

# ON KAWAMATA'S THEOREM

OSAMU FUJINO

ABSTRACT. We give an alternate proof to the main theorem of Kawamata's paper: Pluricanonical systems on minimal algebraic varieties. It is the first rigorous proof of Kawamata's theorem since the original argument contains a mistake. Our proof also works for varieties in class  $\mathcal{C}$ . We note that our proof is completely different from Kawamata's.

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## 1. INTRODUCTION

One of the main purposes of this paper is to cut off a chain of mistakes and troubles caused by [Ka, Theorem 4.3]. We give an alternate proof to the following famous theorem, which we call *Kawamata's theorem* in this paper. This theorem is indispensable for the abundance conjecture.

**Theorem 1.1** (cf. [KMM, Theorem 6-1-11]). *Let  $(X, B)$  be a klt pair, let  $\pi : X \rightarrow S$  be a proper surjective morphism of normal varieties. Assume the following conditions:*

- (a)  $H$  is a  $\pi$ -nef  $\mathbb{Q}$ -Cartier divisor on  $X$ ,
- (b)  $H - (K_X + B)$  is  $\pi$ -nef and  $\pi$ -abundant, and
- (c)  $\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$  and  $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$  for some  $a \in \mathbb{Q}$  with  $a > 1$ , where  $\eta$  is the generic point of  $S$ .

*Then  $H$  is  $\pi$ -semi-ample.*

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It was first proved in [Ka] on the assumption that  $S$  is a point. Kawamata's proof heavily depends on a very technical generalization of Kollár's injectivity theorem on *generalized normal crossing varieties* (see [Ka, Section 4]). Once we adopt this difficult injectivity theorem, X-method works and the proof is essentially the same as the one of Kawamata–Shokurov base point free theorem. Unfortunately, there is an ambiguity in the proof of [Ka, Theorem 4.3] (see [F2, Remark 3.10.3] and 5.1 below). Our proof is completely different from Kawamata's. His proof relies on the theory of mixed Hodge structures for reducible varieties. Our proof grew out from the theory of variation of Hodge structures, especially, Deligne's canonical extensions of Hodge bundles. We note that our method saves Kawamata's theorem but does not recover the results in [Ka, Section 4]. They are completely generalized in [F4, Chapter 2] for *embedded simple normal crossing pairs*. However, [F4] does not recover [Ka, Theorem 4.3]. Compare the arguments in [F4, Chapter 2] with Kawamata's ones. The reader can find a slight generalization of Kawamata's theorem and some other applications of our methods in [F3] and [F5].

We summarize the contents of this paper. In Section 2, we will give an alternate proof of Kawamata's theorem. By using Ambro's formula, we will reduce Kawamata's theorem to a reformulated version of Kawamata–Shokurov base point free theorem. Section 3 is an appendix, where we will quickly review Ambro's formula for the reader's convenience. In Section 4, we will prove Kawamata's theorem for varieties in class  $\mathcal{C}$ , which is [N2, Theorem 5.5]. We separate this section from Section 2 in order not to make needless confusion. In the final section: Section 5, we will make some comments on the topics related to Kawamata's theorem for the coming generation.

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We will work over an algebraically closed field  $k$  of characteristic zero throughout this paper. We adopt the language of *b-divisors* and use the standard notation of the log minimal model program. See, for example, [C].

## 2. PROOF OF KAWAMATA'S THEOREM

The following theorem is a reformulation of Kawamata–Shokurov base point free theorem. The original proof works without any changes (cf. [KMM, Theorem 3-1-1]).

**Theorem 2.1** (Base Point Free Theorem). *Let  $(X, B)$  be a sub klt pair, let  $\pi : X \rightarrow S$  be a proper surjective morphism of normal varieties and  $D$  a  $\pi$ -nef Cartier divisor on  $X$ . Assume the following conditions:*

- (1)  *$rD - (K_X + B)$  is nef and big over  $S$  for some positive integer  $r$ , and*
- (2)  *$\pi_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil + j\overline{D}) \subseteq \pi_* \mathcal{O}_X(jD)$  for any positive integer  $j$ , where  $\mathbf{A}(X, B)$  is the discrepancy  $\mathbb{Q}$ -b-divisor and  $\overline{D}$  is the Cartier closure of  $D$  (see [C, Example 2.3.12 (1) (3)]).*

*Then  $mD$  is  $\pi$ -generated for  $m \gg 0$ , that is, there exists a positive integer  $m_0$  such that for any  $m \geq m_0$  the natural homomorphism  $\pi^* \pi_* \mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD)$  is surjective.*

Before we go to the proof of Theorem 1.1, let us recall the definition of *abundant* divisors, which are called *good* divisors in [Ka]. See [KMM, §6-1].

**Definition 2.2** (Abundant divisor). Let  $X$  be a complete normal variety and  $D$  a  $\mathbb{Q}$ -Cartier nef divisor on  $X$ . We define the *numerical Iitaka dimension* to be

$$\nu(X, D) = \max\{e; D^e \not\equiv 0\}.$$

This means that  $D^{e'} \cdot S = 0$  for any  $e'$ -dimensional subvarieties  $S$  of  $X$  with  $e' > e$  and there exists an  $e$ -dimensional subvariety  $T$  of  $X$  such that  $D^e \cdot T > 0$ . Then it is easy to see that  $\kappa(X, D) \leq \nu(X, D)$ , where  $\kappa(X, D)$  denotes *Iitaka's  $D$ -dimension*. A nef  $\mathbb{Q}$ -divisor  $D$  is said to be *abundant* if the equality  $\kappa(X, D) = \nu(X, D)$  holds. Let  $\pi : X \rightarrow S$  be a proper surjective morphism of normal varieties and  $D$  a  $\mathbb{Q}$ -Cartier divisor on  $X$ . Then  $D$  is said to be  $\pi$ -abundant if  $D|_{X_\eta}$  is abundant, where  $X_\eta$  is the generic fiber of  $\pi$ .

*Proof of Theorem 1.1.* If  $H - (K_X + B)$  is  $\pi$ -big, then the statement follows from the original Kawamata–Shokurov base point free theorem. Thus, from now on, we assume that  $H - (K_X + B)$  is not  $\pi$ -big. Then there exists a diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \mu \downarrow & & \downarrow \varphi \\ X & \xrightarrow{\pi} & S \end{array}$$

which satisfies the following conditions (see [KMM, Proposition 6-1-3 and Remark 6-1-4] or [N1, Lemma 6]):

- (i)  $\mu, f$  and  $\varphi$  are projective morphism,
- (ii)  $Y$  and  $Z$  are non-singular varieties,
- (iii)  $\mu$  is a birational morphism and  $f$  is a surjective morphism having connected fibers,
- (iv) there exists a  $\varphi$ -nef and  $\varphi$ -big  $\mathbb{Q}$ -divisor  $M_0$  on  $Z$  such that

$$\mu^*(H - (K_X + B)) \sim_{\mathbb{Q}} f^*M_0,$$

and

- (v) there is a  $\varphi$ -nef  $\mathbb{Q}$ -divisor  $D$  on  $Z$  such that

$$\mu^*H \sim_{\mathbb{Q}} f^*D.$$

Note that  $f : Y \rightarrow Z$  is the Iitaka fibration with respect to  $H - (K_X + B)$  over  $S$ . We put  $K_Y + B_Y = \mu^*(K_X + B)$  and  $H_Y = \mu^*H$ . We note that  $(Y, B_Y)$  is not necessarily klt but *sub* klt. Thus, we have  $H_Y - (K_Y + B_Y) \sim_{\mathbb{Q}} f^*M_0$  (resp.  $H_Y \sim_{\mathbb{Q}} f^*D$ ), where  $M_0$  (resp.  $D$ ) is a  $\varphi$ -nef and  $\varphi$ -big (resp.  $\varphi$ -nef)  $\mathbb{Q}$ -divisor as we saw in (iv) and (v). Furthermore, we can assume that  $D$  and  $H$  are Cartier divisors and  $H_Y \sim f^*D$  by replacing  $D$  and  $H$  by sufficiently divisible multiples. If we need, we modify  $Y$  and  $Z$  birationally and can assume the following conditions:

- (1)  $K_Y + B_Y \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M)$ , where  $B_Z$  is the *discriminant*  $\mathbb{Q}$ -divisor of  $(Y, B_Y)$  on  $Z$  and  $M$  is the *moduli*  $\mathbb{Q}$ -divisor on  $Z$ ,
- (2)  $(Z, B_Z)$  is a sub klt pair,
- (3)  $M$  is a  $\varphi$ -nef  $\mathbb{Q}$ -divisor on  $Z$ ,
- (4)  $\varphi_*\mathcal{O}_Z(\lceil \mathbf{A}(Z, B_Z) \rceil + j\overline{D}) \subseteq \varphi_*\mathcal{O}_Z(jD)$  for every positive integer  $j$ , and
- (5)  $D - (K_Z + B_Z)$  is  $\varphi$ -nef and  $\varphi$ -big.

Indeed, let  $P \subset Z$  be a prime divisor. Let  $a_P$  be the largest real number  $t$  such that  $(Y, B_Y + tf^*P)$  is sub lc over the generic point of  $P$ . It is obvious that  $a_P = 1$  for all but finitely many prime divisors  $P$  of  $Z$ . We note that  $a_P$  is a positive rational number for any  $P$ . The *discriminant*  $\mathbb{Q}$ -divisor on  $Z$  defined by the following formula

$$B_Z = \sum_P (1 - a_P)P.$$

We note that  $\lfloor B_Z \rfloor \leq 0$ . By the properties (iv) and (v), we can write

$$K_Y + B_Y \sim_{\mathbb{Q}} f^*(M_1)$$

for a  $\mathbb{Q}$ -Cartier divisor  $M_1$  on  $Z$ . We define  $M = M_1 - (K_Z + B_Z)$  and call it the *moduli  $\mathbb{Q}$ -divisor* on  $Z$ , where  $B_Z$  is the discriminant  $\mathbb{Q}$ -divisor defined above. Note that  $M$  is called the *log-semistable part* in [FM, Section 4]. So, the condition (1) obviously holds by the definitions of the discriminant  $\mathbb{Q}$ -divisor  $B_Z$  and the moduli  $\mathbb{Q}$ -divisor  $M$ . If we take birational modifications of  $Y$  and  $Z$  suitably, we have that  $M$  is  $\varphi$ -nef and  $(Z, B_Z)$  is sub klt. Thus we obtain (2) and (3). For the details, see [A1, Theorems 0.2 and 2.7] or Theorem 3.2 below. We note the following lemma (cf. [A1, Lemma 6.2]), which we need to apply [A1, Theorems 0.2 and 2.7] or Theorem 3.2 to  $f : Y \rightarrow Z$  (see the condition (2) in 3.1).

**Lemma 2.3.** *We have  $\text{rank} f_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) = 1$ .*

*Proof of the lemma.* Since  $\mathcal{O}_Z \simeq f_* \mathcal{O}_Y \subseteq f_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner)$ , we know  $\text{rank} f_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) \geq 1$ . Without loss of generality, we can shrink  $S$  and assume that  $S$  is affine. Let  $A$  be a  $\varphi$ -very ample divisor such that  $f_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) \otimes \mathcal{O}_Z(A)$  is  $\varphi$ -generated. Since  $M_0$  is a  $\varphi$ -big  $\mathbb{Q}$ -divisor on  $Z$ , we have  $\mathcal{O}_Z(A) \subset \mathcal{O}_Z(mM_0)$  for a sufficiently large and divisible integer  $m$ . We note that  $\pi_* \mu_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner + \overline{f^*(mM_0)}) \simeq \pi_* \mu_* \mathcal{O}_Y(f^*(mM_0))$ , where  $\overline{f^*(mM_0)}$  is the Cartier closure of  $f^*(mM_0)$  (see [C, Example 2.3.12 (1)]). It is because  $\mu^*(H - (K_X + B)) = H_Y - (K_Y + B_Y) \sim_{\mathbb{Q}} f^* M_0$ . Therefore,

$$\begin{aligned} & \varphi_*(f_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) \otimes \mathcal{O}_Z(A)) \\ \subseteq & \varphi_*(f_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) \otimes \mathcal{O}_Z(mM_0)) \\ \simeq & \varphi_* \mathcal{O}_Z(mM_0). \end{aligned}$$

So,  $\text{rank} f_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) \leq 1$ . We finish the proof of the lemma.  $\square$

We know the following lemma by Lemma 9.2.2 and Proposition 9.2.3 in [A2] (see also Theorem 3.2 (a) below).

**Lemma 2.4.** *We have  $\mathcal{O}_Z(\ulcorner \mathbf{A}(Z, B_Z) \urcorner + j\overline{D}) \subseteq f_* \mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner + j\overline{H_Y})$  for any integer  $j$ .*

Pushing forward it by  $\varphi$ , we obtain that

$$\begin{aligned}
\varphi_* \mathcal{O}_Z(\lceil \mathbf{A}(Z, B_Z) \rceil + j\overline{D}) &\subseteq \varphi_* f_* \mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) \rceil + j\overline{H_Y}) \\
&\simeq \pi_* \mu_* \mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) \rceil + j\overline{H_Y}) \\
&\simeq \pi_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil + j\overline{H}) \\
&\simeq \pi_* \mathcal{O}_X(jH) \\
&\simeq \pi_* \mu_* \mathcal{O}_Y(jH_Y) \\
&\simeq \varphi_* f_* \mathcal{O}_Y(jH_Y) \\
&\simeq \varphi_* \mathcal{O}_Z(jD)
\end{aligned}$$

for any integer  $j$ . Thus, we have (4). The relation  $H_Y - (K_Y + B_Y) \sim_{\mathbb{Q}} f^*(D - (K_Z + B_Z + M))$  implies that  $D - (K_Z + B_Z + M)$  is  $\varphi$ -nef and  $\varphi$ -big. By (3),  $M$  is  $\varphi$ -nef. Therefore,  $D - (K_Z + B_Z) = D - (K_Z + B_Z + M) + M$  is  $\varphi$ -nef and  $\varphi$ -big. It is the condition (5). Apply Theorem 2.1 to  $D$  on  $(Z, B_Z)$ . Then we obtain that  $D$  is  $\varphi$ -semi-ample. This implies that  $H$  is  $\pi$ -semi-ample. We finish the proof.  $\square$

The following corollaries are obvious by Theorem 1.1.

**Corollary 2.5.** *Let  $(X, B)$  be a klt pair, let  $\pi : X \rightarrow S$  be a proper surjective morphism of normal varieties. Assume that  $K_X + B$  is  $\pi$ -nef and  $\pi$ -abundant. Then  $K_X + B$  is  $\pi$ -semi-ample.*

**Corollary 2.6.** *Let  $X$  be a complete normal variety such that  $K_X \sim_{\mathbb{Q}} 0$ . Assume that  $X$  has only klt singularities. Let  $H$  be a nef and abundant  $\mathbb{Q}$ -Cartier divisor on  $X$ . Then  $H$  is semi-ample.*

### 3. APPENDIX: QUICK REVIEW OF AMBRO'S FORMULA

In this appendix, we quickly review Ambro's formula. For the details, see the original paper [A1] or Kollár's survey article [Ko].

**3.1.** Let  $f : X \rightarrow Y$  be a proper surjective morphism of normal varieties. Let  $p : Y \rightarrow S$  be a proper morphism onto a variety  $S$ . Assume the following conditions:

- (1)  $K_X + B$  is  $\mathbb{Q}$ -Cartier and  $(X, B)$  is sub klt over the generic point of  $Y$ ,
- (2)  $\text{rank} f_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) = 1$ , and
- (3)  $K_X + B \sim_{\mathbb{Q}, f} 0$ .

By (3), we can write  $K_X + B \sim_{\mathbb{Q}} f^* D$  for some  $\mathbb{Q}$ -Cartier divisor  $D$  on  $Y$ . Let  $B_Y$  be the *discriminant  $\mathbb{Q}$ -divisor* on  $Y$ . For the definition, see the proof of Theorem 1.1. We put  $M_Y = D - (K_Y + B_Y)$  and call  $M_Y$  the *moduli  $\mathbb{Q}$ -divisor* on  $Y$ . Then we have  $K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y)$ .

Let  $\sigma : Y' \rightarrow Y$  be a proper birational morphism from a normal variety  $Y'$ . Then we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{\mu} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xleftarrow{\sigma} & Y' \end{array}$$

such that

- (i)  $\mu$  is birational morphism from a normal variety  $X'$ ,
- (ii) we put  $K_{X'} + B' = \mu^*(K_X + B)$ . Then we can write  $K_{X'} + B' \sim_{\mathbb{Q}} f'^*(K_{Y'} + B_{Y'} + M_{Y'})$ , where  $B_{Y'}$  is the discriminant  $\mathbb{Q}$ -divisor on  $Y'$  associated to  $f' : X' \rightarrow Y'$ .

Ambro's theorem [A1, Theorems 0.2 and 2.7] says

**Theorem 3.2.** *If we choose  $Y'$  appropriately, then we have the following properties for every proper birational morphism  $\nu : Y'' \rightarrow Y'$  from a normal variety  $Y''$ .*

- (a)  $K_{Y'} + B_{Y'}$  is  $\mathbb{Q}$ -Cartier and  $\nu^*(K_{Y'} + B_{Y'}) = K_{Y''} + B_{Y''}$ . In particular,  $\mathbf{A}(Y', B_{Y'})_{Y''} = -B_{Y''}$ .
- (b) The moduli  $\mathbb{Q}$ -divisor  $M_{Y'}$  is nef over  $S$  and  $\nu^*(M_{Y'}) = M_{Y''}$ .

We note that the nefness of the moduli  $\mathbb{Q}$ -divisor follows from Fujita–Kawamata's semi-positivity theorem. It is a consequence of the theory of variation of Hodge structures. For the details, see, for example, [M, Section 5], [F1, Section 5], or [Ko].

#### 4. KAWAMATA'S THEOREM FOR VARIETIES IN CLASS $\mathcal{C}$

In this section, we treat Nakayama's theorem: [N2, Theorem 5.5], which is Kawamata's theorem for varieties in class  $\mathcal{C}$ . First, let us recall the definition of the varieties in class  $\mathcal{C}$ .

**Definition 4.1** (Class  $\mathcal{C}$ ). A compact complex variety in class  $\mathcal{C}$  is a variety which is dominated by a compact Kähler manifold. It is known that  $X$  is in class  $\mathcal{C}$  if and only if  $X$  is bimeromorphically equivalent to a compact Kähler manifold.

Next, we recall the definitions of the Kähler cone and the nef line bundles on a compact Kähler manifold.

**Definition 4.2** (Kähler cone). Let  $Y$  be a  $d$ -dimensional compact Kähler manifold. We define the Kähler cone  $\text{KC}(Y)$  of  $Y$  to be the set

$$\{[\omega] \in H^{1,1}(Y, \mathbb{R}); \omega \text{ is a Kähler form on } Y.\},$$

where  $H^{1,1}(Y, \mathbb{R}) := H^2(Y, \mathbb{R}) \cap H^{1,1}(Y, \mathbb{C})$ . Then  $\text{KC}(Y)$  is an open convex cone in  $H^{1,1}(Y, \mathbb{R})$ .  $\overline{\text{KC}}(Y)$  is the closure of  $\text{KC}(Y)$  in  $H^{1,1}(Y, \mathbb{R})$ .

Finally, we recall the definitions of the quasi-nef line bundles, the homological Kodaira dimension, and the quasi-nef and abundant line bundles, which were introduced in [N2].

**Definition 4.3** (cf. [N2, Definition 2.4]). Let  $L$  be a line bundle on a compact Kähler manifold  $Y$ .  $L$  is said to be *nef* if the real first Chern class  $c_1(L)$  is contained in  $\overline{\text{KC}}(Y)$ .

**Remark 4.4.** For a new numerical characterization of the Kähler cone of a compact Kähler manifold, see [DP, Main Theorem 0.1]. A nef line bundle on a compact Kähler manifold can be characterized numerically by [DP, Corollaries 0.3 and 0.4].

**Definition 4.5** (cf. [N2, Definition 2.6]). Let  $X$  be a compact complex variety in class  $\mathcal{C}$ . A line bundle  $L$  on  $X$  is called *quasi-nef* if there exists a bimeromorphic morphism  $\mu : Y \rightarrow X$  from a compact Kähler manifold  $Y$  such that  $\mu^*L$  is nef.

**Definition 4.6** (cf. [N2, Definition 2.9]). Let  $L$  be a quasi-nef line bundle on  $X \in \mathcal{C}$ . Take a bimeromorphic morphism  $\mu : Y \rightarrow X$  from a compact Kähler manifold  $Y$  such that  $\mu^*L$  is nef. Then we define

$$\kappa_{\text{hom}}(L) := \max\{l \geq 0; 0 \neq c_1(\mu^*L)^l \in H^{l,l}(Y, \mathbb{R})\}$$

and call it the *homological Kodaira dimension* of  $L$ . It is well-defined, because it is independent of the choice of  $Y$ .

**Definition 4.7** (cf. [N2, Definition 2.11]). Let  $L$  be a line bundle on a compact complex variety  $X$  in class  $\mathcal{C}$ .  $L$  is said to be *big* if  $\kappa(X, L) = \dim X$ . If  $L$  is quasi-nef and  $\kappa(X, L) = \kappa_{\text{hom}}(L)$ , then  $L$  is called *abundant*.

Now, we state the main theorem of this section. It is nothing but [N2, Theorem 5.5]. The reader can find some applications of Theorem 4.8 in [COP].

**Theorem 4.8** (cf. [N2, Theorem 5.5]). *Let  $X$  be a compact normal complex variety in class  $\mathcal{C}$ ,  $B$  an effective  $\mathbb{Q}$ -divisor on  $X$ , and  $H$  a  $\mathbb{Q}$ -Cartier divisor on  $X$ . Then  $H$  is semi-ample under the following conditions:*

- (1)  $(X, B)$  is klt,
- (2)  $H$  is quasi-nef,
- (3)  $H - (K_X + B)$  is quasi-nef and abundant, and



- (4)  $\kappa_{\text{hom}}(aH - (K_X + B)) = \kappa_{\text{hom}}(H - (K_X + B))$  and  $\kappa(X, aH - (K_X + B)) \geq 0$  for some  $a \in \mathbb{Q}$  with  $a > 1$ .

*Sketch of the proof.* First, we recall Nakamaya's result.

**Lemma 4.9** ([N2, Proposition 2.14 and Corollary 2.16]). *There exists the following diagram*

$$X \xleftarrow{\mu} Y \xrightarrow{f} Z,$$

where

- (a)  $Y$  is a compact Kähler manifold and  $\mu$  is a bimeromorphic morphism,
- (b)  $Z$  is a smooth projective variety,
- (c)  $f$  is surjective and has connected fibers,
- (d) there exists a nef and big  $\mathbb{Q}$ -divisor  $M_0$  on  $Z$  such that

$$\mu^*(H - (K_X + B)) \sim_{\mathbb{Q}} f^*M_0,$$

and

- (e) there is a nef  $\mathbb{Q}$ -divisor  $D$  on  $Z$  such that

$$\mu^*H \sim_{\mathbb{Q}} f^*D.$$

We note that  $Z$  is a smooth projective variety.

Let  $f : Y \rightarrow Z$  be the proper surjective morphism from a compact Kähler manifold  $Y$  to a normal projective variety  $Z$  obtained in Lemma 4.9. Let  $B_Y$  be a  $\mathbb{Q}$ -divisor on  $Y$  such that  $K_Y + B_Y = \mu^*(K_X + B)$ . Then we have the following properties:

- (1)  $K_Y + B_Y$  is  $\mathbb{Q}$ -Cartier and  $(Y_z, B_z)$  is sub klt for general  $z \in Z$ , where  $Y_z = f^{-1}(z)$  and  $B_z = B_Y|_{Y_z}$ ,
- (2)  $\text{rank} f_* \mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) \rceil) = 1$ , and
- (3)  $K_Y + B_Y \sim_{\mathbb{Q}, f} 0$ .

We note that (1) is obvious by the definition of  $B_Y$ , (2) follows from the proof of Lemma 2.3, and (3) is also obvious by Lemma 4.9. Under these conditions (1), (2), and (3), Ambro's theorem (see [A1, Theorems 0.2 and 2.7] or Theorem 3.2) holds if we adopt [N3, 3.7. Theorem (4)] in the proof of Ambro's theorem. Note that it is not difficult to modify the arguments in [A1] for our setting. More explicitly, let  $\sigma : Z' \rightarrow Z$  be a proper birational morphism from a normal projective variety  $Z'$ . If we choose  $Z'$  appropriately, then we have the following properties for every proper birational morphism  $\nu : Z'' \rightarrow Z'$  from a normal projective variety  $Z''$ .

- (a)  $K_{Z'} + B_{Z'}$  is  $\mathbb{Q}$ -Cartier and  $\nu^*(K_{Z'} + B_{Z'}) = K_{Z''} + B_{Z''}$ , where  $B_{Z'}$  and  $B_{Z''}$  are the discriminant  $\mathbb{Q}$ -divisors. In particular,  $\mathbf{A}(Z', B_{Z'})_{Z''} = -B_{Z''}$ .

(b) The moduli  $\mathbb{Q}$ -divisor  $M_{Z'}$  is nef and  $\nu^*(M_{Z'}) = M_{Z''}$ .

For the details and the notation, see Section 3.

By applying Ambro's theorem to  $f : Y \rightarrow Z$ , the proof of Theorem 1.1 works without any modifications. We note that  $Z$  is a *projective* variety. Thus, we obtain the semi-ampleness of  $H$ .  $\square$

## 5. COMMENTS FOR THE COMING GENERATION

The results in [Ka] had already been used in various papers. We think that almost all the papers just used the main results of [Ka], that is, Theorems 1.1 and 6.1 in [Ka]. Therefore, by this paper, a great part of the trouble caused by [Ka, Theorem 4.3] were removed. Unfortunately, some authors used the arguments in [Ka]. We give some comments for the coming generation.

**5.1.** As we pointed out in [F2, Remark 3.10.3], the proof of [Ka, Theorem 4.3] is not completed (see also [KMM, Theorem 6-1-6]). We recall the trouble in [Ka] here for the reader's convenience.

We use the same notation as in the proof of Theorem 4.3 in [Ka]. By [Ka, Theorem 3.2],  $'E_1^{p,q} \rightarrow ''E_1^{p,q}$  are zero for all  $p$  and  $q$ . It does not directly say that

$$H^i(X, \mathcal{O}_X(-\lceil L \rceil)) \rightarrow H^i(D, \mathcal{O}_D(-\lceil L \rceil))$$

are zero for all  $i$ . So, the proofs of Theorems 4.4, 4.5, 5.1, and 6.1 in [Ka] do not work. It is because everything depends on Theorem 4.3 in [Ka]. Thus, we have no rigorous proofs for [KMM, Theorems 6-1-8, 6-1-9].

If someone corrects the proof of [Ka, Theorem 4.3], then the following comments are unnecessary.

**5.2.** In [N1], Nakayama obtained the relative version of Kawamata's theorem. The proof given there heavily depends on Kawamata's original proof. So, it does not work by the trouble in [Ka, Theorem 4.3]. Of course, [N1, Theorem 5] is true by our main theorem: Theorem 1.1.

**5.3.** Section 5 in [N2] contains the same trouble. It is because it depends on Kawamata's paper [Ka]. In Section 4, we give a rigorous proof to [N2, Theorem 5.5].

**5.4.** In [Fk], Fukuda obtained a slight generalization of Kawamata's theorem. See [Fk, Proposition 3.3]. In the final step of the proof of [Fk, Proposition 3.3], Fukuda used [Ka, Theorem 5.1]. So, Fukuda's original proof also has some troubles by [Ka, Theorem 4.3]. Fortunately, we can prove a slight generalization of [Fk, Proposition 3.3] in [F3, Section 6].

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DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY  
*E-mail address:* fujino@math.kyoto-u.ac.jp