ON THE FOURTH JOHNSON HOMOMORPHISM OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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Abstract. In this paper we consider the Johnson homomorphism of the automorphism group of a free group with respect to the lower central series of the IA-automorphism group of a free group. In particular, we determine the ratoinal cokernel of the fourth Johnson homomorphism, and show that there appears a new obsturustion for the surjectivity of the Johnson homomorphism. Furthermore we characterize this obstruction using trace maps.

1. Introduction

Let F_n be a free group of rank $n \geq 2$, and $\mathrm{Aut}\,F_n$ the automorphism group of F_n. Let denote $\rho : \mathrm{Aut}\,F_n \to \mathrm{Aut}\,H$ the natural homomorphism induced from the abelianization H of F_n. The kernel of ρ is called the IA-automorphism group of F_n, denoted by IA_n. The IA-automorphism group IA_n reflects many richness and complexity of the structure of $\mathrm{Aut}\,F_n$, and plays important roles on various studies of $\mathrm{Aut}\,F_n$.

Although the study of the IA-automorphism group has a long history since its finitely many generators were obtained by Magnus [12] in 1935, the combinatorial group structure of IA_n is still quite complicated. For instance, any presentation for IA_n is not known in general. Nielsen [18] showed that IA_2 coincides with the inner automorphsim group, hence, is a free group of rank 2. For $n \geq 3$, however, IA_n is much larger than the inner automorphism group $\mathrm{Inn}\,F_n$. Krstić and McCool [11] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not.

The purpose of our research is to clarlify the group structure of IA_n. In particular, we are interested in to determine the graded quotients of the Johnson filtration of $\mathrm{Aut}\,F_n$. The Johnson filtration is a dcending central series

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

consisting of normal subgroups of $\mathrm{Aut}\,F_n$, which first term is IA_n. Then the Johnson homomorphisms

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

are defined on each graded quotient of the Johnson filtration. In particular, they are $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphisms. (For detail, see Subsection 2.4.) The

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study of the Johnson homomorphisms was originally begun in 1980 by D. Johnson [8] who determined the abelianization of the Torelli subgroup of a mapping class group of a surface in [9]. Now, the theory of the Johnson homomorphisms has been developed by many authors, and there is a broad range of results for it. (For example, see [7], [10] and [16].)

Through the images of the Johnson homomorphisms, we can study IA_n using infinitely many pieces of a free abelian group of finite rank. They are regarded as one by one approximations of IA_n, and to clarify the structure of them plays an important role in the study of IA_n. In this paper, in particular, we are interested in the irreducible decomposition of the cokernel of $\tau_{k,\mathbf{Q}} = \tau_k \otimes \mathrm{id}_{\mathbf{Q}}$ as a $\mathrm{GL}(n, \mathbf{Z})$-module. Now, for $1 \leq k \leq 3$, the cokernel of $\tau_{k,\mathbf{Q}}$ is completely determined. (See [1], [21] and [23] for $k = 1$, 2 and 3 respectively.) In general, however it is quite hard problem to solve. One reason for it is that we can not obtain an explicit generating system of each $\mathrm{gr}^k(\mathcal{A}_n)$ easily.

To avoid this difficulty, we consider the lower central series $\mathcal{A}'_n(1) = \mathrm{IA}_n$, $\mathcal{A}'_n(2)$, ... of IA_n. Since the Johnson filtration is central, $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for $k \geq 1$. It is conjectured that $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis who showed $\mathcal{A}'_2(k) = \mathcal{A}_2(k)$ for each $k \geq 1$ and $\mathcal{A}'_3(3) = \mathcal{A}_3(3)$ in [1]. Now, we have $\mathcal{A}'_n(2) = \mathcal{A}_n(2)$ due to Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [10]. (See (1) below.) Furthermore $\mathcal{A}'_n(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [21].

For each $k \geq 1$, set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. Since IA_n is finitely generated as above, each $\mathrm{gr}^k(\mathcal{A}'_n)$ is also fintely generated as an abelian group. Then we can also define the Johnson homomorphisms

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

by an argument similar to that in the definition of τ_k. Since $\mathrm{gr}^k(\mathcal{A}'_n)$ is fintely generated, it is easier to study the cokernel of τ'_k than that of τ_k. Furthermore, It is also important to determine $\mathrm{Coker}(\tau'_k)$ from the view point of the study of the difference between the Johnson filtration and the lower central series of IA_n. In this paper, as a consective result of our research [23], we determine the rational cokernel of the fourth Johnson homomorphism $\tau'_{4,\mathbf{Q}} := \tau'_4 \otimes \mathrm{id}_{\mathbf{Q}}$.

Theorem 1. *(= Theorem 4.1.) For any $n \geq 6$,*

$$\mathrm{Coker}(\tau'_{4,\mathbf{Q}}) = S^4 H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[2,1^2]} \oplus H_{\mathbf{Q}}^{[2,2]}.$$

In the right hand side of the equation above, the first term $S^4 H_{\mathbf{Q}}$ is called the Morita obstruction for the surjectivity of the Johnson homomorphism, which can be detected by the Morita trace $\mathrm{Tr}_{[k]}$. (See Section 4.) The second term is an obstruction which can be detected by the trace map $\mathrm{Tr}_{[2,1^2]}$ constructed in our previous paper [23]. The finial term is an obstruction of new type. In this paper, we construct a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism $\mathrm{Tr}_{[2,2]}$ which detect $H_{\mathbf{Q}}^{[2,2]}$, and call it a trace map for $H_{\mathbf{Q}}^{[2,2]}$. This part is the main purpose of the paper. From Theorem 1, we obtain a lower bound on the rank of the fourth graded quotient of the Johnson filtration of $\mathrm{Aut}\, F_n$.

Corollary 1. *(= Corollary 4.1.) For $n \geq 6$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^4(\mathcal{A}_n)) \geq \frac{1}{5} n^2(n^4 - 1) - \frac{1}{4} n(n+1)(n^2 - n + 2).$$

Here we brief our strategy to show Theorem 1. First, we give an upper bound of the cokernel of $\tau'_{4,\mathbf{Q}}$. Using an explicit generating system of $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$, and reducing some elements of it, we see that $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$ is generated by $r := n(n+1)(n^2-n+2)/4$ elements. Here r is the dimension of $S^4 H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[2,1^2]} \oplus H_{\mathbf{Q}}^{[2,2]}$ as a $\mathbf{Q}$-vector space. Then, we show that there does exist $S^4 H_{\mathbf{Q}}$, $H_{\mathbf{Q}}^{[2,1^2]}$ and $H_{\mathbf{Q}}^{[2,2]}$ in $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$ by detecting them with the trace maps.

This paper consists of five sections. In Section 2, we recall the definition and some properties of the IA-automorphism group and the Johnson homomorphisms of the automorphism group of a free group. In Section 3, we discuss generators of $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$. In Section 4, we define the trace maps, and study some properties of them. Then we show Theorem 1.

Contents

2. Preliminaries

In this section, after fixing notation and conventions, we recall the the IA-automorphism group and the Johnson homomorphisms the automorphism group of a free group.

2.1. **Notation and Conventions.**

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The automorphism group $\mathrm{Aut}\, G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For any $\mathbf{Z}$-module M, we denote $M \otimes_{\mathbf{Z}} \mathbf{Q}$ by the symbol obtained by attaching a subscript $\mathbf{Q}$ to M, like $M_{\mathbf{Q}}$ or $M^{\mathbf{Q}}$. Similarly, for any $\mathbf{Z}$-linear map $f : A \to B$, the induced $\mathbf{Q}$-linear map $A_{\mathbf{Q}} \to B_{\mathbf{Q}}$ is denoted by $f_{\mathbf{Q}}$ or $f^{\mathbf{Q}}$.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. **IA-automorphism group.**

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, \ldots, x_n$. We denote the abelianization of F_n by H, and its dual group by $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. Let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced from the abelianization of F_n. It is easily seen that ρ is surjective. In this paper we identifies $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H as a free abelian group induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of ρ is called the IA-automorphism group of F_n. Magnus [12] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : \begin{cases} x_i & \mapsto {x_j}^{-1} x_i x_j, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct $i,\, j \in \{1, 2, \ldots, n\}$ and

$$K_{ijk} : \begin{cases} x_i & \mapsto x_i x_j x_k {x_j}^{-1} {x_k}^{-1}, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct $i,\, j,\, k \in \{1, 2, \ldots, n\}$ such that $j > k$.

Recently, Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [10] independently showed that the the abelianization of IA_n is a free abelian group, and the Magnus generators above induce a basis of it. More precisely, they showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module. Krstić and McCool [11] showed that IA_3 is not finitely presentable. For $n \geq 4$, however, it is still not known whether IA_n is finitely presentable or not.

2.3. **Associated Lie algebra of a group.**

In this subsection we recall the associated Lie algebra of a group G. In particular, we use the case where $G = F_n$ and IA_n.

Let G be a group, and $\Gamma_G(k)$ the k-th term of the lower central series of G defined by

$$\Gamma_G(1) := G, \quad \Gamma_G(k) := [\Gamma_G(k-1), G], \;\; k \geq 2.$$

For each $k \geq 1$, set $\mathcal{L}_G(k) := \Gamma_G(k)/\Gamma_G(k+1)$ and

$$\mathcal{L}_G := \bigoplus_{k \geq 1} \mathcal{L}_G(k).$$

Then $\mathcal{L}_G$ has a graded Lie algebra structure induced from the commutator bracket on G. We call $\mathcal{L}_G$ the associated Lie algebra of a group G.

For any $g_1, \ldots, g_k \in G$, a commutator of weight k among the components $g_1, \ldots, g_k$ of the type

$$[[\cdots[[g_1, g_2], g_3], \cdots], g_k]$$

with all of its brackets to the left of all the elements occuring is called a simple k-fold commutator, denoted by $[g_1, g_2, \cdots, g_k]$. Then we have

Lemma 2.1. *If a group G is generated by $g_1, \ldots, g_n$, then for each $k \geq 1$, $\mathcal{L}_G(k)$ is generated by (the coset classes of) the simple k-fold commutators*

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}], \quad i_j \in \{1, \ldots, n\}.$$

For a proof, see [14] for example.

If G is a free group F_n of rank n, for simplicity, we write $\Gamma_n(k)$, $\mathcal{L}_n(k)$ and $\mathcal{L}_n$ for $\Gamma_G(k)$, $\mathcal{L}_G(k)$ and $\mathcal{L}_G$ respectively. The Lie algebra $\mathcal{L}_n$ is called the free Lie algebra generated by H. Since the group $\mathrm{Aut}\, F_n$ naturally acts on $\mathcal{L}_n(k)$ for each $k \geq 1$, and since IA_n acts on it trivially, the action of $\mathrm{GL}(n, \mathbf{Z})$ on each $\mathcal{L}_n(k)$ is well-defined.

Let $T(H)$ be the tensor algebra of H over $\mathbf{Z}$. Then the algebra $T(H)$ is the universal envelopping algebra of the free Lie algebra $\mathcal{L}_n$, and the natural map $\mathcal{L}_n \to T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for X, $Y \in \mathcal{L}_n$ is an injective homomorphism between the graded Lie algebras. Hence we also regard $\mathcal{L}_n(k)$ as a submodule of $H^{\otimes k}$ for each $k \geq 1$. (See [22] for basic material concerning free Lie algebra.)

2.4. **Johnson homomorphisms.**

In this subsection, we recall the Johnson homomorphisms. To begin with, we consider a descending filtration of $\mathrm{Aut}\, F_n$ called the Johnosn filtration. For $k \geq 0$, the action of $\mathrm{Aut}\, F_n$ on each nilpotent quotient $F_n/\Gamma_n(k+1)$ of F_n induces a homomorphism

$$\rho^k : \mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

We denote the kernel of ρ^k by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, with $\mathcal{A}_n(1) = \mathrm{IA}_n$. It is called the Johnson filtration of $\mathrm{Aut}\, F_n$. For each $k \geq 1$, the group $\mathrm{Aut}\, F_n$ acts on $\mathcal{A}_n(k)$ by conjugation, and it naturally induces an action of $\mathrm{GL}(n, \mathbf{Z})$ on $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$. The graded sum $\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ has a graded Lie algebra structure induced from the commutator bracket on IA_n.

The graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ are considered as one by one approximations of IA_n, and they have many important information of IA_n. In order to study the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n)$, we define the Johnson homomorphisms of $\mathrm{Aut}\, F_n$ as follows. For each $k \geq 1$, define a homomorphism $\tilde{\tau}_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \;\mapsto\; (x \mapsto x^{-1}x^{\sigma}), \quad x \in H.$$

Then the kernel of $\tilde{\tau}_k$ is just $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

The homomorphsim τ_k is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. It is easily seen that each τ_k is $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism. For the Magnus generators of IA_n, their images by τ_1 are given by

$$\tau_1(K_{ij}) = x_i^* \otimes [x_i, x_j], \quad \tau_1(K_{ijl}) = x_i^* \otimes [x_j, x_l]. \tag{2}$$

Let $\mathrm{Der}\,(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree k part of $\mathrm{Der}\,(\mathcal{L}_n)$ is considered as $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, and we identify them in this paper. Then the sum of the Johnson homomorphisms

$$\tau := \bigoplus_{k \geq 1} \tau_k : \mathrm{gr}(\mathcal{A}_n) \to \mathrm{Der}\,(\mathcal{L}_n)$$

is a graded Lie algebra homomorphism. In fact, if we denote by $\partial\xi$ the element of $\mathrm{Der}\,(\mathcal{L}_n)$ corresponding to an element $\xi \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$, and write the action of $\partial\xi$ on $X \in \mathcal{L}_n$ as $X^{\partial\xi}$ then we have

$$\tau_{k+l}([\sigma, \sigma']) = \tau_k(\sigma)^{\partial\tau_l(\sigma')} - \tau_l(\sigma')^{\partial\tau_k(\sigma)}.$$

for any $\sigma \in \mathcal{A}_n(k)$ and $\sigma' \in \mathcal{A}_n(l)$.

In addition to the Johnson filtration, we consider the lower central series $\mathcal{A}_n'(1) = \mathrm{IA}_n$, $\mathcal{A}_n'(2), \ldots$ of IA_n. Since the Johnson filtration is central, $\mathcal{A}_n'(k) \subset \mathcal{A}_n(k)$ for each $k \geq 1$. It is conjectured that $\mathcal{A}_n'(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis as mentioned above. Now, we have $\mathcal{A}_n'(2) = \mathcal{A}_n(2)$ due to Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [10], and $\mathcal{A}_n'(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [21]. For each $k \geq 1$, set $\mathrm{gr}^k(\mathcal{A}_n') := \mathcal{A}_n'(k)/\mathcal{A}_n'(k+1)$. Similarly to $\mathrm{gr}^k(\mathcal{A}_n)$, $\mathrm{GL}(n, \mathbf{Z})$ naturally acts on $\mathrm{gr}^k(\mathcal{A}_n')$. Moreover, since IA_n is finitely generated by the Magnus generators K_{ij} and K_{ijl}, each $\mathrm{gr}^k(\mathcal{A}_n')$ is also fintely generated by the simple k-fold commutators among the components K_{ij} and K_{ijl} by Lemma 2.1.

A restriction of $\tilde{\tau}_k$ to $\mathcal{A}_n'(k)$ induces a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\tau_k' : \mathrm{gr}^k(\mathcal{A}_n') \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1),$$

and the sum

$$\tau' := \bigoplus_{k \geq 1} \tau_k' : \mathrm{gr}(\mathcal{A}_n') \to \mathrm{Der}\,(\mathcal{L}_n)$$

is also a graded Lie algebra homomorphism. Furthermore, we have

$$\tau_{k+l}'([\sigma, \sigma']) = \tau_k'(\sigma)^{\partial\sigma'} - \tau_l'(\sigma')^{\partial\sigma}.$$

for any $\sigma \in \mathcal{A}_n'(k)$ and $\sigma' \in \mathcal{A}_n'(l)$. Using this formula recursively, we can easily compute $\tau_k(\sigma)$ for any $\sigma \in \mathcal{A}_n'(k)$ from (2). We should remark that in general, it is not known whether τ_k' is injective or not. In this paper, we study the rational Johnson homomorphisms $\tau_{k,\mathbf{Q}}' = \tau_k' \otimes \mathrm{id}_{\mathbf{Q}}$, and give an irreducible decomposition of $\mathrm{Coker}(\tau_{4,\mathbf{Q}}')$ as a $\mathrm{GL}(n, \mathbf{Z})$-module.

3. Generators of $\mathrm{Coker}(\tau_{4,\mathbf{Q}}')$

In this section, we give a generating system of $\mathrm{Coker}(\tau_{4,\mathbf{Q}}')$ consists of $r := n(n+1)(n^2-n+2)/4$ elements. Here r is the dimension of $S^4 H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[2,1^2]} \oplus H_{\mathbf{Q}}^{[2,2]}$ as a $\mathbf{Q}$-vector space. First, we consider a generating system of $\mathrm{Coker}(\tau_k')$ for general $k \geq 2$ and $n \geq k+2$. Then, in Proposition 3.1, we consider $\mathrm{Coker}(\tau_{4,\mathbf{Q}}') = \mathrm{Coker}(\tau_4') \otimes_{\mathbf{Z}} \mathbf{Q}$ for $n \geq 6$ and $k = 4$.

To begin with, using Lemma 2.1, we see that

$$\mathfrak{E} := \{x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \,|\, 1 \leq i,\ i_l \leq n\} \tag{3}$$

generates $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, and hence $\mathfrak{E}$ also generates $\mathrm{Coker}(\tau'_k)$. In the following, we reduce the elements of $\mathfrak{E}$ keeping it beeing a generating system of $\mathrm{Coker}(\tau'_k)$. To do this, we prepare some lemmas.

Lemma 3.1. *For $n \geq 3$ and $k \geq 1$, if $i_l \neq i$ for $1 \leq l \leq k+1$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] = 0 \in \mathrm{Coker}(\tau'_k).$$

Proof. We show the lemma by induction on k. For $k=1$, we have $\tau'_1(K_{ii_1i_2}) = x_i^* \otimes [x_{i_1}, x_{i_2}]$. Assume $k \geq 2$. By the inductive hypothesis, there exists a certain $\sigma \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1}(\sigma) = x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_k}].$$

On the other hand, we have $\tau_1(K_{ii_{k+1}}) = x_i^* \otimes [x_i, x_{i_{k+1}}]$. Then

$$\tau'_k([K_{ii_{k+1}}, \sigma]) = x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}].$$

This completes the proof of Lemma 3.1. $\square$

Let $\mathfrak{F}$ be a subset of $\mathfrak{E}$ consisting of elements $x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}]$ such that there exists a certain $m \in \{1, 2, \ldots, k+1\}$ such that $i_m = i$ and $i_l \neq i$ for $l \neq m$.

Lemma 3.2. *For $n \geq k+1$, $\mathrm{Coker}(\tau'_k)$ is generated by $\mathfrak{F}$.*

Proof. Take any $x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \in \mathfrak{E}$ such that $i_{l_1} = i_{l_2} = i$ for l_1, l_2 such that $l_1 \neq l_2$. Since $n \geq k+1$, there exists a certain $j \in \{1, 2, \ldots, n\}$ such that $j \neq i, i_l$ for $1 \leq 1 \leq k+1$. Set

$$\sigma := \begin{cases} K_{iji_{k+1}}, & i \neq i_{k+1}, \\ K_{ij}^{-1}, & i = i_{k+1}. \end{cases}$$

Then

$$\tau'_1(\sigma) = x_i^* \otimes [x_j, x_{i_{k+1}}].$$

On the other hand, from Lemma 3.1, there exists a certain $\sigma' \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1}(\sigma') = x_j^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_k}].$$

Then, using the Jacobi identity, we obtain

$$\begin{aligned} \tau'_k([\sigma, \sigma']) &= x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \\ &\qquad - \sum_{l=1}^{k} \delta_{ii_l} x_j^* \otimes [x_{i_1}, \ldots, x_{i_{l-1}}, [x_j, x_{i_{k+1}}], x_{i_{l+1}}, \ldots, x_k], \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \\ &\quad - \sum_{l=1}^{k} \delta_{ii_l} \Big(x_j^* \otimes [x_{i_1}, \ldots, x_{i_{l-1}}, x_j, x_{i_{k+1}}, x_{i_{l+1}}, \ldots, x_k] \\ &\qquad\qquad - x_j^* \otimes [x_{i_1}, \ldots, x_{i_{l-1}}, x_{i_{k+1}}, x_j, x_{i_{l+1}}, \ldots, x_k] \Big). \end{aligned}$$

This completes the proof of Lemma 3.2. $\square$

Lemma 3.3. *For $n \geq 3$ and $k \geq 2$, if $i_l \neq i$ for $1 \leq l \leq k$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_i] = 0 \in \mathrm{Coker}(\tau'_k).$$

Proof. We show the lemma by induction on k. For $k=2$, we have

$$\tau_2'([K_{ii_1}, K_{ii_2}]) = x_i^* \otimes [x_i, x_{i_2}, x_{i_1}] - x_i^* \otimes [x_i, x_{i_1}, x_{i_2}] = x_i^* \otimes [x_{i_1}, x_{i_2}, x_i].$$

Assume $k \geq 3$. By the inductive hypothesis, there exists a certain $\sigma \in \mathcal{A}_n'(k-1)$ such that

$$\tau_{k-1}'(\sigma) = x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}, x_i].$$

On the other hand, we have $\tau_1'(K_{ii_k}) = x_i^* \otimes [i, i_k]$. Then

$$\begin{aligned}\tau_k'([K_{ii_k}, \sigma]) &= x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}, x_i, x_{i_k}] - x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}, [x_i, x_{i_k}]], \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_i].\end{aligned}$$

This completes the proof of Lemma 3.3. $\square$

In general for any $x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}}]$, we may assume $i_1 \neq i_2$. Then from the lemmas above, we see that for $n \geq k+1$,

$$\mathfrak{F}' := \{x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}}] \in \mathfrak{F} \mid i_2, i_{k+1} \neq i\}$$

generates $\mathrm{Coker}(\tau_k')$.

Lemma 3.4. *For $n \geq k+2$, if $i \neq i_2, \ldots, i_{k+1}$, then for any $j \in \{1, 2, \ldots, n\}$ such that $j \neq i_l$, i,*

$$x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] = x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_2}, \ldots, x_{i_k}] \in \mathrm{Coker}(\tau_k').$$

Proof. From Lemma 3.1, there exists a certain $\sigma' \in \mathcal{A}_n'(k-1)$ such that

$$\tau_{k-1}'(\sigma') = x_j^* \otimes [x_i, x_{i_2}, \ldots, x_{i_k}].$$

Then, we obtain,

$$\tau_k'([K_{iji_{k+1}}, \sigma']) = x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_2}, \ldots, x_{i_k}].$$

This completes the proof of Lemma 3.4. $\square$

Lemma 3.5. *For $n \geq k+1$, if $i \neq i_2, \ldots, i_{k+1}$, then*

$$x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] = x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}] \in \mathrm{Coker}(\tau_k').$$

Proof. Since $n \geq k+1$, there exists a certain $j \in \{1, 2, \ldots, n\}$ such that $j \neq i, i_l$ for $3 \leq l \leq k+1$. From Lemma 3.4, there exists a certain $\sigma \in \mathcal{A}_n'(k-1)$ such that

$$\tau_{k-1}'(\sigma) = x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_3}, \ldots, x_{i_k}].$$

Then we have

$$\begin{aligned}\tau_k'([\sigma, K_{ii_2}]) &= x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}] \\ &\quad - \delta_{i_2, j} x_i^* \otimes [x_j, x_{i_{k+1}}, x_{i_3}, \ldots, x_{i_k}, x_i], \\ &= x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}]\end{aligned}$$

in $\mathrm{Coker}(\tau_k')$. This completes the proof of Lemma 3.5. $\square$

Lemma 3.6. *For $n \geq k+2$, if $i \neq i_1, i_2, i_4, \ldots, i_{k+1}$, then for any $j \in \{1, 2, \ldots, n\}$ such that $j \neq i_l$, i,*

$$\begin{aligned}x_i^* \otimes &[x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}] \\ &= x_j^* \otimes [x_j, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_2}, x_{i_1}] - x_j^* \otimes [x_j, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_1}, x_{i_2}] \in \mathrm{Coker}(\tau_k').\end{aligned}$$

Proof. From Lemma 3.1, there exist certain $\sigma, \sigma' \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-2}(\sigma) = x_i^* \otimes [x_j, x_{i_4}, \dots, x_{i_{k+1}}],$$
$$\tau'_2(\sigma') = x_j^* \otimes [x_{i_1}, x_{i_2}, x_i].$$

Then, using the Jacobi identity, we have

$$\begin{aligned}\tau'_k([\sigma, \sigma']) &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \dots, x_{i_{k+1}}] - x_j^* \otimes [[x_{i_1}, x_{i_2}], [x_j, x_{i_4}, \dots, x_{i_{k+1}}]],\\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \dots, x_{i_{k+1}}]\\ &\qquad - x_j^* \otimes [x_j, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_2}, x_{i_1}] + x_j^* \otimes [x_j, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_1}, x_{i_2}].\end{aligned}$$

This completes the proof of Lemma 3.6. □

Lemma 3.7. *For $n \geq k+2$, if $i \neq i_1, i_2, \dots, i_{k-1}, i_{k+1}$, then for any $j \in \{1, 2, \dots, n\}$ such that $j \neq i_l$, i,*

$$\begin{aligned}&x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_i, x_{i_{k+1}}]\\ &\qquad = x_j^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_j, x_{i_{k+1}}] \in \mathrm{Coker}(\tau'_k).\end{aligned}$$

Proof. From Lemma 3.1, there exists a certain $\sigma' \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1}(\sigma') = x_j^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_i].$$

Then, we obtain

$$\begin{aligned}\tau'_k([K_{iji_{k+1}}, \sigma']) &= x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_i, x_{i_{k+1}}] - x_j^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, [x_j, x_{i_{k+1}}]],\\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_i, x_{i_{k+1}}]\\ &\qquad - x_j^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_j, x_{i_{k+1}}] + x_j^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_{i_{k+1}}, x_j].\end{aligned}$$

From Lemma 3.3, we obtain Lemma 3.7. □

Next, we consider the case where $k = 3$ and 4.

Lemma 3.8. *For $n \geq 3$, if $i \neq i_1, i_2, i_4$, then*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] = x_i^* \otimes [x_i, x_{i_4}, x_{i_2}, x_{i_1}] - x_i^* \otimes [x_i, x_{i_4}, x_{i_1}, x_{i_2}] \in \mathrm{Coker}(\tau'_3).$$

Proof. From Lemma 3.3, there exists a certain $\sigma \in \mathcal{A}'_n(2)$ such that

$$\tau'_2(\sigma) = x_i^* \otimes [x_{i_1}, x_{i_2}, x_i].$$

Then, we obtain

$$\begin{aligned}\tau'_3([K_{ii_4}, \sigma]) &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] - x_i^* \otimes [x_{i_1}, x_{i_2}, [x_i, x_{i_4}]],\\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] + x_i^* \otimes [x_i, x_{i_4}, [x_{i_1}, x_{i_2}]],\\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] - x_i^* \otimes [x_i, x_{i_4}, x_{i_2}, x_{i_1}] + x_i^* \otimes [x_i, x_{i_4}, x_{i_1}, x_{i_2}].\end{aligned}$$

This completes the proof of Lemma 3.8. □

Lemma 3.9. *For $n \geq 4$, if $i \neq i_1, i_2, i_4$, then*

(1) $x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] - x_i^* \otimes [x_{i_2}, x_{i_4}, x_i, x_{i_1}] = 0 \in \mathrm{Coker}(\tau'_3)$,
(2) $x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] + x_i^* \otimes [x_{i_1}, x_{i_4}, x_i, x_{i_2}] = 0 \in \mathrm{Coker}(\tau'_3)$.

Proof. For the part (1), from Lemma 3.8, we have

$$x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] = x_i^* \otimes [x_i, x_{i_4}, x_{i_2}, x_{i_1}] - x_i^* \otimes [x_i, x_{i_4}, x_{i_1}, x_{i_2}],$$
$$x_i^* \otimes [x_{i_2}, x_{i_4}, x_i, x_{i_1}] = x_i^* \otimes [x_i, x_{i_1}, x_{i_4}, x_{i_2}] - x_i^* \otimes [x_i, x_{i_1}, x_{i_2}, x_{i_4}]$$

in $\mathrm{Coker}(\tau_3')$. From Lemma 3.5, we obtain the part (1). Similarly, we obtain the part (2). $\square$

Lemma 3.10. *If* $n \geq 6$, $i_3 \neq i_5$ *and* $i \neq i_1, i_2, i_3, i_5$,

$$\begin{aligned} x_i^*\otimes&[x_{i_1}, x_{i_2}, x_{i_3}, x_i, x_{i_5}] \\ &= x_i^* \otimes [x_{i_3}, x_{i_5}, x_i, x_{i_1}, x_{i_2}] - x_i^* \otimes [x_{i_3}, x_{i_5}, x_i, x_{i_2}, x_{i_1}] \in \mathrm{Coker}(\tau_4'). \end{aligned}$$

Proof. Since $n \geq 6$, there exists a certain j such that $j \neq i_l, i$. From Lemma 3.9, there exists some $\sigma \in \mathcal{A}_n'(3)$ such that

$$\tau_3'(\sigma) = x_i^* \otimes [x_j, x_{i_3}, x_i, x_{i_5}] - x_i^* \otimes [x_{i_3}, x_{i_5}, x_i, x_j].$$

Then,

$$\begin{aligned} \tau_4'([\sigma, K_{ji_1i_2}]) &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_i, x_{i_5}] - x_i^* \otimes [x_{i_3}, x_{i_5}, x_i, [x_{i_1}, x_{i_2}]], \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_i, x_{i_5}] \\ &\qquad + x_i^* \otimes [x_{i_3}, x_{i_5}, x_i, x_{i_2}, x_{i_1}] - x_i^* \otimes [x_{i_3}, x_{i_5}, x_i, x_{i_1}, x_{i_2}]. \end{aligned}$$

This completes the proof of Lemma 3.10. $\square$

Using the Lemmas above, we show the main proposition of this section.

Proposition 3.1. *For* $n \geq 6$, $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$ *is generated by* r *elements.*

Proof. To begin with, we recall that $\mathfrak{F}'$ generates $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$. First, from Lemma 3.5, if $i \neq i_2, \ldots, i_5$, then

$$\begin{aligned} x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] &= x_i^* \otimes [x_i, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_2}] \\ &= x_i^* \otimes [x_i, x_{i_4}, x_{i_5}, x_{i_2}, x_{i_3}] = x_i^* \otimes [x_i, x_{i_5}, x_{i_2}, x_{i_3}, x_{i_4}]. \end{aligned}$$

Furtheremore, from Lemma 3.4, these elements do not depend on the choice of i such that $i \neq i_l$. Hence we can set

$$s(i_2, i_3, i_4, i_5) := x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] \in \mathrm{Coker}(\tau'_{4,\mathbf{Q}}).$$

Similarly, from Lemma 3.7, if $i \neq i_1, i_2, i_3, i_5$, we can set

$$t(i_1, i_2, i_3, i_5) := x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_i, x_{i_5}] \in \mathrm{Coker}(\tau'_{4,\mathbf{Q}}),$$

and from Lemma 3.6, if $i \neq i_1, i_2, i_4, i_5$, we can set

$$\begin{aligned} u(i_1, i_2, i_4, i_5) &:= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, x_{i_5}], \\ &= s(i_4, i_5, i_2, i_1) - s(i_4, i_5, i_1, i_2) \in \mathrm{Coker}(\tau'_{4,\mathbf{Q}}). \end{aligned}$$

Then we have

$$t(i_1, i_2, i_3, i_5) = -t(i_2, i_1, i_3, i_5), \tag{4}$$
$$t(i_1, i_2, i_3, i_5) = t(i_3, i_5, i_1, i_2), \tag{5}$$
$$t(i_1, i_2, i_3, i_5) - t(i_1, i_3, i_2, i_5) + t(i_1, i_5, i_2, i_3) = 0. \tag{6}$$

The equation (4) is trivial. Since from Lemma 3.10,

$$\begin{aligned} t(i_1,i_2,i_3,i_5) &= u(i_3,i_5,i_1,i_2) - u(i_3,i_5,i_2,i_1) \\ &= s(i_1,i_2,i_5,i_3) - s(i_1,i_2,i_3,i_5) - s(i_2,i_1,i_5,i_3) + s(i_2,i_1,i_3,i_5), \\ t(i_3,i_5,i_1,i_2) &= u(i_1,i_2,i_3,i_5) - u(i_1,i_2,i_5,i_3), \\ &= s(i_3,i_5,i_2,i_1) - s(i_3,i_5,i_1,i_2) - s(i_5,i_3,i_2,i_1) + s(i_5,i_3,i_1,i_2), \end{aligned}$$

we obtain (5), and similarly (6). In particular, from (4) and (5),

$$t(i_1,i_2,i_3,i_5) = -t(i_1,i_2,i_5,i_3) \tag{7}$$

and $t(i_1,i_2,i_3,i_3) = 0 \in \mathrm{Coker}(\tau'_{4,\mathbf{Q}})$. Using these relatons we see that any $t(i_1,i_2,i_3,i_5)$ is contained in the subvector space V of $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$ generated by

$$\{t(i_1,i_2,i_3,i_5) \,|\, i_1 < i_2 \leq i_5,\ i_1 \leq i_3 < i_5\}$$

which consists of $n^2(n^2-1)/12$ elements. In fact, for any $t(i_1,i_2,i_3,i_5)$, using (4) and (5), we may assume $i_1 \leq i_2, i_3, i_5$. If $i_3 = i$ or $i_5 = i$, using (5) if necessary, we obtain $t(i_1,i_2,i_3,i_5) \in V$. If $i_3, i_5 \neq i$, using (6) if necessary, we see that $t(i_1,i_2,i_3,i_5)$ is written as a linear combination of $t(j_1,j_2,j_3,j_5)$ for $j_1 < j_2 \leq j_3, j_5$. Then using (7), we obtain $t(i_1,i_2,i_3,i_5) \in V$.

Next we consider the quotient vector space $\overline{V} := \mathrm{Coker}(\tau'_{4,\mathbf{Q}})/V$. We write $\doteq$ for the equality in $\overline{V}$ to distinguish from that in $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$. The quotient space $\overline{V}$ is generated by $s(i_2,i_3,i_4,i_5)$s and $u(i_1,i_2,i_4,i_5)$s. In $\overline{V}$, we have

$$u(i_1,i_2,i_4,i_5) \doteq -u(i_2,i_1,i_4,i_5), \tag{8}$$

$$u(i_1,i_2,i_4,i_5) \doteq u(i_1,i_2,i_5,i_4), \tag{9}$$

$$u(i_1,i_2,i_4,i_5) + u(i_1,i_4,i_5,i_2) + u(i_1,i_5,i_2,i_4) \doteq 0. \tag{10}$$

The equation (8) is trivial, and (9) follows from Lemma 3.10. Similar to (6), we obtain (10). From (9) and (10), if we set $i_2 = i_4$,

$$2u(i_1,i_2,i_2,i_5) \doteq -u(i_1,i_5,i_2,i_2), \tag{11}$$

and if we set $i_2 = i_4 = i_5$,

$$u(i_1,i_2,i_2,i_2) \doteq 0 \tag{12}$$

respectively. Using these relatons we see that any $u(i_1,i_2,i_4,i_5)$ is contained in the subvector space W of $\overline{V}$ generated by

$$\{u(i_1,i_2,i_4,i_5) \,|\, i_1 > i_2 > i_4 \leq i_5\}$$

which consists of $n(n+1)(n-1)(n-2)/8$ elements. In fact, for any $u(i_1,i_2,i_4,i_5)$ we may assume $i_1 > i_2$ by using (8). If $i_2 > i_4$ or $i_2 > i_5$, then $u(i_1,i_2,i_4,i_5) \in W$ by (9). Assume $i_4, i_5 \geq i_2$. If $i_4 = i_2$ or $i_5 = i_2$, using (11), (12) and (8) if necessary, we see $u(i_1,i_2,i_4,i_5) \in W$. If $i_4, i_5 \neq i_2$, using (10), (8) and (9), we also see $u(i_1,i_2,i_4,i_5) \in W$.

Finally, we consider the quotient vector space $\overline{W} := \overline{V}/W$. We write $\mathrel{\overset{\circ}{=}}$ for the equality in $\overline{W}$ to distinguish from that in $\overline{V}$. The quotient space $\overline{W}$ is generated by $s(i_2,i_3,i_4,i_5)$s. In $\overline{W}$, we have

$$s(i_2,i_3,i_4,i_5) \mathrel{\overset{\circ}{=}} s(i_2,i_3,i_5,i_4)$$

from Lemma 3.6. This shows that for any element γ of the symmetric group $\mathfrak{S}_4$ of degree 4,

$$s(i_{\gamma(1)}, i_{\gamma(2)}, i_{\gamma(3)}, i_{\gamma(4)}) \overset{\circ}{=} s(i_1, i_2, i_3, i_4).$$

Hence $\overline{W}$ is generated by

$$\{s(i_1, i_2, i_3, i_4) \,|\, i_1 \leq i_2 \leq i_3 \leq i_4\}$$

which consists of $n(n+1)(n+2)(n+3)/24$ elements.

Therefore we conclude that $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$ is generated by r elements. This completes the proof of Proposition 3.1. $\square$

This proposition shows that r is an upper bound of the dimension of $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$ as a $\mathbf{Q}$-vector space. In the next section, we show that it is just r.

4. Trace maps

In this section we consider to detect the irreducible $\mathrm{GL}(n,\mathbf{Z})$-module $S^4 H_{\mathbf{Q}}$, $H_{\mathbf{Q}}^{[2,1^2]}$ and $H_{\mathbf{Q}}^{[2,2]}$ in $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$. To do this we construct $\mathrm{GL}(n,\mathbf{Z})$-homomorphisms called trace maps. Here we use some basic facts of the representation theory of $\mathrm{GL}(n,\mathbf{Z})$. The reader is referred to, for example, Fulton-Harris [6] and Fulton [5].

To begin with, for $k \geq 1$ and $1 \leq l \leq k+1$, let $\varphi_l^k : H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$ be the contraction defined by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_l}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_{l-1}} \otimes x_{j_{l+1}} \otimes \cdots \otimes x_{j_{k+1}}.$$

For the natural embedding $\iota^{k+1} : \mathcal{L}_n(k+1) \to H^{\otimes(k+1)}$, we obtain a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism

$$\Phi_l^k = \varphi_l^k \circ (id_{H^*} \otimes \iota^{k+1}) : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We also call the map Φ_l^k contraction. We often write Φ_l for Φ_l^k for simplicity.

Next, we consdier the trace maps. For each $k \geq 1$, and any partition λ of k, we denote by H^λ the Schur-Weyl module of H corresponding to the partition λ of k. For example, the modules $H^{[k]}$ and $H^{[1^k]}$ are the symmetric product $S^k H$ and the exterior product $\Lambda^k H$ respectively. Let $f_\lambda : H^{\otimes k} \to H^\lambda$ be a natural projection. Using the contractions Φ_l^k and the projections f_λ, we obtain a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism from the target of the Johnson homomorphisms to the Schur-Weyl module H^λ.

The most important homomorphism is

$$\mathrm{Tr}_{[k]} = f_{[k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H,$$

called the Morita trace, where $f_{[k]} : H^{\otimes k} \to S^k H$ is a natural projection defiend by

$$f_{[k]}(x_{i_1} \otimes \cdots \otimes x_{i_k}) = x_{i_1} \cdots x_{i_k}.$$

The Morita trace was introduced with remarkable pioneer works by Shigeyuki Morita who showed that $\mathrm{Tr}_{[k]}$ is surjective and vanishes on the image of the Johnson homomorphism τ_k for $n \geq 3$ and $k \geq 2$. This shows that $S^k H_{\mathbf{Q}}$ appears in the irreducible decomposition of $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$ and $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ as a $\mathrm{GL}(n,\mathbf{Z})$-module.

Second, we consider

$$\mathrm{Tr}_{[2,1^{k-2}]} := (id_H \otimes f_{[1^{k-1}]}) \circ \Phi_2^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H \otimes_{\mathbf{Z}} \Lambda^{k-1} H,$$

called the trace map for $H_{\mathbf{Q}}^{[2,1^{k-2}]}$, where $f_{[1^k]} : H^{\otimes k} \to \Lambda^k H$ is a natural projection defiend by

$$f_{[1^k]}(x_{i_1} \otimes \cdots \otimes x_{i_k}) = x_{i_1} \wedge \cdots \wedge x_{i_k}.$$

Let I be the $\mathrm{GL}(n, \mathbf{Z})$-submodule of $H \otimes_{\mathbf{Z}} \Lambda^{k-1} H$ defined by

$$I = \langle x \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge y + y \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge x \mid x, y, z_t \in H \rangle.$$

In our previous paper [23], we showed that $\mathrm{Im}\,(\mathrm{Tr}^{\mathbf{Q}}_{[2,1^{k-1}]}) = I_{\mathbf{Q}}$ and $\mathrm{Tr}_{[2,1^{k-2}]}$ vanishes on τ'_k for any even $k \geq 4$ and $n \geq k+1$. Now, using Pieri's formula (See [6], for example.), we have $H_{\mathbf{Q}} \otimes_{\mathbf{Z}} \Lambda^{k-1} H_{\mathbf{Q}} \cong H_{\mathbf{Q}}^{[2,1^{k-2}]} \oplus \Lambda^k H_{\mathbf{Q}}$. For even k, since $I_{\mathbf{Q}}$ is contained in the kernel of a natural map $H_{\mathbf{Q}} \otimes_{\mathbf{Z}} \Lambda^{k-1} H_{\mathbf{Q}} \to \Lambda^k H_{\mathbf{Q}}$ defined by

$$x \otimes y_1 \wedge \cdots \wedge y_{k-1} \mapsto x \wedge y_1 \wedge \cdots \wedge y_{k-1},$$

we obtain $I_{\mathbf{Q}} \cong H_{\mathbf{Q}}^{[2,1^{k-2}]}$. This shows that $H_{\mathbf{Q}}^{[2,1^{k-2}]}$ appears in the irreducible decomposition of $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ as a $\mathrm{GL}(n, \mathbf{Z})$-module for any even $k \geq 4$ and $n \geq k+1$.

Next we consider to detect $H_{\mathbf{Q}}^{[2,2]}$ in $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$. To begin with, we remark that $H^{[2,2]}$ is a free abelian group of rank $n^2(n^2-1)/12$. In this paper we identify $H^{[2,2]}$ with a quotient of $\Lambda^2 H \otimes \Lambda^2 H$ by the submodule generated by

- $(v \wedge w) \otimes (x \wedge y) - (x \wedge y) \otimes (v \wedge w)$,
- $(v \wedge w) \otimes (x \wedge y) - (x \wedge w) \otimes (v \wedge y) - (v \wedge x) \otimes (w \wedge y)$

for any $v, w, x, y \in H$. (For details, see [5].) In $H^{[2,2]}$, we write $(a \wedge b) \cdot (c \wedge d)$ for the coset class of $(a \wedge b) \otimes (c \wedge d)$ for simplicity. Then a basis of $H^{[2,2]}$ is given by

$$\{(i_1 \wedge i_2) \cdot (i_3 \wedge i_5) \,|\, i_1 < i_2 \leq i_5,\ i_1 \leq i_3 < i_5\}.$$

For $i = 1, 2$, let $f_i : H^{\otimes 4} \to H^{[2,2]}$ be a projection defined by

$$f_i(a \otimes b \otimes c \otimes d) = \begin{cases} (a \wedge c) \cdot (b \wedge d), & i = 1, \\ (a \wedge d) \cdot (b \wedge c), & i = 2. \end{cases}$$

Then set

$$\mathrm{Tr}_{[2,2]} := f_1 \circ \Phi_1^4 - 2(f_2 \circ \Phi_1^4) : H^* \otimes_{\mathbf{Z}} H^{\otimes 5} \to H^{[2,2]}.$$

We call it the trace map for $H^{[2,2]}$.

Proposition 4.1. *For $n \geq 3$, $\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}$ is surjective.*

Proof. For any i, i_1, i_2, i_3, i_5, we have

$$\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}(i^* \otimes [i_1, i_2, i_3, i, i_5]) = -6(i_1 \wedge i_2) \cdot (i_3 \wedge i_5).$$

In general, this element is non-trivial in $H_{\mathbf{Q}}^{[2,2]}$. Since $H_{\mathbf{Q}}^{[2,2]}$ is a irreducible $\mathrm{GL}(n, \mathbf{Z})$-module, we see $\mathrm{Im}(\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}) = H_{\mathbf{Q}}^{[2,2]}$. $\square$

To show that $\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}$ vanishes on the image of $\tau'_{4,\mathbf{Q}}$, we need to prepare generators of $\mathrm{Im}(\tau'_{4,\mathbf{Q}})$. First, we consider generators of $\mathrm{Im}(\tau'_{3,\mathbf{Q}})$. Let $\mathfrak{C}$ be a subset of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(4)$ consisting of

(C1): $x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}]$,
(C2): $x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_i]$,

(C3): $x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}] - x_j^* \otimes [x_j, x_{i_3}, x_{i_4}, x_{i_2}]$,
(C4): $x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}] - x_i^* \otimes [x_i, x_{i_3}, x_{i_4}, x_{i_2}]$,
(C5): $x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] - x_j^* \otimes [x_j, x_{i_4}, x_{i_2}, x_{i_1}] + x_j^* \otimes [x_j, x_{i_4}, x_{i_1}, x_{i_2}]$,
(C6): $x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_i] - x_j^* \otimes [x_j, x_i, x_{i_2}, x_{i_1}] + x_j^* \otimes [x_j, x_i, x_{i_1}, x_{i_2}]$,
(C7): $x_i^* \otimes [x_i, x_{i_2}, x_i, x_{i_4}] - 2x_j^* \otimes [x_j, x_{i_4}, x_{i_2}, x_i] + x_j^* \otimes [x_j, x_{i_4}, x_i, x_{i_2}]$,
(C8): $x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_i] - x_j^* \otimes [x_j, x_i, x_{i_2}, x_{i_3}]$,
(C9): $x_i^* \otimes [x_i, x_{i_2}, x_i, x_i] - 2x_j^* \otimes [x_j, x_i, x_{i_2}, x_i] + x_j^* \otimes [x_j, x_i, x_i, x_{i_2}]$

where $i \neq i_l$ and $j \neq i, i_l$ in each of elements above.

Lemma 4.1. *For $n \geq 5$, $\mathrm{Im}(\tau'_{3,\mathbf{Q}})$ is generated by $\mathfrak{C}$.*

Proof. First we show (C1), ... , (C9) belong to $\mathrm{Im}(\tau'_{3,\mathbf{Q}})$. From Lemmas 3.1, 3.3, 3.4 and 3.5, we obtain (C1), (C2), (C3), (C4) $\in \mathrm{Im}(\tau'_{3,\mathbf{Q}})$ respectively. Furthermore, using an argument similar to that in Lemma 3.6, we obtain (C5), (C6), (C7), (C8), (C9) $\in \mathrm{Im}(\tau'_{3,\mathbf{Q}})$. The details are left to the reader as exercises.

Next we consider the quotient space D of $H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_n^{\mathbf{Q}}(4)$ by a subspace generated by (C1), ... , (C9). From (C1), (C2), (C5), ... , (C9), D is generated by $\{x_i^* \otimes [x_i, x_{i_1}, x_{i_2}, x_{i_3}] \,|\, i_l \neq i\}$. Using (C4), if $i \neq i_2, i_3, i_4$, then

$$x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}] = x_i^* \otimes [x_i, x_{i_3}, x_{i_4}, x_{i_2}] = x_i^* \otimes [x_i, x_{i_4}, x_{i_2}, x_{i_3}].$$

Furtheremore, from (C3), these elements do not depend on the choice of i such that $i \neq i_l$. Hence we can set

$$s(i_2, i_3, i_4) := x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}] \in D.$$

Then it is easily seen that D is generated by

$$\{s(j_1, j_2, j_3) \,|\, j_1 < j_2, j_3\} \cup \{s(j_1, j_1, j_2) \,|\, j_1 \leq j_2\}$$

which consists of $n(n^2+2)/3$ elements. On the other hand, by our previous result, the dimension of $\mathrm{Coker}(\tau'_{3,\mathbf{Q}})$ as a $\mathbf{Q}$-vector space is just $n(n^2+2)/3$. (See [23].) This shows that $D = \mathrm{Im}(\tau'_{3,\mathbf{Q}})$. $\square$

Since $\tau'_{\mathbf{Q}} := \bigoplus_{k \geq 1} \tau'_{k,\mathbf{Q}}$ is a graded Lie algebra homomorphism, from Lemma 4.1 and (2), we see that $\mathrm{Im}(\tau'_{4,\mathbf{Q}})$ is generated by

$$\mathfrak{D} := \{[f, x_p^* \otimes [x_p, x_q]],\ [f, x_p^* \otimes [x_q, x_r]] \,|\, f \in \mathfrak{C},\ \ p \neq q \neq r \neq p\}$$

where the bracket of the outeside in the elements above means the Lie bracket of the derivation algebra $\mathrm{Der}(\mathcal{L}_n)$.

Proposition 4.2. *For $n \geq 6$, $\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}$ vanishes on $\mathrm{Im}(\tau'_{4,\mathbf{Q}})$.*

Proof. It suffices to show that $\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}$ vanishes on $\mathfrak{D}$. We show this by direct computation. Since there are too much computation, we give a few examples of them here.

Step I.

First we show $\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([f, x_p^* \otimes [x_p, x_q]]) = 0$ and $\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([f, x_p^* \otimes [x_q, x_r]]) = 0$ for $f =$ (C1), ... , (C5). Then using these results, we show the other cases. To begin with, observe

$$\begin{aligned}
[(\mathrm{C1}), x_p^* \otimes [x_q, x_r]] =& \delta_{pi_1} i^* \otimes [x_q, x_r, x_{i_2}, x_{i_3}, x_{i_4}] + \delta_{pi_2} x_i^* \otimes [x_{i_1}, [x_q, x_r], x_{i_3}, x_{i_4}] \\
&+ \delta_{pi_3} x_i^* \otimes [x_{i_1}, x_{i_2}, [x_q, x_r], x_{i_4}] + \delta_{pi_4} x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, [x_q, x_r]] \\
&- \delta_{qi} x_p^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_r] + \delta_{ri} x_p^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_q].
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([(\mathrm{C1}), x_p^* \otimes [x_q, x_r]]) \\
&\quad = \delta_{pi_1}\delta_{qi}((x_r \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_{i_4}) - (x_r \wedge x_{i_4}) \cdot (x_{i_2} \wedge x_{i_3})) \\
&\quad + \delta_{pi_1}\delta_{ri}(-(x_q \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_{i_4}) + (x_q \wedge x_{i_4}) \cdot (x_{i_2} \wedge x_{i_3})) \\
&\quad + \delta_{pi_2}\delta_{qi}(-(x_r \wedge x_{i_1}) \cdot (x_{i_3} \wedge x_{i_4}) + (x_r \wedge x_{i_4}) \cdot (x_{i_1} \wedge x_{i_3})) \\
&\quad + \delta_{pi_2}\delta_{ri}((x_q \wedge x_{i_1}) \cdot (x_{i_3} \wedge x_{i_4}) - (x_q \wedge x_{i_4}) \cdot (x_{i_1} \wedge x_{i_3})) \\
&\quad + \delta_{pi_3}\delta_{qi}(3(x_{i_1} \wedge x_{i_2}) \cdot (x_r \wedge x_{i_4})) + \delta_{pi_3}\delta_{ri}(-3(x_{i_1} \wedge x_{i_2}) \cdot (x_q \wedge x_{i_4})) \\
&\quad + \delta_{pi_4}\delta_{qi}(-6(x_{i_1} \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_r)) + \delta_{pi_4}\delta_{ri}(6(x_{i_1} \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_q)) \\
&\\
&\quad + \delta_{qi}\delta_{pi_1}(-(x_{i_2} \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_r) + (x_{i_2} \wedge x_r) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\quad + \delta_{qi}\delta_{pi_2}((x_{i_1} \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_r) - (x_{i_1} \wedge x_r) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\quad + \delta_{qi}\delta_{pi_3}(3(x_{i_1} \wedge x_{i_2}) \cdot (x_{i_4} \wedge x_r)) + \delta_{qi}\delta_{pi_4}(6(x_{i_1} \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_r)) \\
&\\
&\quad + \delta_{ri}\delta_{pi_1}((x_{i_2} \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_q) - (x_{i_2} \wedge x_q) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\quad + \delta_{ri}\delta_{pi_2}(-(x_{i_1} \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_q) + (x_{i_1} \wedge x_q) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\quad + \delta_{ri}\delta_{pi_3}(-3(x_{i_1} \wedge x_{i_2}) \cdot (x_{i_4} \wedge x_q)) + \delta_{ri}\delta_{pi_4}(-6(x_{i_1} \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_q)) \\
&=0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
[(\mathrm{C3}), x_p^* \otimes [x_p, x_q]] =& \delta_{pi} i^* \otimes [x_i, x_q, x_{i_2}, x_{i_3}, x_{i_4}] + \delta_{pi_2} x_i^* \otimes [x_i, [x_{i_2}, x_q], x_{i_3}, x_{i_4}] \\
&+ \delta_{pi_3} x_i^* \otimes [x_i, x_{i_2}, [x_{i_3}, x_q], x_{i_4}] + \delta_{pi_4} x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, [x_{i_4}, x_q]] \\
&- \delta_{pj} j^* \otimes [x_j, x_q, x_{i_3}, x_{i_4}, x_{i_2}] - \delta_{pi_3} x_j^* \otimes [x_j, [x_{i_3}, x_q], x_{i_4}, x_{i_2}] \\
&- \delta_{pi_4} x_j^* \otimes [x_j, x_{i_3}, [x_{i_4}, x_q], x_{i_2}] - \delta_{pi_2} x_j^* \otimes [x_j, x_{i_3}, x_{i_4}, [x_{i_2}, x_q]] \\
&- \delta_{pi} x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}, x_q] + \delta_{qi} x_p^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}, x_p] \\
&+ \delta_{pj} x_j^* \otimes [x_j, x_{i_3}, x_{i_4}, x_{i_2}, x_q] - \delta_{qj} x_p^* \otimes [x_j, x_{i_3}, x_{i_4}, x_{i_2}, x_p]
\end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([(\mathrm{C3}), x_p^* \otimes [x_p, x_q]]) \\
&\qquad = \delta_{pi}((x_q \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_{i_4}) - (x_q \wedge x_{i_4}) \cdot (x_{i_2} \wedge x_{i_3})) \\
&\qquad + \delta_{pi_2}(3(x_{i_2} \wedge x_q) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\qquad + \delta_{pi_2}\delta_{qi}((x_{i_2} \wedge x_i) \cdot (x_{i_3} \wedge x_{i_4}) - (x_{i_2} \wedge x_{i_4}) \cdot (x_i \wedge x_{i_3})) \\
&\qquad + \delta_{pi_3}(-3(x_{i_2} \wedge x_{i_4}) \cdot (x_{i_3} \wedge x_q)) + \delta_{pi_3}\delta_{qi}(-3(x_i \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\qquad + \delta_{pi_4}(3(x_{i_2} \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_q)) + \delta_{pi_4}\delta_{qi}(6(x_i \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\qquad \\
&\qquad - \delta_{pj}((x_q \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_{i_2}) - (x_q \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\qquad - \delta_{pi_3}(3(x_{i_3} \wedge x_q) \cdot (x_{i_4} \wedge x_{i_2})) \\
&\qquad - \delta_{pi_3}\delta_{qi}((x_{i_3} \wedge x_j) \cdot (x_{i_4} \wedge x_{i_2}) - (x_{i_3} \wedge x_{i_2}) \cdot (x_j \wedge x_{i_4})) \\
&\qquad - \delta_{pi_4}(-3(x_{i_3} \wedge x_{i_2}) \cdot (x_{i_4} \wedge x_q)) - \delta_{pi_4}\delta_{qi}(-3(x_j \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_{i_2})) \\
&\qquad - \delta_{pi_2}(3(x_{i_3} \wedge x_{i_4}) \cdot (x_{i_2} \wedge x_q)) - \delta_{pi_2}\delta_{qi}(6(x_j \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_{i_2})) \\
&\qquad \\
&\qquad - \delta_{pi}((x_{i_2} \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_q) - (x_{i_2} \wedge x_q) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\qquad + \delta_{qi}\delta_{pi_2}(-(x_i \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_{i_2}) + (x_i \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\qquad + \delta_{qi}\delta_{pi_3}(-3(x_i \wedge x_{i_2}) \cdot (x_{i_4} \wedge x_{i_3})) + \delta_{qi}\delta_{pi_4}(-6(x_i \wedge x_{i_2}) \cdot (x_{i_3} \wedge x_{i_4})) \\
&\qquad \\
&\qquad + \delta_{pj}((x_{i_3} \wedge x_{i_4}) \cdot (x_{i_2} \wedge x_q) - (x_{i_3} \wedge x_q) \cdot (x_{i_4} \wedge x_{i_2})) \\
&\qquad - \delta_{qj}\delta_{pi_3}(-(x_j \wedge x_{i_4}) \cdot (x_{i_2} \wedge x_{i_3}) + (x_j \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_{i_2})) \\
&\qquad - \delta_{qj}\delta_{pi_4}(-3(x_j \wedge x_{i_3}) \cdot (x_{i_2} \wedge x_{i_4})) - \delta_{qj}\delta_{pi_2}(-6(x_j \wedge x_{i_3}) \cdot (x_{i_4} \wedge x_{i_2})), \\
&\qquad = 0.
\end{aligned}
$$

By an argument similar to the above, we can show $\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([f, x_p^* \otimes [x_p, x_q]]) = 0$ and $\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([f, x_p^* \otimes [x_q, x_r]]) = 0$ for $f =$ (C1), ... , (C5). The calcilations are left to the reader as exercises.

Step II.

Next, we consider (C6)

$$x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_i] - x_j^* \otimes [x_j, x_i, x_{i_2}, x_{i_1}] + x_j^* \otimes [x_j, x_i, x_{i_1}, x_{i_2}],$$

and $[(\mathrm{C6}), x_p^* \otimes [x_p, x_q]]$. To begin with, observe

$$
\begin{aligned}
&[x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_i], x_p^* \otimes [x_p, x_q]] \\
&\quad = \delta_{pi_1} x_i^* \otimes [[x_p, x_q], x_{i_2}, x_i, x_i] + \delta_{pi_2} x_i^* \otimes [x_{i_1}, [x_p, x_q], x_i, x_i] \\
&\qquad + \delta_{pi}(x_i^* \otimes [x_{i_1}, x_{i_2}, [x_p, x_q], x_i] + x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, [x_p, x_q]]) \\
&\qquad - \delta_{pi} x_p^* \otimes [[x_{i_1}, x_{i_2}, x_i, x_i], x_q] - \delta_{qi} x_p^* \otimes [x_p, [x_{i_1}, x_{i_2}, x_i, x_i]]).
\end{aligned}
$$

Since $n \geq 6$, there exist j such that $j \neq i,\ i_1,\ i_2,\ p,\ q$. Then

$$\begin{aligned}
&\Phi_1([x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_i], x_p^* \otimes [x_p, x_q]]) \\
&\quad = \delta_{pi_1}\big(\Phi_1(x_i^* \otimes [[x_p, x_q], x_{i_2}, x_j, x_i])\big|_{j=i} + \Phi_1(x_i^* \otimes [[x_p, x_q], x_{i_2}, x_i, x_j])\big|_{j=i} \\
&\qquad\qquad - \delta_{qi}\Phi_1(x_i^* \otimes [[x_p, x_i], x_{i_2}, x_j, x_j])\big|_{j=i}\big) \\
&\qquad + \delta_{pi_2}\big(\Phi_1(x_i^* \otimes [x_{i_1}, [x_p, x_q], x_j, x_i])\big|_{j=i} + \Phi_1(x_i^* \otimes [x_{i_1}, [x_p, x_q], x_i, x_j])\big|_{j=i} \\
&\qquad\qquad - \delta_{qi}\Phi_1(x_i^* \otimes [x_{i_1}, [x_p, x_i], x_j, x_j])\big|_{j=i}\big) \\
&\qquad + \delta_{pi}\big(\Phi_1(x_i^* \otimes [x_{i_1}, x_{i_2}, [x_i, x_q], x_j])\big|_{j=i} + \Phi_1(x_i^* \otimes [x_{i_1}, x_{i_2}, [x_j, x_q], x_i])\big|_{j=i} \\
&\qquad\qquad + \Phi_1(x_i^* \otimes [x_{i_1}, x_{i_2}, x_j, [x_i, x_q]])\big|_{j=i} + \Phi_1(x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, [x_j, x_q]])\big|_{j=i}\big) \\
&\qquad - \delta_{pi}\big(\Phi_1(x_i^* \otimes [[x_{i_1}, x_{i_2}, x_i, x_j], x_q])\big|_{j=i} + \Phi_1(x_i^* \otimes [[x_{i_1}, x_{i_2}, x_j, x_i], x_q])\big|_{j=i}\big) \\
&\qquad - \delta_{qi}\big(\Phi_1(x_p^* \otimes [x_p, [x_{i_1}, x_{i_2}, x_i, x_j]])\big|_{j=i} + \Phi_1(x_p^* \otimes [x_p, [x_{i_1}, x_{i_2}, x_j, x_i]])\big|_{j=i} \\
&\qquad\qquad - \delta_{pi_1}\Phi_1(x_p^* \otimes [x_p, [x_{i_1}, x_{i_2}, x_j, x_j]])\big|_{j=i} \\
&\qquad\qquad - \delta_{pi_2}\Phi_1(x_p^* \otimes [x_p, [x_{i_1}, x_{i_2}, x_j, x_j]])\big|_{j=i}\big)
\end{aligned}$$

where $v|_{j=i}$ means an element obtained from v by rewritting x_j as x_i whenever x_j appears. It is easily seen that the element above is equal to

$$\begin{aligned}
&\Phi_1([x_i^* \otimes [x_{i_1}, x_{i_2}, x_j, x_i], x_p^* \otimes [x_p, x_q]])\big|_{j=i} \\
&\quad + \Phi_1([x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_j], x_p^* \otimes [x_p, x_q]])\big|_{j=i} \\
&\quad + \delta_{pi}\Phi_1([x_i^* \otimes [x_{i_1}, x_{i_2}, x_j, x_i], x_j^* \otimes [x_j, x_q]])\big|_{j=i} \\
&\quad + \delta_{pi}\Phi_1([x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_j], x_j^* \otimes [x_j, x_q]])\big|_{j=i} \\
&\quad - \delta_{qi}\delta_{pi_1}\Phi_1([x_i^* \otimes [x_{i_1}, x_{i_2}, x_j, x_j], x_{i_1}^* \otimes [x_j, x_i]])\big|_{j=i} \\
&\quad - \delta_{qi}\delta_{pi_2}\Phi_1([x_i^* \otimes [x_{i_1}, x_{i_2}, x_j, x_j], x_{i_2}^* \otimes [x_j, x_i]])\big|_{j=i}.
\end{aligned}$$

Hence, using the results obtained in Step I, we see

$$\begin{aligned}
&\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_i], x_p^* \otimes [x_p, x_q]]) \\
&\quad = \mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([x_j^* \otimes [x_j, x_i, x_{i_2}, x_{i_1}], x_p^* \otimes [x_p, x_q]]) \\
&\qquad - \mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([x_j^* \otimes [x_j, x_i, x_{i_1}, x_{i_2}], x_p^* \otimes [x_p, x_q]]).
\end{aligned}$$

This shows

$$\mathrm{Tr}^{\mathbf{Q}}_{[2,2]}([(\mathrm{C6}), x_p^* \otimes [x_p, x_q]] = 0.$$

By an argument similar to that in the above, we can show the other cases. The calculations are left to the reader as exercises. $\square$

From Propositions 3.1 and 4.2, we obtain

Theorem 4.1. *For any $n \geq 6$,*

$$\mathrm{Coker}(\tau'_{4,\mathbf{Q}}) = S^4 H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[2,1^2]} \oplus H_{\mathbf{Q}}^{[2,2]}.$$

Finally, we give a lower bound on the rank of the fourth graded quotient of the Johnson filtration of $\mathrm{Aut}\, F_n$.

Corollary 4.1. *For $n \geq 6$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^4(\mathcal{A}_n)) \geq \frac{1}{5}n^2(n^4-1) - \frac{1}{4}n(n+1)(n^2-n+2).$$

Proof. In general, since $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for any $k \geq 1$, it follows from

$$\begin{aligned}\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^4(\mathcal{A}_n)) &= \mathrm{rank}_{\mathbf{Z}}(\mathrm{Im}(\tau_4)) \geq \mathrm{rank}_{\mathbf{Z}}(\mathrm{Im}(\tau'_4)) \\ &= \dim_{\mathbf{Q}}(\mathrm{Im}(\tau'_{4,\mathbf{Q}})) \\ &= \dim_{\mathbf{Q}}(H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}^{\mathbf{Q}}_n(5)) - \dim_{\mathbf{Q}}(\mathrm{Coker}(\tau'_{4,\mathbf{Q}})).\end{aligned}$$

This completes the proof of Corollary 4.1. $\square$

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