THE JOHNSON FILTRATION OF THE MCCOOL STABILIZER SUBGROUP OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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Abstract. Let F_n be a free group of rank n with basis $x_1, x_2, \ldots, x_n$. We denote the subgroup of the automorphism group of a free group consisting of automorphisms which fix each of $x_2, \ldots, x_n$ by S_n. In this paper, we call S_n the McCool subgroup. Let IS_n be a subgroup of S_n consisting of automorphisms which induce the identity on the abelianization of the free group. The main purpose of the paper is to show the Johnson filtration of the automorphism group of a free group restricted to IS_n coincides with its lower central series. Then, we study the second integral homology group of IS_n through the second and third Johnson homomorphisms of S_n.

1. Introduction

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, x_2, \ldots, x_n$, and $F_n = \Gamma_n(1)$, $\Gamma_n(2)$, $\ldots$ its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n which induce the identity on the quotient group $F_n/\Gamma_n(k+1)$. The group $\mathcal{A}_n(1)$ is called the IA-automorpshim group and also denoted by IA_n. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, called the Johnson filtration of $\mathrm{Aut}\, F_n$. The Johnson filtration of $\mathrm{Aut}\, F_n$ was originally introduced in 1963 with a remarkable pioneer work by Andreadakis [1] who showed that $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, $\ldots$ is a descending central series of $\mathcal{A}_n(1)$, and that for each $k \geq 1$ the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. In general, to determine the structure of $\mathrm{gr}^k(\mathcal{A}_n)$ plays important roles on the study of the algebraic structure of $\mathrm{Aut}\, F_n$. For $1 \leq k \leq 3$, the rank of $\mathrm{gr}^k(\mathcal{A}_n)$ is determined. Andreadakis [1] computed the rank of $\mathrm{gr}^1(\mathcal{A}_n)$. Moreover, by an independent works of Cohen-Pakianathan [5, 6], Farb [7] and Kawazumi [13], it is known that $\mathrm{gr}^1(\mathcal{A}_n)$ is isomorphic to the abelianization of IA_n. For $k = 2$ and 3, the rank of $\mathrm{gr}^k(\mathcal{A}_n)$ is determined by Pettet [24] and Satoh [26] respectively. For $k \geq 4$, however, it seems that there are few results for the structure of $\mathrm{gr}^k(\mathcal{A}_n)$.

In this paper, we consider certain subgroups of $\mathrm{Aut}\, F_n$ and restrict the Johnson filtration to them. Let S_n be the subgroup of $\mathrm{Aut}\, F_n$ consisting of automorphisms which fix each of $x_2, \ldots, x_n$. We call S_n the McCool stabilizer subgroup of $\mathrm{Aut}\, F_n$. Let

2000 *Mathematics Subject Classification.* 20F28(Primary), 20J06(Secondly).

Key words and phrases. automrophism group of a free group, IA-automorphism group, Johnson homomorphism, second homology group.

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IS_n a subgroup of S_n consisting of automorphisms which which induce the identity on the abelianization of F_n. The groups S_n and IS_n were first studied by McCool. He [16] gave a finite presentation of S_n, and showed that IS_n is not finitely presentable. Furthermore, he [16] also gave an infinite presentation of IS_n. Set $\mathcal{S}_n(k) := \mathcal{A}_n(k) \cap \mathrm{S}_n$ for each $k \geq 0$. Then $\mathcal{S}_n(0) = \mathrm{S}_n$ and $\mathcal{S}_n(1) = \mathrm{IS}_n$. We call a descending central filtration

$$\mathrm{S}_n = \mathcal{S}_n(0) \supset \mathcal{S}_n(1) \supset \mathcal{S}_n(2) \supset \cdots$$

the Johnson filtration of S_n. Set $\mathrm{gr}^k(\mathcal{S}_n) := \mathcal{S}_n(k)/\mathcal{S}_n(k+1)$. The first purpose of the paper is to determine the structure of $\mathrm{gr}^k(\mathcal{S}_n)$.

In order to study $\mathrm{gr}^k(\mathcal{S}_n)$, we consider the Johnson homomorphisms of S_n. Let H be the abelianization of F_n and $H^* = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H. Let $\mathcal{L}_n = \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ be the free graded Lie algebra generated by H and $r_n(k)$ the rank of $\mathcal{L}_n(k)$ as a free abelian group. Then for each $k \geq 1$, a $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphim

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

is defined. (For definition, see Subsection 2.4.) This is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. Historically, the study of the Johnson homomorphisms was begun in 1980 by D. Johnson [11]. He studied the Johnson homomorphism of a mapping class group of a closed oriented surface, and determined the abelianization of the Torelli group. (See [12].) There is a broad range of remarkable results for the Johnson homomorphisms of a mapping class group. (For example, see [8] and [19].) We denote by τ_k^S the restriction of the Johnson homomorphism τ_k to $\mathrm{gr}^k(\mathcal{S}_n) \subset \mathrm{gr}^k(\mathcal{A}_n)$, and call it the Johnson homomorphism of S_n. Then we completely determine the image of τ_k^S.

Theorem 1. *(= Theorem 3.1 and Corllary 3.2.) For each $k \geq 1$, the image of τ_k^S is isomorphic to $\mathcal{L}_{n-1}(k+1) \oplus \mathcal{L}_{n-1}(k)$, and hence*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{S}_n)) = r_{n-1}(k+1) + r_{n-1}(k).$$

In the study of the Johnson filtration of $\mathrm{Aut}\, F_n$, it would be also interesting to determine whether $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, ... coincides with the lower central series $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, ... of $\mathcal{A}_n(1)$ or not. Andreadakis [1] showed that $\mathcal{A}_2(k) = \mathcal{A}'_2(k)$ and $\mathcal{A}_3(3) = \mathcal{A}'_3(3)$. From the results due to Cohen-Pakianathan [5, 6], Farb [7] and Kawazumi [13], we have $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ for $n \geq 3$. Furthermore, Pettet [24] obtained that $\mathcal{A}'_n(3)$ has finite index in $\mathcal{A}_n(3)$. Now it is conjectured by Andreadakis that $\mathcal{A}_n(k) = \mathcal{A}'_n(k)$ for any $n \geq 3$ and $k \geq 3$. In this paper, we show that the Johnson filtration $\mathcal{S}_n(1)$, $\mathcal{S}_n(2)$, ... coincides with the lower central series of $\mathcal{S}'_n(1)$, $\mathcal{S}'_n(2)$, ... of IS_n. Namely,

Theorem 2. *(= Theorem 3.2.) For each $k \geq 1$, we have $\mathcal{S}_n(k) = \mathcal{S}'_n(k)$.*

By a work of McCool [16], it is known that there is a surjective homomorpshim from IS_n to a free group W of rank $n-1$. (See Subsection 2.5.) This homomorphism induces surjective homomorphisms between the k-th terms of the lower central series of them for each $k \geq 1$. For $k \geq 2$, since the k-th term of the lower central series of a free group of rank ≥ 2 is a free group of infinite rank, we see that $H_1(\mathcal{S}_n(k), \mathbf{Z})$ contains a free abelian group of infinite rank. In particular, we obtain

Theorem 3. *(= Theorem 3.3.) For each $k \geq 2$, $H_1(\mathcal{S}_n(k), \mathbf{Q})$ is infinitly generated as a $\mathbf{Q}$-vector space.*

Next, we consider the integral second homology and cohomology groups of IS_n. In general, since no presentation of IA_n is obtained, it is difficult to study the second homology of IA_n. Bestvina-Bux-Margalit [3] showed that $H_2(\mathrm{IA}_3, \mathbf{Z})$ is not finitely generated. By a recent work of Pettet [24], the image of the cup product of the first cohomology groups of IA_n in the second cohomology is determined. However, it is not known whether $H_2(\mathrm{IA}_n, \mathbf{Z})$ is finitely generated or not for $n \geq 4$. On the other hand, McCool [16] showed that IS_n is finitely generated by automorphisms K_{1i} for $2 \leq i \leq n$ and K_{1ij} for $2 \leq j < i \leq n$, (For details, see Subsection 2.2.), but is not finitely presentable. Hence, it seems that the structure of the second homology group of IS_n are not so simple. In this paper, we study non-trivial second homology classes of IS_n which can be detected using the second and third Johnson homomorphisms. In particular, we show

Theorem 4. *(= Theorem 4.1.) $H_2(\mathrm{IS}_n, \mathbf{Z})$ contains a free abelian group of rank*

$$\frac{1}{24}n(n-1)(n-2)(n^2+6n-15).$$

To show this, we study the structure of the second homology group of IS_n using combinatorical group theory. Let F be a free group generated by K_{1i} and K_{1jk}. The rank of F is $m := n(n-1)/2$. Let $\varphi : F \to \mathrm{IS}_n$ a natural surjection and R the kernel of φ. Then considering the homological five-term exact sequence of a group extension

$$1 \to R \to F \xrightarrow{\varphi} \mathrm{IS}_n \to 1,$$

we see $H_1(R, \mathbf{Z})_{\mathrm{IS}_n} \cong H_2(\mathrm{IS}_n, \mathbf{Z})$. (For example, see theorem 8.1 in [10] for the five-term exact sequence of a group extension.) For the lower central series $\Gamma_F(1) \supset \Gamma_F(k) \supset \cdots$ of F, set $R_k := R \cap \Gamma_F(k)$ and $\overline{R}_k := R/R_k$. Then we have a surjective homorphism

$$\psi_k : H_1(R, \mathbf{Z})_{\mathrm{IS}_n} \to H_1(\overline{R}_{k+1}, \mathbf{Z})_{\mathrm{IS}_n}.$$

Our strategy is to detect non-trivial second homology classes of IS_n through ψ_k by studying the structure of the target of ψ_k. In particular, we determine the structure of $H_1(\overline{R}_{k+1}, \mathbf{Z})_{\mathrm{IS}_n}$ for $k = 2$ and 3. (See Subsection 4.2.)

On the other hand, by a cohomological argument similar to above, we also have $H^1(R, \mathbf{Z})^{\mathrm{IS}_n} \cong H^2(\mathrm{IS}_n, \mathbf{Z})$ and an injective homomorphism

$$\psi^k : H^1(\overline{R}_{k+1}, \mathbf{Z})^{\mathrm{IS}_n} \to H^1(R, \mathbf{Z})^{\mathrm{IS}_n}.$$

Hence we can consider $H^1(\overline{R}_{k+1}, \mathbf{Z})^{\mathrm{IS}_n}$ as a subgroup of $H^2(\mathrm{IS}_n, \mathbf{Z})$. In Subsection 4.3, we show $H^1(\overline{R}_3, \mathbf{Z})^{\mathrm{IS}_n} = H^1(\overline{R}_3, \mathbf{Z})$ and

Theorem 5. *(= Lemma 4.1 and Proposition 4.2.) For $n \geq 3$,*

(1) $H^1(\overline{R}_3, \mathbf{Z}) \cong \mathbf{Z}^{\oplus (n-1)(n-2)(n-3)(3n+4)/24}$.
(2) $H^1(\overline{R}_3, \mathbf{Z})$ *is the image of the cup product*

$$\cup : \Lambda^2 H^1(\mathrm{IS}_n, \mathbf{Z}) \to H^2(\mathrm{IS}_n, \mathbf{Z}).$$

In Section 2, we recall the definition and some properties of the IA-automorphism group, the Johnson homomorphisms and the McCool stabilizer subgroup of the automorphism group of a free group. In Section 3, we determine the image of the Johnson homomorphism of IS_n and show that the Johnson filtration of IS_n is its lower central series. In Section 4, we study the second homology and cohomology of IS_n.

Contents

2. Preliminaries

In this section, we recall the definition and some properties of the IA-automorphism group, the Johnson homomorphisms and the McCool stabilizer subgroup of the automorphism group of a free group.

2.1. **Notation.**

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group automorphism group $\mathrm{Aut}\, G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ without no confusion.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. **IA-automorphism group.**

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, \ldots, x_n$. We denote the abelianization of F_n by H, and its dual group by $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. Let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced from the abelianization of F_n. In this paper we identifies $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H as a free abelian group induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of

ρ is called the IA-automorphism group of F_n. It is well known due to Nielsen [21] that IA_2 coincides with the inner automorphsim group $\mathrm{Inn}\, F_2$ of F_2. Namely, IA_2 is a free group of rank 2. However, IA_n for $n \geq 3$ is much larger than the inner automorphism group $\mathrm{Inn}\, F_n$. Indeed, Magnus [15] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : \begin{cases} x_i & \mapsto {x_j}^{-1} x_i x_j, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct $i,\ j \in \{1, 2, \ldots, n\}$ and

$$K_{ijk} : \begin{cases} x_i & \mapsto x_i x_j x_k {x_j}^{-1} {x_k}^{-1}, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct $i,\ j,\ k \in \{1, 2, \ldots, n\}$ such that $j > k$.

For any $n \geq 3$, although a generating set of IA_n is well known as above, any presentation of IA_n is still not known. For $n = 3$, Krstić and McCool [14] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is also not known whether IA_n is finitely presentable or not. Recently, Cohen-Pakianathan [5, 6], Farb [7] and Kawazumi [13] independently showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module.

2.3. **Free Lie algebra.**

In this subsection we recall the definition and some properties of the free Lie algebra, which are required to define and study the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by the rule

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2.$$

We denote by $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ the graded quotient of the lower central series of F_n, and by $\mathcal{L}_n := \bigoplus_{k\geq 1} \mathcal{L}_n(k)$ the associated graded sum. Since the group $\mathrm{Aut}\, F_n$ naturally acts on $\mathcal{L}_n(k)$ for each $k \geq 1$, and since IA_n acts on it trivially, the action of $\mathrm{GL}(n, \mathbf{Z})$ on each $\mathcal{L}_n(k)$ is well-defined. Furthermore, the graded sum $\mathcal{L}_n$ naturally has a graded Lie algebra structure induced from the commutator bracket on F_n, and called the free Lie algebra generated by H. (See [25] for basic material concerning free Lie algebra.) It is classically well known due to Witt [27] that each $\mathcal{L}_n(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank

$$r_n(k) := \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}} \tag{2}$$

where μ is the Möbius function. For example, the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathcal{L}_n(k)$ for $1 \leq k \leq 3$ is given by

$$\mathcal{L}_n(1) = H, \quad \mathcal{L}_n(2) = \Lambda^2 H,$$
$$\mathcal{L}_n(3) = (H \otimes_{\mathbf{Z}} \Lambda^2 H) \big/ \langle x \otimes y \wedge z + y \otimes z \wedge x + z \otimes x \wedge y \mid x, y, z \in H \rangle.$$

It is well known that Hall [9] gave a basis of the free abelian group $\mathcal{L}_n(k)$ with basic commutators. The basic commutators have both weight and ordering. They are defined inductively as follows: First, the basic commutators of weight 1 are $x_1, \dots, x_n$ and the ordering is $x_1 < \dots < x_n$. Now assume we have defined the basic commutators together with their ordering for all weights less than k. Then the basic commutators of weight k are the elements of the form $c = [c_1, c_2]$ where c_1, c_2 are the basic commutators of weight k_1, k_2 such that

- $k = k_1 + k_2$,
- $c_1 > c_2$,
- If $c_1 = [c_3, c_4]$, then $c_4 \leq c_2$.

Furthermore, we define the ordering satisfying:

- Any basic commutators of weight k is greater than those of lower weight,
- Two basic commutators of weight k are ordered lexicographically.

For example, $[x_i, x_j]$ for $i > j$, and $[[x_i, x_j], x_k]$ for $i > j \leq k$ are basic commutators of weight 2 and 3 respectively. Hall [9] showed that the set of basic commutators of weight k forms a basis of $\mathcal{L}_n(k)$ as a free abelian group. In this paper, we fix this basis of $\mathcal{L}_n(k)$, and call it the Hall's basis.

Next, we consider an embeddings of the free Lie algebra into the tensor algebra. Let $T(H)$ be the tensor algebra of H over $\mathbf{Z}$. Then $T(H)$ is the universal envelopping algebra of the free Lie algebra $\mathcal{L}_n$, and the natural map $\iota : \mathcal{L}_n \to T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for X, $Y \in \mathcal{L}_n$ is an injective graded Lie algebra homomorphism. We denote by ι_k be the homomorphism of degree k part of ι, and consider $\mathcal{L}_n(k)$ as a submodule $H^{\otimes k}$ through ι_k.

Finally, we consider a Lie subalgebra of $\mathcal{L}_n$ generated by $x_2, \dots, x_n$. Let F' be a subgroup of F_n generated by $x_2, x_3, \dots, x_n$. The group F' is a free group of rank $n-1$. We denote the lower central series of F' by $\Gamma'(1), \Gamma'(2), \cdots$, and denote its graded quotient by $\mathcal{L}'(l) := \Gamma'(l)/\Gamma'(l+1)$ for each $l \geq 1$. Then $\mathcal{L}'(l) \cong \mathcal{L}_{n-1}(l)$ as an abelian group. The restriction of a natural inclusion map $F' \hookrightarrow F_n$ to $\Gamma'(k)$ induces a homomorphism $\alpha_k : \mathcal{L}'(k) \to \mathcal{L}_n(k)$. Since the set of all basic commutators of weight k among the components $x_2, x_3, \dots, x_n$, which is a basis of $\mathcal{L}'(k)$, is embedded into the set of the basic commutators of weight k in $\mathcal{L}_n(k)$ by α_k, the homomorphism α_k is injective. In this paper, we identify $\mathcal{L}'(k)$ with $\alpha_k(\mathcal{L}'(k))$. In particular, $\mathcal{L}'(k)$ is considered as a direct summand of $\mathcal{L}_n(k)$ as an abelian group.

Here we remark $\Gamma_n(k) \cap F' = \Gamma'(k)$ for each $k \geq 1$. To prove this, it suffices to show $\Gamma_n(k) \cap F' \subset \Gamma'(k)$. Suppose $x \in \Gamma_n(k) \cap F'$ and $x \notin \Gamma'(k)$. Then there exists some $l \in \{1, 2, \dots, k-1\}$ such that $x \in \Gamma'(l)$ and $x \notin \Gamma'(l+1)$. On the other hand, since $x \in \Gamma_n(k)$, $x = 0$ in $\mathcal{L}'(l) \subset \mathcal{L}_n(l)$. Hence $x \in \Gamma'(l+1)$. This is a contradiction.

2.4. **Johnson homomorphisms.**

In this subsection, we recall the definition and some properties of the Johnson homomorphisms. To begin with, we consider a descending filtration of $\mathrm{Aut}\, F_n$ called the Johnosn filtration. For $k \geq 0$, the action of $\mathrm{Aut}\, F_n$ on each nilpotent quotient $F_n/\Gamma_n(k+1)$ induces a homomorphism

$$\rho^k : \mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

The map ρ^0 is trivial, and $\rho^1 = \rho$. We denote the kernel of ρ^k by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, with $\mathcal{A}_n(1) = IA_n$. We call it the Johnson filtration of $\mathrm{Aut}\, F_n$. For each $k \geq 1$, the group $\mathrm{Aut}\, F_n$ acts on $\mathcal{A}_n(k)$ by conjugation, and it naturally induces an action of $\mathrm{GL}(n, \mathbf{Z})$ on $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$. The graded sum $\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ has a graded Lie algebra structure induced from the commutator bracket on IA_n.

To study the $\mathrm{GL}(n, \mathbf{Z})$-module structure of each graded quotient $\mathrm{gr}^k(\mathcal{A}_n)$, we define the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. For each $k \geq 1$, define a homomorphism $\mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \;\mapsto\; (x \mapsto x^{-1}x^{\sigma}), \quad x \in H.$$

Then the kernel of this homomorphism is just $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)).$$

The homomorphsim τ_k is called the k-th Johnson homomorphism of $\mathrm{Aut}\, G$. It is easily seen that each τ_k is $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism. Since each Johnson homomorphism τ_k is injective, to determine the cokernel of τ_k is an important problem on the study of the structure of $\mathrm{gr}^k(\mathcal{A}_n)$ as a $\mathrm{GL}(n, \mathbf{Z})$-module.

Andreadakis [1] showed that the first Johnson homomorphism τ_1 is surjective by computing the image of the generators of IA_n above. It is well known that τ_1 is nothing but the abelianization of IA_n by Cohen-Pakianathan [5, 6], Farb [7] and Kawazumi [13]. Recently, Pettet [24] determined the cokernel of $\tau_{2,\mathbf{Q}}$, and in our previous paper [26], we determined those of τ_2 and $\tau_{3,\mathbf{Q}}$. For $k \geq 4$, however, the $\mathrm{GL}(n, \mathbf{Z})$-module structure of the cokernel of τ_k is not determined.

2.5. McCool stabilizer subgroup.

Here we consider the McCool stabilizer subgroup. Let S_n be the subgroup of $\mathrm{Aut}\, F_n$ consisting of automorphisms which fix each of $x_2, \ldots, x_n$. We call S_n the McCool stabilizer subgroup. We denote the intersection of S_n with IA_n by IS_n. McCool [16] showed that IS_n is finitely generated but not finitely presentable. He [16] also gave an inifinite presentation of IS_n.

For any $i \in \{2, \ldots, n\}$, let v_i be the automorphism of F_n which send x_1 to $x_1 x_i$ and fix the other generators x_t. The subgroup V of $\mathrm{Aut}\, F_n$ generated by all v_i is a free group of rank $n-1$. The subgroup W of IA_n generated by all K_{1i} is also a free group of rank $n-1$. Then McCool [16] showed that IS_n is a semidirect product of $[V, V]$ by W. Namely, we have a split group extension

$$1 \to [V, V] \to \mathrm{IS}_n \to W \to 1. \tag{3}$$

Furthermore, he [16] showed that $[V, V]$ is the normal closure of $\{K_{1ij} \,|\, i > j\}$ in IS_n, and IS_n is generated by K_{1i} and K_{1ij}. Thus, considering a homomorphism $\mathrm{IS}_n \hookrightarrow \mathrm{IA}_n \to \mathrm{IA}^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H$, we see that $H_1(\mathrm{IS}_n, \mathbf{Z})$ is a free abelian group of rank $n(n-1)/2$ with basis $\{K_{1i}, K_{1jk} \,|\, 2 \leq i, j, k \leq n,\ j > k\}$. In the following, we fix this basis and its dual basis $\{K_{1i}^*, K_{1jk}^*\}$ of $H^1(\mathrm{IS}_n, \mathbf{Z})$.

In this paper, we mainly consider the Johnson filtration of $\mathrm{Aut}\, F_n$ restricted to S_n. Namely, set $\mathcal{S}_n(k) := \mathcal{A}_n(k) \cap \mathrm{S}_n$ for each $k \geq 0$. Then $\mathcal{S}_n(0) = \mathrm{S}_n$ and $\mathcal{S}_n(1) = \mathrm{IS}_n$. We call a descending central filtration

$$\mathrm{S}_n = \mathcal{S}_n(0) \supset \mathcal{S}_n(1) \supset \mathcal{S}_n(2) \supset \cdots$$

the Johnson filtration of S_n. Set $\mathrm{gr}^k(\mathcal{S}_n) := \mathcal{S}_n(k)/\mathcal{S}_n(k+1)$. We denote by τ_k^S the restriction of the Johnson homomorphism τ_k to $\mathrm{gr}^k(\mathcal{S}_n) \subset \mathrm{gr}^k(\mathcal{A}_n)$, and call it Johnson homomorphism of S_n.

Let $\mathcal{S}_n'(1) \supset \mathcal{S}_n'(2) \supset \cdots$ be the lower central series of $\mathrm{IS}_n = \mathcal{S}_n(1)$, and set $\mathrm{gr}^k(\mathcal{S}_n') := \mathcal{S}_n'(k)/\mathcal{S}_n'(k+1)$. Then we obtain a homomorphism $\nu_k : \mathrm{gr}^k(\mathcal{S}_n') \to \mathrm{gr}^k(\mathcal{S}_n)$ induced from the inclusion $\mathcal{S}_n'(k) \hookrightarrow \mathcal{S}_n(k)$ for each $k \geq 1$.

3. The Johnson filtration of IS_n

In the following, we always assume $n \geq 3$. In this section, we determine the image of the Johnson homomorphism τ_k^S, and show that the Johnson filtaration $\mathcal{S}_n(1) \supset \mathcal{S}_n(2) \supset \cdots$ coincides with the lower central series of IS_n. Let T_k be a $\mathbf{Z}$-submudule of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ consisting of all elements type of $x_1^* \otimes A$ where $A \in \mathcal{L}'(k+1)$. Let E_k be a $\mathbf{Z}$-submodule of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ consisting of all elements type of $x_1^* \otimes [B, x_1]$ where $B \in \mathcal{L}'(k)$.

Lemma 3.1. *For any $k \geq 1$, as an abelian groups, we have isomorphisms*

(1) $T_k \cong \mathcal{L}'(k+1)$,
(2) $E_k \cong \mathcal{L}'(k)$.

Furthermore, the sum $T_k + E_k$ in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ is a direct sum.

Proof. The part (1) is trivial. For the part (2), let $f_k : \mathcal{L}'(k) \to E_k$ be a homomorphism defined by $f_k(B) := x_1^* \otimes [B, x_1]$ for any $B \in \mathcal{L}'(k)$. We construct the inverse of f_k as follows. First, using a contraction map, we define a homomorphisms $\mu^k : H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$ by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto -x_i^*(x_{j_1}) \cdot x_{j_2} \otimes \cdots \otimes x_{j_{k+1}},$$

and

$$\Phi^k := \mu^k \circ (id_{H^*} \otimes \iota_n^{k+1}) : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We denote the restriction of Φ^k to E_k by g_k. Then identifying $\mathcal{L}'(k)$ with its image of an injective homomorphism $\mathcal{L}'(k) \xrightarrow{\alpha_k} \mathcal{L}_n(k) \xrightarrow{\iota_k} H^{\otimes k}$, we obtain a homomorphism

$$g_k : E_k \to \mathcal{L}'(k). \tag{4}$$

It is easily seen that g_k is the inverse homomorphism of f_k. This shows the part of (2).

Finally, we show that the sum $T_k + E_k$ in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ is a direct sum. Let $\gamma \in T_k \cap E_k$. Since $\gamma \in T_k$, $\Phi^k(\gamma) = 0$. Hence $g_k(\gamma) = 0$. Since g_k is injective, we have $\gamma = 0$. This completes the proof of Lemma 3.1. □

For any $a \in F'$, let v_a be an automorphism of F_n which maps x_1 to $x_1 a$ and fix the other generators x_t. Then a map $\psi_V : F' \to V$ defined by $\psi_V(a) := v_a$ is an isomorphism. Similarly, For any $b \in F'$, let w_b be an automorphism of F_n which maps x_1 to $b^{-1} x_1 b$ and fix the other generators x_t. Then a map $\psi_W : F' \to W$ defined by $\psi_W(b) := w_b$ is also an isomorphism.

Lemma 3.2. *For each $k \geq 1$, we have:*

(1) *For any $b \in \Gamma'(k)$, $w_b \in \mathcal{S}'_n(k)$.*
(2) *For any $a \in \Gamma'(k+1)$, $v_a \in \mathcal{S}'_n(k)$.*

Proof. The part (1) is immediately follows from the restriction of ψ_W to $\Gamma'(k)$. For the part (2), it suffices to show the lemma in the case where a is a simple $(k+1)$-fold commutator

$$[a_1, \ldots, a_{k+1}] := [[\cdots[[a_1, a_2], a_3], \ldots], a_n] \in \Gamma'(k+1)$$

for any $a_i \in F'$ since such commutators generate $\Gamma'(k+1)$ and ψ_V is a homomorphism. We use the induction on $k \geq 1$. Suppose $k = 1$. For any $a \in \Gamma'(2)$, considering the isomorphism ψ_V, we obtain $v_a \in [V, V] \subset \mathrm{IS}_n = \mathcal{S}'_n(1)$. Assume $k \geq 2$. By the inductive hypothesis, there exists some $v_{a'} \in \mathcal{S}'_n(k-1)$ for $a' := [a_1, \ldots, a_k]$. From the part (1), we see $w_{a_{k+1}} \in \mathcal{S}'_n(1)$. Then we obtain $v_a = [w_{a_{k+1}}^{-1}, v_{a'}^{-1}] \in \mathcal{S}'_n(k)$. This completes the proof of Lemma 3.2. $\square$

Theorem 3.1. *For each $k \geq 1$, the image of τ_k^S is $T_k \oplus E_k$ in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$.*

Proof. Let denote the image of τ_k^S by $\mathrm{Im}(\tau_k^S)$. First, we show $\mathrm{Im}(\tau_k^S) \subset T_k \oplus E_k$. For any $\sigma \in \mathcal{S}_n(k)$, by the split extension (3), there are $v \in [V, V]$ and $w \in W$ such that $\sigma = vw$. Set $x_1^v := x_1 x$ and $x_1^w := y^{-1} x_1 y$ where $x, y \in F'$. Then $x_1^{-1} x_1^\sigma = [x_1^{-1}, y^{-1}]x \in \Gamma_n(k+1)$. Here we show $y \in \Gamma'(k)$. If $y \notin \Gamma'(k)$, there is some $l \in \{1, \ldots, k-1\}$ such that $y \in \Gamma'(l)$ and $y \notin \Gamma'(l+1)$. Then $\sigma \in \mathcal{S}_n(l)$ since $[x_1^{-1}, y^{-1}]$ and $[x_1^{-1}, y^{-1}]x$ are in $\Gamma_n(l+1)$. Thus, in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(l+1)$,

$$0 = \tau_l^S(\sigma) = x_1^* \otimes x_1^{-1} x_1^\sigma = x_1^* \otimes ([x_1^{-1}, y^{-1}]x) = x_1^* \otimes [x_1, y] + x_1^* \otimes x.$$

Since $x_1^* \otimes [x_1, y] \in E_l$ and $x_1^* \otimes x \in T_l$, we have $x_1^* \otimes [x_1, y] = x_1^* \otimes x = 0$ in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(l+1)$ by Lemma 3.1. In particular, considering the isomorphism $g_l : E_l \to \mathcal{L}'(l)$ defined in (4), we obtain $y = 0 \in \mathcal{L}'(l)$. Hence $y \in \Gamma'(l+1)$. This is a contradiction. Therefore we conclude $y \in \Gamma'(k)$ and $x \in \Gamma'(k+1)$, and hence $\mathrm{Im}(\tau_k^S) \subset T_k \oplus E_k$.

Next we show $\mathrm{Im}(\tau_k^S) \supset T_k \oplus E_k$. For any element $x_1^* \otimes A \in T_k$, let $a \in \Gamma'(k+1)$ represent A. Then $v_a \in \mathcal{S}'_n(k)$ by Lemma 3.2, and $\tau_k^S(v_a) = x_1^* \otimes A$. Similarly, for any element $x_1^* \otimes [B, x_1] \in E_k$, let $b \in \Gamma'(k)$ represent B. Then $w_b \in \mathcal{S}'_n(k)$ by Lemma 3.2, and $\tau_k^S(w_b^{-1}) = x_1^* \otimes [B, x_1]$. This completes the proof of Theorem 3.1. $\square$

As a corollary, since $\mathcal{L}'(k) \cong \mathcal{L}_{n-1}(k)$ as an abelian group, we have

Corollary 3.1. *For each $k \geq 1$, $\mathrm{rank}_{\mathbf{Z}}(\mathrm{Im}(\tau_k^S)) = r_{n-1}(k+1) + r_{n-1}(k)$.*

Furthermore, observing the latter part of the proof of Theorem 3.1 we see that $\tau_k^S \circ \nu_k : \mathrm{gr}^k(\mathcal{S}'_n) \to T_k \oplus E_k$ is surjective. Since τ_k^S is injective, we have

Corollary 3.2. *For each $k \geq 1$, the natural homomorphism $\nu_k : \mathrm{gr}^k(\mathcal{S}'_n) \to \mathrm{gr}^k(\mathcal{S}_n)$ is surjective.*

Next we show that each ν_k is injective. Then we obtain $\mathcal{S}_n(k) = \mathcal{S}'_n(k)$ for each $k \geq 1$ by an inductive argument. Here we introduce generators of $\mathcal{S}'_n(k)$.

Lemma 3.3. *For each $k \geq 1$, $\mathcal{S}'_n(k)$ is generated by automorphisms*

(G1) $v_a \in [V, V]$ *for $a \in \Gamma'(k+1)$,* (G2) $w_b \in W$ *for* $b \in \Gamma'(k)$.

Proof. We prove the lemma by the induction on $k \geq 1$. For $k = 1$, it is clear from (3). Suppose that $k \geq 2$. Then $\mathcal{S}'_n(k)$ is generated by elements $[\sigma, \tau]$ for any generators $\sigma \in \mathcal{S}'_n(l)$ and $\tau \in \mathcal{S}'_n(m)$ such that $l + m = k$. By the inductive hypothesis, we may assume that σ and τ are automorphisms of type (G1) or type (G2). If $(\sigma, \tau) = (v_a, v_{a'})$ then $[\sigma, \tau] = v_{[a'^{-1}, a^{-1}]}$. Similarly, if $(\sigma, \tau) = (v_a, w_b)$ or (w_b, v_a) then $[\sigma, \tau] = v_{[b^{-1}, a^{-1}]}$ or $v_{[a^{-1}, b^{-1}]}$ respectively. Finally, if $(\sigma, \tau) = (w_b, w_{b'})$ then $[\sigma, \tau] = w_{[b'^{-1}, b^{-1}]}$. This completes the proof of Lemma 3.3. $\square$

Proposition 3.1. *For each* $k \geq 1$, $\mathrm{gr}^k(\mathcal{S}'_n)$ *is a free group of rank* $r_{n-1}(k+1) + r_{n-1}(k)$.

Proof. It suffices to show that $\mathrm{gr}^k(\mathcal{S}'_n)$ is generated by $r_{n-1}(k+1) + r_{n-1}(k)$ elements since there is a surjective homomorphism ν_k from $\mathrm{gr}^k(\mathcal{S}'_n)$ to a free abelian group of rank $r_{n-1}(k+1) + r_{n-1}(k)$.

Let $c_1, \ldots, c_p \in \Gamma'(k+1)$ be the basic commutators of weight $k+1$ among the component $x_2, \ldots, x_n$ such that $c_1 < \cdots < c_j$. Choose an automorphism $v_{c_i} \in \mathcal{S}'_n(k)$ for each i, and fix it. Similarly, let $d_1, \ldots, d_q$ be the basic commutators of weight k among the component $x_2, \ldots, x_n$ such that $d_1 < \cdots < d_q$. Choose an automorphisms $w_{d_j} \in \mathcal{S}'_n(k)$ for each j, and fix. We show v_{c_i}, $1 \leq i \leq p$ and w_{d_j}, $1 \leq j \leq q$ generate $\mathrm{gr}^k(\mathcal{S}'_n)$. By Lemma 3.3, we have the generators v_a and w_b of $\mathcal{S}'_n(k)$. For any v_a, we can write

$$a = c_1^{e_1} \cdots c_p^{e_p} a'$$

for some $e_i \in \mathbf{Z}$ and $a' \in \Gamma'(k+2)$ since $\{c_1, \ldots, c_p\}$ is a basis of $\mathcal{L}'(k+1)$. Using Lemma 3.2, we see

$$v_a v_{c_1}^{-e_1} \cdots v_{c_p}^{-e_p} = v_{a'} \in \mathcal{S}'_n(k+1).$$

This shows that $v_a = v_{c_1}^{e_1} \cdots v_{c_p}^{e_p} \in \mathrm{gr}^k(\mathcal{S}'_n)$. Similarly, for any w_b, we can write

$$b = d_1^{f_1} \cdots d_q^{f_q} b'$$

for some $f_j \in \mathbf{Z}$ and $b' \in \Gamma'(k+1)$. Using Lemma 3.2, we see

$$w_b w_{d_1}^{-f_1} \cdots w_{d_q}^{-f_q} = w_{b'} \in \mathcal{S}'_n(k+1).$$

This shows that $w_b = w_{d_1}^{f_1} \cdots w_{d_q}^{f_q} \in \mathrm{gr}^k(\mathcal{S}'_n)$. Therefore we conclude that v_{c_i}, $1 \leq i \leq p$ and w_{d_j}, $1 \leq j \leq q$ generate $\mathrm{gr}^k(\mathcal{S}'_n)$. This completes the proof of Proposition 3.1. $\square$

From this proposition, we see that each ν_k is injective, and hence is an isomorphism. Then we obtain:

Theorem 3.2. *For each* $k \geq 1$, *we have* $\mathcal{S}_n(k) = \mathcal{S}'_n(k)$.

Proof. We prove the theorem by induction on k. By definition, we have $\mathcal{S}_n(1) = \mathcal{S}'_n(1)$. Suppose $k \geq 2$. By the inductive hypothesis, we have $\mathcal{S}_n(k-1) = \mathcal{S}'_n(k-1)$, and hence $\mathrm{gr}^{k-1}(\mathcal{S}_n) = \mathcal{S}'_n(k-1)/\mathcal{S}_n(k)$. Then, since $\nu_{k-1} : \mathrm{gr}^{k-1}(\mathcal{S}'_n) \to \mathcal{S}'_n(k-1)/\mathcal{S}_n(k)$ is an isomorphism, we obtain $\mathcal{S}_n(k) = \mathcal{S}'_n(k)$. This completes the proof of Theorem 3.2. $\square$

In general, for each $k \geq 2$, to determine whether $H_1(\mathcal{A}_n(k), \mathbf{Z})$ is finitely generated or not is difficult problem. On the other hand, we see $H_1(\mathcal{S}_n(k), \mathbf{Z})$ is not finitely generated. Moreover we obtain:

Theorem 3.3. *For each $k \geq 2$, $H_1(\mathcal{S}_n(k), \mathbf{Q})$ is infinitly generated as a $\mathbf{Q}$-vector space.*

Proof. Let $\mathrm{IS}_n \to W$ be the surjective homomorphism defined in (3). By restricting it to $\mathcal{S}'_n(k)$, we obtain a surjective homomorphism $\mathcal{S}'_n(k) \to W(k)$ where $W(k)$ means the k-th subgroup of the lower central series of the free group W. Since $W(k)$ is a free group of infinite rank for $k \geq 2$, its abelianization is a free abelian group of infinite rank. This shows that $H_1(\mathcal{S}_n(k), \mathbf{Z})$ contains a free abelian group of infinite rank. This completes the proof of Therem 3.3. □

4. Second (co)homology of IS_n

In [16], McCool showed that IS_n is not finitely presentable. Hence, the structure of the integral second homology group of IS_n is not so simple. In this section, we study non-trivial second homology and cohomology classes of IS_n which can be detected using the second and third Johnson homomorphisms. To begin with, we consider a free group generated by K_{1i} and K_{1jk}, and study its subgroup consisting of relators among the generators K_{1i} and K_{1jk}. In Subsection 4.3, we consider the second cohomology group.

4.1. **Minimal presentation of** IS_n.

Let F be a free group on $\{K_{1i}, K_{1jk} \,|\, 1 \leq i, j, k \leq n-1,\ k < j\}$. The rank of F is $m := n(n-1)/2$. Let $\varphi : F \to \mathrm{IS}_n$ be a natural surjection and R the kernel of φ. Then we have a minimal presentation of IS_n

$$1 \to R \to F \xrightarrow{\varphi} \mathrm{IS}_n \to 1. \tag{5}$$

The word "minimal" means that the number of generators is minimal among any presentation of IS_n. Since the abelianization of IS_n is a free abelian group with basis $\{K_{1i}, K_{1jk} \,|\, 1 \leq i, j, k \leq n-1,\ k < j\}$, the induced homomorphism

$$\varphi_* : H_1(F, \mathbf{Z}) \to H_1(\mathrm{IS}_n, \mathbf{Z})$$

is an isomorphism. Hence considering the homological five-term exact sequence

$$0 = H_2(F, \mathbf{Z}) \to H_2(\mathrm{IS}_n, \mathbf{Z}) \to H_1(R, \mathbf{Z})_{\mathrm{IS}_n} \to H_1(F, \mathbf{Z}) \to H_1(\mathrm{IS}_n, \mathbf{Z}) \to 0.$$

of (5), we obtain an isomorphism

$$H_2(\mathrm{IS}_n, \mathbf{Z}) \cong H_1(R, \mathbf{Z})_{\mathrm{IS}_n}.$$

(For example, see theorem 8.1 in [10] for the five-term exact sequence of a group extension.) In order to detect non-trivial elements of $H_1(R, \mathbf{Z})_{\mathrm{IS}_n}$, we consider the graded quotients of a descending filtration of R induced from the lower central series of F. Let $F = \Gamma_F(1) \supset \Gamma_F(k) \supset \cdots$ be the lower central series of F. Set $\mathcal{L}_F(k) = \Gamma_F(k)/\Gamma_F(k+1)$ for each $k \geq 1$. Then $\mathcal{L}_F(k)$ is a free abelian group of rank $r_m(k)$. We define a linear ordering among the generators of F such that

- $K_{1i} > K_{ij}$ if $i > j$,
- $K_{1ij} > K_{1kl}$ if $(i, j) > (k, l)$,
- $K_{1ij} > K_{1k}$ for any i, j and k,

and fix this ordering. Here $(i, j) > (k, l)$ denotes the lexicographic ordering defined by

$$(i, j) > (k, l) \iff i > k, \ \text{ or } \ i = k \ \text{ and } \ j > l.$$

We use this ordering to define the Hall's basis of $\mathcal{L}_F(k)$ later.

Let $R_1 \supset R_2 \supset \cdots$ be a descending filtration of R defined by $R_k := R \cap \Gamma_F(k)$ for each $k \geq 1$. Then $R_k = R$ for $1 \leq k \leq 2$. For each $k \geq 1$, let

$$\varphi_k : \mathcal{L}_F(k) \to \mathrm{gr}^k(\mathcal{S}'_n)$$

be a homomorphism induced from the natural projection $\varphi : F \to \mathrm{IS}_n$. Observing $R_k/R_{k+1} \cong (R_k \Gamma_F(k+1))/\Gamma_F(k+1)$, we have an exact sequence

$$0 \to R_k/R_{k+1} \xrightarrow{\epsilon_k} \mathcal{L}_F(k) \xrightarrow{\varphi_k} \mathrm{gr}^k(\mathcal{S}'_n) \to 0. \tag{6}$$

Since the Johnson homomorphism τ_k is injective, the module R_k/R_{k+1} is also characterized as the kernel of the composite map $\tau_k \circ \varphi_k$. In this sequence, all three groups are free abelian groups. For $1 \leq k \leq 3$, their ranks are given as follows:

k	$r_m(k)$	$\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{S}'_n))$
1	$n(n-1)/2$	$n(n-1)/2$
2	$n(n^2-1)(n-2)/8$	$(n-1)(n-2)(2n+3)/6$
3	$n(n^2-1)(n-2)(n^2-n+2)/24$	$n(n-1)(n-2)(3n+1)/12$

k	$\mathrm{rank}_{\mathbf{Z}}(R_k/R_{k+1})$
1	0
2	$(n-1)(n-2)(n-3)(3n+4)/24$
3	$n^2(n-1)(n-2)(n^2-5)/24$

Set $\overline{R}_k := R/R_k$. Consider the right action of F on R, defined by

$$r \cdot x := x^{-1}rx, \quad r \in R, \;\; x \in F.$$

Then the natural projection $R \to \overline{R}_{k+1}$ induces a surjective homomorphism

$$\psi_k : H_1(R, \mathbf{Z})_{\mathrm{IS}_n} \to H_1(\overline{R}_{k+1}, \mathbf{Z})_{\mathrm{IS}_n}.$$

Our strategy is to detect non-trivial second homology classes of IS_n by studying the structure of each $H_1(\overline{R}_{k+1}, \mathbf{Z})_{\mathrm{IS}_n}$. In the paper, we especially consider the case where $k = 2$ and 3.

4.2. The structure of $(\overline{R}_3)_{\mathrm{IS}_n}$ and $(\overline{R}_4)_{\mathrm{IS}_n}$.

Let us consider the case where $k = 2$. With the notation above, we see $H_1(\overline{R}_3, \mathbf{Z})_{\mathrm{IS}_n} = H_1(\overline{R}_3, \mathbf{Z}) = \overline{R}_3$ since IS_n acts on $\overline{R}_3$ trivially. First, we give a basis of $\overline{R}_3$. In the following, we often identify $\overline{R}_3$ with $R\,\Gamma_F(3)/\Gamma_F(3) \subset \mathcal{L}_F(2)$ by the second isomorphism theorem in group theory. Let $\mathcal{B}$ be a subset of $\mathcal{L}_F(2)$ consisting of

$$R_{ijk} := [K_{1ij}, K_{1k}] + [K_{1jk}, K_{1i}] - [K_{1ik}, K_{1j}]$$

for $i > j > k$, and

$$S_{ijkl} := [K_{1ij}, K_{1kl}]$$

for $(i,j) > (k,l)$. Then we have:

Lemma 4.1. *The set $\mathcal{B}$ is a basis of $\overline{R}_3$ as a free abelian group.*

Proof. It is easily seen that $\tau_2^S \circ \varphi_2(R_{ijk}) = 0$ and $\tau_2^S \circ \varphi_2(S_{ijkl}) = 0$. Since $\tau_2^S : \mathrm{gr}^2(\mathcal{S}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3)$ is injective, we see $R_{ijk}, S_{ijkl} \in R/R_3 = \ker(\varphi_2)$. Let M be a $\mathbf{Z}$-submodule of $\mathcal{L}_F(2)$ generated by $\mathcal{B}$. We show the quotient module $\mathcal{L}_F(2)/M$ is a free abelian group of rank $(n-1)(n-2)(2n+3)/6$. Since there is a surjective homomorphism $\mathcal{L}_F(2)/M \to \mathcal{L}_F(2)/\overline{R}_3 \cong \mathbf{Z}^{\oplus(n-1)(n-2)(2n+3)/6}$, we can write $\mathcal{L}_F(2)/M \cong \mathbf{Z}^{\oplus r} \oplus$ (torsion part) for some $r \geq (n-1)(n-2)(2n+3)/6$.

Recall that the Hall's basis of $\mathcal{L}_F(2)$ is given by

- $[K_{1i}, K_{ij}]$ if $i > j$,
- $[K_{1ij}, K_{1kl}]$ if $(i,j) > (k,l)$,
- $[K_{1ij}, K_{1k}]$ for any i, j and k such that $i > j$.

These elements also generate $\mathcal{L}_F(2)/M$. On the other hand, in $\mathcal{L}_F(2)/M$, we can reduce the generators $[K_{1ij}, K_{1kl}]$ by S_{ijkl}. Furthermore, we can also reduce the generators $[K_{1ij}, K_{1k}]$ for $i > j > k$ by R_{ijk} remaining the generators $[K_{1ij}, K_{1k}]$ for $i > j \leq k$. Consequently, $\mathcal{L}_F(2)/M$ is generated by $[K_{1i}, K_{1j}]$ for $i > j$, and $[K_{1ij}, K_{1k}]$ for $i > j \leq k$. The number of such elements is just $(n-1)(n-2)(2n+3)/6$. Hence we see $\mathcal{L}_F(2)/M$ is a free abelian group of rank $(n-1)(n-2)(2n+3)/6$. Therefore we obtain $M = \overline{R}_3$. Finally, it is easily seen that the order of the set $\mathcal{B}$ is just $(n-1)(n-2)(n-3)(3n+4)/24$. Hence $\mathcal{B}$ is a basis of $\overline{R}_3$. This completes the proof of Lemma 4.1. $\square$

Next, we consider the case where $k = 3$, namely, the structure of $H_1(\overline{R}_4, \mathbf{Z})_{\mathrm{IS}_n}$. We show that $H_1(\overline{R}_4, \mathbf{Z})_{\mathrm{IS}_n}$ is a direct sum of $\overline{R}_3$ and $R_3/([F,R]R_4)$, and give a set of generators of $R_3/([F,R]R_4)$. Considering the long exact sequence of an exact sequence

$$0 \to R_3/R_4 \to \overline{R}_4 \to \overline{R}_3 \to 0$$

of IS_n-modules, we obtain

$$H_1(\mathrm{IS}_n, \overline{R}_3) \xrightarrow{\delta} (R_3/R_4)_{\mathrm{IS}_n} \to (\overline{R}_4)_{\mathrm{IS}_n} \to (\overline{R}_3)_{\mathrm{IS}_n} \to 0. \tag{7}$$

(For example, see proposition 6.1 in [4] for the long exact sequence.) Since $(R_3/R_4)_{\mathrm{IS}_n} = R_3/R_4$, and since $(\overline{R}_3)_{\mathrm{IS}_n} = \overline{R}_3$ is a free abelian group, we have

$$(\overline{R}_4)_{\mathrm{IS}_n} \cong \overline{R}_3 \oplus \mathrm{Coker}(\delta)$$

as an abelian group. Then,

Lemma 4.2. $\mathrm{Coker}(\delta) = R_3/([F,R]R_4)$.

Proof. First, we characterize the image of δ using a bilinear map

$$[\ ,\ ] : \mathcal{L}_F(2) \otimes \mathcal{L}_F(1) \to \mathcal{L}_F(3), \quad x \otimes y \mapsto [x,y]$$

induced from the commutator bracket in F. Restricting $[\ ,\ ]$ to $\overline{R}_3 \otimes \mathcal{L}_F(1)$, we obtain

$$[\ ,\ ]_R : \overline{R}_3 \otimes \mathcal{L}_F(1) \to R_3/R_4.$$

By the definition of the connecting homomorphism δ, we have $\mathrm{Im}(\delta) = \mathrm{Im}([\ ,\ ]_R)$. This completes of the proof of the Lemma 4.2. $\square$

Now, we study the structure of the abelian group $R_3/([F,R]R_4)$. To begin with, we consider a basis of R_3/R_4. Let $\mathcal{C}$ be a subset of $\mathcal{L}_F(3)$ consisting of

(i) $[[K_{1i}, K_{1j}], K_{1ij}]$ for $i > j$,
(ii) $[[K_{1i}, K_{1j}], K_{1kl}] + [[K_{1k}, K_{1l}], K_{1ij}]$ for $(i,j) > (k,l)$,
(iii) $[R_{ijk}, K_{1l}]$ for $i > j > k \leq l$,

(iv) $[[K_{1ij}, K_{1k}], K_{1lm}]$ for any i, j, k, l and m,
(v) $[[K_{1ij}, K_{1kl}], K_{1mp}]$ for $(i,j) > (k,l) \leq (m,p)$
where all indeces are elements of $\{2, 3, \ldots, n\}$.

Lemma 4.3. *The set $\mathcal{C}$ is a basis of R_3/R_4 as a free abelian group.*

Proof. It is easily seen that $\mathcal{C} \subset R_3/R_4$. Let N be a $\mathbf{Z}$-submodule of $\mathcal{L}_F(3)$ generated by $\mathcal{C}$. We show the quotient module $\mathcal{L}_F(3)/N$ is a free abelian group of rank $n(n-1)(n-2)(3n+1)/12$. Since there is a surjective homomorphism $\mathcal{L}_F(3)/N \to \mathcal{L}_F(3)/(R_3/R_4) \cong \mathbf{Z}^{\oplus n(n-1)(n-2)(3n+1)/12}$, we can write $\mathcal{L}_F(3)/N \cong \mathbf{Z}^{\oplus r} \oplus$ (torsion part) for some $r \geq n(n-1)(n-2)(3n+1)/12$.

Recall that the Hall's basis of $\mathcal{L}_F(3)$ is given by
(i)' $[[K_{1i}, K_{1j}], K_{1k}]$ for $i > j \leq k$,
(ii)' $[[K_{1i}, K_{1j}], K_{1kl}]$ for $i > j$,
(iii)' $[[K_{1ij}, K_{1k}], K_{1l}]$ for $k \leq l$,
(iv)' $[[K_{1ij}, K_{1k}], K_{1lm}]$ for any i, j, k, l and m,
(v)' $[[K_{1ij}, K_{1kl}], K_{1mp}]$ for $(i,j) > (k,l) \leq (m,p)$.
These elements also generate $\mathcal{L}_F(3)/N$. We reduce these generators of $\mathcal{L}_F(3)/N$ using $\mathcal{C}$. First, from (iv) and (v), we can reduce the generators (iv)' and (v)' . Next we consider (iii)'. If $j > k$, using (iii), we have

$$[[K_{1ij}, K_{1k}], K_{1l}] = -[[K_{1jk}, K_{1i}], K_{1l}] + [[K_{1ik}, K_{1j}], K_{1l}] \in \mathcal{L}_F(3)/N.$$

Furthermore, if $i > l$, using the Jacobi's identity, we have

$$[[K_{1jk}, K_{1i}], K_{1l}] = [[K_{1i}, K_{1l}], K_{1jk}] - [[K_{1jk}, K_{1l}], K_{1i}].$$

Similarly, if $j > l$,

$$[[K_{1ik}, K_{1j}], K_{1l}] = [[K_{1j}, K_{1l}], K_{1ik}] - [[K_{1ik}, K_{1l}], K_{1j}].$$

This shows that each generator (iii)' is written as a sum of the generators (iii)' for $i > j \leq k \leq l$ and (ii)'. Finally, from (i) and (ii), we can reduce the generators (ii)' for $(i,j) \leq (k,l)$ by remaining (ii)' for $(i,j) > (k,l)$. Therefore, $\mathcal{L}_F(3)/N$ is generated by $[[K_{1i}, K_{1j}], K_{1k}]$ for $i > j \leq k$, $[[K_{1i}, K_{1j}], K_{1kl}]$ for $(i,j) > (k,l)$ and $[[K_{1ij}, K_{1k}], K_{1l}]$ for $i > j \leq k \leq l$. The number of such elements is just $n(n-1)(n-2)(3n+1)/12$. Hence we see $\mathcal{L}_F(3)/N$ is a free abelian group of rank $n(n-1)(n-2)(3n+1)/12$. Therefore we have $N = R_3/R_4$. It is easily seen that the order of the set $\mathcal{C}$ is just $n^2(n-1)(n-2)(n^2-5)/24$. Hence $\mathcal{C}$ is a basis of R_3/R_4 This completes the proof of Lemma 4.3. $\square$

Next we consider $R_3/([F,R]R_4)$. Let $\mathcal{C}_1$ be a subset of R_3/R_4 consisting of

$$[[K_{1i}, K_{1j}], K_{1ij}]$$

for $i > j$,

$$[[K_{1i}, K_{1j}], K_{1kl}] + [[K_{1k}, K_{1l}], K_{1ij}]$$

for $(i,j) > (k,l)$,

$$[[K_{1tp}, K_{1r}], K_{1sq}],\ [[K_{1tp}, K_{1s}], K_{1rq}],\ [[K_{1tq}, K_{1s}], K_{1rp}],$$
$$[[K_{1sp}, K_{1t}], K_{1rq}],\ [[K_{1sq}, K_{1t}], K_{1rp}]$$

for $2 \leq p < q < r < s < t \leq n$,

$$[[K_{1sp}, K_{1q}], K_{1rp}],\ [[K_{1sp}, K_{1q}], K_{1rq}],\ [[K_{1sq}, K_{1q}], K_{1rp}],$$
$$[[K_{1sq}, K_{1r}], K_{1sp}],\ [[K_{1sq}, K_{1r}], K_{1qp}],\ [[K_{1sq}, K_{1r}], K_{1rp}],$$
$$[[K_{1sp}, K_{1r}], K_{1rq}],\ [[K_{1sp}, K_{1r}], K_{1qp}],\ [[K_{1rq}, K_{1s}], K_{1rp}],$$
$$[[K_{1sq}, K_{1s}], K_{1rp}],\ [[K_{1sp}, K_{1s}], K_{1rq}],\ [[K_{1rp}, K_{1s}], K_{1qp}]$$

for $2 \leq p < q < r < s \leq n$,

$$[[K_{1rp}, K_{1p}], K_{1qp}],\ [[K_{1rp}, K_{1q}], K_{1qp}],\ [[K_{1rp}, K_{1q}], K_{1rp}],$$
$$[[K_{1rq}, K_{1q}], K_{1qp}],\ [[K_{1rq}, K_{1q}], K_{1rp}],\ [[K_{1qp}, K_{1r}], K_{1qp}],$$
$$[[K_{1rp}, K_{1r}], K_{1qp}],\ [[K_{1rq}, K_{1r}], K_{1qp}],\ [[K_{1rq}, K_{1r}], K_{1rp}]$$

for $2 \leq p < q < r \leq n$ and

$$[[K_{1qp}, K_{1p}], K_{1qp}],\ [[K_{1qp}, K_{1q}], K_{1qp}]$$

for $2 \leq p < q \leq n$. Let $\mathcal{C}_2$ be a subset of R_3/R_4 consisting of

$$[[K_{1tp}, K_{1s}], K_{1rq}] - [[K_{1tq}, K_{1s}], K_{1rp}] + [[K_{1tr}, K_{1s}], K_{1qp}]$$

for $2 \leq p < q < r < s < t \leq n$. Then we show:

Proposition 4.1. *$R_3/([F,R]R_4) \cong \mathbf{Z}^{\oplus(n-1)(n-2)(n^3+3n^2-10n+12)/24} \oplus (\mathbf{Z}/2\mathbf{Z})^{\oplus\binom{n-1}{5}}$. Furtheremore, $\mathcal{C}_1$ is a basis of the free part, and $\mathcal{C}_2$ is a basis of the torsion part as a $\mathbf{Z}/2\mathbf{Z}$-vector space.*

Proof. From Lemma 4.1, $([F,R]R_4)/R_4 \subset R_3/R_4$ is generated by

$$[R_{ijk}, K_{1l}],\ [R_{ijk}, K_{1lm}],\ [S_{ijkl}, K_{1m}] \text{ and } [S_{ijkl}, K_{1mp}]. \tag{8}$$

Hence $R_3/([F,R]R_4)$ has a presentation as an abelian group with generators (i), ... , (v) of $\mathcal{C}$ subject to relators (8). We reduce the generators and the relators of this presentation using Tietze transformations.

First, we consider $[S_{ijkl}, K_{1mp}]$. If $(k,l) > (m,p)$, by the Jacobi identity, we have

$$[S_{ijkl}, K_{1mp}] = -[S_{klmp}, K_{1ij}] + [S_{ijmp}, K_{1kl}].$$

Hence we can remove $[S_{ijkl}, K_{1mp}]$ for $(k,l) > (m,p)$ from the set of relators, and may assume that $(i,j) > (k,l) \leq (m,p)$. Then using this relator, we can reduce the generator (v).

Next we consider $[R_{ijk}, K_{1l}]$. Suppose $l < k$. By the Jacobi identity,

$$\begin{aligned}[R_{ijk}, K_{1l}] = &- [[K_{1k}, K_{1l}], K_{1ij}] + [[K_{1ij}, K_{1l}], K_{1k}] \\ &- [[K_{1i}, K_{1l}], K_{1jk}] + [[K_{1jk}, K_{1l}], K_{1i}] \\ &+ [[K_{1j}, K_{1l}], K_{1ik}] - [[K_{1ik}, K_{1l}], K_{1j}],\end{aligned}$$

and

$$\begin{aligned}&= [R_{jkl}, K_{1i}] + [R_{ijl}, K_{1k}] - [R_{ikl}, K_{1j}] \\ &\quad - ([[K_{1i}, K_{1j}], K_{1kl}] + [[K_{1k}, K_{1l}], K_{1ij}]) - ([[K_{1i}, K_{1l}], K_{1jk}] + [[K_{1j}, K_{1k}], K_{1il}]) \\ &\quad + ([[K_{1i}, K_{1k}], K_{1jl}] + [[K_{1j}, K_{1l}], K_{1ik}]).\end{aligned}$$

Hence we remove the relator $[R_{ijk}, K_{1l}]$ for $l < k$. Then using $[R_{ijk}, K_{1l}]$ for $k \leq l$, we reduce the generator (iii).

Finally, we consider the generators $[[K_{1ij}, K_{ik}], K_{1lm}]$. To begin with, using the relator $[R_{ijk}, K_{1lm}]$, we reduce the generator $[[K_{1ij}, K_{1k}], K_{1lm}]$ for $j > k$. From the Jacobi identity, the relation $[S_{ijkl}, K_{1m}] = 0$ is equivalent to

$$[[K_{1ij}, K_{1k}], K_{1lm}] = [[K_{1lm}, K_{1k}], K_{1ij}],$$

and to

(R1) If $k \geq j$ and $k \geq m$,

$$[[K_{1ij}, K_{ik}], K_{1lm}] = [[K_{1lm}, K_{1k}], K_{1ij}],$$

(R2) If $j \leq k < m$,

$$[[K_{1ij}, K_{ik}], K_{1lm}] = -[[K_{1mk}, K_{1l}], K_{1ij}] + [[K_{1lk}, K_{1m}], K_{1ij}],$$

(R3) If $m \leq k < j$,

$$-[[K_{1jk}, K_{1i}], K_{1lm}] + [[K_{1ik}, K_{1j}], K_{1lm}] = [[K_{1lm}, K_{1k}], K_{1ij}],$$

(R4) If $k < j$ and $k < m$,

$$\begin{aligned}-[[K_{1jk}, K_{1i}], K_{1lm}] &+ [[K_{1ik}, K_{1j}], K_{1lm}] \\ &= -[[K_{1mk}, K_{1l}], K_{1ij}] + [[K_{1lk}, K_{1m}], K_{1ij}].\end{aligned}$$

By (R1), we reduce the generator $[[K_{1ij}, K_{ik}], K_{1lm}]$ for $k \geq j$, k and $(i, j) < (l, m)$. Since the relation (R3) is obtained from (R2) by exchanging the role of j and m, we remove the relation (R3). From (R2), we reduce the generator $[[K_{1ij}, K_{ik}], K_{1lm}]$ for $k < m$. Then (R4) is rewritten as follows:

(R5) If $k < m < l \leq j < i$,

$$\begin{aligned}-[[K_{1jk}, K_{1i}], &K_{1lm}] + [[K_{1ik}, K_{1j}], K_{1lm}] \\ &= [[K_{1jl}, K_{1i}], K_{1mk}] - [[K_{1il}, K_{1j}], K_{1mk}] - [[K_{1jm}, K_{1i}], K_{1lk}] \\ &\quad + [[K_{1im}, K_{1j}], K_{1lk}].\end{aligned}$$

(R6) If $k < m \leq j \leq l \leq i$ or $k < m \leq j < i \leq l$,

$$\begin{aligned}-[[K_{1lm}, K_{1i}], &K_{1jk}] + [[K_{1ik}, K_{1j}], K_{1lm}] \\ &= -[[K_{1ij}, K_{1l}], K_{1mk}] - [[K_{1lk}, K_{1i}], K_{1jm}] + [[K_{1im}, K_{1j}], K_{1lk}].\end{aligned}$$

(R7) If $k < j \leq m < l \leq i$ or $k < j \leq m \leq i \leq l$,

$$\begin{aligned}-[[K_{1lm}, K_{1i}], K_{1jk}] &- [[K_{1ik}, K_{1l}], K_{1mj}] + [[K_{1ik}, K_{1m}], K_{1lj}] \\ &= -[[K_{1ij}, K_{1l}], K_{1mk}] + [[K_{1ij}, K_{1m}], K_{1lk}].\end{aligned}$$

(R8) If $k < j < i \leq m < l$,

$$\begin{aligned}[[K_{1mi}, K_{1l}], &K_{1jk}] - [[K_{1li}, K_{1m}], K_{1jk}] - [[K_{1mj}, K_{1l}], K_{1ik}] \\ &+ [[K_{1lj}, K_{1m}], K_{1ik}] = -[[K_{1mk}, K_{1l}], K_{1ij}] + [[K_{1lk}, K_{1m}], K_{1ij}].\end{aligned}$$

Since (R7) and (R8) are obtained from (R6) and (R5) respectively by exchanging the role of indeces, we remove these relations.

Using (R5) and (R6), we reduce the generators $[[K_{1ij}, K_{1k}], K_{1lm}]$ for $i > j \leq k$, $l > m \leq k$ and $(i, j) \geq (l, m)$. Set $I := \{i, j, k, l, m\}$. We denote the order of the set I by $\sharp I$.

- Case I. $\sharp I = 5$.

For $p, q, r, s, t \in \{2, 3, \ldots, n\}$ such that $p < q < r < s < t$, let $\{i, j, k, l, m\} = \{p, q, r, s, t\}$. Then there are eight types of generators $[[K_{1ij}, K_{1k}], K_{1lm}]$:

$$\begin{aligned}
&[[K_{1tp}, K_{1r}], K_{1sq}],\ [[K_{1tq}, K_{1r}], K_{1sp}],\\
&[[K_{1tp}, K_{1s}], K_{1rq}],\ [[K_{1tq}, K_{1s}], K_{1rp}],\ [[K_{1tr}, K_{1s}], K_{1qp}],\\
&[[K_{1sp}, K_{1t}], K_{1rq}],\ [[K_{1sq}, K_{1t}], K_{1rp}],\ [[K_{1sr}, K_{1t}], K_{1qp}].
\end{aligned}$$

On the other hand, if we rewrite (R5) and (R6) using p, q, r, s, t, we obtain the following relations. For (R5), we have $(i, j, k, l, m) = (t, s, p, r, q)$ and

$$\begin{aligned}
-[[K_{1sp}, K_{1t}], K_{1rq}] + [[K_{1tp}, K_{1s}], K_{1rq}] &= [[K_{1sr}, K_{1t}], K_{1qp}]\\
&\quad - [[K_{1tr}, K_{1s}], K_{1qp}] - [[K_{1sq}, K_{1t}], K_{1rp}] + [[K_{1tq}, K_{1s}], K_{1rp}].
\end{aligned} \tag{9}$$

For (R6), we have $(i, j, k, l, m) = (t, r, p, s, q)$ or (s, r, p, t, q), and

$$\begin{aligned}
-[[K_{1sq}, K_{1t}], K_{1rp}] + [[K_{1tp}, K_{1r}], K_{1sq}] &= -[[K_{1tr}, K_{1s}], K_{1qp}]\\
&\quad - [[K_{1sp}, K_{1t}], K_{1rq}] + [[K_{1tq}, K_{1r}], K_{1sp}]
\end{aligned} \tag{10}$$

or

$$\begin{aligned}
-[[K_{1tq}, K_{1s}], K_{1rp}] + [[K_{1tq}, K_{1r}], K_{1sp}] &= -[[K_{1sr}, K_{1t}], K_{1qp}]\\
&\quad - [[K_{1tp}, K_{1s}], K_{1rq}] + [[K_{1tp}, K_{1r}], K_{1sq}]
\end{aligned} \tag{11}$$

respectively. Then considering (9) + (10), we obtain

$$\begin{aligned}
&[[K_{1sp}, K_{1t}], K_{1rq}] - [[K_{1sq}, K_{1t}], K_{1rp}] + [[K_{1sr}, K_{1t}], K_{1qp}]\\
&\quad = -[[K_{1tp}, K_{1s}], K_{1rq}] + [[K_{1tq}, K_{1s}], K_{1rp}] - [[K_{1tr}, K_{1s}], K_{1qp}].
\end{aligned} \tag{12}$$

Similarly, considering (9) + (10) + (11), we obtain

$$2([[K_{1tp}, K_{1s}], K_{1rq}] - [[K_{1tq}, K_{1s}], K_{1rp}] + [[K_{1tr}, K_{1s}], K_{1qp}]) = 0 \tag{13}$$

Using Tietze transformation, we replace the relations (9), (10) and (11) by (10), (12) and (13). Furthermore, by (10) and (12), we reduce the generator $[[K_{1tq}, K_{1r}], K_{1sp}]$ and $[[K_{1sr}, K_{1t}], K_{1qp}]$. Finally, we replace the generator $[[K_{1tr}, K_{1s}], K_{1qp}]$ by

$$[[K_{1tp}, K_{1s}], K_{1rq}] - [[K_{1tq}, K_{1s}], K_{1rp}] + [[K_{1tr}, K_{1s}], K_{1qp}].$$

- Case II. $\sharp I = 4$.

For $p, q, r, s \in \{2, 3, \ldots, n\}$ such that $p < q < r < s$, let $\{i, j, k, l, m\} = \{p, q, r, s\}$. Then there are fifteen types of generators $[[K_{1ij}, K_{1k}], K_{1lm}]$:

$$\begin{gathered}
[[K_{1sp}, K_{1q}], K_{1rp}],\ [[K_{1sp}, K_{1q}], K_{1rq}],\ [[K_{1sq}, K_{1q}], K_{1rp}], \\
[[K_{1sq}, K_{1r}], K_{1sp}],\ [[K_{1sq}, K_{1r}], K_{1qp}],\ [[K_{1sq}, K_{1r}], K_{1rp}], \\
[[K_{1sp}, K_{1r}], K_{1rq}],\ [[K_{1sp}, K_{1r}], K_{1qp}],\ [[K_{1sr}, K_{1r}], K_{1qp}], \\
[[K_{1rq}, K_{1s}], K_{1rp}],\ [[K_{1sq}, K_{1s}], K_{1rp}],\ [[K_{1sp}, K_{1s}], K_{1rq}], \\
[[K_{1rp}, K_{1s}], K_{1qp}],\ [[K_{1rq}, K_{1s}], K_{1qp}],\ [[K_{1sr}, K_{1s}], K_{1qp}].
\end{gathered}$$

We rewrite (R5) and (R6) using p, q, r, s. For (R5), we have $(i, j, k, l, m) = (s, r, p, r, q)$ and

$$[[K_{1sr}, K_{1r}], K_{1qp}] = -[[K_{1sp}, K_{1r}], K_{1rq}] + [[K_{1sq}, K_{1r}], K_{1rp}]. \tag{14}$$

For (R6), we consider the case where $k < m \leq j \leq l \leq i$ and $k < m \leq j < i \leq l$. Since $\sharp I = 4$, only one equlity in each of inequlites holds. For $k < m = j < l < i$, we have $(i, j, k, l, m) = (s, q, p, r, q)$ and

$$\begin{aligned}
[[K_{1rq}, K_{1s}], K_{1qp}] = [[K_{1sp}, K_{1q}], K_{1rq}] &- [[K_{1sq}, K_{1q}], K_{1rp}] \\
&+ [[K_{1sq}, K_{1r}], K_{1qp}].
\end{aligned} \tag{15}$$

For $k < m < j = l < i$ and $(i, j, k, l, m) = (s, r, p, r, q)$, we obtain a relaton equivalent to (14). For $k < m < j < l = i$, we have $(i, j, k, l, m) = (s, r, p, s, q)$ and

$$[[K_{1sr}, K_{1s}], K_{1qp}] = -[[K_{1sp}, K_{1s}], K_{1rq}] + [[K_{1sq}, K_{1s}], K_{1rp}]. \tag{16}$$

For $k < m = j < i < l$ and $(i, j, k, l, m) = (r, q, p, s, q)$, we obtain a relaton equivalent to (15). Then, by (14), (15) and (16), we reduce the generators $[[K_{1sr}, K_{1r}], K_{1qp}]$, $[[K_{1rq}, K_{1s}], K_{1qp}]$ and $[[K_{1sr}, K_{1s}], K_{1qp}]$.

- Case III. $\sharp I = 3$.

For $p, q, r \in \{2, 3, \ldots, n\}$ such that $p < q < r$, let $\{i, j, k, l, m\} = \{p, q, r\}$. There are nine types of generators $[[K_{1ij}, K_{1k}], K_{1lm}]$:

$$\begin{gathered}
[[K_{1rp}, K_{1p}], K_{1qp}],\ [[K_{1rp}, K_{1q}], K_{1qp}],\ [[K_{1rp}, K_{1q}], K_{1rp}], \\
[[K_{1rq}, K_{1q}], K_{1qp}],\ [[K_{1rq}, K_{1q}], K_{1rp}],\ [[K_{1qp}, K_{1r}], K_{1qp}], \\
[[K_{1rp}, K_{1r}], K_{1qp}],\ [[K_{1rq}, K_{1r}], K_{1qp}],\ [[K_{1rq}, K_{1r}], K_{1rp}].
\end{gathered}$$

We rewrite (R6) using p, q, r. Since $\sharp I = 3$, there is no relation obtained from (R5). For (R6), two equlities hold in the inequalities $k < m \leq j \leq l \leq i$ and $k < m \leq j < i \leq l$. Then, from both cases, we obtain trivial relations.

- Case IV. $\sharp I = 2$.

For $p, q \in \{2, 3, \ldots, n\}$ such that $p < q$, let $\{i, j, k, l, m\} = \{p, q\}$. There are two types of generators $[[K_{1ij}, K_{1k}], K_{1lm}]$:

$$[[K_{1qp}, K_{1p}], K_{1qp}],\ [[K_{1qp}, K_{1q}], K_{1qp}].$$

By an argument similar to that in Case III, we see that all relations obtained from (R6) are trivial.

From the argument above, we obtain a presentation of $R_3/([F,R]R_4)$ as an abelian group with generating set $\mathcal{C}_1$ and $\mathcal{C}_2$ subject to relations (13). Hence we have

$$R_3/([F,R]R_4) \cong \mathbf{Z}^{\oplus(n-1)(n-2)(n^3+3n^2-10n+12)/24} \oplus (\mathbf{Z}/2\mathbf{Z})^{\oplus\binom{n-1}{5}}.$$

This completes the proof of Proposition 4.1. □

Since there is a surjective homomorphism

$$H_2(\mathrm{IS}_n, \mathbf{Z}) \xrightarrow{\cong} H_1(R,\mathbf{Z})_{\mathrm{IS}_n} \xrightarrow{\psi_3} H_1(\overline{R}_4,\mathbf{Z})_{\mathrm{IS}_n} \cong \overline{R}_3 \oplus R_3/([F,R]R_4),$$

observing the rank of the free part of the target, we have:

Theorem 4.1. *$H_2(\mathrm{IS}_n,\mathbf{Z})$ contains a free abelian group of rank*

$$\frac{1}{24}n(n-1)(n-2)(n^2+6n-15).$$

Considering $(\mathbf{Z}/2\mathbf{Z})^{\oplus\binom{n-1}{5}} \subset R_3/([F,R]R_4)$, we can also detect other non-trivial elments in $H_2(\mathrm{IS}_n,\mathbf{Z})$. However, it seems to be difficult problem to determine whether these elements are torsion in $H_2(\mathrm{IS}_n,\mathbf{Z})$ or not.

4.3. $H^1(\overline{R}_3,\mathbf{Z})$ and cup products.

In this subsection, we consider the second cohomology of IS_n. By an argument similar to above, considering the cohomological five-term exact sequence

$$0 \to H^1(\mathrm{IS}_n,\mathbf{Z}) \to H^1(F,\mathbf{Z}) \to H^1(R,\mathbf{Z})^{\mathrm{IS}_n} \to H^2(\mathrm{IS}_n,\mathbf{Z}) \to H^2(F,\mathbf{Z}) = 0.$$

of (5), we obtain

$$H^2(\mathrm{IS}_n,\mathbf{Z}) \cong H^1(R,\mathbf{Z})^{\mathrm{IS}_n}.$$

Furthermore, the natural projection $R \to \overline{R}_{k+1}$ induces an injective homomorphism

$$\psi^k : H^1(\overline{R}_{k+1},\mathbf{Z})^{\mathrm{IS}_n} \to H^1(R,\mathbf{Z})^{\mathrm{IS}_n}.$$

Hence we can consider $H^1(\overline{R}_{k+1},\mathbf{Z})^{\mathrm{IS}_n}$ as a subgroup of $H^2(\mathrm{IS}_n,\mathbf{Z})$. In particular, we obtain

$$\begin{aligned} H^1(\overline{R}_4,\mathbf{Z})^{\mathrm{IS}_n} &= \mathrm{Hom}_{\mathbf{Z}}((\overline{R}_4)_{\mathrm{IS}_n},\mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}((\overline{R}_3),\mathbf{Z}) \oplus \mathrm{Hom}_{\mathbf{Z}}(R_3/([F,R]R_4),\mathbf{Z}), \\ &\subset H^2(\mathrm{IS}_n,\mathbf{Z}). \end{aligned}$$

This shows

Corollary 4.1. *$H^2(\mathrm{IS}_n,\mathbf{Z})$ contains a free abelian group of rank*

$$\frac{1}{24}n(n-1)(n-2)(n^2+6n-15).$$

Finally, we characterize $H^1(\overline{R}_3,\mathbf{Z}) \subset H^2(\mathrm{IS}_n,\mathbf{Z})$ using the cup product of the first cohomology classes of IS_n. Namely, we prove:

Proposition 4.2. *The image of the cup product*

$$\cup : \Lambda^2 H^1(\mathrm{IS}_n,\mathbf{Z}) \to H^2(\mathrm{IS}_n,\mathbf{Z})$$

is $H^1(\overline{R}_3,\mathbf{Z})$.

Proof. First, considering the cohomological five-term exact sequence of

$$1 \to \mathcal{S}'_n(2) \to \mathrm{IS}_n \xrightarrow{p} \mathrm{IS}_n^{\mathrm{ab}} \to 1, \tag{17}$$

we have

$$0 \to H^1(\mathrm{IS}_n^{\mathrm{ab}}, \mathbf{Z}) \to H^1(\mathrm{IS}_n, \mathbf{Z}) \to H^1(\mathcal{S}'_n(2), \mathbf{Z})^{\mathrm{IS}_n} \to H^2(\mathrm{IS}_n^{\mathrm{ab}}, \mathbf{Z}) \to H^2(\mathrm{IS}_n, \mathbf{Z}).$$

Since $H^1(\mathrm{IS}_n^{\mathrm{ab}}, \mathbf{Z}) \cong H^1(\mathrm{IS}_n, \mathbf{Z})$, and $H^1(\mathcal{S}'_n(2), \mathbf{Z})^{\mathrm{IS}_n} = H^1(\mathrm{gr}^2(\mathcal{S}'_n), \mathbf{Z})$, we obtain an exact sequence

$$0 \to H^1(\mathrm{gr}^2(\mathcal{S}'_n), \mathbf{Z}) \to H^2(\mathrm{IS}_n^{\mathrm{ab}}, \mathbf{Z}) \to H^2(\mathrm{IS}_n, \mathbf{Z}).$$

Since $H_1(\mathrm{IS}_n, \mathbf{Z})$ is free abelian group of finite rank, we have a natural isomorphism $H^2(\mathrm{IS}_n^{\mathrm{ab}}, \mathbf{Z}) \cong \Lambda^2 H^1(\mathrm{IS}_n, \mathbf{Z})$. Then the image of $p^* : H^2(\mathrm{IS}_n^{\mathrm{ab}}, \mathbf{Z}) \to H^2(\mathrm{IS}_n, \mathbf{Z})$ is regarded as that of the cup product $\cup : \Lambda^2 H^1(\mathrm{IS}_n, \mathbf{Z}) \to H^2(\mathrm{IS}_n, \mathbf{Z})$.

On the other hand, we also consider a five-term exact sequence

$$\begin{aligned} 0 \to H^1(\mathrm{gr}^2(\mathcal{S}'_n), \mathbf{Z}) \to H^1(\mathcal{L}_F(2), \mathbf{Z}) \to H^1(\overline{R}_3, \mathbf{Z})^{\mathcal{L}_F(2)} \\ \to H^2(\mathrm{gr}^2(\mathcal{S}'_n), \mathbf{Z}) \to H^2(\mathcal{L}_F(2), \mathbf{Z}) \end{aligned}$$

of (6) for $k = 2$. Since $\mathcal{L}_F(2)$ acts on $\overline{R}_3$ trivially, we have $H^1(\overline{R}_3, \mathbf{Z})^{\mathcal{L}_F(2)} = H^1(\overline{R}_3, \mathbf{Z})$. Furthermore, since $\mathrm{gr}^2(\mathcal{S}'_n) = \mathrm{gr}^2(\mathcal{S}_n)$ is a free abelian group, the second homomorphism $H^1(\mathcal{L}_F(2), \mathbf{Z}) \to H^1(\overline{R}_3, \mathbf{Z})$ is surjective. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\mathrm{gr}^2(\mathcal{S}'_n), \mathbf{Z}) & \xrightarrow{\mathrm{tg}} & H^2(\mathrm{IS}_n^{\mathrm{ab}}, \mathbf{Z}) & \xrightarrow{p^*} & H^2(\mathrm{IS}_n, \mathbf{Z}) & & \\ & & \| & & \downarrow \mu & & & & \\ 0 & \longrightarrow & H^1(\mathrm{gr}^2(\mathcal{S}'_n), \mathbf{Z}) & \xrightarrow{\varphi_2^*} & H^1(\mathcal{L}_F(2), \mathbf{Z}) & \xrightarrow{\epsilon_2^*} & H^1(\overline{R}_3, \mathbf{Z}) & \longrightarrow & 0 \end{array}$$

where tg is the transgression and μ is a natural isomorphism. Hence we obtain $\mathrm{Im}(\cup) = \mathrm{Im}(p^*) \cong \mathrm{Im}(\epsilon_2^*)$. This completes the proof of Proposition 4.2. □

Let $\{R^*_{ijk}, S^*_{ijkl}\}$ be the dual basis of $H^1(\overline{R}_3, \mathbf{Z})$ to $\mathcal{B}$. Finally, we explicitly write down each of R^*_{ijk} and S^*_{ijkl} as a cup product of some first cohomology classes of IS_n.

Lemma 4.4. *We have*

$$R^*_{ijk} = K^*_{1ij} \cup K^*_{1k} = K^*_{1jk} \cup K^*_{1i} = -K^*_{1ik} \cup K^*_{1j}$$

and

$$S^*_{ijkl} = K^*_{1ij} \cup K^*_{1kl}$$

in $H^2(\mathrm{IS}_n, \mathbf{Z})$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} H^2(\mathrm{IS}_n^{\mathrm{ab}}, \mathbf{Z}) & \xrightarrow{p^*} & H^2(\mathrm{IS}_n, \mathbf{Z}) \\ \downarrow \mu & & \uparrow \\ H^1(\mathcal{L}_F(2), \mathbf{Z}) & \xrightarrow{\epsilon_2^*} & H^1(\overline{R}_3, \mathbf{Z}) \end{array}$$

By a basic argument, we see that $\epsilon_2^* \circ \mu(K^*_{1ij} \cup K^*_{1k}) = \epsilon_2^*([K_{1ij}, K_{1k}]^*) = R^*_{ijk}$. Hence we obtain $R^*_{ijk} = K^*_{1ij} \cup K^*_{1k} \in H^1(\overline{R}_3, \mathbf{Z}) \subset H^2(\mathrm{IS}_n, \mathbf{Z})$. Similarly, the other equations are obtained. □.

5. Acknowledgments

This research is supported by JSPS Research Fellowship for Young Scientists.

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