldsymbol{\alpha})| \leq \mathrm{const}_M \left\|\boldsymbol{\alpha}\right\| \quad \text{over } \bigcup_{\gamma,s} (B(x_{\gamma,s}, b) \cup B(y_{\gamma,s}, b)).$$

Therefore $\left\|d^+_{\boldsymbol{A}'} j_2(\boldsymbol{\alpha})\right\|_{BL^p} \leq \mathrm{const}_M \cdot b^{4/p} \left\|\boldsymbol{\alpha}\right\|$. □

Fix $z \in X \setminus \{x_s, y_s | s \in S\}$. We take $b = 4N\sqrt{\lambda} > 0$ so small that the balls of radius b around x_s and y_s ($s \in S$) don't contain z. Let z_γ ($\gamma \in \Gamma$) be the point in X_γ corresponding to z. We define $\Omega(\mathrm{ad}\boldsymbol{E})_0$ as the set of $\chi \in \Omega(\mathrm{ad}\boldsymbol{E})$ satisfying $\chi(z_\gamma) = 0$ for all $\gamma \in I$ (i.e., $[E_\gamma, A_\gamma] \in M_1$). Here we consider $z_\gamma \in X''_\gamma \subset X^{\sharp(\Gamma,S)}$.

Lemma 6.2. *For any* $\chi \in \Omega^0(\mathrm{ad}\boldsymbol{E})_0$, $\boldsymbol{v} \in \boldsymbol{V}$ *and* $\boldsymbol{\alpha} \in \boldsymbol{H}$,

$$\left\|\chi\right\|_{\mathcal{C}^0} + \left\|\boldsymbol{v}\right\| + \left\|\boldsymbol{\alpha}\right\| \leq \mathrm{const}_M \left\|d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha})\right\|_{BL^q}.$$

Proof. For each $\gamma \in \Gamma$ we define a section χ_γ of $\mathrm{ad}E_\gamma$ over X''_γ by

$$\chi_\gamma := \chi + \sum_s (\beta'_\gamma - 1) v_{\gamma,s} + \sum_s (1 - \beta'_\gamma) v_{\gamma s^{-1},s}.$$

We have $d_{A'_\gamma}\chi_\gamma = d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v})$ over X''_γ and $\chi_{\gamma s} - \chi_\gamma = v_{\gamma,s}$ over $X''_\gamma \cap X''_{\gamma s}$. We have $\|\chi\|_{\mathcal{C}^0} + \|\boldsymbol{v}\| \leq (4|S|+3)\sup_\gamma \|\chi_\gamma\|_{\mathcal{C}^0(X''_\gamma)}$. If $\gamma \in I$, then $\chi_\gamma(z_\gamma) = 0$. If $\gamma \notin I$, then A_γ is irreducible. Therefore

$$\begin{aligned}\|\chi_\gamma\|_{\mathcal{C}^0(X''_\gamma)} + \|\alpha_\gamma\|_{L^q(X_\gamma)} &\leq \mathrm{const}_M \left\|d_{A'_\gamma}\chi_\gamma + \alpha_\gamma\right\|_{L^q(X''_\gamma, g_\gamma)}, \\ &\leq \mathrm{const}'_M \|d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha})\|_{BL^q} + \mathrm{const}_M \|\alpha_\gamma - j_2(\boldsymbol{\alpha})\|_{L^q(X''_\gamma, g_\gamma)}.\end{aligned}$$

By using the argument in the proof of Lemma 6.1, we get

$$\|\alpha_\gamma - j_2(\boldsymbol{\alpha})\|_{L^q(X''_\gamma, g_\gamma)} \leq \mathrm{const}_M \cdot b^{4/q}\|\boldsymbol{\alpha}\|.$$

Since $b \ll 1$, we get the above conclusion. $\square$

Let $\chi \in \Omega^0(\mathrm{ad}\boldsymbol{E})_0$ and $\xi \in \Omega^+(\mathrm{ad}\boldsymbol{E})$ be smooth (not necessarily compact supported) 0-form and self-dual form valued in $\mathrm{ad}\boldsymbol{E}$ over $X^{\sharp(\Gamma,S)}$, and let $\boldsymbol{v} \in \boldsymbol{V}$ and $\boldsymbol{\alpha} \in \boldsymbol{H}$. We define the norm $\|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)\|_{B_1}$ by

$$\|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)\|_{B_1} := \|d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha})\|_{BL^q} + \|\xi\|_{BL^p}. \tag{46}$$

Lemma 6.2 shows that this becomes a norm. (Of course, its value might be infinity.) We define the Banach space B_1 as the completion of the space of $(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi) \in \Omega^0(\mathrm{ad}\boldsymbol{E})_0 \oplus \boldsymbol{V} \oplus \boldsymbol{H} \oplus \Omega^+(\mathrm{ad}\boldsymbol{E})$ of $\|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)\|_{B_1} < \infty$ in the norm $\|\cdot\|_{B_1}$:

$$B_1 := \overline{\{(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi) \in \Omega^0(\mathrm{ad}\boldsymbol{E})_0 \oplus \boldsymbol{V} \oplus \boldsymbol{H} \oplus \Omega^+(\mathrm{ad}\boldsymbol{E}) |\ \|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)\|_{B_1} < \infty\}},$$

where the overline means the completion in the norm $\|\cdot\|_{B_1}$. Let $\omega \in \Omega^1(\mathrm{ad}\boldsymbol{E})$ be a smooth 1-form valued in $\mathrm{ad}\boldsymbol{E}$ over $X^{\sharp(\Gamma,S)}$. We define the norm $\|\omega\|_{B_2}$ by setting

$$\|\omega\|_{B_2} := \|\omega\|_{BL^q} + \left\|d^+_{\boldsymbol{A}'}\omega\right\|_{BL^p}. \tag{47}$$

We define the Banach space B_2 as the completion of the space of $\omega \in \Omega^1(\mathrm{ad}\boldsymbol{E})$ of $\|\omega\|_{B_2} < \infty$ in the norm $\|\cdot\|_{B_2}$:

$$B_2 := \overline{\{\omega \in \Omega^1(\mathrm{ad}\boldsymbol{E}) |\ \|\omega\|_{B_2} < \infty\}}.$$

Let $P = P_\theta : BL^p \to BL^q$ be the map defined in Lemma 4.1. P is a right inverse of $d^+_{\boldsymbol{A}'}$. We define a linear map $T : B_1 \to B_2$ by

$$T(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi) := d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha}) + P(\xi). \tag{48}$$

Proposition 6.3. *T is a bounded linear operator. Moreover there exists a positive constant K depending only on M such that for any $(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi) \in B_1$*

$$\|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)\|_{B_1} \leq K \|T(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)\|_{B_2}.$$

Proof. Set $\omega := T(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)$. We have (using $d^+_{\boldsymbol{A}'}j_1(v) = 0$)

$$d^+_{\boldsymbol{A}'}\omega = [F^+_{\boldsymbol{A}'}, \chi] + d^+_{\boldsymbol{A}'}j_2(\boldsymbol{\alpha}) + \xi.$$

Form Lemma 6.1 and 6.2, T is bounded. From this equation (remember $\|F^+(\boldsymbol{A}')\|_{BL^p} \leq \mathrm{const}_M b^{4/p}$)

$$\begin{aligned}\|\xi\|_{BL^p} &\leq \|\omega\|_{B_2} + \mathrm{const}_M \cdot b^{4/p}(\|\chi\|_{\mathcal{C}^0} + \|\boldsymbol{\alpha}\|),\\ &\leq \|\omega\|_{B_2} + \mathrm{const}_M \cdot b^{4/p} \|d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha})\|_{BL^q},\\ &= \|\omega\|_{B_2} + \mathrm{const}_M \cdot b^{4/p} \|\omega - P(\xi)\|_{BL^q},\\ &\leq (1+\mathrm{const}_M \cdot b^{4/p}) \|\omega\|_{B_2} + \mathrm{const}_M \cdot b^{4/p} \|\xi\|_{BL^p}.\end{aligned}$$

Since $b \ll 1$, we get $\|\xi\|_{BL^p} \leq 2\|\omega\|_{B_2}$ and

$$\|d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha})\|_{BL^p} = \|\omega - P(\xi)\|_{BL^p} \leq \mathrm{const}_M \|\omega\|_{B_2}.$$

□

This result shows that T is an embedding. Indeed we want to prove that T is an isomorphism. Donaldson-Kronheimer [5, Section 7.2.5] proves a similar result by using the index theorem. But we cannot use the index theorem and must prove it by a direct analysis.

Let $\omega \in B_2$ and set $\omega' := \omega - Pd^+_{\boldsymbol{A}'}\omega$. We have $d^+_{\boldsymbol{A}'}\omega' = 0$ and $\|\omega'\|_{BL^q} \leq \mathrm{const}_M \|\omega\|_{B_2}$. Consider $\beta_\gamma\omega'$ on X_γ for each $\gamma \in \Gamma$. We have $d^+_{A_\gamma}\{\beta_\gamma\omega' - P_\gamma d^+_{A_\gamma}(\beta_\gamma\omega')\} = 0$ where $P_\gamma = (d^+_{A_\gamma})^*(d^+_{A_\gamma}(d^+_{A_\gamma})^*)^{-1}$ is the right inverse of $d^+_{A_\gamma}$. Then there uniquely exist $\chi_\gamma \in \Omega^0(\mathrm{ad}E_\gamma)$ and $\alpha_\gamma \in H^1_{A_\gamma}$ such that

$$d_{A_\gamma}\chi_\gamma + \alpha_\gamma = \beta_\gamma\omega' - P_\gamma d^+_{A_\gamma}(\beta_\gamma\omega'),$$

and $\chi_\gamma(z_\gamma) = 0$ if $\gamma \in I$. (If $\gamma \in I$, then $H^1_{A_\gamma} = 0$ and $\alpha_\gamma = 0$.) Since $d^+_{\boldsymbol{A}'}\omega' = 0$,

$$P_\gamma d^+_{A_\gamma}(\beta_\gamma\omega') = P_\gamma(d\beta_\gamma \wedge \omega')^+ + P_\gamma[(A_\gamma - A'_\gamma) \wedge (\beta_\gamma\omega')]^+. \tag{49}$$

A difficulty comes from the term $P_\gamma(d\beta_\gamma \wedge \omega')^+$. The term $P_\gamma[(A_\gamma - A'_\gamma) \wedge (\beta_\gamma\omega)]^+$ can be easily estimated:

$$\begin{aligned}\left\|P_\gamma[(A_\gamma - A'_\gamma) \wedge (\beta_\gamma\omega')]^+\right\|_{L^q(X_\gamma, g_\gamma)} &\leq \mathrm{const}_M \left\|A_\gamma - A'_\gamma\right\|_{L^4} \|\beta_\gamma\omega'\|_{L^q(X'_\gamma, g_\gamma)},\\ &\leq \mathrm{const}_M \cdot b^2 N^{2-8/q} \|\omega\|_{B_2}.\end{aligned} \tag{50}$$

Here we have used (10) and (15). If we choose $b^2N^{2-8/q} \ll 1$, then this is a good estimate. But a similar estimation gives

$$\left\|P_\gamma(d\beta_\gamma \wedge \omega')^+\right\|_{L^q(X_\gamma, g_\gamma)} \leq \mathrm{const}_M \cdot N^{2-8/q} \|\omega\|_{B_2}.$$

Since $N \gg 1$ and $2 - 8/q > 0$, this is not a small term. We will come back to this point later. We have $(x_\gamma(z_\gamma) = 0$ for $\gamma \in \Gamma)$

$$\|\chi_\gamma\|_{\mathcal{C}^0} + \|\alpha_\gamma\|_{L^q(X_\gamma, g_\gamma)} \leq \mathrm{const}_M \left\|d_{A_\gamma}\chi_\gamma + \alpha_\gamma\right\|_{L^q(X_\gamma, g_\gamma)} \leq \mathrm{const}_M \cdot N^{2-8/q} \|\omega\|_{B_2}. \tag{51}$$

Set $\chi := \sum_\gamma \beta'_\gamma \chi_\gamma \in \Omega^0(\mathrm{ad}\boldsymbol{E})_0$. Then we have the following equation (using $\beta'_\gamma\beta_\gamma = \beta'_\gamma$):

$$
\begin{aligned}
& d_{\boldsymbol{A}'}\chi - \sum_\gamma d\beta'_\gamma \otimes \chi_\gamma + \sum_\gamma \beta'_\gamma \alpha_\gamma \\
& = \omega' + \sum_\gamma \left(\beta'_\gamma[A_\gamma - A'_\gamma, \chi_\gamma] - \beta'_\gamma P_\gamma[(A_\gamma - A'_\gamma)\wedge(\beta_\gamma\omega')]^+\right) - \sum_\gamma \beta'_\gamma P_\gamma (d\beta_\gamma \wedge \omega')^+.
\end{aligned} \tag{52}
$$

From (51),

$$
\left\|[A_\gamma - A'_\gamma, \chi_\gamma]\right\|_{L^q(X''_\gamma, g)} \leq \left\|A_\gamma - A'_\gamma\right\|_{L^q(X''_\gamma, g_\gamma)} \|\chi_\gamma\|_{\mathcal{C}^0} \leq \mathrm{const}_M \cdot b^{1+4/q} N^{2-8/q} \|\omega\|_{B_2}. \tag{53}
$$

$$
\begin{aligned}
\left\|P_\gamma[(A_\gamma - A'_\gamma)\wedge(\beta_\gamma\omega')]^+\right\|_{L^q(X''_\gamma, g)} &\leq \mathrm{const}_M \left\|A_\gamma - A'_\gamma\right\|_{L^4} \|\beta_\gamma \omega\|_{L^q(X'_\gamma, g_\gamma)}, \\
&\leq \mathrm{const}_M \cdot b^2 N^{2-8/q} \|\omega\|_{B_2}.
\end{aligned} \tag{54}
$$

Hence

$$
\begin{aligned}
\left\|\sum_\gamma \left(\beta'_\gamma[A_\gamma - A'_\gamma, \chi_\gamma] - \beta'_\gamma P_\gamma[(A_\gamma - A'_\gamma)\wedge(\beta_\gamma\omega')]^+\right)\right\|_{BL^q} &\leq \mathrm{const}_M \cdot b^{1+4/q} N^{2-8/q} \|\omega\|_{B_2}, \\
&\leq \mathrm{const}_M \cdot b N^2 \|\omega\|_{B_2}.
\end{aligned} \tag{55}
$$

The estimation of the term $\sum_\gamma \beta'_\gamma P_\gamma (d\beta_\gamma \wedge \omega')^+$ needs the following lemma:

Lemma 6.4. *Let $0 < \delta < 1$, and f be a L^p-function in $\mathbb{R}^4$ satisfying*

$$
\mathrm{supp} f \subset \{x | \sqrt{\lambda}/N \leq |x| \leq \sqrt{\lambda}/N^{1-\delta}\},
$$

where $N \gg 1$. Set

$$
F(x) := \int_{\mathbb{R}^4} \frac{f(y)}{|x-y|^3} dy.
$$

Then we have

$$
\left(\int_{|x|\geq\sqrt{\lambda}/2} |F(x)|^q dx\right)^{1/q} \leq \mathrm{const} \cdot N^{-4(1-1/p)(1-\delta)} \|f\|_{L^p(\mathbb{R}^4)}.
$$

Here remember that $2 < p < 4$, $q > 4$ and $1 - 4/p = -4/q$.

Proof. Using a scale change, we suppose $\lambda = 1$ without loss of generality. If $|x| \geq 1/2$ and $|y| \leq N^{-1+\delta}$, then (using $N \gg 1$ and $-1+\delta < 0$) $|x-y| \geq |x| - N^{-1+\delta} \geq |x|/2$. Then $1/|x-y|^3 \leq 2^3/|x|^3$. Hence for $|x| \geq 1/2$

$$
|F(x)| \leq \frac{2^3}{|x|^3} \int_{\mathbb{R}^4} |f(y)| dy.
$$

Using $q > 4$, we get

$$
\int_{|x|\geq 1/2} |F(x)|^q dx \leq 2^{3q} \int_{1/2}^{\infty} \frac{dr}{r^{3q-3}} \left(\int_{\mathbb{R}^4} |f(y)| dy\right)^q \leq \mathrm{const} \left(\int_{\mathbb{R}^4} |f(y)| dy\right)^q.
$$

$$\left(\int_{|x|\geq 1/2}|F(x)|^q dx\right)^{1/q} \leq \text{const}\int_{\mathbb{R}^4}|f(y)|dy,$$
$$\leq \text{const}\,\|f\|_{L^p(\mathbb{R}^4)}\left(\text{vol}\,(\text{supp}f)\right)^{1-1/p},$$
$$\leq \text{const}\cdot N^{-4(1-\delta)(1-1/p)}\,\|f\|_{L^p(\mathbb{R}^4)}\,.$$

□

We use this lemma for $\delta = 1/6$.

Lemma 6.5.

$$\left\|P_\gamma(d\beta_\gamma\wedge\omega')^+\right\|_{L^q(X''_\gamma, g_\gamma)} \leq \text{const}_M\cdot N^{-1/2}\,\|\omega\|_{B_2}\,,$$

Proof. Set $\sigma := (d\beta_\gamma\wedge\omega')^+$ and $\delta = 1/6$. There exists $r_0 > 0$ (independent of λ and N) such that the metric g_γ is flat over the balls $B_s = B(x_{\gamma,s}, r_0)$ and $B'_s = B(y_{\gamma,s}, r_0)$ ($s\in S$). We assume $r_0 \gg b = 4N\sqrt{\lambda}$. Set $B := \bigcup(B_s\cup B'_s)$. We define the annulus region A_s and A'_s by $A_s := B(x_{\gamma,s}, \sqrt{\lambda}/N^{1-\delta})\setminus \bar{B}(x_{\gamma,s}, \sqrt{\lambda}/N)$ and $A'_s := B(y_{\gamma,s}, \sqrt{\lambda}/N^{1-\delta})\setminus \bar{B}(y_{\gamma,s}, \sqrt{\lambda}/N)$, and set $A := \bigcup(A_s\cup A'_s)$. Remember that $\text{supp}\,\sigma\subset \text{supp}(d\beta_\gamma)\subset A$ by Lemma 3.4.

$P_\gamma\sigma$ can be expressed by using the Green kernel:

$$P_\gamma\sigma(x) = \int_A G(x,y)\sigma(y)d\text{vol}(y),$$

where the volume form $d\text{vol}(y) = d\text{vol}_{g_\gamma}(y)$ is defined by using the metric g_γ. The Green kernel $G(x,y)$ has a singularity of degree 3 along the diagonal (cf. Donaldson [4, p. 310]):

$$|G(x,y)| \leq \text{const}_M/d(x,y)^3,$$

where $d(x,y)$ is the distance on X_γ defined by g_γ.

$$\int_{X''_\gamma}|P_\gamma\sigma|^q d\text{vol} = \int_{X''_\gamma\setminus B}|P_\gamma\sigma|^q d\text{vol} + \int_{B\cap X''_\gamma}|P_\gamma\sigma|^q d\text{vol}.$$

The first term can be easily estimated:

$$\int_{X''_\gamma\setminus B}|P_\gamma\sigma|^q d\text{vol} \leq \text{const}_M\int_{X''_\gamma}d\text{vol}(x)\left(\int_A|\sigma(y)|d\text{vol}(y)\right)^q,$$
$$\leq \text{const}_M\cdot(\text{vol}A)^{q(1-1/p)}\left(\int_A|\sigma(y)|^p d\text{vol}(y)\right)^{q/p},$$
$$\leq \text{const}_M\cdot(\lambda^2 N^{-4(1-\delta)})^{q(1-1/p)}\left(\int_A|\sigma(y)|^p d\text{vol}(y)\right)^{q/p}.$$

From Lemma 6.4,

$$\left(\int_{B\cap X''_\gamma}|P_\gamma\sigma|^q d\text{vol}\right)^{1/q} \leq \text{const}_M\cdot N^{-4(1-\delta)(1-1/p)}\left(\int_A|\sigma(y)|^p d\text{vol}(y)\right)^{1/p}.$$

Hence

$$\|P_\gamma\sigma\|_{L^q(X''_\gamma,g_\gamma)} \le \mathrm{const}_M \cdot N^{-4(1-\delta)(1-1/p)} \|\sigma\|_{L^p(X'_\gamma,g_\gamma)}.$$

From (10),

$$\|\sigma\|_{L^p(X'_\gamma,g_\gamma)} \le \|d\beta_\gamma\|_{L^4} \|\omega'\|_{L^q(X'_\gamma,g_\gamma)} \le \mathrm{const}_M \cdot N^{2-8/q} \|\omega'\|_{L^q(X'_\gamma,g)} \le \mathrm{const}_M \cdot N^{2-8/q} \|\omega\|_{B_2}.$$

We have $1-4/p = -4/q$, $2<p<4$ and $\delta = 1/6$. Then $2-8/q-4(1-\delta)(1-1/p) = 4\delta(1-1/p)-4/p < -1/2$. Therefore we get the above conclusion. □

From the above Lemma, we get

$$\left\|\sum_\gamma \beta'_\gamma P_\gamma(d\beta_\gamma\wedge\omega')^+\right\|_{BL^q} \le \mathrm{const}_M \cdot N^{-1/2}\|\omega\|_{B_2}.$$

From the equation (52) and the estimate (55), we get

$$\left\|d_{\boldsymbol{A}'}\chi - \sum_\gamma d\beta'_\gamma\otimes\chi_\gamma + \sum_\gamma \beta'_\gamma\alpha_\gamma - \omega'\right\|_{BL^q} \le \mathrm{const}_M(bN^2+N^{-1/2})\|\omega\|_{B_2}. \tag{56}$$

Let $W_{\gamma,s} := B(x_{\gamma,s}, 2\sqrt{\lambda})\setminus B(x_{\gamma,s},\sqrt{\lambda}/2)\subset X_\gamma$ be the "neck" region ($\gamma\in\Gamma$, $s\in S$). Since $d\beta'_\gamma = -d\beta'_{\gamma s}$ over $W_{\gamma,s}$, the term $-\sum_\gamma d\beta'_\gamma\otimes\chi_\gamma$ can be expressed by

$$-\sum_\gamma d\beta'_\gamma\otimes\chi_\gamma = \sum_{\gamma,s} d\beta'_\gamma\otimes(-\chi_\gamma+\chi_{\gamma s})|_{W_{\gamma,s}}.$$

We have $d_{A_\gamma}\chi_\gamma = \beta_\gamma\omega' - P_\gamma d^+_{A_\gamma}(\beta_\gamma\omega') - \alpha_\gamma$. Since $\beta_\gamma = 1$ over the neck $W_{\gamma,s}$, we have

$$d_{\boldsymbol{A}'}\chi_\gamma = \omega' - P_\gamma d^+_{A_\gamma}(\beta_\gamma\omega') - \alpha_\gamma + [A'_\gamma - A_\gamma, \chi_\gamma] \quad \text{on } W_{\gamma,s}.$$

Therefore on the neck $W_{\gamma,s}$

$$\begin{aligned} &d_{\boldsymbol{A}'}(\chi_\gamma - \chi_{\gamma s}) \\ &= -P_\gamma d^+_{A_\gamma}(\beta_\gamma\omega') - \alpha_\gamma + [A'_\gamma - A_\gamma,\chi_\gamma] + P_{\gamma s}d^+_{A_{\gamma s}}(\beta_{\gamma s}\omega') + \alpha_{\gamma s} - [A'_{\gamma s} - A_{\gamma s},\chi_{\gamma s}]. \end{aligned} \tag{57}$$

As in (54) and Lemma 6.5,

$$\left\|P_\gamma d^+_{A_\gamma}(\beta_\gamma\omega')\right\|_{L^q(W_{\gamma,s},g)} \le \mathrm{const}_M\cdot(N^{-1/2}+b^2N^{2-8/q})\|\omega\|_{B_2}.$$

From (53),

$$\left\|[A_\gamma - A'_\gamma,\chi_\gamma]\right\|_{L^q(X''_\gamma,g)} \le \mathrm{const}_M\cdot b^{1+4/q}N^{2-8/q}\|\omega\|_{B_2}.$$

From (51) and an (elliptic) estimate $\|\alpha_\gamma\|_{L^\infty(X_\gamma,g_\gamma)} \le \mathrm{const}_M\|\alpha_\gamma\|_{L^q(X_\gamma,g_\gamma)}$,

$$\begin{aligned} \|\alpha_\gamma\|_{L^q(W_{\gamma,s},g)} &\le (\mathrm{const}\cdot\lambda^2)^{1/q}\|\alpha_\gamma\|_{L^\infty(X_\gamma,g_\gamma)}, \\ &\le \mathrm{const}_M\cdot\lambda^{2/q}\|\alpha_\gamma\|_{L^q(X_\gamma,g_\gamma)} \le \mathrm{const}_M\cdot\lambda^{2/q}N^{2-8/q}\|\omega\|_{B_2}. \end{aligned}$$

In the same way, we get the estimates of the other terms in the right-hand-side of (57). Then

$$\|d_{\boldsymbol{A}'}(\chi_\gamma - \chi_{\gamma s})\|_{L^q(W_{\gamma,s},g)} \le \mathrm{const}_M\cdot(N^{-1/2}+bN^2+\lambda^{2/q}N^2)\|\omega\|_{B_2}.$$

Let $v_{\gamma,s} \in (\mathrm{ad}E_\gamma)_{x_{\gamma,s}} \cong (\mathrm{ad}E_{\gamma s})_{y_{\gamma s,s}}$ be the mean value of $\chi_{\gamma s} - \chi_\gamma$ over the neck $W_{\gamma,s}$. Using the Sobolev embedding $L_1^q \hookrightarrow \mathcal{C}^{0,1-4/q}$ (Hölder space), we get

$$\begin{aligned}\|\chi_{\gamma s} - \chi_\gamma - v_{\gamma,s}\|_{\mathcal{C}^0(W_{\gamma,s})} &\le \mathrm{const}\cdot \lambda^{1/2-2/q}\|d_{\boldsymbol{A}'}(\chi_\gamma - \chi_{\gamma s})\|_{L^q(W_{\gamma,s},g)},\\ &\le \mathrm{const}_M \cdot \lambda^{1/2-2/q}(N^{-1/2} + bN^2 + \lambda^{2/q}N^2)\|\omega\|_{B_2}.\end{aligned}$$

Set $\boldsymbol{v} := (v_{\gamma,s})_{\gamma,s} \in \boldsymbol{V}$.

(58)

$$\begin{aligned}\left\|\sum_\gamma d\beta'_\gamma \otimes \chi_\gamma + j_1(\boldsymbol{v})\right\|_{BL^q} &= \left\|\sum_{\gamma,s} d\beta'_\gamma \otimes (v_{\gamma,s} - \chi_{\gamma s} + \chi_\gamma)|_{W_{\gamma,s}}\right\|_{BL^q},\\ &\le \mathrm{const}_M \cdot \frac{1}{\sqrt{\lambda}}(\sqrt{\lambda})^{4/q}\lambda^{1/2-2/q}(N^{-1} + bN^2 + \lambda^{2/q}N^2)\|\omega\|_{B_2},\\ &\le \mathrm{const}_M(N^{-1/2} + bN^2 + \lambda^{2/q}N^2)\|\omega\|_{B_2}.\end{aligned}$$

Set $\boldsymbol{\alpha} := (\alpha_\gamma)_\gamma \in \boldsymbol{H}$. From (51), $\|\boldsymbol{\alpha}\| = \sup\|\alpha_\gamma\|_{L^q} \le \mathrm{const}_M \cdot N^{2-8/q}\|\omega\|_{B_2}$. Using Lemma 6.1, we get ($b = 4N\sqrt{\lambda}$)

$$\begin{aligned}\left\|j_2(\boldsymbol{\alpha}) - \sum_{\gamma\in\Gamma}\beta'_\gamma\alpha_\gamma\right\|_{BL^q} &\le \mathrm{const}_M \cdot b^{4/q}\|\boldsymbol{\alpha}\| \le \mathrm{const}_M \cdot b^{4/q}N^{2-8/q}\|\omega\|_{B_2},\\ &\le \mathrm{const}_M \cdot \lambda^{2/q}N^2\|\omega\|_{B_2}.\end{aligned}$$

Using this and (58) in the estimate (56), we get

$$\|d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha}) - \omega'\|_{BL^q} \le \mathrm{const}_M \cdot (N^{-1/2} + bN^2 + \lambda^{2/q}N^2)\|\omega\|_{B_2}.$$

We have $d^+_{\boldsymbol{A}'}(d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha}) - \omega') = [F^+_{\boldsymbol{A}'}, \chi] + d^+_{\boldsymbol{A}'}j_2(\boldsymbol{\alpha})$. Using $\|F^+_{\boldsymbol{A}'}\|_{BL^p} \le \mathrm{const}_M \cdot b^{4/p}$, (51) and Lemma 6.1, we get

$$\begin{aligned}\|[F^+_{\boldsymbol{A}'}, \chi] + d^+_{\boldsymbol{A}'}j_2(\boldsymbol{\alpha})\|_{BL^p} &\le \mathrm{const}_M \cdot b^{4/p}N^{2-8/q}\|\omega\|_{B_2} + \mathrm{const}_M \cdot b^{4/p}\|\boldsymbol{\alpha}\|,\\ &\le \mathrm{const}_M \cdot \lambda^{2/q}N^2\|\omega\|_{B_2}.\end{aligned}$$

Thus we conclude that

$$\|d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha}) - \omega'\|_{B_2} \le \mathrm{const}_M \cdot (N^{-1/2} + bN^2 + \lambda^{2/q}N^2)\|\omega\|_{B_2}.$$

We define a bounded linear operator $T' : B_2 \to B_1$ by $T'(\omega) := (\chi, \boldsymbol{v}, \boldsymbol{\alpha}, d^+_{\boldsymbol{A}'}\omega)$. Remember $\omega' = \omega - Pd^+_{\boldsymbol{A}'}\omega$. The above shows $\|TT'(\omega) - \omega\|_{B_2} \le \mathrm{const}_M \cdot (N^{-1/2} + bN^2 + \lambda^{2/q}N^2)\|\omega\|_{B_2}$. Therefore if we choose λ and N appropriately, then $(TT')^{-1}$ exists and $T'(TT')^{-1}$ becomes a right inverse of T. In particular, T becomes surjective and hence isomorphic (see Proposition 6.3). Then we get the following.

Proposition 6.6. *There are $N_0 > 0$ and $\lambda_0(N) > 0$ such that if $N \ge N_0$ and $\lambda \le \lambda_0(N)$ then $T : B_1 \to B_2$ (given in (48)) is an isomorphism and satisfies*

$$\|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)\|_{B_1} \le K\|T(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \xi)\|_{B_2},$$

where K is a positive constant depending only on M.

6.2. **Some continuities.** Let's recall our situation. Γ is a finitely generated group and S is its finite generating set which does not contain the identity element e. The group Γ can be considered as a metric space endowed with the (left-invariant) word distance by S: For $\gamma, \gamma' \in \Gamma$,

$$d_S(\gamma, \gamma') := \min\{n \geq 0 | \exists \gamma_1, \cdots, \gamma_n \in S \cup S^{-1} : \gamma^{-1}\gamma' = \gamma_1 \cdots \gamma_n\}$$

For a subset $\Omega \subset \Gamma$ and an integer $d > 0$, we set

$$B_d(\Omega) := \{\gamma \in \Gamma | \exists \gamma' \in \Omega : d_S(\gamma, \gamma') \leq d\}.$$

We define a open set $X_\Omega \subset X$ by

$$X_\Omega := \bigcup_{\gamma \in \Omega} X''_\gamma.$$

Let $\theta_i = (E_{i\gamma}, A_{i\gamma}, \rho_{i\gamma,s})_{\gamma \in \Gamma, s \in S}$ $(i = 1, 2)$ be two gluing data, i.e., $E_{i\gamma}$ is a principal $SU(2)$-bundle over X_γ and $A_{i\gamma}$ is an ASD connection on $E_{i\gamma}$ satisfying $[E_{i\gamma}, A_{i\gamma}] \in M$. $\rho_{i\gamma,s} : (E_{i\gamma})_{x_{\gamma,s}} \to (E_{i\gamma s})_{y_{\gamma s,s}}$ is an $SU(2)$-isomorphism. For each $i = 1, 2$, we have the operator $P_i : BL^p(\Omega^+(\mathrm{ad}\boldsymbol{E}_i)) \to BL^q(\Omega^1(\mathrm{ad}\boldsymbol{E}_i))$ which is a right inverse of $d^+_{\boldsymbol{A}'_i}$ by Lemma 4.1. Let $\Omega \subset \Gamma$ be a finite set. We want to compare the operators P_1 and P_2 over X_Ω. Suppose that there is an integer $d > 0$ such that $E_{1\gamma} = E_{2\gamma}$, $A_{1\gamma} = A_{2\gamma}$ for $\gamma \in B_d(\Omega)$ and $\rho_{1\gamma,s} = \rho_{2\gamma,s}$ for $\gamma \in B_d(\Omega)$ and $s \in S$ with $\gamma s \in B_d(\Omega)$. Then we can naturally consider that $\boldsymbol{E}_1 = \boldsymbol{E}_2$ and $\boldsymbol{A}'_1 = \boldsymbol{A}'_2$ over $X_{B_d(\Omega)}$.

Lemma 6.7. *Let $\xi_i \in BL^p_i$ $(i = 1, 2)$. We denote $\xi_i|_{X_{B_d(\Omega)}}$ as the restriction of ξ_i to $X_{B_d(\Omega)}$ (and we extend it to $X^{\sharp(\Gamma,S)}$ by zero). Then for each $\gamma \in \Omega$*

$$\|P_1(\xi_1) - P_2(\xi_2)\|_{L^q(X''_\gamma, g)} \leq \mathrm{const}_M \left\|\xi_1|_{B_d(\Omega)} - \xi_2|_{B_d(\Omega)}\right\|_{BL^p} + \mathrm{const}_M \cdot 2^{-d}(\|\xi_1\|_{BL^p} + \|\xi_2\|_{BL^p}),$$

where const_M *are positive constants depending only on M. (Especially they are independent of Ω and the integer $d > 0$.) In particular, if $\xi_1|_{X_{B_d(\Omega)}} = \xi_2|_{X_{B_d(\Omega)}}$ then*

$$\|P_1(\xi_1) - P_2(\xi_2)\|_{L^q(X''_\gamma, g)} \leq \mathrm{const}_M \cdot 2^{-d}(\|\xi_1\|_{BL^p} + \|\xi_2\|_{BL^p}).$$

Proof.

$$\begin{aligned} P_i(\xi_i) &= Q_i(1 + R_i)^{-1}\xi_i, \\ &= Q_i(1 - R_i + R_i^2 - \cdots + (-1)^{d-1}R_i^{d-1})\xi_i + (-1)^d Q_i R_i^d (1 + R_i)^{-1}\xi_i. \end{aligned}$$

From the definitions of the operators Q and R in Section 4, we have $Q_i R_i^k \xi_i = Q_i R_i^k(\xi_i|_{X_{B_d(\Omega)}})$ and $Q_2 R_2^k(\xi_2|_{X_{B_d(\Omega)}}) = Q_1 R_1^k(\xi_2|_{X_{B_d(\Omega)}})$ over X_Ω for $k \leq d - 1$. (These follows from the fact that Q_i and R_i have "one-step propagation".) Therefore for $\gamma \in \Omega$

$$\|P_1(\xi_1) - P_2(\xi_2)\|_{L^q(X''_\gamma, g)} \leq \mathrm{const}_M \left\|\xi_1|_{B_d(\Omega)} - \xi_2|_{B_d(\Omega)}\right\|_{BL^p} + \mathrm{const}_M \cdot 2^{-d}(\|\xi_1\|_{BL^p} + \|\xi_2\|_{BL^p}).$$

Here we have used $\|R_i\| \leq 1/2$ (see Lemma 4.1). □

The following will be used in Section 7.

Corollary 6.8. *For any $\varepsilon > 0$, there exists $d = d(M, \varepsilon) > 0$ satisfying the following: Let $\Omega \subset \Gamma$ be a finite subset. If $E_{1\gamma} = E_{2\gamma}$, $A_{1\gamma} = A_{2\gamma}$ for all $\gamma \in B_d(\Omega)$ and $\rho_{1\gamma,s} = \rho_{2\gamma,s}$ for all $\gamma \in B_d(\Omega)$ and $s \in S$ with $\gamma s \in B_d(\Omega)$, then for any $\gamma \in \Omega$*

$$\|\boldsymbol{A}(\theta_1) - \boldsymbol{A}(\theta_2)\|_{L^q(X''_\gamma, g)} < \varepsilon.$$

Proof. $\boldsymbol{\xi}_i = \boldsymbol{\xi}(\theta_i)$ satisfies $(i = 1, 2)$

$$\boldsymbol{\xi}_i + (P_i(\boldsymbol{\xi}_i) \wedge P_i(\boldsymbol{\xi}_i))^+ = -F^+(\boldsymbol{A}'_i).$$

Let m and d_0 be (large) positive integers which will be fixed later. Set $d := md_0$ and suppose that $\theta_1 = \theta_2$ over $B_d(\Omega)$. Since we have $\boldsymbol{A}'_1 = \boldsymbol{A}'_2$ over $B_d(\Omega)$, we have

$$\begin{aligned} \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 &= (P_2(\boldsymbol{\xi}_2) \wedge P_2(\boldsymbol{\xi}_2))^+ - (P_1(\boldsymbol{\xi}_1) \wedge P_1(\boldsymbol{\xi}_1))^+, \\ &= ((P_2(\boldsymbol{\xi}_2) - P_1(\boldsymbol{\xi}_1)) \wedge P_2(\boldsymbol{\xi}_2))^+ + (P_1(\boldsymbol{\xi}_1) \wedge ((P_2(\boldsymbol{\xi}_2) - P_1(\boldsymbol{\xi}_1)))^+, \end{aligned} \tag{59}$$

over $B_d(\Omega)$. For $k = 1, 2, \cdots, m$, we set

$$a_k := \sup_{\gamma \in B_{kd_0}(\Omega)} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{L^p(X''_\gamma, g)}.$$

From Remark 4.4, Lemma 6.7 and (59), we have

$$a_k \leq \mathrm{const}_M \cdot b^2 \sup_{\gamma \in B_{kd_0}(\Omega)} \|P_2(\boldsymbol{\xi}_2) - P_1(\boldsymbol{\xi}_1)\|_{L^q(X''_\gamma, g)} \leq \mathrm{const}_M \cdot b^2 (a_{k+1} + b^{4/p} 2^{-d_0}),$$

where const_M is independent of k and Ω. Since $b > 0$ is sufficiently small, we have

$$a_k \leq 2^{-1} a_{k+1} + 2^{-d_0}.$$

Hence $a_1 \leq 2^{-m+1} a_m + 2^{-d_0+1}$. We have $a_m \leq \|\boldsymbol{\xi}_1\|_{BL^p} + \|\boldsymbol{\xi}_2\|_{BL^p} \leq \mathrm{const}_M b^{4/p} \ll 1$. Hence

$$a_1 \leq 2^{-m+1} + 2^{-d_0+1}.$$

We have $\boldsymbol{A}_1 - \boldsymbol{A}_2 = P_1(\boldsymbol{\xi}_1) - P_2(\boldsymbol{\xi}_2)$ over $B_d(\Omega)$, and for any $\gamma \in \Omega$ (using Lemma 6.7)

$$\|P_1(\boldsymbol{\xi}_1) - P_2(\boldsymbol{\xi}_2)\|_{L^q(X''_\gamma, g)} \leq \mathrm{const}_M (a_1 + b^{4/p} \cdot 2^{-d_0}).$$

We choose m and d_0 sufficiently large. Then for $\gamma \in \Gamma$

$$\|P_1(\boldsymbol{\xi}_1) - P_2(\boldsymbol{\xi}_2)\|_{L^q(X''_\gamma, g)} < \varepsilon.$$

□

Let $[\theta_n] = [(E_{n\gamma}, A_{n\gamma}, \rho_{n\gamma,s})_{\gamma \in \Gamma, s \in S}] \in \mathrm{GlD}$ $(n = 1, 2, 3, \cdots)$ be the sequence of the equivalence classes of gluing data. Since M is compact, if we take a subsequence, this sequence (pointwisely) converges to a gluing data $[\theta] = [(E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma \in \Gamma, s \in S}]$ in the following sense (cf. Section 7). For each $\gamma \in \Gamma$ there exists $n_0(\gamma) > 0$ and a sequence of gauge transformations $g_{n\gamma} : E_{n\gamma} \to E_\gamma$ $(n \geq n_0(\gamma))$ such that $g_{n\gamma}(A_{n\gamma})$ converges to A_γ (in the $\mathcal{C}^\infty$-topology) and $g_{n\gamma s} \rho_{n\gamma,s} g_{n\gamma}^{-1}$ converges to $\rho_{\gamma,s}$ as $n \to \infty$.

For each $\gamma \in \Gamma$ we can assume that, for $n \geq n_0(\gamma)$, $E_{n\gamma} = E_\gamma$, $A_{n\gamma} = A_\gamma + \tilde{\alpha}_{n\gamma}$ and $\rho_{n\gamma,s} = \rho_{\gamma,s} e^{v_{n\gamma,s}}$ $(s \in S)$ where $\alpha_{n\gamma} \in H^1_{A_\gamma}$ and $v_{n\gamma,s} \in (\mathrm{ad}E)_{x_{\gamma,s}}$. (See Section 3.4.2 and

5.1). Moreover we have $\lim_{n\to\infty} \|\alpha_{n\gamma}\|_{L^q} = 0$ and $\lim_{n\to\infty} |v_{n\gamma,s}| = 0$. Therefore the bundle map $h_{n\gamma} : E_{n\gamma} = E_\gamma \to E_\gamma$ given in (34) can be defined for $n \geq n_1(\gamma)$. Here $n_1(\gamma)$ is an appropriate large number with $n_1(\gamma) \geq n_0(\gamma)$, $n_0(\gamma s^{-1})$ $(s \in S)$. For each $n \geq 1$ there exist a (possibly empty) finite subset $\Omega_n \subset \Gamma$ (each $\gamma \in \Omega_n$ satisfies $n \geq n_1(\gamma)$) such that we can define a bundle map $h_n : \boldsymbol{E}_n|_{X_{\Omega_n}} \to \boldsymbol{E}|_{X_{\Omega_n}}$ by gluing these $h_{n\gamma}$. We can take these Ω_n so that $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$ and $\bigcup_{n\geq 1} \Omega_n = \Gamma$.

Let $\xi_n \in BL^p(\Omega^+(\mathrm{ad}\boldsymbol{E}_n))$ $(n = 1, 2, \cdots)$ and suppose $\sup_n \|\xi_n\|_{BL^p} < \infty$. For each finite subset $\Omega \subset \Gamma$, $L^p(X_\Omega)$ is a reflexive Banach space. Hence if we take a subsequence of $\{\xi_n\}$, there exists $\xi \in BL^p(\Omega^+(\mathrm{ad}\boldsymbol{E}))$ with $\|\xi\|_{BL^p} \leq \sup_n \|\xi_n\|_{BL^p}$ such that, for any finite subset $\Omega \subset \Gamma$, $h_n(\xi_n)|_\Omega$ weakly converges to $\xi|_\Omega$ in $L^p(X_\Omega)$.

Lemma 6.9. *In the above situation, $h_n(P_{\theta_n}(\xi_n))|_{X_\Omega}$ weakly converges to $P_\theta(\xi)|_{X_\Omega}$ in $L^q(X_\Omega)$ as $n \to \infty$ for any finite subset $\Omega \subset \Gamma$.*

Proof. Take $\varepsilon > 0$ and $\eta \in (L^q(X_\Omega))^* = L^{q'}(X_\Omega)$ $(1/q + 1/q' = 1)$. Let $d > 0$ be a large integer which will be fixed later. Set $\xi'_n := \xi_n|_{B_d(\Omega)} = 1_{X_{B_d(\Omega)}} \cdot \xi_n$ and $\xi''_n := \xi_n - \xi'_n$ where $1_{X_{B_d(\Omega)}}$ is the characteristic function of $X_{B_d(\Omega)}$. We also define $\xi' := \xi|_{B_d(\Omega)}$ and $\xi'' := \xi - \xi'$. $h_n(\xi'_n)$ weakly converges to ξ' in $L^p(X_{B_d(\Omega)})$. Then $P(h_n(\xi'_n))$ weakly converges to $P(\xi')$ in BL^q. Set $P'_n := h_n \circ P_n \circ h_n^{-1} : L^p(X_{B_d(\Omega)}, \Omega^+(\mathrm{ad}\boldsymbol{E})) \to BL^q$. $(P_n := P_{\theta_n}.)$ We have

$$h_n(P_n(\xi_n)) = (P'_n(h_n\xi'_n) - P(h_n\xi'_n)) + P(h_n\xi'_n) + h_n(P_n(\xi''_n)). \tag{60}$$

From Lemma 6.7, $\|h_n(P_n(\xi''_n))\|_{L^q(X_\Omega)} \leq \mathrm{const}_{\Omega,M} \cdot 2^{-d} \|\xi_n\|_{BL^p}$ and $\|P(\xi'')\|_{L^q(X_\Omega)} \leq \mathrm{const}_{\Omega,M} \cdot 2^{-d} \|\xi\|_{BL^p} \leq \mathrm{const}_{\Omega,M} \cdot 2^{-d} \sup_n \|\xi_n\|_{BL^p}$.

The term $(P'_n(h_n\xi'_n) - P(h_n\xi'_n))$ can be evaluated by using Lemma 5.1 and 6.7 as follows. Define (for $n \gg 0$) a gluing data $\hat{\theta}_n = (\hat{E}_{n\gamma}, \hat{A}_{n\gamma}, \hat{\rho}_{n\gamma,s})_{\gamma\in\Gamma, s\in S}$ by $(\hat{E}_{n\gamma}, \hat{A}_{n\gamma}, \hat{\rho}_{n\gamma,s}) = (E_{n\gamma}, A_{n\gamma}, \rho_{n\gamma,s})$ for $(\gamma, s) \in B_{d+1}(\Omega) \times S$ and $(\hat{E}_{n\gamma}, \hat{A}_{n\gamma}, \hat{\rho}_{n\gamma,s}) = (E_\gamma, A_\gamma, \rho_{\gamma,s})$ otherwise. Lemma 6.7 gives $(\hat{P}'_n := h_n \circ P_{\hat{\theta}_n} \circ h_n^{-1})$

$$\left\| P'_n(h_n\xi'_n) - \hat{P}'_n(h_n\xi'_n) \right\|_{L^q(X_\Omega, g)} \leq \mathrm{const}_{\Omega,M} \cdot 2^{-d} \|\xi_n\|_{BL^p}.$$

Lemma 5.1 gives $(n \gg 1)$

$$\left\| \hat{P}'_n(h_n\xi'_n) - P(h_n\xi'_n) \right\|_{BL^q} \leq \mathrm{const}_M \cdot \sup_{\gamma\in B_{d+1}(\Omega), s\in S} \{d_{L^q}([A_{n\gamma}], [A_\gamma]) + |\rho_{n\gamma,s} - \rho_{\gamma,s}|\} \|\xi_n\|_{BL^p}$$

Therefore for $\eta \in (L^q(X_\Omega))^*$

$$\begin{aligned} |\langle h_n(P_n(\xi_n)) - P(\xi), \eta\rangle| &\leq |\langle P(h_n\xi'_n) - P(\xi'), \eta\rangle| + \mathrm{const}_{\Omega,M} \cdot \|\eta\| \, 2^{-d} \sup_m \|\xi_m\|_{BL^p}, \\ &+ \mathrm{const}_{\Omega,M} \cdot \|\eta\| \sup_{\gamma\in B_{d+1}(\Omega), s\in S} \{d_{L^q}([A_{n\gamma}], [A_\gamma]) + |\rho_{n\gamma,s} - \rho_{\gamma,s}|\} \sup_m \|\xi_m\|_{BL^p} \end{aligned}$$

We choose $d > 0$ so that $\mathrm{const}_{\Omega,M} \cdot \|\eta\| \, 2^{-d} \sup_n \|\xi_n\|_{BL^p} \leq \varepsilon/3$. We can choose $n_1 > 0$ so that for $n \geq n_1$

$$\mathrm{const}_{\Omega,M} \cdot \|\eta\| \sup_{\gamma\in B_{d+1}(\Omega), s\in S} \{d_{L^q}([A_{n\gamma}], [A_\gamma]) + |\rho_{n\gamma,s} - \rho_{\gamma,s}|\} \cdot \sup_m \|\xi_m\|_{BL^p} \leq \varepsilon/3.$$

Since $P(h_n\xi'_n)$ weakly converges to $P(\xi')$ in BL^q, we can choose n_2 so that for $n \geq n_2$

$$|\langle P(h_n\xi'_n) - P(\xi'), \eta\rangle| \leq \varepsilon/3.$$

Therefore for $n \geq \max(n_1, n_2)$

$$|\langle h_n(P_n(\xi_n)) - P(\xi), \eta\rangle| \leq \varepsilon.$$

Thus $\lim_{n\to\infty}\langle h_n(P_n(\xi_n)), \eta\rangle = \langle P(\xi), \eta\rangle$. This means that $h_n(P_n(\xi_n))|_\Omega$ weakly converges to $P(\xi)|_\Omega$ in $L^q(X_\Omega)$. □

6.3. **Proof of Surjectivity.** Let (E_1, A_1) and (E_2, A_2) be two pairs of a principal $SU(2)$-bundle over X and an ASD connection on it. We define the L^q-distance between their gauge equivalence classes by (recall $q > 4$)

$$d_{L^q}([E_1, A_1], [E_2, A_2]) := \inf_{g: E_1 \to E_2} \|A_2 - g(A_1)\|_{L^q(X)},$$

where g runs over bundle isomorphisms between E_1 and E_2. If E_1 and E_2 are not isomorphic, then we set $d_{L^q}([E_1, A_1], [E_2, A_2]) := \infty$. Recall that M denotes a set of gauge equivalence classes of (E, A) satisfying the conditions (a), (b), (c) in the beginning of Section 3.2. Let $\mathcal{L} \subset M$ be a subset such that there exists $\delta > 0$ satisfying $B_\delta(\mathcal{L}) \subset M$. Here $B_\delta(\mathcal{L}) \subset M$ means that if a pair (E, A) of a principal $SU(2)$-bundle E over X and an ASD connection A on E satisfies $d_{L^q}([E, A], [F, B]) \leq \delta$ for some $[F, B] \in \mathcal{L}$ then $[E, A] \in M$. We define $\mathrm{GlD}(\mathcal{L}) \subset \mathrm{GlD}$ by

$$\mathrm{GlD}(\mathcal{L}) := \{[(E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma, s\in S}] \in \mathrm{GlD} |\, [E_\gamma, A_\gamma] \in \mathcal{L} \text{ for all } \gamma \in \Gamma\}$$

Let $\mathcal{B}$ be the set of all gauge equivalence classes of (F, B) where F is a principal $SU(2)$-bundle over $X^{\sharp(\Gamma,S)}$ and B is a connection on it. By using the cut-off construction in Section 3.3, we have the map:

$$J : \mathrm{GlD} \to \mathcal{B}, \quad [\theta] \mapsto [\boldsymbol{E}(\theta), \boldsymbol{A}'(\theta)].$$

For $[F_i, B_i] \in \mathcal{B}$ $(i = 1, 2)$, we define their BL^q-distance by

$$d_{BL^q}([F_1, B_1], [F_2, B_2]) := \inf_{g: F_1 \to F_2} \|B_2 - g(B_1)\|_{BL^q}.$$

(This may be $+\infty$.) For $\nu > 0$ we define a subset $U(\mathcal{L}, \nu) \subset \mathcal{B}$ by

$$U(\mathcal{L}, \nu) := \{[F, B] \in \mathcal{B} |\, d_{BL^q}([F, B], J(\mathrm{GlD}(\mathcal{L}))) < \nu, \left\|F_B^+\right\|_{BL^p} < \nu^{3/2}\}.$$

Here $d_{BL^q}([F, B], J(\mathrm{GlD}(\mathcal{L}))) < \nu$ means that there exists a gluing data $[\theta] \in \mathrm{GlD}(\mathcal{L})$ such that $d_{BL^q}([F, B], [\boldsymbol{E}(\theta), \boldsymbol{A}'(\theta)]) < \nu$. The following lemma will be used in Section 8. (This is essentially given in Donaldson-Kronheimer [5, Lemma (7.2.43)].)

Lemma 6.10. *There exists $b_0 = b_0(M, \nu)$ and $\nu' = \nu'(\nu)$ such that if $b = N\sqrt{\lambda} \leq b_0$ and $[F, B] \in \mathcal{B}$ satisfies for all $\gamma \in \Gamma$*

$$\inf_{[E,A]\in\mathcal{L}} d_{L^q}([F|_{X''_\gamma}, B|_{X''_\gamma}], [E|_{X''_\gamma}, A|_{X''_\gamma}]) < \nu', \quad \left\|F_B^+\right\|_{BL^p} < \nu^{3/2},$$

then we have $[F,B]\in U(\mathcal{L},\nu)$

Proof. There are $[E_\gamma, A_\gamma]\in M$ $(\gamma\in\Gamma)$ and bundle maps $g_\gamma : F|_{X''_\gamma}\to E_\gamma|_{X''_\gamma}$ such that

$$\|g_\gamma(B)-A_\gamma\|_{L^q(X''_\gamma,g)}<\nu'.$$

From (14) we get

$$\left\|A_\gamma - A'_\gamma\right\|_{L^q(X''_\gamma,g)}\leq \mathrm{const}_M\cdot b^{1+4/q}.$$

Hence

$$\left\|g_\gamma(B)-A'_\gamma\right\|_{L^q(X''_\gamma,g)}<\nu'+\mathrm{const}_M\cdot b^{1+4/q}.$$

For each $\gamma\in\Gamma$ and $s\in S$, we set $h_{\gamma,s}:=g_{\gamma s}g_\gamma^{-1}: E_\gamma\to E_{\gamma s}$ over the "neck" $W_{\gamma,s}:=X''_\gamma\cap X''_{\gamma s}$ Then $\left\|h_{\gamma,s}(A'_\gamma)-A'_{\gamma s}\right\|_{L^q(W_{\gamma,s},g)}\leq 2(\nu'+\mathrm{const}_M\cdot b^{1+4/q}) =: \varepsilon$. In the exponential gauges of A_γ around $x_{\gamma,s}$ and $y_{\gamma,s}$, the connection matrix $A'_\gamma=0$ over the necks. Therefore, in these gauges, $\|dh_{\gamma,s}\|_{L^q(W_{\gamma,s},g)}\leq\varepsilon$. Using the Sobolev embedding $L^q_1\hookrightarrow C^{0,1-4/q}$, we get

$$|h_{\gamma,s}(x)-h_{\gamma,s}(y)|\leq \mathrm{const}\cdot\varepsilon|x-y|^{1-4/q},$$

for any $x,y\in W_{\gamma,s}$. (The above "const" does not depend on λ.) Since the right-hand-side is sufficiently small, there is $\rho_{\gamma,s}:(E_\gamma)_{x_{\gamma,s}}\to(E_{\gamma s})_{y_{\gamma s,s}}$ such that $h_{\gamma,s}=\rho_{\gamma,s}e^{u_{\gamma,s}}$ and $u_{\gamma,s}$ satisfies

$$\|du_{\gamma,s}\|_{L^q(W_{\gamma,s},g)}\leq\mathrm{const}\cdot\varepsilon,\quad |u_{\gamma,s}|\leq\mathrm{const}\cdot\lambda^{1/2-2/q}\varepsilon. \tag{61}$$

Set $\theta:=(E_\gamma, A_\gamma,\rho_{\gamma,s})_{\gamma,s}$.

We define $k_\gamma: E_\gamma|_{X''_\gamma}\to E_\gamma|_{X''_\gamma}$ as follows; k_γ is equal to $e^{(1-\beta'_\gamma)u_{\gamma,s}}$ around the points $x_{\gamma,s}$, and equal to $e^{-(1-\beta'_\gamma)\tilde{u}_{\gamma s^{-1},s}}$ $(\tilde{u}_{\gamma,s}=\rho_{\gamma s^{-1},s}u_{\gamma s^{-1},s}\rho^{-1}_{\gamma s^{-1},s})$ around the points $y_{\gamma,s}$. k_γ is equal to 1 outside the "neck" regions. We set $\tilde{g}_\gamma:=k_\gamma g_\gamma: F|_{X''_\gamma}\to E_\gamma|_{X''_\gamma}$. These compatibly (i.e. $\tilde{g}_{\gamma s}=\rho_{\gamma,s}\tilde{g}_\gamma$) define $\tilde{g}:F\to\boldsymbol{E}(\theta)$. We have $\tilde{g}_\gamma(B)-A'_\gamma = k_\gamma(g_\gamma(B)-A'_\gamma)+k_\gamma(A'_\gamma)-A'_\gamma$. From (61) we have $\left\|k_\gamma(A'_\gamma)-A'_\gamma\right\|_{L^q(W_{\gamma,s},g)}\leq\mathrm{const}\cdot\varepsilon$. (Note that $|d\beta'_\gamma\otimes u_{\gamma,s}|\leq\mathrm{const}\cdot\lambda^{-2/q}\varepsilon$ and hence $\left\|d\beta'_\gamma\otimes u_{\gamma,s}\right\|_{L^q(W_{\gamma,s},g)}\leq\mathrm{const}\cdot\varepsilon$.) Therefore we have

$$\|\tilde{g}(B)-\boldsymbol{A}'(\theta)\|_{BL^q}\leq\mathrm{const}\cdot\varepsilon.$$

□

Recall that $\mathcal{L}\subset M$ satisfies $B_\delta(\mathcal{L})\subset M$ where $B_\delta(\mathcal{L})$ is the δ-neighborhood of $\mathcal{L}$ with respect to the L^q-distance.

Theorem 6.11. *There are* $\nu_0(\delta)>0$, $N_0>0$ *and* $\lambda_0(N,\nu,\delta)>0$ *satisfying the following: If* $\nu\leq\nu_0(\delta)$, $N\geq N_0$ *and* $\lambda\leq\lambda(N,\nu,\delta)$ *then for any* $[F,B]\in U(\mathcal{L},\nu)$ *there exist* $[\theta]\in\mathrm{GlD}$ *and* $\xi\in BL^p(\Omega^+(\boldsymbol{E}(\theta)))$ *satisfying*

$$[F,B]=[\boldsymbol{E},\boldsymbol{A}'(\theta)+P_\theta(\xi)],\quad \|\xi\|_{BL^p}\leq 3\nu^{3/2}.$$

In particular if $[F,B]\in U(\mathcal{L},\nu)$ and B is an ASD connection, then there exists $[\theta]\in \mathrm{GlD}$ satisfying $[F,B]=[\boldsymbol{E}(\theta),\boldsymbol{A}(\theta)]$ (see Proposition 4.3 and the statement of the uniqueness of $\boldsymbol{\xi}$ there).

We will prove this theorem by using the continuity method developed in Donaldson-Kronheimer [5, Section 7.2.4, 7.2.5].

Let $[F,B]\in U(\mathcal{L},\nu)$. There is $[F,\boldsymbol{B}']\in J(\mathrm{GlD}(\mathcal{L}))$ satisfying $B=\boldsymbol{B}'+\boldsymbol{b}$ with $\|\boldsymbol{b}\|_{BL^q}<\nu$. For $t\in[0,1]$ we set $B_t:=\boldsymbol{B}'+t\boldsymbol{b}$. For small $\nu>0$ and $b=4N\sqrt{\lambda}>0$, all B_t are contained in $U(\mathcal{L},\nu)$; In fact, when $t=0$, $\|F^+(B_0)\|_{BL^p}=\|F^+(\boldsymbol{B}')\|_{BL^p}\leq \mathrm{const}_M\cdot b^{4/p}<\nu^{3/2}$. For $t\in(0,1]$, $F^+(B_t)=tF^+(B)+(1-t)F^+(\boldsymbol{B}')+(t^2-t)(\boldsymbol{b}\wedge\boldsymbol{b})^+$.

$$\begin{aligned}\left\|F^+(B_t)\right\|_{BL^p}&\leq t\left\|F^+(B)\right\|_{BL^p}+(1-t)\left\|F^+(\boldsymbol{B}')\right\|_{BL^p}+\mathrm{const}\cdot(t-t^2)\left\|\boldsymbol{b}\right\|_{BL^q}^2,\\ &<t\cdot\nu^{3/2}+\mathrm{const}_M\cdot(1-t)(b^{4/p}+\nu^2)\leq\nu^{3/2}.\end{aligned}$$

Hence $[F,B_t]\in U(\mathcal{L},\nu)$.

Let $\varepsilon>0$ be a small number which will be fixed later. Let $S\subset[0,1]$ be the set of $t\in[0,1]$ such that there exist a gluing data θ_t, $\xi_t\in BL^p(\Omega^+(\boldsymbol{E}(\theta_t)))$ and a gauge transformation $u_t:F\to\boldsymbol{E}(\theta_t)$ satisfying

$$u_t(B_t)=\boldsymbol{A}'(\theta_t)+P_{\theta_t}(\xi_t),\quad \|\xi_t\|_{BL^p}<\varepsilon. \tag{62}$$

We have $0\in S$. From this equation, we have $u_t(F^+(B_t))=F^+(\boldsymbol{A}'(\theta_t))+\xi_t+(P_{\theta_t}\xi_t\wedge P_{\theta_t}\xi_t)^+$. Hence

$$\begin{aligned}\|\xi_t\|_{BL^p}&\leq\left\|F^+(B_t)\right\|_{BL^p}+\left\|F^+(\boldsymbol{A}'(\theta_t))\right\|_{BL^p}+\mathrm{const}_M\cdot\|\xi_t\|_{BL^p}^2,\\ &\leq\nu^{3/2}+\mathrm{const}'_M\cdot b^{4/p}+\mathrm{const}_M\cdot\varepsilon\left\|\xi_t\right\|_{BL^p}.\end{aligned}$$

We choose $\varepsilon>0$ so that $\mathrm{const}_M\cdot\varepsilon\leq 1/2$. Then

$$\|\xi_t\|_{BL^p}\leq 2(\nu^{3/2}+\mathrm{const}'_M b^{4/p}).$$

We choose ν and b sufficiently small so that $2(\nu^{3/2}+\mathrm{const}'_M b^{4/p})\leq 3\nu^{3/2}\leq\varepsilon/2$. Therefore we get

$$\|\xi_t\|_{BL^p}\leq 3\nu^{3/2}\leq\varepsilon/2. \tag{63}$$

In particular, from the open condition $\|\xi_t\|_{BL^p}<\varepsilon$, we have deduced the closed condition $\|\xi_t\|_{BL^p}\leq\varepsilon/2$.

Now we will prove that S is a closed set in $[0,1]$. Let $t_n\in S$ ($n=1,2,3,\cdots$) be a sequence converging to $t\in[0,1]$. Set $\theta_n=\theta_{t_n}=(E_{n\gamma},A_{n\gamma},\rho_{n\gamma,s})_{\gamma\in\Gamma,s\in S}$. We have $u_n(B_{t_n})=\boldsymbol{A}'_n+P_n(\xi_n)$ with $\|\xi_n\|_{BL^p}\leq\varepsilon/2$. From the argument before Lemma 6.9, using some gauge transformations, we can suppose that θ_n converges to $\theta=(E_\gamma,A_\gamma,\rho_{\gamma,s})_{\gamma\in\Gamma,s\in S}$ as follows; There is $n_0(\gamma)>0$ for each $\gamma\in\Gamma$ such that $E_{n\gamma}=E_\gamma$, $A_{n\gamma}=A_\gamma+\tilde{\alpha}_{n\gamma}$ ($\alpha_{n\gamma}\in H^1_{A_\gamma}$), $\rho_{n\gamma,s}=\rho_{\gamma,s}e^{v_{n\gamma,s}}$ for $n\geq n_0(\gamma)$, and $\alpha_{n\gamma}$ and $v_{n\gamma,s}$ converge to 0 as $n\to\infty$. Moreover there exist $\xi\in BL^p(\Omega^+(\mathrm{ad}\boldsymbol{E}))$, an exhausting sequence $\Omega_1\subset\Omega_2\subset\Omega_3\subset\cdots\subset\Gamma$ and bundle maps $h_n:\boldsymbol{E}_n|_{X_{\Omega_n}}\to\boldsymbol{E}|_{X_{\Omega_n}}$ such that $h_n(\xi_n)|_{X_\Omega}$ weakly converges to $\xi|_{X_\Omega}$ in

$L^p(X_\Omega)$ for any finite subset $\Omega \subset \Gamma$. From $\|\xi_n\|_{BL^p} \leq \varepsilon/2$, we have $\|\xi\|_{BL^p} \leq \varepsilon/2$. Set $g_n := h_n \circ u_n$ (over X_{Ω_n}). Then

$$g_n(B_n) = h_n(\boldsymbol{A}'_n) + h_n(P_n(\xi_n)) \quad \text{over } X_{\Omega_n}.$$

For any finite subset $\Omega \subset \Gamma$, the right-hand-side of this equation weakly converges to $\boldsymbol{A}'(\theta) + P_\theta(\xi)$ in $L^q(X_\Omega)$ (Lemma 6.9). On the other hand, if we take a subsequence, there exists a bundle map g defined over $X^{\sharp(\Gamma,S)}$ such that g_n weakly converges to g in $L^q_1(X_\Omega)$ for each finite subset $\Omega \subset \Gamma$. Then we get

$$g(B_t) = \boldsymbol{A}'(\theta) + P_\theta(\xi), \quad \|\xi\|_{BL^p} \leq \varepsilon/2 < \varepsilon.$$

This shows $t \in S$. Thus S is a closed set in $[0,1]$.

Next we will prove that S is open in $[0,1]$. Suppose that the equation (62) holds at some $t \in [0,1]$. Then $\boldsymbol{A}' = \boldsymbol{A}'(\theta_t)$ satisfies $d_{BL^q}([\boldsymbol{A}'], [\boldsymbol{B}']) \leq \nu + \mathrm{const} \cdot \varepsilon$. Therefore if we choose b, ε and ν small enough, then $\theta_t = (E_\gamma, A_\gamma, \rho_{\gamma,s})_{\gamma\in\Gamma, s\in S}$ satisfies

$$B_{\delta/2}([E_\gamma, A_\gamma]) \subset M, \tag{64}$$

for every $\gamma \in \Gamma$. (Recall that $B_\delta(\boldsymbol{\mathcal{L}}) \subset M$.)

Consider the following map:

$$G : B_1 \to B_2, \quad (\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta) \mapsto e^{-\chi}(\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}} + P_{\boldsymbol{v},\boldsymbol{\alpha}}(\eta + \xi_t)) - u_t(B_t),$$

where B_1 and B_2 denote the Banach spaces defined in Section 6.1. Of course, we consider only very small $(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta) \in B_1$. $\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}}$ and $P_{\boldsymbol{v},\boldsymbol{\alpha}}$ is the connection and the operator defined in Section 5.1. $\boldsymbol{A}'_{0,0} = \boldsymbol{A}' = \boldsymbol{A}'(\theta_t)$ and $P_{0,0} = P_{\theta_t}$. We have $G(0) = 0$. If we prove that the derivative of G at the origin $(dG)_0 : B_1 \to B_2$ is isomorphic, then the inverse mapping theorem and (64) implies that $t \in S$ is an inner point. $(dG)_0 : B_1 \to B_2$ is given by

$$(dG)_0(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta) = T(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta) + [P(\xi_t), \chi] + \partial_{\boldsymbol{v}} P_{\boldsymbol{v},0}(\xi_t) + \partial_{\boldsymbol{\alpha}} P_{0,\boldsymbol{\alpha}}(\xi_t),$$

where $T(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta) = d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha}) + P(\eta)$ is the operator given in (48) and $P = P_{0,0} = P_{\theta_t}$. Proposition 6.6 shows that T is an isomorphism satisfying $\|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta)\|_{B_1} \leq K \|T(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta)\|_{B_2}$. Therefore if we prove $\|T - (dG)_0\| < K^{-1}$, then $(dG)_0$ is an isomorphism.

We have $\|[P(\xi_t), \chi]\|_{BL^q} \leq \mathrm{const}_M \cdot \|\xi_t\|_{BL^p} \|\chi\|_{\mathcal{C}^0} \leq \mathrm{const}_M \cdot \varepsilon \|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta)\|_{B_1}$ from Lemma 6.2 and (46). We have $d^+_{\boldsymbol{A}'}[P(\xi_t), \chi] = [\xi_t, \chi] - [P(\xi_t) \wedge d_{\boldsymbol{A}'}\chi]^+$. Hence

$$\left\|d^+_{\boldsymbol{A}'}[P(\xi_t), \chi]\right\|_{BL^p} \leq \|\chi\|_{\mathcal{C}^0} \|\xi_t\|_{BL^p} + \|P(\xi_t)\|_{BL^q} \|d_{\boldsymbol{A}'}\chi\|_{BL^4}.$$

We have $\|d_{\boldsymbol{A}'}\chi\|_{BL^4} \leq \|d_{\boldsymbol{A}'}\chi + j_1(\boldsymbol{v}) + j_2(\boldsymbol{\alpha})\|_{BL^4} + \|j_1(\boldsymbol{v})\|_{BL^4} + \|j_2(\boldsymbol{\alpha})\|_{BL^4}$. Recall the equation (45) ($j_1(\boldsymbol{v}) = d\beta'_\gamma \otimes v_{\gamma,s}$ over $\Omega(x_{\gamma,s})$) and the fact that $\left\|d\beta'_\gamma\right\|_{L^4}$ is a constant independent of the parameters λ and N (because of the conformal invariance of the L^4-norms of 1-forms; see the argument before (17)). Hence (using Lemma 6.2)

$$\|j_1(\boldsymbol{v})\|_{BL^4} \leq \mathrm{const} \|\boldsymbol{v}\| \leq \mathrm{const}_M \|(\chi, \boldsymbol{v}, \boldsymbol{\alpha}, \eta)\|_{B_1}. \tag{65}$$

From Lemma 6.1, $\|j_2(\boldsymbol{\alpha})\|_{BL^4} \leq \mathrm{const}\cdot\|j_2(\boldsymbol{\alpha})\|_{BL^q} \leq \mathrm{const}_M\cdot\|\boldsymbol{\alpha}\| \leq \mathrm{const}_M\cdot\|(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_1}$. Hence $\left\|d^+_{\boldsymbol{A}'}[P(\xi_t),\chi]\right\|_{BL^p} \leq \mathrm{const}_M\cdot\varepsilon\,\|(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_1}$. Thus $\|[P(\xi_t),\chi]\|_{B_2} \leq \mathrm{const}_M\cdot \varepsilon\,\|(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_1}$.

By using an argument similar to that in Lemma 5.1, we have

$$\|\partial_{\boldsymbol{v}}P_{\boldsymbol{v},0}(\xi_t)\|_{BL^q} + \|\partial_{\boldsymbol{\alpha}}P_{0,\boldsymbol{\alpha}}(\xi_t)\|_{BL^q} \leq \mathrm{const}_M\cdot\varepsilon(\|\boldsymbol{v}\|+\|\boldsymbol{\alpha}\|) \leq \mathrm{const}_M\cdot\varepsilon\,\|(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_1}.$$

Differentiating the equation $d^+_{\boldsymbol{A}'_{\boldsymbol{v},\boldsymbol{\alpha}}}P_{\boldsymbol{v},\boldsymbol{\alpha}}(\xi_t) = \xi_t$ with respect to $\boldsymbol{v}$, we have $d^+_{\boldsymbol{A}'}\partial_{\boldsymbol{v}}P_{\boldsymbol{v},0}(\xi_t) = -(j_1(\boldsymbol{v})\wedge P(\xi_t))^+$. Using the above (65), we have

$$\left\|d^+_{\boldsymbol{A}'}\partial_{\boldsymbol{v}}P_{\boldsymbol{v},0}(\xi_t)\right\|_{BL^p} \leq \|j_1(\boldsymbol{v})\|_{BL^4}\,\|P(\xi_t)\|_{BL^q} \leq \mathrm{const}_M\cdot\varepsilon\,\|(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_1}.$$

Similarly

$$\left\|d^+_{\boldsymbol{A}'}\partial_{\boldsymbol{\alpha}}P_{0,\boldsymbol{\alpha}}(\xi_t)\right\|_{BL^p} \leq \mathrm{const}_M\cdot\varepsilon\,\|\boldsymbol{\alpha}\| \leq \mathrm{const}_M\cdot\varepsilon\,\|(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_1}.$$

Therefore

$$\|\partial_{\boldsymbol{v}}P_{\boldsymbol{v},0}(\xi_t)\|_{B_2} + \|\partial_{\boldsymbol{\alpha}}P_{0,\boldsymbol{\alpha}}(\xi_t)\|_{B_2} \leq \mathrm{const}_M\cdot\varepsilon\,\|(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_1}.$$

Thus

$$\|(dG)_0(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta) - T(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_2} \leq \mathrm{const}_M\cdot\varepsilon\,\|(\chi,\boldsymbol{v},\boldsymbol{\alpha},\eta)\|_{B_1}.$$

Hence if we choose ε small enough, then $(dG)_0$ is an isomorphism. This shows that S is open in $[0,1]$. S is a non-empty open closed set in $[0,1]$. Thus $S=[0,1]$. In particular we have $1\in S$. This proves Theorem 6.11.

7. Estimation of the mean dimension

As in the previous sections, M denotes a set of equivalence classes of (E,A) (E is a principal $SU(2)$-bundle over X and A is an ASD connection on it) which satisfies the conditions (a), (b), (c) in the beginning of Section 3.2. GlD $=$ GlD$_M$ is the set of the equivalence classes of M-gluing data defined in Definition 3.2. GlD is endowed with the topology of "point-wise convergence" as follows. A sequence $[\theta_n] = [E_{n\gamma}, A_{n\gamma}, \rho_{n\gamma,s}]_{\gamma\in\Gamma,s\in S}$ in GlD ($n\geq 1$) converges to $[\theta] = [E_\gamma, A_\gamma, \rho_{\gamma,s}]_{\gamma\in\Gamma,s\in S}$ if the following condition is satisfied. For any finite subset $\Omega\subset\Gamma$, there exist $n_0(\Omega)>0$ and bundle isomorphisms $g_{n\gamma}: E_{n\gamma}\to E_\gamma$ for $n\geq n_0(\Omega)$ and $\gamma\in\Omega$ such that $g_{n\gamma}(A_{n\gamma})$ converges to A_γ (as $n\to\infty$) in the $\mathcal{C}^\infty$-topology for all $\gamma\in\Omega$, and that $g_{n\gamma s}\rho_{n\gamma,s}g_{n\gamma}^{-1}: (E_\gamma)_{x_{\gamma,s}} \to (E_{\gamma s})_{y_{\gamma s,s}}$ converges to $\rho_{\gamma,s}$ for any (γ,s) with $\gamma,\gamma s\in\Omega$. This topology is metrizable and compact (because we suppose that M is compact). Γ continuously acts on GlD by (this is a right action)

$$[E_\gamma, A_\gamma, \rho_{\gamma,s}]_{\gamma\in\Gamma,s\in S}\cdot g := [E_{g\gamma}, A_{g\gamma}, \rho_{g\gamma,s}]_{\gamma\in\Gamma,s\in S},$$

where we naturally consider that $(E_{g\gamma}, A_{g\gamma})$ is a data defined on X_γ and that $\rho_{g\gamma,s}$ is a map from $(E_{g\gamma})_{x_{\gamma,s}}$ to $(E_{g\gamma})_{y_{\gamma s,s}}$. A distance on GlD is given as follows: For $n\geq 1$,

$[\theta]=[E_\gamma, A_\gamma, \rho_{\gamma,s}]_{\gamma\in\Gamma, s\in S}$ and $[\theta']=[F_\gamma, B_\gamma, \rho'_{\gamma,s}]_{\gamma\in\Gamma, s\in S}$ in GlD, we define $\delta_n([\theta],[\theta'])$ by

$$\delta_n([\theta],[\theta']) := \inf_{g_\gamma: E_\gamma\to F_\gamma (\gamma\in B_n)} \left(\sum_{\gamma\in B_n} \|g_\gamma(A_\gamma)-B_\gamma\|_{L^\infty(X)} + \sum_{\gamma,\gamma s\in B_n} |g_{\gamma s}\rho_{\gamma,s}g_\gamma^{-1}-\rho'_{\gamma,s}|\right),$$

where B_n is the n-ball (with respect to the word distance) centered at the origin in Γ and g_γ ($\gamma\in B_n$) runs over bundle isomorphisms between E_γ and F_γ. If E_γ is not isomorphic to F_γ for some $\gamma\in B_n$, then we set $\delta_n([\theta],[\theta']) := +\infty$. We define a distance $d([\theta],[\theta'])$ by

$$d([\theta],[\theta']) := \sum_{n\geq 1}\frac{1}{2^n}\frac{\delta_n([\theta],[\theta'])}{1+\delta_n([\theta],[\theta'])}.$$

We define the space $\mathcal{M}(\mathrm{GlD}) = \mathcal{M}(\mathrm{GlD}_M)$ by

$$\mathcal{M}(\mathrm{GlD}) := \{[\boldsymbol{E}(\theta), \boldsymbol{A}(\theta)] \,|\, [\theta]\in \mathrm{GlD}\}.$$

$\mathcal{M}(\mathrm{GlD})$ is endowed with the topology of $\mathcal{C}^\infty$-convergence on compact subsets in $X^{\sharp(\Gamma,S)}$. This topology is metrizable. Γ continuously acts on $\mathcal{M}(\mathrm{GlD})$ by (3). The map

$$\mathrm{GlD}\to\mathcal{M}(\mathrm{GlD}), \quad [\theta]\mapsto [\boldsymbol{E}(\theta), \boldsymbol{A}(\theta)],$$

is Γ-equivariant.

Lemma 7.1. *The above map* GlD $\to \mathcal{M}(\mathrm{GlD})$ *is a* Γ*-homeomorphism.*

Proof. Proposition 5.5 shows that the map is bijective. Since GlD is compact, it is enough to prove that the map is continuous.

Let $\varepsilon>0$ and $\Omega\subset\Gamma$ be a finite subset. Let $\theta_1 = (E_{1\gamma}, A_{1\gamma}, \rho_{1\gamma,s})_{\gamma\in\Gamma,s\in S}$ and $\theta_2 = (E_{2\gamma}, A_{2\gamma}, \rho_{2\gamma,s})_{\gamma\in\Gamma,s\in S}$ be two gluing data. Let $d=d(M,\varepsilon)$ be the positive constant given by Corollary 6.8. Suppose that $E_{1\gamma}=E_{2\gamma}$ for $\gamma\in B_{d+1}(\Omega)$ and that $\|A_{1\gamma}-A_{2\gamma}\|_{L^q}$ and $|\rho_{1\gamma,s}-\rho_{2\gamma,s}|$ ($\gamma\in B_d(\Omega), s\in S$) are sufficiently small. We define another gluing data $\theta' := (E'_\gamma, A'_\gamma, \rho'_{\gamma,s})_{\gamma\in\Gamma,s\in S}$ by $(E'_\gamma, A'_\gamma, \rho'_{\gamma,s}) = (E_{1\gamma}, A_{1\gamma}, \rho_{1\gamma,s})$ for $\gamma\in B_d(\Omega)$ and $(E'_\gamma, A'_\gamma, \rho'_{\gamma,s}) = (E_{2\gamma}, A_{2\gamma}, \rho_{2\gamma,s})$ for $\gamma\in\Gamma\setminus B_d(\Omega)$. From Corollary 6.8, we have (for $\gamma\in\Omega$)

$$\|\boldsymbol{A}(\theta_1)-\boldsymbol{A}(\theta')\|_{L^q(X''_\gamma,g)}<\varepsilon.$$

On the other hand, for all $\gamma\in\Gamma$ and $s\in S$ we have $E_{2\gamma}=E'_\gamma$, $\left\|A_{2\gamma}-A'_\gamma\right\|_{L^q(X''_\gamma,g)}\ll 1$ and $|\rho_{2\gamma,s}-\rho'_\gamma|\ll 1$. Therefore (using the arguments in Section 5.1)

$$d_{BL^q}([\boldsymbol{A}(\theta_2)],[\boldsymbol{A}(\theta')])<\varepsilon.$$

Thus there exists a bundle map g from $\boldsymbol{E}(\theta_1)$ to $\boldsymbol{E}(\theta_2)$ over Ω such that for all $\gamma\in\Omega$

$$\|g(\boldsymbol{A}(\theta_1))-\boldsymbol{A}(\theta_2)\|_{L^q(X''_\gamma,g)}<2\varepsilon.$$

This shows that GlD $\to\mathcal{M}(\mathrm{GlD})$ is continuous. $\square$

In the rest of this section we assume that Γ is amenable. Let $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$ be an amenable sequence in Γ. This sequence satisfies (for any $r > 0$)

$$|B_r(\Omega_n)|/|\Omega_n| \to 1 \quad (n \to \infty), \tag{66}$$

where $B_r(\Omega_n)$ is the r-neighborhood of Ω_n. For each Ω_n we define the distance $d_{\Omega_n}([\theta], [\theta'])$ on GlD by

$$d_{\Omega_n}([\theta], [\theta']) := \max_{g \in \Omega_n} d([\theta].g, [\theta'].g).$$

Proposition 7.2.

$$\dim(\mathcal{M}(\mathrm{GlD}) : \Gamma) \leq 3|S| + \dim M,$$

where $\dim M$ *denotes the (topological) covering dimension of* M*.*

Proof. From Lemma 7.1, it is enough to prove that

$$\dim(\mathrm{GlD} : \Gamma) \leq 3|S| + \dim M.$$

Fix any $\varepsilon > 0$. Take $n_0 = n_0(\varepsilon) > 0$ satisfying

$$\sum_{n \geq n_0} \frac{1}{2^n} < \varepsilon.$$

For any finite set $\Omega \subset \Gamma$, we define $B_{-1}(\Omega)$ as the set of $\gamma \in \Omega$ satisfying $\gamma s \in \Omega$ for all $s \in S$. We define a finite dimensional compact metrizable space $\mathrm{GlD}|_\Omega$ by

$$\mathrm{GlD}|_\Omega := \{\left((E_\gamma, A_\gamma)_{\gamma \in \Omega},\, (\rho_{\gamma,s})_{\gamma \in B_{-1}(\Omega), s \in S}\right) \mid [E_\gamma, A_\gamma] \in M,\, \rho_{\gamma,s} : (E_\gamma)_{x_s} \to (E_{\gamma s})_{y_s}\}/\sim,$$

where the equivalence relation $\sim$ is defined as follows. $\theta = \left((E_\gamma, A_\gamma)_{\gamma \in \Omega},\, (\rho_{\gamma,s})_{\gamma \in B_{-1}(\Omega), s \in S}\right)$ is equivalent to $\theta' = \left((F_\gamma, B_\gamma)_{\gamma \in \Omega},\, (\rho'_{\gamma,s})_{\gamma \in B_{-1}(\Omega), s \in S}\right)$ if there exist $g_\gamma : E_\gamma \to F_\gamma$ $(\gamma \in \Omega_n)$ such that $g_\gamma(A_\gamma) = B_\gamma$ for all $\gamma \in \Omega$ and $\rho'_{\gamma,s} g_\gamma = g_{\gamma s} \rho_{\gamma,s}$ for all $(\gamma, s) \in B_{-1}(\Omega_n) \times S$.

There is a natural projection $\mathrm{GlD} \to \mathrm{GlD}|_\Omega$. Consider the following projection map:

$$\mathrm{GlD}|_\Omega \to M^\Omega.$$

The topological dimension of each fiber of this map is $\leq 3|\Omega||S|$. Hence

$$\dim \mathrm{GlD}|_\Omega \leq |\Omega| \dim M + 3|\Omega||S|.$$

For each Ω_n in the amenable sequence, consider the following map

$$p : \mathrm{GlD} \to \mathrm{GlD}|_{B_{n_0}(\Omega_n)}.$$

If $p([\theta]) = p([\theta'])$, then we have

$$d_{\Omega_n}([\theta], [\theta']) < \varepsilon.$$

Therefore (see Appendix B)

$$\mathrm{Widim}_\varepsilon(\mathrm{GlD}, d_{\Omega_n}) \leq \dim \left(\mathrm{GlD}|_{B_{n_0}(\Omega_n)}\right) \leq |B_{n_0}(\Omega_n)|(\dim M + 3|S|).$$

Using (66), we get

$$\mathrm{Widim}_\varepsilon(\mathrm{GlD} : \Gamma) = \lim_{n \to \infty} \mathrm{Widim}_\varepsilon(\mathrm{GlD}, d_{\Omega_n})/|\Omega_n| \leq \dim M + 3|S|.$$

This holds for any $\varepsilon > 0$. Thus we get the above result. $\square$

Let $[E, A] \in M$. We call this point a regular point of M if $[E, A] \in M_0$ and there is $\delta > 0$ such that, for any ASD connection B on E satisfying $d_{L^q}([A], [B]) \leq \delta$, the pair $[E, B]$ is contained in M_0. Remember that for any $[E, A] \in M_0$ the connection A is irreducible (see (b) in the beginning of Section 3.2) and then its isotropy group is $\{\pm 1\}$.

Proposition 7.3. *Let $[E, A]$ be a regular point of M, then we have*

$$\dim(\mathcal{M}(\mathrm{GlD}) : \Gamma) \geq 3|S| + \dim H^1_A.$$

Proof. We will prove $\dim(\mathrm{GlD} : \Gamma) \geq 3|S| + \dim H^1_A$. There is a compact neighborhood K of the origin in H^1_A such that, for all $\alpha \in K$, $[E, A + \tilde{\alpha}] \in M_0$ and the map $K \ni \alpha \mapsto [E, A + \tilde{\alpha}] \in M_0$ is injective. Here $A + \tilde{\alpha}$ is the ASD connection introduced in Section 3.4.2. Let $\mathrm{Hom}_{SU(2)}(E_{x_s}, E_{y_s})$ be the space of $SU(2)$-isomorphisms between the fibers E_{x_s} and E_{y_s} $(s \in S)$. Let $L_s \subset \mathrm{Hom}_{SU(2)}(E_{x_s}, E_{y_s})$ be a compact set such that L_s is homeomorphic to a three dimensional ball and that, for any $u, v \in L_s$, we have $u \neq -v$. Then we have a natural Γ-equivariant continuous map

$$\left(K \times \prod_{s \in S} L_s\right)^{\Gamma} \to \mathrm{GlD}.$$

From the conditions of K and L_s, this map is injective. Therefore

$$\dim(\mathrm{GlD} : \Gamma) \geq \dim\left(\left(K \times \prod_{s \in S} L_s\right)^{\Gamma} : \Gamma\right) = \dim\left(K \times \prod_{s \in S} L_s\right) = \dim H^1_A + 3|S|.$$

$\square$

8. Proof of Theorem 2.3

In this and the next sections we set $X := S^4$ with the metric h satisfying the conditions (i), (ii) in the beginning of Section 2.1. Let $0 \leq c < \overline{c} < +\infty$ and $d \in (2, +\infty]$. As in Section 2.1 we define $M = M_{S^4}(\overline{c}, d)$ as the space of the gauge equivalence classes of (E, A) where E is a principal $SU(2)$-bundle over S^4 and A is an ASD connection on it satisfying $\|F_A\|_{L^d(S^4, h)} \leq \overline{c}$. We set $M_1 := \{[S^4 \times SU(2), \text{the product connection}]\}$ and $M_0 := M \setminus M_1$.

If A is an ASD connection on (S^4, h), then A is also ASD with respect to the standard metric on S^4 because h is conformally equivalent to the standard one. Therefore all non-flat ASD connections on (S^4, h) are regular. In particular, $M = M_0 \sqcup M_1$ satisfies the conditions (a), (b), (c) in the beginning of Section 3.2. We consider the gluing data space $\mathrm{GlD} = \mathrm{GlD}_M$ for this $M = M_{S^4}(\overline{c}, d)$.

Recall that $\mathcal{M}(c,d)$ is the space of $[E,A]$ where E is a principal $SU(2)$-bundle over $(S^4)^{\sharp(\Gamma,S)}$ and A is an ASD connection on it satisfying (2):

$$\|F_A\|_{L^d(X''_\gamma,g)} \leq c \quad \text{for all } \gamma \in \Gamma.$$

(Here $X''_\gamma = U_\gamma$ in the notation of Section 2.1.)

Proposition 8.1. *There are $N_0(c,\bar{c},d) > 0$ and $\lambda_0(c,\bar{c},d,N) > 0$ such that if $N \geq N_0(c,\bar{c},d)$ and $\lambda \leq \lambda_0(c,\bar{c},d,N)$ then*

$$\mathcal{M}(c,d) \subset \mathcal{M}(\mathrm{GlD}),$$

i.e., for any $[E,A] \in \mathcal{M}(c,d)$ there exists $[\theta] \in \mathrm{GlD}$ satisfying $[E,A] = [\boldsymbol{E}(\theta), \boldsymbol{A}(\theta)]$.

We need some preliminary results for the proof of this proposition. Our argument is based on Donaldson-Kronheimer [5, Section 7.3]. The following proposition is given in [5, Proposition 7.3.3], and we omit the proof.

Proposition 8.2. *Let $T > 0$ and $k > 0$. Consider $(-T,T) \times S^3$ with the usual product metric. There are positive constants η and $C = C(k)$ (η is independent of T and k, and C is independent of T) such that if an ASD connection A on $(-T,T) \times S^3$ satisfies*

$$\|F(A)\|_{L^2}^2 := \int_{(-T,T)\times S^3} |F(A)|^2 d\mathrm{vol} \leq \eta^2,$$

then

$$|F(A)| \leq C e^{2(|t|-T)} \|F(A)\|_{L^2},$$

for all $(t,\theta) \in (-T,T) \times S^3$ with $|t| \leq T - k$.

Using the stereographic projection, we can translate this proposition to a result on the Euclidean space:

Corollary 8.3. *Let $\sigma > 0$ and $\lambda > 0$ with $\lambda/\sigma \leq \sqrt{\lambda}/2$. Set $k := 0.9$ and*

$$\Omega := \{x \in \mathbb{R}^4 |\, k\lambda/\sigma < |x| < k^{-1}\sigma\}.$$

There exist $\eta > 0$ and $C > 0$ (independent of σ and λ) such that if an ASD connection A on Ω satisfies

$$\|F(A)\|_{L^2}^2 := \int_\Omega |F(A)|^2 d\mathrm{vol} \leq \eta^2,$$

then

$$|F(A)| \leq \frac{C}{\sigma^2} \|F(A)\|_{L^2} \quad (\sqrt{\lambda}/2 \leq |x| \leq \sigma).$$

Moreover A can be represented by the connection matrix satisfying

$$|A| \leq \frac{C|x|}{\sigma^2} \|F(A)\|_{L^2} \quad (\sqrt{\lambda}/2 \leq |x| \leq \sigma).$$

Proof. See Donaldson-Kronheimer [5, p. 314]. $\square$

Lemma 8.4. *For any $\nu > 0$ there is $\lambda_0 > 0$ such that if $\lambda \leq \lambda_0$ then all $[E, B] \in \mathcal{M}(c, d)$ satisfies*

$$\inf_{[A]\in M_{S^4}(c,d)} d_{L^q}([B|_{X''_\gamma}], [A|_{X''_\gamma}]) < \nu \quad \text{for all } \gamma \in \Gamma.$$

Note that $[A]$ runs over $M_{S^4}(c, d)$ (not $M = M_{S^4}(\bar{c}, d)$).

Proof. We can prove this lemma by using the argument in [5, Section 7.3.4]. For $\varepsilon > 0$ we set (cf. Section 3.4.3)

$$X_\varepsilon := S^4 \setminus \left(\bigcup_{s\in S} \bar{B}(x_s, \varepsilon) \cup \bar{B}(y_s, \varepsilon) \right).$$

Suppose the above statement is false. Then there are $\nu > 0$ and a decreasing sequence $\lambda_1 > \lambda_2 > \lambda_3 > \cdots \to 0$ and ASD connections B_n on $X_{\lambda_n/\sigma}$ (σ is a small positive constant chosen below) satisfying

$$\|F(B_n)\|_{L^d(X_{\sqrt{\lambda_n}/2}, g)} \leq c, \tag{67}$$

$$\inf_{[A]\in M_{S^4}(c,d)} d_{L^q}([B_n|_{X_{\sqrt{\lambda_n}/2}}], [A|_{X_{\sqrt{\lambda_n}/2}}]) \geq \nu. \tag{68}$$

Let $\Omega'_n(x_s)$ and $\Omega'_n(y_s)$ ($s \in S$) be the annulus regions (in X) around x_s and y_s of inner radius $= k\lambda_n/\sigma$ and outer radius $= \sigma$. Since $d > 2$, we can choose $\sigma > 0$ so small that all B_n and $[A] \in M_{S^4}(c, d)$ have curvatures of L^2-norm $\leq \eta$ over each $\Omega'_n(x_s)$ and $\Omega'_n(y_s)$. (η is a positive constant given in Corollary 8.3.)

From (67) and $d > 2$, the Uhlenbeck compactness implies that (if we choose a subsequence) there exists $[A] \in M_{S^4}(c, d)$ such that $[B_n]$ converges to $[A]$ in the $\mathcal{C}^\infty$-topology over compact subsets in $X \setminus \{x_s, y_s | \, s \in S\}$.

On the other hand, from Corollary 8.3, B_n and A can be represented over $\Omega'_n(x_s)$ and $\Omega'_n(y_s)$ by the connection matrices satisfying

$$|B_n| \leq \text{const} \cdot |x|, \quad |A| \leq \text{const} \cdot |x| \quad (\sqrt{\lambda_n}/2 \leq |x| \leq \sigma),$$

where "const" is a positive constant independent of λ_n. This estimate and the $\mathcal{C}^\infty$-convergence mentioned above imply

$$d_{L^q}([B_n|_{X_{\sqrt{\lambda_n}/2}}], [A|_{X_{\sqrt{\lambda_n}/2}}]) \to 0.$$

This contradicts (68). □

Proof of Proposition 8.1. Set $\mathcal{L} := M_{S^4}(c, d) \Subset M = M_{S^4}(\bar{c}, d)$. For any $\nu > 0$, Lemma 6.10 and Lemma 8.4 implies (for appropriate N and λ) $\mathcal{M}(c, d) \subset U(\mathcal{L}, \nu)$. Then, from Theorem 6.11, for any $[E, A] \in \mathcal{M}(c, d)$ there exists $[\theta] \in \mathrm{GlD}_M$ satisfying $[E, A] = [\boldsymbol{E}(\theta), \boldsymbol{A}(\theta)]$. □

Proof of Theorem 2.3. (i) Take $\bar{c}$ such that $0 \leq c < \bar{c} < c_0(d)$. Then $M := M_{S^4}(\bar{c}, d) = M_1$ i.e., M consists only of the product connection. Then all $\boldsymbol{A}(\theta)$ ($[\theta] \in \mathrm{GlD}_M$) become flat connections. (See Remark 4.5.) Since we have $\mathcal{M}(c,d) \subset \mathcal{M}(\mathrm{GlD}_M)$ (for appropriate N and λ), $\mathcal{M}(c,d)$ is equal to the moduli space of flat $SU(2)$-connections.
(ii) Fix $N = N_0(c, \bar{c}, d)$ (the constant in Proposition 8.1). Using Propositions 7.2 and 8.1, we get ($\lambda \ll 1$)

$$\dim(\mathcal{M}(c,d) : \Gamma) \leq \dim(\mathcal{M}(\mathrm{GlD}) : \Gamma) \leq 3|S| + \dim M_{S^4}(\bar{c}, d).$$

□

9. Proof of Theorem 2.4

In this section we suppose $X = S^4$ and $d \in (2, +\infty)$. Let $0 < \underline{c} < c < +\infty$ and set $c' := (\underline{c} + c)/2$ ($\underline{c} < c' < c$). We also suppose that $\dim M_{S^4}(\underline{c}, d) > 0$. Then there exists $[E_0, A_0] \in M_{S^4}(\underline{c}, d)$ such that A_0 is a regular ASD connection and $\dim H^1_{A_0} \geq \dim M_{S^4}(\underline{c}, d)$. $[E_0, A_0]$ becomes a regular point of $M' := M_{S^4}(c', d)$. (See Proposition 7.3.)

Proposition 9.1. *There is $b_0(c, c', d) > 0$ such that if $b = 4N\sqrt{\lambda} \leq b_0(c, c', d)$ then*

$$\mathcal{M}(\mathrm{GlD}_{M'}) \subset \mathcal{M}(c,d),$$

i.e., for any $[\theta] \in \mathrm{GlD}_{M'}$ we have

$$\|F(\boldsymbol{A}(\theta))\|_{L^d(X''_\gamma, g)} \leq c \quad \text{for all } \gamma \in \Gamma.$$

Proof. We use an argument similar to that in the proof of Lemma 8.4. Set $\varepsilon := (c - c')/2 = (c - \underline{c})/4$. Suppose the above statement is false. Then there are parameters λ_n and $N_n \geq N_0(M')$ ($n = 1, 2, 3, \cdots$) ($N_0(M')$ is the constant given by Proposition 4.3) satisfying $b_n := 4N_n\sqrt{\lambda_n} \to 0$, and M'-gluing data $\theta_n = (E_{n\gamma}, A_{n\gamma}, \rho_{n\gamma,s})_{\gamma \in \Gamma, s \in S}$ such that for some $\gamma_n \in \Gamma$

$$\left\|F(\boldsymbol{A}^{(n)}(\theta_n))\right\|_{L^d(X''_{\gamma_n}, g)} > c,$$

where $\boldsymbol{A}^{(n)} := \boldsymbol{A}^{(n)}(\theta_n)$ is $\boldsymbol{A}(\theta_n)$ for the parameters $\lambda = \lambda_n$ and $N = N_n$. Using the Γ-equivariance, we can assume that $\gamma_n = e$ (the identity element of Γ). Taking a subsequence, we can also assume that $E_{ne} = E_{me}$ ($=: E$) for $m \geq n \gg 1$ and A_{ne} converges to $A \in M'$ (an ASD connection on E) in the C^∞-topology. Since $b_n \to 0$, $\boldsymbol{A}^{(n)}$ converges to A up to gauge equivalence in the $\mathcal{C}^\infty$-topology over compact subsets in $X_e \setminus \{x_{e,s}, y_{e,s} | \, s \in S\}$

We have a uniform upper-bound on $\left\|F(\boldsymbol{A}^{(n)})\right\|_{BL^p}$ by Proposition 4.6. Then, from $p > 2$ and Corollary 8.3, there exists $\sigma > 0$ such that in the Euclidean coordinates around $x_{e,s}$ and $y_{e,s}$ we have $|F(\boldsymbol{A}^{(n)})|(x) \leq \mathrm{const}_\sigma$ for $\sqrt{\lambda_n}/2 < |x| < \sigma$ (for all n). Hence for a sufficiently small $\sigma' < \sigma$ we have (recall: $\varepsilon = (c - c')/2$)

$$\int_{\sqrt{\lambda_n}/2 < |x| < \sigma'} |F(\boldsymbol{A}^{(n)})|^d \leq \varepsilon^d.$$

(Here we have used $d < +\infty$.) $|F(\boldsymbol{A}^{(n)})|$ uniformly converges to $|F(A)|$ over $X_{e,\sigma'}$. ($X_{e,\sigma}$ is the complement of the balls $B(x_{e,s},\sigma')$ and $B(y_{e,s},\sigma')$ in $X_e = S^4_e$.) Hence for $n \gg 1$

$$\left\|F(\boldsymbol{A}^{(n)})\right\|_{L^d(X_{e,\sigma'},g)} \leq c' + \varepsilon.$$

Therefore for $n \gg 1$

$$\left\|F(\boldsymbol{A}^{(n)})\right\|_{L^d(X''_e,g)} \leq c' + \varepsilon + \varepsilon = c.$$

This contradicts the assumption. □

Proof of Theorem 2.4. Using Propositions 7.3 and 9.1, we get ($\lambda \ll 1$)

$$\dim(\mathcal{M}(c,d):\Gamma) \geq \dim(\mathcal{M}(\mathrm{GlD}_{M'}):\Gamma) \geq 3|S| + \dim H^1_{A_0} \geq 3|S| + \dim M_{S^4}(\underline{c},d).$$

□

Appendix A. Proof of the compactness of $\mathcal{M}(c,d)$

The purpose of this appendix is to prove the following proposition:

Proposition A.1. *Let $d > 2$. Let X be a connected, oriented (possibly non-compact) Riemannian 4-manifold without boundary, and E be a principal $SU(2)$-bundle over X. Let $\{A_n\}_{n\geq 1}$ be a sequence of ASD connections on E such that for any compact set $K \subset X$ we have*

$$\sup_{n\geq 1} \|F(A_n)\|_{L^d(K)} < \infty.$$

Then there exist a subsequence (we also denote it by $\{A_n\}_{n\geq 1}$), a sequence of gauge transformations $g_n : E \to E$ and an ASD connection A on E such that $g_n(A_n)$ converges to A in the $\mathcal{C}^\infty$-topology over every compact set in X.

This proposition follows from Donaldson-Kronheimer [5, Proposision (4.4.9)]. But, unfortunately, I think that the proof of [5, Proposision (4.4.9)] contains a gap. I think that the proof of [5, Lemma (4.4.5)] is not correct (cf. [17, the footnote in p. 5]). The proof given below essentially uses the fact that the gauge group is $SU(2)$, and I don't know whether the same result holds for more general gauge group. In Section 2, we introduce the moduli space $\mathcal{M}(c,d)$ and state that this space is compact in the topology defined there. This compactness easily follows from the above proposition.

If X is a compact manifold, then the above statement is certainly well-known (see [17, Theorem E]). So we assume that X is non-compact below.

Lemma A.2. *Let $Y \subset X$ be a compact submanifold with boundary in X satisfying $H^4(X,Y;\mathbb{Z}) = 0$. Let g be a gauge transformation of E over Y. (Strictly speaking, we suppose that g is defined smoothly over a neighborhood of Y.) Then g can be extended to a gauge transformation of E defined over X.*

Proof. In this proof, we use the fact that the gauge group is $SU(2)$. Since X is non-compact, E is isomorphic to the product bundle $X \times SU(2)$. Hence g can be identified with the map from Y to $SU(2)$. $SU(2) \cong S^3$ is 2-connected and $\pi_3(SU(2)) = \mathbb{Z}$. So the obstruction on the extension of g to X is contained in $H^4(X, Y; \mathbb{Z})$. But this is 0. Hence g can be extended over X. □

From the second countability of X, there exists a sequence of compact submanifolds with boundary Y_k $(k \geq 1)$ such that

$$Y_1 \Subset Y_2 \Subset Y_3 \Subset \cdots, \quad X = \bigcup_{k\geq 1} Y_k.$$

Lemma A.3. *We can choose the above sequence so that $H^4(X, Y_k; \mathbb{Z}) = 0$ for all $k \geq 1$.*

Proof. Let $X \setminus \mathrm{int}(Y_k) = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_N$ be the decomposition into the sum of the connected components. N is less than or equal to the number of the connected components of ∂Y_k. Each X_n is a (possibly non-compact) submanifold in X with non-empty boundary $(\partial Y_k = \partial X_1 \sqcup \cdots \sqcup \partial X_N)$. We have (by the excision theorem)

$$H^4(X, Y_k; \mathbb{Z}) = \prod_{n=1}^{N} H^4(X_n, \partial X_n; \mathbb{Z}).$$

If X_n is non-compact, then we get $H^4(X_n, \partial X_n; \mathbb{Z}) = 0$. Hence if we can arrange Y_k so that all X_n becomes non-compact, then we get $H^4(X, Y_k; \mathbb{Z}) = 0$. This can be easily achieved as follows: Suppose that X_1, X_2, $\cdots$, X_m are compact and that X_{m+1}, X_{m+2}, $\cdots$, X_N are non-compact. Then we set $Y'_k := Y_k \cup X_1 \cup X_2 \cup \cdots \cup X_m$. Y'_k also becomes a compact submanifold in X, and $X \setminus \mathrm{int}(Y'_k) = X_{m+1} \cup X_{m+2} \cup \cdots \cup X_N$. Since each X_n $(n \geq m+1)$ is non-compact, we get $H^4(X, Y'_k; \mathbb{Z}) = 0$. □

We suppose that the sequence Y_k satisfies $H^4(X, Y_k; \mathbb{Z}) = 0$.

Proof of Proposition A.1. Using a collar neighborhood of ∂Y_k, we can construct a open set $U_k \supset Y_k$ (in X) which is diffeomorphic to $Y_k \cup_{\partial Y_k} \partial Y_k \times [0, 1)$. We can arrange them so that $U_1 \subset U_2 \subset U_3 \subset \cdots$. Using [17, Thoerem E'] for these U_k and a diagonal process, we get a subsequence (we also denote it by $\{A_n\}$) satisfying the following: For each $k \geq 1$ there exist a sequence of gauge transformations $g_n^{(k)}$ on U_k and an ASD connection $A^{(k)}$ defined over U_k such that $g_n^{(k)}(A_n)$ converges to $A^{(k)}$ $(n \to \infty)$ in the $\mathcal{C}^\infty$-topology over compact subsets in U_k.

For each $k \geq 1$, $A^{(k+1)}$ is gauge equivalent to $A^{(k)}$ over U_k. Hence there exists a gauge transformation h_k defined over U_k satisfying $h_k(A^{(k+1)}) = A^{(k)}$ on U_k. Using Lemma A.2, we have a gauge transformation h'_k defined over X which is equal to h_k over a neighborhood of Y_k. We have $h'_k(A^{(k+1)}) = A^{(k)}$ over a neighborhood of Y_k. We define an ASD connection A over X by setting $A := A^{(1)}$ over Y_1 and $A := h'_1 \circ h'_2 \circ \cdots \circ h'_{k-1}(A^{(k)})$ over Y_k $(k \geq 2)$. This definition is well-defined.

For each $k \geq 1$, by applying Lemma A.2 to the sequence $\{h'_1 \circ h'_2 \circ \cdots \circ h'_{k-1} \circ g_n^{(k)}\}_{n\geq 1}$, we get a sequence of gauge transformations $\{u_n^{(k)}\}_{n\geq 1}$ defined over X such that $u_n^{(k)} = h'_1 \circ h'_2 \circ \cdots \circ h'_{k-1} \circ g_n^{(k)}$ over Y_k. Then $u_n^{(k)}(A_n)$ converges to A $(n \to \infty)$ in the $\mathcal{C}^\infty$-topology over Y_k. In particular, there exists $n_k \geq 1$ satisfying

$$\left\|A - u_{n_k}^{(k)}(A_{n_k})\right\|_{\mathcal{C}^k(Y_k)} \leq 1/k.$$

(Here $\|\cdot\|_{\mathcal{C}^k(Y_k)}$ is a $\mathcal{C}^k$-norm over Y_k defined by using a fixed connection on E.) We can choose the above n_k so that $n_1 < n_2 < n_3 < \cdots$. Then $u_{n_k}^{(k)}(A_{n_k})$ converges to A $(k \to \infty)$ in the $\mathcal{C}^\infty$-topology over every compact set in X. □

Appendix B. Review of mean dimension

We review the definitions and basic properties of mean dimension. For the detail, see Gromov [8], Lindenstrauss-Weiss [10] and Lindenstrauss [9].

Let (X, d) be a compact metric space and $\varepsilon > 0$. Let Y be a topological space and $f : X \to Y$ a continuous map. We call f an ε-embedding if we have $\mathrm{Diam} f^{-1}(y) \leq \varepsilon$ for any $y \in Y$. For example, consider $[0,1] \times [0,\varepsilon]$ with the standard Euclidean distance, and let $f : [0,1] \times [0,\varepsilon] \to [0,1]$ be the natural projection. Then f is an ε-embedding.

We define $\mathrm{Widim}_\varepsilon(X, d)$ as the minimum integer $n \geq 0$ such that there exist an n-dimensional polyhedron P and an ε-embedding $f : X \to P$. For example, we have $\mathrm{Widim}_\varepsilon([0,1] \times [0,\varepsilon], \text{the Euclidean distance}) = 1$ for $\varepsilon < 1$. The following is one of the most basic examples (see Gromov [8, p. 332]). For its proof, see Lindenstrauss-Weiss [10, Lemma 3.2] or Tsukamoto [14, Example 4.1].

Example B.1. Consider $[0,1]^N$ with the ℓ^∞-distance $d_\infty(x, y) = \max_i |x_i - y_i|$. For $\varepsilon < 1$ we have

$$\mathrm{Widim}_\varepsilon([0,1]^N, d_\infty) = N.$$

Let Γ be a finitely generated infinite group with a finite generating set S. Γ is equipped with the word distance: for $\gamma, \gamma' \in \Gamma$, we define $d(\gamma, \gamma')$ as the minimum integer $n \geq 0$ such that there exist $\gamma_1, \cdots, \gamma_n$ in $S \cup S^{-1}$ satisfying $\gamma^{-1}\gamma' = \gamma_1 \cdots \gamma_n$.

For a finite subset $\Omega \subset \Gamma$ and $r > 0$, we define the r-boundary $\partial_r \Omega \subset \Gamma$ as the set of $\gamma \in \Gamma$ such that the r-ball $B(\gamma, r)$ around γ has non-empty intersection with both Ω and $\Gamma \setminus \Omega$. Let $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$ be a sequence of finite subsets in Γ. We call this sequence an amenable sequence if for any $r > 0$ we have $|\partial_r \Omega_n| / |\Omega_n| \to 0$ as n goes to ∞. We call Γ an amenable group if it has an amenable sequence. In this appendix we assume that Γ is an amenable group with an amenable sequence $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$. For example, $\mathbb{Z}$ is an amenable group with the amenable sequence $\Omega_n := \{0, 1, \cdots, n\}$.

Suppose that Γ continuously (not necessarily isometrically) acts on X. (We suppose that the action is a right-action.) For a finite subset $\Omega \subset \Gamma$ we define the distance $d_\Omega(\cdot, \cdot)$

by setting

$$d_\Omega(x,y) := \sup_{\gamma\in\Omega} d(x.\gamma, y.\gamma),$$

for $x, y \in X$. For $\varepsilon > 0$ we define $\mathrm{Widim}_\varepsilon(X:\Gamma)$ by

$$\mathrm{Widim}_\varepsilon(X:\Gamma) := \lim_{n\to\infty} \mathrm{Widim}_\varepsilon(X, d_{\Omega_n})/|\Omega_n|.$$

This limit always exists, and it is independent of the choice of amenable sequences. (see Gromov [8, pp. 336-338] and Lindenstrauss-Weiss [10, Appendix]). We define the mean dimension $\dim(X:\Gamma)$ by

$$\dim(X:\Gamma) := \lim_{\varepsilon\to 0} \mathrm{Widim}_\varepsilon(X:\Gamma).$$

(This might be infinity.) The value of $\dim(X:\Gamma)$ is a topological invariant. That is, if two distances d and d' on X defines the same topology, then we have $\dim((X,d):\Gamma) = \dim((X,d'):\Gamma)$. The following is the most basic result. (See Gromov [8, p. 340] and Lindenstrauss-Weiss [10, Proposition 3.1, 3.3].)

Example B.2. Let K be a compact metric space and set $X := K^\Gamma$. Γ acts on X by the shift action: for $x = (x_\gamma)_{\gamma\in\Gamma} \in X$ and $g \in \Gamma$ we set

$$x.g = (y_\gamma)_{\gamma\in\Gamma}, \quad y_\gamma := x_{g\gamma}.$$

Then we have

$$\dim(X:\Gamma) \le \dim K,$$

where $\dim K$ denotes the topological covering dimension of K. Moreover if K is a finite polyhedron, then we have

$$\dim(X:\Gamma) = \dim K.$$

Proof. Set $N := \dim K$ and we suppose $\mathrm{Diam} K = 1$ for simplicity. Let $w : \Gamma \to \mathbb{R}_{>0}$ be a positive function satisfying $w(e) = 1$ (e is the identity element of Γ) and $\sum_{\gamma\in\Gamma} w(\gamma) = 2$. We define the distance $d(x,y)$ ($x, y \in X$) by setting

$$d(x,y) := \sum_{\gamma\in\Gamma} w(\gamma) d(x_\gamma, y_\gamma).$$

For $\varepsilon > 0$, let $r > 0$ be a positive number such that the sum of $w(\gamma)$ over $\gamma \in \Gamma \setminus B(e,r)$ is less than ε. Then for any Ω_n, the natural projection

$$\varphi : X \to K^{\Omega_n \cup \partial_r \Omega_n}$$

satisfies $\mathrm{Diam}(\varphi^{-1}(y), d_{\Omega_n}) < \varepsilon$ for any $y \in K^{\Omega_n\cup\partial_r\Omega_n}$. Therefore

$$\mathrm{Widim}_\varepsilon(X, d_{\Omega_n}) \le N|\Omega_n \cup \partial_r\Omega_n|.$$

Since $\lim_{n\to\infty} |\Omega_n\cup\partial_r\Omega_n|/|\Omega_n| = 1$, we have $\mathrm{Widim}_\varepsilon(X:\Gamma) \le N$. Hence $\dim(X:\Gamma) \le N$.

Next we suppose K is a polyhedron. Then there exists a topological embedding $[0,1]^N \hookrightarrow K$. So we can assume $K = [0,1]^N$ with the ℓ^∞-distance. There exists a distance non-decreasing continuous map from $([0,1]^{N|\Omega_n|}, d_{\ell^\infty})$ to (X, d_{Ω_n}). Then for $\varepsilon < 1$

$$\mathrm{Widim}_\varepsilon(X, d_{\Omega_n}) \geq \mathrm{Widim}_\varepsilon([0,1]^{N|\Omega_n|}, d_{\ell^\infty}) = N|\Omega_n|.$$

Here we have used the result of Example B.1. Hence we get $\mathrm{Widim}_\varepsilon(X : \Gamma) \geq N$ for $\varepsilon < 1$. Thus $\dim(X : \Gamma) = N$. □

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