Bishop-Gromov Type Inequality on Ricci Limit Spaces

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Abstract

In this paper, we study limit spaces of a sequence of complete n-dimensional Riemannian manifolds whose Ricci curvatures have definite lower bound. we will give several measure theoretical properties of such limit spaces.

1 Introduction

In this paper, we study a pointed metric space (Y, y) that is pointed Gromov-Hausdorff limit of a sequence of complete, pointed, connected n-dimensional Riemannian manifolds, $\{(M_i, m_i)\}_i$, with $Ric_{M_i} \geq -(n-1)$. (We call a such metric space (Y, y) *Ricci limit space* in this paper. See [13].) In the papers [4], [5], [6], J. Cheeger and T. H. Colding studied such limit spaces, showed many important results. There exists a Borel measure on a Ricci limit space (Y, y), υ that is called by *limit measure.* (See Definition 2.3.) They developed the structure theory by using the limit measure υ and results in [2], [3], [7]. Most of this paper, we will study measure theoretical properties on Ricci limit spaces. In another paper [12], we will discuss several application of the results in this paper to low dimensional Ricci limit spaces.

First, we study a cut locus on Ricci limit spaces in section 3. We prove that the measure of cut locus is equal to zero. (See Theorem 3.2.) We will study cut locus as geometric approach in [12]. We also give a relationship between "the limit space of cut locuses" and "cut locus of the limit space". See Theorem 3.5.

J. Cheeger and T. H. Colding defined the measure of codimension one of υ, υ_{-1} in [5]. (See Definition 4.1 for the definition of υ_{-1}.) We will give several properties of υ_{-1}. For example, we will show the Bishop-Gromov type inequality for υ_{-1};

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THEOREM 1.1. *Let (Y, y) be a Ricci limit space. Then, there exists a positive constant $C(n) > 0$ depending only on n, such that for every positive numbers $0 < s < t < \infty$, every point $x \in Y$ and for every Borel set $A \subset \partial B_t(x)$,*

$$\frac{\upsilon_{-1}(A)}{\mathrm{vol}\partial B_t(\underline{p})} \leq C(n) \frac{\upsilon_{-1}(\partial B_s(x) \cap C_x(A))}{\mathrm{vol}\partial B_s(\underline{p})}$$

holds.

Here, $\underline{p}$ is a point in standard n-dimensional hyperbolic space $\mathbf{H}^n(-1)$ and $C_x(A) = \{z \in Y|$ There exists $w \in A$ such that $\overline{x,z} + \overline{z,w} = \overline{x,w}$ holds.$\}$. ($\overline{x,z}$ is the distance between x and z on Y.) This is like Laplacian comparison theorem on Riemannian manifolds. See Theorem 1.2 in [12] for a geometric application of Theorem 1.1 to low dimensional Ricci limit spaces.

We will also show some finiteness result (Theorem 4.2) and non-zero property for υ_{-1} (Corollary 4.7). It means that the measure υ_{-1} is a good measure on the set $\partial B_r(x) \setminus C_x$. Here, C_x is the cut locus of $x \in Y$. These properties are similar to that on Riemannian manifolds.

We will give a relationship between the limit measure υ and the measure υ_{-1} in section 5. Theorem 5.2 is like co-area formula for Lipshictz maps on Euclidean spaces. (See 3.2.12. Theorem in [8].)

Finally, we also consider the subset of Ricci limit space (Y, y), $A_Y(\alpha)$ consists of points $x \in Y$ satisfying $\upsilon(B_r(x)) \sim r^\alpha$ as $r \to 0$. (See Definition 6.1.) We can regard the limit measure υ on $A_Y(\alpha)$ as α-dimensional Hausdorff measure $\mathcal{H}^\alpha$ in some sense. We will give an upper bound of Hausdorff dimension of the set. As a corollary, we will give an easy proof of Corollary 6.4.

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2 Notation

In this section, we recall some fundamental notion on metric spaces and the notion of Ricci limit spaces. (See [4].)

DEFINITION 2.1. We say that a metric space X is *proper* if every bounded closed set is compact. A metric space X is said to be *geodesic space* if for every points $x_1, x_2 \in X$, there exists an isometric embedding $\gamma : [0, \overline{x_1, x_2}] \to X$ such that $\gamma(0) = x_1$, $\gamma(\overline{x_1, x_2}) = x_2$. Here $\overline{x_1, x_2}$ is the distance between x_1 and x_2 on X. (We say that γ is *minimal geodesic from x_1 to x_2*.)

For proper geodesic space X, a point $x \in X$, a set $A \subset X$, and for positive numbers $0 < r < R$, we use the following notations; $B_r(x) = \{z \in X | \overline{x,z} < r\}$, $\overline{B}_r(x) = \{z \in X | \overline{x,z} \le r\}$, $A_{r,R}(x) = \overline{B}_R(x) \setminus B_r(x)$, $\partial B_r(x) = \{z \in X | \overline{x,z} = r\}, C_x(A) = \{z \in X |$ There exists $w \in A$ such that $\overline{x,z} + \overline{z,w} = \overline{x,w}$ holds.$\}$. Throughout the paper, we fix a positive integer $n > 0$.

Definition 2.2. Let (Y, y) be a pointed proper geodesic space ($y \in Y$), $K \in \mathbf{R}$ a real number. We say that (Y, y) is (n, K)-*Ricci limit space* if there exists a sequence of real numbers $K_i \in \mathbf{R}$ and a sequence of pointed, complete, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $Ric_{M_i} \ge K_i(n-1)$, such that K_i converges to K and that (M_i, m_i) converges to (Y, y) as $i \to \infty$ in the sense of pointed Gromov-Hausdorff topology.

Here, for a sequence of pointed proper geodesic space $\{(X_i, x_i)\}_i$, we say that (X_i, x_i) *converges to a pointed proper geodesic space* (X_∞, x_∞) *in the sense of Gromov-Hausdorff topology* if there exist sequences of positive numbers ϵ_i, $R_i > 0$ and exists a sequence of maps $\phi_i : (B_{R_i}(x_i), x_i) \to (B_{R_i}(x_\infty), x_\infty)$ such that ϵ_i converges to 0, R_i converges to ∞, $|\overline{z_i, w_i} - \overline{\phi_i(z_i), \phi_i(w_i)}| < \epsilon_i$ holds for every points $z_i, w_i \in B_{R_i}(x_i)$, and that $B_{\epsilon_i}(\mathrm{Image}(\phi_i)) \supset B_{R_i}(X_\infty)$ holds. (We say that ϕ_i *is* ϵ_i*-Gromov-Hausdorff approximation.*) Then for a sequence of points $z_i \in X_i$ such that the set $\{\overline{x_i, z_i} | i \in \mathbf{N}\}$ is bounded set in $\mathbf{R}$, we say that z_i *converges to a point* $z_\infty \in X_\infty$ *in the sense of Gromov-Hausdorff topology* if $\overline{\phi_i(z_i), z_\infty} < \epsilon_i$. (We denote it by either $z_i \to z_\infty$ or $\overline{z_i, z_\infty} < \epsilon_i$.)

We remark that for every $K \ne 0$ and every (n, K)-Ricci limit space (Y, y), there exists a sequence of complete, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $Ric_{M_i} \ge K(n-1)$, such that (M_i, m_i) converges to (Y, y) by rescaling. Throughout the paper, (Y, y) is always $(n, -1)$-Ricci limit space and is *not* a single point. More simply, we say that (Y, y) is *Ricci limit space*.

We shall give the definition of *limit measure*. The measure is useful tool for studying Ricci limit spaces.

Definition 2.3. Let υ be a Borel measure on Y. We say that υ is *limit measure* if there exists a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $Ric_{M_i} \ge -(n-1)$, such that (M_i, m_i) converges to (Y, y) and that for every positive number $r > 0$ and every points $x \in Y$, $\hat{m}_j \in M_j$ satisfying $\hat{m}_j \to x$ in the sense of pointed Gromov-Hausdorff topology,

$$\frac{\mathrm{vol}(B_r(\hat{m}_j))}{\mathrm{vol} B_1(m_j)} \to \upsilon(B_r(x))$$

holds. We say that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}B_1(m_j))$ *converges to* (Y, y, υ) *in the sense of measured Gromov-Hausdorff topology.*

There exists a limit measure on Y. (See Theorem 1.6, Theorem 1.10 in [4] and see [9].) It is *not* an unique in generally. (See Example 1.24 in [4].) Throughout the paper, υ is always fixed limit measure on Y.

3 Cut Locus

In this section, we study a cut locus on Ricci limit spaces.

3.1 Measure of cut locus

First, we give the definition of cut locus.

DEFINITION 3.1. For proper geodesic space X and every $w \in X$, we put $C_w = \{x \in X|$ For every point $z \in Y \setminus x$, $\overline{w,x} + \overline{x,z} - \overline{w,z} > 0$ holds.$\}$. (If X is a single point, then $C_x = \emptyset$.) We say that C_w is *cut locus of w.*

The following theorem is main result in this subsection.

THEOREM 3.2. *We have $\upsilon(C_w) = 0$ for every point $w \in Y$.*

PROOF. We shall give a proof of the case $w = y$ only. There exists a sequence of complete pointed, connected n-dimensional Riemannian manifolds, $\{(M_j, m_j)\}_j$ such that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}B_1(m_j))$ converges to (Y, y, υ) in the sense of measured Gromov-Hausdorff topology. For every positive number $r > 0$ and every positive integer $N \in \mathbf{N}$, we put $C_y(r) = \{x \in Y|$ For every $z \in Y \setminus B_r(x)$, $\overline{y,x} + \overline{x,z} - \overline{y,z} > 0$ holds.$\}$ and $C_y(r,N) = \{x \in Y|$ For every $z \in Y \setminus B_r(x)$, $\overline{y,x} + \overline{x,z} - \overline{y,z} \geq N^{-1}$ holds.$\}$. By the definition, $C_y(r,N)$ is compact.

CLAIM 3.3. *We have $C_y(r) = \bigcup_{N \in \mathbf{N}} C_y(r,N)$.*

It suffices to see that $C_y(r) \subset \bigcup_{N \in \mathbf{N}} C_y(r,N)$. This proof is by contradiction. We assume that there exists $x \in C_y(r) \setminus \bigcup_{N \in \mathbf{N}} C_y(r,N)$. Then, for every positive integer N, there exists a point $y_N \in Y \setminus B_r(x)$ such that $\overline{y,x} + \overline{x,y_N} - \overline{y,y_N} < N^{-1}$ holds. Clearly, for every positive integer N, there exists a point $z_N \in \partial B_r(x)$ such that $\overline{x,z_N} + \overline{z_N,y_N} = \overline{x,y_N}$ holds. Then, by triangle inequality, we have $\overline{y,x} + \overline{x,z_N} - \overline{y,z_N} < N^{-1}$. Since $\partial B_r(x)$ is compact, there exists a subsequence $\{z_{k(N)}\}_N$ and a point $z_\infty \in \partial B_r(x)$ such that $z_{k(N)}$ converges to z_∞ in Y. Therefore, we have $\overline{y,x} + \overline{x,z_\infty} = \overline{y,z_\infty}$. This contradicts the assumption. Thus we have the Claim 3.3.

By the definition, we have $C_y = \bigcap_{r>0} C_y(r)$. We fix a positive number $r > 0$ and a positive integer $N \in \mathbf{N}$. Let $l \in \mathbf{N}$ be a positive integer, $\delta > 0$ a sufficiently small positive number satisfying $0 < \delta << 2^{-l}, r, N^{-1}$. Let $\{x_i\}_{i=1}^k$ be a maximal 100δ-separated set on the set $C_y(r,N) \cap A_{2^{-l},2^l}(y)$. For every positive integers $i, j > 0$ $(1 \leq i \leq k)$, we take $x_i(j) \in M_j$ such that $x_i(j)$ converges to x_i as $j \to \infty$ in the sense of pointed Gromov-Hausdorff topology. In general, for a complete pointed, connected n-dimensional Riemannian manifold, (M, m), we put $S_m M = \{u \in T_m M | |u| = 1\}$ and $t(u) = \sup\{t \in \mathbf{R}_{>0} | \exp_m su \in M \setminus C_m$ holds for every positive number $0 < s < t\}$ for every $u \in S_m M$. For every positive numbers $0 < r_1 < r_2$ and $\eta > 0$, we also put $X(m, r_1, r_2, \eta) = \{\exp_m tu \in M | u \in S_m M, t(u) - \eta \leq t < t(u), \exp_m tu \in A_{r_1,r_2}(m)\}$.

Claim 3.4. *We have* $\bigcup_{i=1}^k B_{10\delta}(x_i(j)) \setminus C_{m_j} \subset X(m_j, 2^{-l-1}, 2^{l+1}, 100r)$ *for every sufficiently large* j.

We take $x \in B_{10\delta}(x_i(j)) \setminus C_{m_j}$. For every point $z \in M_j \setminus B_{40r}(x)$, we take $w \in Y$ such that $\overline{z, w} < \epsilon_j$ in the sense of pointed Gromov-Hausdorff topology. ($\epsilon_j \to 0$) Then, we have

$$\begin{aligned} \overline{m_j, x} + \overline{x, z} - \overline{m_j, z} &\geq \overline{m_j, x_i(j)} + \overline{x_i(j), z} - \overline{m_j, z} - 100\delta \\ &\geq \overline{y, x_i} + \overline{x_i, w} - \overline{y, w} - 100\delta - 10\epsilon_j \qquad (*) \end{aligned}$$

and $\overline{w, x_i} \geq \overline{z, x_i(j)} - \epsilon_j \geq \overline{z, x} - \overline{x, x_i(j)} - \epsilon_j \geq 40r - 50\delta - \epsilon_j > 30r$. By the definition of x_i, we have

$$(*) \geq N^{-1} - 100\delta - 10\epsilon_j \geq (2N)^{-1} > 0.$$

Thus there exist $u \in S_{m_j} M_j$ and positive number $t > 0$ such that $t(u) - 50r \leq t < t(u)$ and $x = \exp_{m_j} tu$ hold. Since $\overline{x, x_i(j)} < 10\delta$ holds, we have $x \in A_{2^{-l-1},2^{l+1}}(m_j)$. Therefore, we have $x \in X(m_j, 2^{-l-1}, 2^{l+1}, 100r)$. Hence, we have Claim 3.4.

Since $\{B_{10\delta}(x_i(j))\}_i$ are pairwise disjoint for every sufficiently large j, we have

$$\sum_{i=1}^k \underline{\mathrm{vol}} B_{10\delta}(x_i(j)) \leq \underline{\mathrm{vol}} X(m_j, 2^{-l-1}, 2^{l+1}, 100r).$$

Here, $\underline{\mathrm{vol}} = \mathrm{vol}/\mathrm{vol} B_1(m_j)$. By the proof of Lemma 2.16 in [4], there exists a positive constant $C = C(l, n) > 0$ depending only on l, n, such that $\underline{\mathrm{vol}} X(m_j, 2^{-l-1}, 2^{l+1}, 100r) \leq C(l, n) r$ holds. Thus, we have

$$\begin{aligned} \upsilon(C_y(r, N) \cap A_{2^{-l},2^l}(y)) &\leq \sum_{i=1}^k \upsilon(B_{1000\delta}(x_i)) \\ &\leq C \sum_{i=1}^k \upsilon(B_{10\delta}(x_i)) \\ &\leq Cr. \end{aligned}$$

Therefore, by letting $\delta \to 0$, $N \to \infty$, $r \to 0$, and $l \to \infty$, we have $\upsilon(C_y) = 0$. $\square$

We remark that $\mathcal{WE}_0(w) \subset C_w$ holds for every $w \in Y$. (See Definition 2.10 in [4] for the definition of $\mathcal{WE}_0(w)$.) Therefore, Theorem 3.2 differs Proposition 2.13 in [4].

3.2 Convergence of cut locuses

In this subsection, we give a relationship between "the limit space of cut locuses" and "the cut locus of the limit space". Roughly speaking, we will show "the limit space of cut locuses" contains "the cut locus of the limit space". Let $\{(M_i, m_i)\}_i$ be a sequence of complete pointed, connected n-dimensional Riemannian manifolds with $Ric_{M_i} \geq -(n-1)$. For every positive number $R > 0$, the sequence $(\overline{B}_{2R}(m_i) \cap (C_{m_i} \cup m_i), m_i)_{i \in \mathbf{N}}$ is precompact in the sense of pointed Gromov-Hausdorff topology. We assume that there exist a pointed proper geodesic space (Y, y) and a pointed compact metric space (X_R, x_R) such that $(\overline{B}_{2R}(m_i) \cap (C_{m_i} \cup m_i), m_i)$ conveges to (X_R, x_R) and that (M_i, m_i) converges to (Y, y).

Theorem 3.5. *Under the notation above, there exists an isometric embedding* $\Phi : (\overline{B}_R(y) \cap (C_y \cup y), y) \to (X_R, x_R)$.

Proof. First, we shall prove that for every finite points $x_1, x_2, \cdots, x_N \in C_y \cap \overline{B}_R(y)$, there exists an isometric embedding $\phi_N : (\{x_1, x_2, \cdots, x_N, y\}, y) \to (X_R, x_R)$. We fix a finite points $x_1, x_2, \cdots, x_N \in C_y \cap \overline{B}_R(y)$. For every sufficiently large $k \in \mathbf{N}$, there exists a positive number $\tau > 0$ such that $\overline{y, x_i} + \overline{x_i, x} - \overline{y, x} \geq \tau$ holds for every $1 \leq i \leq N$ and every point $x \in \overline{B}_{10R}(y) \setminus B_{k^{-1}}(x_i)$. We take ϵ_i-Gromov-Hausdorff approximations ($\epsilon_i \to 0$), $\phi_i : (\overline{B}_{2R}(m_i), m_i) \to (\overline{B}_{2R}(y), y)$, $\hat{\phi}_i : (\overline{B}_{2R}(y), y) \to (\overline{B}_{2R}(m_i), m_i)$, $\psi_i : (\overline{B}_{2R}(m_i) \cap (C_{m_i} \cup m_i), m_i) \to (X_R, x_R)$ and $\hat{\psi}_i : (X_R, x_R) \to (\overline{B}_{2R}(m_i) \cap (C_{m_i} \cup m_i), m_i)$ such that $\overline{\phi_i \circ \hat{\phi}_i, \mathrm{id}} < \epsilon_i$, $\overline{\hat{\phi}_i \circ \phi_i, \mathrm{id}} < \epsilon_i$, $\overline{\psi_i \circ \hat{\psi}_i, \mathrm{id}} < \epsilon_i$ hold and that $\overline{\hat{\psi}_i \circ \psi_i, \mathrm{id}} < \epsilon_i$ holds. Here, the inequality $\overline{\phi_i \circ \hat{\phi}_i, \mathrm{id}} < \epsilon_i$ means that $\overline{\phi_i \circ \hat{\phi}_i(x), x} < \epsilon_i$ holds for every $x \in \overline{B}_{2R}(y)$. We have $\overline{m_i, \hat{\phi}_i(x_j)} + \overline{\hat{\phi}_i(x_j), z_i} - \overline{m_i, z_i} > \tau/100$ for every sufficiently large i, every $1 \leq j \leq N$ and every point $z_i \in B_{2R}(m_i) \setminus B_{100k^{-1}}(\hat{\phi}_i(x_j))$. Thus, there exists a point $x_j(i,k) \in C_{m_i} \cap \overline{B}_{2R}(m_j)$ such that $\overline{\hat{\phi}_i(x_j), x_j(i,k)} < 100k^{-1}$ holds. Without loss of generality, we can assume that the sequence $\{\psi_i(x_j(i,k))\}_i$ is a Cauchy sequence in X_R for every $1 \leq j \leq N$. We put $x(j,k) = \lim_{i \to \infty} \psi_i(x_j(i,k))$. Similarly, without loss of generality, we can assume that the sequence $\{x(j,k)\}_k$ is a Cauchy sequence for every j. We put $x(j, \infty) = \lim_{i \to \infty} x(j,k)$ and put $\phi_N(x_j) = x(j, \infty)$. Then we have an isometric embedding $\phi_N : (\{x_1, x_2, \cdots, x_N, y\}, y) \to (X_R, x_R)$.

By using ϕ_N and diagonal argument, it is not difficult to construct the map Φ. $\square$

Clearly, in general, the cut locus of the limit space is *not* isometric to the limit space of cut locuses. For example, consider the situation that the flat torus $\mathbf{S}^1(r) \times \mathbf{S}^1$ converges to $\mathbf{S}^1$ as $r \to 0$. Here, $\mathbf{S}^1(r) = \{x \in \mathbf{R}^2 | |x| = r\}$.

4 The measure of codimension one

In this section, we recall the definition of the measure υ_{-1} on Ricci limit spaces, and give several properties of υ_{-1}.

4.1 Definition and finiteness

First, we recall the definition of υ_{-1}. (See (2.1) and (2.2) in [5].)

DEFINITION 4.1. For positive numbers $\beta, \delta > 0$ and a subset $A \subset Y$, we put

$$(\upsilon_{-\beta})_\delta(A) = \inf\{\Sigma_{i\in I} r_i^{-\beta}\upsilon(B(x_i))|\ \sharp I \le \aleph_0,\ A \subset \bigcup_{i\in I} B_{r_i}(x_i),\ r_i < \delta\},$$

$$\upsilon_{-\beta}(A) = \lim_{\delta\to 0}(\upsilon_{-\beta})_\delta(A).$$

By Caratheodory criterion, $\upsilon_{-\beta}$ is a Borel measure on Y. We remark that $\upsilon_{-1}(x) > 0$ holds if and only if $\liminf_{r\to 0}\upsilon(B_r(x))/r > 0$ holds. The following theorem is main result in this subsection. (We will give a result where sharpens the conclusion in the following theorem later. See Corollary 5.5.) This theorem is used in the proof of Theorem 1.1.

THEOREM 4.2. *There exists a positive constant $C(n) > 0$ depending only on n such that for every positive number $t > 0$ and every point $x \in Y$,*

$$\frac{\upsilon_{-1}(\partial B_t(x))}{\upsilon(B_t(x))} \le C(n)\frac{\mathrm{vol}(\partial B_t(\underline{p}))}{\mathrm{vol}B_t(\underline{p})}$$

holds. Here, $\underline{p}$ is a point in standard n-dimensional hyperbolic space $\mathbf{H}^n(-1)$. Especially, we have $\upsilon_{-1}(\partial B_t(x)) < \infty$.

PROOF. We can assume that $\partial B_t(x) \neq \emptyset$. There exists a sequence of complete pointed connected n-dimensional Riemannian manifolds $\{(M_j, m_j)\}_j$ with $Ric_{M_j} \ge -(n-1)$ such that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}B_1(m_j))$ converges to (Y, y, υ) in the sense of measured Gromov-Hausdorff topology. We fix a sufficiently small positive number $0 < \delta << t$. Let $\{x_i\}_{i=1}^N$ be a maximal 100δ-separated set on $\partial B_t(x)$. For every positive integers $i, j > 0$ $(1 \le i \le N)$, we take $x(j), x_i(j) \in M_j$ such that $x_i(j)$ converges to x_i as $j \to \infty$ and that $x(j)$ converges to x as $j \to \infty$. We put $S_j^i = \{u \in S_{x(j)}M_j|$There exists $0 < s < t(u)$ such that $\exp_{x(j)} su \in B_\delta(x_i(j))$ holds$\}$. We also put $I_j^i(u) = \{s \in (0, t(u))|\exp_{x(j)} su \in B_\delta(x_i(j))\}$ for $u \in S_i^j$, $\underline{k}(t) = \sinh(t)$ and put $\theta(t,u) = t^{n-1}(\det(g_{ij}|_{\exp_{x(j)} tu}))^{\frac{1}{2}}$. Here, $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ where $(x_1, x_2, \dots, x_n)$ is a normal coordinate around $x(j)$. Then, we have

$$\begin{aligned}
\mathrm{vol}B_\delta(x_i(j)) &= \int_{S^i_j}\int_{I^i_j(u)} \theta(s,u)dsdu \\
&\le \int_{S^i_j}\int_{I^i_j(u)} \theta(t-2\delta,u)\frac{\underline{k}^{n-1}(s)}{\underline{k}^{n-1}(t-2\delta)}dsdu \\
&\le 2\int_{S^i_j}\int_{I^i_j(u)} \theta(t-2\delta,u)dsdu \\
&\le 4\delta\int_{S^i_j} \theta(t-2\delta,u)du \\
&\le 4\delta\mathrm{vol}\big(\partial B_{t-2\delta}(x(j))\cap C_{x(j)}(B_\delta(x_i(j)))\big).
\end{aligned}$$

Since the set $\{\partial B_{t-2\delta}(x(j))\cap C_x(B_\delta(x_i(j)))\}_i$ are pairwise disjoint for every sufficiently large j, we have

$$\sum_{i=1}^N \delta^{-1}\underline{\mathrm{vol}}B_\delta(x_i(j)) \le 4\underline{\mathrm{vol}}(\partial B_{t-2\delta}(x(j))\setminus C_{x(j)}).$$

By Bishop-Gromov volume comparison theorem, we have

$$\begin{aligned}
\underline{\mathrm{vol}}(\partial B_{t-2\delta}(x(j))\setminus C_{x(j)}) &= \frac{\mathrm{vol}B_{t-2\delta}(x(j))}{\mathrm{vol}B_1(m_j)}\frac{\mathrm{vol}(\partial B_{t-2\delta}(m_j)\setminus C_{x(j)})}{\mathrm{vol}B_{t-2\delta}(x(j))} \\
&\le \frac{\mathrm{vol}B_{t-2\delta}(x(j))}{\mathrm{vol}B_1(m_j)}\frac{\mathrm{vol}\partial B_{t-2\delta}(\underline{p})}{\mathrm{vol}B_{t-2\delta}(\underline{p})}.
\end{aligned}$$

Thus, we have

$$\sum_{i=1}^N \delta^{-1}\underline{\mathrm{vol}}B_\delta(x_i(j)) \le 5\frac{\mathrm{vol}B_{t-2\delta}(x(j))}{\mathrm{vol}B_1(m_j)}\frac{\mathrm{vol}\partial B_{t-2\delta}(\underline{p})}{\mathrm{vol}B_{t-2\delta}(\underline{p})}.$$

By letting $j\to\infty$, we have

$$\begin{aligned}
(\upsilon_{-1})_{1000\delta}(\partial B_t(x)) &\le \sum_{i=1}^N (1000\delta)^{-1}\upsilon(B_{1000\delta}(x_i)) \\
&\le C(n)\sum_{i=1}^N \delta^{-1}\upsilon(B_\delta(x_i)) \\
&\le C(n)\upsilon(B_t(x))\frac{\mathrm{vol}\partial B_{t-2\delta}(\underline{p})}{\mathrm{vol}B_{t-2\delta}(\underline{p})}.
\end{aligned}$$

Therefore, by letting $\delta\to 0$, we have

$$\upsilon_{-1}(\partial B_t(x)) \le C(n)\upsilon(B_t(x))\frac{\mathrm{vol}\partial B_t(\underline{p})}{\mathrm{vol}B_t(\underline{p})}.$$

□

We shall state the following proposition. See (4.3) in [6].

Proposition 4.3. *We assume that $\partial B_1(y) \neq \emptyset$. Then for every positive number $R > 0$ and every point $x \in B_R(y)$, we have*

$$\upsilon(B_s(x)) \leq C(R, n)s$$

for every positive number $0 < s < 1$. Here, $C(R, n) > 0$ is a positive constant depending only on R and n.

Proof. By an argument simular to the proof of Proposition 5.2 in [12]. $\square$

As a corollary of Theorem 4.2 and Proposition 4.3, we have an upper bound for υ_{-1};

Corollary 4.4. *We assume that $\partial B_1(y) \neq \emptyset$. Then for every positive number $R > 0$ and every point $x \in B_R(y)$, we have*

$$\upsilon_{-1}(\partial B_s(x)) \leq C(R, n).$$

for every positive number $0 < s < 1$.

4.2 Bishop-Gromov type inequality

In this subsection, we shall give a proof of Theorem 1.1.

Proof of Theorem 1.1. First, we assume that A is compact. There exists a sequence of complete pointed connected n-dimensional Riemannian manifolds $\{(M_j, m_j)\}_j$ with $Ric_{M_j} \geq -(n-1)$ such that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}B_1(m_j))$ converges to (Y, y, υ) in the sense of measured Gromov-Hausdorff topology. We fix a sufficiently small positive number $\delta > 0$ and put $C_x(A, s, \delta) = \{z \in \partial B_s(x) |$ There exists $p \in \partial B_{t-100\delta}(x) \cap \overline{B}_{1000\delta}(A)$ such that $\overline{x, z} + \overline{z, p} - \overline{x, p} \leq \delta$ holds.$\}$. Clearly, $C_x(A, s, \delta)$ is compact and $\bigcap_{\delta > 0} C_x(A, s, \delta) = \partial B_s(x) \cap C_x(A)$ holds. Let $\epsilon > 0$ be a positive number satisfying $\epsilon << s, t-s, \delta$. There exists a set $\{B_{r_i}(x_i)\}_{i=1}^N$ such that $|\upsilon_{-1}(C_x(A, s, \delta)) - \Sigma_{i=1}^N r_i^{-1}\upsilon(B_{r_i}(x_i))| < \epsilon$, $C_x(A, s, \delta) \subset \bigcup_{i=1}^N B_{r_i}(x_i)$, $0 < r_i < \min\{r, \tau, t-s, \delta\}/1000$ hold for every i, and that $C_x(A, s, \delta) \cap B_{r_i}(x_i) \neq \emptyset$ holds for every i. We put $\hat{\tau} = \min_{1 \leq i \leq N}\{r_i\}/1000$. For every positive integer $j \in \mathbf{N}$, let $x(j)$, $x_i(j)$ be points in M_j satisfying $\overline{x_i(j), x_i} < \epsilon_j$, $\overline{x(j), x} < \epsilon_j$ in the sense of pointed Gromov-Hausdorff topology ($\epsilon_j \to 0$). For every positive integer $i, j > 0$ $(1 \leq i \leq N)$, we put $S_i^j = \{u \in S_{x(j)}M_j |$ There exists $0 < t < t(u)$ such that $\exp_{x(j)} tu \in B_{4r_i}(x_i(j))$ holds.$\}$ and $\hat{S}_i^j = \{u \in S_i^j | t(u) > t - 100\delta\}$. We also put $I_i^j(u) = \{t \in (0, t(u)) | \exp_{x(j)} tu \in B_{4r_i}(x_i(j))\}$ and $\hat{I}_i^j(u) = B_{r_i}(I_i^j(u))$ $(\subset (0, t(u)))$ for

$u \in \hat{S}_i^j$. Then, we have

$$\begin{aligned}
\mathrm{vol} B_{10r_i}(x_i(j)) &\geq \int_{\hat{S}_i^j} \int_{\hat{I}_i^j} \theta(r,u) dr du \\
&\geq \int_{\hat{S}_i^j} \int_{\hat{I}_i^j} \underline{k}^{n-1}(r) \frac{\theta(t-100\delta, u)}{\underline{k}^{n-1}(t-100\delta)} dr du \\
&\geq \frac{\underline{k}^{n-1}(s-100\delta)}{\underline{k}^{n-1}(t-100\delta)} r_i \int_{\hat{S}_i^j} \theta(t-100\delta, u) du \\
&= \frac{\underline{k}^{n-1}(s-100\delta)}{\underline{k}^{n-1}(t-100\delta)} r_i \mathrm{vol}(\partial B_{t-100\delta}(x(j)) \cap S_{x(j)}(B_{4r_i}(x_i(j))) \setminus C_{x(j)}).
\end{aligned}$$

Here, $S_{x(j)}(B_{4r_i}(x_i(j))) = \{\alpha \in M_j \setminus C_{x(j)} |$ There exists $\beta \in B_{4r_i}(x_i(j))$ such that $\overline{x(j), \beta} + \overline{\beta, \alpha} = \overline{x(j), \alpha}$ holds or $\overline{x(j), \alpha} + \overline{\alpha, \beta} = \overline{x(j), \beta}$ holds.$\}$. Therefore, we have

$$\sum_{i=1}^N r_i^{-1} \underline{\mathrm{vol}} B_{10r_i}(x_i(j)) \geq \frac{\underline{k}^{n-1}(s-100\delta)}{\underline{k}^{n-1}(t-100\delta)} \sum_{i=1}^N \underline{\mathrm{vol}}(\partial B_{t-100\delta}(m_j) \cap S_{x(j)}(B_{4r_i}(x_i(j))) \setminus C_{x(j)}).$$

Let $\{\hat{x}_i\}_{i=1}^{\hat{N}}$ be a maximal 10000δ-separated set on A and $\hat{x}_i(j) \in M_j$ a point satisfying $\overline{\hat{x}_i(j), \hat{x}_i} < \epsilon_j$.

Claim 4.5. *For every sufficiently large $j > 0$, every point $z_j \in B_{\delta/10}(\hat{x}_i(j))$ and every minimal geodesic from $x(j)$ to z_j, $\gamma : [0, \overline{x(j), z_j}] \to M_j$, we have*

$$\mathrm{Image}(\gamma) \cap (\bigcup_{i=1}^N B_{4r_i}(x_i(j))) \neq \emptyset.$$

Let $m_j(s+\hat{\tau}) \in \mathrm{Image}(\gamma)$ be a point satisfying $\overline{m_j, m_j(s+\hat{\tau})} = s+\hat{\tau}$. We take points $\tilde{m}_j(s+\hat{\tau})$, $\tilde{z}_j \in Y$ such that $\overline{\tilde{m}_j(s+\hat{\tau}), m_j(s+\hat{\tau})} < \epsilon_j$ holds and that $\overline{\tilde{z}_j, z_j} < \epsilon_j$ holds. Then, we have $\overline{x, \tilde{m}_j(s+\hat{\tau})} + \overline{\tilde{m}_j(s+\hat{\tau}), \tilde{z}_j} - \overline{x, \tilde{z}_j} < 10\epsilon_j$. There exists $w \in Y$ such that $\overline{x, w} + \overline{w, \tilde{m}_j(s+\tilde{\tau})} = \overline{x, \tilde{m}_j(s+\tilde{\tau})}$, $\overline{w, \tilde{m}_j(s+\tilde{\tau})} < 2\tilde{\tau}$ hold and that $w \in \partial B_s(x)$ holds. Since $\epsilon_j << \tilde{\tau}$, we have $\overline{x, \tilde{m}_j(s+\hat{\tau})} + \overline{\tilde{m}_j(s+\hat{\tau}), \hat{x}_i} - \overline{x, \hat{x}_i} \leq \delta$. Hence, we have $w \in C_x(A, s, \delta)$. Therefore, there exists $1 \leq i \leq N$ such that $w \in B_{r_i}(x_i)$ holds. Therefore we have $\tilde{m}_j(s+\tilde{\tau}) \in B_{2r_i}(x_i)$ and $m_j(s+\tilde{\tau}) \in B_{4r_i}(x_i(j))$. Thus, we have Claim 4.5.

For every ball $B_\delta(\hat{x}_i(j))$, we put $\acute{S}_i^j = \{u \in S_{x(j)} M_j |$ There exists $0 < t < t(u)$ such that $\exp_{m_j} tu \in B_\delta(\hat{x}_i(j))$ holds.$\}$ and $\acute{I}_i^j(u) = \{t \in (0, t(u)) | \exp_{x(j)} tu \in B_\delta(\hat{x}_i(j))\}$ for

$u \in \acute{S}_i^j$. Then, we have

$$\begin{aligned}
\mathrm{vol} B_\delta(\hat{x}_i(j)) &= \int_{\acute{S}_i^j} \int_{\acute{I}_i^j} \theta(r,u) dr du \\
&\leq \int_{\acute{S}_i^j} \int_{\acute{I}_i^j} \underline{k}^{n-1}(r) \frac{\theta(t-100\delta, u)}{\underline{k}^{n-1}(t-100\delta)} dr du \\
&\leq 2 \frac{\underline{k}^{n-1}(t+100\delta)}{\underline{k}^{n-1}(t-100\delta)} \delta \int_{\acute{S}_i^j} \theta(t-100\delta, u) du \\
&= 2 \frac{\underline{k}^{n-1}(t+100\delta)}{\underline{k}^{n-1}(t-100\delta)} \delta \mathrm{vol}\big(\partial B_{t-100\delta}(X(j)) \cap C_{x(j)}(B_\delta(\hat{x}_i(j)))\big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sum_{i=1}^{\hat{N}} \delta^{-1} \underline{\mathrm{vol}} B_\delta(\hat{x}_i(j)) &\leq 2 \sum_{i=1}^{\hat{N}} \frac{\underline{k}^{n-1}(t+100\delta)}{\underline{k}^{n-1}(t-100\delta)} \underline{\mathrm{vol}}\big(\partial B_{t-100\delta}(x(j)) \cap C_{x(j)}(B_\delta(\hat{x}_i(j)))\big) \\
&= 2 \frac{\underline{k}^{n-1}(t+100\delta)}{\underline{k}^{n-1}(t-100\delta)} \underline{\mathrm{vol}}\big(\partial B_{t-100\delta}(x(j)) \cap C_{x(j)}(\bigsqcup_{i=1}^{\hat{N}} B_\delta(\hat{x}_i(j)))\big).
\end{aligned}$$

By Claim 4.5, we have

$$\partial B_{t-100\delta}(m_j) \cap C_{x(j)}(\bigsqcup_{i=1}^{\hat{N}} B_\delta(\hat{x}_i(j))) \subset \partial B_{t-100\delta}(x(j)) \cap S_{x(j)}(\bigcup_{i=1}^{N} B_{4r_i}(x_i(j))).$$

Thus, we have

$$\sum_{i=1}^{\hat{N}} \delta^{-1} \underline{\mathrm{vol}} B_\delta(\hat{x}_i(j)) \leq 3 \frac{\underline{k}^{n-1}(t-100\delta)}{\underline{k}^{n-1}(s-100\delta)} \sum_{i=1}^{N} r_i^{-1} \underline{\mathrm{vol}} B_{10r_i}(x_i(j)).$$

Therefore, by letting $j \to \infty$, we have

$$\begin{aligned}
(\upsilon_{-1})_{10^6\delta}(A) &\leq \sum_{i=1}^{\hat{N}} (10^5\delta)^{-1} \upsilon(B_{10^5\delta}(\hat{x}_i)) \\
&\leq C(n) \sum_{i=1}^{\hat{N}} \delta^{-1} \upsilon(B_\delta(\hat{x}_i)) \\
&\leq C(n) \frac{\underline{k}^{n-1}(t-100\delta)}{\underline{k}^{n-1}(s-100\delta)} \sum_{i=1}^{N} r_i^{-1} \upsilon(B_{10r_i}(x_i)) \\
&\leq C(n) \frac{\underline{k}^{n-1}(t-100\delta)}{\underline{k}^{n-1}(s-100\delta)} (\upsilon_{-1}(C_x(A,s,\delta)) + \epsilon).
\end{aligned}$$

By letting $\epsilon \to 0$ and $\delta \to 0$, we have Theorem 1.1 for every compact set A. By standard argument in measure theory, it is easy to prove Theorem 1.1 for every Borel set A. $\square$

There exist several applications to one dimensional Ricci limit spaces as a corollary of Theorem 1.1. See [12].

4.3 Non-zero

In this subsection, we give a non-zero property for υ_{-1}. First, we shall prove the following theorem;

THEOREM 4.6. *There exists a positive constant $C(n) > 0$ depending only on n such that for every positive numbers $0 < s \le r < t$, every point $x \in Y$ and every Borel set $A \subset A_{r,t}(x)$,*

$$\frac{\upsilon(A)}{\mathrm{vol}B_t(\underline{p}) - \mathrm{vol}B_r(\underline{p})} \le C(n)\frac{\upsilon_{-1}(\partial B_s(x) \cap C_x(A))}{\mathrm{vol}\partial B_s(\underline{p})}$$

holds. Especially, if $\upsilon(A) > 0$ holds, then $\upsilon_{-1}(\partial B_s(x) \cap C_x(A)) > 0$ holds.

PROOF. Without loss of generality, we can assume that $s < r$ holds and A is compact. We fix a sufficiently small positive number δ and put $C_x(A, s, \delta) = \{z \in \partial B_s(x) |$There exists $\alpha \in Y$ such that $\overline{\alpha, A} \le \delta$ and $\overline{x, z} + \overline{z, \alpha} - \overline{x, \alpha} \le \delta$ hold.$\}$. We remark that $C_x(A, s, \delta)$ is a compact set and that $\bigcap_{\delta > 0} C_x(A, s, \delta) = \partial B_s(x) \cap C_x(A)$ holds. Let $\epsilon > 0$ be a positive number. There exists $\{B_{r_i}(x_i)\}_{i=1}^N$ such that $|\upsilon_{-1}(C_x(A, s, \delta)) - \Sigma_{i=1}^N r_i^{-1} \upsilon(B_{r_i}(x_i))| < \epsilon$, $C_x(A, s, \delta) \subset \bigcup_{i=1}^N B_{r_i}(x_i)$, $0 < r_i < \min\{r, \tau, t - s, \delta\}/1000$ hold for every i, and that $C_x(A, s, \delta) \cap B_{r_i}(x_i) \neq \emptyset$ holds. By taking a maximal 100δ - separated set on A and by an argument simular to the proof of Theorem 1.1, we have

$$\upsilon(A) \le C(n)\frac{\mathrm{vol}B_{t+100\delta}(\underline{p}) - \mathrm{vol}B_{r-100\delta}(\underline{p})}{\mathrm{vol}\partial B_{s-100\delta}(\underline{p})}(\upsilon_{-1}(C_x(A, s, \delta)) + \epsilon).$$

Therefore, by letting $\epsilon \to 0$, $\delta \to 0$, we have Theorem 4.6. □

Next corollary is a non-zero property for υ_{-1};

COROLLARY 4.7. *Let x be a point in Y and $R > 0$ a positive number. We assume that $\partial B_R(x) \setminus C_x \neq \emptyset$. Then for every $z \in \partial B_R(x) \setminus C_x$ and every positive number $\epsilon > 0$, $\upsilon_{-1}(B_\epsilon(z) \cap \partial B_R(x) \setminus C_x) > 0$ holds.*

PROOF. There exist a sufficiently small positive number $0 < \tau < \epsilon/1000$ and a point $w \in Y$ such that $\overline{x, z} + \overline{z, w} = \overline{x, w}$ and $\overline{z, w} = \tau$ hold. Then, since $\partial B_R(x) \cap C_x(B_{\tau/1000}(w)) \subset B_\epsilon(z) \cap \partial B_R(x) \setminus C_x$, we have Corollary 4.7 by Theorem 4.6. □

Finally, we shall give the following theorem.

COROLLARY 4.8. *For every points $x, z \in Y$ such that $x \neq z$, the following conditions are equivalent:*

1. *$\upsilon(C_x(\{z\})) > 0$ holds.*

2. *$\upsilon_{-1}(\partial B_t(x) \cap C_x(\{z\})) > 0$ holds for every $0 < t < \overline{x, z}$.*

3. $\upsilon_{-1}(\partial B_t(x)\cap C_x(\{z\}))>0$ *holds for some* $0<t<\overline{x,z}$*.*

PROOF. First, we assume that $\upsilon(C_x(\{z\}))>0$ holds. We put $r=\overline{x,z}>0$. There exists a positive integer $N\in\mathbf{N}$ such that $\upsilon(C_x(\{z\})\cap A_{N^{-1}r,\,(N+1)^{-1}r}(x))>0$ holds. Thus, by Theorem 4.6, we have $\upsilon_{-1}(\partial B_t(x)\cap C_x(\{z\}))>0$ for every $0<t<(N+1)^{-1}r$. Since $\partial B_s(x)\cap C_x(\{z\})=\partial B_{r-s}(z)\cap C_z(\{x\})$ holds for every $0<s<r$, by Theorem 1.1, we have $\upsilon_{-1}(\partial B_t(x)\cap C_x(\{z\}))>0$ for every $0<t<r$.

Next, we assume that $\upsilon(C_x(\{z\}))=0$. Then, by Corollary 5.5, there exists $t\in(0,\overline{x,z})$ such that $\upsilon_{-1}(\partial B_t(x)\cap C_x(\{z\}))=0$ holds. □

5 Co-area formula for distance function

In this section, we give a relationship between the limit measure υ and the measure υ_{-1}. Let x be a point in Y and $A\subset Y$ a subset. We define $\Phi_A:\mathbf{R}_{\geq 0}\to\mathbf{R}_{\geq 0}$ by

$$\Phi_A(t)=\upsilon_{-1}(\partial B_t(x)\cap A).$$

PROPOSITION 5.1. *For every Borel set $A\subset Y$, the map Φ_A is a Lebesgue measurable function.*

We will give a proof of Proposition 5.1 in Appendix. The following theorem is main result in this section.

THEOREM 5.2. *Let x be a point in Y. There exists the non-negative valued function $f\in L_\infty(Y)$ and a positive constant $C(n)>0$ depending only on n, such that $f|_U\not\equiv 0$ holds for every open set $U\subset Y$, $|f|_{L_\infty}\leq C(n)$ holds, and*

$$\int_0^\infty\int_{\partial B_t(x)\setminus C_x} g\,d\upsilon_{-1}dt=\int_Y gf\,d\upsilon$$

holds for every $g\in L_1(Y)$.

PROOF. There exists a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_j,m_j)\}_j$ with $Ric_{M_j}\geq-(n-1)$ such that $(M_j,m_j,\mathrm{vol}/\mathrm{vol}B_1(m_j))$ converges to (Y,y,υ) in the sense of measured Gromov-Hausdorff topology. For every positive number $\tau>0$, we put $\mathcal{D}_\tau=\{w\in Y|$ There exists $z\in Y\setminus B_\tau(x)$ such that $\overline{x,w}+\overline{w,z}=\overline{x,z}$ holds. $\}$. Clearly, $\mathcal{D}_\tau$ is a closed set and $\bigcup_{\tau>0}\mathcal{D}_\tau=Y\setminus C_x$. We fix $\tau>0$. Let $s,t,r,R,\delta>0$ be positive numbers satisfying $0<\delta<<\tau,s$ and $\delta<<r<t<R$. We assume that $A_{r,R}(x)\neq\emptyset$. We take a point $w\in A_{r,R}(x)$. Let $\{x_i\}_{i=1}^N$ be a maximal 100δ-separated set on $\partial B_t(x)\cap\overline{B}_s(w)$. We take a positive number $\hat{t}>0$ such that $|t-\hat{t}|\leq\delta$ and $\hat{t}\in[r,R]$ hold.

Claim 5.3. *We have $\partial B_{\hat t}(x)\cap \mathcal{D}_\tau \cap \overline{B}_{s-100\delta}(w)\subset \bigcup_{i=1}^N B_{300\delta}(x_i)$.*

Let z be a point in $\partial B_{\hat t}(x)\cap \mathcal{D}_\tau\cap \overline{B}_{s-100\delta}(w)$. First, we assume that $\hat t\geq t$. Then there exists $\alpha\in \partial B_t(x)\cap \overline{B}_s(w)$ such that $\overline{x,\alpha}+\overline{\alpha,z}=\overline{x,z}$ and $\overline{\alpha,z}\leq\delta$ hold. Thus, there exists a positive integer $1\leq i\leq N$ such that $\alpha\in B_{250\delta}(x_i)$ holds. Therefore, we have $z\in B_{300\delta}(x_i)$.

Next, we assume that $\hat t<t$. Since $\delta<<\tau$, there exists $\alpha\in\partial B_t(x)\cap\overline{B}_s(w)$ such that $\overline{x,z}+\overline{z,\alpha}=\overline{x,\alpha}$ and $\overline{\alpha,z}\leq\delta$ hold. Thus there exists a positive integer $1\leq i\leq N$ such that $\alpha\in B_{200\delta}(x_i)$ holds. Hence, we have $z\in B_{300\delta}(x_i)$. Therefore, we have Claim 5.3.

For every positive integers $i,j>0$ $(1\leq i\leq N)$, let $x_i(j),x(j)\in M_j$ be points satisfying $\overline{x_i(j),x_i}<\epsilon_j$ and $\overline{x(j),x}<\epsilon_j$ $(\epsilon_j\to 0)$. We put $S^i_j=\{u\in S_{x(j)}M_j|$ There exists $0<t<t(u)$ such that $\exp_{x(j)}tu\in B_\delta(x_i(j))$ holds.$\}$ and $I_i(u)=\{t\in(0,t(u))|\exp_{x(j)}tu\in B_\delta(x_i(j))\}$ for $u\in S^i_j$. Then, we have

$$\begin{aligned}
\mathrm{vol}B_\delta(x_i(j))&=\int_{S^i_j}\int_{I_i(u)}\theta(\hat s,u)d\hat s du\\
&\leq\int_{S^i_j}\int_{I_i(u)}\underline{k}^{n-1}(\hat s)\frac{\theta(\hat t-10\delta,u)}{\underline{k}^{n-1}(\hat t-10\delta)}d\hat s du\\
&\leq 2\int_{S^i_j}\int_{I_i(u)}\theta(\hat t-10\delta,u)d\hat s du\\
&\leq 5\delta\int_{S^i_j}\theta(\hat t-10\delta,u)du\\
&\leq 5\delta\mathrm{vol}\big(\partial B_{\hat t-10\delta}(x(j))\cap C_{x(j)}(B_\delta(x_i(j)))\cap B_{20\delta}(x_i(j))\setminus C_{x(j)}\big).
\end{aligned}$$

Claim 5.4. *For every $i_1,i_2\in\{1,2,\cdots,N\}$ such that $i_1\neq i_2$, for every sufficiently large integer j, we have $C_{x(j)}(B_{2\delta}(x_{i_1}(j)))\cap C_{x(j)}(B_{2\delta}(x_{i_1}(j)))\cap B_{20\delta}(x_{i_2}(j))=\phi$.*

Assume that the assertion were false. We take $z_j\in C_{x(j)}(B_{2\delta}(x_{i_1}(j)))\cap C_{x(j)}(B_{2\delta}(x_{i_2}(j)))\cap B_{20\delta}(x_{i_2}(j))$. There exist $y_{i_1}(j)\in B_{2\delta}(x_{i_1}(j))$, $y_{i_2}(j)\in B_{2\delta}(x_{i_2}(j))$ such that $\overline{x(j),z_j}+\overline{z_j,y_{i_1}(j)}=\overline{x(j),y_{i_1}(j)}$, $\overline{x(j),z_j}+\overline{z_j,y_{i_2}(j)}=\overline{x(j),y_{i_2}(j)}$ hold. Then, by triangle inequality, we have

$$\begin{aligned}
\overline{x_{i_1}(j),x_{i_2}(j)}&\leq\overline{x_{i_1}(j),y_{i_1}(j)}+\overline{y_{i_1}(j),z_j}+\overline{z_j,y_{i_2}(j)}+\overline{y_{i_2}(j),x_{i_2}(j)}\\
&\leq 2\delta+\overline{y_{i_1}(j),z_j}+\overline{z_j,y_{i_2}(j)}+2\delta\\
&\leq 4\delta+t+5\delta-\overline{x(j),z_j}+\overline{z_j,y_{i_2}(j)}\\
&\leq 9\delta+t-(\overline{x(j),y_{i_2}(j)}-\overline{z_j,y_{i_2}(j)})+\overline{z_j,y_{i_2}(j)}\\
&\leq 9\delta+t-\overline{x(j),y_{i_2}(j)}+50\delta\\
&\leq 9\delta+5\delta+50\delta=64\delta.
\end{aligned}$$

Thus, we have $\overline{x_{i_1}, x_{i_2}} < 70\delta$. This is contradiction. Therefore, we have Claim 5.4.

Let $w(j) \in M_j$ be a point satisfying $\overline{w(j), w} < \epsilon_j$. By Claim 5.4 and $B_{20\delta}(x_i(j)) \subset B_{s+100\delta}(w(j))$, we have

$$\sum_{i=1}^{N} \underline{\mathrm{vol}} B_\delta(x_i(j)) \leq 10\delta \underline{\mathrm{vol}}\big(\partial B_{\hat{t}-10\delta}(x(j)) \cap B_{s+100\delta}(w(j)) \setminus C_{x(j)}\big).$$

On the other hand, for every sufficiently large j, we have

$$|\sum_{i=1}^{N} \upsilon(B_\delta(x_i)) - \sum_{i=1}^{N} \underline{\mathrm{vol}} B_\delta(x_i(j))| < \delta^2.$$

Therefore, for every sufficiently large j, we have

$$\begin{aligned}
&(\upsilon_{-1})_{1000\delta}(\partial B_{\hat{t}}(x) \cap \overline{B}_{s-100\delta}(w) \cap \mathcal{D}_\tau) \\
&\leq \sum_{i=1}^{N} (1000\delta)^{-1} \upsilon(B_{1000\delta}(x_i)) \\
&\leq C(n) \sum_{i=1}^{N} \delta^{-1} \upsilon(B_\delta(x_i)) \\
&\leq C(n)(\delta + \sum_{i=1}^{N} \delta^{-1} \underline{\mathrm{vol}} B_\delta(x_i(j))) \\
&\leq C(n)\delta + C(n)\underline{\mathrm{vol}}(\partial B_{\hat{t}-10\delta}(x(j)) \cap B_{s+100\delta}(w(j)) \setminus C_{x(j)}). \qquad (*)
\end{aligned}$$

Let $\{t_i\}_{i=1}^k \subset [r, R]$ be a subset satisfying $[r, R] \subset \bigcup_{i=1}^k B_{\delta/2}(t_i)$. For every $i = 1, 2, \cdots, k$, we have that inequality $(*)$ holds for every sufficiently large integer j and every $\hat{t} \in [r, R]$ satisfying $|\hat{t} - t_i| < \delta$. Hence, inequality $(*)$ holds for every sufficiently large j and every $\hat{t} \in [r, R]$. Therefore, for such sufficiently large integer j, we have

$$\begin{aligned}
&\int_r^R (\upsilon_{-1})_{1000\delta}\big(\partial B_{\hat{t}}(x) \cap \overline{B}_{s-\tau}(w) \cap \mathcal{D}_\tau\big) d\hat{t} \\
&\leq \int_r^R (\upsilon_{-1})_{1000\delta}\big(\partial B_{\hat{t}}(x) \cap \overline{B}_{s-100\delta}(w) \cap \mathcal{D}_\tau\big) d\hat{t} \\
&\leq C(n)(R-r)\delta + C(n) \int_r^R \underline{\mathrm{vol}}\big(\partial B_{\hat{t}-10\delta}(x(j)) \cap B_{s+100\delta}(w(j)) \setminus C_{x(j)}\big) d\hat{t} \\
&\leq C(n)(R-r)\delta + C(n) \int_{\tau-10\delta}^{R-10\delta} \underline{\mathrm{vol}}\big(\partial B_\alpha(x(j)) \cap B_{s+100\delta}(w(j)) \setminus C_{x(j)}\big) d\alpha \\
&\leq C(n)(R-r)\delta + C(n)\underline{\mathrm{vol}} B_{s+100\delta}(w(j)).
\end{aligned}$$

By letting $j \to \infty$, $\delta \to 0$, $R \to \infty$, $r \to 0$ and letting $\tau \to 0$, we have

$$\int_0^\infty \upsilon_{-1}(\partial B_{\hat{t}}(x) \cap \overline{B}_s(w) \setminus C_x) d\hat{t} \leq C(n)\upsilon(\overline{B}_s(w)).$$

Since the map $\hat{\Psi} : \mathcal{B}(Y) \to \mathbf{R} \cup \{\infty\}$,

$$\hat{\Psi}(A) = \int_0^\infty \upsilon_{-1}(\partial B_t(x) \cap A \setminus C_x)dt$$

is an additive set function on $\mathcal{B}(Y)$. Here $\mathcal{B}(Y) = \{A \in 2^Y | A \text{ is a Borel set of } Y\}$. By standard argument in measure theory, for every Borel set $A \in \mathcal{B}(Y)$, we have

$$\int_0^\infty \upsilon_{-1}(\partial B_t(x) \cap A \setminus C_x)dt \leq C(n)\upsilon(A).$$

By Radon-Nikodym theorem, we have Theorem 5.2. □

We give next inequality where sharpens the conclusion in Theorem 4.2.

COROLLARY 5.5. *For every positive numbers* $0 < r_1 < r_2 \leq R$, *every point* $x \in Y$ *and every Borel set* $A \subset \partial B_R(x)$,

$$\frac{\upsilon_{-1}(A)}{\mathrm{vol}\partial B_R(\underline{p})} \leq C(n)\frac{\upsilon(A_{r_1,\ r_2}(x) \cap C_x(A))}{\mathrm{vol}B_{r_2}(\underline{p}) - \mathrm{vol}B_{r_1}(\underline{p})}$$

holds.

PROOF. It follows from Theorem 1.1 and Theorem 5.2, immediately. □

6 Ahlfors α-regular set and the Hausdorff dimension

We consider a set that on the set, limit measure υ is equivalent to some Hausdorff measure.

DEFINITION 6.1. For positive numbers $\alpha \geq 0$, $C > 1$, we put

$$A_Y(\alpha, C) = \{x \in Y | C^{-1}s^\alpha \leq \upsilon(B_s(x)) \leq Cs^\alpha \text{ for every } 0 < s < 1\},$$

$$A_Y(\alpha) = \bigcup_{C>1} A_Y(\alpha, C).$$

We call the set $A_Y(\alpha)$ *Ahlfors α-regular set.*

Note that $A_Y(\alpha, C)$ is a compact set. Next, we shall give a notion of tangent cone.

DEFINITION 6.2. Let (W, w), (Z, z) be pointed proper geodesic spaces. We say that (W, w) is *tangent cone at* $\alpha \in Z$ if there exists a sequence of positive numbers $r_i > 0$ such that r_i converges to 0 and that rescaled pointed proper geodesic spaces $(Z, r_i^{-1}d_Z, \alpha)$ converges to (W, w) in the sence of pointed Gromov-Hausdorff topology. Here, d_Z is the metric (distance function) on Z.

We shall give an upper bound of Hausdorff dimension of Ahlfors α-regular set.

THEOREM 6.3. *We have* $\dim_{\mathcal{H}} A_Y(\alpha) \le [\alpha]$ *for every positive number* $\alpha > 0$. *Here* $[\alpha] = \sup\{k \in \mathbf{Z} | k \le \alpha\}$.

PROOF. This proof is by contradiction. We assume that $\dim_{\mathcal{H}} A_Y(\alpha) > [\alpha]$ holds. Then, there exist a sufficiently small positive number $0 < \beta < 1$ and a positive number $C > 1$ such that $\mathcal{H}^{\alpha+\beta}(A_Y(\alpha, C)) > 0$ holds. By density result of Geometric measure theory, there exist $x \in Y$, a tangent cone at x, $(T_xY, 0_x)$, and exists a sequence of positive numbers $r_i > 0$ such that r_i converges to 0, $\lim_{i \to 0} \mathcal{H}^{\alpha+\beta}_{\infty}(\overline{B}_{r_i}(x))/{r_i}^{\alpha+\beta} > 0$ holds and that $(Y, r_i^{-1} d_Y, x)$ converges to $(T_xY, 0_x)$. (For example, see (1.39) in [5] for the definition of $(\alpha+\beta)$-dimensional spherical Hausdorff content, $\mathcal{H}^{\alpha+\beta}_{\infty}$.) Without loss of generality, we can assume that there exist a compact metric space Z, a limit measure υ_{∞} on $(T_xY, 0_x)$, positive number $\hat{C} > 1$ and exists an isometric embedding $\phi : Z \to A_{T_xY}(\alpha, \hat{C}) \cap \overline{B}_1(0_x)$ for υ_{∞} such that $H^{\alpha+\beta}(Z) > 0$ holds and that $(\overline{B}_{r_i}(x) \cap A_Y(\alpha, C), r_i^{-1} d_Y)$ converges to Z in the sense of Gromov-Hausdorff topology. Especially, $\mathcal{H}^{\alpha+\beta}(\overline{B}_1(0_x) \cap A_{T_xY}(\alpha, \hat{C})) > 0$ holds. By Proposition 2.5 in [5], we have $\mathcal{H}^{\alpha+\beta}(\overline{B}_1(0_x) \cap A_{T_xY}(\alpha, \hat{C}) \setminus \mathcal{WD}_0(0_x)) > 0$. (See Definition 2.10 in [4] for the definition of $\mathcal{WD}_0(x)$.) We put $(Y_1, y_1) = (T_xY, 0_x)$. Then, there exist a point $z \in A_{Y_1}(\alpha, \hat{C}) \setminus \mathcal{WD}_0(y_1)$, a sequence of positive numbers $s_i > 0$ and a pointed proper geodesic space (W, w) such that s_i converges to 0, $\lim_{i \to 0} \mathcal{H}^{\alpha+\beta}_{\infty}(\overline{B}_{s_i}(z))/s_i^{\alpha+\beta} > 0$ and $(Y_1, s_i^{-1} d_{Y_1}, z)$ converges to $(\mathbf{R} \times W, (0, w))$.

By iterating this argument, there exist an iterated tangent cone of Y, (T, t), a limit measure $\tilde{\upsilon}_{\infty}$ on (T, t), a positive constant $\tilde{C} > 1$ and a proper geodesic space X such that $\mathcal{H}^{\alpha+\beta}(\overline{B}_1(t) \cap A_T(\alpha, \tilde{C})) > 0$ holds for $\tilde{\upsilon}_{\infty}$, and that T is isometric to $\mathbf{R}^{[\alpha]+1} \times X$ holds. Therefore, there exists a point $w \in T$ such that $\liminf_{r \to 0} \tilde{\upsilon}_{\infty}(B_r(w))/r^{\alpha} > 0$ holds. This contradicts Proposition 1.35 in [4]. $\square$

Next Corollary follows from Theorem 5.5 in [6], immediately. We shall give an alternative proof.

COROLLARY 6.4. *We assume that* $\upsilon(A_Y(\alpha)) > 0$ *holds. Then* α *is an integer.*

PROOF. By the assumption, we have $\mathcal{H}^{\alpha}(A(\alpha)) > 0$. Hence, $\dim_{\mathcal{H}} A(\alpha) \ge \alpha$. Therefore, by Theorem 6.3, we have $\alpha = [\alpha]$. $\square$

7 Appendix: A proof of Proposition 5.1

In this section, we shall prove Proposition 5.1. We fix positive numbers $0 < r < R$. For every $t \in \mathbf{Q}_{>0}$, let $\{x_i^t\}_{i \in \mathbf{N}}$ be a countable dense set of $\partial B_t(x)$. For every positive integer $N \in \mathbf{N}$ and every positive number $\delta > 0$, we put $\mathcal{B} = \{\overline{B}_s(x_i^t) | i \in \mathbf{N},\ s, t \in \mathbf{Q}_{>0}\}$, $\mathcal{B}_{\delta}^N = \{(\overline{B}_{r_i}(x_i))_{i=1,2,\cdots,N} \in \mathcal{B}^N | \overline{B}_{r_i}(x_i) \in \mathcal{B}, r_i < \delta\}$ and put $\mathcal{B}_{\delta} = \bigcup_{N \in \mathbf{N}} \mathcal{B}_{\delta}^N$. Clearly, these are countable sets.

LEMMA 7.1. *Let $A \subset Y$ be a compact set. Then the function $t \mapsto (\upsilon_{-1})_\delta(\partial B_t(x) \cap A)$ is a Borel fuction for every positive number $\delta > 0$. Especially, the map $\Phi_A|_{[r,R]}$ is a Borel function.*

PROOF. For every $F = (\overline{B}_{r_i}(x_i))_{i=1,2,\cdots,N} \in \mathcal{B}_\delta$, we define a map Ψ_F from $[r,\ R]$ to $\mathbf{R} \cup \{\infty\}$ by $\Psi_F(t) = \sum_{i=i}^N r_i^{-1}\upsilon(B_{r_i}(x_i))$ if $\partial B_t(x) \cap A \subset \bigcup_{i=1}^N \overline{B}_{r_i}(x_i)$ holds, $\Psi_F(t) = \infty$ if otherwise. Since $\partial B_t(x) \cap A$ is a compact set, Ψ_F is a Borel function. Therefore, $\Psi = \inf_{F \in \mathcal{B}_\delta} \Psi_F$ is a Borel function. By the definition of $(\upsilon_{-1})_\delta$, we have $\Psi(t) = (\upsilon_{-1})_\delta(\partial B_t(x) \cap A)$. □

Therefore, we have the following corollary.

COROLLARY 7.2. *Let $O \subset Y$ be a open set. Then the map $\Phi_O|_{[r,R]}$ is a Borel function.*

Here we put $\sigma = \{A \in \mathcal{B}(Y)|$ For every positive number $\epsilon > 0$, there exist a sequence of compact sets $K_i \subset A$, a sequence of open sets $A \subset O_i$ and exists a Lebesgue measurable set $E_\epsilon \subset [r,\ R]$ such that $\mathcal{H}^1([r,\ R] \setminus E_\epsilon) < \epsilon$ holds and that $\sup_{t \in E_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap A \setminus K_i)$ and $\sup_{t \in E_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap O_i \setminus A)$ converge to 0 as $i \to \infty$. $\}$. Note that for every sets $A_i \in \sigma$, $\Phi_A|_{[r,\ R]}$ is a Lebesgue measurable function for every set $A = \bigcup_{i \in \mathbf{N}} A_i$.

LEMMA 7.3. *σ is σ - algebra.*

PROOF. It suffices to show $\bigcup_{i \in \mathbf{N}} A_i \in \sigma$ for every sets $A_i \in \sigma$. We take a sequence $A_i \in \sigma$. Let $\epsilon > 0$ be a positive number. For every $i \in \mathbf{N}$, there exist a sequence of compact sets $K_i(j) \subset A_i$, a sequence of open sets $A_i \subset O_i(j)$, and exists a Lebesgue measurable set $E_\epsilon(i) \subset [r, R]$ such that $\mathcal{H}^1([r, R] \setminus E_\epsilon(i)) < 2^{-i}\epsilon$ holds and that $\sup_{t \in E_\epsilon(i)} \upsilon_{-1}(\partial B_t(x) \cap O_i(j) \setminus A_i)$ and $\sup_{t \in E_i(l)} \upsilon_{-1}(\partial B_t(x) \cap A_i \setminus K_i(j))$ converge to 0 as $j \to \infty$. Thus, for every $l \in \mathbf{N}$, there exists a sufficiently large integer $N(l) \in \mathbf{N}$ such that for every $1 \leq i \leq l$, $\sup_{t \in E_\epsilon(i)} \upsilon_{-1}(\partial B_t(x) \cap A_i \setminus K_i(N(l))) \leq l^{-1}2^{-i}$ holds. Since $\upsilon_{-1}(\partial B_t(x) \cap (\bigcup_{i=1}^l A_i))$ converges to $\upsilon_{-1}(\partial B_t(x) \cap (\bigcup_{i \in \mathbf{N}} A_i))$ as $l \to \infty$ for every $t \in [r, R]$, by Egoroff's theorem, there exists a Lebesgue measurable set $E_\epsilon \subset [r, R]$ such that $\mathcal{H}^1([r, R] \setminus E_\epsilon) < \epsilon$ holds and that $\sup_{t \in E_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap (\bigcup_{i \in \mathbf{N}} A_i \setminus \bigcup_{i=1}^l A_i))$ converges to 0 as $l \to \infty$. We put $\hat{E}_\epsilon = \bigcap_{i \in \mathbf{N}} E_\epsilon(i) \cap E_\epsilon$. Then, we have, $\mathcal{H}^1([r, R] \setminus \hat{E}_\epsilon) \leq \sum_{i \in \mathbf{N}} \mathcal{H}^1([r, R] \setminus E_\epsilon(i)) + \mathcal{H}^1([r, R] \setminus E_\epsilon) < 2\epsilon$. We also put a compact set $\hat{K}_l = \bigcup_{i=1}^l K_i(N(l))$. Then, $\sup_{t \in \hat{E}_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap (\bigcup_{i \in \mathbf{N}} A_i \setminus \hat{K}_l))$ converges to 0 as $l \to \infty$. For every integers $l, i \in \mathbf{N}$, there exists a sufficiently large $j(l,i) \in \mathbf{N}$ such that $\sup_{t \in E_\epsilon(i)} \upsilon_{-1}(\partial B_t(x) \cap (O_i(j(l,i)) \setminus A_i)) < l^{-1}2^{-i}$ holds. We put a open set $O_l = \bigcup_{i \in \mathbf{N}} O_i(j(l,i))$. Then $\sup_{t \in \hat{E}_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap (O_l \setminus \bigcup_{i \in \mathbf{N}} A_i))$ converges to 0 as $l \to \infty$. Therefore $\bigcup_{i \in \mathbf{N}} A_i \in \sigma$ holds. □

LEMMA 7.4. *$\sigma = \mathcal{B}(Y)$ holds.*

Proof. For every open set $O \subset Y$, there exists a sequence of compact sets $K_i \subset O$ such that $\bigcup_{i\in\mathbf{N}} K_i = O$. By Egoroff's theorem, for every positive number $\epsilon > 0$, there exists a Lebesgue measurable set $E_\epsilon \subset [r, R]$ such that $\sup_{t\in E_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap O \setminus K_i))$ converges to 0 as $i \to \infty$. Thus, $O \in \sigma$. Therefore we have Lemma 7.4 $\square$

Proposition 5.1 follows from these lemma above, immediately.

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