An alternative proof of Berg and Nikolaev's characterization of CAT(0)-spaces via quadrilateral inequality*†

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Abstract

We give a short alternative proof of Berg and Nikolaev's recent theorem on a characterization of CAT(0)-spaces via quadrilateral inequality.

1 Introduction

The aim of this article is to present a short alternative proof of Berg and Nikolaev's recent striking theorem:

Theorem 1.1 ([BN, Theorem 6]) *Let (X, d) be a geodesic space. Then the following are equivalent*:

(i) (X, d) *is a* CAT(0)*-space.*

(ii) *Any four points* $w, x, y, z \in X$ *satisfy the* quadrilateral inequality

$$d(w, y)^2 + d(x, z)^2 \leq d(w, x)^2 + d(x, y)^2 + d(y, z)^2 + d(z, w)^2. \tag{1.1}$$

This theorem gives an appropriate answer to the long-standing question (cf. [Gr]): how to characterize CAT(0)-spaces in such a way that it makes sense in discrete (non-geodesic) spaces? We also mention that there is a connection with the geometry of Banach spaces as (1.1) is what Enflo called the *roundness* 2 (see [BL] and [OP]). We refer to [BN] for more details and histrical background.

Example 1.2 Besides CAT(0)-spaces, there are further simple examples satisfying (1.1).

(a) *Ultrametric spaces* (X, d) (i.e., $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ holds for all $x, y, z \in X$).

(b) The metric space $(X, d^{1/2})$ for given any metric space (X, d).

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Our proof of Theorem 1.1 is based on simple calculations which have a somewhat similar flavor to Berg and Nikolaev's original proof, but much shorter and clearer. The point is to compare the original triangle with the subdivided ones (see Lemma 2.2(ii)).

2 Proof of Theorem 1.1

Let (X, d) be a metric space. A rectifiable curve $\gamma : [0, 1] \longrightarrow X$ is called a *geodesic* if it is locally minimizing and parametrized proportionally to the arclength. If in addition γ is globally minimizing, then we call it a *minimal geodesic*. We say that (X, d) is a *geodesic space* if any two points $x, y \in X$ can be joined by a minimal geodesic. A geodesic space (X, d) is called a CAT(0)-*space* if we have, for any $x, y, z \in X$ and any minimal geodesic $\gamma : [0, 1] \longrightarrow X$ from y to z,

$$d\big(x, \gamma(1/2)\big)^2 \le \frac{1}{2}d(x, y)^2 + \frac{1}{2}d(x, z)^2 - \frac{1}{4}d(y, z)^2. \tag{2.1}$$

We refer to [BH] and [BBI] for the fundamentals of CAT(0)-spaces. It is easy to see that any CAT(0)-space satisfies the quadrilateral inequality (1.1), we just apply (2.1) to two triangles (w, x, z) and (y, x, z). Thus the remainder of the paper is devoted to the derivation of (i) from (ii) in Theorem 1.1.

Assume that a geodesic space (X, d) satisfies (1.1). Then any two points $x, y \in X$ are connected by a unique minimal geodesic. Indeed, given two minimal geodesics $\gamma, \eta : [0, 1] \longrightarrow X$ from x to y, applying (1.1) to $(x, \gamma(1/2), y, \eta(1/2))$ immediately implies $\gamma(1/2) = \eta(1/2)$. Hence $\gamma(m/2^n) = \eta(m/2^n)$ for any $m = 0, 1, \dots, 2^n$ and finally $\gamma = \eta$ on $[0, 1]$. Henceforce, we denote by $\gamma_{xy} : [0, 1] \longrightarrow X$ the unique minimal geodesic from x to y.

Lemma 2.1 *For any $x, y, z \in X$, we have $d(\gamma_{xy}(1/2), \gamma_{xz}(1/2)) \le d(y, z)$.*

Proof. Put $y' := \gamma_{xy}(1/2)$ and $z' := \gamma_{xz}(1/2)$. Applying (1.1) to (x, y', y, z') yields

$$d(x, y)^2 + d(y', z')^2 \le \frac{1}{2}d(x, y)^2 + d(y, z')^2 + \frac{1}{4}d(x, z)^2.$$

We similarly obtain

$$d(x, z)^2 + d(y', z')^2 \le \frac{1}{2}d(x, z)^2 + d(y', z)^2 + \frac{1}{4}d(x, y)^2$$

and hence

$$2d(y', z')^2 + \frac{1}{4}d(x, y)^2 + \frac{1}{4}d(x, z)^2 \le d(y, z')^2 + d(y', z)^2. \tag{2.2}$$

We next apply (1.1) to (y, z, z', y') and find

$$d(y, z')^2 + d(y', z)^2 \le d(y, z)^2 + \frac{1}{4}d(x, y)^2 + \frac{1}{4}d(x, z)^2 + d(y', z')^2. \tag{2.3}$$

Combining (2.2) and (2.3) completes the proof. $\square$

Lemma 2.2 (i) *For any $w, x, y, z \in X$, we have*

$$d(w,y)^2 + d(x,z)^2 \leq 2d(w,x)^2 + d(x,y)^2 + \frac{1}{2}d(y,z)^2 + d(z,w)^2.$$

(ii) *For any $x, y, z \in X$, it holds that*

$$\begin{aligned} &2d(x,y)^2 + 2d(x,z)^2 - d(y,z)^2 - 4d\big(x, \gamma_{yz}(1/2)\big)^2 \\ &\geq 2\big\{2d(y',y)^2 + 2d(y',z)^2 - d(y,z)^2 - 4d\big(y', \gamma_{yz}(1/2)\big)^2\big\}, \end{aligned}$$

where we put $y' := \gamma_{xy}(1/2)$.

Proof. (i) Put $v := \gamma_{yz}(1/2)$. Then the claim immediately follows from (1.1) applied to (w, x, y, v) as well as (w, x, v, z). (ii) Apply (i) to $(\gamma_{yz}(1/2), y', x, z)$. $\square$

Given arbitrary $x, y, z \in X$, put $w := \gamma_{yz}(1/2)$ and $y_n := \gamma_{yx}(2^{-n})$ for each $n \in \mathbb{N}$. Then our goal is to show

$$2d(x,y)^2 + 2d(x,z)^2 - d(y,z)^2 - 4d(x,w)^2 \geq 0. \tag{2.4}$$

We first apply Lemma 2.2(ii) repeatedly to see

$$\begin{aligned} &2d(x,y)^2 + 2d(x,z)^2 - d(y,z)^2 - 4d(x,w)^2 \\ &\geq 2^n\{2d(y_n,y)^2 + 2d(y_n,z)^2 - d(y,z)^2 - 4d(y_n,w)^2\}. \end{aligned} \tag{2.5}$$

Then we apply Lemma 2.2(ii) in the other direction and find, putting $v_n := \gamma_{y_n z}(1/2)$,

$$\begin{aligned} &2d(y_n,y)^2 + 2d(y_n,z)^2 - d(y,z)^2 - 4d(y_n,w)^2 \\ &\geq 2\{2d(v_n,y)^2 + 2d(v_n,z)^2 - d(y,z)^2 - 4d(v_n,w)^2\}. \end{aligned} \tag{2.6}$$

Now, it follows from (1.1) for (v_n, y, w, z) that

$$d(v_n,w)^2 + d(y,z)^2 \leq d(v_n,y)^2 + d(v_n,z)^2 + \frac{1}{2}d(y,z)^2.$$

Combining this with (2.5), (2.6) and Lemma 2.1 implies

$$\begin{aligned} &2d(x,y)^2 + 2d(x,z)^2 - d(y,z)^2 - 4d(x,w)^2 \\ &\geq -2^{n+2}d(v_n,w)^2 \geq -2^{n+2}d(y_n,y)^2 = -2^{2-n}d(x,y)^2. \end{aligned}$$

Letting n go to infinity shows (2.4) and completes the proof of Theorem 1.1.

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