BASE POINT FREE THEOREM FOR LOG CANONICAL PAIRS

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Abstract. We give a new proof to the base point free theorem for log canonical pairs.

Contents

1. Introduction

In this paper, we give a new proof to the base point free theorem for log canonical pairs. The main theorem of this paper is as follows.

Theorem 1.1. *Let X be a normal projective variety and B an effective $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let L be a nef Cartier divisor on X. Assume that $aL - (K_X + B)$ is ample for some $a > 0$. Then the linear system $|mL|$ is base point free for $m \gg 0$, that is, there is a positive integer m_0 such that $|mL|$ is base point free for any $m \geq m_0$.*

It is a very special case of [A, Theorem 5.1]. His proof depends on the theory of *quasi-log varieties*. For the details of quasi-log varieties, see [F1]. The proof given here does not need the theory of quasi-log varieties. We just need the generalized Kollár's torsion-free and vanishing theorems. See Theorem 2.1 below.

We explain our proof more precisely. By Shokurov's concentration method and a generalized Kollár's vanishing theorem, we obtain a correct generalization of Shokurov's non-vanishing theorem for log canonical pairs. By our non-vanishing theorem, we can create a new log

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canonical center, and apply the non-vanishing theorem again to this new log canonical center. Then we obtain the base point free theorem for log canonical pairs. The reader will find that our proof is very similar to the original proof for klt pairs. In some sense, the proof given in Section 3 is more natural than the original one. Anyway, we do not have to discuss difficult vanishing and torsion-free theorems for reducible varieties.

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

Notation. Let X be a normal variety and B an effective $\mathbb{Q}$-divisor such that K_X+B is $\mathbb{Q}$-Cartier. Then we can define the *discrepancy* $a(E,X,B)\in\mathbb{Q}$ for any prime divisor E *over* X. If $a(E,X,B)\geq -1$ (resp. >-1) for any E, then (X,B) is called *log canonical* (resp. *kawamata log terminal*). We sometimes abbreviate log canonical (resp. kawamata log terminal) to *lc* (resp. *klt*).

Assume that (X,B) is log canonical. If E is a prime divisor over X such that $a(E,X,B)=-1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X,B), where $c_X(E)$ is the closure of the image of E on X.

Let (X,B) be a log canonical pair and M an effective $\mathbb{Q}$-divisor on X. The *log canonical threshold* of (X,B) with respect to M is defined by

$$c=\sup\{t\in\mathbb{R}\,|\,(X,B+cM)\text{ is log canonical}\}.$$

We can easily check that c is a rational number and that $(X,B+cM)$ is lc but not klt.

Let (X,B) be a log canonical pair. Then a *stratum* of (X,B) denotes X itself or an lc center of (X,B).

Let Y be a smooth variety and T a simple normal crossing divisor on Y. Then a *stratum* of T means an lc center of the pair (Y,T).

Let r be a rational number. The integral part $\llcorner r\lrcorner$ is the largest integer $\leq r$ and the fractional part $\{r\}$ is defined by $r-\llcorner r\lrcorner$. We put $\ulcorner r\urcorner=-\llcorner -r\lrcorner$ and call it the round-up of r. For a $\mathbb{Q}$-divisor $D=\sum_{i=1}^{r}d_iD_i$, where D_i is a prime divisor for any i and $D_i\neq D_j$ for $i\neq j$, we call D a *boundary* $\mathbb{Q}$-divisor if $0\leq d_i\leq 1$ for any i. We note that $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. We put $\llcorner D\lrcorner=\sum\llcorner d_i\lrcorner D_i$, $\ulcorner D\urcorner=\sum\ulcorner d_i\urcorner D_i$, $\{D\}=\sum\{d_i\}D_i$, $D^{<1}=\sum_{d_i<1}d_iD_i$, and $D^{=1}=\sum_{d_i=1}D_i$.

We write $\mathrm{Bs}|L|$ to denote the *base locus* of the linear system $|L|$.

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2. Preliminaries

In this section, we collect preliminary results for the reader's convenience. The next theorem is a very special case of [A, Theorem 3.2].

Theorem 2.1 (Torsion-freeness and vanishing theorem)**.** *Let Y be a smooth projective variety and B a boundary $\mathbb{Q}$-divisor such that* $\mathrm{Supp}B$ *is simple normal crossing. Let $f : Y \to X$ be a projective morphism and L a Cartier divisor on Y such that $H \sim_{\mathbb{Q}} L-(K_Y+B)$ is f-semi-ample.*

- (i) *Every non-zero local section of $R^qf_*\mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y, B).*
- (ii) *Assume that $H \sim_{\mathbb{Q}} f^*H'$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H' on X. Then $H^p(X, R^qf_*\mathcal{O}_Y(L)) = 0$ for any $p > 0$ and $q \geq 0$.*

The proof of Theorem 2.1 is not difficult. For a short, easy, and almost self-contained proof, see [F2]. As an application of Theorem 2.1, we prepare the following powerful vanishing theorem. It will play basic roles for the study of log canonical pairs.

Theorem 2.2 (cf. [A, Theorem 4.4])**.** *Let X be a normal projective variety and B a boundary $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let D be a Cartier divisor on X. Assume that $D-(K_X+B)$ is ample. Let $\{C_i\}$ be* any *set of lc centers of the pair (X, B). We put $W = \bigcup C_i$ with a reduced scheme structure. Then we have*

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(D)) = 0, \quad H^i(X, \mathcal{O}_X(D)) = 0,$$

and

$$H^i(W, \mathcal{O}_W(D)) = 0$$

for any $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. In particular, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(W, \mathcal{O}_W(D))$$

is surjective.

Proof. Let $f : Y \to X$ be a resolution such that $\mathrm{Supp}f_*^{-1}B \cup \mathrm{Exc}(f)$, where $\mathrm{Exc}(f)$ is the exceptional locus of f, is a simple normal crossing divisor. We can further assume that $f^{-1}(W)$ is a simple normal crossing divisor on Y. We can write

$$K_Y + B_Y = f^*(K_X + B).$$

Let T be the union of the irreducible components of $B_Y^{=1}$ that are mapped into W by f. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A-T) \to \mathcal{O}_Y(A) \to \mathcal{O}_T(A) \to 0,$$

where $A = \ulcorner -(B_Y^{<1}) \urcorner$. Note that A is an effective f-exceptional divisor. We obtain the following long exact sequence

$$\begin{aligned} 0 \to f_*\mathcal{O}_Y(A-T) \to f_*\mathcal{O}_Y(A) \to f_*\mathcal{O}_T(A) \\ \xrightarrow{\delta} R^1 f_*\mathcal{O}_Y(A-T) \to \cdots . \end{aligned}$$

Since

$$A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) = -(K_Y + B_Y) \sim_{\mathbb{Q}} -f^*(K_X + B),$$

any non-zero local section of $R^1 f_*\mathcal{O}_Y(A-T)$ contains in its support the f-image of some strata of $(Y, \{B_Y\} + B_Y^{=1} - T)$ by Theorem 2.1 (i). On the other hand, $W = f(T)$. Therefore, the connecting homomorphism δ is a zero map. Thus, we have a short exact sequence

$$0 \to f_*\mathcal{O}_Y(A-T) \to \mathcal{O}_X \to f_*\mathcal{O}_T(A) \to 0.$$

So, we obtain $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ and $f_*\mathcal{O}_Y(A-T) \simeq \mathcal{I}_W$, the defining ideal sheaf of W. The isomorphism $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ plays crucial roles. Thus we write it as a lemma.

Lemma 2.3. *We have $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$. It obviously implies that $f_*\mathcal{O}_T \simeq \mathcal{O}_W$.*

Since

$$f^*D + A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) \sim_{\mathbb{Q}} f^*(D - (K_X + B)),$$

and

$$f^*D + A - (K_Y + \{B_Y\} + B_Y^{=1}) \sim_{\mathbb{Q}} f^*(D - (K_X + B)),$$

we have

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(D)) \simeq H^i(X, f_*\mathcal{O}_Y(A-T) \otimes \mathcal{O}_X(D)) = 0$$

and

$$H^i(X, \mathcal{O}_X(D)) \simeq H^i(X, f_*\mathcal{O}_Y(A) \otimes \mathcal{O}_X(D)) = 0$$

for any $i > 0$ by Theorem 2.1 (ii). By the long exact sequence

$$\begin{aligned} \cdots \to H^i(X, \mathcal{O}_X(D)) \to H^i(W, \mathcal{O}_W(D)) \\ \to H^{i+1}(X, \mathcal{I}_W \otimes \mathcal{O}_X(D)) \to \cdots , \end{aligned}$$

we have $H^i(W, \mathcal{O}_W(D)) = 0$ for any $i > 0$. We finish the proof. □

As a corollary, we can easily check the following result (cf. [A, Propositions 4.7 and 4.8]).

Theorem 2.4. *Let X be a normal projective variety and B an effective $\mathbb{Q}$-divisor such that (X, B) is log canonical. Then we have the following properties.*

(1) *(X, B) has at most finitely many lc centers.*
(2) *An intersection of two lc centers is a union of lc centers.*
(3) *Any union of lc centers of (X, B) is semi-normal.*
(4) *Let $x \in X$ be a closed point such that (X, B) is lc but not klt at x. Then there is a unique minimal lc center W_x passing through x, and W_x is normal at x.*

Proof. We use the notation in the proof of Theorem 2.2. (1) is obvious. (3) is also obvious by Lemma 2.3 since T is a simple normal crossing divisor. Let C_1 and C_2 be two lc centers of (X, B). We fix a closed point $P \in C_1 \cap C_2$. It is enough to find an lc center C such that $P \in C \subset C_1 \cap C_2$. We put $W = C_1 \cup C_2$. By Lemma 2.3, we obtain $f_*\mathcal{O}_T \simeq \mathcal{O}_W$. This means that $f : T \to W$ has connected fibers. We note that T is a simple normal crossing divisor on Y. Thus, there exist irreducible components T_1 and T_2 of T such that $T_1 \cap T_2 \cap f^{-1}(P) \neq \emptyset$ and that $f(T_i) \subset C_i$ for $i = 1, 2$. Therefore, we can find an lc center C with $P \in C \subset C_1 \cap C_2$. We finish the proof of (2). Finally, we will prove (4). The existence and the uniqueness of the minimal lc center follow from (2). We take the unique minimal lc center $W = W_x$ passing through x. By Lemma 2.3, we have $f_*\mathcal{O}_T \simeq \mathcal{O}_W$. By shrinking W around x, we can assume that every stratum of T dominates W. Thus, $f : T \to W$ factors through the normalization W^ν of W. Since $f_*\mathcal{O}_T \simeq \mathcal{O}_W$, we obtain that $W^\nu \to W$ is an isomorphism. So, we obtain (4). □

3. Proof of the main theorem

In this section, we prove Theorem 1.1. I think Proposition 3.1 is a correct generalization of Shokurov's non-vanishing theorem for log canonical pairs.

Proposition 3.1 (Non-vanishing theorem)**.** *On the same assumption as in* Theorem 1.1*, the base locus of the linear system $|mL|$ contains no lc centers of (X, B) for $m \gg 0$.*

First, we give a proof to Theorem 1.1 by using Proposition 3.1.

Proof of Theorem 1.1. If L is numerically trivial, then

$$h^0(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$$

by the vanishing theorem (cf. Theorem 2.2). Thus, L is linearly trivial. In this case, $|mL|$ is free for any $m \gg 0$. So, from now on, we can assume that L is not numerically trivial.

We assume that (X, B) is klt. Let $x \in X$ be a general smooth point. Then we can find an effective $\mathbb{Q}$-divisor M on X such that

$$M \sim_{\mathbb{Q}} lL - (K_X + B)$$

for some large integer l and that $\mathrm{mult}_x M > n = \dim X$. It is well known as Shokurov's concentration method. See, for example, [KM, 3.5 Step 2]. Let c be the log canonical threshold of (X, B) with respect to M. By the construction, we have $0 < c < 1$. Then

$$(a - ac + cl)L - (K_X + B + cM) \sim_{\mathbb{Q}} (1 - c)(aL - (K_X + B))$$

is ample. Therefore, by replacing B with $B + cM$, a with $a - ac + cl$, we can assume that (X, B) is lc but not klt.

From now on, we assume that (X, B) is lc but not klt and that L is not numerically trivial. By Proposition 3.1, we can take general members $D_1, \cdots, D_{n+1} \in |p^{m_1}L|$ for some prime integer p and a positive integer m_1. Since $D_1, \cdots, D_{n+1}$ are general, $(X, B + D_1 + \cdots + D_{n+1})$ is lc outside $\mathrm{Bs}|p^{m_1}L|$. It is easy to see that $(X, B + D)$, where $D = D_1 + \cdots + D_{n+1}$, is not lc at the generic point of any irreducible component of $\mathrm{Bs}|p^{m_1}L|$. Let c be the log canonical threshold of (X, B) with respect to D. Then $(X, B + cD)$ is lc but not klt, and $0 < c < 1$. We note that

$$(c(n + 1)p^{m_1} + a)L - (K_X + B + cD) \sim_{\mathbb{Q}} aL - (K_X + B)$$

is ample. By the construction, there exists an lc center of $(X, B + cD)$ contained in $\mathrm{Bs}|p^{m_1}L|$. By Proposition 3.1, we can find $m_2 > m_1$ such that $\mathrm{Bs}|p^{m_2}L| \subsetneq \mathrm{Bs}|p^{m_1}L|$. By the noetherian induction, there exists m_k such that $\mathrm{Bs}|p^{m_k}L| = \emptyset$. Let p' be a prime integer such that $p' \neq p$. Then, by the same argument, we can prove $\mathrm{Bs}|p'^{m_{k'}}L| = \emptyset$ for some positive integer $m_{k'}$. So, there exists a positive number m_0 such that $|mL|$ is free for any $m \geq m_0$. $\square$

Let us go to the proof of Proposition 3.1.

Proof of Proposition 3.1. Let W be a minimal lc center of (X, B). If $L|_W$ is numerically trivial, then we have

$$h^0(W, \mathcal{O}_W(\pm L)) = \chi(W, \mathcal{O}_W(\pm L)) = \chi(W, \mathcal{O}_W) = h^0(W, \mathcal{O}_W) = 1$$

by the vanishing theorem (see Theorem 2.2). Therefore, $L|_W$ is linearly trivial. In particular, $|mL|_W|$ is free for any $m > 0$. On the other hand,

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(W, \mathcal{O}_W(mL))$$

is surjective for any $m \geq a$ by Theorem 2.2. Thus, $\mathrm{Bs}|mL|$ does not contain W for any $m \geq a$.

Assume that $L|_W$ is not numerically trivial. Let $x \in W$ be a general smooth point. If l is a sufficiently large integer, then we can find an effective Cartier divisor N on W such that $N \sim b(lL - (K_X + B))$ with $\mathrm{mult}_x N > b \dim W$ for some positive integer b by Shokurov's concentration method. If b is sufficiently large and divisible, then $\mathcal{I}_W \otimes \mathcal{O}_X(b(lL - (K_X + B)))$ is generated by global sections and $H^1(X, \mathcal{I}_W \otimes \mathcal{O}_X(b(lL - (K_X + B)))) = 0$ since $lL - (K_X + B)$ is ample, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. By using the following short exact sequence

$$\begin{aligned} 0 &\to H^0(X, \mathcal{I}_W \otimes \mathcal{O}_X(b(lL - (K_X + B)))) \\ &\to H^0(X, \mathcal{O}_X(b(lL - (K_X + B)))) \\ &\to H^0(W, \mathcal{O}_W(b(lL - (K_X + B)))) \to 0, \end{aligned}$$

we can find an effective $\mathbb{Q}$-divisor M on X with the following properties.

(i) $M|_W$ is an effective $\mathbb{Q}$-divisor such that $\mathrm{mult}_x M|_W > \dim W$.
(ii) $M \sim_{\mathbb{Q}} lL - (K_X + B)$ for some large positive integer l.
(iii) $(X, B + M)$ is lc outside W.

We take the log canonical threshold c of (X, B) with respect to M. Then $(X, B + cM)$ is lc but not klt. By the above construction, we have $0 < c < 1$. By replacing (X, B) with $(X, B + cM)$ as in the proof of Theorem 1.1, we can find a smaller lc center W' of $(X, B + cM)$ contained in W. By repeating this process, we reach the situation where $L|_W$ is numerically trivial.

Anyway, we proved that $\mathrm{Bs}|mL|$ contains no lc centers of (X, B) for $m \gg 0$. □

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