# ON INJECTIVITY, VANISHING AND TORSION-FREE THEOREMS

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Abstract. This short paper is an advertisement of the recent big progress on injectivity, vanishing, and torsion-free theorems introduced by Florin Ambro. These results will be indispensable for the log minimal model program in the coming decade.

## 1. Introduction

In this short paper, we will discuss Ambro's formulation of Kollár's injectivity, vanishing, and torsion-free theorems. These new results will play important roles in the log minimal model program (LMMP, for short) for log canonical pairs in the coming decade.

In 1980's, the X-method initiated by Kawamata, sophisticated and repeatedly used by many minimal modelers, for example, Reid, Shokurov, and others, was the most important technique in the LMMP. It is and will be a basic argument in the theory of higher dimensional algebraic varieties.

Around 1990, Nadel introduced his multiplier ideal sheaves and proved a vanishing theorem related to Nadel's multiplier ideal sheaves. Now, it is called Nadel's vanishing theorem. We note that in the geometric situation the Nadel vanishing theroem is essentially the same as the Kawamata–Viehweg vanishing theorem. This vanishing theorem was used for geometric applications by, for example, Demailly, Tsuji, Siu, and others. In particular, Siu invented very clever applications of Nadel's multiplier ideal sheaves and the Ohsawa–Takegoshi $L^2$-extension theorem. It led us to the proof of the flip theorem by Hacon and McKernan based on Shokurov's framework.

The main difference between the traditional X-method, which is a clever application of the Kawamata–Viehweg vanishing theorem, and the arguments based on the Nadel vanishing theorem is as follows. Let

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$X$ be a variety and $\mathcal{H}$ a linear system on $X$. In the traditional X-method, we take blow-ups of $X$ and resolve the singularities of $X$ and $\mathcal{H}$ first. Then we apply the Kawamata–Viehweg vanishing theorem. In the arguments based on the Nadel vanishing theorem, we construct a multiplier ideal sheaf $\mathcal{I}$ corresponding to the singularities of $X$ and $\mathcal{H}$. Then we use the Nadel vanishing theorem for the line bundles tensored by $\mathcal{I}$. Each method has its own advantage. We do not discuss the details here. See, for example, [KM, Chapter 3] and [L, Part Three].

The theory of quasi-log varieties introduced in [A] also has its advantage to investigate linear systems on varieties. I think the results obtained in [A] have a strong possibility that they will cause new developments in the LMMP. Unfortunately, his results are not popular yet. They are not known even to experts. Therefore, I would like to advertise the injectivity, vanishing, and torsion-free theorems in [A]. We only treat a special case of his results for simplicity. However, it is sufficient for many applications in the LMMP. We will give a proof of them on the assumption that the considered variety is smooth. The reader can understand the main idea of the proof is not difficult. I hope that this paper will motivate the reader to study these new techniques. For a systematic and thorough treatment of this topic, see [F, Chapter 2].

Anyway, this paper is a gentle introduction to Chapter 2 in [F], which is very technical and not so easy to read.

We summarize the contents of this paper. In Section 2, we will state Ambro's formulation of Kollár's results in a simplified form. Section 3 is a short review of the Hodge theoretic aspect of the injectivity theorem. In Section 4, we give a proof to the results in Section 2 on the assumption that the considered variety is smooth. The reader will find that we need no new ideas. In Section 5, we will explain applications of our new vanishing theorem. The first one is the extension theorem from lc centers. It is very strong and can not be reached by the Kawamata–Viehweg–Nadel vanishing theorem. The final theorem is the Kodaira vanishing theorem for log canonical pairs.

**Notation.** Let $X$ be a normal variety and $B$ an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then we can define the discrepancy $a(E, X, B) \in \mathbb{Q}$ for any prime divisor $E$ over $X$. If $a(E, X, B) \geq -1$ for any $E$, then $(X, B)$ is called *log canonical*. We sometimes abbreviate log canonical to *lc*. Assume that $(X, B)$ is log canonical. If $E$ is a prime divisor over $X$ such that $a(E, X, B) = -1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of $(X, B)$, where $c_X(E)$ is the closure of the image of $E$ on $X$.

Let $r$ be a rational number. The integral part $\llcorner r \lrcorner$ is the largest integer $\leq r$ and the fractional part $\{r\}$ is defined by $r - \llcorner r \lrcorner$. We put $\ulcorner r \urcorner = -\llcorner -r \lrcorner$ and call it the round-up of $r$. For a $\mathbb{Q}$-divisor $D = \sum_{i=1}^{r} d_i D_i$, where $D_i$ is a prime divisor for any $i$ and $D_i \neq D_j$ for $i \neq j$, we call $D$ a *boundary* $\mathbb{Q}$-divisor if $0 \leq d_i \leq 1$ for any $i$. We note that $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. We put $\llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i$, $\ulcorner D \urcorner = \sum \ulcorner d_i \urcorner D_i$, $\{D\} = \sum \{d_i\} D_i$, $D^{<1} = \sum_{d_i<1} d_i D_i$, and $D^{=1} = \sum_{d_i=1} D_i$.

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

## 2. New theorems for simple normal crossing pairs

In this section, we state the vanishing and torsion-free theorems in [A] in a simplified form without proofs. First, we fix the setting.

**2.1** (Setting)**.** Let $Y$ be a simple normal crossing divisor on a smooth variety $M$ and let $D$ be a boundary $\mathbb{Q}$-divisor on $M$ such that $\mathrm{Supp}(D + Y)$ is simple normal crossing and that $D$ and $Y$ have no common irreducible components. We put $B = D|_Y$ and consider the pair $(Y, B)$. Let $\nu : Y^{\nu} \to Y$ be the normalization. We put $K_{Y^{\nu}} + \Theta = \nu^*(K_Y + B)$. A *stratum* of $(Y, B)$ is an irreducible component of $Y$ or the image of some lc center of $(Y^{\nu}, \Theta)$. When $Y$ is smooth and $B$ is a boundary $\mathbb{Q}$-divisor on $Y$ such that $\mathrm{Supp} B$ is simple normal crossing, we put $M = Y \times \mathbb{A}^1$ and $D = B \times \mathbb{A}^1$. Then $(Y, B) \simeq (Y \times \{0\}, B \times \{0\})$ satisfies the above conditions.

The following theorem is a special case of [A, Theorem 3.2].

**Theorem 2.2.** *Let $(Y, B)$ be as above. Let $f : Y \to X$ be a proper morphism and $L$ a Cartier divisor on $Y$.*

(1) *Assume that $H \sim_{\mathbb{Q}} L - (K_Y + B)$ is $f$-semi-ample. Then every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the $f$-image of some strata of $(Y, B)$.*

(2) *Let $\pi : X \to V$ be a proper morphism and assume that $H \sim_{\mathbb{Q}} f^* H'$ for some $\pi$-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H'$ on $X$. Then, $R^q f_* \mathcal{O}_Y(L)$ is $\pi_*$-acyclic, that is, $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for any $p > 0$.*

This theorem is very powerful and will play crucial roles in the LMMP for log canonical pairs. We will prove it on the assumption that $Y$ is *smooth* in Section 4.

## 3. Hodge theoretic aspect

In this section, we will prove the following injectivity theorem. It is essentially the same as [EV, 3.2. Theorem. c), 5.1. b)]. We use the classical topology throughout this section.

**Proposition 3.1** (Fundamental injectivity theorem)**.** *Let $X$ be a projective smooth variety and $S+B$ a boundary $\mathbb{Q}$-divisor on $X$ such that the support of $S+B$ is simple normal crossing and that $\llcorner S+B\lrcorner = S$. Let $L$ be a Cartier divisor on $X$ and let $D$ be an effective Cartier divisor whose support is contained in* $\operatorname{Supp}B$. *Assume that $L\sim_{\mathbb{Q}} K_X+S+B$. Then the natural homomorphisms*

$$H^q(X,\mathcal{O}_X(L))\to H^q(X,\mathcal{O}_X(L+D)),$$

*which are induced by the inclusion $\mathcal{O}_X\to\mathcal{O}_X(D)$, are injective for all $q$.*

Before we prove Proposition 3.1, let us recall some results on the Hodge theory.

**3.2.** Let $V$ be a smooth projective variety and $\Sigma$ a simple normal crossing divisor on $V$. Let $\iota: V\backslash\Sigma\to V$ be the natural open immersion. Then $\iota_!\mathbb{C}_{V\backslash\Sigma}$ is quasi-isomorphic to the complex $\Omega^\bullet_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma)$. By this quasi-isomorphism, we can construct the following spectral sequence

$$E^{pq}_1=H^q(V,\Omega^p_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma))\Rightarrow H^{p+q}_c(V\setminus\Sigma,\mathbb{C}).$$

By the Serre duality, the right hand side $H^q(V,\Omega^p_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma))$ is dual to $H^{n-q}(V,\Omega^{n-p}_V(\log\Sigma))$, where $n=\dim V$. By the Poincaré duality, $H^{p+q}_c(V\setminus\Sigma,\mathbb{C})$ is dual to $H^{2n-(p+q)}(V\setminus\Sigma,\mathbb{C})$. Therefore,

$$\dim H^k_c(V\setminus\Sigma,\mathbb{C})=\sum_{p+q=k}\dim H^q(V,\Omega^p_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma))$$

by Deligne (cf. [D, Corollaire (3.2.13) (ii)]). Thus, the above spectral sequence degenerates at $E_1$. We will use this $E_1$-degeneration in the proof of Proposition 3.1. By the above $E_1$-degeneration, we obtain

$$H^k_c(V\setminus\Sigma,\mathbb{C})\simeq\bigoplus_{p+q=k}H^q(V,\Omega^p_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma)).$$

In particular, the natural inclusion $\iota_!\mathbb{C}_{V\backslash\Sigma}\subset\mathcal{O}_V(-\Sigma)$ induces surjections

$$H^p_c(V\setminus\Sigma,\mathbb{C})\simeq H^p(V,\iota_!\mathbb{C}_{V\backslash\Sigma})\to H^p(V,\mathcal{O}_V(-\Sigma))$$

for any $p$.

*Proof of* Proposition 3.1. We put $\mathcal{L} = \mathcal{O}_X(L - K_X - S)$. Let $\nu$ be the smallest positive integer such that $\nu L \sim \nu(K_X + S + B)$. In particular, $\nu B$ is an integral Weil divisor. We take the $\nu$-fold cyclic cover $\pi' : Y' = \mathrm{Spec}_X \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i} \to X$ associated to the section $\nu B \in |\mathcal{L}^{\nu}|$. More precisely, let $s \in H^0(X, \mathcal{L}^{\nu})$ be a section whose zero divisor is $\nu B$. Then the dual of $s : \mathcal{O}_X \to \mathcal{L}^{\nu}$ defines a $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i}$. Let $Y \to Y'$ be the normalization and $\pi : Y \to X$ the composition morphism. For the details, see [EV, 3.5. Cyclic covers]. We can take a finite cover $\varphi : V \to Y$ such that $V$ is smooth, $\varphi$ is a Kummer cover, and $T$ is a simple normal crossing divisor on $V$, where $\psi = \pi \circ \varphi$ and $T = \psi^* S$, by Kawamata's covering trick (cf. [EV, 3.17. Lemma]).

We can decompose $\psi_* \Omega^{\bullet}_V(\log T) \otimes \mathcal{O}_V(-T)$ and $\psi_*(\iota_! \mathbb{C}_{V \setminus T})$ into eigen components. We have that

$$\mathcal{C} \xrightarrow{qis} \Omega^{\bullet}_X(\log(S + B)) \otimes \mathcal{L}^{-1}(-S)$$

is a direct summand of

$$\psi_*(\iota_! \mathbb{C}_{V \setminus T}) \xrightarrow{qis} \psi_* \Omega^{\bullet}_V(\log T) \otimes \mathcal{O}_V(-T),$$

where *qis* means an quasi-isomorphism. The $E_1$-degeneration of the spectral sequence

$$\begin{aligned} E_1^{pq} \quad &= \quad H^q(V, \Omega^p_V(\log T) \otimes \mathcal{O}_V(-T)) \\ &\Rightarrow \quad \mathbb{H}^{p+q}(V, \Omega^{\bullet}_V(\log T) \otimes \mathcal{O}_V(-T)) \simeq H^{p+q}(V, \iota_! \mathbb{C}_{V \setminus T}) \end{aligned}$$

implies the $E_1$-degeneration of

$$\begin{aligned} E_1^{pq} &= H^q(X, \Omega^p_X(\log(S + B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\Rightarrow \mathbb{H}^{p+q}(X, \Omega^{\bullet}_X(\log(S + B)) \otimes \mathcal{L}^{-1}(-S)) \simeq H^{p+q}(X, \mathcal{C}) \end{aligned}$$

Therefore, the inclusion $\mathcal{C} \subset \mathcal{L}^{-1}(-S)$ induces surjections

$$H^p(X, \mathcal{C}) \to H^p(X, \mathcal{L}^{-1}(-S)).$$

We can check the following simple property by seeing the monodromy action of the Galois group of $\psi : V \to X$ on $\mathcal{C}$ around $\mathrm{Supp} B$.

**Corollary 3.3** (cf. [KM, Corollary 2.54]). *Let $U \subset X$ be a connected open set such that $U \cap \mathrm{Supp} B \neq \emptyset$. Then $H^0(U, \mathcal{C}|_U) = 0$.*

This property is utilized via the following fact. The proof is obvious.

**Lemma 3.4** (cf. [KM, Lemma 2.55]). *Let $F$ be a sheaf of Abelian groups on a topological space $X$ and $F_1, F_2 \subset F$ subsheaves. Let $Z \subset X$ be a closed subset. Assume that*

1. *$F_2|_{X \setminus Z} = F|_{X \setminus Z}$, and*
2. *if $U$ is connected, open and $U \cap Z \neq \emptyset$, then $H^0(U, F_1|U) = 0$.*

*Then $F_1$ is a subsheaf of $F_2$.*

As a corollary, we obtain:

**Corollary 3.5** (cf. [KM, Corollary 2.56]). *Let $M \subset \mathcal{L}^{-1}(-S)$ be a subsheaf such that $M|_{X\setminus \mathrm{Supp}B} = \mathcal{L}^{-1}(-S)|_{X\setminus \mathrm{Supp}B}$. Then the injection*

$$\mathcal{C} \to \mathcal{L}^{-1}(-S)$$

*factors as*

$$\mathcal{C} \to M \to \mathcal{L}^{-1}(-S).$$

*Therefore,*

$$H^i(X, M) \to H^i(X, \mathcal{L}^{-1}(-S))$$

*is surjective for every $i$.*

*Proof.* The first part is clear from Corollary 3.3 and Lemma 3.4. This implies that we have maps

$$H^i(X, \mathcal{C}) \to H^i(X, M) \to H^i(X, \mathcal{L}^{-1}(-S)).$$

As we saw above, the composition is surjective. Hence so is the map on the right. $\square$

Therefore, $H^q(X, \mathcal{L}^{-1}(-S-D)) \to H^q(X, \mathcal{L}^{-1}(-S))$ is surjective for any $q$. By the Serre duality, we obtain

$$H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S)) \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S+D))$$

is injective for any $q$. This means that

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

is injective for any $q$. $\square$

## 4. New theorems for smooth varieties

In this section, we prove Theorem 2.2 on the assumption that $Y$ is a smooth projective variety and $V$ is a point. First, we prove a generalization of Kollár's injectivity theorem (cf. [A, Theorem 3.1]). It is an easy consequence of Proposition 3.1 and will produce the desired torsion-free and vanishing theorems.

**Theorem 4.1** (Injectivity theorem). *Let $X$ be a smooth projective variety and $S+B$ a boundary $\mathbb{Q}$-divisor such that* $\mathrm{Supp}(S+B)$ *is simple normal crossing and that $\llcorner S+B \lrcorner = S$. Let $L$ be a Cartier divisor on $X$ and $D$ an effective Cartier divisor that contains no lc centers of $(X, S+B)$. Assume the following conditions.*

(i) $L \sim_{\mathbb{Q}} K_X + S + B + H$,
(ii) *$H$ is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, and*

(iii) *$tH \sim_{\mathbb{Q}} D+D'$ for some positive rational number $t$, where $D'$ is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor that contains no lc centers of $(X, S+B)$.*

*Then the homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

*which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all $q$.*

*Proof.* We can take a resolution $f : Y \to X$ such that $f$ is an isomorphism outside $\mathrm{Supp}(D+D'+B)$, and that the union of the support of $f^*(S+B+D+D')$ and the exceptional locus of $f$ has a simple normal crossing support on $Y$. Let $B'$ be the strict transform of $B$ on $Y$. We write $K_Y + S' + B' = f^*(K_X + S + B) + E$, where $S'$ is the strict transform of $S$, and $E$ is $f$-exceptional. It is easy to see that $E_+ = \ulcorner E \urcorner \geq 0$. We put $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that $E_+$ is Cartier and $E_-$ is $\mathbb{Q}$-Cartier. Since $f^*H$ is semi-ample, we can write $f^*H \sim_{\mathbb{Q}} aH'$, where $0 < a < 1$ and $H'$ is a general Cartier divisor on $Y$. We put $B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D+D') + (1-\varepsilon)aH'$ for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{Q}} K_Y + S' + B''$. By the construction, $\llcorner B'' \lrcorner = 0$, the support of $S' + B''$ is simple normal crossing on $Y$, and $\mathrm{Supp} B'' \supset \mathrm{Supp} f^*D$. So, Proposition 3.1 implies that the homomorphisms $H^q(Y, \mathcal{O}_Y(L')) \to H^q(Y, \mathcal{O}_Y(L' + f^*D))$ are injective for all $q$. By Lemma 4.2 below, $R^q f_* \mathcal{O}_Y(L') = 0$ for any $q > 0$ and it is easy to see that $f_* \mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By the Leray spectral sequence, the homomorphisms $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ are injective for all $q$. $\square$

Let us recall the following well-known easy lemma.

**Lemma 4.2.** *Let $V$ be a smooth projective variety and $B$ a boundary $\mathbb{Q}$-divisor on $V$ such that* $\mathrm{Supp}B$ *is simple normal crossing. Let $f : V \to W$ be a projective birational morphism onto a variety $W$. Assume that $f$ is an isomorphism at the generic point of any lc center of $(V, B)$ and that $D$ is a Cartier divisor on $V$ such that $D - (K_V + B)$ is nef. Then $R^i f_* \mathcal{O}_V(D) = 0$ for any $i > 0$.*

*Proof.* We use the induction on the number of irreducible components of $\llcorner B \lrcorner$ and on the dimension of $V$. If $\llcorner B \lrcorner = 0$, then the lemma follows from the Kawamata–Viehweg vanishing theorem (cf. [KM, Corollary 2.68]). Therefore, we can assume that there is an irreducible divisor $S \subset \llcorner B \lrcorner$. We consider the following short exact sequence

$$0 \to \mathcal{O}_V(D-S) \to \mathcal{O}_V(D) \to \mathcal{O}_S(D) \to 0.$$

By induction, we see that $R^if_*\mathcal{O}_V(D-S)=0$ and $R^if_*\mathcal{O}_S(D)=0$ for any $i>0$. Thus, we have $R^if_*\mathcal{O}_V(D)=0$ for $i>0$. $\square$

The next theorem is the main result of this section (cf. [A, Theorem 3.2]).

**Theorem 4.3** (Torsion-free and vanishing theorems). *Let $Y$ be a smooth projective variety and $S+B$ a boundary $\mathbb{Q}$-divisor such that* $\mathrm{Supp}(S+B)$ *is simple normal crossing and that $\llcorner S+B\lrcorner=S$. Let $f:Y\to X$ be a projective morphism and $L$ a Cartier divisor on $Y$ such that $H\sim_{\mathbb{Q}}L-(K_Y+S+B)$ is $f$-semi-ample.*

(i) *Every non-zero local section of $R^qf_*\mathcal{O}_Y(L)$ contains in its support the $f$-image of some strata of $(Y,S+B)$.*

(ii) *Assume that $H\sim_{\mathbb{Q}}f^*H'$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H'$ on $X$. Then $H^p(X,R^qf_*\mathcal{O}_Y(L))=0$ for any $p>0$ and $q\geq 0$.*

*Proof.* We can assume that $H$ is semi-ample by replacing $L$ (resp. $H$) with $L+f^*A'$ (resp. $H+f^*A'$), where $A'$ is a very ample Cartier divisor on $X$. Assume that $R^qf_*\mathcal{O}_Y(L)$ has a local section whose support does not contain the image of any $(Y,S+B)$-stratum. Then we can find a very ample Cartier divisor $A$ with the following properties.

(a) $f^*A$ contains no lc centers of $(Y,S+B)$, and

(b) $R^qf_*\mathcal{O}_Y(L)\to R^qf_*\mathcal{O}_Y(L)\otimes\mathcal{O}_X(A)$ is not injective.

We can assume that $H-f^*A$ is semi-ample by replacing $L$ (resp. $H$) with $L+f^*A$ (resp. $H+f^*A$). If necessary, we replace $L$ (resp. $H$) with $L+f^*A''$ (resp. $H+f^*A''$), where $A''$ is a very ample Cartier divisor on $X$. Then, we have

$$H^0(X,R^qf_*\mathcal{O}_Y(L))\simeq H^q(Y,\mathcal{O}_Y(L))$$

and

$$H^0(X,R^qf_*\mathcal{O}_Y(L)\otimes\mathcal{O}_X(A))\simeq H^q(Y,\mathcal{O}_Y(L+f^*A)).$$

We obtain that

$$H^0(X,R^qf_*\mathcal{O}_Y(L))\to H^0(X,R^qf_*\mathcal{O}_Y(L)\otimes\mathcal{O}_X(A))$$

is not injective by (b) if $A''$ is sufficiently ample. So,

$$H^q(Y,\mathcal{O}_Y(L))\to H^q(Y,\mathcal{O}_Y(L+f^*A))$$

is not injective. It contradicts Theorem 4.1. We finish the proof of (i).

Let us go to the proof of (ii). We take a general member $A\in|mH'|$, where $m$ is a sufficiently large and divisible integer, such that $A'=f^*A$

and $R^qf_*\mathcal{O}_Y(L+A')$ is $\Gamma$-acyclic for all $q$. By (i), we have the following short exact sequences,

$$0 \to R^qf_*\mathcal{O}_Y(L) \to R^qf_*\mathcal{O}_Y(L+A') \to R^qf_*\mathcal{O}_{A'}(L+A') \to 0.$$

for any $q$. Note that $R^qf_*\mathcal{O}_{A'}(L+A')$ is $\Gamma$-acyclic by induction on $\dim X$ and $R^qf_*\mathcal{O}_Y(L+A')$ is also $\Gamma$-acyclic by the above assumption. Thus, $E_2^{pq} = 0$ for $p \geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{array}{ccc} E_2^{pq} = H^q(X, R^qf_*\mathcal{O}_Y(L)) & \Longrightarrow & H^{p+q}(Y, \mathcal{O}_Y(L)) \\ \Big\downarrow{\scriptstyle \varphi^{pq}} & & \Big\downarrow{\scriptstyle \varphi^{p+q}} \\ \overline{E}_2^{pq} = H^p(X, R^qf_*\mathcal{O}_Y(L+A')) & \Longrightarrow & H^{p+q}(Y, \mathcal{O}_Y(L+A')) \end{array}$$

We note that $\varphi^{1+q}$ is injective by Theorem 4.1. We have $E_2^{1q} \to H^{1+q}(Y, \mathcal{O}_Y(L))$ is injective by the fact that $E_2^{pq} = 0$ for $p \geq 2$. We also have that $\overline{E}_2^{1q} = 0$ by the above assumption. Therefore, we obtain $E_2^{1q} = 0$ since the injection $E_2^{1q} \to H^{1+q}(Y, \mathcal{O}_Y(L+A'))$ factors through $\overline{E}_2^{1q}$. This implies that $H^p(X, R^qf_*\mathcal{O}_Y(L)) = 0$ for any $p > 0$. $\square$

## 5. Applications

In this final section, we give easy applications of our new vanishing theorem. The next theorem is enough powerful and can not be obtained by the classical approaches.

**Theorem 5.1** (cf. [A, Theorem 4.4])**.** *Let $X$ be a normal projective variety and $B$ a boundary $\mathbb{Q}$-divisor on $X$ such that $(X, B)$ is log canonical. Let $L$ be a Cartier divisor on $X$. Assume that $L - (K_X + B)$ is ample. Let $\{C_i\}$ be* any *set of lc centers of the pair $(X, B)$. We put $W = \bigcup C_i$ with a reduced scheme structure. Then we have*

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) = 0$$

*for any $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of $W$ on $X$. In particular, the restriction map*

$$H^0(X, \mathcal{O}_X(L)) \to H^0(W, \mathcal{O}_W(L))$$

*is surjective. Therefore, if $(X, B)$ has a zero-dimensional lc center, then the linear system $|L|$ is not empty and the base locus of $|L|$ contains no zero-dimensional lc centers of $(X, B)$.*

*Proof.* Let $f : Y \to X$ be a resolution such that $\mathrm{Supp} f_*^{-1}B \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor. We can further assume that $f^{-1}(W)$

is a simple normal crossing divisor on $Y$. We can write

$$K_Y + B_Y = f^*(K_X + B).$$

Let $T$ be the union of the irreducible components of $B_Y^{=1}$ that are mapped into $W$ by $f$. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A-T) \to \mathcal{O}_Y(A) \to \mathcal{O}_T(A) \to 0,$$

where $A = \ulcorner -(B_Y^{<1}) \urcorner$. Note that $A$ is an effective $f$-exceptional divisor. We obtain the following long exact sequence

$$\begin{aligned} 0 &\to f_*\mathcal{O}_Y(A-T) \to f_*\mathcal{O}_Y(A) \to f_*\mathcal{O}_T(A) \\ &\xrightarrow{\delta} R^1 f_*\mathcal{O}_Y(A-T) \to \cdots . \end{aligned}$$

Since

$$A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) = -(K_Y + B_Y) \sim_{\mathbb{Q}} -f^*(K_X + B),$$

any non-zero local section of $R^1 f_*\mathcal{O}_Y(A-T)$ contains in its support the $f$-image of some strata of $(Y, \{B_Y\} + B_Y^{=1} - T)$ by Theorem 4.3 (i). On the other hand, $W = f(T)$. Therefore, the connecting homomorphism $\delta$ is a zero map. Thus, we have a short exact sequence

$$0 \to f_*\mathcal{O}_Y(A-T) \to \mathcal{O}_X \to f_*\mathcal{O}_T(A) \to 0.$$

So, we obtain $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ and $f_*\mathcal{O}_Y(A-T) \simeq \mathcal{I}_W$, the defining ideal sheaf of $W$. The isomorphism $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ plays crucial roles in the theory of quasi-log varieties. So, we proved it here. Since

$$f^*L + A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) \sim_{\mathbb{Q}} f^*(L - (K_X + B)),$$

we have

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) \simeq H^i(X, f_*\mathcal{O}_Y(A-T) \otimes \mathcal{O}_X(L)) = 0$$

for any $i > 0$ by Theorem 4.3 (ii). We finish the proof. □

We close this paper with the Kodaira vanishing theorem for log canonical pairs. For a more general result containing the Kawamata–Viehweg vanishing theorem, see [F, Theorem 2.48].

**Theorem 5.2** (Kodaira vanishing theorem for lc pairs)**.** *Let $X$ be a normal projective variety and $B$ a boundary $\mathbb{Q}$-divisor on $X$ such that $(X, B)$ is log canonical. Let $L$ be a $\mathbb{Q}$-Cartier Weil divisor on $X$ such that $L - (K_X + B)$ is ample. Then $H^q(X, \mathcal{O}_X(L)) = 0$ for any $q > 0$.*

*Proof.* Let $f : Y \to X$ be a resolution of $(X, B)$ such that $K_Y = f^*(K_X + B) + \sum_i a_i E_i$ with $a_i \geq -1$ for any $i$ and that $\operatorname{Supp} \sum E_i$ is simple normal crossing. We can assume that $\sum_i E_i \cup \operatorname{Supp} f^*L$ is a simple normal crossing divisor on $Y$. We put $E = \sum_i a_i E_i$ and $F = \sum_{a_j = -1} (1 - b_j) E_j$, where $b_j = \operatorname{mult}_{E_j}\{f^*L\}$. We note that $A =$

$L-(K_X+B)$ is ample by the assumption. So, we have $f^*A = f^*L - f^*(K_X+B) = \ulcorner f^*L+E+F \urcorner - (K_Y+F+\{-(f^*L+E+F)\})$. We can easily check that $f_*\mathcal{O}_Y(\ulcorner f^*L+E+F\urcorner) \simeq \mathcal{O}_X(L)$ and that $F+\{-(f^*L+E+F)\}$ has a simple normal crossing support and is a boundary $\mathbb{Q}$-divisor on $Y$. By Theorem 4.3 (ii), we obtain that $\mathcal{O}_X(L)$ is $\Gamma$-acyclic. Thus, we have $H^q(X, \mathcal{O}_X(L)) = 0$ for any $q>0$. □

The reader can find more advanced topics and many other applications in [F]. As we pointed out before, this paper is a gentle introduction to Chapter 2 in [F]. We recommend the reader to see [F].

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