FIRST COHOMOLOGY GROUPS OF THE AUTOMORPHISM GROUP OF A FREE GROUP WITH COEFFICIENTS IN THE ABELIANIZATION OF THE IA-AUTOMORPHISM GROUP

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Abstract. We compute a twisted first cohomology group of the automorphism group of a free group with coefficients in the abelianization V of the IA-automorphism group of a free group. In particular, we show it is generated by two crossed homomorphisms constructed by the Magnus representation and the Magnus expansion due to Morita and Kawazumi respectively. Then we determine the first homology group of the automorphism group of a free group with coefficients in the dual group V^* of V.

1. Introduction

Let F_n be a free group of rank $n \geq 2$ with basis $x_1, \ldots, x_n$, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. The study of the (co)homology groups of $\mathrm{Aut}\, F_n$ has been developped for these twenty years by many authors with various kinds of methods. There are some remarkable results for the (co)homology groups of $\mathrm{Aut}\, F_n$ with trivial coefficients. For example, Gersten [6] showed $H_2(\mathrm{Aut}\, F_n, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 5$, and Hatcher and Vogtmann [7] showed $H_q(\mathrm{Aut}\, F_n, \mathbf{Q}) = 0$ for $n \geq 1$ and $1 \leq q \leq 6$, except for $H_4(\mathrm{Aut}\, F_4, \mathbf{Q}) = \mathbf{Q}$. Furthermore, recently Galatius [5] showed that the stable integral homology groups of $\mathrm{Aut}\, F_n$ are ismorphic to those of the symmetric group $\mathfrak{S}_n$ of degree n. In particular, this shows that the stable rational homology groups of $\mathrm{Aut}\, F_n$ are trivial.

On the other hand, in contrast to the (co)homology groups with trivial cofficients, those with twisted cofficients are still far from well understood. Let H be the abelianization of F_n and $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H. Then $\mathrm{Aut}\, F_n$ naturally acts on them. With respect to these cofficients, in our previous paper [20, 22], we compute the first and the rational second homology groups of $\mathrm{Aut}\, F_n$. Hatcher and Wahl [8] showed that the stable homology groups of $\mathrm{Aut}\, F_n$ with coefficients in H are trivial using the stability of the homology groups of the mapping class groups of certain 3-manifolds. There are, however, few computation for twisted (co)homology groups of $\mathrm{Aut}\, F_n$ other than the above. In our research we are interested in not only to compute twisted (co)homology groups of $\mathrm{Aut}\, F_n$ but also to find non-trivial twisted (co)homology classes of it.

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Now, let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced from the abelianization of F_n. The kernel of ρ is called the IA-automorphism group of F_n, denoted by IA_n. If we denote the abelianization of IA_n by V simply, $\mathrm{Aut}\, F_n$ naturally acts on V by the conjugation. Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [9] independently showed that V is isomorphic to $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ as a $\mathrm{GL}(n, \mathbf{Z})$-module. In particular, it is known that this abelianization is induced from the first Johnson homomorphism τ_1 of $\mathrm{Aut}\, F_n$.

By recent studies of twisted (co)homology groups of $\mathrm{Aut}\, F_n$ by Morita [14] and Kawazumi [9], we see that there are two intrinsic construction of crossed homomorphisms from $\mathrm{Aut}\, F_n$ into $V_L := V \otimes_{\mathbf{Z}} L$ for any principal ideal domain L. One is obtained from the Magnus representation of $\mathrm{Aut}\, F_n$ due to Morita [14], denoted by f_M, and the other is obtained from the Magnus expansion of F_n due to Kawazumi [9], denoted by f_K. (For details, see Section 3.) A construction of a crossed homomorphism with the Magnus representation introduced by Morita was originally appeared in his study of twisted first cohomology group of the mapping class group of a surface. Then, using his idea, we [20] showed that $H^1(\mathrm{Aut}\, F_n, H) \cong \mathbf{Z}$ is also generated by a crossed homomorphism obtained from the Magnus representation. On the other hand, it is known with a recent remarkable work by Kawazumi [9] that there is a crossed homomorphism τ_1^θ from $\mathrm{Aut}\, F_n$ into $H^{\otimes 2}$, called the first Johnson map induced by the Magnus expansion θ. Furthermore, he introduced a series of non-trivial cohomology classes h_p in $H^p(\mathrm{Aut}\, F_n, H^* \otimes_{\mathbf{Z}} H^{\otimes p+1})$ for any $p \geq 1$, and showed that h_1 is obtained from τ_1^θ.

In this paper, we determine the first stable homology group of $\mathrm{Aut}\, F_n$ with coefficients in $V^* := \mathrm{Hom}_{\mathbf{Z}}(V, \mathbf{Z})$.

Theorem 1. *(= Theorem 4.1.) For $n \geq 5$,*

$$H_1(\mathrm{Aut}\, F_n, V^*) = \mathbf{Z}^{\oplus 2}.$$

In order to show this, we compute the first cohomology groups of $\mathrm{Aut}\, F_n$ with coefficients in $V_L := V \otimes_{\mathbf{Z}} L$ for any principal ideal domain L, and show

$$H^1(\mathrm{Aut}\, F_n, V_L) = L^{\oplus 2}$$

for $n \geq 5$ using the Nielsen's presentation for $\mathrm{Aut}\, F_n$. In particular, we show that the crossed homomorphisms f_M and f_K generate $H^1(\mathrm{Aut}\, F_n, V_L)$. Furthermore, in Section 5, we see that the first Johnson homomorphism τ_1 does not extend to $\mathrm{Aut}\, F_n$ as a crossed homomorphism.

At the end of the paper, we consider some quotient groups of $\mathrm{Aut}\, F_n$. In particular, we determine the twisted first homology groups of the outer automorphism group $\mathrm{Out}\, F_n$ as follows:

Theorem 2. *(= Theorem 5.1.) For $n \geq 5$,*

$$H_1(\mathrm{Out}\, F_n, V^*) \cong \begin{cases} \mathbf{Z}, & n : \text{even}, \\ \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, & n : \text{odd}. \end{cases}$$

This paper consists of six sections. In Section 2, we fix some notation and convensions. Then we recall the Nielsen's finite presentation for the automorphism group of a free group. In Section 3, we construct two crossed homomorphisms f_M and f_K from $\mathrm{Aut}\, F_n$ into V_L for any principal ideal domain L. In Section 4, we directly compute the twisted first cohomology groups of $\mathrm{Aut}\, F_n$ using the Nielsen's presentation. Then, using the universal coefficients theorem, we determine the twisted first homology group. In Section 5, we compute the twisted first homology groups of the general linear group $\mathrm{GL}(n, \mathbf{Z})$, the outer automorphism group $\mathrm{Out}\, F_n$.

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2. Preliminaries

In this section, after fixing some notation and coneventions, we recall the Nielsen's finite presentation for the automorphism group of a free group which is used to compute the first cohomology groups of $\mathrm{Aut}\, F_n$. Then we also recall the IA-automorphism group of a free group and its abelianization.

2.1. **Notation and conventions.**

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group $\mathrm{Aut}\, F_n$ of F_n acts on F_n from the right. For any $\sigma \in \mathrm{Aut}\, F_n$ and $x \in G$, the action of σ on x is denoted by x^{σ}.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- Let L be an arbitary commutative ring. For any $\mathbf{Z}$-module M, we denote $M \otimes_{\mathbf{Z}} L$ by the symbol obtained by attaching a subscript L to M, like M_L or M^L. Similarly, for any $\mathbf{Z}$-linear map $f : A \to B$, the induced L-linear map $A_L \to B_L$ is denoted by f_L or f^L.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

- For a group G and a G-module M, we set

$$\mathrm{Cros}(G,M) := \{f : G \to M \mid f : \text{crossed homomorphism}\},$$
$$\mathrm{Prin}(G,M) := \{g : G \to M \mid g : \text{principal homomorphism}\}.$$

2.2. **Nielsen's Presentation.** For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, \ldots, x_n$. Let P, Q, S and U be automorphisms of F_n defined as follows:

	x_1	x_2	x_3	$\cdots$	x_{n-1}	x_n
P	x_2	x_1	x_3	$\cdots$	x_{n-1}	x_n
Q	x_2	x_3	x_4	$\cdots$	x_n	x_1
S	x_1^{-1}	x_2	x_3	$\cdots$	x_{n-1}	x_n
U	x_1x_2	x_2	x_3	$\cdots$	x_{n-1}	x_n

In 1924, Nielsen [18] obtained a first finite presentation for $\mathrm{Aut}\, F_n$.

Theorem 2.1 (Nielsen [18])**.** *For $n \geq 2$, $\mathrm{Aut}\, F_n$ is generated by P, Q, S and U subject to relators:*

(N1): P^2, Q^n, S^2,
(N2): $(QP)^{n-1}$,
(N3): $(PSPU)^2$,
(N4): $[P, Q^{-l}PQ^l]$, $\quad 2 \leq l \leq n/2$,
(N5): $[S, Q^{-1}PQ]$, $[S, QP]$,
(N6): $(PS)^4$,
(N7): $[U, Q^{-2}PQ^2]$, $[U, Q^{-2}UQ^2]$, $\quad n \geq 3$,
(N8): $[U, Q^{-2}SQ^2]$, $[U, SUS]$,
(N9): $[U, QPQ^{-1}PQ]$, $[U, PQ^{-1}SUSQP]$,
(N10): $[U, PQ^{-1}PQPUPQ^{-1}PQP]$,
(N11): $U^{-1}PUPSUSPS$,
(N12): $(PQ^{-1}UQ)^2UQ^{-1}U^{-1}QU^{-1}$.

Let H be the abelianization of F_n, and $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H. Let $e_1, \ldots, e_n$ be the basis of H induced from $x_1, \ldots, x_n$, and $e_1^*, \ldots, e_n^*$ its dual basis of H^*. Here we remark the actions of the generators P, Q, S and U on e_i's and e_i^*'s. In this paper, any $\mathrm{Aut}\, F_n$-module is considered as a left module. Hence the actions of P, Q, S and U on e_k's and e_k^*'s are given by

$$P \cdot e_k = \begin{cases} e_2, & k=1, \\ e_1, & k=2, \\ e_k, & k \neq 1,2, \end{cases} \qquad P \cdot e_k^* = \begin{cases} e_2^*, & k=1, \\ e_1^*, & k=2, \\ e_k^*, & k \neq 1,2, \end{cases}$$
$$Q \cdot e_k = \begin{cases} e_n, & k=1, \\ e_{k-1}, & k \neq 1, \end{cases} \qquad Q \cdot e_k^* = \begin{cases} e_n^*, & k=1, \\ e_{k-1}^*, & k \neq 1, \end{cases}$$
$$S \cdot e_k = \begin{cases} -e_1, & k=1, \\ e_k, & k \neq 1, \end{cases} \qquad S \cdot e_k^* = \begin{cases} -e_1^*, & k=1, \\ e_k^*, & k \neq 1, \end{cases}$$
$$U \cdot e_k = \begin{cases} e_1 - e_2, & k=1, \\ e_k, & k \neq 1, \end{cases} \qquad U \cdot e_k^* = \begin{cases} e_2^* + e_1^*, & k=2, \\ e_k^*, & k \neq 2. \end{cases}$$

2.3. **IA-automorphism group.**

Fixing the basis $e_1, \ldots, e_n$ of H, we identify $\mathrm{Aut}\, H$ with $\mathrm{GL}(n, \mathbf{Z})$. Then, the kernel of the natural homomorphism $\rho : \mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbf{Z})$ induced from the abelianization of F_n is called the IA-automorphism group of F_n, and denoetd by IA_n. Magnus [11] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : \begin{cases} x_i & \mapsto {x_j}^{-1} x_i x_j, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct i, $j \in \{1, 2, \ldots, n\}$ and

$$K_{ijk} : \begin{cases} x_i & \mapsto x_i x_j x_k {x_j}^{-1} {x_k}^{-1}, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct i, j, $k \in \{1, 2, \ldots, n\}$ such that $j > k$.

Recently, Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [9] independently determined the abelianization of IA_n. More precisely, they showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module. This abelianization is induced from the first Johnson homomorphism

$$\tau_1 : \mathrm{IA}_n \to H^* \otimes_{\mathbf{Z}} \Lambda^2 H.$$

(For a basic material concerning the Johnson homomorphism, see [14] and [21] for example.) We identify $\mathrm{IA}_n^{\mathrm{ab}}$ with $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ through τ_1. Then, we verify that (the coset classes of) the Magnus generators

$$K_{ij} = e_i^* \otimes e_i \wedge e_j, \quad K_{ijk} = e_i^* \otimes e_j \wedge e_k$$

form a basis of $\mathrm{IA}_n^{\mathrm{ab}}$ as a free abelian group. In this paper, for simplicity, we write V for $\mathrm{IA}_n^{\mathrm{ab}}$, and set

$$\boldsymbol{e}^i_{j,k} := e_i^* \otimes e_j \wedge e_k$$

for any i, j and k. Moreover, we use

$$I := \{(i, j, k) \,|\, 1 \leq i \leq n, \quad 1 \leq j < k \leq n\}$$

as an index set of a basis of V.

Finally, we recall the inner automorphism group. For each $1 \leq i \leq n$, set

$$\iota_i := K_{1i} K_{2i} \overset{\check{i}}{\cdots} K_{ni} \in \mathrm{IA}_n,$$

and let $\mathrm{Inn}\, F_n$ be a subgroup of IA_n generated by ι_i for $1 \leq i \leq n$. Then the group $\mathrm{Inn}\, F_n$, called the inner automorphism group of F_n, is also a free group with basis ι_1, $\ldots$, ι_n. We remark that the inclusion $\mathrm{Inn}\, F_n \hookrightarrow \mathrm{IA}_n$ induces a $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphism

$$H = (\mathrm{Inn}\, F_n)^{\mathrm{ab}} \to \mathrm{IA}_n^{\mathrm{ab}} = H^* \otimes_{\mathbf{Z}} \Lambda^2 H.$$

3. Construction of crossed homomorphisms

In this section, for any principal ideal domain L, we introduce two crossed homomorphisms f_M and f_K from $\mathrm{Aut}\,F_n$ into V_L, due to Morita [14] and Kawazumi [9] respectively. We remark that in their papers, the action of $\mathrm{Aut}\,F_n$ on F_n is considered as the left one. Therefore, in the following, whenever we consider the left action of $\mathrm{Aut}\,F_n$ on F_n, we use $\sigma(x) := x^{\sigma^{-1}}$ for any $\sigma \in \mathrm{Aut}\,F_n$ and $x \in F_n$.

3.1. **Morita's construction.** First we construct a crossed homomorphism f_M from $\mathrm{Aut}\,F_n$ into V using the Magnus representation of $\mathrm{Aut}\,F_n$ due to Morita [14]. Let

$$\frac{\partial}{\partial x_j} : \mathbf{Z}[F_n] \longrightarrow \mathbf{Z}[F_n]$$

be the Fox free derivations for $1 \leq j \leq n$. (For a basic material concerning with the Fox derivation, see [1] for example.) Let $^- \colon \mathbf{Z}[F_n] \to \mathbf{Z}[F_n]$ be the antiautomorphism induced from the map $F_n \ni y \mapsto y^{-1} \in F_n$, and $\mathfrak{a} : \mathbf{Z}[F_n] \to \mathbf{Z}[H]$ the ring homomorphism induced from the abelianization $F_n \to H$. For any $A = (a_{ij}) \in \mathrm{GL}(n, \mathbf{Z}[F_n])$, let $A^{\mathfrak{a}}$ be the matrix $(a_{ij}^{\mathfrak{a}}) \in \mathrm{GL}(n, \mathbf{Z}[H])$. Then a map

$$r_M : \mathrm{Aut}\,F_n \longrightarrow \mathrm{GL}(n, \mathbf{Z}[H])$$

defined by

$$\sigma \mapsto \left(\overline{\frac{\partial \sigma(x_j)}{\partial x_i}} \right)^{\mathfrak{a}}$$

is called the Magnus representation of $\mathrm{Aut}\,F_n$. We remark that r_M is not a homomorphism but a crossed homomorpshim. Namely, r_M satisfies

$$r_M(\sigma\tau) = r_M(\sigma) \cdot r_M(\tau)^{\sigma *}$$

for any σ, $\tau \in \mathrm{Aut}\,F_n$ where $r_M(\tau)^{\sigma *}$ denotes the matrix obtained from $r_M(\tau)$ by applying a ring homomorphism $\sigma_* : \mathbf{Z}[H] \to \mathbf{Z}[H]$ induced from σ on each entry.

Now, for any $\sigma \in \mathrm{Aut}\,F_n$, we write $\mathrm{sgn}(\sigma)$ for $\det \circ \rho(\sigma) \in \{\pm 1\}$. We define a map $f_M : \mathrm{Aut}\,F_n \longrightarrow \mathbf{Z}[H]$ by

$$\sigma \mapsto \mathrm{sgn}(\sigma)\ \det(r_M(\sigma)).$$

The map f_M is also crossed homomorphism. Furthermore, observing the images of the generators of $\mathrm{Aut}\,F_n$, we verify that the image of f_M is contained in H. Then, composing f_M with a natural homomorpshism $H \to V \to V_L$ induced from the inclusion $\mathrm{Inn}\,F_n \hookrightarrow \mathrm{IA}_n$, we obtain an element in $\mathrm{Cros}(\mathrm{Aut}\,F_n, V_L)$, also denoted by f_M.

3.2. **Kawazumi's construction.** Next, we construct a crossed homomorphism f_K from $\mathrm{Aut}\,F_n$ into V_L using the Magnus expansion of F_n due to Kawazumi [9]. Let $\widehat{T}$ be the complete tensor algebra generated by H. For any Magnus expansion $\theta : F_n \to \widehat{T}$, Kawazumi define a map

$$\tau_1^\theta : \mathrm{Aut}\,F_n \to H^* \otimes_{\mathbf{Z}} H^{\otimes 2}$$

called the first Johnson map induced by the Magnus expansion θ. The map τ_1^θ satisfies

$$\tau_1^\theta(\sigma)([x]) = \theta_2(x) - |\sigma|^{\otimes 2}\theta_2(\sigma^{-1}(x))$$

for any $x \in F_n$, where $[x]$ denotes the coset class of x in H, $\theta_2(x)$ is the projection of $\theta(x)$ in $H^{\otimes 2}$, and $|\sigma|^{\otimes 2}$ denotes the automorphism of $H^{\otimes 2}$ induced by $\sigma \in \mathrm{Aut}\,F_n$. In particular, he showed that τ_1^θ is a crossed homomorphism, and τ_1^θ does not depend on the choice of the Magnus expansion θ. Then, composing τ_1^θ with a natural projection $H^* \otimes_{\mathbf{Z}} H^{\otimes 2} \to H^* \otimes_{\mathbf{Z}} \Lambda^2 H \to V_L$ for any principal ideal domain L, we obtain a crossed homomorphism $f_K \in \mathrm{Cros}(\mathrm{Aut}\, F_n, V_L)$.

Observing the images of the Nielsen's generators, we have

$$f_M(\sigma) := \begin{cases} -(\boldsymbol{e}_{1,2}^2 + \boldsymbol{e}_{1,3}^3 + \cdots + \boldsymbol{e}_{1,n}^n), & \sigma = S, \\ 0, & \sigma = P, Q, U \end{cases}$$

and

$$f_K(\sigma) := \begin{cases} -\boldsymbol{e}_{1,2}^1, & \sigma = U, \\ 0, & \sigma = P, Q, S. \end{cases}$$

Now, consider an element

$$\boldsymbol{a} := \sum_{(i,j,k)\in I} a_{j,k}^i \boldsymbol{e}_{j,k}^i \in V_L,$$

such that

$$a_{j,k}^i := \begin{cases} 0, & i \neq j,\ k, \\ 1, & i = k, \\ -1, & i = j, \end{cases}$$

and let $f_{\boldsymbol{a}} \in \mathrm{Prin}(\mathrm{Aut}\, F_n, V_L)$ be a principal homomorphism associated to $\boldsymbol{a}$. If we set $f_N := 2f_M - f_K - f_{\boldsymbol{a}}$, then f_N is a crossed homomorphism such that

$$f_N(\sigma) := \begin{cases} \boldsymbol{e}_{2,3}^3 + \boldsymbol{e}_{2,4}^4 + \cdots + \boldsymbol{e}_{2,n}^n, & \sigma = U, \\ 0, & \sigma = P, Q, S. \end{cases}$$

We use f_M and f_N to determine the first cohomology group $H^1(\mathrm{Aut}\, F_n, V_L)$ in Section 4.

4. Computation of first (co)homology groups

Let L be an arbitary principal ideal domain. We denote $V \otimes_{\mathbf{Z}} L$ by V_L. In this section, to begin with, we compute $H^1(\mathrm{Aut}\, F_n, V_L)$ for $n \geq 5$. Then, using the universal coefficients theorem, we determine $H_1(\mathrm{Aut}\, F_n, V^*)$ where $V^* := \mathrm{Hom}_{\mathbf{Z}}(V, \mathbf{Z})$ is the dual group of V.

In order to compute the first cohomology groups, we use the Nielsen's presentation for $\mathrm{Aut}\, F_n$. Let F be a free group with basis P, Q, S and U, and $\varphi : F \to \mathrm{Aut}\, F_n$ the natural projection. Then the kernel R of φ is a normal closure of the relators (N1), ..., (N12). Considering the five-term exact sequence of the Lyndon-Hochshild-Serre sepectral sequence of the group extension

$$1 \to R \to F \to \mathrm{Aut}\, F_n \to 1,$$

we obtain

$$0 \to H^1(\mathrm{Aut}\, F_n, V_L) \to H^1(F, V_L) \to H^1(R, V_L).$$

Observing this sequence at the cocycle level, we also obtain an exact sequence

$$0 \to \mathrm{Cros}(\mathrm{Aut}\, F_n, V_L) \to \mathrm{Cros}(F, V_L) \xrightarrow{\iota^*} \mathrm{Cros}(R, V_L)$$

where ι^* is a map induced from the inclusion $\iota : R \hookrightarrow F$. Hence we can consider $\mathrm{Cros}(\mathrm{Aut}\, F_n, V_L)$ as a subgroup consisting of elements of $\mathrm{Cros}(F, V_L)$ which are killed by ι^*. Using the relators of the Nielsen's presentation, we obtain

Proposition 4.1. *For $n \geq 5$,*

$$H^1(\mathrm{Aut}\, F_n, V_L) = L^{\oplus 2}.$$

Proof of Proposition 4.1. For any $\sigma \in F$ and crossed homomorphism $f \in \mathrm{Cros}(F, V_L)$, set

$$f(\sigma) := \sum_{(i,j,k)\in I} a^i_{j,k}(\sigma) \boldsymbol{e}^i_{j,k} \in V_L$$

for $a^i_{j,k}(\sigma) \in L$. Since F is a free group generated by P, Q, S and U, a crossed homomorphism f is completely determined by $a^i_{j,k}(\sigma)$ for $\sigma = P$, Q, S and U. More precisely, a map

$$\mathrm{Cros}(F, V_L) \to L^{\oplus 2n^2(n-1)}$$

defined by

$$f \mapsto \Big(a^i_{j,k}(P),\ a^i_{j,k}(Q),\ a^i_{j,k}(S),\ a^i_{j,k}(U) \Big)_{(i,j,k)\in I}$$

is an isomorphism. Through this map, we identify $\mathrm{Cros}(F, V_L)$ with $L^{\oplus 2n^2(n-1)}$.

Let $f \in \mathrm{Cros}(\mathrm{Aut}\, F_n, V_L) \subset \mathrm{Cros}(F, V_L)$, Using $\iota^*(f) = 0$, and the relators of the Nielsen's presentation, we find some linear relations among $a^i_{j,k}(\sigma)$ for $\sigma = P$, Q, S and U. Then we verify that such $f \in \mathrm{Cros}(\mathrm{Aut}\, F_n, V_L)$ is characterized by at most

$$\begin{aligned} &a^i_{j,k}(Q),\ \ 1 \leq i \leq n-1,\ 1 \leq j < k \leq n, \\ &a^1_{j,k}(U),\ \ 1 \leq j < k \leq n, \\ &a^2_{1,2}(S),\ \ a^3_{2,3}(U). \end{aligned} \tag{2}$$

Later, we see that these elements uniquely determines a crossed homomorphism from $\mathrm{Aut}\, F_n$ into V_L.

Let W be a quotient L-module of a free L-module spanned by $a^i_{j,k}(\sigma)$ for $\sigma = P$, Q, S and U, and $(i,j,k) \in I$ by a submodule generated by all linear relations obtained from $\iota^*(f) = 0$. Then each of the coefficients of $\boldsymbol{e}^i_{j,k}$ in $f(\sigma)$ is considered as an element in W. We denote by $\overline{W}$ the quotient module of W by the submodule generated by all elements in (2). We use $\doteq$ for the equality in $\overline{W}$. Furthermore, we also write $\boldsymbol{a} \doteq \boldsymbol{a}'$ if both of $\boldsymbol{a}$ and $\boldsymbol{a}'$ are in a submodule of V_L generated by $\tau f(\sigma)$ for $\sigma, \tau \in \mathrm{Aut}\, F_n$, and if each of the coefficients of $\boldsymbol{e}^i_{j,k}$ in $\boldsymbol{a}$ is equal to that in $\boldsymbol{a}'$ in $\overline{W}$.

Step I.
From the relation $Q^n = 1$, we obtain

$$(1 + Q + Q^2 + \cdots + Q^{n-1}) f(Q) \doteq 0.$$

For any $1 \leq j < k \leq n$, observing the coefficient of $\boldsymbol{e}^n_{j,k}$ in the equation above, we see

$$\begin{aligned} &a^n_{j,k}(Q) + a^1_{j+1,k+1}(Q) + a^2_{j+2,k+2}(Q) \cdots + a^{n-k}_{j+n-k,n}(Q) \\ &\qquad - a^{n-k+1}_{1,j+n-k+1}(Q) - \cdots - a^{n-j}_{k-j,n}(Q) + a^{n-j+1}_{1,k-j+1}(Q) + \cdots + a^{n-1}_{j-1,k-1}(Q) \doteq 0, \end{aligned}$$

and hence $a^n_{j,k}(Q) \doteq 0$.

Step II.

Next we consider some relations among $a^i_{j,k}(P)$. Observing the coefficients of $\boldsymbol{e}^1_{j,k}$, $\boldsymbol{e}^1_{1,k}$, $\boldsymbol{e}^1_{1,2}$, $\boldsymbol{e}^1_{2,k}$ and $\boldsymbol{e}^i_{1,k}$ in an equation $(P+1)f(P)=0$ induced from a relation $P^2=1$, we obtain

(3) $$a^1_{j,k}(P)+a^2_{j,k}(P)\doteq 0, \quad 3\le j<k\le n,$$

(4) $$a^1_{1,k}(P)+a^2_{2,k}(P)\doteq 0, \quad 3\le k\le n,$$

(5) $$a^1_{1,2}(P)-a^2_{1,2}(P)\doteq 0,$$

(6) $$a^1_{2,k}(P)+a^2_{1,k}(P)\doteq 0, \quad 3\le k\le n,$$

(7) $$a^i_{1,k}(P)+a^i_{2,k}(P)\doteq 0, \quad 3\le i,k\le n$$

respectively. We use these relations later.

Step III.

Here we show $a^i_{j,k}(U)\doteq 0$ for any $(i,j,k)\in I$. First, Observing the coefficients of $\boldsymbol{e}^1_{1,2}$ and $\boldsymbol{e}^i_{2,k}$ for $3\le i,k\le n$ in an equation

$$(SUS-1)f(U)=(U-1)f(SUS)$$

induced from a relation $[U,SUS]=1$, we have

(8) $$a^2_{1,2}(U)\doteq 0 \text{ and } a^i_{1,k}(U)\doteq 0$$

respectively.

For $3\le l\le n-1$, we see

$$(Q^{-(l-1)}UQ^{l-1}-1)f(U)=(U-1)f(Q^{-(l-1)}UQ^{l-1})$$

from a relation $[U,Q^{-(l-1)}UQ^{l-1}]=1$. Since $a^i_{j,k}(Q)\doteq 0$ for any i, j and k, we obtain

(9) $$(Q^{-(l-1)}UQ^{l-1}-1)f(U)\doteq (U-1)Q^{-(l-1)}f(U).$$

Similarly, from a relation $[U,Q^{-(l-1)}PQ^{l-1}]=1$ for $3\le l\le n-1$, we obtain

(10) $$(Q^{-(l-1)}PQ^{l-1}-1)f(U)\doteq (U-1)Q^{-(l-1)}f(P).$$

Observing the coefficients of $\boldsymbol{e}^{l+1}_{2,l}$ in (9), we see $a^2_{1,n+2-l}(U)\doteq 0$ for $3\le l\le n-1$, and hence $a^2_{1,k}(U)\doteq 0$ for $3\le k\le n-1$. Similarly, from the coefficients of $\boldsymbol{e}^2_{1,n-1}$ in (10) for $l=n-1$, we see $a^2_{1,n}(U)\doteq a^2_{1,n-1}(U)\doteq 0$.

Next, observing the coefficients of $\boldsymbol{e}^1_{1,k}$ and $\boldsymbol{e}^1_{2,k}$ for $3\le k\le n$ in

(11) $$(PSPU+1)f(PSPU)=0$$

induced from a relation $(PSPU)^2=1$, and using (4), we obatin

(12) $$2a^1_{1,k}(U)-a^1_{2,k}(P)+a^2_{1,k}(P)+a^1_{2,k}(S)-a^2_{1,k}(U)\doteq 0,$$

(13) $$a^1_{1,k}(U)-a^1_{2,k}(P)+a^2_{1,k}(P)+a^1_{2,k}(S)-a^2_{1,k}(U)-a^2_{2,k}(U)\doteq 0.$$

Then, considering (12) − (13), we have $a^2_{2,k}(U)\doteq -a^1_{1,k}(U)\doteq 0$ for $3\le k\le n$.

From the coefficients of $\boldsymbol{e}^2_{j,l+1}$ in (9), we see $a^2_{j,l}(U)\doteq 0$ for $3\le j<l\le n-1$. Similarly, from $n\ge 5$ and the coefficients of $\boldsymbol{e}^2_{l+1,n}$ in (9), we see $a^2_{l,n}(U)\doteq 0$ for $3\le l\le n-2$. Then, from the coefficient of $\boldsymbol{e}^2_{n-2,n}$ in (10) for $l=n-2$, we see

$a^2_{n-1,n}(U) \doteq a^2_{n-2,n}(U) \doteq 0$. On the other hand, observing the coefficients of $\boldsymbol{e}^l_{1,2}$ in (9), we see $a^{l+1}_{1,2}(U) \doteq 0$ for $3 \le l \le n-1$. Then, from the coefficient of $\boldsymbol{e}^3_{1,2}$ in (10) for $l=3$, we see $a^3_{1,2}(U) \doteq a^4_{1,2}(U) \doteq 0$.

Now, from the coefficients of $\boldsymbol{e}^l_{j,k}$ in (10), we see $a^{l+1}_{j,k}(U) \doteq a^l_{j,k}(U)$ for $3 \le l \le n-1$ and $j,k \ne l+1$. This shows that

$$a^3_{j,k}(U) \doteq a^4_{j,k}(U) \doteq \cdots \doteq a^{j-1}_{j,k}(U), \quad 4 \le j < k \le n, \tag{14}$$

$$a^j_{j,k}(U) \doteq a^{j+1}_{j,k}(U) \doteq \cdots \doteq a^{k-1}_{j,k}(U), \quad 3 \le j < k \le n, \tag{15}$$

$$a^k_{j,k}(U) \doteq a^{k+1}_{j,k}(U) \doteq \cdots \doteq a^n_{j,k}(U), \quad 3 \le j < k \le n. \tag{16}$$

On the other hand, from the coefficients of $\boldsymbol{e}^l_{l,l+1}$, $\boldsymbol{e}^l_{j,l+1}$, $\boldsymbol{e}^l_{j,l}$, $\boldsymbol{e}^l_{l,k}$ and $\boldsymbol{e}^l_{l+1,k}$ in (10), we obtain

$$a^{l+1}_{l,l+1}(U) \doteq -a^l_{l,l+1}(U), \quad 3 \le l \le n-1, \tag{17}$$

$$a^{l+1}_{j,l}(U) \doteq a^l_{j,l+1}(U), \quad 3 \le j < l \le n-1, \tag{18}$$

$$a^{l+1}_{j,l+1}(U) \doteq a^l_{j,l}(U), \quad 4 \le j+1 < l \le n-1, \tag{19}$$

$$a^{l+1}_{l+1,k}(U) \doteq a^l_{l,k}(U), \quad 3 \le l < k-1 \le n-1, \tag{20}$$

$$a^{l+1}_{l,k}(U) \doteq a^l_{l+1,k}(U), \quad 3 \le l < k-1 \le n-1 \tag{21}$$

respectively. Hence for $4 \le j < k \le n$, we see

$$a^{j-1}_{j,k}(U) \overset{(21)}{\doteq} a^j_{j-1,k}(U) \overset{(15)}{\doteq} a^{j-1}_{j-1,k}(U) \overset{(20)}{\doteq} a^j_{j,k}(U),$$

and for $3 \le j < k \le n$, if $k \ne j+1$,

$$a^{k-1}_{j,k}(U) \overset{(18)}{\doteq} a^k_{j,k-1}(U) \overset{(16)}{\doteq} a^{k-1}_{j,k-1}(U) \overset{(19)}{\doteq} a^k_{j,k}(U).$$

If $k=j+1$, $a^{k-1}_{j,k}(U) \overset{(17)}{\doteq} -a^k_{j,k}(U)$. Therefore it suffices to show $a^{i_0}_{j,k}(U) \doteq 0$ for some $1 \le i_0 \le n$ to show $a^i_{j,k}(U) \doteq 0$ for $3 \le i \le n$ and $3 \le j < k \le n$. Observing the coefficients of $\boldsymbol{e}^l_{j,k}$ in (9), we see $a^{l+1}_{j,k}(U) \doteq 0$ for $3 \le l \le n-1$ and $j,k \ne l+1$. If $n \ge 6$, or $n=5$ and $(j,k) \ne (4,5)$ then there exists some l such that $3 \le l \le n-1$ and $l+1 \ne j,k$. In this case, we can take $i_0 := l+1$. If $n=5$ and $(j,k)=(4,5)$, we see $a^3_{4,5}(U) \overset{(21)}{\doteq} a^4_{3,5}(U) \doteq 0$ for $i_0 = 3$.

By the coefficients of $\boldsymbol{e}^l_{2,k}$, $\boldsymbol{e}^l_{2,l+1}$ and $\boldsymbol{e}^3_{2,l+1}$ in (9),

$$a^{l+1}_{2,k}(U) \doteq a^1_{*,*}(U) \doteq 0, \quad 3 \le l \le n-1, \text{ and } k \ne l+1, \tag{22}$$

$$a^{l+1}_{2,l+1}(U) \doteq a^l_{2,l}(U) + a^{l+1}_{2,l}(U) + a^1_{2,*}(U) \overset{(22)}{\doteq} a^l_{2,l}(U), \quad 3 \le l \le n-1, \tag{23}$$

$$a^3_{2,l}(U) \doteq -a^{n+4-l}_{2,n+2-l}(U), \quad 4 \le l \le n-1 \tag{24}$$

respectively. Hence, from (23), we see $a^n_{2,n}(U) \doteq a^{n-1}_{2,n-1}(U) \doteq \cdots \doteq a^3_{2,3}(U) \doteq 0$. From (22) and (24), we see $a^3_{2,l}(U) \doteq 0$ for $4 \le l \le n-1$.

Finally, we show $a^3_{2,n}(U) \doteq 0$. From a relation $[U, Q^{-(n-2)}PUP^{-1}Q^{(n-2)}] = 1$, we have

$$\begin{aligned}&(Q^{-(n-2)}PUP^{-1}Q^{(n-2)} - 1)f(U) \\ &\qquad \doteq (U-1)Q^{-(n-2)}(f(P) + Pf(U) - PUP^{-1}f(P)).\end{aligned}$$

By the coefficients of $\boldsymbol{e}^3_{2,n-1}$ in the equation above, we see $a^3_{2,n}(U) \doteq -a^5_{1,3}(U) \doteq 0$. Therefore we obtain $a^i_{j,k}(U) \doteq 0$ for any $(i,j,k) \in I$.

Step IV.

Here we show $a^i_{j,k}(P) \doteq 0$ for any $(i,j,k) \in I$. From Step II, it suffices to consider the case where $i \neq 2$. First, we show $a^1_{j,k}(P) \doteq 0$ for $1 \leq j < k \leq n$.

From a relation $(PQ^{-1}UQ)^2 = UQ^{-1}UQU^{-1}$ and the results above, we obtain $(1 + PQ^{-1}UQ)f(PQ^{-1}UQ) \doteq 0$, and hence

$$(1 + PQ^{-1}UQ)f(P) \doteq 0. \tag{25}$$

Observing the coefficients of $\boldsymbol{e}^2_{2,3}$ and $\boldsymbol{e}^2_{3,k}$ in (25), we have

$$\begin{aligned}a^2_{2,3}(P) + a^1_{1,3}(P) - a^1_{1,2}(P) &\doteq 0, \\ a^2_{3,k}(P) + a^1_{3,k}(P) - a^1_{2,k}(P) &\doteq 0, \quad 4 \leq k \leq n.\end{aligned}$$

Using (4) and (3), we obtain $a^1_{1,2}(P) \doteq 0$ and $a^1_{2,k}(P) \doteq 0$ for $4 \leq k \leq n$. From a relation $[P, Q^{-(l-1)}UQ^{l-1}] = 1$ for $3 \leq l \leq n-1$, we have

$$(Q^{-(l-1)}UQ^{l-1} - 1)f(P) \doteq 0. \tag{26}$$

By the coefficients of $\boldsymbol{e}^1_{2,4}$ in the equation above for $l = 3$, we see $a^1_{2,3}(P) \doteq 0$. Furthermore, from the coefficients of $\boldsymbol{e}^1_{1,l+1}$ in (26), we see

$$a^1_{1,l}(P) \doteq 0, \quad 3 \leq l \leq n-1.$$

From a relation $(QP)^{n-1} = 1$, we obtain

$$(1 + QP + \cdots + (QP)^{n-2})(f(Q) + Qf(P)) \doteq 0,$$

and hence

$$(1 + QP + \cdots + (QP)^{n-2})Qf(P) \doteq 0. \tag{27}$$

By the coefficients of $\boldsymbol{e}^1_{1,2}$ in (27), we see

$$a^2_{2,3}(P) + a^2_{2,4}(P) + \cdots + a^2_{2,n}(P) - a^2_{1,2}(P) \doteq 0.$$

Then using (4) and (5), we obtain $a^1_{1,n}(P) \doteq 0$.

Now, we consider $a^1_{j,k}(P)$ for $3 \leq j < k \leq n$. From the coefficients of $\boldsymbol{e}^1_{j,l+1}$ in (26), we obtain

$$a^1_{j,l}(P) \doteq a^{n+2-l}_{1,*}(U) - a^{n+3-l}_{1,*}(U) \overset{(8)}{\doteq} 0, \quad 3 \leq j < l \leq n-1. \tag{28}$$

On the other hand, from the coefficients of $\boldsymbol{e}^1_{l+1,n}$ in (26), we see $a^1_{j,n}(P) \doteq 0$ for $3 \leq j \leq n-2$. Furthermore, observing the coefficient of $\boldsymbol{e}^1_{n-1,n}$ in (27), we see

$$-a^2_{1,2}(P) + a^2_{1,3}(P) + a^2_{3,4}(P) + \cdots + a^2_{n-2,n-1}(P) + a^2_{n-1,n}(P) \doteq 0,$$

and hence $a^1_{n-1,n}(P) \doteq -a^2_{n-1,n}(P) \doteq 0$. Therefore we have $a^1_{j,k}(P) \doteq a^2_{j,k}(P) \doteq 0$ for any $1 \le j < k \le n$.

Next, we show $a^i_{1,k}(P) \doteq 0$ for $3 \le i \le n$ and $2 \le k \le n$. Observing the coefficients of $\boldsymbol{e}^l_{1,k}$, $\boldsymbol{e}^l_{1,l+1}$ and $\boldsymbol{e}^3_{1,l+1}$ in (26), we see

$$a^{l+1}_{1,k}(P) \doteq 0, \quad 3 \le l \le n-1,\ 3 \le k \ne l+1, \tag{29}$$

$$a^{l+1}_{1,l+1}(P) \doteq a^l_{1,l}(P) - a^{l+1}_{1,l}(P) \overset{(29)}{\doteq} a^l_{1,l}(P), \quad 3 \le l \le n-1, \tag{30}$$

$$a^3_{1,l}(P) \doteq 0, \quad 3 \le l \le n-1 \tag{31}$$

respectively. From these equations, it suffices to show that $a^3_{1,2}(P) \doteq a^3_{1,3}(P) \doteq 0$. By the coefficients of $\boldsymbol{e}^3_{1,2}$ in (27), we see

$$a^4_{2,3}(P) + a^5_{2,6}(P) + \cdots + +a^n_{2,n-1}(P) + a^1_{2,n}(P) - a^3_{1,2}(P) \doteq 0,$$

and hence $a^3_{1,2}(P) \doteq 0$. On the other hand, by the coefficients of $\boldsymbol{e}^1_{2,3}$ in (25), we see

$$a^1_{2,3}(P) + a^2_{1,3}(P) - a^2_{1,2}(P) - a^3_{1,2}(P) + a^3_{1,3}(P) \doteq 0,$$

and hence $a^3_{1,3}(P) \doteq 0$.

Finally, we show $a^i_{j,k}(P) \doteq 0$ for $3 \le i \le n$ and $3 \le j < k \le n$. Observing the coefficients of $\boldsymbol{e}^l_{j,k}$, $\boldsymbol{e}^l_{j,l+1}$ and $\boldsymbol{e}^l_{l+1,k}$ in (26), we see

$$a^{l+1}_{j,k}(P) \doteq 0, \quad 3 \le l \le n-1,\ j,k \ne l+1, \tag{32}$$

$$a^{l+1}_{j,l+1}(P) - a^l_{j,l}(P) \doteq 0, \quad 3 \le l \le n-1,\ j < l \tag{33}$$

$$a^{l+1}_{l+1,k}(P) - a^l_{l,k}(P) \doteq 0, \quad 3 \le l \le n-1,\ l+1 < k \tag{34}$$

respectively. From (33) and (34), we have

$$\begin{aligned} a^n_{j,n}(P) &\doteq a^{n-1}_{j,n-1}(P) \doteq \cdots \doteq a^{j+1}_{j,j+1}(P) \\ &\doteq -a^j_{j,j+1}(P) \doteq -a^{j-1}_{j-1,j+1}(P) \doteq \cdots \doteq -a^3_{3,j+1}(P) \end{aligned}$$

for $3 \le j \le n-1$. Hence it suffices to show $a^3_{j,k}(P) \doteq 0$ for $3 \le j < k \le n$. Then from the coefficients of $\boldsymbol{e}^3_{j,l+1}$ in (26), we see $a^3_{j,l}(P) \doteq 0$ for $3 \le j < l \le n-1$. Furthermore, from the coefficients of $\boldsymbol{e}^3_{l+1,n}$ in (26), we see $a^3_{l,n}(P) \doteq 0$ for $3 \le l \le n-2$. On the other hand, by the coefficients of $\boldsymbol{e}^3_{n-1,n}$ in (27), we obtain

$$-a^4_{1,n}(P) + a^5_{1,3}(P) + \cdots + a^n_{n-3,n-2}(P) + a^1_{n-2,n-1}(P) + a^3_{n-1,n}(P) \doteq 0,$$

and hence $a^3_{n-1,n}(P) \doteq 0$. Therefore we obtain $a^i_{j,k}(P) \doteq 0$ for any $(i,j,k) \in I$.

Step V.

Here we show $a^i_{j,k}(S) \doteq 0$ for any $(i,j,k) \in I$. First, observing the coefficients of $\boldsymbol{e}^i_{j,k}$ for $2 \le i \le n$ and $2 \le j < k \le n$ in the equations $(1+S)f(S) = 0$ induced from a relation $S^2 = 1$ and

$$(1 + SU + SUSP)f(S) \doteq 0 \tag{35}$$

induced from a relation $SUSPS = (UP)^{-2}$, we obtain $2a^i_{j,k}(S) \doteq 0$ and $3a^i_{j,k}(S) \doteq 0$ respectively. Hence $a^i_{j,k}(S) \doteq 0$ for $2 \le i \le n$ and $2 \le j < k \le n$. Similarly, observing the coefficients of $\boldsymbol{e}^1_{1,2}$ in the two equation above, we also obtain $a^1_{1,2}(S) \doteq 0$. From the coefficients of $\boldsymbol{e}^1_{1,k}$ in an equation

$$(QP-1)f(S) \doteq 0 \tag{36}$$

induced from $[S, QP] = 1$, we see

$$a^1_{1,k+1}(S) - a^1_{1,k}(S) \doteq 0, \quad 2 \le k \le n-1,$$

and hence $a^1_{1,n}(S) \doteq a^1_{1,n-1}(S) \doteq \cdots \doteq a^1_{1,2}(S) \doteq 0$.

Next, for $3 \le k \le n$, from the coefficients $\boldsymbol{e}^1_{1,k}$ and $\boldsymbol{e}^2_{1,k}$ in (35), we obatain

$$\begin{aligned} a^1_{2,k}(S) &\doteq a^2_{1,k}(S), \quad 3 \le k \le n, \\ a^2_{1,k}(S) &\doteq 0, \quad 3 \le k \le n \end{aligned}$$

respectively. From the coefficients $\boldsymbol{e}^i_{1,k}$, $\boldsymbol{e}^1_{j,k}$, we see

$$a^{i+1}_{1,k+1}(S) \doteq a^i_{1,k}(S), \quad 2 \le i, k \le n-1, \tag{37}$$

$$a^{i+1}_{1,2}(S) \doteq a^i_{1,n}(S), \quad 2 \le i \le n-1, \tag{38}$$

$$a^1_{j+1,k+1}(S) \doteq a^1_{j,k}(S), \quad 2 \le j < k \le n-1, \tag{39}$$

$$a^1_{2,j+1}(S) \doteq -a^1_{j,n}(S), \quad 2 \le j \le n-1. \tag{40}$$

Using (37) and (38), for any $2 \le i, k \le n-1$, we see $a^i_{1,k}(S) \doteq a^2_{1,*}(S) \doteq 0$. Similarly, we obtain $a^1_{j,k}(S) \doteq 0$ for any $2 \le j < k \le n-1$ from (39) and (40). Therefore we obtain $a^i_{j,k}(S) \doteq 0$ for any $(i,j,k) \in I$.

From the argument above, we verify that any $a^i_{j,k}(\sigma) \in W$ belongs to the submodule generated by (2). This shows that a map

$$\Phi : \mathrm{Cros}(\mathrm{Aut}\, F_n, V_L) \to L^{\oplus(n^3-n^2+4)/2}$$

defined by

$$f \mapsto \Big((a^i_{j,k}(Q))_{i\neq n,\, 1\le j<k\le n},\ (a^1_{j,k}(U))_{1\le j<k\le n},\ a^2_{1,2}(S),\ a^3_{2,3}(U) \Big)$$

is injective. Let $W' = L^{\oplus(n^3-n^2+4)/2}$ be the target of the map above. Then we consider $\mathrm{Cros}(\mathrm{Aut}\, F_n, V_L)$ as a submodule of W'.

Step VI. $W'/\mathrm{Prin}(\mathrm{Aut}\, F_n, V_L)$.

Here, we show that $W'/\mathrm{Prin}(\mathrm{Aut}\, F_n, V_L)$ is a free L-module of rank 2. For any element

$$\boldsymbol{a} := \sum_{(i,j,k)\in I} a^i_{j,k} \boldsymbol{e}^i_{j,k} \in V_L,$$

let $f_{\boldsymbol{a}} : F \to V$ be the principal homomorphism associated to a. Namely, $f_a(\sigma) = \sigma\cdot a - a$ for any $\sigma \in \mathrm{Aut}\, F_n$. Then we have

$$\begin{aligned} f_{\boldsymbol{a}}(Q) = &\sum_{i\neq n,\, 1\le j<k\le n-1} (a^{i+1}_{j+1,k+1} - a^i_{j,k})\boldsymbol{e}^i_{j,k} + \sum_{1\le i,j\le n-1} (-a^{i+1}_{1,j+1} - a^i_{j,n})\boldsymbol{e}^i_{j,n} \\ &+ \sum_{1\le j<k\le n} (a^1_{j+1,k+1} - a^n_{j,k})\boldsymbol{e}^n_{j,k} + \sum_{1\le j\le n-1} (-a^1_{1,j+1} - a^n_{j,n})\boldsymbol{e}^n_{j,n} \end{aligned}$$

and

$$\begin{aligned} f_{\boldsymbol{a}}(U) = a^1_{1,2}\boldsymbol{e}^1_{1,2} + \sum_{3\le k\le n} a^2_{1,k}\boldsymbol{e}^1_{1,k} \\ + \sum_{3\le k\le n}(a^2_{2,k} - a^1_{1,k} - a^2_{1,k})\boldsymbol{e}^1_{2,k} + \sum_{3\le j<k\le n} a^2_{j,k}\boldsymbol{e}^1_{j,k} \\ + \sum_{3\le k\le n} -a^2_{1,k}\boldsymbol{e}^2_{2,k} + \sum_{3\le i,k\le n} -a^i_{1,k}\boldsymbol{e}^i_{2,k}. \end{aligned}$$

In order to determine the L-module structure of $W'/\mathrm{Prin}(\mathrm{Aut}\,F_n, V_L)$, it suffices to find the elementary divisors of a matrix:

$$A := \begin{array}{c} \\ a^1_{j,k} \\ a^2_{j,k} \\ \vdots \\ a^n_{j,k} \end{array} \begin{array}{c} \begin{array}{ccccccc} a^1_{j,k}(Q) & a^2_{j,k}(Q) & \cdots & a^{n-1}_{j,k}(Q) & a^1_{j,k}(U) & a^2_{1,2}(S) & a^3_{2,3}(U) \end{array} \\ \begin{pmatrix} A^{1,1} & A^{1,2} & \cdots & A^{1,n-1} & A^{1,n} & A^{1,n+1} & A^{1,n+2} \\ A^{2,1} & A^{2,2} & \cdots & A^{2,n-1} & A^{2,n} & A^{2,n+1} & A^{2,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{n,1} & A^{n,2} & \cdots & A^{n,n-1} & A^{n,n} & A^{n,n+1} & A^{n,n+2} \end{pmatrix} \end{array}$$

where each $A^{p,q}$ is a block matrix defined by as follows. First, we consdier the case where $1 \le q \le n-1$. For $\sigma = P, Q, S$ and U, set

$$f_{\boldsymbol{a}}(\sigma) := \sum_{(i,j,k)\in I} a^i_{j,k}(\sigma)\boldsymbol{e}^i_{j,k} \in V_L.$$

Then, for any $1 \le j_2 < k_2 \le n$, we have

$$a^q_{j_2,k_2}(Q) = \sum_{(p,j_1,k_1)\in I} C^{p,q}_{(j_1,k_1),\,(j_2,k_2)} a^p_{j_1,k_1}$$

for some $C^{p,q}_{(j_1,k_1),\,(j_2,k_2)} \in L$. Then

$$A^{p,q} := \begin{array}{c} \\ a^p_{1,2} \\ a^p_{1,3} \\ \vdots \\ a^p_{n-1,n} \end{array} \begin{array}{c} \begin{array}{cccc} a^q_{1,2}(Q) & a^q_{1,3}(Q) & \cdots & a^q_{n-1,n}(Q) \end{array} \\ \begin{pmatrix} C^{p,q}_{(1,2),\,(1,2)} & C^{p,q}_{(1,2),\,(1,3)} & \cdots & C^{p,q}_{(1,2),\,(n-1,n)} \\ C^{p,q}_{(1,3),\,(1,2)} & C^{p,q}_{(1,3),\,(1,3)} & \cdots & C^{p,q}_{(1,3),\,(n-1,n)} \\ \vdots & \vdots & \vdots & \vdots \\ C^{p,q}_{(n-1,n),\,(1,2)} & C^{p,q}_{(n-1,n),\,(1,3)} & \cdots & C^{p,q}_{(n-1,n),\,(n-1,n)} \end{pmatrix} \end{array}$$

where the rows are indexed by $a^q_{j,k}(Q)$s according to the usual lexicographic order on the set $\{(j,k)\,|\,1 \le j < k \le n\}$. Similarly, the columns are indexed by $a^p_{j,k}$s. By an argument similar to the above, the block matrices $A^{p,n}$, $A^{p,n+1}$ and $A^{p,n+2}$ for $1 \le p \le n$ are defined from $a^1_{j,k}(U)$s, $a^2_{1,2}(S)$ and $a^3_{2,3}(U)$ respectively.

Set

$$A' := \begin{array}{c} \\ a^1_{j,k} \\ a^2_{j,k} \\ \vdots \\ a^n_{j,k} \end{array} \begin{array}{c} \begin{array}{ccccc} a^1_{j,k}(Q) & a^2_{j,k}(Q) & \cdots & a^{n-1}_{j,k}(Q) & a^1_{j,k}(U) \end{array} \\ \begin{pmatrix} A^{1,1} & A^{1,2} & \cdots & A^{1,n-1} & A^{1,n} \\ A^{2,1} & A^{2,2} & \cdots & A^{2,n-1} & A^{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{n,1} & A^{n,2} & \cdots & A^{n,n-1} & A^{n,n} \end{pmatrix} \end{array}.$$

In the following, we obtain that all elementary divisors of A are equal to $1 \in L$ by showing that A' can be transformed into the identity matrix with only column basic transformations. Then we conclude $W'/\mathrm{Prin}(\mathrm{Aut}\, F_n, V_L) \cong L^{\oplus 2}$.

As first transformation of A', we consider the $a^1_{j,k}(U)$ column of A'. Add the $a^1_{1,k}(U)$ column to $a^1_{2,k}(U)$ column for $3 \leq k \leq n$, and minus $a^1_{1,k}(Q)$ column from $a^1_{2,k}(U)$ column for $3 \leq k \leq n$. Then by minusing the $a^1_{1,2}(U)$ column from the $a^1_{2,n}$ column. we obtain

$$\begin{array}{c|ccc} & a^1_{1,k}(U) & a^1_{2,k}(U) & a^1_{*,k}(U) \\ \hline a^1_{j,k} & O & O & O \\ a^2_{1,k} & E & O & O \\ a^2_{2,k} & O & X & O \\ a^2_{*,k} & O & O & E \\ a^*_{j,k} & O & O & O \end{array}$$

where

$$X = \begin{array}{c} \\ a^2_{2,3} \\ a^2_{2,4} \\ \vdots \\ \vdots \\ a^2_{2,n} \end{array} \begin{array}{c} \begin{array}{ccccc} a^1_{2,3}(U) & a^1_{2,4}(U) & \cdots & a^1_{2,n-1}(U) & a^1_{2,n}(U) \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & & & 0 \\ 0 & -1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & 0 & & -1 & 1 \end{pmatrix} \end{array}.$$

It is easily seen that X can be transformed into the identity matrix with the basic transformation with respect to the $a^1_{2,*}(U)$ column.

Next, we consider the $a^1_{j,k}(Q)$ and $a^2_{j,k}(Q)$ columns. Using the results above, we can transform the $a^1_{j,k}(Q)$ and $a^2_{j,k}(Q)$ columns into

$$\begin{array}{c} \\ a^1_{j,k} \\ a^*_{j,k} \end{array} \begin{array}{c} a^1_{j,k}(Q) \\ \begin{pmatrix} E \\ O \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \\ a^1_{j,k} \\ a^2_{j,k} \\ a^3_{j,k} \\ a^*_{j,k} \end{array} \begin{array}{c} a^2_{j,k}(Q) \\ \begin{pmatrix} O \\ O \\ E \\ O \end{pmatrix} \end{array}.$$

respectively. For any $2 \leq p \leq n-2$, assume that the $a^p_{j,k}(Q)$ column has transformed into

$$\begin{array}{cc} & a^p_{j,k}(Q) \\ \begin{array}{c} a^*_{j,k} \\ a^{p+1}_{j,k} \\ a^*_{j,k} \end{array} & \left(\begin{array}{c} O \\ E \\ O \end{array} \right) \end{array},$$

Then we can easily transform the $a^{p+1}_{j,k}(Q)$ column into

$$\begin{array}{cc} & a^{p+1}_{j,k}(Q) \\ \begin{array}{c} a^*_{j,k} \\ a^{p+2}_{j,k} \\ a^*_{j,k} \end{array} & \left(\begin{array}{c} O \\ E \\ O \end{array} \right) \end{array}.$$

Therefore we conclude that A' can be transformed into the identity matrix with only column basic transformations, and hence all elementary divisors of A are equal to $1 \in L$.

Finally, considering the crossed homomorphisms f_M, $f_N \in \mathrm{Cros}(F, V_L)$ defined in Section 3, we see Φ is surjective, and hence

$$H^1(\mathrm{Aut}\, F_n, V_L) \cong L^{\oplus 2}.$$

This copmpletes the proof of Proposition 4.1. $\square$

From Proposition 4.1, we see that the crossed homomorphisms f_M and f_K generate $H^1(\mathrm{Aut}\, F_n, V_L)$ for $n \geq 5$. Now, we compute the first homology group of $\mathrm{Aut}\, F_n$ with coefficients in the dual group $V^* := \mathrm{Hom}_{\mathbf{Z}}(V, \mathbf{Z})$ of V. To begin with, we prepare the following lemma.

Lemma 4.1. *For $n \geq 2$,*

$$H_0(\mathrm{Aut}\, F_n, V^*) = 0.$$

Proof of Lemma 4.1. We consider $H_0(\mathrm{Aut}\, F_n, V^*)$ as the abelian group of coinvarints of V^* by the action of $\mathrm{Aut}\, F_n$. We write $\equiv$ for the equality in $H_0(\mathrm{Aut}\, F_n, V^*)$. Let $\hat{\boldsymbol{e}}^i_{j,k}$ be the dual basis of V^* with respect to the basis $\boldsymbol{e}^i_{j,k}$ of V. Since

$$\hat{\boldsymbol{e}}^i_{j,k} \equiv Q \cdot \hat{\boldsymbol{e}}^i_{j,k} = \begin{cases} \hat{\boldsymbol{e}}^{i-1}_{j-1,k-1}, & 2 \leq i \leq n, \ 2 \leq j < k \leq n, \\ -\hat{\boldsymbol{e}}^{i-1}_{k-1,n}, & 2 \leq i \leq n, \ 1 = j < k \leq n, \end{cases}$$

it suffices to show $\hat{\boldsymbol{e}}^1_{j,k} \equiv 0$ for any $1 \leq j < k \leq n$. This is obtained from

$$\begin{aligned}
&\hat{\boldsymbol{e}}^1_{j,k} \equiv P \cdot \hat{\boldsymbol{e}}^1_{j,k} = \hat{\boldsymbol{e}}^2_{j,k} = \hat{\boldsymbol{e}}^1_{j,k} - U \cdot \hat{\boldsymbol{e}}^1_{j,k} \equiv 0, \quad 3 \leq j < k \leq n, \\
&\hat{\boldsymbol{e}}^1_{2,k} \equiv P \cdot \hat{\boldsymbol{e}}^1_{2,k} = \hat{\boldsymbol{e}}^2_{1,k} = \hat{\boldsymbol{e}}^1_{1,k} - U \cdot \hat{\boldsymbol{e}}^1_{1,k} \equiv 0, \quad 3 \leq k \leq n, \\
&\hat{\boldsymbol{e}}^1_{1,2} \equiv P \cdot \hat{\boldsymbol{e}}^1_{1,2} = -\hat{\boldsymbol{e}}^2_{1,2} = U \cdot \hat{\boldsymbol{e}}^1_{1,2} - \hat{\boldsymbol{e}}^1_{1,2} \equiv 0, \\
&\hat{\boldsymbol{e}}^1_{1,k} \equiv (QP) \cdot \hat{\boldsymbol{e}}^1_{1,k} = \hat{\boldsymbol{e}}^1_{1,k+1}, \quad 2 \leq k \leq n-1.
\end{aligned}$$

This completes the proof of Lemma 4.1. $\square$

Theorem 4.1. *For $n \geq 5$,*

$$H_1(\mathrm{Aut}\, F_n, V^*) \cong \mathbf{Z}^{\oplus 2}.$$

Proof of Theorem 4.1. For any principal ideal domain L, considering the universal coefficients theorem, we obtain

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_0(\mathrm{Aut}\, F_n, V^*), L) \to H^1(\mathrm{Aut}\, F_n, V_L) \to \mathrm{Hom}_{\mathbf{Z}}(H_1(\mathrm{Aut}\, F_n, V^*), L) \to 0.$$

From Lemma 4.1,

$$\mathrm{Hom}_{\mathbf{Z}}(H_1(\mathrm{Aut}\, F_n, V^*), L) \cong H^1(\mathrm{Aut}\, F_n, V_L). \tag{41}$$

Since $\mathrm{Aut}\, F_n$ and V^* are finitely generated, the first homology group $H_1(\mathrm{Aut}\, F_n, V^*)$ is also finitely generated abelian group. Hence, we can set

$$H_1(\mathrm{Aut}\, F_n, V^*) \cong \mathbf{Z}^{\oplus m} \oplus \mathbf{Z}/p_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p_l\mathbf{Z}$$

for some integer $m \geq 0$ and prime numbers $p_1, \cdots, p_l$. If we set $L = \mathbf{Z}$, from (41) and Proposition 4.1, we see $m = 2$. On the other hand, if we set $L = \mathbf{Z}/p\mathbf{Z}$ for any prime p, we see $l = 0$. This completes the proof of Theorem 4.1. $\square$

5. Some quotient groups of $\mathrm{Aut}\, F_n$

In this section, we consider twisted first homology groups of some quotient groups of $\mathrm{Aut}\, F_n$.

5.1. General linear group.

Using the results for $\mathrm{Aut}\, F_n$, we compute the first homology group of $\mathrm{GL}(n, Z)$ with coefficients in V^*. To begin with, we consider the restriction of the crossed homomorphisms f_M and f_K to IA_n. By an easy calculation, we have

$$f_M(K_{ij}) = -(\boldsymbol{e}^1_{1,j} + \boldsymbol{e}^2_{2,j} + \overset{\check{j}}{\cdots} + \boldsymbol{e}^n_{n,j}), \quad f_K(K_{ij}) = 2\boldsymbol{e}^i_{i,j} \tag{42}$$

for $1 \leq i \neq j \leq n$, and

$$f_M(K_{ijk}) = 0, \quad f_K(K_{ijk}) = 2\boldsymbol{e}^i_{j,k} \tag{43}$$

for $1 \leq j < k \leq n$ and $i \neq j, k$. We left the calculations to the reader as an excercise. Then we have

Proposition 5.1. *For $n \geq 5$,*

$$H^1(\mathrm{GL}(n, \mathbf{Z}), V_L) = \begin{cases} L, & L = \mathbf{Z} \text{ or } 1/2 \in L, \\ \mathbf{Z}/2\mathbf{Z}, & L = \mathbf{Z}/2^l\mathbf{Z}, \ l \geq 1. \end{cases}$$

Proof of Proposition 5.1. Considering the five-term exact sequence of

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbf{Z}) \to 1,$$

we have

$$0 \to H^1(\mathrm{GL}(n, \mathbf{Z}), V_L) \xrightarrow{\mu} H^1(\mathrm{Aut}\, F_n, V_L) \xrightarrow{\nu} H^1(\mathrm{IA}_n, V_L)^{\mathrm{GL}(n,\mathbf{Z})}.$$

From (42) and (43), if $L = \mathbf{Z}$ or L contains $1/2$, $\nu(f_1)$ and $\nu(f_2)$ are linearly independent in $H^1(\mathrm{IA}_n, V_L)^{\mathrm{GL}(n,\mathbf{Z})}$. Hence, the kernel of μ is trivial. If $L = \mathbf{Z}/2^l\mathbf{Z}$, for a, $b \in \mathbf{Z}$, $af_M + bf_K \in \mathrm{Ker}(\nu) \iff 2^l | a$ and $2^{l-1} | b$. This completes the proof of Proposition 5.1. $\square$

From (42) and (43), we see that the first Johnson homomorphism τ_1 does not extend to $\mathrm{Aut}\, F_n$ as a crossed homomorphism. If not, suppose there exists a crossed homomorphism $f \in \mathrm{Cros}(\mathrm{Aut}\, F_n, V)$ such that $f \equiv \tau_1$ on IA_n. Then f is cohomologous to

$af_M + bf_K$ for some a, $b \in \mathbf{Z}$ by Proposition 4.1. Observing the both images of them on K_{ijk}, we see that it is impossible.

By Proposition 5.1 and an argument similar to that in Theorem 4.1, we also obtain

Corollary 5.1. *For $n \geq 5$,*

$$H_1(\mathrm{GL}(n, \mathbf{Z}), V^*) \cong \mathbf{Z}/2\mathbf{Z}.$$

5.2. **Outer automorphsim group.** Here, we determine the first homology group of the outer automorpshim group $\mathrm{Out}\, F_n := \mathrm{Aut}\, F_n / \mathrm{Inn}\, F_n$ of a free group with coefficients in V^*.

Proposition 5.2. *For $n \geq 5$,*

(1) *If n is even,*

$$H^1(\mathrm{Out}\, F_n, V_L) = L.$$

(2) *If n is odd,*

$$H^1(\mathrm{Out}\, F_n, V_L) = \begin{cases} L, & L = \mathbf{Z} \ \text{ or } \ 1/2 \in L, \\ L \oplus \mathbf{Z}/2\mathbf{Z}, & L = \mathbf{Z}/2^l\mathbf{Z}, \ \ l \geq 1. \end{cases}$$

Proof of Proposition 5.2. Considering the five-term exact sequence of

$$1 \to \mathrm{Inn}\, F_n \to \mathrm{Aut}\, F_n \to \mathrm{Out}\, F_n \to 1,$$

we have

$$0 \to H^1(\mathrm{Out}\, F_n, V_L) \to H^1(\mathrm{Aut}\, F_n, V_L) \xrightarrow{\alpha} H^1(\mathrm{Inn}\, F_n, V_L)^{\mathrm{Out}\, F_n}.$$

Observing

$$\alpha(f_M) = (n-1) \sum_{i=1}^{n} \iota_i^* \otimes (\boldsymbol{e}_{1,i}^1 + \boldsymbol{e}_{2,i}^2 + \overset{\check{i}}{\cdots} + \boldsymbol{e}_{n,i}^n),$$

$$\alpha(f_K) = 2 \sum_{i=1}^{n} \iota_i^* \otimes (\boldsymbol{e}_{1,i}^1 + \boldsymbol{e}_{2,i}^2 + \overset{\check{i}}{\cdots} + \boldsymbol{e}_{n,i}^n),$$

we obtain the required results. This completes the proof of Proposition 5.2. □

This induces

Theorem 5.1. *For $n \geq 5$,*

$$H_1(\mathrm{Out}\, F_n, V^*) \cong \begin{cases} \mathbf{Z}, & n : \text{even}, \\ \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, & n : \text{odd}. \end{cases}$$

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