On the dichotomy in the heat kernel two sided estimates

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Contents

1 Preliminaries

Let (M, d) be a locally compact separable metric space and μ be a Radon measure on M with full support. Assume that there exists a heat kernel $\{p_t\}_{t>0}$ on M:
Definition. A family $\{p_t\}_{t>0}$ of Borel functions $p_t(x, y)$ on $M \times M$ is called a *heat kernel* if the following conditions are satisfied, for all $x, y \in M$ and all $s, t > 0$:

(i) Positivity: $p_t(x, y) \geq 0$.

(ii) The total mass inequality

$$\int_M p_t(x, y) d\mu(y) \leq 1. \tag{1.1}$$

(iii) Symmetry: $p_t(x,y) = p_t(y,x)$.

(iv) Semigroup property:

$$p_{s+t}(x,y) = \int_M p_s(x,z)p_t(z,y)d\mu(z). \tag{1.2}$$

(v) Approximation of identity: for any $u \in L^2(M,\mu)$

$$\int_M p_t(x,y)u(y)d\mu(y) \xrightarrow{L^2} u(x) \quad \text{as } t \to 0+. \tag{1.3}$$

Any heat kernel gives rise to the *heat semigroup* $\{P_t\}_{t>0}$ where P_t is the operator in L^2 defined by

$$P_t u(x) = \int_M p_t(x,y)u(y)d\mu(y). \tag{1.4}$$

The semigroup identity (1.2) implies that $P_tP_s = P_{t+s}$, that is, the family $\{P_t\}_{t>0}$ is a *semigroup.* It follows from (1.3) that

$$s\text{-}\lim_{t\to 0} P_t = \mathrm{I},$$

where I is the identity operator in L^2 and s-lim stands for *strong* limit. Hence, $\{P_t\}_{t>0}$ is a strongly continuous, self-adjoint, contraction semigroup in L^2.

Given the semigroup $\{P_t\}_{t>0}$, define the *infinitesimal generator* $\mathcal{L}$ of the semigroup by

$$\mathcal{L}u := \lim_{t\to 0} \frac{u - P_t u}{t}, \tag{1.5}$$

where the limit is understood in the L^2-norm. The *domain* $\mathrm{dom}(\mathcal{L})$ of the generator $\mathcal{L}$ is the space of functions $u \in L^2$ for which the limit in (1.5) exists. By the Hille–Yosida theorem, $\mathrm{dom}(\mathcal{L})$ is dense in L^2. Furthermore, $\mathcal{L}$ is a self-adjoint, positive definite operator, which immediately follows from the fact that the semigroup $\{P_t\}$ is self-adjoint and contractive. Moreover, we have

$$P_t = \exp(-t\mathcal{L}), \tag{1.6}$$

where the right hand side is understood in the sense of spectral theory.

Let a heat kernel p_t on (M,d,μ) satisfy the following two-sided estimate for all $x, y \in M$ and all $t \in (0,\infty)$:

$$\frac{c_1}{t^{\alpha/\beta}}\Phi\left(c_2\frac{d(x,y)}{t^{1/\beta}}\right) \le p_t(x,y) \le \frac{c_3}{t^{\alpha/\beta}}\Phi\left(c_4\frac{d(x,y)}{t^{1/\beta}}\right) \tag{1.7}$$

where α, β are positive constants and Φ is a non-negative monotone decreasing function on $[0,\infty)$. There are two very important classes of heat kernels that satisfy (1.7). One is the heat kernel of diffusions on various fractals, where the function Φ is of the form

$$\Phi(s) = \exp(-s^\gamma), \tag{1.8}$$

for some $\gamma > 0$ (see [2] and the references therein). The Gauss-Weierstrass heat kernel on $\mathbb{R}^d$ is included in this class, in which case $\alpha = d, \beta = \gamma = 2$. The other is the heat kernel of stable-like processes, where the function Φ is of the form

$$\Phi(s) = (1+s)^{-\alpha-\beta}, \tag{1.9}$$

(see [5] and the references therein). The heat kernel of the symmetric β-stable process on $\mathbb{R}^d$ is included in this class, in which case $\alpha = d, 0 < \beta < 2$.

The nature of the parameters α and β is important. The parameter α turns out to be the Hausdorff dimension of M. The parameter β is called the walk dimension of the heat kernel p_t. This terminology is from the following observation: if the heat kernel p_t is the transition density of a Markov process X_t on M, then under mild assumptions about Φ, (1.7) implies that the average time t needed for the process X_t to move away to a distance r from the origin is of the order r^β (see, for example, [2, Lemma 3.9]).

As mentioned above, there are important classes of heat kernels that satisfy (1.7). One can then ask the following natural question:

Is there a heat kernel p_t satisfying (1.7), where Φ is different from (1.8) and (1.9)?

The main purpose of this paper is to answer this question. In Theorem 3.4, we will show under mild assumptions that the shape of Φ for any heat kernel p_t satisfying (1.7) is either (1.8) or (1.9). Our approach is analytic, which does not depend on the existence of the process X_t.

2 Dirichlet form associated with a heat kernel

Let (M, d, μ) be a metric measure space with a heat kernel $\{p_t\}_{t>0}$, and let $\{P_t\}_{t>0}$ be the heat semigroup defined by (1.4). For any $t > 0$, we define a quadratic form $\mathcal{E}_t$ on L^2 by

$$\mathcal{E}_t[u] := \left(\frac{u - P_t u}{t}, u\right), \tag{2.10}$$

where $(\cdot,\cdot)$ is the inner product in L^2. An easy computation shows that $\mathcal{E}_t$ can be equivalently defined by

$$\mathcal{E}_t[u] = \frac{1}{2t}\int_M\int_M |u(x)-u(y)|^2\, p_t(x,y)d\mu(y)d\mu(x). \tag{2.11}$$

In terms of the spectral resolution $\{E_\lambda\}$ of the generator $\mathcal{L}$, $\mathcal{E}_t$ can be expressed as follows

$$\mathcal{E}_t[u] = \int_0^\infty \frac{1-e^{-t\lambda}}{t} d\|E_\lambda u\|_2^2,$$

which implies that $\mathcal{E}_t[u]$ is decreasing in t (indeed, this is an elementary exercise to show that the function $t \mapsto \frac{1-e^{-t\lambda}}{t}$ is decreasing).

Let us define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}[u] := \lim_{t\to 0+} \mathcal{E}_t[u] = \int_0^\infty \lambda\, d\|E_\lambda u\|_2^2 \tag{2.12}$$

(where the limit may be $+\infty$ since $\mathcal{E}[u] \geq \mathcal{E}_t[u]$) and its *domain* $\mathcal{D}(\mathcal{E})$ by

$$\mathcal{D}(\mathcal{E}) := \{u \in L^2 : \mathcal{E}[u] < \infty\}.$$

It is clear from (2.11) and (2.12) that $\mathcal{E}_t$ and $\mathcal{E}$ are positive definite.

It is easy to see from (2.12) that $\mathcal{D}(\mathcal{E}) = \mathrm{dom}(\mathcal{L}^{1/2})$. The domain $\mathcal{D}(\mathcal{E})$ is dense in L^2 because $\mathcal{D}(\mathcal{E})$ contains $\mathrm{dom}(\mathcal{L})$. Indeed, if $u \in \mathrm{dom}(\mathcal{L})$ then using (1.5) and (2.10), we obtain

$$\mathcal{E}[u] = \lim_{t \to 0} \mathcal{E}_t[u] = (\mathcal{L}u, u) < \infty. \tag{2.13}$$

The quadratic form $\mathcal{E}[u]$ extends to a bilinear form $\mathcal{E}(u, v)$ by the polarization identity

$$\mathcal{E}(u, v) = \frac{1}{2}(\mathcal{E}[u+v] - \mathcal{E}[u-v]).$$

It follows from (2.13) that $\mathcal{E}(u, v) = (\mathcal{L}u, v)$ for all $u, v \in \mathrm{dom}(\mathcal{L})$. The space $\mathcal{D}(\mathcal{E})$ is naturally endowed with the inner product

$$[u, v] := (u, v) + \mathcal{E}(u, v). \tag{2.14}$$

It is possible to show that the form $\mathcal{E}$ is *closed*, that is the space $\mathcal{D}(\mathcal{E})$ is *Hilbert*.

It is easy to see from (1.4) and the definition of a heat kernel that the semigroup $\{P_t\}$ is *Markovian*, that is $0 \leq u \leq 1$ implies $0 \leq P_t u \leq 1$. This implies that the form $\mathcal{E}$ satisfies the Markov property, that is $u \in \mathcal{D}(\mathcal{E})$ implies $v := \min(u_+, 1) \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}[v] \leq \mathcal{E}[u]$. Hence, $\mathcal{E}$ is a *Dirichlet form*.

We say that $\mathcal{E}$ is *local* if supp u and supp v are disjoint compact sets for $u, v \in \mathcal{D}(\mathcal{E})$, then $\mathcal{E}(u, v) = 0$. $\mathcal{E}$ is called *stochastically complete* if $P_t 1 = 1$ for all $t > 0$, that is, the equality holds in (1.1).

A Dirichlet form $\mathcal{E}$ is said to be *regular* if there exists a subspace $\mathcal{C} \subset \mathcal{D}(\mathcal{E}) \cap C_0(M)$ such that $\mathcal{C}$ is dense in $\mathcal{D}(\mathcal{E})$ with $[\cdot]$-norm and dense in $C_0(M)$ with uniform norm. (Here $C_0(M)$ is the space of continuous compactly supported functions on M.) When $\mathcal{E}$ is *regular*, there is a corresponding Markov process X_t which is furthermore a Hunt process. As we mentioned in the first section, we do *not* assume $\mathcal{E}$ to be regular throughout this paper.

3 Main result

Fix two positive parameters α and β and a monotone decreasing function $\Phi : [0, +\infty) \to [0, +\infty)$ such that $\Phi(c) > 0$ for some $c > 0$.

Lemma 3.1 *Assume that $\{p_t\}$ is a heat kernel on (M, d, μ) such that, for all $x, y \in M$ and $t > 0$,*

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right), \tag{3.1}$$

for some $C > 0$. Then either the associated Dirichlet form $\mathcal{E}$ is local or

$$\Phi(s) \geq \frac{c}{(1+s)^{\alpha+\beta}} \tag{3.2}$$

for some $c > 0$.

Proof. Consider the form $\mathcal{E}_t$ which is given by

$$\mathcal{E}_t(u,v)=\frac{1}{2t}\int_M\int_M (u(x)-u(y))\,(v(x)-v(y))\,p_t(x,y)d\mu(y)d\mu(x). \tag{3.3}$$

Let $u,v\in L^1(M,\mu)$ be two non-negative functions with disjoint supports $A=\operatorname{supp} u$ and $B=\operatorname{supp} v$, and set

$$r=d(A,B)>0. \tag{3.4}$$

The integrand in (3.3) vanishes if either both x,y are outside A or both x,y are outside B. Hence, we can restrict the integration to the domain where one of the variables x,y is in A and the other is in B. Hence, we obtain, using the symmetry of the heat kernel,

$$\begin{aligned}\mathcal{E}_t(u,v) &= -\frac{1}{2t}\int_A\int_B u(x)v(y)\,p_t(x,y)d\mu(y)d\mu(x)\\ &\quad -\frac{1}{2t}\int_B\int_A u(y)v(x)\,p_t(x,y)d\mu(y)d\mu(x)\\ &= -\frac{1}{t}\int_A\int_B u(x)v(y)\,p_t(x,y)d\mu(y)d\mu(x). \end{aligned}\tag{3.5}$$

If $x\in A$ and $y\in B$ then $d(x,y)\geq r$ and

$$p_t(x,y)\leq\frac{C}{t^{\alpha/\beta}}\Phi\left(\frac{r}{t^{1/\beta}}\right).$$

Therefore, (3.5) implies

$$|\mathcal{E}_t(u,v)|\leq\frac{C}{t^{1+\alpha/\beta}}\Phi\left(\frac{r}{t^{1/\beta}}\right)\|u\|_{L^1}\|v\|_{L^1}. \tag{3.6}$$

If (3.2) fails then there exists a sequence $\{s_k\}\to\infty$ such that

$$s_k^{\alpha+\beta}\Phi(s_k)\to 0\quad\text{as } k\to\infty.$$

Define a sequence $\{t_k\}$ from the condition

$$s_k=\frac{r}{t_k^{1/\beta}}.$$

Then

$$s_k^{\alpha+\beta}\Phi(s_k)=\frac{r^{\alpha+\beta}}{t_k^{1+\alpha/\beta}}\Phi\left(\frac{r}{t_k^{1/\beta}}\right)\to 0\quad\text{as } k\to\infty,$$

and (3.6) implies that

$$\mathcal{E}_{t_k}(u,v)\to 0\quad\text{as } k\to\infty. \tag{3.7}$$

Therefore, if $\operatorname{supp} u$ and $\operatorname{supp} v$ are disjoint compact sets for $u,v\in\mathcal{D}(\mathcal{E})$, then we can take $r>0$ as in (3.4), so by (3.7), $\mathcal{E}(u,v)=\lim_{k\to\infty}\mathcal{E}_{t_k}(u,v)=0$. Hence the locality of $\mathcal{E}$ follows. ■

Lemma 3.2 *Assume that $\{p_t\}$ is a heat kernel on (M,d,μ) such that, for all $x,y\in M$ and $t>0$,*

$$p_t(x,y)\geq \frac{c}{t^{\alpha/\beta}}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right), \tag{3.8}$$

for some $c>0$. Then

$$\Phi(s)\leq \frac{C}{(1+s)^{\alpha+\beta}} \tag{3.9}$$

for some $C>0$.

Proof. Let u be a non-constant function from $L^2(M,\mu)$. Choose a ball $Q\subset M$ where u is non-constant and let $a>b$ be two real values such that the sets

$$A=\{x\in Q: u(x)\geq a\} \text{ and } B=\{x\in Q: u(x)\leq b\}$$

have positive measures. If the diameter of Q is D then, by (3.8), we have for all $x,y\in Q$

$$p_t(x,y)\geq \frac{c}{t^{\alpha/\beta}}\Phi\left(\frac{D}{t^{1/\beta}}\right)$$

whence by (3.3)

$$\begin{aligned}\mathcal{E}(u,u) &\geq \mathcal{E}_t(u,u)\geq \frac{1}{2t}\int_A\int_B (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x)\\ &\geq (a-b)^2\mu(A)\mu(B)\frac{c}{2t^{1+\alpha/\beta}}\Phi\left(\frac{D}{t^{1/\beta}}\right)\\ &= \frac{c'}{t^{1+\alpha/\beta}}\Phi\left(\frac{D}{t^{1/\beta}}\right) \end{aligned} \tag{3.10}$$

where $c'>0$. If (3.9) fails then there exists a sequence $\{s_k\}\to\infty$ such that

$$s_k^{\alpha+\beta}\Phi(s_k)\to\infty \quad \text{as } k\to\infty.$$

Define a sequence $\{t_k\}$ from the condition

$$s_k=\frac{D}{t_k^{1/\beta}}.$$

Then

$$\frac{1}{t_k^{1+\alpha/\beta}}\Phi\left(\frac{D}{t_k^{1/\beta}}\right)=D^{-(\alpha+\beta)}s_k^{\alpha+\beta}\Phi(s_k)=\longrightarrow\infty \quad \text{as } k\to\infty,$$

and (3.10) yields $\mathcal{E}(u,u)=\infty$. Therefore, the domain of the form $\mathcal{E}$ contains only constants. Note that $L^2(M,\mu)$ does not consist of only constants. (Indeed, since μ is a Radon measure on M with full support, it is enough to check that M consists of more than one point. By (3.8), $p_t(x,x)\to\infty$ as $t\to 0$, so if $M=\{x\}$, this contradicts (1.1).) Thus, the fact that the form $\mathcal{E}$ contains only constants contradicts the fact that this domain is dense in $L^2(M,\mu)$. ■

We say that (M,d) satisfies the chain condition if there exists a (large) constant C such that for any two points $x,y\in M$ and for any positive integer n, there exists a sequence $\{x_i\}_{i=0}^n$ of points in M such that $x_0=x$, $x_n=y$, and

$$d(x_i,x_{i+1})\le C\frac{d(x,y)}{n},\qquad \text{for all } i=0,1,\cdots,n-1.$$

In the following, we denote $\Phi(s)\simeq f(s)$ if there exist constants $c_1,c_2>0$ such that $c_1f(s)\le\Phi(s)\le c_2f(s)$ for all $s>0$. Similarly, we denote $\Phi(s)\asymp f(Cs)g(cs)$ if there exist constants $c_1,\cdots,c_4>0$ such that $f(c_1s)g(c_2s)\le\Phi(s)\le f(c_3s)g(c_4s)$ for all $s>0$.

Corollary 3.3 *If the following estimate holds for all $x,y\in M$ and $t>0$,*

$$p_t(x,y)\asymp\frac{C}{t^{\alpha/\beta}}\Phi\left(c\frac{d(x,y)}{t^{1/\beta}}\right)\tag{3.11}$$

then either the Dirichlet form $\mathcal{E}$ is local or

$$\Phi(s)\simeq\frac{1}{(1+s)^{\alpha+\beta}}.\tag{3.12}$$

Proof. Indeed, if $\mathcal{E}$ is non-local then Φ satisfies (3.2) and (3.9), whence the claim follows. ■

Theorem 3.4 *Let the metric space (M,d) satisfy the chain condition, the heat kernel be stochastically complete, and (3.11) hold with some $\alpha,\beta>0$ and Φ. Then $\beta\le\alpha+1$,*

$$\mu(B(x,r))\simeq r^\alpha,\tag{3.13}$$

and the following dichotomy holds:

- *either the Dirichlet form $\mathcal{E}$ is local, $\beta\ge2$, and $\Phi(s)\asymp C\exp\left(-cs^{\frac{\beta}{\beta-1}}\right)$.*
- *or the Dirichlet form $\mathcal{E}$ is non-local and $\Phi(s)\simeq(1+s)^{-(\alpha+\beta)}$.*

Proof. By Lemma 3.2, we have the upper bound

$$\Phi(s)\le\frac{C}{(1+s)^{\alpha+\beta}}.\tag{3.14}$$

In particular, this implies

$$\int_0^\infty s^{\alpha-1}\Phi(s)\,ds<\infty.\tag{3.15}$$

By [9, Theorem 3.2] (see also [6]), the estimate (3.11) with a function Φ satisfying (3.15) and the stochastic completeness imply (3.13). By [9, Corollary 3.3], we have $\operatorname{diam}(M)=\infty$. Furthermore, as it follows from the proof of [9, Theorem 3.2], for any $\varepsilon>0$ there is $\delta>0$ such that

$$\int_{B(x,r)^c}p_t(x,y)\,d\mu(y)\le\varepsilon,\tag{3.16}$$

provided $t\le(\delta r)^\beta$.

Also, (3.11) with (3.15) and the chain condition imply that $\beta \le \alpha+1$ (see [9, Theorem 4.8(ii)] and [6])

If the form $\mathcal{E}$ is non-local, then by Lemma 3.1 and Lemma 3.2, Φ satisfies (3.12), which finishes the proof in this case. Assume now that $\mathcal{E}$ is local. In this case, we will show that (3.16) implies that $\beta \ge 1$ and, for all $t, r > 0$ and $x \in M$,

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \le \begin{cases} C \exp\left(-c\left(\frac{r^\beta}{t}\right)^{\frac{1}{\beta-1}}\right), & \text{if } \beta > 1, \\ C \exp\left(-c\left(\frac{r}{t}\right)\right), & \text{if } \beta = 1. \end{cases} \tag{3.17}$$

Indeed, for each $\beta > 0$, using (4.21) and (4.22) in [8], letting $k \to \infty$ and then replacing $2r$ by r, we have

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \le C \exp\left(\lambda t - c_1 \lambda^{1/\beta} r\right), \qquad \text{for all } \lambda \ge c_2 r^{-\beta}. \tag{3.18}$$

(Note that the arguments in [8] do not require the regularity of the Dirichlet form.) When $\beta < 1$, take $\lambda = c_3 (r/t)^{\beta/(\beta-1)}$, where $c_3 - c_1 c_3^{1/\beta} = -1$. Then, $\lambda \ge c_2 r^{-\beta}$ is equivalent to $t \ge c_4 r^\beta$ for some $c_4 > 0$, so we obtain

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \le C \exp\left(-\left(\frac{t}{r^\beta}\right)^{\frac{1}{1-\beta}}\right) \qquad \text{for } \ t \ge c_4 r^\beta. \tag{3.19}$$

On the other hand, by the lower bound of (3.11), for $t = M r^\beta$ and $y \in B(x, 2r)$, we have

$$p_t(x,y) \ge \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{2r}{t^{1/\beta}}\right) = \frac{C}{M^{\alpha/\beta} r^\alpha} \Phi\left(\frac{2c}{M^{1/\beta}}\right) \ge \frac{C'}{M^{\alpha/\beta} r^\alpha} \tag{3.20}$$

when M is large enough, since Φ is monotone decreasing and $\Phi(a) > 0$ for some $a > 0$. Integrating (3.20) over $y \in B(x,2r) \setminus B(x,r)$ and using (3.13), we have that the left hand side of (3.19) is greater than or equal to $C'' M^{-\alpha/\beta}$. This is a contradiction when M is very large, because the right hand side of (3.19) is $C \exp(-M^{1/(1-\beta)})$ for $t = M r^\beta$. So, we obtain $\beta \ge 1$. Now, applying (3.18) with $\lambda = c(r/t)^{\beta/(\beta-1)}$ when $\beta > 1$ and with $\lambda = ct^{-1}$ when $\beta = 1$, we obtain (3.17) for $t \le c' r^\beta$. (3.17) is always true for $t \ge c' r^\beta$ by adjusting C, so the proof of (3.17) is completed.

Now, $x, y \in M$, $t > 0$, and for $r = \frac{1}{2} d(x,y)$,

$$\begin{aligned} p_t(x,y) &= \int_M p_{t/2}(x,z)\, p_{/2}(z,y)\, d\mu(z) \\ &\le \int_{B(x,r)^c \cup B(y,r)^c} p_{t/2}(x,z)\, p_{/2}(z,y)\, d\mu(z) \\ &\le \sup_{z \in M} p_{t/2}(z,y) \int_{B(x,r)^c} p_{t/2}(x,z)\, d\mu(z) + \sup_{z \in M} p_{t/2}(x,z) \int_{B(y,r)^c} p_{t/2}(y,z)\, d\mu(z). \end{aligned}$$

Since by (3.11) $p_t(x,y) \le C t^{-\alpha/\beta}$ for all $x, y \in M$ and $t > 0$, combining this with (3.17) we obtain,

$$p_t(x,y) \le \begin{cases} \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right), & \text{if } \beta > 1, \\ \frac{C}{t^\alpha} \exp\left(-c\left(\frac{r}{t}\right)\right), & \text{if } \beta = 1. \end{cases} \tag{3.21}$$

Now, by [9, Theorem 4.8(i)], the estimates (3.11) and (3.21) imply $\beta \geq 2$. (Note that the arguments in [9] do not require the regularity of the Dirichlet form.)

On the other hand, when $\beta \geq 2$ (in fact $\beta > 1$ is enough), the standard chain argument (see [9, Corollary 3.5]) shows that the lower bound in (3.11) implies the lower bounds

$$p_t(x,y) \geq \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right).$$

Combining these estimates with (3.11), we obtain

$$\Phi(s) \asymp C \exp\left(-cs^{\frac{\beta}{\beta-1}}\right),$$

with $\beta \geq 2$, which finishes the proof. ■

Remark. 1) This theorem excludes discrete cases. Indeed, for the discrete cases, (3.11) does not hold for very small t. For example, continuous time simple random walk on $\mathbb{Z}^d$ satisfies (3.11) with $\alpha = d, \beta = 2$ and $\Phi(s) \asymp C\exp(-cs^2)$ for $d(x,y) \vee 1 \leq t$, but (3.11) does not hold for $t << 1$.
2) In this theorem, we assume (3.11) for all $x, y \in M$ and $t > 0$. But if the Dirichlet form $\mathcal{E}$ is regular, then we can relax this part of the assumption and need only to assume (3.11) for μ-a.e. $x, y \in M$ and all $t > 0$. See [3, Theorem 2.1] and [4].
3) In the case of a local form, we obtain the relations between α and β

$$2 \leq \beta \leq \alpha + 1. \tag{3.22}$$

By [1], any couple of α, β in this range can be realized for the above heat kernel estimates. In the case of a non-local form, we have instead the range

$$0 < \beta \leq \alpha + 1.$$

Any couple in the range $0 < \beta < \alpha + 1$ can be realized. Indeed, if $\mathcal{L}$ is the generator of diffusion with parameters α and β from the range (3.22) then $\mathcal{L}^\delta$, $\delta \in (0,1)$, generated a Hunt process with the walk dimension $\beta' = \delta\beta$ and the same α, so that β' can take any value from $(0, \alpha+1)$. We do not know whether $\beta = \alpha + 1$ can occur for non-local processes or not.

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