

Brownian survival and Lifshitz tail in Frenkel disorder

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Abstract

We consider the annealed asymptotics for the survival probability of Brownian motion among randomly distributed traps. The configuration of traps is given by independent displacements of the lattice points. We determined the asymptotics for the logarithm of the survival probability up to multiplicative constant. As applications, we show the Lifshitz tail effect of the density of states of associated random Schrödinger operator and intermittency for the parabolic Anderson problem.

1 Introduction

We consider the annealed asymptotics for the survival probability of Brownian motion among randomly distributed traps. This problem for the Poissonian configuration of traps was firstly investigated by Donsker and Varadhan [3] and later by Sznitman [16] with generalization of the shape of each traps. Sznitman also generalized the configuration to Gibbsian point processes in [18].

In this article, we discuss another model where traps are attached around a randomly perturbed lattice. To be more precise, our process is the killed Brownian motion whose generator is

$$H_\xi = -\frac{1}{2}\Delta + \sum_{q \in \mathbb{Z}^d} W(\cdot - q - \xi_q), \quad (1)$$

where $(\xi_q)_{q \in \mathbb{Z}^d}$ is a collection of i.i.d. random vectors and W is a compactly supported nonnegative function. We allow W to take the value ∞ , which means imposing Dirichlet boundary condition on $\{W = \infty\}$. If $W \equiv \infty$ on its support, we call the traps *hard*. The random potential in (1) is a model of the “Frenkel disorder” in solid state physics and is called “random displacement model” in the theory of random Schrödinger operator. For such models with *bounded* displacements, there are some results concerning the spectral properties of the generator. Kirsch and Martinelli [9] discussed the existence of band gaps and Klopp [10] proved spectral localization in a semi-classical limit. More recently, Baker, Loss and Stolz [1] studied which configuration minimizes the bottom of the spectrum of (1). On the other hand, there are few results when displacements are *unbounded*, which is the object of this article. In the future paper [5], we will discuss the similar model with non-compactly supported potentials and negative potentials. We will also discuss in [5] the one dimensional result which is not discussed in the present article.

There are at least three important aspects of the survival probability. The first is as the partition function for the Brownian motion conditioned to survive. Actually, some detailed studies on the surviving Brownian motion were developed after [16]. (See e.g. [17] and [12] for path localization results.) The second is as the Laplace transform of the density of states. It is well known that one can derive the asymptotic behavior of the density of states near the

bottom of the spectrum from the survival asymptotics using an exponential Tauberian theorem. See e.g. Fukushima [4], Nakao [11], and Sznitman [16] for this way of studies on the density of states. The last is as the solution to the parabolic Anderson problem. The quenched survival probability for the Brownian motion starting from x is expressed by a Feynman-Kac functional. From the expression, we can identify it with the solution of the heat equation associated with H_ξ . Therefore, the annealed asymptotics of the survival probability gives the moment asymptotics of the solution.

Now we describe the settings precisely. Let $((\xi_q)_{q \in \mathbb{Z}^d}, \mathbb{P}_\theta)$ be \mathbb{R}^d -valued i.i.d. random variables with the distribution

$$\mathbb{P}_\theta(\xi_q \in dx) = N(d, \theta) \exp\{-|x|^\theta\} dx, \quad (2)$$

where $N(d, \theta)$ is the normalizing. Although our proof need such an assumption only on the tail, we assume ξ_q to have the precise density (2) for simplicity. The value of θ is related to the strength of the disorder: large θ implies the weak disorder and small θ implies the converse. Given random vectors, we define the perturbed lattice by $\xi = \sum_{q \in \mathbb{Z}^d} \delta_{q+\xi_q}$ and let $V(\cdot, \xi)$ be the random potential in (1). We denote by Ξ the sample space of ξ , the space of simple pure point measures on \mathbb{R}^d . We use the notation $((B_t)_{t \geq 0}, P_x)$ for the standard Brownian motion which is independent of ξ . The entrance time to a closed set F and the exit time from an open set U is denoted by H_F and T_U , respectively. Then the survival probability, our main object of this article, is described as follows:

$$S_t = \mathbb{E}_\theta \otimes E_0 \left[\exp \left\{ - \int_0^t V(B_s, \xi) ds \right\} \right]. \quad (3)$$

Intuitively, this quantity seems to decay exponentially since traps are distributed almost uniformly in the space. However, the decay rate should be slower than the periodic case since large *trap free regions* caused by disorder helps Brownian survival. We make a remark on the starting point of the Brownian motion before stating the results. Since our trap field is *not* \mathbb{R}^d -translation invariant but \mathbb{Z}^d -shift invariant, the asymptotics of the survival probability may depend on the starting point. However, it will be clear from the proof that all the results stated in this article do not depend on the starting point. For this reason, we shall only consider the Brownian motion starting from the origin.

We discuss the long time asymptotics of $\log S_t$, instead of S_t itself, in this article. The results are slightly different in two and higher dimension. We start with two dimensional case.

Theorem 1. *For $d = 2$ and $\theta > 0$, we have*

$$\log S_t \asymp -t^{\frac{2+\theta}{4+\theta}} (\log t)^{-\frac{\theta}{4+\theta}} \quad (4)$$

as $t \rightarrow \infty$. Here $f(t) \asymp g(t)$ means that $f(t)/g(t)$ is positive and bounded away from both 0 and ∞ .

It is worth mentioning that perturbed lattice has an interesting aspect in this case. Let $Z_{\mathbb{C}}$ be the flat chaotic analytic zero points (CAZP), the zero points of the Gaussian entire function $f_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} a_n z^n / \sqrt{n!}$ where $(a_n)_{n=0}^{\infty}$ is i.i.d. standard complex Gaussian variables. Sodin and Tsirelson [15] proved that there exists a collection of random variables $(\zeta_q)_{q \in \mathbb{Z}^2}$ such that $\sum_{q \in \mathbb{Z}^2} \delta_{\sqrt{\pi}q + \zeta_q}$ has the same distribution as $Z_{\mathbb{C}}$. Though $(\zeta_q)_{q \in \mathbb{Z}^2}$ is not an independent family, it is invariant under lattice shifts and its distribution has a Gaussian upper bound for the tail. Therefore, our model with parameter $\theta = 2$ can be regarded as a toy model for the flat CAZP. Indeed, Sodin and Tsirelson called our model “the second toy model” in [15]. Next, we state the result for higher dimensions.

Theorem 2. For $d \geq 3$ and $\theta > 0$, we have

$$\log S_t \asymp -t^{\frac{d^2+2\theta}{d^2+2d+2\theta}} \quad (5)$$

as $t \rightarrow \infty$.

Our results say that the survival probability decays faster than in Poissonian case. This is quite natural since the perturbed lattice is more *ordered* than the Poisson point process and therefore there are fewer large *trap free regions*. Furthermore, we have following simple but interesting observations.

Remarks. (*weak and strong disorder limits*)

- (i) $\lim_{\theta \rightarrow \infty} \frac{d^2+2\theta}{d^2+2d+2\theta} = 1$, which is the same power as in boundedly perturbed lattice traps.
- (ii) $\lim_{\theta \rightarrow 0} \frac{d^2+2\theta}{d^2+2d+2\theta} = \frac{d}{d+2}$, which is the same power as in the Poissonian traps (see [3]).

It is also possible to show the convergence of the law of ξ under \mathbb{P}_θ , to a boundedly perturbed lattice as $\theta \rightarrow \infty$, and to the Poisson point process as $\theta \rightarrow 0$. We shall prove these convergence results in the Appendix I. It will be clear from the proof that we can make ξ converge to the perfect lattice as $\theta \rightarrow \infty$ by replacing the density (2) of ξ_q by $\exp\{-(1+|x|)^\theta\}$. Since such a change does not affect any other results as mentioned before, our model can be regarded as an interpolation between perfect crystal and completely disordered media.

Let us briefly explain the construction of the article. We prove Theorems 1 and 2 in Section 2. Our strategy to prove the survival asymptotics is based on the idea in [16] rather than the one in [3]. The first step is a reduction to a certain variational problem. In this step, we use a coarse graining method which is slightly altered version of Sznitman's "method of enlargement of obstacles". The second step is the analysis of the variational problem. However, we reverse the order and analyze the variational problem first since it gives the correct scale which we need in the coarse graining. In Section 3, we give two applications of the survival asymptotics. The first is the Lifshitz tail effect on the density of states of H_ξ , which says that the spectrum of H_ξ is exponentially thin around the bottom. The second is the intermittency of the solution of the heat equation associated with H_ξ , which implies the strong inhomogeneity of the solution.

2 Proof of the survival asymptotics

2.1 Rough procedure

We explain the rough procedure of the proof in this section. First of all, we slightly modify the random potential as follows:

$$V(x, \xi) = \sum_{q \in \mathbb{Z}^d} h \cdot 1_{\{\xi(C(\epsilon q, \epsilon)) \geq 1\}} 1_{C(q, L)}(x) + \infty \cdot 1_{\mathcal{T}^c}(x), \quad (6)$$

where $C(y, l) = y + [-l/2, l/2]^d$ and $\mathcal{T} = (-t, t)^d$. This new potential bounds the original one from both above and below in \mathcal{T} by taking small ϵ and varying $h \in (0, \infty]$ and $L > 0$. Moreover, the restriction on \mathcal{T} does not affect the results since $P_0(T_{\mathcal{T}} \leq t)$ decays exponentially in t . Therefore it is sufficient to prove the survival asymptotics for the modified potential (6). Hereafter we take $\epsilon, h, L = 1$ so that $V(x, \xi) = 1_{\text{supp}V(\cdot, \xi)}(x)$ for simplicity. We start with

following obvious lower and upper bounds.

Lower bound: For any $U \in \mathcal{S} = \{\text{possible shape of } \text{supp } V(\cdot, \xi)\}_{\xi \in \Xi}$,

$$S_t \geq \mathbb{P}_\theta(\xi(U^c) = 0) E_0 \left[\exp \left\{ - \int_0^t 1_U(B_s) ds \right\}; T_{\mathcal{T}} > t \right]. \quad (7)$$

Upper bound: By summing over $U \in \mathcal{S}$, we obtain

$$\begin{aligned} S_t &\leq \sum_{U \in \mathcal{S}} \mathbb{P}_\theta(\xi(U^c) = 0) E_0 \left[\exp \left\{ - \int_0^t 1_U(B_s) ds \right\} \right] \\ &\leq \#\mathcal{S} \sup_{U \in \mathcal{S}} \mathbb{P}_\theta(\xi(U^c) = 0) E_0 \left[\exp \left\{ - \int_0^t 1_U(B_s) ds \right\}; T_{\mathcal{T}} > t \right]. \end{aligned} \quad (8)$$

Here we have $\#\mathcal{S} < \infty$ thanks to above modification and therefore the upper bound makes sense. However, there still remains a problem since we have too many configurations: $\#\mathcal{S} \sim 2^{td}$. We shall remedy this situation by reducing $\#\mathcal{S}$ to the small order using a coarse graining method. Once $\#\mathcal{S}$ is shown to be negligible, the proof of the survival asymptotics is reduced to the analysis of the variational problem

$$\sup_{U \in \mathcal{S}} \mathbb{P}_\theta(\xi(U^c) = 0) E_0 \left[\exp \left\{ - \int_0^t 1_U(B_s) ds \right\}; T_{\mathcal{T}} > t \right]. \quad (9)$$

As we announced in the introduction, we shall analyze this variational problem in Section 2.2 and give the coarse graining scheme in Section 2.3. Finally, we shall patch them together in Section 2.4 to complete the proof.

Remark. For $\log S_t$ with above modified potential, we can derive finer asymptotics than Theorems 1 and 2. We shall state it at the end of Section 2 since it requires the notation defined in the proof.

2.2 Analysis of the variational problem

In this section, we analyze the variational problem (9) and find correct scale. It is well known that the Brownian expectation part is controlled by the principal eigenvalue $\lambda_1(U)$ of the Dirichlet-Schrödinger operator $-1/2\Delta + 1_U$ in \mathcal{T} :

$$\log E_0 \left[\exp \left\{ - \int_0^t 1_U(B_s) ds \right\}; T_{\mathcal{T}} > t \right] \sim -\lambda_1(U)t \quad (t \rightarrow \infty). \quad (10)$$

On the other hand, we use next lemma to control the hole probability of perturbed lattice.

Lemma 1. *There exists $M_1(\epsilon) > 0$ ($\epsilon \in (0, 1)$) such that for any $U \in \mathcal{S}$,*

$$\mathbb{P}_\theta(\xi(U^c) = 0) \leq M_1(\epsilon)^{|U^c|} \exp \left\{ - (1 - \epsilon) \int_{U^c} \text{dist}(q, \partial U)^\theta dx \right\} \quad (11)$$

Especially, if $\{U_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ satisfies $\int_{U_n^c} \text{dist}(q, \partial U_n)^\theta dx / |U_n^c| \rightarrow \infty$, we have

$$\log \mathbb{P}_\theta(\xi(U_n^c) = 0) \leq - \int_{U_n^c} \text{dist}(q, \partial U_n)^\theta dx (1 + o(1)) \quad (12)$$

as $n \rightarrow \infty$.

Proof. Let $\epsilon \in (0, 1)$ be fixed. We consider the probability of a necessary condition:

$$\begin{aligned}
& \mathbb{P}_\theta \left(|\xi_q| > \text{dist}(q, \partial U) \text{ for all } q \in U^c \cap \mathbb{Z}^d \right) \\
&= \prod_{q \in U^c \cap \mathbb{Z}^d} \int_{|x| > \text{dist}(q, \partial U)} N(d, \theta) \exp\{-|x|^\theta\} dx \\
&= \prod_{q \in U^c \cap \mathbb{Z}^d} \sigma_d \int_{\text{dist}(q, \partial U)}^\infty N(d, \theta) r^{d-1} \exp\{-r^\theta\} dr \\
&\leq \prod_{q \in U^c \cap \mathbb{Z}^d} M_1(\epsilon) \int_{\text{dist}(q, \partial U)}^\infty (1 - \epsilon) \theta r^{\theta-1} \exp\{-(1 - \epsilon)r^\theta\} dr \\
&= M_1(\epsilon)^{\#U^c \cap \mathbb{Z}^d} \exp\left\{ -(1 - \epsilon) \sum_{q \in U^c \cap \mathbb{Z}^d} \text{dist}(q, \partial U)^\theta \right\}.
\end{aligned} \tag{13}$$

Here σ_d is the surface area of the unit sphere in \mathbb{R}^d and

$$M_1(\epsilon) = \frac{N(d, \theta) \sigma_d}{(1 - \epsilon) \theta} \sup_{r > 1/2} r^{d-\theta} \exp\{-\epsilon r^\theta\} < \infty. \tag{14}$$

Finally, it is not difficult to see

$$\sup_{U \in \mathcal{S}, q \in U^c \cap \mathbb{Z}^d} \left\{ \int_{C(q, 1)} \text{dist}(x, \partial U)^\theta dx - \text{dist}(q, \partial U)^\theta \right\} < \infty. \tag{15}$$

Thus we can replace the sum by the integral by making $M_1(\epsilon)$ larger if necessary. \square

Though we have not discussed above, it is also possible to show a lower bound similar to (12) under some additional assumptions. (See Proposition 7 in Section 2.4.) Thus we get roughly

$$\log \mathbb{P}_\theta(\xi(U) = 0) \sim - \int_U \text{dist}(x, \partial U)^\theta dx \tag{16}$$

for large and not very thin U . If we pretend to have (16) for all U , we can rewrite our variational problem as

$$\begin{aligned}
& \log \sup_{U \in \mathcal{S}} \mathbb{P}_\theta(\xi(U^c) = 0) E_0 \left[\exp \left\{ - \int_0^t 1_U(B_s) ds \right\}; T_T > t \right] \\
&\sim - \inf_{U \in \mathcal{S}} \left\{ \lambda_1(U) t + \int_U \text{dist}(x, \partial U)^\theta dx \right\}.
\end{aligned} \tag{17}$$

It is easy to see that the infimum of (17) is achieved by large U when t is large. Thus it is convenient to introduce a scaling $U = rU_r$ by a factor $r > 0$. Under this scaling, the right hand side of (17) takes the form

$$- \inf_{U_r \in \mathcal{S}_r} \left\{ \lambda_1^r(U_r) t r^{-2} + r^{d+\theta} \int_{U_r} \text{dist}(x, \partial U_r)^\theta dx \right\}. \tag{18}$$

Here $\mathcal{S}_r = \{r^{-1}U; U \in \mathcal{S}\}$ and $\lambda_1^r(U_r)$ is the principal eigenvalue of the scaled Dirichlet-Schrödinger operator $-1/2\Delta + r^2 1_{U_r}$ in $\mathcal{T}_r = r^{-1}\mathcal{T}$.

Now, if we only considered *regular* U_r 's for which

$$\lambda_1^r(U_r) \asymp 1 \quad \text{and} \quad \int_{U_r^c} \text{dist}(x, \partial U_r)^\theta dx \asymp 1, \quad (19)$$

then the optimal scale would be $r = t^{1/(d+\theta+2)}$. However, this scale gives wrong magnitude $t^{(d+\theta)/(d+\theta+2)}$. The key observation to find the correct scale is that we can easily decrease the value of the integral $\int_{U_r^c} \text{dist}(x, \partial U_r)^\theta dx$. For instance, let us consider a domain with many tiny holes

$$U_r^c = (-1, 1)^d \setminus \bigcup_{q \in \mathbb{Z}^d} C(\delta(r)q, r^{-1}). \quad (20)$$

Then we have

$$\int_{U_r^c} \text{dist}(x, \partial U_r)^\theta dx \asymp \delta(r)^\theta, \quad (21)$$

which goes to 0 if $\delta(r) \rightarrow 0$, as $r \rightarrow \infty$. Of course, such a domain with too small $\delta(r)$ have large principal eigenvalue $\lambda_1^r(U_r)$ and thus should be unimportant for the variational problem (18). What we want to know is how small we can take $\delta(r)$ while keeping the control of $\lambda_1^r(U_r)$. The solution for this example with hard traps (i.e. $h = \infty$) was given by Rauch and Taylor [13]. They showed that

$$\delta(r) = \begin{cases} (\log r)^{-\frac{1}{2}} & (d = 2), \\ r^{-\frac{d-2}{d}} & (d \geq 3), \end{cases} \quad (22)$$

is critical for whether $\lambda_1^r(U_r) \rightarrow \infty$ or not. This critical regimes are called ‘‘constant capacity regime’’ (see Section 3.2.B in [19]). The next proposition is a generalization of above criticality.

Proposition 3. *Let $\delta(r)$ be as in (22). There exists a function $M_2(\epsilon) \rightarrow \infty$ ($\epsilon \rightarrow 0$) such that if $U_r \subset \mathcal{S}_r$ satisfies*

$$\# \left\{ q \in U_r^c \cap \frac{1}{r} \mathbb{Z}^d; \text{dist}(q, \partial U_r) \geq \epsilon \delta(r) \right\} < \epsilon r^d, \quad (23)$$

then $\lambda_1^r(U_r) > M_2(\epsilon)$. In particular, we have

$$\inf_{r \geq 1, U \in \mathcal{S}} \left\{ \lambda_1^r(U_r) + \delta(r)^{-\theta} \int_{U_r^c} \text{dist}(x, \partial U_r)^\theta dx \right\} > 0. \quad (24)$$

Proof. We first recall that the principal eigenvalue can be expressed by the Dirichlet form

$$\lambda_1^r(U_r) = \int_{\mathcal{T}_r} \frac{1}{2} |\nabla \psi_r|^2(x) + r^2 1_{U_r}(x) \psi_r^2(x) dx \quad (25)$$

using associated L^2 -normalized eigenfunction ψ_r . Our basic strategy is estimating the right hand side by patching local estimates. For the local estimates, we use following lemma.

Lemma 2. *There exists $c_1(d) > 0$ such that*

$$\frac{1}{\|\phi\|_2^2} \int_{C(\epsilon\delta(r)i, 2\epsilon\delta(r))} \frac{1}{2} |\nabla \phi|^2(x) + r^2 1_{C(y, \frac{1}{r})}(x) \phi^2(x) dx \geq c_1(d) \epsilon^{-d}. \quad (26)$$

for any $i \in \mathbb{Z}^d$, $C(y, \frac{1}{r}) \subset C(\epsilon\delta(r)i, 2\epsilon\delta(r))$, $\epsilon > 0$ and $\phi \in W^{1,2}(C(\epsilon\delta(r)i, 2\epsilon\delta(r)))$.

Proof. This estimate can be found in Theorem 1.3 of [2]. We also refer the reader to Taylor's earlier work [20] for the case $d \geq 3$. \square

Now we show how to patch the local estimates. Let $\epsilon > 0$ be small and $\mathcal{I}(r)$ be the collection of $i \in \mathbb{Z}^d$ for which $C(\epsilon\delta(r)i, \epsilon\delta(r))$ intersects both U_r and U_r^c . Then, for large r , each $C(\epsilon\delta(r)i, 2\epsilon\delta(r))$ ($i \in \mathcal{I}(r)$) contains at least one $1/r$ -box $\subset U_r$. Therefore for all $i \in \mathcal{I}(r)$, we have

$$\frac{\int_{C(\epsilon\delta(r)i, 2\epsilon\delta(r))} \frac{1}{2} |\nabla \psi_r|^2(x) + r^2 1_{U_r}(x) \psi_r^2(x) dx}{\int_{C(\epsilon\delta(r)i, 2\epsilon\delta(r))} \psi_r^2(x) dx} \geq c_1(d) \epsilon^{-d} \quad (27)$$

by using Lemma 2 with $\phi = \psi_r|_{C(\epsilon\delta(r)i, 2\epsilon\delta(r))}$. Moreover, since there exists $m(d) \in \mathbb{N}$ such that any $x \in \mathbb{R}^d$ is contained in at most $m(d)$ different $C(\epsilon\delta(r)i, 2\epsilon\delta(r))$, we find

$$\begin{aligned} & \int_{\mathcal{I}} \frac{1}{2} |\nabla \psi_r|^2(x) + r^2 1_{U_r}(x) \psi_r^2(x) dx \\ & \geq m(d)^{-1} \sum_{i \in \mathcal{I}(r)} \int_{C(\epsilon\delta(r)i, 2\epsilon\delta(r))} \frac{1}{2} |\nabla \psi_r|^2(x) + r^2 1_{U_r}(x) \psi_r^2(x) dx. \end{aligned} \quad (28)$$

On the other hand, it is easy to see that

$$\bigcup_{q \in U_r^c \cap \frac{1}{r}\mathbb{Z}^d; \text{dist}(q, \partial U_r) < \epsilon\delta(r)} C(q, \frac{1}{r}) \subset \bigcup_{i \in \mathcal{I}(r)} C(\epsilon\delta(r)i, 2\epsilon\delta(r)) \quad (29)$$

for large r . From this and the assumption (23), it follows

$$\left| \mathcal{I}_r \setminus \bigcup_{i \in \mathcal{I}(r)} C(\epsilon\delta(r)i, 2\epsilon\delta(r)) \right| \leq \epsilon \quad (30)$$

when r is sufficiently large. Therefore,

$$1 = \|\psi_r\|_2^2 \leq \sum_{i \in \mathcal{I}(r)} \int_{C(\epsilon\delta(r)i, 2\epsilon\delta(r))} \psi_r^2(x) dx + \|\psi_r\|_\infty^2 \epsilon. \quad (31)$$

We consider the case $\|\psi_r\|_\infty \leq \epsilon^{-1/4}$ first. In this case, we have

$$\lambda_1^r(U_r) \geq \frac{m(d)^{-1} c_1(d) \epsilon^{-d} \sum_{i \in \mathcal{I}(r)} \int_{C(\epsilon\delta(r)i, 2\epsilon\delta(r))} \psi_r^2(x) dx}{\sum_{i \in \mathcal{I}(r)} \int_{C(\epsilon\delta(r)i, 2\epsilon\delta(r))} \psi_r^2(x) dx + \epsilon^{1/2}} \quad (32)$$

by substituting (28) and (31) into (25). The right hand side is greater than $(2m(d))^{-1} c_1(d) \epsilon^{-d}$ when $\epsilon \leq 1/4$. Next, we consider the case $\|\psi_r\|_\infty > \epsilon^{-1/4}$. This case is easier since we know an L^∞ -bound for the normalized eigenfunction (see (3.1.55) of [19])

$$\|\psi_r\|_\infty \leq c_2(d) \lambda_1^r(U_r)^{d/4} \quad (33)$$

which gives $\lambda_1^r(U_r) \geq c_2(d)^{-4/d} \epsilon^{-1/d}$. Combining the estimates in the two cases, we get

$$\lambda_1^r(U_r) \geq ((2m(d))^{-1} c_1(d) \epsilon^{-d}) \wedge (c_2(d)^{-4/d} \epsilon^{-1/d}) \quad (34)$$

and the proof is finished. \square

From this proposition, we know that the correct scale r should satisfy $tr^{-2} \asymp r^{d+\theta}\delta(r)^\theta$. It can be written by simple functions of t as follows:

$$r = \begin{cases} t^{\frac{1}{4+\theta}} (\log t)^{\frac{\theta}{8+2\theta}} & (d = 2), \\ t^{\frac{d}{d^2+2d+2\theta}} & (d \geq 3). \end{cases} \quad (35)$$

For these scales, tr^{-2} gives the correct magnitudes

$$tr^{-2} = \begin{cases} t^{\frac{2+\theta}{4+\theta}} (\log t)^{-\frac{\theta}{4+\theta}} & (d = 2), \\ t^{\frac{d^2+2\theta}{d^2+2d+2\theta}} & (d \geq 3). \end{cases} \quad (36)$$

2.3 Coarse graining

In this section, we give the coarse graining scheme which reduces the combinatorial complexity of configurations by replacing dense traps by a large box-shaped traps. Throughout this section, we are dealing with scaled traps with the correct scale r in (35). The scaled configuration of points is denoted by ξ_r .

We take a positive number $\eta \in (0, 1)$ so small as to satisfy

$$\eta^2 + \left(\frac{d-2}{2} + \frac{\theta}{d} \right) \eta < \frac{\theta}{d^2} \quad (37)$$

and let

$$\gamma = \frac{d-2}{d} + \frac{2\eta}{d} < 1. \quad (38)$$

We further introduce notation concerning diadic decomposition of \mathbb{R}^d . Let \mathcal{I}_k be the collection of indices of the form

$$\dot{i} = (i_0, i_1, \dots, i_k) \in \mathbb{Z}^d \times (\{0, 1\}^d)^k. \quad (39)$$

We associate to above index \dot{i} a box:

$$C_{\dot{i}} = q_{\dot{i}} + 2^{-k}[0, 1]^d \quad \text{where} \quad q_{\dot{i}} = i_0 + 2^{-1}i_1 + \dots + 2^{-k}i_k. \quad (40)$$

For $\dot{i} \in \mathcal{I}_k$ and $k' \leq k$, we define the truncation

$$[\dot{i}]_{k'} = (i_0, i_1, \dots, i_{k'}). \quad (41)$$

The notation $\dot{i} \preceq \dot{i}'$ means that \dot{i} is a truncation of \dot{i}' . Finally, we introduce

$$n_\beta(r) = \left\lceil \beta \frac{\log r}{\log 2} \right\rceil \quad (42)$$

for $\beta > 0$ so that $2^{-n_\beta-1} < r^{-\beta} \leq 2^{-n_\beta}$.

Now we give the precise definition of ‘‘dense traps’’ in the first paragraph.

Definition 1. We call C_q ($q \in \mathbb{Z}^d$) a density box if all $C_{\dot{i}}$'s ($\dot{i} \in \mathcal{I}_{n_\beta}, q \preceq \dot{i}$) satisfy following:

$$\text{for at least half of } \dot{i}' \succeq \dot{i} \ (\dot{i}' \in \mathcal{I}_{n_\beta}), \ q_{\dot{i}'} + 2^{-n_\beta-1}[0, 1]^d \text{ contain a point of } \xi_r. \quad (43)$$

The union of all density boxes is denoted by $\underline{\mathcal{D}}_r(\xi)$.

In [19], Sznitman defined density boxes in a different way and proved that they can be replaced by hard traps. We shall prove that our density set is a subset of Sznitman's one to use the result in [19]. We start by recalling Sznitman's definition of the density set and the result on the principal eigenvalue. For $\xi_r = \sum_q \delta_{x_q}$ and $\dot{u} \in \mathcal{I}_k$,

$$K_{\dot{u}} = 2^k \left(\bigcup_{x_q \in C_{\dot{u}}} \overline{B}(x_q, \sqrt{d}/r) \right) \quad (44)$$

is called the skeleton of traps. Sznitman defined the density box as follows:

Definition 2. (pp. 150-152 in [19]) $C_{\dot{u}}$ ($\dot{u} \in \mathcal{I}_{n_\gamma}$) is called a density box if the quantitative Wiener criterion:

$$\sum_{1 \leq k \leq n_\gamma} \text{cap}(K_{[\dot{u}]_k}) \geq \delta n_\gamma. \quad (45)$$

holds for some $\delta > 0$. Here $\text{cap}(\cdot)$ denotes the capacity relative to $1 - \Delta/2$ when $d = 2$ and $-\Delta/2$ when $d \geq 3$. The union of all density boxes is denoted by $\mathcal{D}_r(\xi)$.

The next theorem enables us to replace density boxes by large box-shaped traps.

Spectral control. (Theorem 4.2.3 in [19]) There exists $\rho > 0$ such that for all $M > 0$ and sufficiently large r ,

$$\sup_{\xi \in \Xi} (\lambda_1^r(r^{-1} \text{supp } V(\cdot, \xi), \mathcal{R}_r(\xi)) \wedge M - \lambda_1^r(r^{-1} \text{supp } V(\cdot, \xi)) \wedge M) \leq r^{-\rho}, \quad (46)$$

where $\mathcal{R}_r(\xi) = \mathcal{I}_r \setminus \mathcal{D}_r(\xi)$ and $\lambda_1^r(U, R)$ denotes the principal eigenvalue of Dirichlet-Schrödinger operator $-1/2\Delta + r^2 \cdot 1_U$ in R .

As announced before, we show the next proposition to apply this theorem to the density set.

Proposition 4. $\underline{\mathcal{D}}_r(\xi) \subset \mathcal{D}_r(\xi)$. Accordingly, $\underline{\mathcal{R}}_r(\xi) \stackrel{\text{def}}{=} \mathcal{I}_r \setminus \underline{\mathcal{D}}_r(\xi) \supset \mathcal{R}_r(\xi)$

Proof. Let C_q be a density box. We check the quantitative Wiener criterion (45) for all $\dot{u} \succeq q$ ($\dot{u} \in \mathcal{I}_{n_\gamma}$) by showing

$$\text{cap}(K_{[\dot{u}]_k}) \geq c_3(d) \quad \text{for all } k \leq n_{\eta\gamma}. \quad (47)$$

To get the lower bound for the capacity, we use following variational characterization:

$$\text{cap}(K) = \sup \left\{ \left(\iint g(x, y) \nu(dx) \nu(dy) \right)^{-1}; \nu \in \mathcal{M}_1(K) \right\} \quad (48)$$

where $\mathcal{M}_1(K)$ denotes the set of probability measure supported on K and $g(\cdot, \cdot)$ the Green function corresponding to $1 - \Delta/2$ when $d = 2$ and to $-\Delta/2$ when $d \geq 3$. Thus it suffices to find a $\nu_k \in \mathcal{M}_1(K_{[\dot{u}]_k})$ satisfying

$$\iint g(x, y) \nu_k(dx) \nu_k(dy) \leq c_3(d)^{-1} \quad (49)$$

for each $k \leq n_{\eta\gamma}$.

Now, note that (43) remains valid for $[\dot{u}]_k$ instead of $\dot{u} \in \mathcal{I}_{n_\gamma}$ as long as $k \leq n_{\eta\gamma}$. Therefore for such k , we can find a collection of points

$$\{x_m \in q_{\dot{u}_m} + 2^{-n_\gamma-1}[0, 1]^d; \dot{u}_m \in \mathcal{I}_{n_\gamma} \text{ are distinct.}\}_{1 \leq m \leq n} \subset \text{supp } \xi_r \quad (50)$$

whose cardinality $n \geq 2^{d(n_\gamma - k) - 1}$. We denote by e_m and cap_m respectively the equilibrium measure and the capacity of $2^k \bar{B}(x_m, \sqrt{d}/r)$ and let

$$\nu_k = \frac{\sum_{m=1}^n e_m}{\sum_{m=1}^n \text{cap}_m} \in \mathcal{M}_1(K_{[\hat{u}]_k}). \quad (51)$$

Let us show this ν_k satisfies (49). We use the fact $\iint g(x, y) e_m(dx) e_m(dy) = \text{cap}_m$ to obtain

$$\begin{aligned} & \iint g(x, y) \nu_k(dx) \nu_k(dy) \\ &= \left(\sum_{m=1}^n \text{cap}_m \right)^{-2} \left(\sum_{m=1}^n \iint g(x, y) e_m(dx) e_m(dy) + \sum_{l \neq m} \iint g(x, y) e_l(dx) e_m(dy) \right) \\ &\leq \left(\sum_{m=1}^n \text{cap}_m \right)^{-1} + \text{const}(d) \iint_{(0,1)^d \times (0,1)^d} g(x, y) dx dy. \end{aligned} \quad (52)$$

Since the last term is a constant depending only on d , it suffices for (49) to show $\sum_{m=1}^n \text{cap}_m \rightarrow \infty$ ($r \rightarrow \infty$). If we note that cap_m is just the capacity of a ball with radius $2^k \sqrt{d}/r$, we find

$$\sum_{m=1}^n \text{cap}_m \geq \begin{cases} c_4(d=2)(\log(2^{-k}r))^{-1} 2^{d(n_\gamma - k) - 1} & (d=2), \\ c_4(d)(2^k/r)^{d-2} 2^{d(n_\gamma - k) - 1} & (d \geq 3). \end{cases} \quad (53)$$

When $d \geq 3$ and $1 \leq k \leq n_\gamma$, the right hand side is larger than

$$\begin{aligned} c_4(d)r^{2-d}2^{dn_\gamma - 2k - 1} &\geq c_4(d)r^{d-2+d\gamma - 2n_\gamma} / 8 \\ &\geq c_4(d)r^{2\gamma(1-\eta)} / 8 \\ &\rightarrow \infty \quad (r \rightarrow \infty). \end{aligned} \quad (54)$$

Here we have used $2^{-n_\beta - 1} < r^{-\beta} \leq 2^{-n_\beta}$ for $\beta > 0$ in the first inequality. The case $d = 2$ can be treated by the same way and the proof of Proposition 4 is completed. \square

Now we turn on to the estimate on the number of non-density boxes in \mathcal{T}_r . It is clear from the definition that the number should be very small. However, we need a quantitative estimate for the coarse graining to go well. We pick a positive parameter

$$\chi \in \left(2\eta^2 + \left(d - 2 + \frac{2\theta}{d} \right) \eta, \frac{2\theta}{d^2} \right) \quad (55)$$

so that

$$d(1 - \eta\gamma) + (1 - \gamma)\theta + \chi > d + \frac{2\theta}{d}, \quad (56)$$

$$d(1 + \chi) < d + \frac{2\theta}{d}. \quad (57)$$

It is easy to see from (37) that such a choice of χ is possible. Thanks to the relation (56), the right hand side of the next proposition is

$$o\left(\exp\left\{-r^{d+\frac{2\theta}{d}}\right\}\right) = o\left(\exp\left\{-t^{\frac{d^2+2\theta}{d^2+2d+2\theta}}\right\}\right) \quad (58)$$

Proposition 5.

$$\mathbb{P}_\theta(|\underline{\mathcal{R}}_r(\xi)| \geq r^\chi) \leq \exp \left\{ -c_5(d)r^{d(1-\eta\gamma)+(1-\gamma)\theta+\chi} \right\}. \quad (59)$$

Proof. Throughout the proof, $c_5(d) > 0$ is a constant whose value may change line by line. We start with an estimate on the probability of $C_q \not\subset \underline{\mathcal{D}}_r(\xi)$. To this end, we consider the following necessary condition:

$$\begin{aligned} &\text{There exists a } \dot{u} \succeq q \ (\dot{u} \in \mathcal{I}_{n_{\eta\gamma}}) \text{ such that for at least half } \dot{u}' \succeq \dot{u} \ (\dot{u}' \in \mathcal{I}_{n_\gamma}), \\ &r^{-1}q + r^{-1}\xi_q \notin q_{\dot{u}'} + 2^{-n_\gamma-1}[0, 1]^d \text{ for all } r^{-1}q \in r^{-1}\mathbb{Z}^d \cap (q_{\dot{u}'} + 2^{-n_\gamma-1}[0, 1]^d). \end{aligned} \quad (60)$$

Note that the condition in the second line is independent in $\dot{u}' \in \mathcal{I}_{n_\gamma}$. From this and a calculation similar to the proof of Lemma 1, it follows

$$\begin{aligned} \mathbb{P}_\theta(C_q \not\subset \underline{\mathcal{D}}_r(\xi)) &\leq 2^{dn_{\eta\gamma}} \left(\frac{2^{d(n_\gamma-n_{\eta\gamma})}}{2^{d(n_\gamma-n_{\eta\gamma})-1}} \right) \exp \left\{ -c_5(d)r^{(1-\gamma)(d+\theta)} \right\}^{2^{d(n_\gamma-n_{\eta\gamma})-1}} \\ &\leq \exp \left\{ -c_5(d)r^{(1-\gamma)(d+\theta)+d\gamma(1-\eta)} \right\} \end{aligned} \quad (61)$$

for large r . Since the condition (60) itself is independent in $q \in \mathbb{Z}^d$, we have

$$\begin{aligned} \mathbb{P}_\theta(|\mathcal{I}_r \setminus \underline{\mathcal{D}}_r(\xi)| \geq r^\chi) &\leq (2t)^{dr^\chi} \exp \left\{ -c_5(d)r^{(1-\gamma)(d+\theta)+d\gamma(1-\eta)} \right\}^{r^\chi} \\ &\leq \exp \left\{ -c_5(d)r^{d(1-\eta\gamma)+(1-\gamma)\theta+\chi} \right\}, \end{aligned} \quad (62)$$

which is the desired estimate. \square

Finally, we bound the cardinality of

$$\begin{aligned} \underline{\mathcal{S}}_r &= \left\{ (\underline{\mathcal{R}}_r(\xi), r^{-1}\text{supp } V(\cdot, \xi) \cap \underline{\mathcal{R}}_r(\xi)); \right. \\ &\quad \left. \xi \in \Xi, \underline{\mathcal{R}}_r(\xi) \text{ is connected, } |\underline{\mathcal{R}}_r(\xi)| < r^\chi \right\}. \end{aligned} \quad (63)$$

An elementary counting shows that $\#\underline{\mathcal{S}}_r$ is at most

$$\begin{aligned} r^\chi (2t)^{dr^\chi} (2^{r^d})^{r^\chi} &= \exp \left\{ r^{d+\chi} \log 2(1 + o(1)) \right\} \\ &= \exp \left\{ o \left(t^{\frac{d^2+2\theta}{d^2+2d+2\theta}} (\log t)^{-\frac{\theta}{4+\theta}} \right) \right\}, \end{aligned} \quad (64)$$

where the second line comes from the relation (57).

2.4 Patching estimates

We complete the proof of survival asymptotics in this section. Throughout this section, we use the correct scale r in (35). Let $\epsilon > 0$ be an arbitrary small number and

$$M_r = \inf_{(R_r, U_r) \in \underline{\mathcal{S}}_r} \left\{ \lambda_1^r(U_r, R_r) + \delta(r)^{-\theta} \int_{R_r \setminus U_r} \text{dist}(x, \partial(R_r \setminus U_r))^\theta dx \right\}. \quad (65)$$

We know $\inf_{r \geq 1} M > 0$ from Proposition 3. Moreover, we can prove $\sup_{r \geq 1} M < \infty$ by substituting the punched domain (20) with (22) to $R_r \setminus U_r$. We postpone the proof of this fact to the Appendix II.

Upper bound: We use (3.1.9) of [19] which claims that for bounded open $U \subset \mathbb{R}^d$ and bounded function $V \geq 0$,

$$\begin{aligned} & \sup_{x \in U} E_x \left[\exp \left\{ - \int_0^t V(B_s) ds \right\}; T_U > t \right] \\ & \leq c(d)(1 + (\lambda_V(U)t)^{d/2}) \exp \{-\lambda_V(U)t\}, \end{aligned} \quad (66)$$

where $\lambda_V(U)t$ is the principal eigenvalue of the Dirichlet-Schrödinger operator $-1/2\Delta + V$ in U . It follows from this result that

$$E_0 \left[\exp \left\{ - \int_0^t V(B_s) ds \right\}; T_U > t \right] \leq c(d, \epsilon) \exp \{-(1 - \epsilon)\lambda_V(U)t\}, \quad (67)$$

for any $\epsilon \in (0, 1)$, where $c(d, \epsilon) = \sup_{\lambda > 0} c(d)(1 + \lambda^{d/2}) \exp \{-\epsilon\lambda\}$. Then, using Spectral control (46) and Proposition 5, we have

$$\begin{aligned} S_t & \leq c(d, \epsilon) \mathbb{E}_\theta [\exp \{-(1 - \epsilon)\lambda_1(\text{supp } V(\cdot, \xi))t\}] \\ & \leq c(d, \epsilon) \mathbb{E}_\theta [\exp \{-(1 - \epsilon)(\lambda_1^r(r^{-1}\text{supp } V(\cdot, \xi), \underline{\mathcal{R}}_r(\xi)) \wedge M_r - r^{-\rho})tr^{-2}\}]; \\ & \quad |\underline{\mathcal{R}}_r(\xi)| < r^\chi] + \mathbb{P}_\theta(|\underline{\mathcal{R}}_r(\xi)| \geq r^\chi) \\ & \leq c(d, \epsilon) \#\underline{\mathcal{S}}_r \sup_{(R_r, U_r) \in \underline{\mathcal{S}}_r} \mathbb{P}_\theta(\xi_r(R_r \setminus U_r) = 0) \\ & \quad \times \exp \{-(1 - \epsilon)(\lambda_1^r(U_r, R_r) \wedge M_r - r^{-\rho})tr^{-2}\} + o\left(\exp\left\{-t^{\frac{d^2+2\theta}{d^2+2d+2\theta}}\right\}\right) \end{aligned} \quad (68)$$

for large t , where we have used the fact that the principal eigenvalue is the infimum of that over the connected components to assume R_r to be connected. Since the factor $\#\underline{\mathcal{S}}_r$ (by (64)) and the second term is negligible compared with the results, we focus on the variational problem. Before applying Lemma 1 to the hole probability term, we see that $R_r \setminus U_r$ is *bulky* when $(R_r, U_r) \in \underline{\mathcal{S}}_r$. (Note that the right hand side of following proposition is much larger than $r^{-\theta}|R_r \setminus U_r|$ thanks to (37).)

Proposition 6. *For any $(R_r, U_r) \in \underline{\mathcal{S}}_r$, let $W_r = R_r \setminus U_r$. Then we have*

$$\int_{W_r} \text{dist}(x, \partial W_r)^\theta dx \geq c_6(d, \theta) r^{-\gamma(\theta+d\eta)} |W_r| \quad (69)$$

for large r .

Proof. By the definition of the density box, each $C_q \subset R_r$ contains a $C_{\dot{a}}$ ($\dot{a} \in \mathcal{I}_{n_\eta\gamma}$) such that half of $\{q_{\dot{a}'} + 2^{-n_\gamma-1}[1/4, 3/4]^d\}_{\dot{a} \leq \dot{a}' \in \mathcal{I}_{n_\eta\gamma}}$ do not intersect with U for large r . Therefore, the number of such $q_{\dot{a}'} + 2^{-n_\gamma-1}[1/4, 3/4]^d$ in the whole R_r is larger than $2^{-d-1}2^{dn_\gamma-dn_\eta\gamma}|R_r|$. Thus we find

$$\begin{aligned} & \int_{W_r} \text{dist}(x, \partial W_r)^\theta dx \\ & \geq 2^{-d-1}2^{dn_\gamma-dn_\eta\gamma}|R_r| \int_{2^{-n_\gamma-1}[1/4, 3/4]^d} \text{dist}(x, \partial W_r)^\theta dx \\ & \geq c_6(d, \theta) r^{d\gamma(1-\eta)} r^{-\gamma(d+\theta)} |W_r|, \end{aligned} \quad (70)$$

which is the desired inequality. \square

Using the relation $tr^{-2} = r^{d+\theta}\delta(r)^\theta$, Lemma 1, and Proposition 6, we obtain

$$\begin{aligned}
& \frac{1}{tr^{-2}} \log \sup_{(R_r, U_r) \in \underline{\mathcal{S}}_r} \mathbb{P}_\theta(\xi_r(W_r) = 0) \exp \left\{ -(1-\epsilon)(\lambda_1^r(U_r, R_r) \wedge M_r - r^{-\rho})tr^{-2} \right\} \\
& \leq - (1-\epsilon) \inf_{(R_r, U_r) \in \underline{\mathcal{S}}_r} \left\{ \lambda_1^r(U_r, R_r) \wedge M_r - r^{-\rho} \right. \\
& \quad \left. + \delta(r)^{-\theta} \int_{W_r} \text{dist}(x, \partial W_r)^\theta dx - (r\delta(r))^{-\theta} |W_r| \log M_1(\epsilon) \right\} \\
& \leq - (1-\epsilon)M_r
\end{aligned} \tag{71}$$

for large r , making ϵ slightly larger in the last line. \square

Lower bound: We start with the following trivial bound:

$$\begin{aligned}
\log S_t & \geq \log \sup_{(R_r, U_r) \in \underline{\mathcal{S}}'_r} \mathbb{P}_\theta(\xi_r(R_r \setminus U_r) = 0) \\
& \quad \times E_0 \left[\exp \left\{ - \int_0^{tr^{-2}} r^2 \cdot 1_{U_r}(B_s) ds \right\}; T_{R_r} > tr^{-2} \right],
\end{aligned} \tag{72}$$

where

$$\underline{\mathcal{S}}'_r = \left\{ (r^{-1}q + R_r, r^{-1}q + U_r); q \in \mathbb{Z}^d, (R_r, U_r) \in \underline{\mathcal{S}}_r \right\}. \tag{73}$$

The role of this extension of $\underline{\mathcal{S}}_r$ will be clear in the proof of Proposition 8. To rewrite the right hand side, we first show following lower bound on the hole probability of $W_r = R_r \setminus U_r$ for $(R_r, U_r) \in \underline{\mathcal{S}}'_r$.

Proposition 7. *For any $(R_r, U_r) \in \underline{\mathcal{S}}'_r$, we have*

$$\mathbb{P}_\theta(\xi_r(W_r) = 0) \geq \exp \left\{ - r^{d+\theta} \int_{W_r} \text{dist}(x, \partial W_r)^\theta dx - c_\tau(d, \theta) r^{d(1+\chi)} \right\} \tag{74}$$

for large r .

Proof. Since R_r is connected and $|R_r| < r^\chi$, $R_r \subset C(q, 2r^\chi)$ for some $q \in \mathbb{Z}^d$ by the definition of $\underline{\mathcal{S}}_r$. Without loss of generality, we can assume $q = 0$. We consider a sufficient condition for $\{\xi(rW_r) = 0\}$ in unscaled picture:

$$\begin{aligned}
\mathbb{P}_\theta(\xi_r(W_r) = 0) & = \mathbb{P}_\theta(\xi(rW_r) = 0) \\
& \geq \prod_{q \in \mathbb{Z}^d \cap rW_r} \mathbb{P}_\theta(q + \xi_q \in \text{the nearest } C_{q'} \not\subset rW_r) \\
& \quad \times \prod_{q \in \mathbb{Z}^d \cap C(0, 4r^{1+\chi}) \setminus rW_r} \mathbb{P}_\theta(q + \xi_q \in C_q) \\
& \quad \times \prod_{q \in \mathbb{Z}^d \setminus C(0, 4r^{1+\chi})} \mathbb{P}_\theta(|\xi_q| \leq |q|/2).
\end{aligned} \tag{75}$$

The first factor of right hand side is bounded from below by

$$\begin{aligned}
& c_8(d, \theta)^{|rW_r|} \exp \left\{ - \sum_{q \in rW_r \cap \mathbb{Z}^d} \text{dist}(q, \partial(rW_r))^\theta \right\} \\
& \geq \exp \left\{ - \sum_{q \in rW_r \cap \mathbb{Z}^d} \text{dist}(q, \partial(rW_r))^\theta - r^{d(1+\chi)} |\log c_8(d, \theta)| \right\}
\end{aligned} \tag{76}$$

where $c_8(d, \theta) = N(d, \theta) \inf\{\exp\{\text{dist}(0, q)^\theta - \text{dist}(0, x)^\theta\}; q \in \mathbb{Z}^d, x \in C(q)\}$. The sum in the right hand side can be replaced by the integral as in Lemma 1, by adding some constant to $|\log c_8(d, \theta)|$. Moreover, the second factor is bounded from below by

$$\mathbb{P}_\theta(q + \xi_q \in C_q)^{4d_r d(1+\chi)} \quad (77)$$

and the third factor is a convergent infinite product. Therefore, (74) holds for large sufficiently $c_7(d, \theta)$. \square

Note that the second term in the right hand side of (74) is negligible compared with $tr^{-2} = r^{d+\theta}\delta(r)^\theta$ thanks to (57).

Next, we shall rewrite the Brownian motion part of (72). Though the result seems to be natural, the proof is rather complicated.

Proposition 8. *For arbitrary small $\epsilon > 0$, we have*

$$\frac{1}{tr^{-2}} \log S_t \geq -(1 + \epsilon) \inf_{(R_r, U_r) \in \underline{\mathcal{S}}'_r} \left\{ \lambda_1^r(U_r, R_r) + \delta(r)^{-\theta} \int_{R_r \setminus U_r} \text{dist}(x, \partial(R_r \setminus U_r))^\theta dx \right\} \quad (78)$$

for large t .

Proof. It is clear that the functional in above infimum is invariant under $r^{-1}\mathbb{Z}^d$ -shift. If we also recall that $\underline{\mathcal{S}}'_r$ contains only finite pairs of sets modulo $r^{-1}\mathbb{Z}^d$ -shift, it follows that we can pick $(R, U) \in \underline{\mathcal{S}}'_r$ which attain the infimum in the right hand side. We write the L^1 -normalized positive eigenfunction corresponding to $\lambda_1^r(U, R)$ by ϕ . Since $\text{supp } \phi \subset R$, there exists a box $C(r^{-1}q, r^{-1})$ where

$$\int_{C(r^{-1}q, r^{-1})} \phi(x) dx \geq r^{-d-\chi}. \quad (79)$$

We can assume $q = 0$ by the shift invariance and the extension of $\underline{\mathcal{S}}_r$ to $\underline{\mathcal{S}}'_r$. We also introduce a slightly modified pair of sets $(R^*, U^*) \in \underline{\mathcal{S}}'_r$ defined by

$$R^* = R \cup C(0, 2r^{-1}) \quad \text{and} \quad U^* = U \setminus C(0, 2r^{-1}). \quad (80)$$

This pair approximates the the infimum in the right hand side of (78) since

$$\begin{aligned} & \int_{R^* \setminus U^*} \text{dist}(x, \partial(R^* \setminus U^*))^\theta dx \\ & \leq \int_{R \setminus U} \text{dist}(x, \partial(R \setminus U))^\theta dx + \int_{C(0, 2r^{-1})} \text{dist}(x, \partial C(0, 2r^{-1}))^\theta dx \\ & \leq \int_{R \setminus U} \text{dist}(x, \partial(R \setminus U))^\theta dx + 2^{d+\theta} r^{-d-\theta} \int_{C(0, 1)} \text{dist}(x, \partial C(0, 1))^\theta dx \end{aligned} \quad (81)$$

and the second term in the right hand side is $o(\delta(r)^\theta)$ as $t \rightarrow \infty$. Now we substitute (R^*, U^*) into (72) and use Proposition 7 to obtain

$$\begin{aligned} \frac{1}{tr^{-2}} \log S_t & \geq -(1 + \epsilon) \delta(r)^{-\theta} \int_{R^* \setminus U^*} \text{dist}(x, \partial(R^* \setminus U^*))^\theta dx \\ & \quad + \log E_0 \left[\exp \left\{ - \int_0^{tr^{-2}} r^2 \cdot 1_{U^*}(B_s) ds \right\}; T_{R^*} > tr^{-2} \right]. \end{aligned} \quad (82)$$

We introduce some more notations to proceed the proof. Let $p_{R,U}(t, x, y)$ denote the integral kernel of the Feynman-Kac semigroup defined by

$$E_x \left[f(B_t) \exp \left\{ - \int_0^t r^2 \cdot 1_U(B_s) ds \right\}; T_R > t \right] \text{ for } f \in L^2(R) \quad (83)$$

and $p_C(t, x, y)$ the transition kernel of the killed Brownian motion when exiting $C(0, 2r^{-1})$. We also need the following uniform upper bound on $\|\phi\|_\infty$.

Lemma 3.

$$\|\phi\|_\infty \leq \exp \left\{ 2 \sup_{r \geq 1} M_r \right\} < \infty \quad (84)$$

Proof. Since $p_{R,U}(t, x, y)$ is smaller than the usual heat kernel, we have $p_{R,U}(1, \cdot, \cdot) < 1$ and therefore

$$\begin{aligned} \phi_{R,U}(x) &= \exp \{ \lambda_1^r(U, R) \} \int_R p_{R,U}(1, x, y) \phi(y) dy \\ &< \exp \{ \lambda_1^r(U, R) \} \int_R \phi(y) dy \end{aligned} \quad (85)$$

for all $x \in R$. The rest is easy from the definition of M_r , (81) and $\|\phi\|_1 = 1$. \square

Using the fact $p_C(t, x, y), p_{R,U}(t, x, y) < p_{R^*, U^*}(t, x, y)$ and the Chapman-Kolmogorov identity, we have

$$\begin{aligned} &E_0 \left[\exp \left\{ - \int_0^{tr^{-2}} r^2 \cdot 1_{U^*}(B_s) ds \right\}; T_{R^*} > tr^{-2} \right] \\ &\geq \int_{C(0, 2r^{-1})} p_C(r^{-1}, 0, x) \int_R p_{R,U}(tr^{-2} - r^{-1}, x, y) \frac{\phi(y)}{\|\phi\|_\infty} dy dx \\ &\geq \|\phi\|_\infty^{-1} \inf_{x \in C(0, r^{-1})} p_C(r^{-1}, 0, x) \exp \{ -\lambda_1^r(U, R) tr^{-2} \} \int_{C(0, r^{-1})} \phi(x) dx \\ &\geq \text{const}(d) r^{-d-\chi} \exp \{ -\lambda_1^r(U, R) tr^{-2} \}, \end{aligned} \quad (86)$$

where we have used Lemma 3 for the first factor in the third line, a scaling argument for the second factor, and (79) for the last factor.

Coming back to (82), we can conclude

$$\frac{1}{tr^{-2}} \log S_t \geq -(1 + \epsilon) \left\{ \lambda_1^r(U, R) + \delta(r)^{-\theta} \int_{R^* \setminus U^*} \text{dist}(x, \partial(R^* \setminus U^*))^\theta dx \right\} \quad (87)$$

and this completes the proof of Proposition 8 in view of (81). \square

Now, note that $\underline{\mathcal{S}}'_r$ in the right hand side of (78) can be replaced by $\underline{\mathcal{S}}_r$ since both terms in the infimum is invariant under $r^{-1}\mathbb{Z}^d$ -shift. Then, the right hand side of (78) equals $-(1 + \epsilon)M_r$ and the proof of the lower bound is completed. \square

Remark. We have shown the following asymptotics which is finer than the results stated in the first section.

Theorem 9. For modified potential (6),

$$\frac{1}{tr^{-2}} \log S_t \sim - \inf_{(R_r, U_r) \in \underline{\mathcal{S}}_r} \left\{ \lambda_1^r(U_r, R_r) + \delta(r)^{-\theta} \int_{R_r \setminus U_r} \text{dist}(x, \partial(R_r \setminus U_r))^\theta dx \right\}. \quad (88)$$

We have proved this result for the parameters $\epsilon, h, L = 1$ above and it is routine to extend (88) for other parameters with appropriate changes on the notation. Though it has been shown only for the modified traps (6), it seems not so far from the original model at least in the case of hard traps. Indeed, for the hard traps, the modification is equivalent to discretize ξ_q as

$$\mathbb{P}_\theta(\xi_q \in dx) = N_{\text{disc}}(d, \theta) \sum_{q \in \mathbb{Z}^d} \exp\{-|q|^\theta\} \delta_q(dx). \quad (89)$$

However, we still do not know whether the right hand side of (88) converges to a limit or not when $r \rightarrow \infty$.

3 Applications

3.1 Lifshitz tail

In this section, we discuss the asymptotic behavior of the density of states of H_ξ which is defined by the thermodynamic limit

$$\ell(d\lambda) = \lim_{N \rightarrow \infty} \frac{1}{(2N)^d} \sum_{i \geq 1} \delta_{\lambda_i^D(H_\xi \text{ in } (-N, N)^d)}(d\lambda). \quad (90)$$

Here $\lambda_i^D(H_\xi \text{ in } (-N, N)^d)$ is the i -th smallest Dirichlet eigenvalue of H_ξ in $(-N, N)^d$. It is well known that above limit exists in the sense of vague convergence and that its Laplace transform can be expressed as

$$\int_0^\infty e^{-t\lambda} \ell(d\lambda) = (2\pi t)^{-\frac{d}{2}} \int_{[0,1]^d} \mathbb{E}_\theta \otimes E_x \left[\exp \left\{ - \int_0^t V(B_s, \xi) ds \right\} \middle| B_t = x \right] dx \quad (91)$$

using Brownian bridge measure. As one can expect from this expression, it is not difficult to see that the right hand side admits essentially same upper and lower bounds as S_t (see e.g. the discussion in [16]):

$$\log \int_0^\infty e^{-t\lambda} \ell(d\lambda) \asymp \begin{cases} -t^{\frac{2+\theta}{4+\theta}} (\log t)^{-\frac{\theta}{4+\theta}} & (d = 2), \\ -t^{\frac{d^2+2\theta}{d^2+2d+2\theta}} & (d \geq 3), \end{cases} \quad (92)$$

as $t \rightarrow \infty$. From this asymptotics and the exponential Tauberian theorem due to Kasahara [8], we find following asymptotics for $\ell([0, \lambda])$ as $\lambda \rightarrow 0$.

Corollary 1. Let ℓ denote the density of states of H_ξ . For any $\theta > 0$,

$$\log \ell([0, \lambda]) \asymp \begin{cases} -\lambda^{-1-\frac{\theta}{2}} (\log \frac{1}{\lambda})^{-\frac{\theta}{2}} & (d = 2), \\ -\lambda^{-\frac{d}{2}-\frac{\theta}{d}} & (d \geq 3). \end{cases} \quad (93)$$

This result says that the density of states is exponentially thin around the bottom of the spectrum, which is called ‘‘the Lifshitz tail effect’’.

3.2 Intermittency

We consider the solution of the initial value problem

$$\frac{\partial}{\partial t} u(t, x) = H_\xi u(t, x) \text{ with } u(0, \cdot) \equiv 1, \quad (94)$$

which is called *parabolic Anderson problem*. The solution u_ξ of this equation is known to admit Feynman-Kac representation (see e.g. chapter 1 of [19]) and therefore we can identify S_t as $\mathbb{E}_\theta[u_\xi(t, 0)]$. We analogously write the p -th moment by $S_{t,p} = \mathbb{E}_\theta[u_\xi(t, 0)^p]$. Then, the solution u_ξ is said to be “intermittent” if

$$\frac{S_{t,q}^{1/q}}{S_{t,p}^{1/p}} \xrightarrow{t \rightarrow \infty} \infty \text{ when } p < q. \quad (95)$$

The intermittency is usually regarded as an evidence of the strong inhomogeneity of the solution field. Indeed, if one considers a function consists of a few high peaks, its L^q -norm tends to be much larger than its L^p -norm for $p < q$. For more on the intermittency, see for instance [6].

We shall prove the intermittency for our model in the following slightly weaker form.

Corollary 2. *Let $S_{t,p}$ be as above. If $1 \leq p < q$ and $q/p \geq 2^{d+\theta+2}$, we have*

$$\frac{S_{t,q}^{1/q}}{S_{t,p}^{1/p}} \rightarrow \infty \text{ as } t \rightarrow \infty. \quad (96)$$

Proof. We prove this result only for the modified potential with the parameters $\epsilon, h, L = 1$. The extension to general situation is not difficult.

Note that it suffices to prove the result for $q = 2^{d+\theta+2}p$ by the monotonicity of $S_{t,p}^{1/p}$ in $p \geq 1$. The key observation is that we can prove

$$\frac{1}{tr^{-2}} \log S_{t,p} \sim - \inf_{(R_r, U_r) \in \underline{\mathcal{S}}_r} \left\{ p \lambda_1^r(U_r, R_r) + \delta(r)^{-\theta} \int_{R_r \setminus U_r} \text{dist}(x, \partial(R_r \setminus U_r))^\theta dx \right\} \quad (97)$$

by exactly the same argument as for Theorem 9. Then, using spatial scaling by the factor $p^{1/(d+\theta+2)}$, we find that the right hand side equals

$$-p^{\frac{d+\theta}{d+\theta+2}} \inf_{(R_r, U_r) \in \underline{\mathcal{S}}_{r,p}} \left\{ \lambda_1^r(U_r, R_r) + \delta(r)^{-\theta} \int_{R_r \setminus U_r} \text{dist}(x, \partial(R_r \setminus U_r))^\theta dx \right\}, \quad (98)$$

where $\underline{\mathcal{S}}_{r,p} = \{(p^{-1/(d+\theta+2)}R_r, p^{-1/(d+\theta+2)}U_r); (R_r, U_r) \in \underline{\mathcal{S}}_r\}$. Now we use the relation $q = 2^{d+\theta+2}p$ to see $\underline{\mathcal{S}}_{r,p} \subset \underline{\mathcal{S}}_{r,q}$ and consequently, the infimum in the right hand side is larger than that over $\underline{\mathcal{S}}_{r,q}$. (To be honest, above inclusion holds only for the pair (R_r, U_r) for which $|R_r|$ is not too large. However, we omit this point since we can see that such pairs are inessential to the infimum by the same argument as for Proposition 5.) Therefore, we obtain

$$\limsup_{t \rightarrow \infty} \frac{\log S_{t,q}^{1/q}}{\log S_{t,p}^{1/p}} \leq \left(\frac{p}{q}\right)^{\frac{2}{d+\theta+2}}, \quad (99)$$

and (96) follows immediately. \square

Appendix I

We discuss here the convergences of the perturbed lattice as a point process. When we consider weak convergences, we regard $(\mathbb{P}_\theta)_{\theta>0}$ as probability measures on Ξ equipped with the vague topology. Let \mathbb{P}_∞ denotes the perturbed lattice with the perturbation variables distributed uniformly on $B(0, 1)$ and \mathbb{P}_0 the Poisson point process with unit intensity.

Theorem 10. \mathbb{P}_θ converges weakly to \mathbb{P}_L ($L \in \{0, \infty\}$) as $\theta \rightarrow L$.

To prove this theorem, we use following result concerning the convergence of point processes. (See Theorem 4.7 of [7].)

Lemma 4. Let $(\mathbb{P}_\theta)_{\theta \in [0, \infty]}$ be a family of probability measures on Ξ . Suppose that following two conditions hold for any bounded Borel set $B \subset \mathbb{R}^d$:

$$\begin{aligned} \text{(i)} \quad & \lim_{\theta \rightarrow L} \mathbb{P}_\theta(\xi(B) = 0) = \mathbb{P}_L(\xi(B) = 0), \\ \text{(ii)} \quad & \limsup_{\theta \rightarrow L} \mathbb{E}_\theta[\xi(B)] \leq \mathbb{E}_L[\xi(B)]. \end{aligned} \tag{100}$$

Then \mathbb{P}_θ converges weakly to \mathbb{P}_L as $\theta \rightarrow L$.

Proof of Theorem 10. We consider the limit $\theta \rightarrow \infty$ first. In this case, the law of each ξ_q converges to the uniform distribution on $B(0, 1)$. Moreover, we have

$$\mathbb{P}_\theta(q + \xi_q \in B) \leq |B|N(d, \theta) \exp\{-\text{dist}(q, B)^\theta\} \tag{101}$$

for any bounded $B \subset \mathbb{R}^d$. This implies that the law of $\xi(B)$ is essentially determined by finite ξ_q 's when θ is large. From these facts, it is easy to verify the conditions (i) and (ii) in Lemma 4 and we have desired convergence.

Next, we turn to more subtle case $\theta \rightarrow 0$. We first verify the condition (i), that is, $\lim_{\theta \rightarrow 0} \mathbb{P}_\theta(\xi(B) = 0) = e^{-|B|}$. Let us take $M > 0$ so large that $B \subset [-M, M]^d$. Then it follows

$$\sup_{x \in B, q \notin [-2M, 2M]^d} \left| |x - q|^\theta - |q|^\theta \right| \leq 2\theta M^\theta \tag{102}$$

for $\theta < 1$, from the mean value theorem. Therefore, for any $\epsilon > 0$, we have

$$1 - \epsilon < \frac{\int_B \exp\{-|x - q|^\theta\} dx}{|B| \exp\{-|q|^\theta\}} < 1 + \epsilon \tag{103}$$

for all $q \notin [-2M, 2M]^d$ when θ is sufficiently small. The right inequality in (103) gives us the upper bound

$$\begin{aligned} \mathbb{P}_\theta(\xi(B) = 0) &= \prod_{q \in \mathbb{Z}^d} \left(1 - N(d, \theta) \int_B \exp\{-|x - q|^\theta\} dx \right) \\ &\leq \prod_{q \notin [-2M, 2M]^d} \left(1 - (1 - \epsilon)N(d, \theta)|B| \exp\{-|q|^\theta\} \right). \end{aligned} \tag{104}$$

Using $1 - a \leq e^{-a}$ in the right hand side, we get

$$\begin{aligned} \limsup_{\theta \rightarrow 0} \mathbb{P}_\theta(\xi(B) = 0) &\leq \limsup_{\theta \rightarrow 0} \exp\left\{ -(1 - \epsilon)N(d, \theta)|B| \sum_{q \notin [-2M, 2M]^d} \exp\{-|q|^\theta\} \right\} \\ &= \exp\{-(1 - \epsilon)|B|\}. \end{aligned} \tag{105}$$

Here, the second line comes from the fact $N(d, \theta) \sum_{q \notin [-2M, 2M]^d} \exp\{-|q|^\theta\} \rightarrow 1$ ($\theta \rightarrow 0$), which can be verified by the same way as (103). For the lower bound, we use the right inequality in (103) as follows:

$$\begin{aligned} \mathbb{P}_\theta(\xi(B) = 0) &= \prod_{q \in \mathbb{Z}^d} \left(1 - N(d, \theta) \int_B \exp\{-|x - q|^\theta\} dx \right) \\ &\geq \prod_{q \in [-2M, 2M]^d} \left(1 - N(d, \theta) \int_B \exp\{-|x - q|^\theta\} dx \right) \\ &\quad \times \prod_{q \notin [-2M, 2M]^d} \left(1 - (1 + \epsilon)N(d, \theta)|B| \exp\{-|q|^\theta\} \right). \end{aligned} \quad (106)$$

Since $N(d, \theta) \rightarrow 0$ ($\theta \rightarrow 0$), the first factor in the right hand side goes to 1 and also

$$\sup_{q \in \mathbb{Z}^d} (1 + \epsilon)N(d, \theta)|B| \exp\{-|q|^\theta\} \rightarrow 0 \quad \text{as } \theta \rightarrow 0. \quad (107)$$

Thus we can use $1 - a \geq e^{-(1+\epsilon)a}$, which is valid only for small $a > 0$, in the second factor and get

$$\begin{aligned} \liminf_{\theta \rightarrow 0} \mathbb{P}_\theta(\xi(B) = 0) &\geq \liminf_{\theta \rightarrow 0} \exp \left\{ -(1 + \epsilon)^2 N(d, \theta) |B| \sum_{q \notin [-2M, 2M]^d} \exp\{-|q|^\theta\} \right\} \\ &= \exp\{-(1 + \epsilon)^2 |B|\}. \end{aligned} \quad (108)$$

Now that we have (105) and (108) for arbitrary $\epsilon > 0$, the condition (i) is verified.

Next, we proceed to check the condition (ii), $\limsup_{\theta \rightarrow 0} \mathbb{E}_\theta[\xi(B)] \leq |B|$. Using the right inequality in (103), we find

$$\begin{aligned} \mathbb{E}_\theta[\xi(B)] &= \sum_{q \in \mathbb{Z}^d} \mathbb{P}_\theta(q + \xi_q \in B) \\ &= N(d, \theta) \sum_{q \in \mathbb{Z}^d} \int_B \exp\{-|x - q|^\theta\} dx \\ &\leq (1 - \epsilon)^{-1} |B| N(d, \theta) \left(\sum_{q \notin [-2M, 2M]^d} \exp\{-|q|^\theta\} + (4M)^d \right). \end{aligned} \quad (109)$$

Since the right hand side of this inequality goes to $(1 - \epsilon)^{-1} |B|$ as $\theta \rightarrow 0$, we have done. \square

Appendix II

Let U_r be the punched domain (20) in the constant capacity regime (22). We shall prove here that the principal Dirichlet eigenvalue $\lambda_1^D(-1/2\Delta \text{ in } U_r)$ remains bounded as $r \rightarrow \infty$. This result for the case $d = 3$ is presented in Theorem 22.1 of [14]. Since the same proof directly applies to all $d \geq 3$, we restrict the discussion on $d = 2$. Let ψ be the L^2 -normalized principal Dirichlet eigenfunction in $(-1, 1)^d$ and

$$\phi_r(x) = \prod_{q \in \mathbb{Z}^d} \left(\frac{\log|x - \delta(r)q| - \log(1/r)}{\log(\delta(r)/2) - \log(1/r)} \right)_+ \wedge 1. \quad (110)$$

Then it easily follows that for arbitrary small $\epsilon > 0$,

$$\inf \left\{ \phi_r(x); (-1, 1)^d \setminus \bigcup_{q \in \mathbb{Z}^d} C(\delta(r)q, \epsilon\delta(r)) \right\} \rightarrow 1 \quad (111)$$

as $r \rightarrow \infty$. Moreover, it is not difficult to show that both $\|(\nabla\psi)\phi_r\|_2$ and $\|\psi(\nabla\phi_r)\|_2$ are bounded. We combine these three estimates to bound the right hand side of

$$\lambda_1^D(-1/2\Delta \text{ in } U_r) \leq \frac{1}{\|\psi\phi_r\|_2^2} \int_{U_r} \frac{1}{2} |\nabla(\psi\phi_r)|^2(x) dx \quad (112)$$

and get the desired result.

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References

- [1] J. Baker, M. Loss, and G. Stolz. Minimizing the ground state energy of an electron in a randomly deformed lattice, 2007, math-ph/07073988.
- [2] I. Ben-Ari. The Asymptotic Shift for the Principal Eigenvalue for Second Order Elliptic Operators on Bounded Domains under Various Boundary Conditions in the Presence of Small Obstacles. *to appear in Israel Journal of Mathematics*, 2008.
- [3] M. D. Donsker and S. R. S. Varadhan. Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.*, 28(4):525–565, 1975.
- [4] M. Fukushima. On the spectral distribution of a disordered system and the range of a random walk. *Osaka J. Math.*, 11:73–85, 1974.
- [5] R. Fukushima and N. Ueki. Classical and quantum behavior of the integrated density of states for a randomly perturbed lattice. *in preparation*, 2008.
- [6] J. Gärtner and S. A. Molchanov. Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.*, 132(3):613–655, 1990.
- [7] O. Kallenberg. *Random measures*. Akademie-Verlag, Berlin, fourth edition, 1986.
- [8] Y. Kasahara. Tauberian theorems of exponential type. *J. Math. Kyoto Univ.*, 18(2):209–219, 1978.
- [9] W. Kirsch and F. Martinelli. On the spectrum of Schrödinger operators with a random potential. *Comm. Math. Phys.*, 85(3):329–350, 1982.
- [10] F. Klopp. Localization for semiclassical continuous random Schrödinger operators. II. The random displacement model. *Helv. Phys. Acta*, 66(7-8):810–841, 1993.

- [11] S. Nakao. On the spectral distribution of the Schrödinger operator with random potential. *Japan. J. Math. (N.S.)*, 3(1):111–139, 1977.
- [12] T. Povel. Confinement of Brownian motion among Poissonian obstacles in \mathbf{R}^d , $d \geq 3$. *Probab. Theory Related Fields*, 114(2):177–205, 1999.
- [13] J. Rauch and M. Taylor. Potential and scattering theory on wildly perturbed domains. *J. Funct. Anal.*, 18:27–59, 1975.
- [14] B. Simon. *Functional integration and quantum physics*. AMS Chelsea Publishing, Providence, RI, second edition, 2005.
- [15] M. Sodin and B. Tsirelson. Random complex zeroes. II. Perturbed lattice. *Israel J. Math.*, 152:105–124, 2006.
- [16] A.-S. Sznitman. Lifschitz tail and Wiener sausage. I, II. *J. Funct. Anal.*, 94(2):223–246, 247–272, 1990.
- [17] A.-S. Sznitman. On the confinement property of two-dimensional Brownian motion among Poissonian obstacles. *Comm. Pure Appl. Math.*, 44(8-9):1137–1170, 1991.
- [18] A.-S. Sznitman. Brownian survival among Gibbsian traps. *Ann. Probab.*, 21(1):490–508, 1993.
- [19] A.-S. Sznitman. *Brownian motion, obstacles and random media*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
- [20] M. E. Taylor. Scattering length and perturbations of $-\Delta$ by positive potentials. *J. Math. Anal. Appl.*, 53(2):291–312, 1976.