Homotopy nilpotency in localized SU(n)

Daisuke Kishimoto

April 10, 2008

Abstract

We determine homotopy nilpotency of the $p$-localized SU($n$) when $p$ is a quasi-regular prime in the sense of [9]. As a consequence, we see that it is not a monotonic decreasing function in $p$.

1 Introduction

Let $G$ be a compact Lie group and let $-(p)$ stand for the $p$-localization in the sense of [2]. In [7], McGibbon asked:

Question 1.1. For which primes $p$ is $G(p)$ homotopy commutative?

He answered this question when $G$ is simply connected. For example, he showed that SU($n$) is homotopy commutative if and only if $p > 2n$. Later, in [8], he studied higher homotopy commutativity of $p$-local finite loop spaces and, motivated by this work, Saumell [11] considered the above question by replacing homotopy commutativity with higher homotopy commutativity in the sense of Williams [14]. For example, she showed that if $p > kn$, then SU($n$) is a $C_k$-space in the sense of Williams [14].

On the contrary, one can ask:

Question 1.2. How far from homotopy commutative is $G(p)$ for a given prime $p$?

In [5], Kaji and the author approached this question by considering homotopy nilpotency which is defined as follows, where we treat only group-like spaces (See [15] for a general definition). Let $X$ be a group-like space, that is, $X$ satisfies all the axioms of groups up to homotopy, and let $\gamma : X \times X \to X$ be the commutator map of $X$. We write the $n$-iterated commutator map $\gamma \circ (1 \times \gamma) \circ \cdots \circ (1 \times \cdots \times 1 \times \gamma) : X^{n+1} \to X$ by $\gamma_n$, where $X^{n+1}$ is the direct product of $(n + 1)$-copies of $X$. We say that $X$ is homotopy nilpotent of class $n$, denoted $\text{nil} X = n$, if $\gamma_n \simeq *$ and $\gamma_{n-1} \not\simeq *$. Namely, $\text{nil} X = n$ means that $X$ is a nilpotent group of class $n$ up
to homotopy. Then one can say that \( \text{nil} X \) tells how far from homotopy commutative \( X \) is. Note that we normalize homotopy nilpotency such that \( \text{nil} X = 1 \) if and only if \( X \) is homotopy commutative. Then, rewriting the above result of McGibbon, we have

\[
\text{nil SU}(n)_{(p)} = 1 \text{ if and only if } p > 2n. \tag{1.1}
\]

In [5], Kaji and the author determined \( \text{nil} X \) for a \( p \)-compact group \( X \) when \( p \) is a regular prime, that is, \( X \) has the homotopy type of the direct product of localized spheres. For example, they showed

\[
\text{nil SU}(n)_{(p)} = \begin{cases} 
2 & \frac{3}{2}n < p < 2n \\
3 & n \leq p \leq \frac{3}{2}n
\end{cases} \tag{1.2}
\]

when \( p \) is odd, and \( \text{nil SU}(2)_{(2)} = 2 \).

The aim of this article is to determine \( \text{nil SU}(n)_{(p)} \) when \( p \) is a quasi-regular prime in the sense of [9], that is, \( \text{SU}(n)_{(p)} \) has the homotopy type of the \( p \)-localization of the direct product of spheres and sphere bundles over spheres. The result is:

**Theorem 1.1.** Let \( p \) be a prime greater than 5. Then we have:

1. \( \text{nil SU}(n)_{(p)} = 3 \) if \( p = n + 1 \) or \( \frac{n}{2} < p \leq \frac{2n+1}{3} \).
2. \( \text{nil SU}(n)_{(p)} = 2 \) if \( \frac{2n+1}{3} < p \leq n - 2 \).

Since the homotopy type of \( \text{SU}(n)_{(p)} \) gets easier as \( p \) increases, it is natural to expect that \( \text{nil SU}(n)_{(p)} \) is a monotonic decreasing function in \( p \). Actually, (1.1) and (1.2) give some evidence for this expectation. However, Theorem 1.1 shows this is faulse in almost all cases as follows.

In [10], it is shown that

\[
\frac{x}{\log x} < \pi(x) < 1.25506 \frac{x}{\log x}
\]

for \( x \geq 17 \), where \( \pi(x) \) is the prime counting function. This implies that there is a prime in \( (\frac{2n+1}{3}, n) \) if \( n \geq 81 \). Then, together with a case by case analysis for \( n \geq 80 \), we obtain:

**Corollary 1.1.** For \( n = 9 \) or \( n \geq 13 \), \( \text{nil SU}(n)_{(p)} \) is not a monotonic decreasing function in \( p \).

In what follows, we will make the conventions: For a map \( f : X \to Y \), \( f_* : [A, X] \to [A, Y] \) and \( f^* : [Y, B] \to [X, B] \) mean the induced maps. If a map \( f : X \to Y_1 \times Y_2 \) satisfies \( \pi_1 \circ f \simeq * \), then we say that \( f \) falls into \( Y_2 \), where \( \pi_1 \) is the first projection. We often assume that the above \( f \) is a map from \( X \) into \( Y_2 \). We denote the adjoint congruence \( [X, \Omega Y] \xrightarrow{\text{ad}} [\Sigma X, Y] \) by \( \text{ad} \). When \( X \) is group-like, we always assume that the homotopy set \( [A, X] \) is a group by the pointwise multiplication and we denote by \( 0 \) unity of this group which is the constant map. We denote the order of an element \( x \) of a group by \( \text{ord}(x) \).
2 Homotopy groups of $B_n$

Hereafter, let $p$ denote an odd prime and put $2 \leq t \leq p$. Each space and map is always assumed to be localized at the prime $p$.

Let us first recall basic results on the $p$-component of the homotopy groups of spheres.

**Theorem 2.1** ([12, Chapter XIII]).

1. $\pi_{2n-1+k}(S^{2n-1}) \cong \begin{cases} \mathbb{Z}/p & k = 2i(p - 1) - 1, i = 1, \ldots, p - 1 \\ \mathbb{Z}/p & k = 2i(p - 1) - 2, i = n, \ldots, p - 1 \\ 0 & \text{other } 1 \leq k \leq 2p(p - 1) - 3 \end{cases}$

2. Let $\alpha_1(3)$ be a generator of $\pi_2(S^3)$ and let $\alpha_i(3) = \{\alpha_{i-1}(3), p, \alpha_1(2i(p - 1) + 2)\}_1 \in \pi_{2i(p - 1) + 2}(S^3)$ for $i = 2, \ldots, p - 1$. Then $\pi_{2n+2i(p-1)-2}(S^{2n-1})$ is generated by $\alpha_i(2n - 1) = \Sigma^{2n-4} \alpha_i(3)$. 

3. $\pi_{2i(p-1)+1}(S^3)$ is generated by $\alpha_1(3) \circ \alpha_{i-1}(2p)$ for $i = 2, \ldots, p - 1$. 

4. $\Sigma^2 : \pi_{2n+2i(p-1)-3}(S^{2n-1}) \to \pi_{2n+2i(p-1)-1}(S^{2n+1})$ is the zero map for $i = n, \ldots, p - 1$. In particular, $\alpha_i(n) \circ \alpha_j(n + 2i(p - 1) - 1) = 0$ for $i + j < p$ and $n \geq 5$.

Let $B_n$ be the $S^{2n-1}$-bundle over $S^{2n+2p-3}$ such that

$$H^*(B_n; \mathbb{Z}/p) = \Lambda(\bar{x}_{2n-1}, \mathbb{F}^1 \bar{x}_{2n-1}),$$

where $|x_{2n-1}| = 2n - 1$. Namely, $B_n$ is induced from the sphere bundle $S^{2n-1} \to O(2n + 1)/O(2n - 1) \to S^{2n}$ by $\frac{1}{2} \alpha_1(2n)$ as in [9]. Recall that we have a cell decomposition

$$B_n(p) = S^{2n-1} \cup_{\alpha_1(2n+1)} e^{2n+2p-3} \cup e^{4n+2p-4}.$$ 

Let $A_n$ denote the $(4n + 2p - 5)$-skeleton of $B_n$, that is, $A_n = S^{2n-1} \cup_{\alpha_1(2n-1)} e^{2n+2p-3}$. In particular, we have

$$A_n = \Sigma^{2n-4} A_2. \quad (2.1)$$

It follows from a result of McGibbon [6] that the cofiber sequence $S^{2n-1} \to A_n \to S^{2n+2p-3}$ splits after a suspension, that is,

$$\Sigma B_n \simeq \Sigma A_n \vee S^{4n+2p-3}. \quad (2.2)$$
Mimura and Toda [9] showed that SU(n) has the homotopy type of the direct product of odd spheres and $B_k$’s if and only if $p > \frac{n}{2}$. We shall be concerned with SU(n) for $\frac{n}{2} < p < n$, equivalently, SU($p + t - 1$) since $2 \leq t \leq p$. In this case, we have a homotopy equivalence

$$SU(p + t - 1) \simeq B_2 \times \cdots \times B_t \times S^{2t+1} \times \cdots \times S^{2p-1}.$$  

We compute the homotopy groups of $B_n$ following Mimura and Toda [9] in a slightly larger range than [9]. Consider the homotopy exact sequence of the fibration $S^{2n-1} \to B_n \to S^{2n+2p-3}$. Then the connecting homomorphism $\delta : \pi_\ast(S^{2n+2p-3}) \to \pi_{\ast-1}(S^{2n-1})$ is given by

$$\delta(\Sigma x) = \alpha_1(2n - 1) \circ x \quad (2.3)$$

Then, by Theorem 2.1, we obtain $\pi_\ast(B_2)$ for $\ast \leq 2p(p - 1)$. In particular, each map $S^m \to B_2$ for $2p + 2 \leq m \leq 2p(p - 1)$ lifts to $S^3 \subset B_2$. It also follows from Theorem 2.1 that, for $n \geq 3$ and $i = 2, \ldots, p - 1$, we have the short exact sequence

$$0 \to \pi_\ast(S^{2n-1}) \to \pi_\ast(B_n) \to \pi_\ast(S^{2n+2p-3}) \to 0$$

for $2n + 2p - 2 \leq \ast \leq 2n + 2p(p - 1) - 4$. Then we have only to consider the case that $\ast = 2n + 2i(p - 1) - 2$ for $i = 2, \ldots, p - 1$. Let $i_n : S^{2n-1} \to A_n$ and $j_n : A_n \to B_n$ be the inclusions and let $q_n : A_n \to S^{2n+2p-3}$ be the pinch map. Consider the following commutative diagram in which the lower horizontal sequence is the exact sequence (2) and we put $k = 2n + 2i(p - 1) - 2$.

$$\begin{array}{ccc}
\pi_k(S^{2n-1}) & \xrightarrow{i_{\ast \ast}} & \pi_k(A_n) \xrightarrow{q_{\ast \ast}} \pi_k(S^{2n+2p-3}) \\
0 & \xrightarrow{j_{\ast \ast}} & \pi_k(B_n) \xrightarrow{\pi_k(S^{2n+2p-3})} 0 \\
\end{array}$$

Note that a coextension $\alpha_{i-1}(2n + 2p - 4) : S^{2n+2i(p-1)-2} \to A_n = S^{2n-1} \cup_{\alpha_{i-1}(2n-1)} e^{2n+2p-3}$ satisfies

$$q_{\ast \ast}(\alpha_{i-1}(2n + 2p - 4)) = -\alpha_{i-1}(2n + 2p - 3)$$

and

$$\alpha_{i-1}(2n + 2p - 4) \circ p = -i_{\ast \ast}(\{\alpha_{i-1}(2n - 1), \alpha_{i-1}(2n + 2p - 4), p\})_1$$

$$= i_{\ast \ast}(\frac{1}{2}(\alpha_{i-1}(2n - 1), p, \alpha_{i-1}(2n + 2p - 4)))_1$$

$$= -i_{\ast \ast}(\frac{1}{2}\alpha_{i}(2n - 1))$$

(See [12, p.179]). Then (2) does not split for $\ast = 2n + 2i(p - 1) - 2$ and hence we have obtained that $\pi_{2n+2i(p-1)-2}(B_n) \cong \mathbb{Z}/p^2$. Moreover, it is generated by $j_{\ast \ast}(\alpha_{i-1}(2n + 2p - 4))$.  

4
In particular, each map $S^m \to B_n$ which is of order $p$ for $2n + 2p - 2 \leq m \leq 2n + 2p(p-1) - 4$ lifts to $S^{2n-1} \subset B_n$. Summarizing, we have calculated:

**Proposition 2.1.**

1. $\pi_{3+k}(B_2) \cong \begin{cases} \mathbb{Z}/p & k = 2i(p-1) - 1, i = 2, \ldots, p-1 \\ \mathbb{Z}_{(p)} & k = 2p-2 \\ 0 & \text{other } 1 \leq k \leq 2(p-1) - 3 \end{cases}$

2. For $n \geq 3$, $\pi_{2n-1+k}(B_n) \cong \begin{cases} \mathbb{Z}/p & k = 2i(p-1) - 1, i = 2, \ldots, p-1 \\ \mathbb{Z}_{(p)} & k = 2i(p-1) - 2, i = n, \ldots, p-1 \\ \mathbb{Z}_{(p)} & k = 2p-2 \\ 0 & \text{other } 1 \leq k \leq 2(p-1) - 3 \end{cases}$

3. For $2p + 2 \leq m \leq 2p(p-1)$, each map $S^m \to B_2$ lifts to $S^3 \subset B_2$.

4. For $n \geq 3$ and $2n + 2p - 2 \leq m \leq 2n + 2p(p-1) - 4$, each map $S^m \to B_n$ of order $p$ lifts to $S^{2n-1} \subset B_n$.

By Theorem 2.1 and Proposition 2.1 we can see the homotopy groups of SU($p + t - 1$) in a range. It will be useful to list up the non-trivial odd homotopy groups of SU($p + t - 1$).

**Corollary 2.1.** Let $p \geq 7$ and $2(p + t) - 1 \leq k \leq 12p - 1$. Then $\pi_k(\text{SU}(p + t - 1)) = 0$ unless $k$ is odd and not in the following table. Moreover, each element of $\pi_{2k-1}(\text{SU}(p + t - 1))$ can be compressed into $S^n \subset \text{SU}(p + t - 1)$ for $n$ in the following table.

<table>
<thead>
<tr>
<th>$k$</th>
<th>6p − 3</th>
<th>8p − 5</th>
<th>8p − 3</th>
<th>10p − 7</th>
<th>10p − 5</th>
<th>10p − 3</th>
<th>12p − 9</th>
<th>12p − 7</th>
<th>12p − 5</th>
<th>12p − 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3 Homotopy nilpotency and Samelson products

Let $X$ be a group-like space. For a map $f : A \to X$ we write by $-f$ the composition $A \xrightarrow{f} X \xrightarrow{\iota} X$, where $\iota : X \to X$ is the homotopy inversion.

Since the pinch map $X^{n+1} \to X^{(n+1)}$ induces a monomorphism $[X^{(n+1)}, X] \to [X^{n+1}, X]$ as in [15, Lemma 1.3.5], the $n$-iterated commutator map of $X$ vanishes if and only if so does the $n$-iterated Samelson product $\langle 1_X, \langle \cdots \langle 1_X, 1_X \rangle \cdots \rangle \rangle$, where $X^{(n+1)}$ is the smash product of $(n+1)$-copies of $X$.
Suppose that $X = X_1 \times \cdots \times X_n$ as spaces, not as group-like spaces. We denote the inclusion $X_k \to X$ and the projection $X \to X_k$ by $i_k$ and $p_k$ respectively for $k = 1, \ldots, n$. Note that $1_X = (i_1 \circ p_1) \cdots (i_n \circ p_n)$, the pointwise multiplication. Kaji and the author [5] showed that the $n$-iterated commutator map of $X$ lies in the commutator subgroup of $[X^{n+1}, X]$ and, by an easy commutator calculus and the above observation, it was obtained:

**Proposition 3.1.** $\text{nil } X < k$ if and only if $\langle \theta_1, \langle \cdots \langle \theta_k, \theta_{k+1} \rangle \cdots \rangle \rangle = 0$ for each $\theta_1, \ldots, \theta_{k+1} \in \{\pm i_1, \ldots, \pm i_n\}$.

We produce formulae for Samelson products which will be useful for our purpose.

**Proposition 3.2.** Let $X$ be a group-like space and let $\theta_i : V_i \to X$ for $i = 1, 2, 3$.

1. If $\langle \pm \theta_1, \langle \pm \theta_2, \pm \theta_3 \rangle \rangle = \langle \pm \theta_2, \langle \pm \theta_3, \pm \theta_1 \rangle \rangle = 0$, then $\langle \pm \theta_3, \langle \pm \theta_1, \pm \theta_2 \rangle \rangle = 0$.

2. $\langle \theta_1, \theta_2 \rangle = 0$ implies $\langle \theta_1, -\theta_2 \rangle = 0$.

3. Let $\theta'_3 : V_3 \to X$. If $\langle \theta_1, \langle \theta_2, \theta_3 \rangle \rangle = \langle \theta_1, \langle \theta_2, \theta'_3 \rangle \rangle = \langle \theta_3, \langle \theta_2, \theta'_3 \rangle \rangle = 0$, then $\langle \theta_1, \langle \theta_2, \theta_3 \theta'_3 \rangle \rangle = 0$.

4. Suppose that $X = X_1 \times \cdots \times X_n$ as spaces and denote by $i_k$ and $p_k$ the inclusion $X_k \to X$ and the projection $X \to X_k$ respectively for $k = 1, \ldots, n$. Then $\langle \theta_1, i_k \circ p_k \circ \theta_2 \rangle = 0$ for $k = 1, \ldots, n$ implies $\langle \theta_1, \theta_2 \rangle = 0$.

**Proof.** 1. Recall first the Hall-Witt formula of groups. Let $G$ be a group and let $[-, -]$ denote the commutator of $G$. Then we have the Hall-Witt formula:

$$[y, [z, x^{-1}]]^x [x, [y, z^{-1}]]^z [z, [x, y^{-1}]]^y = 1$$

for $x, y, z \in G$, where $x^y = yxy^{-1}$.

Let $q_i : V_1 \times V_2 \times V_3 \to V_i$ be the $i$-th projection for $i = 1, 2, 3$. Put $\bar{q}_i = \theta_i \circ q_i$ for $i = 1, 2, 3$.

For $\sigma \in \Sigma_3$, we define $\sigma : V_1 \wedge V_2 \wedge V_3 \to V_{\sigma(1)} \wedge V_{\sigma(2)} \wedge V_{\sigma(3)}$ by $\sigma(v_1, v_2, v_3) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$. Then we have

$$[\bar{\theta}_{\sigma(1)}, [\bar{\theta}_{\sigma(2)}, \bar{\theta}_{\sigma(3)}]] = \sigma^{-1} \circ q^* (\langle \theta_1, \langle \theta_2, \theta_3 \rangle \rangle),$$

where $[-, -]$ denotes the commutator in the group $[V_1 \times V_2 \times V_3, X]$ and $q : X^3 \to X^{(3)}$ is the pinch map. Hence, by hypothesis, we have $[\pm \bar{\theta}_1, [\pm \bar{\theta}_2, \pm \bar{\theta}_3]] = [\pm \bar{\theta}_2, [\pm \bar{\theta}_3, \pm \bar{\theta}_1]] = 0$ and it follows from the Hall-Witt formula that $[\pm \bar{\theta}_3, [\pm \bar{\theta}_1, \pm \bar{\theta}_2]] = 0$. Since $\sigma^{-1}$ and $q^*$ are monic, we have $\langle \pm \theta_3, \langle \pm \theta_1, \pm \theta_2 \rangle \rangle = 0$.
2. This follows from the fact $1_X = (i_1 \circ p_1) \cdots (i_n \circ p_n)$ and the formula

$$[x, yz] = [x, y][x, z]^y$$

for $x, y \in G$.

3. This also follows from the above formula.

4. This follow from the formulae

$$[x, [y, zw]] = [x, [y, z]][x, [z, [y, w]]][y, z][x, [y, w]]$$

for $x, y, z, w \in G$ respectively. \qed

We denote the inclusions $S^{2i-1} \to SU(p+t-1)$, $A_j \to SU(p+t-1)$ and $B_j \to SU(p+t-1)$ by $\epsilon_i$, $\lambda_j$ and $\bar{\lambda}_j$ respectively for $2 \leq i \leq p$ and $2 \leq j \leq t$. We also denote by $\pi_i$ the projections $SU(p+t-1) \to B_i$ for $2 \leq i \leq t$ and $SU(p+t) \to S^{2i-1}$ for $t+1 \leq i \leq p$.

Let $W = A_2 \vee \cdots \vee A_t \vee S^{2t+1} \vee \cdots \vee S^{2p-1}$ and let $j = \lambda_2 \vee \cdots \vee \lambda_t \vee \epsilon_{t+1} \vee \cdots \vee \epsilon_p : W \to SU(p+t-1)$. By (2.2) there is a homotopy retraction $r : \Sigma SU(p+t-1) \to \Sigma W$ of $j$ and as in [7] we can see that there is a self-homotopy equivalence $f : SU(p+t-1) \to SU(p+t-1)$ such that the following square diagram is homotopy commutative.

$$\begin{array}{ccc}
\Sigma SU(p+t-1) & \xrightarrow{f} & \Sigma SU(p+t-1) \\
\downarrow r & & \downarrow \text{adj} \\
\Sigma W & \xrightarrow{\text{adj}} & BSU(p+t-1)
\end{array}$$

Then, for any map $g : \Sigma A \to SU(p+t-1)$, the Whitehead product $[\pm \text{ad} \bar{\lambda}_i, g] = 0$ if and only if $[\pm \text{ad} \lambda_i, g] = 0$. By adjointness of Whitehead products and Samelson products, we have established:

**Proposition 3.3.** For any map $f : X \to SU(p+t)$ and each $i = 1, \ldots, t$ the Samelson product $\langle \pm \bar{\lambda}_i, f \rangle = 0$ if and only if $\langle \pm \lambda_i, f \rangle = 0$. In particular, $\langle \pm \bar{\lambda}_k, \pm \bar{\lambda}_l \rangle = 0$ if and only if $\langle \pm \lambda_k, \pm \lambda_l \rangle = 0$.

**4 Computing the Samelson products**

Let $\Lambda = \{\epsilon_2, \ldots, \epsilon_p, \lambda_2, \ldots, \lambda_t\}$ and $\bar{\Lambda} = \{\bar{\lambda}_2, \ldots, \bar{\lambda}_t\}$, and let $\pm \Lambda = \{\pm \epsilon_2, \ldots, \pm \epsilon_p, \pm \lambda_2, \ldots, \pm \lambda_t\}$ and $\pm \bar{\Lambda} = \{\pm \bar{\lambda}_2, \ldots, \pm \bar{\lambda}_t\}$. We write the domain of $\theta \in \pm \Lambda$ or $\pm \bar{\Lambda}$ by $X(\theta)$. For example, if $\theta = \lambda_i$, then $X(\theta) = A_i$. For $\theta \in \pm \Lambda$ or $\pm \bar{\Lambda}$, we write $|\theta| = i$ if $\theta = \pm \epsilon_i, \pm \lambda_i$ or $\pm \bar{\lambda}_i$. 7
By Proposition 3.1, it is sufficient to calculate the iterated Samelson products \( \langle \theta_1, \langle \cdots \langle \theta_n, \theta_{n+1} \rangle \cdots \rangle \) for \( \theta_1, \ldots, \theta_{n+1} \in \pm \Lambda \) in determining nilSU\((p+ t - 1)\). To do so, we will use the following result of Hamanaka [3].

**Theorem 4.1** (Hamanaka [3]). Let \( X \) be a CW-complex with \( \dim X \leq 2n + 2p - 4 \). Then there is an exact sequence

\[
\tilde{K}^0(X)_{(p)} \cong \bigoplus_{i=0}^{p-2} H^{2n+2i}(X, \mathbb{Z}_{(p)}) \to [X, U(n)]_{(p)} \to \tilde{K}^1(X)_{(p)} \to \bigoplus_{i=0}^{p-3} H^{2n+2i+1}(X, \mathbb{Z}_{(p)})
\]

such that:

1. \( \Theta(x) = \bigoplus_{i=0}^{p-2} (n+i)!ch_{n+i}(x)_{(p)} \) for \( x \in \tilde{K}^0(X) \), where \( ch_k \) is the 2\( k \)-dimensional part of the Chern character.

2. For \( f, g \in [X, U(n)]_{(p)} \), the commutator \( [f, g] \) lies in \( \text{Coker}\Theta \) and represented by

\[
\bigoplus_{k=0}^{p-2} \sum_{i+j=1+n+k} f^*(x_{2i-1}) \cup g^*(x_{2j-1}),
\]

where \( x_{2i-1} \in H^{2i-1}(U(n); \mathbb{Z}_{(p)}) \) is the suspension of the Chern class \( c_i \in H^{2i}(BU(n); \mathbb{Z}_{(p)}) \).

As an easy consequence of Theorem 4.1, Hamanaka [3] showed:

**Proposition 4.1.** \( \text{ord}(\langle \pm \epsilon_i, \pm \epsilon_j \rangle) = \begin{cases} 0 & i + j \leq p + t - 1 \\ p & i + j \geq p + t \end{cases} \)

Now let us calculate other Samelson products of \( \pm \epsilon_i \) and \( \pm \lambda_j \) by applying Theorem 4.1. We have that \( H^*(B_n; \mathbb{Z}_{(p)}) = \Lambda(x_{2n-1}, x_{2n+2p-3}) \) such that the mod \( p \) reduction of \( x_{2n-1} \) and \( x_{2n+2p-3} \) are \( \bar{x}_{2n-1} \) and \( \mathbb{P}^1 \bar{x}_{2n-1} \) respectively. Then \( H^*(A_n; \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)} \langle a_{2n-1}, a_{2n+2p-3} \rangle \) such that \( j_n^*(x_i) = a_i \) for \( i = 2n - 1, 2n + 2p - 3 \), where \( R\langle e_1, e_2, \ldots \rangle \) stands for the free \( R \)-module with a basis \( e_1, e_2, \ldots \) and \( j_n : A_n \to B_n \) is the inclusion.

**Lemma 4.1.** For \( n \leq p \), \( \tilde{K}(\Sigma A_n)_{(p)} = \mathbb{Z}_{(p)} \langle \xi_n, \eta_n \rangle \) such that

\[
ch(\xi_n) = \Sigma a_{2n-1} + \frac{1}{p!}\Sigma a_{2n+2p-3}, \quad ch(\eta_n) = \Sigma a_{2n+2p-3}.
\]

**Proof.** Let \( \gamma \) be the canonical line bundle of \( CP^p \) and let \( \epsilon \in \tilde{K}(CP^p) = [CP^p, BU(\infty)] \) be the composite \( CP^p \xrightarrow{q} S^{2p} \xrightarrow{u} BU(\infty) \) for the pinch map \( q : CP^p \to S^{2p} \) and a generator \( u \) of \( \pi_{2p}(BU(\infty)) \). Note that \( \Sigma CP^p \simeq A_2 \vee S^5 \vee \cdots \vee S^{2p-1} \). By using (2.1), we put \( \xi_n \) and \( \eta_n \) to be the pullback of \( \Sigma^{2n-2}\gamma \) and \( \Sigma^{2n-2}\epsilon \) by the inclusion \( \Sigma A_n \to \Sigma^{2n-2}CP^p \). Then Lemma 4.1 follows from an easy calculation of the Chern character of \( \gamma \) and \( \epsilon \). \qed
Proposition 4.2.  1. For \((i, j) \neq (p, t)\), \(\text{ord}(\langle \pm \epsilon_i, \pm \lambda_j \rangle) = \text{ord}(\langle \pm \lambda_i, \pm \epsilon_t \rangle) = \begin{cases}0 & i + j \leq p + 1 \\ p & i + j \geq p + 2.\end{cases}\)

2. For \(i + j \leq t\), \(\text{ord}(\langle \pm \lambda_i, \pm \lambda_j \rangle) = 0.\)

3. Let \(X(i, j)\) be the \((2i + 2j + 4p - 5)\)-skeleton of \(A_i \wedge A_j\), that is, \(A_i \wedge A_j\) minus the top cell. For \((i, j) \neq (p, p)\), \(\text{ord}(\langle \pm \lambda_i, \pm \lambda_j \rangle|_{X(i,j)}) = \begin{cases}0 & i + j \leq p + 1 \\ p & i + j \geq p + 2.\end{cases}\)

Proof. Let \(p_i : X_1 \times X_2 \rightarrow X_i\) be the \(i\)-th projection for \(i = 1, 2\) and let \(q : X_1 \times X_2 \rightarrow X_1 \wedge X_2\) be the pinch map. For \(f_i : X_i \rightarrow U(n), i = 1, 2\), we have

\[ [f_1 \circ p_1, f_2 \circ p_2] = q^*(\langle f_1, f_2 \rangle) \in [X_1 \times X_2, U(n)] \]

as in the proof of Proposition 3.2. Since \(q^*\) is monic, \(\text{ord}(\langle f_1 \circ p_1, f_2 \circ p_2 \rangle) = \text{ord}(\langle f_1, f_2 \rangle).\) Now if the subcomplex \(Y \subset X_1 \times X_2\) satisfies \(\text{dim} Y \leq 2n + 2p - 4\), it follows from Theorem 4.1 that \([f_1 \circ p_1, f_2 \circ p_2]|_q(Y)\) lies in \(\text{Coker} \Theta \) which is represented by \(\bigoplus_{k=0}^{p-2} \sum_{i+j-1=n+k} g^*(f_1^i(x_{2i-1}) \times f_2^j(x_{2j-1}))\), where \(g : Y \rightarrow X_1 \times X_2\) is the inclusion.

Now we calculate \(\langle \epsilon_i, \lambda_j \rangle\). Note that \(U(n) \simeq SU(n) \times S^1\) as H-spaces, here we localize at the odd prime \(p\). Then we have \(\text{ord}(\langle \epsilon_i, \lambda_j \rangle) = \text{ord}(\langle \epsilon_i', \lambda_j' \rangle)\), where \(\epsilon_i'\) and \(\lambda_j'\) is the compositions \(S^{2i-1} \xrightarrow{\epsilon_i} SU(p + t - 1) \hookrightarrow U(p + t - 1)\) and \(A_i \xrightarrow{\lambda_i} SU(p + t - 1) \hookrightarrow U(p + t - 1)\) respectively. Hence we calculate \(\langle \epsilon_i', \lambda_j' \rangle\). Apply Theorem 4.1 to \(X = S^{2i-1} \times A_j\). Then, by Lemma 4.1, the \(2(i + j + p - 2)\)-dimensional part of \(\text{Coker} \Theta\) is

\[ \mathbb{Z}_p \langle s_{2i-1} \times a_{2j+2p-3} \rangle/(\langle (i+j+p-2)! \rangle s_{2i-1} \times a_{2j+2p-3}), \]

where \(s_{2i-1}\) is a generator of \(H^{2i-1}(S^{2i-1}; \mathbb{Z}_p)\). By definition, \(\epsilon'(x_{2i-1}) = s_{2i-1}\) and \(\lambda'_j(x_{2j+2p-3}) = a_{2j+2p-3}\). Then, by the above observation, \(q^*(\langle \epsilon_i', \lambda_j' \rangle) \in \text{Coker} \Theta\) is represented by \(s_{2i-1} \times a_{2j+2p-3}\). Thus we have calculated \(\text{ord}(\langle \epsilon_i', \lambda_j' \rangle)\). Other Samelson products can be analogously calculated. \(\square\)

In what follows we will often use the argument below implicitly.

Proposition 4.3. Let \(X \rightarrow Y \rightarrow Z\) be a cofiber sequence and let \(W\) be a space such that \([Z, W] = \ast\). If a map \(f : Y \rightarrow W\) satisfies \(f|_X = 0\), then \(f = 0\).

Proof. Proposition 4.3 follows from the exact sequence \([Z, W] \rightarrow [Y, W] \rightarrow [X, W]\) induced from the cofiber sequence \(X \rightarrow Y \rightarrow Z\). \(\square\)
By Theorem 2.1 and Proposition 2.1, the Samelson product \( \langle \pm \theta_1, \pm \theta_2 \rangle \) for \( \theta_1, \theta_2 \in \Lambda \) falls to a single \( B_i \) or \( S^{2j-1} \subset SU(p + t - 1) \) for \( i = 2, \ldots, t \) and \( j = t + 1, \ldots, p \). We shall consider the lifting problem of the above \( \langle \pm \theta_1, \pm \theta_2 \rangle \) when it maps to \( B_i \).

Let us first consider \( \langle \pm \epsilon_i, \pm \epsilon_j \rangle \). Note that we can assume \( i + j \geq p + t \) by Proposition 4.1, which implies that \( \langle \pm \epsilon_i, \pm \epsilon_j \rangle \) falls to \( S^{2(i+j-p)+1} \) for \( i + j \leq 2p - 1 \) to \( B_2 \) for \( i = j = p \). Then it is sufficient to look at the case \( i = j = p \). By Proposition 4.1, \( \text{ord}(\langle \pm \epsilon_p, \pm \epsilon_p \rangle) = p \) and then, by Proposition 2.1, \( \langle \pm \epsilon_p, \pm \epsilon_p \rangle \) lifts to \( S^3 \subset B_2 \). Thus we have obtained:

**Proposition 4.4.** \( \langle \pm \epsilon_i, \pm \epsilon_j \rangle \) falls to \( S^{2(i+j-p)+1} \subset SU(p + t - 1) \) if \( p + t \leq i + j \leq 2p - 1 \) and lifts to \( S^3 \subset B_2 \) if \( i + j = 2p \).

Next we consider \( \langle \pm \epsilon_i, \pm \lambda_j \rangle \) and \( \langle \pm \lambda_j, \pm \epsilon_i \rangle \). In the following calculation, we shall assume the homotopy set \( [\Sigma X, Y] \) is a group by the comultiplication of \( \Sigma X \) and the induced map \( (\Sigma f)^* : [\Sigma X', Y] \to [\Sigma X, Y] \) from \( f : X \to X' \) as a group homomorphism. Now we have the exact sequence induced from the cofiber sequence \( S^{2n+2p-5} \xrightarrow{\alpha_1(2n-2)} S^{2n-2} \to C_{\alpha_1(2n-2)} \) for \( n \geq 3 \):

\[
\pi_{2n-1}(S^{2n-1}) \xrightarrow{\alpha_1(2n-1)^*} \pi_{2n+2p-4}(S^{2n-1}) \to [C_{\alpha_1(2n-2)}, S^{2n-1}] \to \pi_{2n-2}(S^{2n-1})
\]

It follows from Theorem 2.1 that \( \alpha_1(2n-1)^* \) is epic and \( \pi_{2n-2}(S^{2n-1}) = 0 \). Then we obtain:

**Proposition 4.5.** For \( n \geq 3 \), \( [C_{\alpha_1(2n-2)}, S^{2n-1}] = 0 \).

**Corollary 4.1.** For \( p + 2 \leq i + j \leq p + t - 1 \), \( \langle \pm \lambda_i, \pm \epsilon_j \rangle \) and \( \langle \pm \epsilon_j, \pm \lambda_i \rangle \) lift to \( S^{2(i+j-p)+1} \subset B_{i+j-p+1} \).

**Proof.** We only give a proof for \( \langle \epsilon_i, \lambda_j \rangle \) since other ones are analogous. It follows from Proposition 2.1 that \( \langle \epsilon_i, \lambda_j \rangle \) falls to \( B_{i+j-p+1} \subset SU(p + t - 1) \). Since \( S^{2i-1} \wedge A_j = C_{\alpha_2(i+j-2)} \), it follows from Proposition 4.5 that \( q_*(\langle \epsilon_i, \lambda_j \rangle) = 0 \), where \( q : B_{i+j-p+1} \to S^{2(i+j-1)} \) is the projection. Then \( \langle \epsilon_i, \lambda_j \rangle \) lifts to \( S^{2(i+j-p)+1} \) and the proof is completed.

Let us describe the above lift \( f : A_i \wedge S^{2j-1} \to S^{2(i+j-p)+1} \) of the Samelson product \( \langle \lambda_i, \epsilon_j \rangle \). Consider the following commutative diagram in which the row and the column sequences are the exact sequences induced from the cofiber sequence \( S^{2n+2p-4} \to C_{\alpha(2n+2p-4)} \xrightarrow{\partial} S^{2n+4p-5} \).
and the fiber sequence $S^{2n-1} \to B_n \to S^{2n+2p-3}$ respectively.

$$\Sigma C_{\alpha(2n+2p-4)}, S^{2n+2p-3} \xrightarrow{\delta} \pi_{2n+4p-6}(S^{2n-1}) \xrightarrow{q^*} [C_{\alpha(2n+2p-4)}, S^{2n-1}] \xrightarrow{i_*} \pi_{2n+2p-4}(S^{2n-1})$$

Let $\bar{p}: C_{\alpha(2n+2p-4)} \to S^{2n+2p-4}$ be an extension of the degree $p$ self-map of $S^{2n+2p-4}$. Then, by (2.3) and [12, Proposition 1.9], we have

$$\delta(\Sigma \bar{p}) = \alpha(2n-1) \circ \bar{p} = q^* (\{\alpha(2n-1), p, \alpha(2n+2p-4)\}) b = q^*(\alpha_2(2n-1)).$$

On the other hand, it follows from Theorem 2.1 that

$$\text{Im} q^* = \mathbb{Z}/p(q^*(\alpha_2(2n-1))).$$

Then we have established that if $f: C_{\alpha(2n+2p-4)} \to S^{2n-1}$ satisfies $f|_{S^{2n+2p-4}} = 0$, then $i_*(f) = 0$. In particular, it follows from Proposition 4.2 that:

**Proposition 4.6.** For $p + 2 \leq i + j \leq p + t - 1$, any lift of $\langle \lambda_i; \epsilon_j \rangle$ to $S^{2(i+j-p)+1} \subset B_{i+j-p+1}$, say $f$, satisfies $f|_{S^{2n-1}} \not\equiv 0$.

Next we consider the lifting problem of $\langle \pm \lambda_i; \pm \lambda_j \rangle$. Recall from [12, Lemma 3.5] that the cell structure of $C_{\alpha_1(n)} \land C_{\alpha_2(n)}$ for $n \geq p$ is given by

$$C_{\alpha_1(n)} \land C_{\alpha_2(n)} = (C_{\alpha_1(2n)} \lor S^{2n+2p-2} ) \cup_{\nu_n} e^{2n+4p-4},$$

where

$$\nu_n = (i_*(\alpha)) + (-1)^n \alpha_1(2n) \lor \alpha_1(2n+2p-2) \quad (4.1)$$

for the inclusion $i: S^{2n} \to C_{\alpha_1(2n)}$ and some $\alpha \in \pi_{2n+4p-5}(S^{2n})$. Since $n \geq p$, it follows from the Serre isomorphism $\pi_s(S^{2n}) \cong \Sigma \pi_{s-1}(S^{2n-1}) \oplus \pi_s(S^{4n-1})$ that $\alpha$ is a multiple of $\alpha_2(2n)$.

We shall identify $A_i \land A_j$ with $C_{\alpha(i+j-1)} \land C_{\alpha(i+j-1)}$. Consider the following commutative diagram in which the row sequences are the exact sequence induced from the cofiber sequence $A_i \land A_j \to S^{2(i+j+2p-3)} \xrightarrow{f} \Sigma X(i,j)$:

$$\begin{array}{c}
\Sigma X(i,j), S^{k-1} \xrightarrow{f^*} \pi_{k+4p-6}(S^{2k-1}) \xrightarrow{\Sigma \nu_n} [A_i \land A_j, S^{k-1}] \\
\Sigma^{2N} \xrightarrow{\Sigma X(i,j), S^{k+2N-1} (\Sigma^{2N} f)^*} \pi_{k+4p-6+2N}(S^{k+2N-1}) \xrightarrow{\Sigma^{2N}} [\Sigma^{2N} (A_i \land A_j), S^{k+2N-1}]
\end{array}$$
where we put $k = 2(i + j)$. When $N$ is large enough, we have $\Sigma^2 N f = \Sigma k + N - 1$. Let $\bar{p} : C_{\alpha_1(2(i+j) - 1)} \to S^{2(i+j-1)}$ be an extension of the degree $p$ self-map of $S^{2(i+j+N-1)}$. Then, by [12, p.179], we have

\[
(\Sigma^2 N f)^* (\Sigma^2 N \bar{p}) = \{ p, \alpha_1(2(i + j + N) - 1), \alpha_1(2(i + j + N + p - 2)) \} \ni \frac{1}{2} \alpha_2(2(i + j + N) - 1)
\]

as in the proof of Proposition 2.1. On the other hand, it follows from Theorem 2.1 that $\Sigma^2 N : \pi_2(i+j+2p-3) (S^{2(i+j-1)}) \to \pi_2(i+j+2p-3) (S^{2(i+j+N-1)})$ is an isomorphism. Thus we have obtained:

**Proposition 4.7.** The inclusion $X(i,j) \to A_i \wedge A_j$ induces an injection $[A_i \wedge A_j, S^{2(i+j)-1}] \to [X(i,j), S^{2(i+j)-1}]$.

**Corollary 4.2.** For $i + j \leq p$, $\langle \pm \lambda_i, \pm \lambda_j \rangle = 0$.

**Proof.** By Proposition 4.2, it is sufficient to consider the case that $t + 1 \leq i + j \leq p$. In this case, $\langle \pm \lambda_i, \pm \lambda_j \rangle$ falls to $S^{2(i+j)-1} \subset SU(p + t - 1)$ and then the proof is completed by Proposition 4.2 and Proposition 4.7.

**Corollary 4.3.** For $p + 1 \leq i + j \leq 2p - 1$, $\langle \pm \lambda_i, \pm \lambda_j \rangle$ can be compressed into $S^{2(i+j-p)+1} \subset SU(p + t - 1)$.

**Proof.** We only show the case of $\langle \lambda_i, \lambda_j \rangle$ since other cases are similar. By Proposition 2.1 and Proposition 2.1, $\langle \lambda_i, \lambda_j \rangle$ falls to $B_{i+j-p+1}$. Put $\langle \lambda_i, \lambda_j \rangle|_{X(i,j)} = f \vee g : X(i,j) = C_{\alpha_1(2(i+j) - 2)} \vee S^{2(i+j+p-2)} \to B_{i+j-p+1}$. By Proposition 4.5, we have $q_*(f) = 0$ for the projection $q : B_{i+j-p+1} \to S^{2(i+j)-1}$. By Proposition 4.2, $f$ is of order at most $p$ and then, by Proposition 2.1, $q_*(g) = 0$.

Thus, by Proposition 4.7, $q_*(\langle \lambda_i, \lambda_j \rangle) = 0$ and this implies that $\langle \lambda_i, \lambda_j \rangle$ lifts to $S^{2(i+j-p)+1} \subset B_{i+j-p+1}$.

5 Upper bound for nil SU($p + t - 1$)

Hereafter, we suppose that $p \geq 7$.

The aim of this section is to show:

**Theorem 5.1.** nil SU($p + t - 1$) $\leq 3$. 

12
First, here is the proof of Theorem 5.1. By Proposition 3.1 and By Proposition 3.3, it is sufficient to show that

\[ \langle \theta_1, \langle \bar{\theta}_2, \langle \bar{\theta}_3, \bar{\theta}_4 \rangle \rangle \rangle = 0 \] for \( \theta_1 \in \pm \Lambda \) and \( \bar{\theta}_2, \bar{\theta}_3, \bar{\theta}_4 \in \pm \bar{\Lambda} \).

Let \( \omega_1 \in \Lambda \) and let \( \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4 \in \bar{\Lambda} \). It follows from Proposition 3.2 that if \( \langle \pm \langle \pm \bar{\omega}_3, \pm \bar{\omega}_4 \rangle, \langle \pm \bar{\omega}_2, \pm \omega_1 \rangle \rangle = \langle \pm \bar{\omega}_2, \pm \omega_1, \pm \langle \pm \bar{\omega}_3, \pm \bar{\omega}_4 \rangle \rangle = 0 \), then \( \langle \pm \omega_1, \langle \pm \langle \pm \bar{\omega}_3, \pm \bar{\omega}_4 \rangle, \pm \bar{\omega}_2 \rangle \rangle = 0 \). By Proposition 3.3, this implies \( \langle \pm \omega_1, \langle \pm \bar{\omega}_2, \langle \pm \bar{\omega}_3, \pm \bar{\omega}_4 \rangle \rangle = 0 \). On the other hand, by Proposition 3.2, if \( \langle \pm \bar{\omega}_3, \langle \pm \omega_4, \pm \langle \pm \bar{\omega}_2, \pm \omega_1 \rangle \rangle \rangle = \langle \pm \bar{\omega}_3, \langle \pm \bar{\omega}_2, \pm \omega_1 \rangle \rangle \rangle = 0 \), then \( \langle \pm \langle \pm \bar{\omega}_2, \pm \omega_1 \rangle, \langle \pm \bar{\omega}_3, \pm \bar{\omega}_4 \rangle \rangle = 0 \). By Proposition 3.2, this implies \( \langle \pm \langle \pm \bar{\omega}_3, \pm \bar{\omega}_4 \rangle, \langle \pm \bar{\omega}_2, \pm \omega_1 \rangle \rangle = 0 \). Thus the proof is completed by the following propositions.

**Proposition 5.1.** \( \langle \theta_1, \langle \theta_2, \langle \theta_3, \theta_4 \rangle \rangle \rangle = 0 \) for \( \theta_1, \theta_2, \theta_3, \theta_4 \in \pm \Lambda \) and \( \bar{\theta}_3, \bar{\theta}_4 \in \pm \bar{\Lambda} \).

**Proposition 5.2.** \( \langle \theta_1, \langle \theta_2, \langle \theta_3, \theta_4 \rangle \rangle \rangle = \langle \theta_1, \langle \theta_2, \langle \theta_3, \theta_4 \rangle \rangle \rangle = 0 \) for \( \theta_1, \theta_2, \theta_3, \theta_4 \in \pm \bar{\Lambda} \) and \( |\theta_3| + |\theta_4| \neq 2p \).

**Proposition 5.3.** \( \langle \pm \lambda_p, \langle \pm \bar{\lambda}_p, \langle \theta_1, \theta_2 \rangle \rangle \rangle = 0 \) for \( \theta_1, \theta_2 \in \pm \bar{\Lambda} \).

We will calculate iterated Samelson products in \( \pm \bar{\Lambda} \) from those in \( \pm \Lambda \) by using the following lemma.

**Lemma 5.1.** Let

\[ X = \bigvee_{i=1}^{n_1} S^{2np-3} \cup \bigcup_{i=1}^{n_2} \bigcup_{i=1}^{2np-3+2(p-1)} \cup \cdots \cup \bigcup_{i=1}^{n_k} \bigcup_{i=1}^{2np-3+2(k-1)(p-1)} \]

and let \( f : X \to SU(p+t-1) \). If \( n+k \leq p \), then \( f \) can be compressed into \( S^{2n-1} \subset SU(p+t-1) \) and \( \Sigma^{2k} f = 0 \).

**Proof.** If \( f \) falls to \( B_n \), it follows from Theorem 2.1 that \( q_*(f) = 0 \) for the projection \( q : B_n \to S^{2n+2p-3} \) and then \( f \) lifts to \( S^{2n-1} \subset B_n \). Thus we assume that \( f \) is a map from \( X \) to \( S^{2n-1} \). Consider the exact sequence induced from the cofiber sequence \( V_{i=1}^{n_1} S^{2np-3} \xrightarrow{j} X \xrightarrow{q} \)

\[ X/(\bigvee_{i=1}^{n_1} S^{2np-3}) = Y : [Y, S^{2n-1}] \xrightarrow{(q)'} [X, S^{2n-1}] \xrightarrow{j'} \bigoplus_{i=1}^{n_1} \pi_{2np-3}(S^{2n-1}) \]

It follows from Theorem 2.1 that \( (\Sigma^2 q)'^*(\Sigma^2 f) = 0 \) and then there exists \( g : \Sigma^2 Y \to S^{2n+1} \) such that \( (\Sigma^2 q)'^*(g) = \Sigma^2 f \). By induction, we obtain \( \Sigma^{2k} f = 0 \). \( \Box \)
Corollary 5.1. Let $X = S^{2n-1}$ or $S^{2n-1} \cup e^{2n+2p-3}$ for $n \leq 5p-3$ and let $f : X \to SU(p+t-1)$. Then $\langle \theta, f \rangle = \langle f, \theta \rangle = 0$ for each $\theta \in \pm \bar{\Lambda}$.

Proof. By Corollary 2.1, we only have to consider the case $2n-1 = 6p-3, 8p-5, 8p-3, 10p-7$. Then it follows from Lemma 5.1 that $f$ can be compressed into $S^5$ or $S^7 \subset SU(p+t-1)$, and that $\Sigma^4f = 0$. By Proposition 4.2, we assume $|\theta| \geq p-2$. Since $p \geq 7$, $X(\theta)$ is a 6-suspension and then $1_{X(\theta)} \wedge f = f \wedge 1_{X(\theta)} = 0$. □

We give candidates for non-zero 2-iterated Samelson products in $\pm \bar{\Lambda}$.

Proposition 5.4. Let $\theta_1, \theta_2, \theta_3 \in \pm \bar{\Lambda}$. If $|\theta_1| + |\theta_2| + |\theta_3| \neq 2p + 1, 2p + 2, 2p + 3, 3p$, then $\langle \theta_1, \langle \theta_2, \theta_3 \rangle \rangle = 0$.

Proof. Suppose that $|\theta_1| + |\theta_2| + |\theta_3| \neq 2p + 1, 2p + 2, 2p + 3, 3p$. By Proposition 3.3, it is sufficient to show that $\langle \theta_1, \langle \theta_2, \theta_3 \rangle \rangle = 0$ for $\theta_1 \in \pm \Lambda$ and $\theta_2, \theta_3 \in \pm \bar{\Lambda}$.

By Corollary 2.1, $\langle \theta_1, \langle \theta_2, \theta_3 \rangle \rangle = 0$ if $\theta_2, \theta_3 \in \pm \Lambda$. Then, by Proposition 3.2 and Proposition 3.3, it is sufficient to show that $\langle \theta_1, \langle \theta_2, \pm \bar{\Lambda} \rangle \rangle = \langle \theta_1, \langle \pm \bar{\Lambda}_i, \theta_2 \rangle \rangle = 0$ for $\theta_1, \theta_2 \in \pm \Lambda$. Since other cases are analogous, we only show $\langle \lambda_i, \langle \lambda_j, \bar{\lambda}_k \rangle \rangle = 0$. When $j \geq 3$, $A_j$ is a suspension by (2.1). Then it follows from (2.2) that $\langle \lambda_j, \bar{\lambda}_k \rangle = \langle \lambda_j, \lambda_k \rangle \vee f : A_j \wedge B_k = (A_j \wedge A_k) \vee (A_j \wedge S^{4k+2p-1}) \to SU(p+t-1)$. By Corollary 2.1, we have $\langle \lambda_i, \langle \lambda_j, \lambda_k \rangle \rangle = 0$ and, by Corollary 5.1, $\langle \lambda_i, f \rangle = 0$. Then we have established $\langle \lambda_i, \langle \lambda_j, \bar{\lambda}_k \rangle \rangle = 0$.

When $j = 2$, we assume $k = p - 1$ or $p$ by Proposition 4.2. It follows from Theorem 2.1 and Proposition 2.1 that $\langle \lambda_2, \bar{\lambda}_{p-1} \rangle$ falls to $B_2$. By Corollary 4.3 and Theorem 2.1, we have $q_*(\langle \lambda_2, \bar{\lambda}_{p-1} \rangle) = 0$ for the projection $q : B_2 \to S^{2p+1}$. Then $\langle \lambda_2, \bar{\lambda}_{p-1} \rangle$ lifts to $f : A_2 \wedge B_{p-1} \to S^9$. Hence, by Proposition 4.2, $\langle \lambda_i, f \rangle = 0$ if $i \leq p - 1$ and this shows that $\langle \lambda_i, \langle \lambda_j, \bar{\lambda}_k \rangle \rangle = 0$ when $(j, k) = (2, p - 1)$. One can analogously show that $\langle \lambda_i, \langle \lambda_j, \bar{\lambda}_k \rangle \rangle = 0$ when $(j, k) = (2, p)$. □

Proof of Proposition 5.3. As in the above proof of Theorem 5.1, Proposition 5.1 implies that it is sufficient to prove $\langle \pm \langle \theta_1, \theta_2 \rangle, \langle \pm \bar{\lambda}_p, \pm \lambda_p \rangle \rangle = 0$.

By Proposition 5.4, we have only to consider the case that $|\theta_1| + |\theta_2| = p + 1, p + 2, p + 3$ or $2p$. When $|\theta_1| + |\theta_2| = p + 1$, $\langle \theta_1, f \rangle$ falls to $B_2 \times S^5 \times S^7$, $B_2 \times B_3 \times S^7$ or $B_2 \times B_3 \times B_4$ by Theorem 2.1 and Proposition 2.1. On the other hand, $\langle \pm \bar{\lambda}_p, \pm \lambda_p \rangle$ falls to $B_2 \times S^5$ or $B_2 \times B_3$ by Theorem 2.1 and Proposition 2.1. Then, by Proposition 3.2, Proposition 4.2 and Corollary 4.2, we have obtained that $\langle \pm \lambda_p, \langle \pm \bar{\lambda}_p, \langle \theta_1, \theta_2 \rangle \rangle \rangle = 0$. Other cases are quite analogous. □

Now we proceed the calculation to show all 3-iterated Samelson products in $\bar{\Lambda}$ vanish. As a first step, we show:
Proposition 5.5. $\langle \theta_1, \langle \theta_2, \langle \theta_3, \theta_4 \rangle \rangle \rangle = 0$ for $\theta_1, \ldots, \theta_4 \in \pm \Lambda$.

Proof. By Proposition 5.4, we assume that $|\theta_2| + |\theta_3| + |\theta_4| = 2p + 1, 2p + 2, 2p + 3$ or $3p$. We only show the case that $(\theta_1, \theta_2, \theta_3) = (\lambda_i, \lambda_j, \lambda_k)$ for $i + j + k = 2p + 3$ since other cases are analogous. By Corollary 2.1, there is a homotopy commutative diagram:

\[
\begin{array}{ccc}
A_i \wedge A_j \wedge A_k & \xrightarrow{\langle \lambda_i, \lambda_j, \lambda_k \rangle} & SU(p + t - 1) \\
q \downarrow & & \downarrow \\
\left( \bigvee_{i=1}^3 S^8_{(p-3)} \right) \cup e^{10p-5} f & \xrightarrow{\langle \theta_3, \pm \lambda_i \rangle \vee f} & SU(p + t - 1),
\end{array}
\]

where $q$ pinches the $(8p - 4)$-skeleton of $A_i \wedge A_j \wedge A_k$. It follows from Lemma 5.1 that $f$ can be compressed into $S^7 \subset SU(p + t - 1)$ and that $\Sigma^1 f = 0$. Then, by Proposition 4.2, we assume that $i \geq p - 2$ and this implies that $f$ is a 6-suspension. Hence we have $1_{A_i} \wedge f = 0$ and this completes the proof.

Corollary 5.2. $\langle \theta_1, \langle \theta_2, \langle \theta_3, \theta_4 \rangle \rangle \rangle = \langle \theta_1, \langle \theta_2, \langle \theta_4, \theta_3 \rangle \rangle \rangle = 0$ for $\theta_1, \theta_3, \theta_3 \in \pm \Lambda$ and $\theta_4 \in \pm \bar{\Lambda}$.

Proof. By Proposition 5.5, we put $\theta_4 = \pm \bar{\lambda}_i$.

We first consider the case that $\theta_3 \neq \pm \lambda_2$. Since $X(\theta_3)$ is a suspension, we have the following homotopy commutative diagram by (2.2).

\[
\begin{array}{ccc}
X(\theta_3) \wedge B_i & \xrightarrow{\langle \theta_3, \pm \bar{\lambda}_i \rangle} & SU(p + t - 1) \\
\downarrow & & \downarrow \\
(X(\theta_3) \wedge A_i) \vee (X(\theta_3) \wedge S^{4i+2p-4}) & \xrightarrow{\langle \theta_3, \pm \lambda_i \rangle \vee f} & SU(p + t - 1)
\end{array}
\]

Then we have

$\langle \theta_1, \langle \theta_2, \langle \theta_3, \pm \bar{\lambda}_i \rangle \rangle \rangle = \langle \theta_1, \langle \theta_2, \langle \theta_3, \pm \lambda_i \rangle \rangle \rangle \vee \langle \theta_1, \langle \theta_2, f \rangle \rangle :$

\[
\bigwedge_{i=1}^3 X(\theta_i) \wedge B_i \simeq \left( \bigwedge_{i=1}^3 X(\theta_i) \wedge A_i \right) \vee \left( \bigwedge_{i=1}^3 X(\theta_i) \wedge S^{4i+2p-4} \right) \rightarrow SU(p + t - 1)
\]

Thus, by Corollary 5.1 and Proposition 5.5, we have established $\langle \theta_1, \langle \theta_2, \langle \theta_3, \pm \bar{\lambda}_i \rangle \rangle \rangle = 0$. It is analogous to show $\langle \theta_1, \langle \theta_2, \langle \pm \bar{\lambda}_i, \theta_3 \rangle \rangle \rangle = 0$.

We next consider the case that $\theta_3 = \pm \lambda_2$. By Corollary 4.2 and Proposition 5.5, we assume that $\theta_4 = \pm \bar{\lambda}_{p-1}$ or $\pm \lambda_p$. It follows from Corollary 4.3 that we also assume $\langle \pm \lambda_2, \pm \bar{\lambda}_i \rangle : A_2 \wedge B_i \rightarrow S^{2(2i+p)+1}$. Then, by (2.2), we have a homotopy commutative diagram:

\[
\begin{array}{ccc}
\Sigma^2 (A_2 \wedge B_i) & \xrightarrow{\Sigma(\pm \lambda_2, \pm \bar{\lambda}_i)} & S^{2(2i+p)+3} \\
\downarrow & & \downarrow \\
\Sigma^2 (A_2 \wedge A_i) \vee \Sigma^2 (A_2 \wedge S^{4i+2p-4}) & \xrightarrow{\Sigma^2(\pm \lambda_2, \pm \bar{\lambda}_i) \vee f} & S^{2(2i+p)+3}
\end{array}
\]
By Proposition 5.4, we also assume that $|\theta_2| + |\theta_3| + |\lambda_i| = 2p + 1, 2p + 2, 2p + 3$ or $3p$ and this implies that $X(\theta_2)$ is a 6-suspension. Then we have

$$\langle \theta_2, (\pm \lambda_2, \pm \lambda_i) \rangle = \langle \theta_2, (\pm \lambda_2, \pm \lambda_i) \rangle \vee (\langle \theta_2, \epsilon_{3+i-p} \rangle \circ (1_{\Sigma-2X(\theta_2)} \wedge f)) : (X(\theta_2) \wedge A_2 \wedge A_i) \vee (X(\theta_2) \wedge A_2 \wedge S^{2(2+i-p)+1}) \to SU(p + t - 1).$$

By Corollary 5.1, we also have $1_{\Sigma-2X(\theta_1) \wedge f} = 0$ and then, by Proposition 5.5, we have obtained $\langle \theta_1, \theta_2, (\pm \lambda_2, \pm \lambda_i) \rangle = 0$. We can similarly see that $\langle \theta_1, \theta_2, (\pm \lambda_i, \pm \lambda_2) \rangle = 0$.

**Proof of Proposition 5.1.** By Proposition 5.5 and Corollary 5.2, we put $\theta_3 = \pm \lambda_i$ and $\theta_4 = \pm \lambda_j$.

Applying the homotopy extension property of the inclusion $\Sigma A_i \wedge A_j \to \Sigma A_i \wedge B_j$, we replace a homotopy retraction $\Sigma A_i \wedge B_j \to \Sigma A_i \wedge A_j$ with a strict retraction. We also replace a homotopy retraction $\Sigma A_i \wedge B_j \to \Sigma A_i \wedge A_j$ with a strict one.

Let $Y(i, j)$ be the $(4i + 4j + 4p - 7)$-skeleton of $B_i \wedge B_j$, that is, $Y(i, j)$ is $B_i \wedge B_j$ minus the top cell. Since we have strict retractions $\Sigma A_i \wedge B_j \to \Sigma A_i \wedge A_j$ and $\Sigma A_i \wedge B_j \to \Sigma A_i \wedge A_j$, the proof of Corollary 5.2 implies that we can choose contractions of $\langle \theta_1, \theta_2, (\pm \lambda_i, \pm \lambda_j) \rangle$ and $\langle \theta_1, \theta_2, (\pm \lambda_i, \pm \lambda_j) \rangle$ to coincide on $X(\theta_1) \wedge X(\theta_2) \wedge A_i \wedge A_j$. Then, by gluing the above contractions, we obtain

$$\langle \theta_1, \theta_2, (\pm \lambda_i, \pm \lambda_j) |_{Y(i,j)} \rangle = 0 \quad (5.1)$$

for $\theta_1, \theta_2 \in \pm \Lambda$.

Now we first consider the case $\theta_2 \neq \pm \lambda_2$. As in the proof of Corollary 5.2, we have

$$\langle \theta_2, (\pm \lambda_i, \pm \lambda_j) \rangle = \langle \theta_2, (\pm \lambda_i, \pm \lambda_j) |_{Y(i,j)} \rangle \vee f : X(\theta_2) \wedge B_i \wedge B_j \simeq (X(\theta_2) \wedge Y(i, j)) \vee (X(\theta_2) \wedge S^{4(i+j+p-2)}) \to SU(p + t - 1).$$

Then, for $(\theta_1, \theta_2, \theta_3) \neq (\pm \lambda_p, \pm \lambda_p, \pm \lambda_p)$, we have $\langle \theta_1, \theta_2, (\pm \lambda_i, \pm \lambda_j) \rangle = 0$ by Corollary 5.1 and (5.1).

By Proposition 2.1, $\langle \pm \lambda_p, \pm \lambda_p \rangle$ falls to $B_2 \times B_3 \subset SU(p + t - 1)$. Then, by Proposition 3.2, it is sufficient to show that $\langle \theta_1, \pm \lambda_p, \lambda_i \circ \pi_i \circ \langle \pm \lambda_p, \pm \lambda_p \rangle \rangle = 0$ for $i = 2, 3$ for $\theta_1 \in \pm \Lambda$.

Analogously to the above case, we have

$$\langle \pm \lambda_p, \lambda_i \circ \pi_i \circ \langle \pm \lambda_p, \pm \lambda_p \rangle \rangle = \langle \pm \lambda_p, \lambda_i \circ \pi_i \circ \langle \pm \lambda_p, \pm \lambda_p \rangle |_{Y(p,p)} \rangle \vee f_i : A_p \wedge B_p \wedge B_p \simeq (A_p \wedge Y(p, p)) \vee (A_p \wedge S^{12p-8}) \to SU(p + t - 1).$$
By (5.1), it is sufficient to show \( \langle \theta_1, f_i \rangle = 0 \) for \( i = 2, 3 \). By [13], we have \( \pi_{14p−9}(S^3) = \pi_{16p−11}(S^3) = 0 \) and then \( \pi_{14p−9}(B_2) = \pi_{16p−11}(B_2) = 0 \) by the homotopy exact sequence of the fibration \( S^3 → B_2 → S^{2p+1} \) and Theorem 2.1. Thus \( f_2 = 0 \). Similarly, we have \( f_3 = 0 \).

We next consider the case \( \theta_2 = ±\lambda_2 \). By Proposition 5.4, we put \( (i, j) = (p − 1, p), (p, p − 1), (p, p) \). When \( (i, j) = (p, p) \), it follows from Proposition 5.4 that \( |\theta_2| = 2 \) or 3. By Proposition 2.1, \( \langle ±\bar{\lambda}_p, ±\bar{\lambda}_p \rangle \) falls to \( B_2 × B_3 \subset SU(p + t − 1) \). Then, by Proposition 3.2 and Corollary 4.2, we have \( \langle \theta_2, \langle ±\bar{\lambda}_p, ±\bar{\lambda}_p \rangle \rangle = 0 \). We shall prove the case \( (i, j) = (p − 1, p) \). The case \( (i, j) = (p, p − 1) \) is quite analogously proved. By Proposition 2.1, \( \langle ±\bar{\lambda}_{p−1}, ±\bar{\lambda}_p \rangle \) falls to \( B_3 \) and then \( \langle ±\lambda_2, ±\bar{\lambda}_{p−1}, ±\bar{\lambda}_p \rangle \) falls to \( B_3 × B_4 \). Moreover, by Theorem 2.1 and Corollary 4.3, we can see that \( \pi_i \circ \langle ±\lambda_2, ±\bar{\lambda}_{p−1}, ±\bar{\lambda}_p \rangle \) can be compressed into \( S^{2i−1} \) for \( i = 3, 4 \). Then, by Proposition 3.2, we assume \( |\theta_1| ≥ p − 2 \) and then \( X(\theta_1) \) is a 6-suspension. By an analogous argument as above, we have

\[
\Sigma^2\pi_1 \circ \langle ±\lambda_2, ±\bar{\lambda}_{p−1}, ±\bar{\lambda}_p \rangle = \Sigma^2\pi_1 \circ \langle ±\lambda_2, ±\bar{\lambda}_{p−1}, ±\bar{\lambda}_p \rangle|_{Y(p−1,p)} \vee f : \\
\Sigma^2 A_2 \wedge B_{p−1} \wedge B_p \simeq \Sigma^2(A_2 \wedge Y(p−1,p)) \vee \Sigma^2(A^2 \wedge S^{12p−12}) \to S^{2i+1}
\]

for \( i = 3, 4 \). Then as in the proof of Corollary 5.2, we have obtained \( \langle \theta_1, \langle ±\lambda_2, ±\bar{\lambda}_{p−1}, ±\bar{\lambda}_p \rangle \rangle = 0 \).

In order to calculate other Samelson products, we will use:

**Lemma 5.2.** Let \( g : V → W_1 \vee W_2 \) and let \( f_i : W_i → X \) for \( i = 1, 2 \). Suppose that \( f_i \circ p_i \circ g = 0 \) for \( i = 1, 2 \) and that \( X \) is an \( H \)-space, where \( p_i : W_1 \vee W_2 → W_i \) is the \( i \)-th projection. Then \( (f_1 \vee f_2) \circ g = 0 \).

**Proof.** Define \( f_1 \cdot f_2 : W_1 × W_2 → X \) by \( f_1 \cdot f_2(x, y) = f_1(x)f_2(y) \) for \( (x, y) ∈ W_1 × W_2 \). Then we have a homotopy commutative diagram

\[
\begin{array}{ccc}
V \xrightarrow{g} W_1 \vee W_2 \xrightarrow{f_1 \vee f_2} X \\
\downarrow j \quad \downarrow j \\
V \xrightarrow{\text{incl}} W_1 × W_2 \xrightarrow{f_1 \cdot f_2} X,
\end{array}
\]

where \( j \) is the inclusion. This completes the proof. \( \square \)

**Proof of Proposition 5.2.** We first consider the case \( \theta_3 ≠ ±\lambda_2 \). If \( \theta_4 = ±\bar{\lambda}_i \), we have

\[
\langle \theta_3, ±\bar{\lambda}_i \rangle = \langle \theta_3, ±\lambda_i \rangle \vee f : \\
X(\theta_3) \wedge B_i \simeq (X(\theta_3) \wedge A_i) \vee (X(\theta_3) \wedge S^{4i+2p−4}) \to SU(p + t − 1)
\]
and \( \langle \pm \tilde{\lambda}_i, \theta_3 \rangle \) has an analogous decomposition. Then, by Corollary 5.1 and Proposition 5.5, it is sufficient to show that \( \langle \theta_1, \langle \pm \tilde{\lambda}_j, \langle \theta_3, \theta_4 \rangle \rangle \rangle = 0 \) for \( \theta_1, \theta_3, \theta_4 \in \pm \Lambda \). Since \( |\theta_3| + |\theta_4| \neq 2p \), we have \( \langle \theta_3, \theta_4 \rangle : X(\theta_3) \land X(\theta_4) \to S^{2(i+j-p)+1} \) by Proposition 4.4, Corollary 4.1 and Corollary 4.3. Since

\[
\langle \pm \tilde{\lambda}_j, \epsilon_{i+j-p+1} \rangle = \langle \pm \lambda_j, \epsilon_{i+j-p+1} \rangle \lor f : B_j \land S^{2(i+j-p+1)-1} \simeq (A_j \land S^{2(i+j-p+1)-1}) \lor S^{2i+6j-3} \to SU(p + t - 1).
\]

Then, by Corollary 5.1, Proposition 5.5 and Lemma 5.2, we have obtained \( \langle \theta_1, \langle \pm \tilde{\lambda}_j, \langle \theta_3, \theta_4 \rangle \rangle \rangle = 0 \).

We next consider the case \( \theta_3 = \pm \lambda_2 \). By Corollary 4.2, \( \theta_4 = \pm \tilde{\lambda}_{p-1} \) or \( \pm \tilde{\lambda}_p \) and then, by Proposition 5.5, \( \theta_2 = \pm \tilde{\lambda}_p \) and \( \theta_2 = \pm \tilde{\lambda}_{p-1} \) or \( \tilde{\lambda}_p \) according as \( \theta_4 = \pm \tilde{\lambda}_{p-1}, \pm \tilde{\lambda}_p \). Now we consider the case \( \theta_4 = \pm \tilde{\lambda}_{p-1} \). By Proposition 2.1, \( \langle \pm \lambda_3, \pm \tilde{\lambda}_{p-1} \rangle \) falls to \( B_2 \subset SU(p + t - 1) \). Moreover, by Theorem 2.1 and Proposition 4.1, \( \langle \pm \lambda_2, \pm \tilde{\lambda}_{p-1} \rangle \) can be compressed into \( S^3 \subset SU(p + t - 1) \).

Since

\[
\langle \pm \tilde{\lambda}_p, \epsilon_2 \rangle = \langle \pm \lambda_p, \epsilon_2 \rangle \lor f : B_p \land S^3 \simeq (A_p \land S^3) \lor (S^6p-4 \land S^3) \to SU(p + t - 1).
\]

By Corollary 5.1, we have \( \langle \theta_1, f \rangle = 0 \). Then, by Lemma 5.2, it is sufficient to show that \( \langle \theta_1, \langle \pm \lambda_p, \langle \pm \lambda_2, \pm \tilde{\lambda}_{p-1} \rangle \rangle \rangle = 0 \) and this is done by Corollary 5.2. \( \langle \theta_1, \langle \pm \lambda_p, \langle \pm \tilde{\lambda}_{p-1}, \pm \lambda_2 \rangle \rangle \rangle = 0 \) is shown in an analogous way.

Let us consider the case \( \theta_4 = \pm \tilde{\lambda}_p \). As above, \( \langle \pm \lambda_2, \pm \tilde{\lambda}_p \rangle \) can be compressed into \( S^5 \times S^7 \) and then, by Proposition 3.2, it is sufficient to show that \( \langle \theta_1, \langle \pm \lambda_2, \epsilon_i \circ \pi_i \circ \langle \pm \lambda_2, \pm \tilde{\lambda}_p \rangle \rangle \rangle = 0 \) for \( i = 3, 4 \). This is done quite analogously to the above case. We can also see that \( \langle \theta_1, \langle \theta_2, \langle \pm \tilde{\lambda}_p, \pm \lambda_2 \rangle \rangle \rangle = 0 \) as well.

\section{Proof of Theorem 1.1}

\subsection{t = 2}

We shall show \( \langle \epsilon_{p-1}, \langle \lambda_2, \epsilon_p \rangle \rangle \neq 0 \) and then, by Theorem 5.1, the proof of Theorem 1.1 is completed. By Theorem 2.1 and Proposition 2.1, \( \langle \lambda_2, \epsilon_p \rangle \) falls to \( S^5 \subset SU(p + 1) \). Since \( \langle \epsilon_2, \epsilon_p \rangle \neq 0 \) by Proposition 4.1, we have \( \langle \lambda_2, \epsilon_p \rangle = a\alpha_1(5) \) for some integer \( a \) such that \( a \neq 0 \) (p), where \( \alpha_1(5) : C_{\alpha_1(2p+2)} \simeq A_2 \land S^{2p-1} \to S^5 \) is an extension of \( \alpha_1(5) \). Analogously, we have \( \langle \epsilon_{p-1}, \epsilon_3 \rangle = b\alpha_1(5) \) for an integer \( b \) such that \( b \neq 0 \) (p). Then, by [12, Proposition 1.9],

\[
\langle \epsilon_{p-1}, \langle \lambda_2, \epsilon_p \rangle \rangle = ab\alpha_1(5) \circ \Sigma^{2p-3} \alpha_1(5) = abq^*(\{\alpha_1(5), \alpha_1(2p + 2), \alpha_1(4p - 1)\}),
\]

18
where \( q : S^{2p-3} \land A_2 \land S^{2p-1} \to S^{6p-3} \) pinches the bottom cell.

Consider the exact sequence induced from the cofiber sequence \( S^{2p-3} \land A_2 \land S^{2p-1} \xrightarrow{q} S^{6p-3} \xrightarrow{\alpha_1(4p)} S^{4p} \):

\[
\pi_{4p}(S^5) \xrightarrow{\alpha_1(4p)^*} \pi_{6p-3}(S^5) \xrightarrow{q^*} [S^{2p-3} \land A_2 \land S^{2p-1}, S^5]
\]

By Theorem 2.1, \( \alpha_1(4p)^* = 0 \) and then \( q^* \) is monic. It is known that \( \{\alpha_1(5), \alpha_1(2p+2), \alpha_1(4p-1)\} \neq 0 \) (See, for example, [4, P. 38]) and thus, by (6.1), we have established \( \langle \epsilon_{p-1}, \langle \lambda_2, \epsilon_p \rangle \rangle \neq 0 \).

**6.2** \[ 3 \leq t \leq \frac{p-1}{2} \]

By Proposition 4.2, possible non-trivial 2-iterated Samelson products in \( \pm \bar{\Lambda} \) are:

1. \( \langle \pm \epsilon_p, \langle \pm \epsilon_p, \pm \epsilon_p \rangle \rangle \).

2. \( \langle \pm \bar{\lambda}_i, \langle \pm \epsilon_j, \pm \epsilon_k \rangle \rangle, \langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle, \langle \pm \epsilon_i, \langle \pm \epsilon_j, \pm \bar{\lambda}_k \rangle \rangle \) for \( i + j + k = 2p + 1, 2p + 2, 2p + 3 \).

We shall show these Samelson products are all trivial and then, by Proposition 3.1, the proof is completed.

1. By the Jacobi identity of Samelson products, we have \( 3 \langle \pm \epsilon_p, \langle \pm \epsilon_p, \pm \epsilon_p \rangle \rangle = 0 \) and then, for \( p > 3 \), \( \langle \pm \epsilon_p, \langle \pm \epsilon_p, \pm \epsilon_p \rangle \rangle = 0 \).

2. By Proposition 3.2, it is sufficient to show \( \langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle \) for \( i + j + k = 2p + 1, 2p + 2, 2p + 3 \). Let us consider \( \langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle \) for \( i + j + k = 2p + 1 \). By (2.2), we have \( \langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle = \langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle \lor \langle \epsilon_i, f \rangle \) for some \( f : S^{4j+2p-4} \land S^{2k-1} \to SU(p+t-1) \). Then, by Corollary 5.1, it is sufficient to show \( \langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle = 0 \).

Let us consider the case \( i + j + k = 2p + 1 \). By Proposition 4.3, \( \langle \pm \lambda_j, \pm \epsilon_k \rangle \) can be compressed into \( S^{2(j+k-p)+1} \subset SU(p+t-1) \) and then we have

\[
\langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle = \langle \pm \epsilon_i, \epsilon_{j+k-p+1} \rangle \lor (1_{S^{2t-1}} \land f),
\]

where \( f : A_j \land S^{2k-1} \to S^{2(j+k-p)+1} \). Since \( i + j + k - p + 1 = p + 2 \leq p + t - 1 \), we have \( \langle \pm \epsilon_i, \epsilon_{j+k-p+1} \rangle = 0 \) and then \( \langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle = 0 \). Analogously, we can see \( \langle \pm \epsilon_i, \langle \pm \epsilon_j, \pm \bar{\lambda}_k \rangle \rangle = 0 \).

When \( i + j + k = 2p + 2, 2p + 3 \), it follows from Corollary 2.1 that \( \langle \pm \epsilon_i, \langle \pm \lambda_j, \pm \epsilon_k \rangle \rangle = 0 \).

**6.3** \[ \frac{p+1}{2} \leq t \leq p \]

Put \( t \neq p \). We shall show \( \langle \lambda_{p-t+1}, \langle \lambda_t, \epsilon_p \rangle \rangle \neq 0 \) and this completes the proof of Theorem 1.1 by Theorem 5.1. Let \( X \) be the \((8p-4)\)-skeleton of \( A_{p-t+1} \land A_t \land S^{2p-1} \), that is, \( A_{p-t+1} \land A_t \land S^{2p-1} \).
minus the top cell. Then, as in section 4, the cofiber sequence $S^{2(p-t) + 1} \wedge A_t \wedge S^{2p-1} \to X \xrightarrow{q} S^{6p-3}$ splits. We denote a homotopy section of $q$ by $s$. Here, note that the map $q$ is the restriction of $q \wedge 1_{A_t \wedge S^{2p-1}} : A_{p-t+1} \wedge A_t \wedge S^{2p-1} \to S^{2(p-t)-1} \wedge A_t \wedge S^{2p-1}$, where $q : A_{p-t+1} \to S^{2(p-t)-1}$ is the pinch map. Then, by Proposition 4.1, we have a homotopy commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{(1_{S^{2(p-t)-1}} \wedge \langle \lambda_t, \epsilon_p \rangle) |_{X}} & A_{p-t+1} \wedge S^{2t+1} \langle \lambda_{p-t+1}, \epsilon_{t+1} \rangle \xrightarrow{q} B_3 \\
\downarrow{\bar{q}} & & \downarrow{j} \\
S^{2(p-t)-1} \wedge S^{2t-1} \wedge S^{2p-1} & \xrightarrow{1_{S^{2(p-t)-1}} \wedge \langle \epsilon_t, \epsilon_p \rangle} & S^{2(p-t)-1} \wedge S^{2t+1} \xrightarrow{f} B_3
\end{array}
$$

By Theorem 2.1 and Proposition 4.1, we have $1_{S^{2(p-t)-1}} \wedge \langle \epsilon_t, \epsilon_p \rangle = a\alpha_1(4p)$ for some integer $a$ such that $a \not= 0 \ (p)$.

Let $\alpha_1(2p + 2) : S^{4p} \to A_3$ be a coextension of $\alpha_1(2p + 2)$. Then, as in section 2, we have $f = b\iota_t(\alpha_1(2p + 2))$ for some integer $b$, where $\iota : A_3 \to B_3$ is the inclusion. Suppose that $b = b'/p$. Then, by [12, Proposition 1.8], we have

$$f = b'\iota_t(\alpha_1(2p + 2) \circ p) = -b'\iota_t \circ j_t(\alpha_1(5), \alpha_1(2p + 2), p) = -\frac{b'}{2} \iota_t \circ j_t(\alpha_2(5)),$$

where $j : S^5 \to A_3$ is the inclusion. In particular, $f$ lifts to $S^5 \subset B_3$ and this contradicts to Proposition 4.6. Thus we have $b \not= 0 \ (p)$.

On the other hand, it follows from [12, Proposition 1.8] that

$$\alpha_1(2p + 2) \circ \alpha_1(2p + 2) = -j_t(\alpha_1(5), \alpha_1(2p + 2), \alpha_1(4p - 1)).$$

It is known that $\{\alpha_1(5), \alpha_1(2p + 2), \alpha_1(4p - 1)\} \not= 0$ (See [4, p.38]) and then we have established

$$f \circ (1_{S^{2(p-t)-1}} \wedge \langle \epsilon_t, \epsilon_p \rangle) = f \circ (1_{S^{2(p-t)-1}} \wedge \langle \epsilon_t, \epsilon_p \rangle) \circ \bar{q} \circ s = \langle \lambda_{p-t+1}, \langle \lambda_t, \epsilon_p \rangle \circ s \not= 0.$$

This implies $\langle \lambda_{p-t+1}, \langle \lambda_t, \epsilon_p \rangle \rangle \not= 0$.

When $t = p$, the proof is completed by the homotopy exact sequence induced from the fiber sequence $SU(2p - 2) \to SU(2p - 1) \to S^{4p-3}$.

References


