Symmetric jump processes: localization, heat kernels, and convergence

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Abstract

Abstract: We consider symmetric processes of pure jump type. We prove local estimates on the probability of exiting balls, the Hölder continuity of harmonic functions and of heat kernels, and convergence of a sequence of such processes.

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1 Introduction

Suppose $J : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is a symmetric function satisfying

$$\frac{c_1}{|y-x|^{\beta_1}} \le J(x,y) \le \frac{c_2}{|y-x|^{\beta_2}}$$

if $|y - x| \le 1$ and 0 otherwise. Define the Dirichlet form

$$\mathcal{E}(f,f) = \int\int (f(y) - f(x))^2 J(x,y)\, dy\, dx, \tag{1.1}$$

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and we take as the domain of $\mathcal{E}$ the closure with respect to the norm $(\|f\|_{L^2(\mathbb{R}^d)} + \mathcal{E}(f,f))^{1/2}$ of the Lipschitz functions with compact support. When $\beta_1 = \beta_2$, the Dirichlet form and associated infinitesimal generator are said to be of fixed order, namely, β_1, while if $\beta_1 < \beta_2$, the generator is of variable order. The variable order case allows for considerable variability in the jump intensities and directions.

In [1] a number of results were proved for the Hunt process X associated with $\mathcal{E}$, including exit probabilities, heat kernel estimates, a parabolic Harnack inequality, and the lack of continuity of harmonic functions. The last is perhaps the most interesting: it was shown that there exist harmonic functions that are not continuous.

This paper could be considered a sequel to [1], although the set of authors for the present paper neither contains nor is contained in the set of authors of [1]. We prove three main results, which we discuss in turn.

First we discuss estimates on exit probabilities. In [1] some estimates were obtained on $\mathbb{P}^x(\tau_{B(x,r)} < t)$. These estimates held for all x, but were very crude, and were not sensitive to the behavior of $J(x,y)$ when y is close to x. We show in the current paper that to a large extent the behavior of these exit probabilities depend on the size of $J(x,y)$ for y near x. We also allow large jumps, which translates to allowing $J(x,y) \neq 0$ for $|y-x| > 1$. In Example 2.3 we show how under some smoothness in J, we can get fairly precise estimates.

Our motivation for obtaining better bounds on exit probabilities is to consider the question of when harmonic functions and the heat kernel are continuous. The example in [1] shows this continuity need not always hold. However, when J possesses a minimal amount of smoothness, we establish that indeed harmonic functions are Hölder continuous, and the heat kernel is also Hölder continuous. The technique for showing the Hölder continuity of harmonic functions is based on ideas from [3], where the non-symmetric case was considered. More interesting is the part of the proof where we show that Hölder continuity of harmonic functions plus global bounds on the heat kernel imply Hölder continuity of the heat kernel. This argument is of independent interest, and should be applicable in many other situations.

Finally, we suppose we have a sequence of functions J_n with corresponding Dirichlet forms and Hunt processes. We show that if the J_n converge weakly to J, and some uniform integrability holds, then the corresponding processes converge. Note only weak convergence of the J_n is needed. This is in contrast to the diffusion case, where it is known that weak convergence is not sufficient, and a much stronger type of convergence of the Dirichlet forms is required; see [9].

Our assumptions and results are stated and proved in the next three sections, the exit probabilities in Section 2, the regularity in Section 3, and the weak convergence in Section 4.

2 Exit probabilities

Suppose $J : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is jointly measurable. We suppose throughout this paper that there exist constants $\kappa_1, \kappa_2, \kappa_3 > 0$ and $\beta_1, \beta_2 \in (0, 2)$ such that

$$\frac{\kappa_1}{|x-y|^{d+\beta_1}} \leq J(x,y) \leq \frac{\kappa_2}{|x-y|^{d+\beta_2}}, \qquad |x-y| \leq 1, \tag{2.1}$$

and

$$\int_{|x-y|>1} J(x,y)\, dy \leq \kappa_3, \qquad x \in \mathbb{R}^d. \tag{2.2}$$

The constants $\beta_1, \beta_2, c_1, c_2, c_3$ play only a limited role in what follows and (2.1) and (2.2) are used to guarantee a certain amount of regularity. Much more important is the α that is introduced in (2.4). Define a Dirichlet form $\mathcal{E} = \mathcal{E}_J$ by

$$\mathcal{E}(f,f) = \int \int (f(y) - f(x))^2 J(x,y)\, dy\, dx, \tag{2.3}$$

where we take the domain to be the closure of the Lipschitz functions with compact support with respect to the norm $(\|f\|_2 + (\mathcal{E}(f,f))^{1/2}$. Let X be the Hunt process associated with the Dirichlet form $\mathcal{E}$. Let $B(x,r)$ denote the open ball of radius r centered at x. The letter c with or without subscripts will denotes constants whose exact values are unimportant and which may change from line to line.

We remark that if we define $J_1(x,y) = J(x,y)1_{(|x-y|\leq 1)}$ and define the corresponding Dirichlet form in terms of J_1, then the Hunt process $X^{(1)}$ corresponding to this Dirichlet form is conservative by [1, Theorem 1.1]. Using a construction due to Meyer (see [1, Remark 3.4] and [2, Section 3.1]) we can use $X^{(1)}$ to obtain X. This is a probabilistic procedure that involves adding jumps. Only finitely many jumps are added in any finite time interval, and we deduce from this construction that X is also conservative.

We now fix $z_0 \in \mathbb{R}^d$ and assume that there exist constants κ_4 and $\alpha \in (0,2)$ such that

$$J(x,y) \geq \kappa_4 |x-y|^{-d-\alpha}, \qquad x,y \in B(z_0,3r). \tag{2.4}$$

Here α may depend on z_0.

Define

$$L_1(x,s) = \int_{|x-w|\geq s} J(x,w)\,dw, \tag{2.5}$$

$$L_2(x,s) = \int_{|x-w|\leq s} |x-w|^2 J(x,w)\,dw, \tag{2.6}$$

and let

$$L(z_0,r) = \sup_{x\in B(z_0,3r)} L_1(x,r) + \sup_{x\in B(z_0,3r)} \sup_{s\leq r} s^d [s^{-2} L_2(x,s)]^{\frac{d+\alpha}{\alpha}}. \tag{2.7}$$

From (2.4) we see that

$$L(z_0,r) \geq c r^{-\alpha}. \tag{2.8}$$

Theorem 2.1 *Suppose* (2.1), (2.2), *and* (2.4) *hold. There exists* c_1 *(depending only on* d, κ_4, *and* α*) such that if* $r \in (0,1)$, *then for* $x \in B(z_0,r)$,

$$\mathbb{P}^x(\tau_{B(x,r)} < t) \leq c_1 t L(z_0,r).$$

Proof. Let x_0, y_0 be fixed, let $R = |y_0 - x_0|$, and suppose $R \geq 18(d+\alpha)r/\alpha$. By (2.8), the result is immediate if $t > r^\alpha$, so let us suppose $t \leq r^\alpha$. Define

$$\widetilde{J}(x,y) = \begin{cases} J(x,y) & \text{if } x,y \text{ are both in } B(z_0,3r) \text{ and } |x-y|<R, \\ \kappa_4|x-y|^{-d-\alpha} & \text{if at least one of } x \text{ and } y \text{ is not in} \\ & \qquad B(z_0,3r) \text{ and } |x-y|<R, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Define

$$\widetilde{J}_\delta(x,y) = \widetilde{J}(x,y) 1_{(|x-y|\leq\delta)},$$

where we will choose $\delta \in [6r, R)$ in a moment. Let $\widetilde{X}$ be the Hunt process corresponding to $\widetilde{J}$ and $\widetilde{X}^{(\delta)}$ the Hunt process associated with $\widetilde{J}_\delta$.

We have the Nash inequality (see, e.g., (3.9) of [1]):

$$\|u\|_2^{2+\frac{2\alpha}{d}} \le c\Big(\int\int_{|x-y|<\delta} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx + \delta^{-\alpha}\|u\|_2^2\Big)\,\|u\|_1^{2\alpha/d}. \quad (2.10)$$

Using (2.4) we obtain from this that

$$\|u\|_2^{2+\frac{2\alpha}{d}} \le c\Big(\int\int (u(x)-u(y))^2 \widetilde{J}_\delta(x,y)\,dy\,dx + \delta^{-\alpha}\|u\|_2^2\Big)\,\|u\|_1^{2\alpha/d}. \quad (2.11)$$

Let

$$\delta = \frac{R\alpha}{3(d+\alpha)}, \quad (2.12)$$

$$N(\delta) = \delta^{-\alpha} + \sup_{x\in B(z_0,3r)} \delta^{-2} L_2(x,\delta), \quad (2.13)$$

$$\lambda = \frac{1}{3\delta}\log(1/(N(\delta)t)). \quad (2.14)$$

Let $\psi(x) = \lambda(R-|x-x_0|)^+$. Set

$$\Gamma(f,f)(x) = \int (f(y)-f(x))^2 \widetilde{J}_\delta(x,y)\,dy.$$

Since $|e^t-1|^2 \le t^2 e^{2t}$, $|\psi(x)-\psi(y)| \le \lambda|x-y|$, and $\widetilde{J}_\delta(x,y)=0$ unless $|x-y|<\delta$, then

$$\begin{aligned} e^{-2\psi(x)}\Gamma(e^\psi,e^\psi)(x) &= \int_{|x-y|\le\delta} \Big(e^{\psi(x)-\psi(y)}-1\Big)^2 \widetilde{J}_\delta(x,y)\,dy \\ &\le e^{2\lambda\delta}\lambda^2 \int_{|x-y|\le\delta} |x-y|^2 \widetilde{J}_\delta(x,y)\,dy. \end{aligned} \quad (2.15)$$

Since $\delta \ge 6r$, then by our definition of $\widetilde{J}$ we have that the integral on the last line of (2.15) is bounded by $\sup_{x\in B(z_0,3r)} L_2(x,\delta) + \delta^{2-\alpha}$. We therefore have

$$\begin{aligned} e^{-2\psi(x)}\Gamma(e^\psi,e^\psi)(x) &\le e^{2\lambda\delta}\lambda^2\delta^2 N(\delta) \\ &\le e^{3\lambda\delta} N(\delta). \end{aligned}$$

We obtain in the same way the same upper bound for $e^{2\psi(x)}\Gamma(e^{-\psi}, e^{-\psi})(x)$. So by [5, Theorem 3.25] we have

$$p_\delta(t, x_0, y_0) \le ct^{-d/\alpha} e^{ct\delta^{-\alpha}} e^{-\lambda R + ce^{3\lambda\delta} N(\delta)t}, \tag{2.16}$$

where p_δ is the transition density for $\widetilde{X}^{(\delta)}$. (Note that by [1, Theorem 3.1], the transition density $p_\delta(t, x, y)$ exists for $x, y \in \mathbb{R}^d \setminus \mathcal{N}$, where $\mathcal{N}$ is a set of capacity zero, called a properly exceptional set. We will take $x_0, y_0 \in \mathbb{R}^d \setminus \mathcal{N}$.) Since $t \le r^\alpha \le c\delta^\alpha$, we then get

$$p_\delta(t, x_0, y_0) \le ct^{-d/\alpha} e^{-\lambda R} = ct^{-d/\alpha} (N(\delta)t)^{R/3\delta}.$$

Our bound now becomes

$$\begin{aligned} p_\delta(t, x_0, y_0) &\le ct^{-d/\alpha} t^{1+\frac{d}{\alpha}} N(\delta)^{(d+\alpha)/\alpha} \\ &= ctN(\delta)^{(d+\alpha)/\alpha}. \end{aligned}$$

Since $\delta \ge 6r$, then

$$\|\widetilde{J} - \widetilde{J}_\delta\|_\infty \le c\delta^{-(d+\alpha)},$$

so by [2, Lemma 3.1] and by (2.12)

$$\begin{aligned} p(t, x_0, y_0) &\le p_\delta(t, x_0, y_0) + ct\delta^{-(d+\alpha)} \\ &\le ct[\sup_{x \in B(z_0, 3r)} \delta^{-2} L_2(x, \delta) + \delta^{-\alpha}]^{\frac{d+\alpha}{\alpha}} + ctR^{-(d+\alpha)}. \end{aligned}$$

Since

$$\begin{aligned} \sup_{x \in B(z_0, 3r)} \delta^{-2} L_2(x, \delta) + \delta^{-\alpha} &\le c\delta^{-2}[\sup_{x \in B(z_0, 3r)} L_2(x, r) + \delta^{2-\alpha}] \\ &\le cR^{-2} \sup_{x \in B(z_0, 3r)} L_2(x, r) + cR^{-\alpha}, \end{aligned}$$

then, because $\widetilde{X}$ is conservative, integrating over $R \ge r/2$ gives us

$$\mathbb{P}^{x_0}(|\widetilde{X}_t - x_0| \ge r/2) \le ctr^d \Big[r^{-2} \sup_{x \in B(z_0, 3r)} L_2(x, r) \Big]^{\frac{d+\alpha}{\alpha}} + ctr^{-\alpha} \le ctL(z_0, r).$$

By [1, Lemma 3.8] we then have

$$\mathbb{P}^{x_0}(\sup_{s \le t} |\widetilde{X}_s - x_0| > r) \le ctL(z_0, r).$$

We now use Meyer's construction to compare $\widetilde{X}$ to X. Using this construction we obtain, for $x \in B(z_0, r)$,

$$\begin{aligned}\mathbb{P}^x(X_s \neq \widetilde{X}_s \text{ for some } s \leq t) &\leq t \sup_{x' \in B(z_0,2r)} \int_{B(z_0,3r)^c} |J(x',y) - \widetilde{J}(x',y)| dy \\ &\leq ctL(z_0, r).\end{aligned}$$

(The first inequality can be obtained by observing the processes X and $\widetilde{X}$ killed on exiting $B(z_0, 2r)$.) Therefore, for $x \in B(z_0, r)$,

$$\begin{aligned}\mathbb{P}^x(\sup_{s \leq t} |X_s - x| > r) &\leq \mathbb{P}^x(\sup_{s \leq t} |\widetilde{X}_s - x| > r) + \mathbb{P}^x(X_s \neq \widetilde{X}_s \text{ for some } s \leq t) \\ &\leq ctL(z_0, r).\end{aligned}$$

$\square$

Corollary 2.2 *Suppose* (2.1) *and* (2.2) *hold. Suppose instead of* (2.4) *we have*

$$J(x,y) \geq \kappa_4 |x-y|^{-d-\alpha} - K(x,y), \qquad x, y \in B(z_0, 3r), \tag{2.17}$$

where

$$\int_{|x-y| \leq \delta} K(x,y)\, dy \leq \kappa_5 \delta^{-\alpha} \tag{2.18}$$

for all $x \in B(z_0, 3r)$ and all $\delta \leq r$. Then the conclusion of Theorem 2.1 still holds.

Proof. The only place the lower bound on $J(x,y)$ plays a role is in deriving (2.11) from (2.10). If we have (2.17) instead of (2.4), then in place of (2.11) we now have

$$\begin{aligned}\|u\|_2^{2+\frac{2\alpha}{d}} \leq c\Big(&\int\int (u(x)-u(y))^2 \widetilde{J}_\delta(x,y)\, dy\, dx \\ &+ \int\int_{|x-y|\leq\delta} (u(x)-u(y))^2 K(x,y)\, dy\, dx + \delta^{-\alpha}\|u\|_2^2 \Big) \|u\|_1^{2\alpha/d}.\end{aligned} \tag{2.19}$$

But by our assumption on $K(x,y)$, the double integral with K in the integrand is bounded by

$$\int \Big(\int_{|x-y|\le\delta} K(x,y)\,dy \Big)\, u(x)^2\,dx \le c\delta^{-\alpha}\|u\|_2^2.$$

$\square$

Example 2.3 Suppose $\varepsilon > 0$ and there exists a function $s : \mathbb{R}^d \to (\varepsilon, 2-\varepsilon)$ such that

$$|s(x) - s(y)| \le c\log(2/|x-y|), \qquad |x-y| < 1. \tag{2.20}$$

Suppose there exist constants c_1, c_2 such that

$$\frac{c_1}{|x-y|^{d+s(x)\wedge s(y)}} \le J(x,y) \le \frac{c_2}{|x-y|^{d+s(x)\vee s(y)}}. \tag{2.21}$$

Suppose further that (2.2) holds. We show that $L(z_0, r)$ is comparable to $r^{-s(z_0)}$ if $r < 1$.

To see this, note that

$$|x-y|^{s(x)-s(y)} \le |x-y|^{-c/\log(2/|x-y|)} \le e^c, \tag{2.22}$$

if $|x-y| \le 1$ and similarly we have

$$|x-y|^{s(x)-s(y)} \ge |x-y|^{c/\log(2/|x-y|)} \ge e^{-c}. \tag{2.23}$$

If we fix x and let

$$M(v) = \sup_{|x-w|=v} J(x,w),$$

then for $v \le 1$

$$\begin{aligned} M(v) &\le \sup_{|x-w|=v} \frac{c}{v^{d+s(x)}} v^{-|s(x)-s(w)|} \\ &\le \frac{c}{v^{d+s(x)}}. \end{aligned}$$

We then estimate for $r \le 1$

$$\begin{aligned} L_2(x,r) &\le c\int_0^r v^2 M(v) v^{d-1}\,dv \\ &\le c\int_0^r v^{1-s(x)}\,dv = cr^{2-s(x)}, \end{aligned}$$

We can similarly obtain a bound for $L_1(x,r)$:

$$\begin{aligned}
L_1(x,r) &\le c\int_r^1 M(v)v^{d-1}\,dv + \int_{|x-w|>1} J(x,w)\,dw\\
&\le c\int_r^1 v^{-1-s(x)}\,dv + c\\
&\le cr^{-s(x)} + c \le cr^{-s(x)}
\end{aligned}$$

if $r \le 1$.

Next, for $x \in B(z_0, 3r)$, we have $r^{-s(x)}$ is comparable to $r^{-s(z_0)}$ for $r \le 1$. To see this,

$$c \le r^{s(x)-s(z_0)} \le r^{-|s(x)-s(z_0)|} \le c'$$

as in (2.22) and (2.23).

If we take α in (2.4) to be $\inf_{x\in B(z_0,3r)} s(x)$, then we conclude

$$L(x,r) \le cr^{-s(z_0)} + cr^{-\alpha} \le cr^{-s(z_0)},$$

so

$$\mathbb{P}^x(\tau_r \le t) \le ctr^{-s(z_0)}, \qquad x \in B(z_0, r).$$

3 Regularity

We suppose throughout this section that (2.1) and (2.2) hold. We suppose in addition first that there exists c such that

$$\int_A J(z,y)\,dy \ge cL(x,r) \tag{3.1}$$

whenever $r \in (0,1)$, $A \subset B(x,3r)$, $|A| \ge \frac{1}{3}|B(x,r)|$, $x \in \mathbb{R}^d$, and $z \in B(x, r/2)$ and second there exist σ and c such that

$$\frac{L_1(x,\lambda r)}{L_1(x,r)} \le c\lambda^{-\sigma}, \qquad x \in \mathbb{R}^d, r \in (0,1), \lambda \in (1, 1/r). \tag{3.2}$$

It is easy to check that (3.1) and (3.2) hold for Example 2.3.

We say a function h is harmonic in a ball $B(x_0, r)$ if $h(X_{t\wedge\tau_{B(x_0,r(1-\varepsilon))}})$ is a $\mathbb{P}^x$ martingale for q.e. x and every $\varepsilon \in (0,1)$.

Theorem 3.1 *Suppose* (2.1)*,* (2.2)*,* (3.1)*, and* (3.2) *hold. There exist* c_1 *and* γ *such that if* h *is bounded in* $\mathbb{R}^d$ *and harmonic in a ball* $B(x_0, r)$*, then*

$$|h(x) - h(y)| \le c_1 \Big(\frac{|x-y|}{r} \Big)^\gamma \|h\|_\infty, \qquad x, y \in B(x_0, r/2). \tag{3.3}$$

Proof. As in [6, 7] we have the Lévy system formula:

$$\mathbb{E}^x \Big[\sum_{s \le T} f(X_{s-}, X_s) \Big] = \mathbb{E}^x \Big[\int_0^T \Big(\int f(X_s, y) J(X_s, y)\, dy \Big)\, ds \Big] \tag{3.4}$$

for any nonnegative f that is 0 on the diagonal, for every bounded stopping time T, and q.e. starting point x. Given this, the proof is nearly identical to that in [3, Theorem 2.2]. □

We obtain a crude estimate on the expectation of the exit times.

Lemma 3.2 *Assume the lower bound of* (2.1)*. Then there exists* c_1 *such that*

$$\mathbb{E}^x \tau_r \le c_1 r^{\beta_1}, \qquad x \in \mathbb{R}^d, r \in (0, 1/2).$$

Proof. The expression $\sum_{s \le t \wedge \tau_r} 1_{(|X_s - X_{s-}| > 2r)}$ is 1 if there is a jump of size at least $2r$ before time $t \wedge \tau_r$, in which case the process exits $B(x, r)$ before or at time t, or 0 if there is no such jump. So

$$\begin{aligned}
\mathbb{P}^x(\tau_r \le t) &\ge \mathbb{E}^x \sum_{s \le t \wedge \tau_r} 1_{(|X_s - X_{s-}| > 2r)} \\
&= \mathbb{E}^x \int_0^{t \wedge \tau_r} \int_{B(x, 2r)^c} J(X_s, y)\, dy\, ds \\
&\ge c r^{-\beta_1} \mathbb{E}^x [t \wedge \tau_r] \\
&\ge c r^{-\beta_1} t \mathbb{P}^x(\tau_r > t),
\end{aligned}$$

using the lower bound of (2.1). Thus

$$\mathbb{P}^x(\tau_r > t) \le 1 - c r^{-\beta_1} t \mathbb{P}^x(\tau_r > t),$$

or $\mathbb{P}^x(\tau_r > t) \le 1/2$ if we take $t = c^{-1} r^{\beta_1}$. This holds for every $x \in \mathbb{R}^d$. Using the Markov property at time mt,

$$\mathbb{P}^x(\tau_r > (m+1)t) \le \mathbb{E}^x[\mathbb{P}^{X_{mt}}(\tau_r > t); \tau_r > mt] \le \tfrac{1}{2}\mathbb{P}^x(\tau_r > mt).$$

By induction $\mathbb{P}^x(\tau_r > mt) \le 2^{-m}$. With this choice of t, our lemma follows. □

We next show λ-potentials are Hölder continuous. Let

$$U^\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt.$$

Proposition 3.3 *Under the same assumption as in Theorem 3.1, there exist $c_1 = c_1(\lambda)$ and γ' such that if f is bounded, then*

$$|U^\lambda f(x) - U^\lambda f(y)| \le c_1 |x-y|^{\gamma'} \|f\|_\infty.$$

Proof. Fix x_0, let $r \in (0, 1/2)$, and suppose $x, y \in B(x_0, r/2)$. By the strong Markov property,

$$\begin{aligned} U^\lambda f(x) = \mathbb{E}^x &\int_0^{\tau_r} e^{-\lambda t} f(X_t)\, dt + \mathbb{E}^x (e^{-\lambda \tau_r} - 1) U^\lambda f(X_{\tau_r}) \\ &+ \mathbb{E}^x U^\lambda f(X_{\tau_r}) \\ = I_1 &+ I_2 + I_3, \end{aligned}$$

and similarly when x is replaced by y. We have by Lemma 3.2

$$|I_1| \le \|f\|_\infty \mathbb{E}^x \tau_r \le c r^{\beta_1} \|f\|_\infty$$

and by the mean value theorem and Lemma 3.2

$$|I_2| \le \lambda \mathbb{E}^x \tau_r \|U^\lambda f\|_\infty \le c r^{\beta_1} \|f\|_\infty,$$

and similarly when x is replaced by y. So

$$|U^\lambda f(x) - U^\lambda f(y)| \le c r^{\beta_1} \|f\|_\infty + |\mathbb{E}^x U^\lambda f(X_{\tau_r}) - \mathbb{E}^y U^\lambda f(X_{\tau_r})|. \tag{3.5}$$

But $z \to \mathbb{E}^z U^\lambda f(X_{\tau_r})$ is bounded in $\mathbb{R}^d$ and harmonic in $B(x_0, r)$, so by Theorem 3.1 the second term in (3.5) is bounded by

$$c\Big(\frac{|x-y|}{r}\Big)^\gamma \|U^\lambda f\|_\infty.$$

If we use $\|U^\lambda f\|_\infty \le \frac{1}{\lambda}\|f\|_\infty$ and set $r = |x-y|^{1/2}$, then

$$|U^\lambda f(x) - U^\lambda f(y)| \le (c|x-y|^{\beta_1/2} + c|x-y|^{\gamma/2})\|f\|_\infty, \tag{3.6}$$

and our result follows. □

Using the spectral theorem, there exists projection operators E_μ on the space $L^2(\mathbb{R}^d, dx)$ such that

$$\begin{aligned} f &= \int_0^\infty dE_\mu(f), \\ P_t f &= \int_0^\infty e^{-\mu t}\, dE_\mu(f), \\ U^\lambda f &= \int_0^\infty \frac{1}{\lambda+\mu}\, dE_\mu(f). \end{aligned} \tag{3.7}$$

Proposition 3.4 *Under the same assumptions as in Theorem 3.1, if f is in L^2, then $P_t f$ is equal a.e. to a function that is Hölder continuous.*

Proof. Write $\langle f, g\rangle$ for the inner product in L^2. Note that in what follows t is fixed. Each of our constants may depend on t. If $X^{(1)}$ is the Hunt process associated with the Dirichlet form defined in terms of the kernel $J_1(x,y) = J(x,y)1_{(|x-y|<1)}$, we know from [1, Theorem 2.1] that $X^{(1)}$ has a transition density $p(t,x,y)$ bounded by c. Using [2, Lemma 3.1] and Meyer's construction (cf. [1, Section 3]), we then can conclude that X also has a transition density bounded by c. Define

$$h = \int_0^\infty (\lambda+\mu)e^{-\mu t}\, dE_\mu(f).$$

Since $\sup_\mu (\lambda+\mu)^2 e^{-2\mu t} \le c$, then

$$\int_0^\infty (\lambda+\mu)^2 e^{-2\mu t}\, d\langle E_\mu(f), E_\mu(f)\rangle \le c\int_0^\infty d\langle E_\mu(f), E_\mu(f)\rangle = c\|f\|_2^2,$$

we see that h is a well defined function in L^2.

Suppose $g \in L^1$. Then $\|P_t g\|_1 \le \|g\|_1$ by Jensen's inequality, and

$$|P_t g(x)| = \Big| \int p(t,x,y) g(y)\, dy \Big| \le c\|g\|_1$$

by the fact that $p(t,x,y)$ is bounded. So $\|P_t g\|_\infty \le c\|g\|_1$, and it follows that $\|P_t g\|_2 \le c\|g\|_1$. Using Cauchy-Schwarz and the fact that

$$\sup_\mu (\lambda+\mu) e^{-\mu t/2} \le c < \infty,$$

we have

$$\begin{aligned}
\langle h, g\rangle &= \int_0^\infty (\lambda+\mu) e^{-\mu t}\, d\langle E_\mu(f), E_\mu(g)\rangle \\
&\le \Big(\int_0^\infty (\lambda+\mu) e^{-\mu t}\, d\langle E_\mu(f), E_\mu(f)\rangle \Big)^{1/2} \\
&\qquad \times \Big(\int_0^\infty (\lambda+\mu) e^{-\mu t}\, d\langle E_\mu(g), E_\mu(g)\rangle \Big)^{1/2} \\
&\le c\Big(\int_0^\infty d\langle E_\mu(f), E_\mu(f)\rangle \Big)^{1/2} \Big(\int_0^\infty e^{-\mu t/2}\, d\langle E_\mu(g), E_\mu(g)\rangle \Big)^{1/2} \\
&= c\|f\|_2 \|P_{t/2} g\|_2 \\
&\le c\|f\|_2 \|g\|_1.
\end{aligned}$$

Taking the supremum over $g \in L^1$ with L^1 norm less than 1, $\|h\|_\infty \le c\|f\|_2$. But by (3.7)

$$U^\lambda h = \int_0^\infty e^{-\mu t}\, dE_\mu(f) = P_t f, \qquad a.e.,$$

and the Hölder continuity of $P_t f$ follows by Proposition 3.3. $\square$

Finally we have

Theorem 3.5 *Under the same assumption as in Theorem 3.1, we can choose $p(t,x,y)$ to be jointly continuous.*

Proof. Fix y and let $f(z) = p(t/2, z, y)$. f is bounded by c (depending on t) and has L^1 norm equal to 1, hence $f \in L^2$ with norm bounded by c. Note

$$P_{t/2} f(x) = \int p(t/2, x, z) f(z)\, dz = \int p(t/2, x, z) p(t/2, z, y) = p(t,x,y).$$

Using Proposition 3.4 shows that $p(t,x,y)$ is Hölder continuous with constants independent of x and y. This and symmetry gives the result. $\square$

Remark 3.6 The argument we gave deriving the Hölder continuity of the transition densities from the boundedness of the transition densities plus the Hölder continuity of harmonic functions holds in much more general contexts than just jump processes in $\mathbb{R}^d$.

4 Convergence

Suppose now that we have a sequence of jump kernels $J^n(x,y)$ satisfying (2.1), (2.2), (3.1), and (3.2) with constants independent of n. Suppose in addition that

$$\lim_{\eta\to 0}\sup_{n,x}\int_{|y-x|\geq \eta^{-1}} J_n(x,y)\,dy\,dx = 0, \tag{4.1}$$

$$\lim_{\eta\to 0}\sup_{n,x}\int_{|y-x|\leq \eta} |y-x|^2 J_n(x,y)\,dy\,dx = 0, \tag{4.2}$$

and for each η

$$J_n(x,y)1_{(\eta,\eta^{-1})}(|y-x|)\,dx\,dy \to J(x,y)1_{(\eta,\eta^{-1})}(|y-x|)\,dx\,dy \tag{4.3}$$

weakly as $n\to\infty$.

Let $\mathcal{E}^n$ be the Dirichlet forms defined in terms of the J^n with P_t^n, U_n^λ, and $\mathbb{P}_n^x$ the associated semigroup, resolvent, and probabilities. Let P_t, U^λ, and $\mathbb{P}^x$ be the semigroup, resolvent, and probabilities corresponding to the Dirichlet form $\mathcal{E}_J$ defined in terms of the kernel J.

Under the above set-up we have

Theorem 4.1 *If f is bounded and continuous, then $P_t^n f$ converges uniformly on compacts to $P_t f$. For each t, for q.e. x, $\mathbb{P}_n^x$ converges weakly to $\mathbb{P}^x$ with respect to the space $D([0,t])$.*

Proof. The first step is to show that any subsequence $\{n_j\}$ has a further subsequence $\{n_{j_k}\}$ such that $U^\lambda_{n_{j_k}} f$ converges uniformly on compacts whenever f is bounded and continuous. The proof of this is very similar to that of [4, Proposition 6.2], and we refer the reader to that paper.

Now suppose we have a subsequence $\{n'\}$ such that the $U^\lambda_{n'} f$ are equicontinuous and converge uniformly on compacts whenever f is bounded and continuous with compact support. Fix such an f and let H be the limit of $U^\lambda_{n'} f$. We will show

$$\mathcal{E}_J(H,g) = \langle f, g\rangle - \lambda \langle H, g\rangle \tag{4.4}$$

whenever g is a Lipschitz function with compact support, where $\mathcal{E}_J$ is the Dirichlet form corresponding to the kernel J. This will prove that H is the λ-resolvent of f with respect to $\mathcal{E}_J$, that is, $H = U^\lambda f$. We can then conclude that the full sequence $U^\lambda_n f$ converges to $U^\lambda f$ whenever f is bounded and continuous with compact support. The assertions about the convergence of P^n_t and $\mathbb{P}^x_n$ then follow as in [4, Proposition 6.2].

So we need to prove H satisfies (4.4). We drop the primes for legibility.

We know

$$\mathcal{E}^n(U^\lambda_n f, U^\lambda_n f) = \langle f, U^\lambda_n f\rangle - \lambda \langle U^\lambda_n f, U^\lambda_n f\rangle. \tag{4.5}$$

Since $\|U^\lambda_n f\|_2 \le (1/\lambda)\|f\|_2$ (by Jensen's inequality), we have by Cauchy-Schwarz that

$$\sup_n \mathcal{E}^n(U^\lambda_n f, U^\lambda_n f) \le c < \infty.$$

Since the $U^\lambda_n f$ are equicontinuous and converge uniformly to H on $B(0,\eta^{-1}) - \overline{B(0,\eta)}$, then

$$\begin{aligned}
\int\int_{\eta<|y-x|<\eta^{-1}} &(H(y)-H(x))^2 J(x,y)\,dy\,dx \\
&\le \limsup_{n\to\infty} \int\int_{\eta<|y-x|<\eta^{-1}} (U^\lambda_n f(y) - U^\lambda_n f(x))^2 J_n(x,y)\,dy\,dx \\
&\le \limsup_n \mathcal{E}^n(U^\lambda_n f, U^\lambda_n f) \le c < \infty.
\end{aligned}$$

Letting $\eta \to 0$, we have

$$\mathcal{E}_J(H,H) < \infty. \tag{4.6}$$

Fix a Lipschitz function g with compact support and choose M large enough so that the support of g is contained in $B(0,M)$. Then

$$\Big| \int \int_{|y-x|\geq \eta^{-1}} (U_n^\lambda f(y) - U_n^\lambda f(x))(g(y)-g(x)) J_n(x,y)\,dy\,dx \Big|$$
$$\leq \Big(\int \int (U_n^\lambda f(y) - U_n^\lambda f(x))^2 J_n(x,y)\,dy\,dx \Big)^{1/2}$$
$$\times \Big(\int \int_{|y-x|\geq \eta^{-1}} (g(y)-g(x))^2 J_n(x,y)\,dy\,dx \Big)^{1/2}.$$

The first factor is $(\mathcal{E}^n(U_n^\lambda f, U_n^\lambda f))^{1/2}$, while the second factor is bounded by

$$\|g\|_\infty \Big(\int_{B(0,M)} \int_{|y-x|\geq \eta^{-1}} J_n(x,y)\,dx\,dy \Big)^{1/2},$$

which, in view of (4.1), will be small if η is small. Similarly,

$$\Big| \int \int_{|y-x|\leq \eta} (U_n^\lambda f(y) - U_n^\lambda f(x))(g(y)-g(x)) J_n(x,y)\,dy\,dx \Big|$$
$$\leq \Big(\int \int (U_n^\lambda f(y) - U_n^\lambda f(x))^2 J_n(x,y)\,dy\,dx \Big)^{1/2}$$
$$\times \Big(\int \int_{|y-x|\leq \eta} (g(y)-g(x))^2 J_n(x,y)\,dy\,dx \Big)^{1/2}.$$

The first factor is as before, while the second is bounded by

$$\|\nabla g\|_\infty \Big(\int_{B(0,M)} \int_{|y-x|\leq \eta} |y-x|^2 J_n(x,y)\,dx\,dy \Big)^{1/2}.$$

In view of (4.2), the second factor will be small if η is small. Similarly, using (4.6), we have

$$\Big| \int \int_{|y-x|\notin(\eta,\eta^{-1})} (H(y)-H(x))(g(y)-g(x)) J(x,y)\,dy\,dx \Big|$$

will be small if η is taken small enough.

By (4.3), (2.1), (2.2), and the fact that the $U_n^\lambda f$ are equicontinuous and converge to H uniformly on compacts,

$$\int \int_{|y-x|\in(\eta,\eta^{-1})} (U_n^\lambda f(y) - U_n^\lambda f(x))(g(y)-g(x)) J_n(x,y)\,dy\,dx$$
$$\to \int \int_{|y-x|\in(\eta,\eta^{-1})} (H(y)-H(x))(g(y)-g(x)) J(x,y)\,dy\,dx.$$

It follows that

$$\mathcal{E}^n(U_n^\lambda f, g) \to \mathcal{E}_J(H, g). \tag{4.7}$$

But

$$\mathcal{E}^n(U_n^\lambda f, g) = \langle f, g\rangle - \lambda \langle U_n^\lambda f, g\rangle \to \langle f, g\rangle - \lambda\langle H, g\rangle.$$

Combining with (4.7) proves (4.4). □

Remark 4.2 One can modify the above proof to obtain a central limit theorem for symmetric Markov chains. Suppose for each n we have a symmetric Markov chain on $n^{-1}\mathbb{Z}^d$ with unbounded range with conductances C^n_{xy}. If ν_n is the measure that gives mass n^{-d} to each point in $n^{-1}\mathbb{Z}^d$, then we can define the Dirichlet form

$$\mathcal{E}_n(f, f) = \sum_{x,y \in n^{-1}\mathbb{Z}^d} (f(x) - f(y))^2 C^n_{xy}$$

with respect to the measure ν_n. Under appropriate assumptions analogous to those in Sections 2 and 3, one can show that the semigroups corresponding to $\mathcal{E}_n$ converge to those of $\mathcal{E}$ and in addition there is weak convergence of the probability laws. Since the details are rather lengthy, we leave this to the interested reader.

Remark 4.3 We can also prove the following approximation of a jump process by Markov chains, which is a generalization of [8, Theorem 2.3]. Suppose $J : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is a symmetric measurable function satisfying (2.1), (2.2), (3.1), and (3.2). Define the conductivity functions $C^n : n^{-1}\mathbb{Z}^d \times n^{-1}\mathbb{Z}^d \to [0, \infty)$ by

$$C^n(x, y) = n^{2d} \int_{|x-\xi|_\infty < \frac{1}{2n}} \int_{|y-\zeta|_\infty < \frac{1}{2n}} J(\xi, \zeta) d\xi\, d\zeta \quad \text{for} \quad x \neq y \in n^{-1}\mathbb{Z}^d,$$

and $C^n(x, x) = 0$, where $|x - y|_\infty = \max_{1 \le i \le d} |x_i - y_i|$. Let X be the Hunt process corresponding to the Dirichlet form given by (1.1). Then the sequence of processes corresponding to C^n converges weakly to X. Given Remark 4.2, the proof is standard.

Remark 4.4 As we mentioned at the beginning of Section 2, the assumptions (2.1) and (2.2) are used to guarantee a certain amount of regularity, namely, conservativeness and the existence of the heat kernel. However, one can relax these assumptions. All of the results in this paper hold if instead of (2.1) and (2.2) we assume (2.2), (2.4) for all $z_0 \in \mathbb{R}^d$ and the following:

$$\int_{|x-y|\leq 1} |x-y|^2 J(x,y)\,dy \leq \kappa_5, \qquad x \in \mathbb{R}^d.$$

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