On Low Dimensional Ricci Limit Spaces

Shouhei Honda

Abstract

In this paper, we will give a classification of limit spaces, of a sequence of Riemannian manifolds with Ricci curvature bounded below, whose Hausdorff dimension is strictly smaller than two.

1 Introduction

In this paper, we study a pointed metric space (Y, y) that is pointed Gromov-Hausdorff limit of a sequence of complete, pointed, connected n-dimensional Riemaniann manifolds, $\{(M_i, m_i)\}_i$, with $Ric_{M_i} \geq -(n-1)$. Here, n is a fixed positive integer. (We call a such metric space (Y, y) *Ricci limit space* in this paper. See [13].) The structure theory was much developed by J. Cheeger and T. H. Colding, and has many important applications to Riemannian manifolds. (See [4, 5, 6].) Most of this paper, we will study the *low* dimensional Ricci limit spaces by using their theory and using several results in [12]. First, we give the classification of one dimensional Ricci limit spaces;

Theorem 1.1. *Let (Y, y) be a Ricci limit space. Then, the following conditions are equivalent:*

1. $1 \leq \dim_H Y < 2$ *holds.*
2. $\mathcal{R}_i = \emptyset$ *holds for every integer* $i \geq 2$
3. $\upsilon(\mathcal{R}_i) = 0$ *holds for every integer* $i \geq 2$
4. Y *is isometric either to* $\mathbf{R}$, *or to* $\mathbf{R}_{\geq 0}$, *or to* $\mathbf{S}^1(r) = \{x \in \mathbf{R}^2 | |x| = r\}$ *for some positive number* $r > 0$, *or to* $[0, l]$ *for some* $l > 0$.

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Here, $\mathcal{R}_i$ is the i-dimensional regular set in Y and υ is a limit measure on Y. (See Definition 2.4 and Definition 2.6.) The proof is used several results on regular set. We will recall them in section 3. In section 4, we will give a necessary and sufficient condition to exist one dimensional piece and prove Theorem 1.1. As a corollary of Theorem 1.1, The Hausdorff dimension of a Ricci limit space (Y, y) such that $\dim_{\mathcal{H}} Y \leq 2$ holds, is an integer. Next, we will study the problem when the limit measure υ is locally equivalent to one dimensional Hausdorff measure $\mathcal{H}^1$. We will give a necessary and sufficient condition that υ is locally equivalent to $\mathcal{H}^1$. See Theorem 5.5 in section 5.

We use the notion of local Hausdorff dimension in several situations of this paper. We define $\dim_{\mathcal{H}}^{\mathrm{loc}} x = \lim_{r\to 0} \dim_{\mathcal{H}} B_r(x)$ for every $x \in Y$ and put $Y(\alpha) = \{x \in Y | \dim_{\mathcal{H}}^{\mathrm{loc}} x = \alpha\}$ for $\alpha \geq 0$. We also define the notion of *Alexandrov point* in section 6. (See Definition 6.2.) Alexandrov points on metric space means that there exists a definite lower bound of sectional curvature around the point in the sence of Alexandrov geometry. We consider the set $\mathrm{Alex}(Y) = \{x \in Y | x \text{ is an Alexandrov point }\}$ under the assumption $\mathcal{R}_1 \neq \emptyset$;

THEOREM 1.2. *Let* (Y, y) *be a Ricci limit space. We assume that* $\mathcal{R}_1 \neq \emptyset$. *Then, we have* $\mathrm{Alex}(Y) = Y(1)$.

We give a corollary of Theorem 1.2. We fix a sufficiently small positive number $\epsilon > 0$. Let Z be a completion of 5-dimensional Riemannian manifold $(\mathbf{R}_{>0} \times \mathbf{S}^4, dr^2 + (r^{1+\epsilon}/2)^2 g_{\mathbf{S}^4})$. Here, $g_{\mathbf{S}^4}$ is the standard Riemannian metric on 4-dimensional unit sphere. This space is a Ricci limit space. (See Example 8.77 in [4].) On the other hand, for positive number $\tau > 0$, Let Z_τ be a space obtained by adjoining a segment $[-\tau, 0]$ to Z at each origins. J. Cheeger and T. H. Colding showed that for every $\tau > 0$, Z_τ is *not* Ricci limit space. This non-existence result also follows from Theorem 1.2. This is an alternative proof. We take two copies of Z, denote them by Z_1, Z_2. (Namely, Z_1 and Z_2 are isometric to Z, respectively.) Let $\hat{Z}$ be a space obtained by adjoining Z_1 to Z_2 at each origins. We prove that $\hat{Z}$ is *not* a Ricci limit space, as a corollary of Theorem 1.2. In section 6, for every positive number $\tau > 0$ and every complete pointed connected k-dimensional Riemannian manifold, (M, m), we will prove that $(M \times Z_\tau, (m, 0))$ is *not* a Ricci limit space. (See Remark 6.7.)

We will also study the problem whether the Hausdorff dimension of the Ricci limit space is an integer. First, under the condition $2 \leq \dim_{\mathcal{H}} Y < 3$, we will prove that $\dim_{\mathcal{H}}(Y \setminus C_x) \leq 2$ holds for every $x \in Y$. Here, C_x is the cut locus of x. (See Theorem 7.4.) In general situation, J. Cheeger and T. H. Colding gave a sufficient condition to satisfy $\dim_{\mathcal{H}} Y \in \mathbf{Z}$. (See Theorem 1.38 in [5].) This condition is called by *polar*, we can rewrite the condition by using cut locus on iterated tangent cones. X. Menguy showed the existence of non-polar Ricci limit space and the Hausdorff dimension is an integer. (See [14].) We will give another sufficient condition to satisfy $\dim_{\mathcal{H}} Y \in \mathbf{Z}$ that is *weaker* condition than polar. (We call the condition *weakly polar.*) Note that the example by X. Menguy has weakly polar condition. We also study the limit space satisfying weakly polar condition. (See Corollary 8.6.)

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2 Notation

We recall some fundamental notion on metric spaces, the notion of Ricci limit spaces and recall that of regular set on Ricci limit spaces. (See [4].)

DEFINITION 2.1. We say that a metric space X is *proper* if every bounded closed set is compact. A metric space X is said to be *geodesic space* if for every points $x_1, x_2 \in X$, there exists an isometric embedding $\gamma : [0, \overline{x_1, x_2}] \to X$ such that $\gamma(0) = x_1$ and $\gamma(\overline{x_1, x_2}) = x_2$ hold. Here $\overline{x_1, x_2}$ is the distance between x_1 and x_2 on X. (We say that γ is *minimal geodesic from x_1 to x_2.*)

For proper geodesic space X, a point $x \in X$, a set $A \subset X$, and for positive number $r > 0$, we use the following notations; $B_r(x) = \{z \in X | \overline{x, z} < r\}$, $\overline{B}_r(x) = \{z \in X | \overline{x, z} \leq r\}$, $\partial B_r(x) = \{z \in X | \overline{x, z} = r\}, C_x(A) = \{z \in X |$ There exists $w \in A$ such that $\overline{x, z} + \overline{z, w} = \overline{x, w}$ holds.$\}$. Throughout the paper, we fix a positive integer $n > 0$.

DEFINITION 2.2. Let (Y, y) be a pointed proper geodesic space ($y \in Y$), $K \in \mathbf{R}$ a real number. We say that (Y, y) is *(n, K)-Ricci limit space* if there exist a sequence of real numbers $K_i \in \mathbf{R}$ and a sequence of pointed, complete, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $Ric_{M_i} \geq K_i(n-1)$, such that K_i converges to K and that (M_i, m_i) converges to (Y, y) as $i \to \infty$ in the sense of pointed Gromov-Hausdorff topology.

Here, for a sequence of pointed proper geodesic space $\{(X_i, x_i)\}_i$, we say that (X_i, x_i) *converges to a pointed proper geodesic space (X_∞, x_∞) in the sense of Gromov-Hausdorff topology* if there exist sequences of positive numbers ϵ_i, $R_i > 0$ and exists a sequence of maps $\phi_i : (B_{R_i}(x_i), x_i) \to (B_{R_i}(x_\infty), x_\infty)$ such that ϵ_i converges to 0, R_i converges to ∞, $|\overline{z_i, w_i} - \overline{\phi_i(z_i), \phi_i(w_i)}| < \epsilon_i$ hold for every points $z_i, w_i \in B_{R_i}(x_i)$, and that $B_{\epsilon_i}(\mathrm{Image}(\phi_i)) \supset B_{R_i}(X_\infty)$ holds. Then for a sequence of points $z_i \in X_i$ such that the set $\{\overline{x_i, z_i} | i \in \mathbf{N}\}$ is bounded set in $\mathbf{R}$, we say that z_i *converges to a point $z_\infty \in X_\infty$*

in the sense of Gromov-Hausdorff topology if $\overline{\phi_i(z_i), z_\infty} < \epsilon_i$. (We denote it by either $z_i \to z_\infty$ or $\overline{z_i, z} < \epsilon_i$.)

We remark that for every $K \neq 0$ and every (n, K)-Ricci limit space (Y, y), there exists a sequence of complete, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $Ric_{M_i} \geq K(n-1)$, such that (M_i, m_i) converges to (Y, y) by rescaling. Throughout the paper, (Y, y) is always $(n, -1)$-Ricci limit space and is *not* a single point. More simply, we say that (Y, y) is *Ricci limit space*.

DEFINITION 2.3. Let (W, w), (Z, z) be pointed proper geodesic spaces. We say that (W, w) is *tangent cone at* $\alpha \in Z$ if there exists a sequence of positive numbers $r_i > 0$ such that r_i converges to 0 and that rescaled pointed proper geodesic spaces $(Z, r_i^{-1} d_Z, \alpha)$ converges to (W, w) in the sence of pointed Gromov-Hausdorff topology. Here, d_Z is the metric (distance function) on Z.

We remark that by Gromov's pre-compactness theorem, for every point $x \in Y$, there exists a tangent cone at x, $(T_x Y, 0_x)$. In generally, it is *not* an unique. (See [15].) Note that for every tangent cone at x, $(T_x Y, 0_x)$, $(T_x Y, 0_x)$ is $(n, 0)$-Ricci limit space. Next, we shall give a filtration of Ricci limit spaces and the notion of regular set. These are defined by J. Cheeger and T. H. Colding in [4]. Throughout this paper, for every metric spaces X_1, X_2, the metric on $X_1 \times X_2$ is always $(d_{X_1}^2 + d_{X_2}^2)^{1/2}$.

DEFINITION 2.4. Let Z be a proper geodesic space. We assume that for every point $\alpha \in Z$, there exists a tangent cone at α, $(T_\alpha Z, 0_\alpha)$. Then, for non-negative integer $k \in \mathbf{Z}_{\geq 0}$, we put

1. $\mathcal{WE}_k(Z) = \{x \in Z |$ There exists a tangent cone at x, $(T_x Z, 0_x)$ and a proper geodesic space W such that $T_x Z$ is isometric to $\mathbf{R}^k \times W$. $\}$,

2. $\mathcal{E}_k(Z) = \{x \in Z |$ For every tangent cone at x, $(T_x Z, 0_x)$, there exists a proper geodesic space W such that $T_x Z$ is isometric to $\mathbf{R}^k \times W$. $\}$,

3. $\underline{\mathcal{WE}}_k(Z) = \{x \in Z |$ There exist a tangent cone at x, $(T_x Z, 0_x)$ and a proper geodesic space W such that W is not a single point and that $T_x Z$ is isometric to $\mathbf{R}^k \times W$. $\}$,

4. $\mathcal{R}_k(Z) = \{x \in Z |$Every tangent cones at x, $(T_x Z, 0_x)$, is isometric to $(\mathbf{R}^k, 0_k)$ $\}$.

   Let $\epsilon > 0$ be a positive number. We also put

5. $(\mathcal{WE}_k)_\epsilon(Z) = \{x \in Z |$ There exist a positive number $0 < r < \epsilon$ and a proper geodesic space (W, w) such that $d_{GH}((\overline{B}_r(x), x), (\overline{B}_r((0_k, w)), (0_k, w))) < \epsilon r$ holds for $\overline{B}_r((0_k, w)) \subset \mathbf{R}^k \times W$. $\}$.

   Here, d_{GH} is the Gromov-Hausdorff distance between pointed compact metric spaces.

For simplification, we use the following notations for Ricci limit space (Y, y); $\mathcal{WE}_k = \mathcal{WE}_k(Y)$, $\mathcal{E}_k = \mathcal{E}_k(Y)$ etc. We call the set $\mathcal{R}_k$ *k-dimensional regular set of Y* and call the set $\mathcal{R} = \bigcup_k \mathcal{R}_k$ *regular set of Y*.

Remark 2.5. By the definition and Gromov's pre-compactness theorem, $(\mathcal{WE}_k)_\epsilon$ is open and $\mathcal{WE}_k = \cap_{\epsilon>0}(\mathcal{WE}_k)_\epsilon$ holds.

We shall give the definition of *limit measure*. The measure is useful tool for studying Ricci limit spaces.

Definition 2.6. Let υ be a Borel measure on Y. We say that υ is *limit measure* if there exists a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $Ric_{M_i} \geq -(n-1)$, such that (M_i, m_i) converges to (Y, y) and that for every positive number $r > 0$ and every points $x \in Y$, $\hat{m}_j \in M_j$ satisfying $\hat{m}_j \to x$ in the sense of pointed Gromov-Hausdorff topology,

$$\frac{\mathrm{vol}(B_r(\hat{m}_j))}{\mathrm{vol} B_1(m_j)} \to \upsilon(B_r(x))$$

holds. Then, we say that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}B_1(m_j))$ *converges to* (Y, y, υ) *in the sense of measured Gromov-Hausdorff topology.*

There exists a limit measure on Y. (See Theorem 1.6, Theorem 1.10 in [4] and see [9].) It is *not* an unique in generally. (See Example 1.24 in [4].) Throughout the paper, υ is always fixed limit measure on Y.

3 Regularity theorem and low dimensinal tangent cone

In this section, we recall several properties of regular set. One of many important results of J. Cheeger and T. H. Colding, is $\upsilon(Y \setminus \mathcal{R}) = 0$. (See Theorem 2.1 in [4].) We shall study that in more detail. Next proposition is a corollary of Lemma 2.5 in [4]. Note that it does *not* follow from the result $\upsilon(Y \setminus \mathcal{R}) = 0$ immediately.

Proposition 3.1. *There exists a positive number $\epsilon(n) > 0$ depending only on n such that $\upsilon(B_r(x) \cap (\bigcup_{j\geq k} R_j)) > 0$ holds for every integer $1 \leq k \leq n$, every point $x \in (\mathcal{WE}_k)_{\epsilon(n)}$ and for every positive number $r > 0$.*

Proof. By Lemma 2.5 in [4], there exist a positive number $\epsilon(n) > 0$ such that $\upsilon(B_r(x) \cap \mathcal{E}_k) > 0$ holds for every integer $1 \leq k \leq n$, every point $x \in (\mathcal{WE}_k)_{\epsilon(n)}$ and every positive number $r > 0$. If $\upsilon(B_r(x) \cap \mathcal{R}_k) > 0$ holds, we have the claim. We assume that

$\upsilon(B_r(x)\cap\mathcal{R}_k)=0$ holds. Then, since $\upsilon(B_r(x)\cap\mathcal{E}_k)\leq\upsilon(B_r(x)\cap\mathcal{R}_k)+\upsilon(B_r(x)\cap\underline{\mathcal{WE}}_k)$, we have $\upsilon(B_r(x)\cap\underline{\mathcal{WE}}_k)>0$. By the result $\upsilon(\underline{\mathcal{WE}}_k\setminus\mathcal{WE}_{k+1})=0$ (Lemma 2.6 in [4]) and the result $\upsilon(\mathcal{WE}_k\setminus\mathcal{E}_k)=0$ (Lemma 2.5 in [4]), we have $\upsilon(B_r(x)\cap\mathcal{E}_{k+1})>0$. Therefore we have completed the proof of Proposition 3.1 by iterating this argument. $\square$

Next proposition is a corollary of Proposition 3.1 and the proof of Lemma 2.6 in [4].

Proposition 3.2. *Let x be a point in $\underline{\mathcal{WE}}_k$. Then we have $\upsilon(B_r(x)\cap\bigcup_{j\geq k+1}\mathcal{R}_j)>0$ for every positive number $r>0$.*

Proof. First, we remark that for every positive numbers $\epsilon,\delta>0$ and *every* point $x\in\underline{\mathcal{WE}}_k$, there exists a positive number $0<s<\epsilon$ such that

$$\frac{\upsilon(B_s(x)\setminus(\mathcal{WE}_{k+1})_\delta)}{\upsilon(B_s(x))}<\epsilon.$$

See (2.42) in [4]. (We remark that this statement does *not* follow from the result $\upsilon(\underline{\mathcal{WE}}_k\setminus\mathcal{WE}_{k+1})=0$ immediately.) We take $\delta=\epsilon(n)$ as in Proposition 3.1. There exists a sequence $x_i\in(\mathcal{WE}_{k+1})_{\epsilon(n)}$ such that x_i converges to x in Y. We take a positive number $s_i>0$ such that $B_{s_i}(x_i)\subset B_r(x)$ holds for every sufficiently large i. Then, by Proposition 3.1, we have $\upsilon(B_{s_i}(x_i)\cap\bigcup_{j\geq k+1}\mathcal{R}_j)>0$. Especially, $\upsilon(B_r(x)\cap\bigcup_{j\geq k+1}\mathcal{R}_j)>0$ holds. $\square$

We will use next corollaries many times in following sections.

Corollary 3.3. *We have $\underline{\mathcal{WE}}_k\subset\cup_{i\geq k+1}\overline{\mathcal{R}}_i$ for every positive integer $k\geq 1$.*

Corollary 3.4. *We have the following statements for every integer $i\geq 1$.*

1. *If $\upsilon(\mathcal{R}_j)=0$ holds for every $j\geq i$, then we have $\mathcal{WE}_j=\phi$ for every $j\geq i$. Especially, we have $\mathcal{R}_j=\emptyset$ for every $j\geq i$.*

2. *If $\upsilon(\mathcal{R}_j)=0$ holds for every $j\geq i+1$, then we have $\underline{\mathcal{WE}}_j=\emptyset$ for every $j\geq i$.*

4 One dimensional Ricci limit spaces

In this section, we give a necessary and sufficient condition for appearing one dimensional piece. (See section 5 in [5] for the definition of one dimensional piece.) As a corollary, we will give the classification of one dimensional Ricci limit spaces. We say that a point $x\in Y$ is *an interior point* on a minimal geodesic $\gamma:[0,l]\to Y$ $(l>0)$ if $x\in\gamma((0,l))$ holds.

Proposition 4.1. *Let x be a point in $\mathcal{R}_1$. Then, x is an interior point on some minimal geodesic.*

PROOF. This proof is by contradiction. Assume that the assertion were false. Let $r_i > 0$ be a sequence of positive numbers such that r_i converges to 0 and that $(Y, r_i^{-1}d_y, x)$ converges to $(\mathbf{R}, 0)$. Then, for every positive integer $i > 0$, there exist points $x_i^-, x_i^+ \in Y$ and a positive number $\epsilon_i > 0$ such that ϵ_i converges to 0, $|\overline{x_i^-, x} - r_i| < \epsilon_i r_i$, $|\overline{x_i^+, x} - r_i| < \epsilon_i r_i$ hold, and that $\overline{x_i^-, x} + \overline{x_i^+, x} - \overline{x_i^-, x_i^+} < \epsilon_i r_i$ holds. We take a minimal geodesic from x_i^- to x_i^+, $\gamma_i : [0, \overline{x_i^-, x_i^+}] \to Y$. We put $s_i = \overline{x, \mathrm{Image}(\gamma_i)}$. Then we have $s_i > 0$ by the assumption. By triangle inequality, we have s_i converges to 0. Without loss of generality, we can assume that $(Y, x, s_i^{-1}d)$ converges to a tangent cone at x, $(T_xY, 0_x)$.

By the construction, there exist $z \in \partial B_1(0_x)$ and an isometric embedding $L : \mathbf{R} \to T_xY$ such that $z \in \mathrm{Image}(L)$ and $0_x \notin \mathrm{Image}(L)$ hold. By applying splitting theorem to (T_xY, z), there exists a proper geodesic space W such that W is not a single point and that T_xY is isometric to $\mathbf{R} \times W$. This contradicts the assumption $x \in \mathcal{R}_1$. $\square$

The following theorem is the geometric necessary and sufficient condition to appear one dimensional piece;

THEOREM 4.2. *Let $x \in Y \setminus \cup_{i \geq 2} \overline{\mathcal{R}}_i$. Then, there exists a positive number $\epsilon > 0$ such that $(B_\epsilon(x), x)$ is isometric either to $((-\epsilon, \epsilon), 0)$ or to $([0, \epsilon), 0)$. Here, $(-\epsilon, \epsilon)$, $[0, \epsilon)$ are intervals in $\mathbf{R}$.*

PROOF. 1. *The case $x \in \mathcal{R}_1$.*

By Proposition 4.1, there exist a sufficiently small positive number $r > 0$, points $x_-, x_+ \in Y$ and a minimal geodesic from x_- to x_+, $\gamma : [0, \overline{x_-, x_+}] \to Y$ such that $\overline{x_-, x} = \overline{x_+, x} = 100r$, $x \in \mathrm{Image}(\gamma)$ hold and that $\overline{B}_{100r}(x) \subset Y \setminus \cup_{i \geq 2} \overline{\mathcal{R}}_i$ holds.

We assume that $\overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma) \neq \emptyset$. we take $z \in \overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma)$. Let $w \in \mathrm{Image}(\gamma)$ be a point such that $\overline{z, w} = \overline{z, \mathrm{Image}(\gamma)} > 0$. Note that $w \in B_{50r}(x)$. We take a minimal geodesic from z to w, $\gamma_1 : [0, \overline{z, w}] \to Y$. For every positive number $0 < \epsilon << \overline{z, \mathrm{Image}(\gamma)}$, let $w(\epsilon) \in \mathrm{Image}(\gamma)$ be a point in $\mathrm{Image}(\gamma_1)$ with $\overline{w, w(\epsilon)} = \epsilon$, and let be $x_-(\epsilon), x_+(\epsilon) \in \mathrm{Image}(\gamma)$ points with $\overline{x_-(\epsilon), w} = \overline{x_+(\epsilon), w} = \epsilon$. By the definition of w, we have $\overline{x_-(\epsilon), w(\epsilon)} = \overline{x_-(\epsilon), w(\epsilon)} + \overline{w(\epsilon), z} - \overline{w(\epsilon), z} \geq \overline{z, w} - \overline{w(\epsilon), z} = \epsilon$. Similarly, we have $\overline{x_+(\epsilon), w(\epsilon)} \geq \epsilon$. Therefore, for every tangent cone at w, $(T_wY, 0_w)$, there exists a proper geodesic space W such that W is not a single point and that T_wY is isometric to $\mathbf{R} \times W$. Thus, we have $w \in \underline{\mathcal{WE}}_1$. By Corollary 3.3, we have $w \in \bigcup_{i \geq 2} \overline{\mathcal{R}}_i$. This contradicts $\mathrm{Image}(\gamma) \subset Y \setminus \bigcup_{i \geq 2} \overline{\mathcal{R}}_i$. Therefore $\overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma) = \emptyset$ holds, we have the assertion.

2. *The case $x \in Y \setminus \mathcal{R}_1$.*

There exist a sufficiently small positive number $r > 0$, a point $x_+ \in Y$ and a minimal geodesic segment from x to x_+, $\gamma : [0, \overline{x, x_+}] \to Y$ such that $\overline{x, x_+} = 100r$

and $B_{100r}(x) \subset Y \setminus \bigcup_{i \geq 2} \overline{\mathcal{R}_i}$ holds. We assume that $B_{10r}(x) \setminus \mathrm{Image}(\gamma) \neq \emptyset$. We take a point $z \in B_{10r}(x) \setminus \mathrm{Image}(\gamma)$. Let $w \in \mathrm{Image}(\gamma)$ be a point satisfying $\overline{z, w} = \overline{z, \mathrm{Image}(\gamma)} > 0$. Note that $w \in B_{50r}(x)$. If $w \neq x$, there exists a positive number $\epsilon > 0$ such that $(B_\epsilon(w), w)$ is isometric to $((-\epsilon, \epsilon), 0)$ by the case 1. This contradicts the fact $\overline{z, w} = \overline{z, \mathrm{Image}(\gamma)}$. Thus, we have $w = x$. For every positive number $0 \leq \epsilon << 100r$, let $x_+(\epsilon) \in \mathrm{Image}(\gamma)$ be a point satisfying $\overline{x, x_+(\epsilon)} = \epsilon$. We take a minimal geodesic segment from z to $x_+(\epsilon)$, $\gamma_\epsilon : [0, \overline{z, x_+(\epsilon)}] \to Y$ for every sufficiently small positive number $0 < \epsilon << r$.

Claim 4.3. *$x \in \mathrm{Image}(\gamma_\epsilon)$ holds.*

This proof is by contradiction. Assume that the assertion were false. We put $t = \inf\{\overline{z, m} \mid m \in \mathrm{Image}(\gamma_\epsilon) \cap \mathrm{Image}(\gamma)\} > 0$. By the definition, we have $\gamma_\epsilon(t) \in \mathrm{Image}(\gamma)$ and $\gamma_\epsilon(s) \notin \mathrm{Image}(\gamma)$ for every $s < t$. Clearly, we have $\gamma_\epsilon(t) \in \mathcal{E}_1$. By $\gamma_\epsilon(t) \notin \underline{\mathcal{WE}}_1$, we have $\gamma_\epsilon(t) \in \mathcal{R}_1$. By the case 1, there exists a positive number $\tau > 0$ such that $(B_\tau(\gamma_\epsilon(t)), \gamma_\epsilon(t))$ is isometric to $((-\tau, \tau), 0)$. This contradicts the fact $\gamma_\epsilon(s) \notin \mathrm{Image}(\gamma)$ for every $s < t$. Therefore we have Claim 4.3.

We have $x \in \mathcal{E}_1$ by Claim 4.3. By $x \notin \underline{\mathcal{WE}}_1$, we have $x \in \mathcal{R}_1$. This contradicts the assumption $x \in Y \setminus \mathcal{R}_1$.

□

We shall define local Hausdorff dimension.

Definition 4.4. For metric space X and a point $x \in X$, we put $\dim_{\mathcal{H}}^{\mathrm{loc}} x = \lim_{r \to 0} \dim_{\mathcal{H}} B_r(x)$. For non-negative number $\alpha \geq 0$, we put $X(\alpha) = \{x \in X | \dim_{\mathcal{H}}^{\mathrm{loc}} x = \alpha\}$.

The following proposition is the necessary and sufficient condition by using local Hausdorff dimension to appear one dimensional piece.

Theorem 4.5. *Let x be a point in Y. Then, $1 \leq \dim_{\mathcal{H}}^{\mathrm{loc}} x < 2$ holds if and only if $x \in Y \setminus \bigcup_{i \geq 2} \overline{\mathcal{R}_i}$ holds.*

Proof. By Theorem 4.2, if $x \in Y \setminus \bigcup_{i \geq 2} \overline{\mathcal{R}_i}$, then $1 \leq \dim_{\mathcal{H}}^{\mathrm{loc}} x < 2$ holds. We assume that there exists an integer $i \geq 2$ such that $x \in \overline{\mathcal{R}_i}$. Then, for every positive number $s > 0$, there exists $z \in B_s(x) \cap \mathcal{R}_i$. By Corollary 1.36 in [5], we have $\dim_{\mathcal{H}} B_t(z) \geq 2$ for every positive number $t > 0$. Especially, $\dim_{\mathcal{H}} B_s(x) \geq i \geq 2$. Therefore, we have $\dim_{\mathcal{H}}^{\mathrm{loc}} x \geq i \geq 2$. □

Theorem 1.1 follows from Corollary 3.4, Theorem 4.2 and Theorem 4.5 immediately. As a corollary of Theorem 1.1, if $\dim_{\mathcal{H}} Y \leq 2$ holds, then $\dim_{\mathcal{H}} Y \in \mathbf{Z}$ holds. Next, we consider the Ahlfors one regular set $A_Y(1) = \{x \in Y | \liminf_{r \to 0} \upsilon(B_r(x))/r > 0\}$. (See section 6 in [12] for the definition of the set $A_Y(\alpha)$ for real number $1 \leq \alpha \leq n$.)

Corollary 4.6. *We assume that* $\upsilon(Y \setminus A_Y(1)) = 0$ *holds. Then we have* $\dim_{\mathcal{H}} Y = 1$.

Proof. By Theorem 3.23 and Theorem 4.6 in [6], we have $\upsilon(\mathcal{R}_i \setminus (\mathcal{R}_i \cap A_Y(i))) = 0$ for every integer $i \geq 1$. Therefore, by the assumption, we have $\upsilon(\mathcal{R}_i) = 0$ for every integer $i \geq 2$. Thus we have Corollary 4.6 by Theorem 1.1. $\square$

5 Equivalence between limit measure and one dimensional Hausdorff Measure

In this section, we consider locally equivalence between υ and $\mathcal{H}^1$. Here, $\mathcal{H}^1$ is the one dimensional Hausdorff measure. We shall give next proposition without the proof because we can prove it by an argument similar to that of construction of limit measure. Note that for every $0 < r < 1$ and every $x \in Y$, the rescaled pointed proper geodesic space $(Y, r^{-1}d_Y, x)$ is Ricci limit space.

Proposition 5.1. *For every positive number* $0 < r < 1$ *and every point* $x \in Y$, *there exists a limit measure* υ_r *on* $(Y, r^{-1}d_Y, x)$ *such that* $\upsilon_r(B_{s_1}^{r^{-1}d_Y}(x_1))\upsilon(B_{s_2 r}(x_2)) = \upsilon_r(B_{s_2}^{r^{-1}d_Y}(x_2))\upsilon(B_{s_1 r}(x_1))$ *holds for every points* $x_1, x_2 \in Y$ *and for every positive numbers* $s_1, s_2 > 0$. *Especially, for every tangent cone at* x, $(T_xY, 0_x)$, *there exists a limit measure* υ_∞ *on* T_xY *and exists a sequence of positive numbers* $r_i > 0$ *such that* r_i *converges to* 0 *and that* $\upsilon(B_{sr_i}(x))/\upsilon(B_{r_i}(x))$ *converges to* $\upsilon_\infty(B_s(0_x))$ *for every positive number* $s > 0$.

We will give a proof of next proposition in Appendix.

Proposition 5.2. *Let* (W, w) *be a pointed proper geodesic space and* $1 \leq k < n$ *a positive integer. We assume that* W *is not a single point and that* $(\mathbf{R}^k \times W, (0_k, w))$ *is* $(n, 0)$*-Ricci limit space. Then, for every limit measure* υ *on* $\mathbf{R}^k \times W$, *there exists a Borel measure* υ_W *on* W *such that* $\upsilon = \mathcal{H}^k \times \upsilon_W$ *holds and that* $\limsup_{\delta \to 0} \upsilon_W(B_\delta(z))/\delta \leq C(n, \mathrm{diam}(W), R) < \infty$ *holds for every positive number* $R > 0$ *and every point* $z \in B_R(w)$. *Here,* $C(n, \mathrm{diam}(W), R) > 0$ *is a positive constant depending only on* $n, \mathrm{diam}(W), R$.

We remark that for $x \in Y$, $\liminf_{r \to 0} \upsilon(B_r(x))/r > 0$ holds if and only if $\upsilon_{-1}(x) > 0$ holds. (See [5], [12] for the definition of the measure υ_{-1} on Y.) We shall give an example of a point $x \in Y$ satisfying $\upsilon_{-1}(x) > 0$.

Proposition 5.3. *Let* x *be a point in* $\mathcal{R}_1$. *Then we have* $\liminf_{r \to 0} \upsilon(B_r(x))/r > 0$.

Proof. Assume that the assertion were false. Hence $\upsilon_{-1}(x) = 0$. Then, by an argument simular to the proof of Proposition 4.1 and by Theorem 3.7 in [5], there exists a tangent cone at x, $(T_xY, 0_x)$, and a proper geodesic space W such that W is not a single point and that T_xY is isometric to $\mathbf{R} \times W$. We take a limit measure υ_∞ on T_xY as in Proposition 5.1. By Proposition 4.3 in [12] and the assumption, we have $(\upsilon_\infty)_{-1}(0_x) > 0$. This contradicts Proposition 5.2. $\square$

We give the definition of locally equivalence between Borel measures.

DEFINITION 5.4. Let X be a topological space and υ, μ be Borel measures on X. We say that υ are *locally equivalent* μ *at* $x \in X$ if there exist a positive number $C > 1$ and an open neighborfood U of x such that $C^{-1}\mu(A) \le \upsilon(A) \le C\mu(A)$ holds for every Borel set $A \subset U$.

Next theorem is the main result in this section.

THEOREM 5.5. *Let x be a point in Y. The following conditions are equivalent:*

1. *The limit measure υ is locally equivalent to $\mathcal{H}^1$ at x.*

2. $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$ *holds and* $1 \le \dim_{\mathcal{H}}^{\mathrm{loc}} x < 2$ *holds.*

PROOF. It suffices to show that if $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$ holds and $1 \le \dim_H x < 2$ holds, then υ is locally equivalent to $\mathcal{H}^1$ at x. We assume that $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$ and $1 \le \dim_H x < 2$ hold. Then, by Theorem 4.2 and Theorem 4.5, There exists a positive number $\epsilon > 0$ such that $(B_\epsilon(x), x)$ is isometric either to $((-\epsilon, \epsilon), 0)$ or to $([0, \epsilon), 0)$. Note that $\liminf_{r\to 0} \upsilon(B_r(z)) \ge \upsilon_{-1}(z)$ holds for every point $z \in Y$. It is not difficult to check the claim by using Theorem 1.1 in [12]. $\square$

We remark that there exist two limit measures υ_1, υ_2 on the 2-Ricci limit space $[0, 1]$ such that υ_1 is locally equivalent to $\mathcal{H}^1$ at 0 and that υ_2 is *not* locally equivalent to $\mathcal{H}^1$ at 0. See Example 1.24 in [4].

6 The structure of spaces with Ahlfors one regular points

In this section, we study Ricci limit space (Y, y) satisfying $A_Y(1) \neq \emptyset$. We give a characterization of one dimensional piece by existence of lower sectional curvature bounds (Theorem 1.2). As a corollary, we have *non*-existence of $Z_\tau, \hat{Z}$ in section 1 as Ricci limit spaces. We also discuss some uniform properties of Hausdorff dimension on Ricci limit spaces.

6.1 A proof of Theorem 1.2

First, we give an example of a point $x \in Y$ satisfying $\upsilon_{-1}(x) = 0$.

PROPOSITION 6.1. *Let x be a point in $\underline{\mathcal{WE}}_1$. Then we have* $\liminf_{r\to 0} \upsilon(B_r(x))/r = 0$.

PROOF. Assume that the assertion were false. By the definition, there exist a tangent cone at x, $(T_xY, 0_x)$ and a proper geodesic space W such that W is not a single point, T_xY is isometric to $\mathbf{R} \times W$. We take a limit measure υ_∞ on T_xY as in Proposition 5.1. By Proposition 4.3 in [12], we have $(\upsilon_\infty)_{-1}(0_x) > 0$. This contradicts Proposition 5.2. $\square$

We shall define the notion of *Alexandrov point*. It means that there exists a lower sectional curvature bound around the point.

DEFINITION 6.2. Let X be a proper geodesic space, x a point in X. We say that x is an *Alexandrov point* if there exist an open neighborhood of x, U, and a negative number $K < 0$ staisfying the following properties; For every points $x_1, x_2, x_3 \in U$ and every point $x_4 \in X$ satisfying $\overline{x_1, x_4} + \overline{x_4, x_2} = \overline{x_1, x_2}$, there exist points $y_1, y_2, y_3, y_4 \in \mathbf{H}^2(K)$ such that $\overline{x_1, x_2} = \overline{y_1, y_2}$, $\overline{x_2, x_3} = \overline{y_2, y_3}$, $\overline{x_3, x_1} = \overline{y_3, y_1}$, $\overline{x_1, x_4} = \overline{y_1, y_4}$ hold and that $\overline{x_3, x_4} \geq \overline{y_3, y_4}$ holds. Here, $\mathbf{H}^2(K)$ is complete, two dimensional Riemannian manifold such that $\pi_1(\mathbf{H}^2(K)) = 1$ holds and that the sectional curvature $K_{\mathbf{H}^2(K)}$ satisfies $K_{\mathbf{H}^2(K)} \equiv K$.

We put $\mathrm{Alex}(Y) = \{x \in Y | x \text{ is an Alexandrov point }\}$. By the definition, the set $\mathrm{Alex}(Y)$ is an open set.

THEOREM 6.3. *We assume that there exists a point $z \in Y$ such that* $\liminf_{r\to 0} \upsilon(B_r(z))/r > 0$ *holds. Let x be a point in Y. Then, one of the following statements* $1, 2$ *occurs.*

1. $\dim_{\mathcal{H}}^{\mathrm{loc}} x = 1$ *holds.*

2. *x is not an Alexandrov point.*

PROOF. This proof is by contradiction. We assume that $\dim_{\mathcal{H}}^{\mathrm{loc}} x > 1$ holds and that x is an Alexandrov point. We consider the metric ball $B_r(x)$ for fixed sufficiently small positive number $r > 0$. Since $\mathrm{Alex}(Y)$ is open, without loss of generality, we can assume that $x \neq z$. Fix a minimal geodesic from x to z, $\gamma : [0, \overline{x, z}] \to Y$. We put $\alpha = \gamma(r)$ and $w = \gamma(\frac{r}{2})$.

CLAIM 6.4. *Let $\hat{\gamma} : [0, \overline{w, z}] \to Y$ be a minimal geodesic from w to z. Then, $\alpha \in$* $\mathrm{Image}(\hat{\gamma})$ *holds.*

Assume that the assertion were false. There exists $s \in [0, \overline{w,z}]$ such that $\gamma(s) \in \partial B_r(x)$ holds. We put $\hat{\alpha} = \gamma(s) (\neq \alpha)$. Then, we have

$$\begin{aligned} 0 \leq \overline{x,w} + \overline{w,\hat{\alpha}} - \overline{x,\hat{\alpha}} &= \overline{x,w} + (\overline{w,\hat{\alpha}} + \overline{\hat{\alpha},z}) - (\overline{x,\hat{\alpha}} + \overline{\hat{\alpha},z}) \\ &\leq \overline{x,w} + \overline{w,z} - \overline{x,z} \\ &= 0. \end{aligned}$$

Therefore, there exists a minimal geodesic from x to $\hat{\alpha}$, $\Gamma : [0, \overline{x,\hat{\alpha}}] \to Y$ such that $w \in \mathrm{Image}(\Gamma)$ holds. This contradicts the assumption $x \in \mathrm{Alex}(Y)$. Thus, we have Claim 6.4.

By Claim 6.4, for every sufficiently small positive number $t > 0$, there exists a point $\alpha_t \in Y$ such that $\partial B_t(w) \cap C_w(\{z\}) = \{\alpha_t\}$. By Theorem 1.1 in [12], we have $\upsilon_{-1}(\alpha_t) > 0$. On the other hand, for the tangent cone at α_t, $(T_{\alpha_t}Y, 0_{\alpha_t})$, there exists a proper geodesic space W such that $T_{\alpha_t}Y$ is isometric to $\mathbf{R} \times W$. By the assumption $\dim_{\mathcal{H}}^{\mathrm{loc}} x > 1$ and the uniform properties of the Hausdorff dimension on Alexandrov spaces, W is not a single point. Therefore, by Proposition 6.1, we have $\upsilon_{-1}(\alpha_t) = 0$. This is contradiction. $\square$

Theorem 1.2 follows from Theorem 4.2, Theorem 4.5, Proposition 5.3 and Theorem 6.3, immediately. Finally, we shall give the following theorem.

THEOREM 6.5. *Let x be a point in Y. We assume that there exist points $w, z \in Y \setminus x$ such that $w \neq z$, $\overline{x,w} + \overline{w,z} = \overline{x,z}$ holds and that $\upsilon(C_w(\{z\})) > 0$ holds. Then, one of the following statements $1, 2$ occurs.*

1. *$\dim_{\mathcal{H}}^{\mathrm{loc}} x = 1$ holds.*

2. *x is not an Alexandrov point.*

PROOF. By an argument simular to the proof of Theorem 6.3 and by using Corollary 4.8 in [12], it is easy to check this assertion. $\square$

6.2 Some uniform properties of Hausdorff dimension

First, we consider an analogous statement to Theorem 1.2 for tangent cones.

THEOREM 6.6. *Let (X, x) be a proper geodesic space, $k \geq 0$ a nonnegative integer. We assume that $(\mathbf{R}^k \times X, (0_k, x))$ is $(n,0)$-Ricci limit space and that there exists $z \in X$ such that $\dim_{\mathcal{H}}^{\mathrm{loc}} z = 1$. Let w be an Alexandrov point in X. Then we have $\dim_{\mathcal{H}}^{\mathrm{loc}} w = 1$. Especially, we have $\mathrm{Alex}(X) = X(1)$.*

Proof. This proof is by contradiction. We assume that $\dim_{\mathcal{H}}^{\mathrm{loc}} w > 1$ holds. By an argument similar to the proof of Theorem 4.5 and by Corollary 3.3, there exists an open neighborhood of z, U such that $U \cap \underline{\mathcal{WE}}_1(X) = \emptyset$. By a similar argument to the proof of Theorem 4.2, there exists a sufficiently small positive number $\epsilon > 0$ such that $(B_\epsilon(z), z)$ is isometric either to $((-\epsilon, \epsilon), 0)$ or to $([0, \epsilon), 0)$. We take a minimal geodesic from z to w, $\gamma : [0, \overline{z, w}] \to X$ and take a sufficiently small positive number $0 < \tau << \epsilon$. We put $\hat{z} = \gamma(\epsilon/2)$ and $\hat{w} = \gamma(\overline{z, w} - \epsilon)$. We take $\hat{x} \in B_\tau(\hat{z})$ and a minimal geodesic from $\hat{x}$ to $\hat{w}$, $\gamma_1 : [0, \overline{\hat{x}, \hat{w}}] \to X$. Then, we have $\gamma_1([0, 2\tau]) \subset \mathrm{Image}(\gamma)$. Let $(v, \hat{x}) \in B_\tau(0_k, \hat{z})$ be a point and $\Gamma : [0, \overline{(v, \hat{x}), (0_k, \hat{w})}] \to \mathbf{R}^k \times X$ a minimal geodesic from $(v, \hat{x})$ to $(0_k, \hat{w})$. We put $\Gamma(t) = (a(t), \hat{\gamma}(t))$. By simple calculation, we have, the map $\Phi(s) = \hat{\gamma}(\overline{(v, \hat{x}), (0_k, \hat{w})} s / \overline{\hat{x}, \hat{w}})$ for $s \in [0, \overline{\hat{x}, \hat{w}}]$, is a minimal geodesic from $\hat{x}$ to $\hat{w}$ on X. We also have $|a(t)| \leq \tau$ for every t. We put $\alpha = \gamma(\overline{z, w} - 2\epsilon) \in X$. Then, by an argument similar to the proof of Theorem 6.3, we have $C_{(0_k, \hat{w})}(B_\tau(0_k, \hat{z})) \cap (B_{\epsilon+\tau}(0_k, \hat{w}) \setminus B_\epsilon(0_k, \hat{w})) \subset B_{2\tau}(0_k, \alpha)$. Therefore, by Bishop-Gromov volume comparison theorem for υ (See $(A.2.2)$ in [4]), we have $\upsilon(B_\tau(0_k, \hat{z})) \leq C(\epsilon, n, \overline{z, x}) \upsilon(B_{2\tau}(0_k, \alpha))$. Here, $C(\epsilon, n, \overline{z, x}) > 0$ is a positive constant depending only on $\epsilon, n, \overline{z, x}$. By Theorem 4.6 in [6], we have $\liminf_{\tau \to 0} \upsilon(B_\tau(0_k, \hat{z}))/\tau^{k+1} > 0$. Therefore, we have $\liminf_{\tau \to 0} \upsilon(B_\tau(0_k, \alpha))/\tau^{k+1} > 0$. Thus, by Proposition 5.1 and Proposition 5.2, there exists a positive constant $C > 1$ such that $C^{-1}\tau^{k+1} \leq \upsilon(B_\tau(0_k, \alpha)) \leq C\tau^{k+1}$ holds for every $0 < \tau < 1$. Therefore, there exist a pointed proper geodesic space (Z_1, z_1), a limit measure $\hat{\upsilon}$ on $T_{(0_k, \alpha)}(\mathbf{R}^k \times X)$, a tangent cone at α, $(T_\alpha X, 0_\alpha)$, and exists a Borel measure on Z_1, υ_{Z_1} such that $T_\alpha X$ is isometric to $\mathbf{R} \times Z_1$, $T_{(0_k, \alpha)}(\mathbf{R}^k \times X)$ is isometric to $\mathbf{R}^{k+1} \times Z_1$, $\hat{\upsilon} = \mathcal{H}^{k+1} \times \upsilon_{Z_1}$ holds and that $\liminf_{\tau \to 0} \hat{\upsilon}(B_\tau(0_k, z_1))/\tau^{k+1} > 0$ holds. On the other hand, by α is an Alexandrov point, Z_1 is not a single point. Therefore, by Proposition 5.2, we have $\liminf_{\tau \to 0} \hat{\upsilon}(B_\tau(0_k, z_1))/\tau^{k+1} = 0$. This is contradiction. Therefore we have $\mathrm{Alex}(X) \subset X(1)$. Next, we take $\beta \in X(1)$. There exists a positive number $\delta > 0$ such that $\dim_{\mathcal{H}} B_\delta(\beta) < 2$ holds. By using Corollary 1.36 in [5], we have $\mathcal{R}_i(X) \cap B_\delta(\beta) = \emptyset$ for every $i \geq 2$. Therefore, by Corollary 3.3, we have $\underline{\mathcal{WE}}_1(X) = \emptyset$. Thus, by a similar argument to the proof of Theorem 4.2, there exists a positive number $r > 0$ such that $(B_r(\beta), \beta)$ is isometric either to $((-r, r), 0)$ or to $([0, r), 0)$. Especially, we have $\beta \in \mathrm{Alex}(X)$. □

Remark 6.7. Let (X, x) be a proper geodesic space. For an open set $U \subset X$, we say that U has *k-dimensional C^∞-Riemannian structure* if for every point $x \in U$, there exist a open set $V \subset U$ and k-dimensional (not complete) Riemannian manifold N such that V is isometric to N as metric spaces. We assume that there exist an integer $k \geq 2$ and open sets $U_1, U_2 \subset X$ such that U_1 has one dimensional C^∞-Riemannian structure and that U_2 has k-dimensional C^∞-Riemannian structure. Then, by a similar argument to the proof of Theorem 6.6, for every $l \in \mathbf{N}$ and every l-dimensional complete C^∞-Riemannian manifold

(M, m), $(M \times X, (m, x))$ is *not* Ricci limit space. For example, $(M \times Z_\tau, (m, 0))$ is not Ricci limit space. Roughly speaking, a reason of the non-existence is "locally Lipschitz properties of exponential map from higher dimensional point".

We say that a proper geodesic space X is *non-baranching* if for every $x \in X$ and every $y \in X \setminus C_x$, there exists an unique minimal geodesic from x to y. Here, C_x is the cut locus of x, $C_x = \{z \in X |$ For every $w \in X \setminus z$, $\overline{x,z} + \overline{z,w} - \overline{x,w} > 0$ holds. $\}$. (If X is a single point, then $C_x = \emptyset$.)

THEOREM 6.8. *We assume that $\mathcal{R}_1 \neq \emptyset$ and Y is non-branching. Then we have $\dim_{\mathcal{H}} Y = 1$.*

PROOF. We fix a point $x \in Y$. First, we shall prove $Y \setminus C_x \subset A_Y(1)$. For every $z \in Y \setminus C_x$, there exists $w \in Y \setminus C_x$ such that $z \neq w$ and $\overline{x,z} + \overline{z,w} = \overline{x,w}$ hold. By the assumpiton of non-branching, there exists an unique minimal geodesic from x to w, $\gamma : [0, \overline{x,w}] \to Y$ such that $x \in \mathrm{Image}(\gamma)$. By Proposition 5.3 and Theorem 1.1 in [12], we have $\upsilon_{-1}(x) > 0$. Therefore, we have $Y \setminus C_x \subset A_Y(1)$. Thus, by Theorem 3.2 in [12], we have $\upsilon(Y \setminus A_Y(1)) = 0$. By Corollary 4.6, we have the assertion. $\square$

7 Two dimensional case

In this section, we study the Hausdorff dimension of the Ricci limit space (Y, y) such that $2 \leq \dim_{\mathcal{H}} Y < 3$ holds.

PROPOSITION 7.1. *Let $s \geq 1$ be a positive number, $U \subset Y$ an open set satisfying $\dim_{\mathcal{H}} U \leq s$, x a point in U, and $(T_x Y, 0_x)$ a tangent cone at x. We assume that there exists a proper geodesic space W such that $T_x Y$ is isometric to $\mathbf{R}^{[s]-1} \times W$. Then, W is isometoric either to a single point, or to $\mathbf{R}$, or to $\mathbf{R}_{\geq 0}$, or to $\mathbf{S}^1(r)$ for some positive number $r > 0$, or to $[0, l]$ for some $l > 0$. Here, $[s] = \max\{k \in \mathbf{Z} | k \leq s\} \in \mathbf{N}$.*

PROOF. First, we shall prove $\underline{\mathcal{WE}}_1(W) = \emptyset$. We assume that $\underline{\mathcal{WE}}_1(W) \neq \emptyset$. Then we have $\underline{\mathcal{WE}}_{[s]}(T_x Y) \neq \emptyset$. Thus, by Corollary 3.3, we have $\mathcal{WE}_{[s]+1}(T_x Y) \neq \emptyset$. Hence, $(\mathcal{WE}_{[s]+1})_\epsilon \cap U \neq \emptyset$ holds for every positive number $\epsilon > 0$. Thus, by Corollary 3.3, there exists an integer $i \geq [s] + 1$ such that $\mathcal{R}_i \cap U \neq \phi$ holds. Therefore, by Corollary 1.36 in [5], we have $\dim_{\mathcal{H}} U \geq i \geq [s] + 1 > s$. This contradicts the assumption. Therefore we have $\underline{\mathcal{WE}}_1(W) = \emptyset$. By using $\underline{\mathcal{WE}}_1(W) = \emptyset$ and an argument simular to the proof of Theorem 4.2, we have Proposition 7.1. $\square$

We shall apply Proposition 7.1 to estimate of Hausdorff dimension of some subset.

COROLLARY 7.2. *Let $s \geq 1$ be a positive number and $U \subset Y$ an open set satisfying* $\dim_{\mathcal{H}} U \leq s$ *Then, we have* $\dim_H \mathcal{E}_{[s]-1} \cap U \leq [s]$.

PROOF. First, we shall prove the following claim.

CLAIM 7.3. *Let X be a proper geodesic space, $A \subset X$ a subset and $s > 0$ a positive number. We assume that for every point $x \in X$ and every sequence of positive numbers $r_i > 0$ such that r_i converges to 0, there exist a subsequence $r_{n(i)} > 0$ and a tangent cone at x, $(T_x X, 0_x)$ such that $(X, r_{n(i)}^{-1} d_X, x)$ converges to $(T_x X, 0_x)$. We also assume that for every point $\alpha \in A$ and for every tangent cone at α, $(T_\alpha X, 0_\alpha)$, $\dim_{\mathcal{H}} T_\alpha X \leq s$ holds. Then, $\dim_{\mathcal{H}} A \leq s$ holds.*

This proof is by contradiction. We assume that $\dim_{\mathcal{H}} A > s$ holds. There exists a positive number $\epsilon > 0$ such that $\dim_{\mathcal{H}} A > s + \epsilon$ holds. By density result in Geometric measure theory, there exist a point $\alpha \in A$ and a sequence of positive number $r_i > 0$, such that r_i converges to 0 and $\lim_{i \to \infty} (\mathcal{H}^{s+\epsilon}_{\infty}(A \cap \overline{B}_{r_i}(\alpha))/{r_i}^{s+\epsilon}) > 0$ holds. (For example, see (1.39) in [5] for the definition of $(s+\epsilon)$-dimensional spherical Hausdorff content, $\mathcal{H}^{s+\epsilon}_{\infty}$.) Without loss of generality, we can assume that there exists a tangent cone at α, $(T_\alpha X, 0_\alpha)$ such that $(X, r_i^{-1} d_X, \alpha)$ converges to $(T_\alpha X, 0_a)$. By the construction, it is not difficult to see that $\mathcal{H}^{s+\epsilon}(\overline{B}_1(0_\alpha)) > 0$ holds. Especially, we have $\dim_{\mathcal{H}} T_\alpha X \geq s + \epsilon > s$. This contradicts the assumption. Therefore, we have Claim 7.3.

By Proposition 7.1, for every point $x \in \mathcal{E}_{[s]-1} \cap U$ and for every tangent cone at x, $(T_x Y, 0_x)$, we have $\dim_{\mathcal{H}} T_x Y \leq [s]$ holds. Therefore we have Corollary 7.2 by Claim 7.3. $\square$

Finally, we consider the condition $2 \leq \dim_{\mathcal{H}} Y < 3$.

COROLLARY 7.4. *We assume that $2 \leq \dim_H Y < 3$ holds. Then, $\dim_H (Y \setminus C_x) \leq 2$ holds for every $x \in Y$.*

PROOF. By $Y \setminus C_x \subset \mathcal{E}_1$ and Corollary 7.2. $\square$

REMARK 7.5. It seems that for every Ricci limit space (Y, y), $\dim_{\mathcal{H}}(Y \setminus C_y) = \dim_{\mathcal{H}} Y$ holds. If it is true, then for every Ricci limit space (Y, y) such that $\dim_{\mathcal{H}} Y \leq 3$ holds, we have $\dim_{\mathcal{H}} Y \in \mathbf{Z}$ by Theorem 1.1 and Corollary 7.4. Moreover, if this conjecture is true, we can prove that for *every* Ricci limit spaces (Y, y), we have $\dim_{\mathcal{H}} Y \in \mathbf{Z}$. See next section.

8 Hausdorff dimension in higher dimensional case

In this section, we study the problem whether the Hausdorff dimension of Y is an integer. J. Cheeger and T. H. Colding gave a sufficient condition for satisfying $\dim_{\mathcal{H}} Y \in \mathbf{Z}$, that is called by *polar*. (See Definition 4.1 in [4] for the definition of polar.) We remark that there exists a non-polar Ricci limit space such that the Hausdorff dimension is an integer. See [14] for the example. We shall give an another sufficient condition for satisfying $\dim_{\mathcal{H}} Y \in \mathbf{Z}$, that contains polar condition.

DEFINITION 8.1. The pointed proper geodesic space (X, x) is called by *iterated tangent cone of Y* if there exists a sequence of pointed proper geodesic spaces $\{(X_i, x_i)\}_{i=0}^N$ such that X_0 is isometric to Y, (X_N, x_N) is isometric to (X, x) and (X_{i+1}, x_{i+1}) is a tangent cone at some point in X_i for every i.

We shall prove next theorem.

THEOREM 8.2. *We assume that for every iterated tangent cone of Y, (X, x), $\dim_{\mathcal{H}}(X \setminus C_x) \geq \dim_{\mathcal{H}} C_x$ holds. (This condition is equivalent to $\dim_{\mathcal{H}}(X \setminus C_x) = \dim_{\mathcal{H}} X$.) Then we have $\dim_{\mathcal{H}} B_R(z) \in \mathbf{Z}$ for every point $z \in Y$ and every positive number $R > 0$. Especially, $\dim_{\mathcal{H}} Y \in \mathbf{Z}$ and $\dim_{\mathcal{H}}^{\mathrm{loc}} z \in \mathbf{Z}$ hold.*

PROOF. First, we take an integer $k > 0$ such that $\dim_{\mathcal{H}} B_R(z) < k+1$ holds. We shall prove $\dim_{\mathcal{H}} B_R(z) \leq k$. By Claim 7.3, it suffices to see that $\dim_{\mathcal{H}} T_z Y \leq k$ holds for every point $z \in Y$ and every tangent cone at z, $(T_z Y, 0_z)$. We fix a tangent cone $(T_z Y, 0_z)$ and put $(Y_1, y_1) = (T_z Y, 0_z)$. By the assumption and Claim 7.3, it suffices to see that $\dim_{\mathcal{H}} T_{z_1} Y_1 \leq k$ holds for every point $z_1 \in Y_1 \setminus C_{y_1}$ and every tangent cone at z_1, $(T_{z_1} Y_1, 0_{z_1})$. We also fix a tangent cone $(T_{z_1} Y_1, 0_{z_1})$ and put $(Y_2, y_2) = (T_{z_1} Y_1, 0_{z_1})$. By the construction, there exists a pointed proper geodesic space (W_2, w_2) such that (Y_2, y_2) is isometric to $(\mathbf{R} \times W_2, (0, w_2))$. Without loss of generality, we can assume that W_2 is not a single point.

CLAIM 8.3. *In general, we have $C_{(0_k, w)} = \mathbf{R}^k \times C_w$ in $\mathbf{R}^k \times W$ for every positive integer $k > 0$ and every pointed proper geodesic space (W, w).*

If W is a single point, then $C_w = \emptyset$ holds, especially, we have Claim 8.2. We assume that W is not a single point. It suffices to see that for every point $(t_k, x) \in \mathbf{R}^k \times W \setminus C_{(0_k, w)}$, $x \in W \setminus C_w$ holds. We can assume that $x \neq w$. By the definition, there exists a point $(s_k, z) \in \mathbf{R}^k \times W$ such that $(s_k, z) \neq (t_k, x)$ and $\overline{(0_k, w), (t_k, x)} + \overline{(t_k, x), (s_k, z)} = \overline{(0_k, w), (s_k, z)}$ hold. We take an isometric embedding $\{w, \ x, \ z\} \to \mathbf{R}^2$. We denote the images by $\hat{w}, \ \hat{x}, \ \hat{z}$, respectively. Then we have $\overline{(0_k, \hat{w}), (t_k, \hat{x})} + \overline{(t_k, \hat{x}), (s_k, \hat{z})} = \overline{(0_k, \hat{w}), (s_k, \hat{z})}$ in $\mathbf{R}^{k+2}$. By simple calculation, we have $\overline{\hat{w}, \hat{x}} + \overline{\hat{x}, \hat{z}} = \overline{\hat{w}, \hat{z}}$ and $\overline{\hat{z}, \hat{x}} > 0$. Therefore, $x \in W \setminus C_w$ holds. Thus we have Claim 8.3.

By the assumption and Claim 8.3, we have $\dim_{\mathcal{H}}(W_2 \setminus C_{w_2}) \geq \dim_{\mathcal{H}} C_{w_2}$. Thus, proving that $\dim_{\mathcal{H}} Y_2 \leq k$ holds, suffices to see that for every point $\hat{w}_2 \in W_2 \setminus C_{w_2}$ and for every tangent cone at $\hat{w}_2$, $(T_{\hat{w}_2} W_2, 0_{\hat{w}})$, $\dim_{\mathcal{H}} T_{\hat{w}_2} W_2 \leq k-1$ holds. We fix a tangent cone $(T_{\hat{w}_2} W_2, 0_{\hat{w}})$ and put $(W_3, w_3) = (T_{\hat{w}_2} W_2, 0_{\hat{w}_2})$. By the construction, there exists a pointed proper geodesic space (W_4, w_4) such that (W_3, w_3) is isometric to $(\mathbf{R} \times W_4, (0, w_4))$. By Claim 7.3, without loss of generality, we can assume that W_4 is not a single point. Since $(\mathbf{R}^2 \times W_4, (0_2, w_4))$ is an iterated tangent cone of Y, by the assumption and Claim 8.3, we have $\dim_{\mathcal{H}}(W_4 \setminus C_{w_4}) \geq \dim_{\mathcal{H}} C_{w_4}$. Therefore, it suffices to see that for every point $\hat{w}_4 \in W_4 \setminus C_{w_4}$ and every tangent cone at $\hat{w}_4$, $(T_{\hat{w}_4} W_4, 0_{\hat{w}_4})$, $\dim_{\mathcal{H}} T_{\hat{w}_4} W_4 \leq k-2$ holds.

We continue this argument and constract pointed proper geodesic space (W_{2k}, w_{2k}) as above. Then, it suffices to see that $\dim_{\mathcal{H}}(W_{2k}, w_{2k}) \leq 0$ holds, i.e. W_{2k} is a single point. We assume that W_{2k} is not a single point. Then, there exist an iterated tangent cone of $B_R(z)$, (X, x), and a proper geodesic space L such that X is isometric to $\mathbf{R}^{k+1} \times L$. Therefore, we have $(\mathcal{WE}_{k+1})_\epsilon \cap B_R(z) \neq \emptyset$ for every positive number $\epsilon > 0$. Thus, by Corollary 3.3, there exists an integer $i \geq k+1$ such that $\mathcal{R}_i \cap B_R(z) \neq \emptyset$. Therefore, by Corollary 1.36 in [5], we have $\dim_{\mathcal{H}} B_R(z) \geq i \geq k+1$. This contradicts the assumption. Therefore, we have $\dim_{\mathcal{H}} B_R(z) \leq k$.

Here, we take an integer $k > 0$ such that $k \leq \dim_{\mathcal{H}} B_R(z) < k+1$ holds. Then, we have the $\dim_{\mathcal{H}} B_R(z) = k$. □

Remark 8.4. By an argument similar to the proof of Theorem 8.2, if for every iterated tangent cone of Y, (X, x), $\dim_{\mathcal{H}}(X \setminus \mathcal{WD}_0(x)) \geq \dim_{\mathcal{H}} \mathcal{WD}_0(x)$ holds, then we have the same conclusion to Theorem 8.2. (See Definition 2.10 in [4] for the definition of $\mathcal{WD}_0(x)$.)

It is not difficult to see that Y is polar if and only if $C_x = \phi$ for every iterated tangent cone of Y, (X, x).

Theorem 8.5. *We assume that for every iterated tangent cone of Y, (X, x),* $\dim_{\mathcal{H}}(X \setminus C_x) = \dim_{\mathcal{H}} X$ *holds. Let $R > 0$ be a positive number, $k > 0$ a positive integer and z a point in Y. We also assmue that* $\dim_{\mathcal{H}} B_R(z) \geq k$ *holds. Then, we have* $\upsilon(B_R(z) \cap (\bigcup_{i \geq k} \mathcal{R}_i)) > 0$.

Proof. We take a sufficiently small positive number $\epsilon > 0$. By the assumption, we have $\mathcal{H}^{k-\epsilon}(B_R(z)) = \infty$. Hence, by an argument simular to the proof of Claim 7.3, there exist a point $x \in B_R(z)$ and a tangent cone at x, $(T_x Y, 0_x)$ such that $\mathcal{H}^{k-\epsilon}(T_x Y) > 0$ holds. We fix a tangent cone $(T_x Y, 0_x)$ and put $(Y_1, y_1) = (T_x Y, 0_x)$. Since $\dim_{\mathcal{H}} Y_1 \geq k - \epsilon > k - 2\epsilon > 0$ and $\dim_{\mathcal{H}}(Y_1 \setminus C_{y_1}) = \dim_{\mathcal{H}} Y_1$, we have $\mathcal{H}^{k-2\epsilon}(Y_1 \setminus C_{y_1}) = \infty$. Similarly, there exist a point $x_1 \in Y_1 \setminus C_{y_1}$ and a tangent cone at x_1, $(T_{x_1} Y_1, 0_{x_1})$ such that $\mathcal{H}^{k-2\epsilon}(T_{x_1} Y_1) > 0$ holds. We put $(Y_2, y_2) = (T_{x_1} Y_1, 0_{x_1})$. By the construction, there exists a pointed proper geodesic space (X_2, x_2) such that (Y_2, y_2) is isometric to $(\mathbf{R} \times X_2, (0, x_2))$. Thus, $\dim_{\mathcal{H}} X_2 \geq k - 1 - 2\epsilon > k - 1 - 3\epsilon > 0$ holds. Therefore, since $\dim_{\mathcal{H}} X_2 = \dim_{\mathcal{H}}(X_2 \setminus C_{x_2})$, we have $\mathcal{H}^{k-1-3\epsilon}(X_2 \setminus C_{x_2}) = \infty$. By an argument similar to that above, there exist a point $\hat{x}_2 \in X_2$ and a tangent cone at $\hat{x}_2$, $(T_{\hat{x}_2} X_2, 0_{\hat{x}_2})$ such that $\mathcal{H}^{k-1-3\epsilon}(T_{\hat{x}_2} X_2) > 0$ holds. We put $(X_3, x_3) = (T_{\hat{x}_2} X_2, 0_{\hat{x}_2})$. By the construction, there exists a pointed proper geodesic space (X_4, x_4) such that (X_3, x_3) is isometric to $(\mathbf{R} \times X_4, (0, x_4))$. Since $(\mathbf{R}^2 \times X_4, (0_2, x_4))$ is an iterated tangent cone of $B_R(z)$, by the assumption, we have $\dim_{\mathcal{H}} X_4 = \dim_{\mathcal{H}}(X_4 \setminus C_{x_4})$ and $\dim_{\mathcal{H}} X_4 \geq k - 2 - 3\epsilon > k - 2 - 4\epsilon$.

We continue this argument and construct a pointed proper geodesic space $(X_{2(k-1)}, x_{2(k-1)})$ as above. By the construction, $(\mathbf{R}^{k-1} \times X_{2(k-1)}, (0_k, x_{2(k-1)}))$ is an iterated tangent cone of $B_R(z)$. We have $\dim_{\mathcal{H}} X_{2(k-1)} \geq k - (k-1) - 2(k-2)\epsilon > 1 - 2(k-1)\epsilon > 0$. Since $X_{2(k-1)}$ is a geodesic space, we have $\dim_{\mathcal{H}} X_{2(k-1)} \geq 1$. Therefore, there exists pointed proper geodesic space (W, w) such that $\mathbf{R}^k \times W$ is an iterated tangent cone of $B_R(z)$. Thus, $(\mathcal{WE}_k)_\epsilon \cap B_R(z) \neq \phi$ holds for every positive number $\epsilon > 0$. Therefore, by Proposition 3.1, we have $\upsilon(B_R(z) \cap (\bigcup_{i \geq k} \mathcal{R}_i)) > 0$. $\square$

Next corollary is main result in this section.

Corollary 8.6. *We assume that for every iterated tangent cone of Y, (X, x), $\dim_{\mathcal{H}}(X \setminus C_x) = \dim_H X$ holds. We take an integer $k > 0$ such that $\mathcal{R}_k \neq \phi$ and $\mathcal{R}_i = \phi$ hold for every integer $i > k$. Then we have the following statements:*

1. $\dim_{\mathcal{H}} Y = k$ *holds.*

2. $\mathcal{H}^k(\mathcal{R}^k) > 0$ *holds.*

3. $\upsilon(\mathcal{R}^k) > 0$ *holds.*

Proof. By Corollary 1.36 in [5], we have $\dim_{\mathcal{H}} Y \geq k$. We assume that $\dim_{\mathcal{H}} Y \geq k+1$ holds. Then, by Theorem 8.5, there exists an integer $i \geq k+1$ such that $\mathcal{R}_i \neq \emptyset$. This contradicts the assumption. Thus we have $\dim_{\mathcal{H}} Y < k+1$. By Theorem 8.2, we have $\dim_{\mathcal{H}} Y = k$. Next, we assume that $\upsilon(\mathcal{R}_k) = 0$ holds. Then $\upsilon(B_r(x) \cap \bigcup_{i \geq k} \mathcal{R}_i) = \upsilon(B_r(x) \cap \mathcal{R}_k) = 0$ for every point $x \in \mathcal{R}_k$ and every positive number $r > 0$. This contradicts Proposition 3.1. Thus, we have $\upsilon(\mathcal{R}_k) > 0$. By Theorem 3.23 and Theorem 4.6 in [6], we have $\mathcal{H}^k(\mathcal{R}^k) > 0$. $\square$

9 Appendix: A proof of Proposition 5.2

In this section, we shall prove Proposition 5.2. First, we give a next lemma without the proof because it is not difficult to prove by simple calculation.

Lemma 9.1. *For every positive numbers $0 < r < R$, there exists a positive constant $C(r,R) > 0$ satisfying the following properties; Let X be a metric space, x_1 a point in X, $k > 0$ a positive integer, $\delta, \epsilon > 0$ positive numbers with $\delta < r/1000$, v_α, v_β points in $\mathbf{R}^k$ and x_2 a point in $\overline{B}_R(x_1)\backslash B_r(x_1)$. We assume that $|v_\alpha| \le 1, |v_\beta| \le 1$ hold in $\mathbf{R}^k$, $\overline{x_\alpha, x_2} \le \delta, \overline{x_\beta, x_2} \le \delta$ hold in X and that $\overline{(0_k, x_1), (v_\alpha, x_\alpha)} + \overline{(v_\alpha, x_\alpha), (v_\beta, x_\beta)} - \overline{(0_k, v_\beta), (v_\beta, x_\beta)} \le \epsilon$ holds in $\mathbf{R}^k \times X$. Then, we have $\overline{(v_\alpha, x_\alpha), (v_\beta, x_\beta)} \le C(r,R)(\delta + \epsilon)$.*

Proof of Proposition 5.2. Without loss of generality, we can assume that $w \neq z$. By the assumption, there exist a sequence of pointed complete connected Riemannian manifolds $\{(M_j, m_j)\}_j$ and a sequence of positive numbers $\epsilon_j > 0$ such that ϵ_j converges to 0, $Ric_{M_j} \ge -\epsilon_j$ holds and $(M_j, m_j, \mathrm{vol}/\mathrm{vol}B_1(m_j))$ converges to $(\mathbf{R}^k \times W, (0_k, w), \upsilon)$ in the sense of measured Gromov-Hausdorff topology. We take a sufficiently small positive number $\delta > 0$. Let $\{(t_i, x_i)\}_{i=1}^N$ be a maximal δ-separated set on $[0,1]^k \times \overline{B}_\delta(z)$. Let $0 < r < 1$ be a positive number satisfying $z \in \overline{B}_R(w) \setminus B_r(w)$. For every positive integers $i, j > 0$ $(1 \le i \le N)$, there exists a point $y_j^i \in M_j$ such that y_j^i converges to (t_i, x_i) as $j \to \infty$ in the sense of pointed Gromov-Hausdorff topology. Thus, for sufficiently large j, $\{\overline{B}_{\delta/3}(y_j^i)\}_i$ is pairwise disjoint in M_j. We put $X_j = \bigcup_i \overline{B}_{\delta/3}(y_j^i)$, $S_{m_j}M_j = \{u \in T_{m_j}M_j \,||u| = 1\}$, $t(u) = \sup\{t \in \mathbf{R}_{>0} | \exp_{m_j} su \in M_j \setminus C_{m_j}$ holds for every positive number $0 < s < t\}$ for $u \in S_{m_j}M_j$, and put $\hat{S}_{m_j}M_j = \{u \in S_{m_j}M_j|$ there exists a positive number $0 < t < t(u)$ such that $\exp_{m_j} tu \in X_j$ holds.$\}$. We also put $A_j(u) = \{t \in (0, t(u)) | \exp_{m_j} tu \in X_j\}$ for $u \in \hat{S}_{m_j}M_j$ and $\theta(t,u) = t^{n-1}(\det(g_{ij}|_{\exp_{m_j} tu}))^{\frac{1}{2}}$. Here, $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ where $(x_1, x_2, \dots, x_n)$ is a normal coordinate around m_j. Then, by Laplacian comparison theorem, we have

$$\begin{aligned}
\mathrm{vol}X_j &= \int_{\hat{S}_{m_j}M_j} \int_{A_j(u)} \theta(t,u)dtdu \\
&\le \int_{\hat{S}_{m_j}} \int_{A_j(u)} \sinh^{n-1}(t) \frac{\theta(\frac{r}{2}, u)}{\sinh^{n-1}(\frac{r}{2})} dtdu \\
&\le \int_{\hat{S}_{m_j}M_j} \frac{\theta(\frac{r}{2}, u)}{\sinh^{n-1}(\frac{r}{2})} \int_{A_j(u)} \sinh^{n-1}(2R+10)dtdu \\
&\le C(n,r,R) \int_{\hat{S}_{m_j}M_j} \theta(\frac{r}{2}, u)\mathcal{H}^1(A_j(u))du.
\end{aligned}$$

Here, $C(n,r,R) > 0$ is a postive constant depending only on n, r, R. We put $a_j(u) = \inf A_j(u)$ and $b_j(u) = \sup A_j(u)$ for $u \in \hat{S}_{m_j}M_j$. Then, by Lemma 9.1, we have $b_j(u) - a_j(u) \le C(r,R)\delta$. Thus

$$\underline{\mathrm{vol}}X_j \le C(r,R)\delta\underline{\mathrm{vol}}(\partial B_{\frac{r}{2}}(m_j) \setminus C_{m_j}).$$

Here, $\underline{\mathrm{vol}} = \mathrm{vol}/\mathrm{vol}B_1(m_j)$. By Bishop-Gromov volume comparison theorem, we have

$\mathrm{vol}(\partial B_{\frac{r}{2}}(m_j) \setminus C_{m_j})/\mathrm{vol}B_{\frac{r}{2}}(m_j) \leq \mathrm{vol}\partial B_{\frac{r}{2}}(\underline{p})/\mathrm{vol}B_{\frac{r}{2}}(\underline{p})$. Thus, we have

$$\sum_{i=1}^{N} \upsilon(B_{\frac{\delta}{3}}(t_i, x_i)) \leq C(n, r, R)\delta.$$

By measured splitting theorem, Proposition 1.35 in [4], there exists a Borel measure on W, υ_W such that $\upsilon = \mathcal{H}^k \times \upsilon_W$ holds. Therefore, by Bishop-Gromov volume comparison theorem for υ (See (1.12) in [4]),

$$\begin{aligned} \upsilon_W(B_\delta(w)) = \upsilon([0,1]^k \times B_\delta(w)) &\leq \sum_{i=1}^{N} \upsilon(B_\delta(t_i, x_i)) \\ &\leq C(n) \sum_{i=1}^{N} \upsilon(B_{\frac{\delta}{3}}(t_i, x_i)) \\ &\leq C(n, r, R)\delta. \end{aligned}$$

Therefore, we have Proposition 5.2. □

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Department of Mathematics
Kyoto University
Kyoto 606-8502
Japan

E-mail address: honda@math.kyoto-u.ac.jp