ASYMPTOTIC DISTRIBUTION OF CRITICAL VALUES

TOMOHiro FUKAYA AND MASAKI TSUKAMOTO

Abstract. Let $X$ be a closed manifold and $f : X \times X \to \mathbb{R}$ be a smooth function. Define $f_n : X^{n+1} \to \mathbb{R}$ by $f_n(x_1, \ldots, x_{n+1}) := \frac{1}{n} \sum f(x_i, x_{i+1})$. We study the asymptotic distribution of the critical values of $f_n$ as $n$ goes to infinity.

1. Main results

Let $X$ be a compact connected smooth manifold of positive dimension and $f : X \times X \to \mathbb{R}$ be a smooth function. For an integer $n \geq 1$ we define $f_n : X^{n+1} \to \mathbb{R}$ by

$$f_n(x_1, x_2, \ldots, x_{n+1}) := \frac{1}{n} \sum_{i=1}^{n} f(x_i, x_{i+1}).$$

Bertelson-Gromov [2] proposed the study of this kind of function (in some more general context). Let $Cr(f_n) \subset \mathbb{R}$ be the set of all critical values of $f_n$. We are interested in the asymptotic behavior of $Cr(f_n)$ as $n \to \infty$. At first sight, the definition of $f_n$ looks simple and this problem seems easy. But if we try to compute the critical values of $f_n$ for some examples, then we soon realize that it is almost impossible to compute them exactly in general.

Set $m_n := \min_{x \in X^{n+1}} f_n(x)$, $M_n := \max_{x \in X^{n+1}} f_n(x)$ and $K := M_1 - m_1 \geq 0$. Let $m_\infty := \lim_{n \to \infty} m_n = \sup_n m_n$ and $M_\infty := \lim_{n \to \infty} M_n = \inf_n M_n$ (See Lemma 3.1 in Section 3). For a positive number $r$ and a closed set $A \subset \mathbb{R}$ we denote $B_r(A)$ as the closed $r$-neighborhood of $A$ in $\mathbb{R}$, i.e.,

$$B_r(A) := \{x \in \mathbb{R} | \exists a \in A : |x - a| \leq r\}.$$ 

Our main theorem is the following.

**Theorem 1.1.** $Cr(f_n)$ is "$K/n$-dense" in the closed interval $[m_n, M_n]$:

$$[m_n, M_n] \subset B_{K/n}(Cr(f_n)).$$

In particular,

$$[m_\infty, M_\infty] \subset B_{K/n}(Cr(f_n)).$$

Date: July 9, 2008.

The first and second authors were supported by Grant-in-Aid for JSPS Fellows (19·3177) and (19·1530) respectively from Japan Society for the Promotion of Science.
Corollary 1.2.

\[ \#Cr(f_n) \geq \frac{M_{\infty} - m_{\infty}}{2K} \cdot n, \]

where \( \#Cr(f_n) \) denotes the number of the critical values of \( f_n \).

In particular, if \( m_{\infty} \neq M_{\infty} \) then we have

\[ \liminf_{n \to \infty} \frac{\#Cr(f_n)}{n} > 0. \]

We have \( f_n(x, x, \cdots, x) = f(x, x) \) for \( x \in X \). Hence for any \( n \geq 1 \)

\[ m_n \leq \min_{x \in X} f(x, x) \leq \max_{x \in X} f(x, x) \leq M_n. \]

Therefore if \( f \) is not constant on the diagonal of \( X \times X \), then \( m_{\infty} \neq M_{\infty} \). Obviously the condition that \( f \) is not constant on the diagonal is a “generic condition”, i.e., the set of such \( f \)’s becomes an open dense subset in \( C^\infty(X \times X) \) (the space of smooth functions in \( X \times X \)) with respect to the \( C^\infty \)-topology. Therefore we get the following result.

Corollary 1.3. There exists an open dense subset \( U \) in \( C^\infty(X \times X) \) such that for any \( f \in U \) we have

\[ \liminf_{n \to \infty} \frac{\#Cr(f_n)}{n} > 0. \]

The following are (easy) examples where we can exactly calculate the critical values of \( f_n \) (cf. [2, Example 10.7]).

Example 1.4. Let \( X = S^1 \) and \( g : S^1 \to [0,1] \) be a smooth function whose critical values are \( 0, 1 \). Define \( f : S^1 \times S^1 \to [0,1] \) by \( f(\theta, \varphi) := g(\theta) \). Then \( m_{\infty} = 0, M_{\infty} = 1 \) and \( Cr(f_n) = \{0,1/n,2/n,\cdots,(n-1)/n,1\} \). Hence \( \#Cr(f_n) = n + 1 \).

On the other hand, if we define \( f : S^1 \times S^1 \to \mathbb{R} \) by \( f(\theta, \varphi) = g(\theta) - g(\varphi) \), then we have \( f(\theta_1, \cdots, \theta_{n+1}) = (g(\theta_1) - g(\theta_{n+1}))/n \). We have \( m_{\infty} = M_{\infty} = 0, Cr(f_n) = \{-1/n, 0, 1/n\} \) and \( \#Cr(f_n) = 3 \).

2. Support of cohomology classes

The essential ingredient of the proof of the main theorem is the notion “support of cohomology classes” used in Bertelson-Gromov [2] (see [2, Notation 4.1]). We review it in this section.

Let \( Y \) be a compact connected manifold and \( a \in H^*(Y) \) be a cohomology class. (In this paper the cohomology \( H^*(Y) \) is defined over a field. If \( Y \) is orientable, then we use the cohomology over \( \mathbb{R} \). If \( Y \) is not orientable, then we use the cohomology over \( \mathbb{Z}/2\mathbb{Z} \).) Let \( U \subset Y \) be an open set. We denote \( \text{supp} \ a \subset U \) if there exists an open set \( V \subset Y \) such that \( Y = U \cup V \) and \( a|_V = 0 \) in \( H^*(V) \). If we use the de Rham cohomology, then \( \text{supp} \ a \subset U \) means that there exists a differential form \( \alpha \) such that \( \text{supp} \ \alpha \subset U \) and \( a = [\alpha] \). The basic property of this notion is the following; For \( a, b \in H^*(Y) \) and open sets \( U, V \subset Y \), if \( \text{supp} \ a \subset U \) and \( \text{supp} \ b \subset V \), then we have \( \text{supp} \ (a \cup b) \subset U \cap V \).
Proof. There are open sets \( U' \) and \( V' \) such that \( Y = U \cup U' = V \cup V' \) and \( a|_{V'} = 0 \) in \( H^*(U') \) and \( b|_{V'} = 0 \) in \( H^*(V') \). From the exact sequence \( H^*(U' \cup V', U') \to H^*(U' \cup V') \to H^*(U') \), there exists \( \alpha \in H^*(U' \cup V', U') \) satisfying \( \alpha|_{U' \cup V'} = a|_{U' \cup V'} \) in \( H^*(U' \cup V') \). In the same way we have \( \beta \in H^*(U' \cup V', V') \) satisfying \( \beta|_{U' \cup V'} = b|_{U' \cup V'} \) in \( H^*(U' \cup V') \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^*(U' \cup V', U') \otimes H^*(U' \cup V', V') & \xrightarrow{\cup} & H^*(U' \cup V', U' \cup V') = 0 \\
\downarrow & & \downarrow \\
H^*(U' \cup V') \otimes H^*(U' \cup V') & \xrightarrow{\cup} & H^*(U' \cup V')
\end{array}
\]

Then \( (a \cup b)|_{U' \cup V'} = (\alpha \cup \beta)|_{U' \cup V'} = 0 \) in \( H^*(U' \cup V') \). Since \( Y = (U \cap V) \cup (U' \cup V') \), this shows \( \text{supp} (a \cup b) \subset U \cap V \). \( \square \)

The following is the most important example. (This is remarked in [2, Remark 10.8].)

Example 2.1. \( \omega \in H^{\dim Y}(Y) \) be a cohomology class of top degree. Then for any non-empty open set \( U \) in \( Y \) we have \( \text{supp} \omega \subset U \).

Lemma 2.2. Let \( c_1 \leq c_2 \) be two real numbers and \( \varphi : Y \to \mathbb{R} \) be a smooth function in \( Y \). For \( a, b \in H^*(Y) \), suppose that \( \text{supp} a \subset \varphi^{-1}(c_1, +\infty) \), \( \text{supp} b \subset \varphi^{-1}(-\infty, c_2) \) and \( a \cup b \neq 0 \). Then the closed interval \([c_1, c_2]\) contains a critical value of \( \varphi \).

Proof. Suppose that there are no critical values in \([c_1, c_2]\). Using the gradient flow of \( \varphi \), we can see that \( \text{supp} b \subset \varphi^{-1}(-\infty, c_2) \) implies \( \text{supp} b \subset \varphi^{-1}(-\infty, c_1) \). (See Milnor [3, Chapter 3, Theorem 3.1].) Then \( \text{supp} (a \cup b) \subset \varphi^{-1}(c_1, +\infty) \cap \varphi^{-1}(-\infty, c_1) = \emptyset \) and hence \( a \cup b = 0 \). (A similar argument is given in [2, p. 34].) \( \square \)

3. Proof of the main results

The argument in this section is partly suggested by the argument of [2, Example 10.7, Remark 10.8]. Set \( f'_n(x) := nf_n(x) = \sum_k f(x_k, x_{k+1}) \), \( m'_n := \min_x f'_n(x) \) and \( M'_n := \max_x f'_n(x) \). We have \( m_n = m'_n/n \) and \( M_n = M'_n/n \). The following lemma is almost trivial.

Lemma 3.1.

\[
m'_{n_1+n_2} \geq m'_{n_1} + m'_{n_2}, \quad M'_{n_1+n_2} \leq M'_{n_1} + M'_{n_2}.
\]

Therefore the limits of \( m_n \) and \( M_n \) exist and

\[
m_\infty = \lim_{n \to \infty} m_n = \sup_{n \geq 1} m_n, \quad M_\infty = \lim_{n \to \infty} M_n = \inf_{n \geq 1} M_n.
\]

Fix \( n > 0 \), and define \( t_k \ (k = 0, 1, \cdots, n) \) by \( t_0 := m_n, \ t_n := M_n \) and

\[
t_k := \frac{k}{n} M_k + \frac{n-k}{n} m_{n-k} = \frac{1}{n} (M'_k + m'_{n-k}) \quad (0 < k < n).
\]

Lemma 3.2. We have \( m_n = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = M_n \) and \( |t_{k+1} - t_k| \leq K/n \).
Proof. Let $M'_k = f'_k(p_1, \cdots, p_{k+1})$ and $m'_{n-k-1} = f'_{n-k-1}(q_1, \cdots, q_{n-k})$. We have

$$M'_{k+1} \geq f'_k(p_1, \cdots, p_{k+1}, q_1) = M'_k + f(p_{k+1}, q_1).$$

$$M'_{k+1} + m'_{n-k-1} \geq M'_k + f(p_{k+1}, q_1) + f'_{n-k-1}(q_1, \cdots, q_{n-k}),$$

$$= M'_k + f'_{n-k}(p_{k+1}, q_1, \cdots, q_{n-k}) \geq M'_k + m'_{n-k}.$$

Therefore $t_{k+1} \geq t_k$. From Lemma 3.1, $M'_{k+1} - M'_k \leq M'_1$ and $m'_{n-k-1} - m'_{n-k} \leq -m'_1$. Hence

$$t_{k+1} - t_k = \frac{1}{n}(M'_{k+1} - M'_k + m'_{n-k-1} - m'_{n-k}) \leq \frac{1}{n}(M'_1 - m'_1) = K/n.$$

\[ \square \]

Lemma 3.3. The closed interval $[t_k, t_{k+1}]$ contains a critical value of $f_n$.

Proof. $t_0 = m_n$ and $t_n = M_n$ are certainly critical values of $f_n$. Therefore we can assume $0 < k < n - 1$. Let $\varepsilon > 0$ be any positive number. Define $\pi_1 : X^{n+1} \to X^{k+1}$ and $\pi_2 : X^{n+1} \to X^{n-k}$ by $\pi_1(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{k+1})$ and $\pi_2(x_1, \cdots, x_{n+1}) = (x_{k+2}, \cdots, x_{n+1})$.

Let $p \in X^{k+1}$ be a point where $f_k$ attains its maximum ($f_k(p) = M_k$), and $\omega \in H^{(k+1)\dim X}(X^{k+1})$ be a non-zero cohomology class of top degree. For any open neighborhood $U$ of $p$ in $X^{k+1}$, we have $\text{supp } \omega \subset U$. Then $a := \pi_1^* \omega \in H^*(X^{n+1})$ satisfies $\text{supp } a \subset U \times X^{n-k}$. If we choose $U$ sufficiently small, then $(x_1, \cdots, x_{n+1}) \in U$ satisfies $f_k(x_1, \cdots, x_{k+1}) > M_k - n\varepsilon/k$. Hence $(x_1, \cdots, x_{n+1}) \in U \times X^{n-k}$ satisfies

$$f_n(x_1, \cdots, x_{n+1}) = \frac{1}{n}(f_k(x_1, \cdots, x_{k+1}) + (n-k)f_{n-k}(x_{k+1}, \cdots, x_{n+1})),$$

$$> \frac{1}{n}\{k(M_k - n\varepsilon/k) + (n-k)m_{n-k}\} = t_k - \varepsilon.$$

Therefore $a \subset U \times X^{n-k} \subset f_n^{-1}(t_{k+1} - \varepsilon, +\infty)$.

Let $q \in X^{n-k}$ be a point where $f_{n-k-1}$ attains its minimum ($f_{n-k-1}(q) = m_{n-k-1}$), and $\eta \in H^{(n-k)\dim X}(X^{n-k})$ be a non-zero cohomology class of top degree. For any open neighborhood $V$ of $q$ in $X^{n-k}$, we have $\text{supp } \eta \subset V$. If we choose $V$ sufficiently small, then we have $X^{k+1} \times V \subset f_n^{-1}(-\infty, t_{k+1} + \varepsilon)$. Then $b := \pi_2^* \eta \in H^*(X^{n+1})$ satisfies $\text{supp } b \subset X^{n-k} \times V \subset f_n^{-1}(-\infty, t_{k+1} + \varepsilon)$. Obviously we have $a \cup b \neq 0$. Then Lemma 2.2 implies that $[t_k - \varepsilon, t_{k+1} + \varepsilon]$ contains a critical value of $f_n$. We can take $\varepsilon > 0$ arbitrarily small and $\text{Cr}(f_n)$ is a compact set. Therefore $[t_k, t_{k+1}] \cap \text{Cr}(f_n) \neq \emptyset$. \[ \square \]

Proof of Theorem 1.1. Since $m_0 = t_0 \leq t_1 \leq \cdots \leq t_n = M_n$, for any $x \in [m_n, M_n]$ there is $k$ such that $x \in [t_k, t_{k+1}]$. From Lemma 3.3, there exists a critical value $c$ of $f_n$ in $[t_k, t_{k+1}]$.

$$|c - x| \leq |t_{k+1} - t_k| \leq K/n.$$

This shows $[m_n, M_n] \subset B_{K/n}(\text{Cr}(f_n))$. \[ \square \]
Proof of Corollary 1.2. We can assume \( \sharp Cr(f_n) < \infty \) and \( m_n \neq M_n \). Let 
\[
Cr(f_n) = \{ m_n = c_0 < c_1 < \cdots < c_a = M_n \}.
\]
From \([m_n, M_n] \subset B_{K/n}(Cr(f_n))\), we have \( |c_{k+1} - c_k| \leq 2K/n \). Hence
\[
M_n - m_n = \sum_{k=0}^{a-1} (c_{k+1} - c_k) \leq a \cdot 2K/n.
\]
Therefore
\[
\sharp Cr(f_n) = a + 1 \geq \frac{M_n - m_n}{2K} \cdot n + 1 \geq \frac{M_\infty - m_\infty}{2K} \cdot n.
\]

References


Tomohiro Fukaya
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
\textit{E-mail address:} tomo_xi@math.kyoto-u.ac.jp

Masaki Tsukamoto
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
\textit{E-mail address:} tukamoto@math.kyoto-u.ac.jp