ASYMPTOTIC DISTRIBUTION OF CRITICAL VALUES

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Abstract. Let X be a closed manifold and $f : X \times X \to \mathbb{R}$ be a smooth function. Define $f_n : X^{n+1} \to \mathbb{R}$ by $f_n(x_1, \cdots, x_{n+1}) := \frac{1}{n} \sum f(x_i, x_{i+1})$. We study the asymptotic distribution of the critical values of f_n as n goes to infinity.

1. Main results

Let X be a compact connected smooth manifold of positive dimension and $f : X \times X \to \mathbb{R}$ be a smooth function. For an integer $n \geq 1$ we define $f_n : X^{n+1} \to \mathbb{R}$ by

$$f_n(x_1, x_2, \cdots, x_{n+1}) := \frac{1}{n} \sum_{i=1}^{n} f(x_i, x_{i+1}). \tag{1}$$

Bertelson-Gromov [2] proposed the study of this kind of function (in some more general context). Let $Cr(f_n) \subset \mathbb{R}$ be the set of all critical *values* of f_n. We are interested in the asymptotic behavior of $Cr(f_n)$ as $n \to \infty$. At first sight, the definition of f_n looks simple and this problem seems easy. But if we try to compute the critical values of f_n for some examples, then we soon realize that it is almost impossible to compute them exactly in general.

Set $m_n := \min_{x \in X^{n+1}} f_n(x)$, $M_n := \max_{x \in X^{n+1}} f_n(x)$ and $K := M_1 - m_1 \geq 0$. Let $m_\infty := \lim_{n\to\infty} m_n = \sup_n m_n$ and $M_\infty := \lim_{n\to\infty} M_n = \inf_n M_n$ (See Lemma 3.1 in Section 3). For a positive number r and a closed set $A \subset \mathbb{R}$ we denote $B_r(A)$ as the closed r-neighborhood of A in $\mathbb{R}$, i.e.,

$$B_r(A) := \{x \in \mathbb{R} | \, \exists a \in A : |x - a| \leq r\}.$$

Our main theorem is the following.

Theorem 1.1. *$Cr(f_n)$ is "K/n-dense" in the closed interval $[m_n, M_n]$:*

$$[m_n, M_n] \subset B_{K/n}(Cr(f_n)).$$

In particular,

$$[m_\infty, M_\infty] \subset B_{K/n}(Cr(f_n)).$$

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Corollary 1.2.
$$\sharp Cr(f_n) \geq \frac{M_\infty - m_\infty}{2K} \cdot n,$$
where $\sharp Cr(f_n)$ *denotes the number of the critical values of* f_n.

In particular, if $m_\infty \neq M_\infty$ then we have
$$\liminf_{n\to\infty} \frac{\sharp Cr(f_n)}{n} > 0.$$
We have $f_n(x, x, \cdots, x) = f(x, x)$ for $x \in X$. Hence for any $n \geq 1$
$$m_n \leq \min_{x\in X} f(x, x) \leq \max_{x\in X} f(x, x) \leq M_n.$$
Therefore if f is not constant on the diagonal of $X \times X$, then $m_\infty \neq M_\infty$. Obviously the condition that f is not constant on the diagonal is a "generic condition", i.e., the set of such f's becomes an open dense subset in $\mathcal{C}^\infty(X \times X)$ (the space of smooth functions in $X \times X$) with respect to the $\mathcal{C}^\infty$-topology. Therefore we get the following result.

Corollary 1.3. *There exists an open dense subset* U *in* $\mathcal{C}^\infty(X \times X)$ *such that for any* $f \in U$ *we have*
$$\liminf_{n\to\infty} \frac{\sharp Cr(f_n)}{n} > 0.$$

The following are (easy) examples where we can exactly calculate the critical values of f_n (cf. [2, Example 10.7]).

Example 1.4. Let $X = S^1$ and $g : S^1 \to [0, 1]$ be a smooth function whose critical values are $0, 1$. Define $f : S^1 \times S^1 \to [0, 1]$ by $f(\theta, \varphi) := g(\theta)$. Then $m_\infty = 0$, $M_\infty = 1$ and $Cr(f_n) = \{0, 1/n, 2/n, \cdots, (n-1)/n, 1\}$. Hence $\sharp Cr(f_n) = n + 1$.

On the other hand, if we define $f : S^1 \times S^1 \to \mathbb{R}$ by $f(\theta, \varphi) = g(\theta) - g(\varphi)$, then we have $f(\theta_1, \cdots, \theta_{n+1}) = (g(\theta_1) - g(\theta_{n+1}))/n$. We have $m_\infty = M_\infty = 0$, $Cr(f_n) = \{-1/n, 0, 1/n\}$ and $\sharp Cr(f_n) = 3$.

2. Support of cohomology classes

The essential ingredient of the proof of the main theorem is the notion "support of cohomology classes" used in Bertelson-Gromov [2] (see [2, Notation 4.1]). We review it in this section.

Let Y be a compact connected manifold and $a \in H^*(Y)$ be a cohomology class. (In this paper the cohomology $H^*(Y)$ is defined over a field. If Y is orientable, then we use the cohomology over $\mathbb{R}$. If Y is not orientable, then we use the cohomology over $\mathbb{Z}/2\mathbb{Z}$.) Let $U \subset Y$ be a open set. We denote $\operatorname{supp} a \subset U$ if there exists an open set $V \subset Y$ such that $Y = U \cup V$ and $a|_V = 0$ in $H^*(V)$. If we use the de Rham cohomology, then $\operatorname{supp} a \subset U$ means that there exists a differential form α such that $\operatorname{supp} \alpha \subset U$ and $a = [\alpha]$. The basic property of this notion is the following; For $a, b \in H^*(Y)$ and open sets $U, V \subset Y$, if $\operatorname{supp} a \subset U$ and $\operatorname{supp} b \subset V$, then we have $\operatorname{supp}(a \cup b) \subset U \cap V$.

Proof. There are open sets U' and V' such that $Y = U \cup U' = V \cup V'$ and $a|_{U'} = 0$ in $H^*(U')$ and $b|_{V'} = 0$ in $H^*(V')$. From the exact sequence $H^*(U' \cup V', U') \to H^*(U' \cup V') \to H^*(U')$, there exists $\alpha \in H^*(U' \cup V', U')$ satisfying $\alpha|_{U' \cup V'} = a|_{U' \cup V'}$ in $H^*(U' \cup V')$. In the same way we have $\beta \in H^*(U' \cup V', V')$ satisfying $\beta|_{U' \cup V'} = b|_{U' \cup V'}$ in $H^*(U' \cup V')$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^*(U' \cup V', U') \otimes H^*(U' \cup V', V') & \xrightarrow{\cup} & H^*(U' \cup V', U' \cup V') = 0 \\ \downarrow & & \downarrow \\ H^*(U' \cup V') \otimes H^*(U' \cup V') & \xrightarrow{\cup} & H^*(U' \cup V') \end{array}$$

Then $(a \cup b)|_{U' \cup V'} = (\alpha \cup \beta)|_{U' \cup V'} = 0$ in $H^*(U' \cup V')$. Since $Y = (U \cap V) \cup (U' \cup V')$, this shows $\operatorname{supp}(a \cup b) \subset U \cap V$. □

The following is the most important example. (This is remarked in [2, Remark 10.8].)

Example 2.1. $\omega \in H^{\dim Y}(Y)$ be a cohomology class of top degree. Then for any non-empty open set U in Y we have $\operatorname{supp} \omega \subset U$.

Lemma 2.2. *Let $c_1 \leq c_2$ be two real numbers and $\varphi : Y \to \mathbb{R}$ be a smooth function in Y. For $a, b \in H^*(Y)$, suppose that $\operatorname{supp} a \subset \varphi^{-1}(c_1, +\infty)$, $\operatorname{supp} b \subset \varphi^{-1}(-\infty, c_2)$ and $a \cup b \neq 0$. Then the closed interval $[c_1, c_2]$ contains a critical value of φ.*

Proof. Suppose that there are no critical values in $[c_1, c_2]$. Using the gradient flow of φ, we can see that $\operatorname{supp} b \subset \varphi^{-1}(-\infty, c_2)$ implies $\operatorname{supp} b \subset \varphi^{-1}(-\infty, c_1)$. (See Milnor [3, Chapter 3, Theorem 3.1].) Then $\operatorname{supp}(a \cup b) \subset \varphi^{-1}(c_1, +\infty) \cap \varphi^{-1}(-\infty, c_1) = \emptyset$ and hence $a \cup b = 0$. (A similar argument is given in [2, p. 34].) □

3. Proof of the main results

The argument in this section is partly suggested by the argument of [2, Example 10.7, Remark 10.8]. Set $f'_n(x) := n f_n(x) = \sum_k f(x_k, x_{k+1})$, $m'_n := \min_x f'_n(x)$ and $M'_n := \max_x f'_n(x)$. We have $m_n = m'_n/n$ and $M_n = M'_n/n$. The following lemma is almost trivial

Lemma 3.1.

$$m'_{n_1+n_2} \geq m'_{n_1} + m'_{n_2}, \quad M'_{n_1+n_2} \leq M'_{n_1} + M'_{n_2}.$$

Therefore the limits of m_n and M_n exist and

$$m_\infty = \lim_{n\to\infty} m_n = \sup_{n\geq 1} m_n, \quad M_\infty = \lim_{n\to\infty} M_n = \inf_{n\geq 1} M_n.$$

Fix $n > 0$, and define t_k $(k = 0, 1, \cdots, n)$ by $t_0 := m_n$, $t_n := M_n$ and

$$t_k := \frac{k}{n} M_k + \frac{n-k}{n} m_{n-k} = \frac{1}{n}(M'_k + m'_{n-k}) \quad (0 < k < n).$$

Lemma 3.2. *We have $m_n = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = M_n$ and $|t_{k+1} - t_k| \leq K/n$.*

Proof. Let $M'_k = f'_k(p_1, \cdots, p_{k+1})$ and $m'_{n-k-1} = f'_{n-k-1}(q_1, \cdots, q_{n-k})$. We have

$$M'_{k+1} \geq f'_{k+1}(p_1, \cdots, p_{k+1}, q_1) = M'_k + f(p_{k+1}, q_1).$$

$$\begin{aligned} M'_{k+1} + m'_{n-k-1} &\geq M'_k + f(p_{k+1}, q_1) + f'_{n-k-1}(q_1, \cdots, q_{n-k}), \\ &= M'_k + f'_{n-k}(p_{k+1}, q_1, \cdots, q_{n-k}) \geq M'_k + m'_{n-k}. \end{aligned}$$

Therefore $t_{k+1} \geq t_k$. From Lemma 3.1, $M'_{k+1} - M'_k \leq M'_1$ and $m'_{n-k-1} - m'_{n-k} \leq -m'_1$. Hence

$$t_{k+1} - t_k = \frac{1}{n}(M'_{k+1} - M'_k + m'_{n-k-1} - m'_{n-k}) \leq \frac{1}{n}(M'_1 - m'_1) = K/n.$$

□

Lemma 3.3. *The closed interval $[t_k, t_{k+1}]$ contains a critical value of f_n.*

Proof. $t_0 = m_n$ and $t_n = M_n$ are certainly critical values of f_n. Therefore we can assume $0 < k < n-1$. Let $\varepsilon > 0$ be any positive number. Define $\pi_1 : X^{n+1} \to X^{k+1}$ and $\pi_2 : X^{n+1} \to X^{n-k}$ by $\pi_1(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{k+1})$ and $\pi_2(x_1, \cdots, x_{n+1}) = (x_{k+2}, \cdots, x_{n+1})$.

Let $p \in X^{k+1}$ be a point where f_k attains its maximum ($f_k(p) = M_k$), and $\omega \in H^{(k+1)\dim X}(X^{k+1})$ be a non-zero cohomology class of top degree. For any open neighborhood U of p in X^{k+1}, we have $\operatorname{supp}\omega \subset U$. Then $a := \pi_1^*\omega \in H^*(X^{n+1})$ satisfies $\operatorname{supp} a \subset U \times X^{n-k}$. If we choose U sufficiently small, then $(x_1, \cdots, x_{k+1}) \in U$ satisfies $f_k(x_1, \cdots, x_{k+1}) > M_k - n\varepsilon/k$. Hence $(x_1, \cdots, x_{n+1}) \in U \times X^{n-k}$ satisfies

$$\begin{aligned} f_n(x_1, \cdots, x_{n+1}) &= \frac{1}{n}(kf_k(x_1, \cdots, x_{k+1}) + (n-k)f_{n-k}(x_{k+1}, \cdots, x_{n+1})), \\ &> \frac{1}{n}\{k(M_k - n\varepsilon/k) + (n-k)m_{n-k}\} = t_k - \varepsilon. \end{aligned}$$

Therefore $\operatorname{supp} a \subset U \times X^{n-k} \subset f_n^{-1}(t_{k+1} - \varepsilon, +\infty)$.

Let $q \in X^{n-k}$ be a point where f_{n-k-1} attains its minimum ($f_{n-k-1}(q) = m_{n-k-1}$), and $\eta \in H^{(n-k)\dim X}(X^{n-k})$ be a non-zero cohomology class of top degree. For any open neighborhood V of q in X^{n-k}, we have $\operatorname{supp}\eta \subset V$. If we choose V sufficiently small, then we have $X^{k+1} \times V \subset f_n^{-1}(-\infty, t_{k+1} + \varepsilon)$. Then $b := \pi_2^*\eta \in H^*(X^{n+1})$ satisfies $\operatorname{supp} b \subset X^{n-k} \times V \subset f_n^{-1}(-\infty, t_{k+1} + \varepsilon)$. Obviously we have $a \cup b \neq 0$. Then Lemma 2.2 implies that $[t_k - \varepsilon, t_{k+1} + \varepsilon]$ contains a critical value of f_n. We can take $\varepsilon > 0$ arbitrarily small and $Cr(f_n)$ is a compact set. Therefore $[t_k, t_{k+1}] \cap Cr(f_n) \neq \emptyset$. □

Proof of Theorem 1.1. Since $m_0 = t_0 \leq t_1 \leq \cdots \leq t_n = M_n$, for any $x \in [m_n, M_n]$ there is k such that $x \in [t_k, t_{k+1}]$. From Lemma 3.3, there exists a critical value c of f_n in $[t_k, t_{k+1}]$.

$$|c - x| \leq |t_{k+1} - t_k| \leq K/n.$$

This shows $[m_n, M_n] \subset B_{K/n}(Cr(f_n))$. □

Proof of Corollary 1.2. We can assume $\sharp Cr(f_n) < \infty$ and $m_n \neq M_n$. Let

$$Cr(f_n) = \{m_n = c_0 < c_1 < \cdots < c_a = M_n\}.$$

From $[m_n, M_n] \subset B_{K/n}(Cr(f_n))$, we have $|c_{k+1} - c_k| \leq 2K/n$. Hence

$$M_n - m_n = \sum_{k=0}^{a-1}(c_{k+1} - c_k) \leq a \cdot 2K/n.$$

Therefore

$$\sharp Cr(f_n) = a + 1 \geq \frac{M_n - m_n}{2K} \cdot n + 1 \geq \frac{M_\infty - m_\infty}{2K} \cdot n.$$

□

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