THE MODULI SPACE OF POLYNOMIAL MAPS OF ONE COMPLEX VARIABLE

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Abstract. This paper, motivated from complex dynamics and algebraic geometry, studies the moduli space of polynomial maps of $\mathbb{C}$, from the standpoint of their fixed-point eigenvalues. We denote by $\widetilde{P}_d$ the set of affine conjugacy classes of $f(z) \in \mathbb{C}[z]$ with $\deg C = d$, and define the map $\Phi_d : \widetilde{P}_d \to \widetilde{\Lambda}_d := \left\{ (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \;\middle|\; \sum_{i=1}^{d} \prod_{j \neq i} (1 - \lambda_j) = 0 \right\} \Big/ \mathfrak{S}_d$ by corresponding each $f(z) \in \widetilde{P}_d$ to the collection of the eigenvalues of the fixed points of f. This paper describes in detail the fiber structure of the map Φ_d.

This map Φ_d is generically finite and has a very beautiful fiber structure. We shall exactly find the cardinality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ for any $\bar{\lambda} = \overline{(\lambda_1, \ldots, \lambda_d)} \in \widetilde{\Lambda}_d$ with $\lambda_i \neq 1$ for $1 \leq i \leq d$. Precisely, the cardinality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ is computed in finite steps only from the two combinatorial data $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ which are defined in Main Theorem 1. The local fiber structure of the map $\Phi_d : \widetilde{P}_d \to \widetilde{\Lambda}_d$ is also completely determined by $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$, which is stated in Main Theorem 2. We also provide some problems on combinatorics.

1. Introduction

In this paper, we shall study the moduli space of polynomial maps of the complex plane $\mathbb{C}$, from the standpoint of their fixed-point eigenvalues. There are two reasons why we have studied this theme. One of them is the interest from complex dynamics. In complex dynamics, the eigenvalues of the fixed points of a polynomial map f of $\mathbb{C}$ play a very important role to characterize the original map f. Hence it is interesting to ask to what extent the fixed-point eigenvalues of f determine the original map f. This paper answers this question; we shall study in detail how many affine conjugacy classes of polynomial maps there are when the eigenvalues of their fixed points are specified (see Main Theorems 1, 3 and Definition 2.1).

The other reason is the interest from algebraic geometry. There have been a huge number of studies on moduli spaces of maps on various spaces. However fixed-point eigenvalues have not been taken notice of in the study of moduli spaces of maps, though it is natural to make the fibration of a moduli space of maps by corresponding each map to the set of its fixed-point eigenvalues. In this paper, we shall make this fibration for the moduli space of polynomial maps of $\mathbb{C}$, and shall find that its fiber structure is very beautiful (see Main Theorem 2). It is also expected that if a moduli space of maps have finite dimension, then the similar results as in this paper hold.

In Introduction, we shall give the exact formulation of the problem, and explain the historical background of this paper. After that, we shall state Main Theorems 1 and 2, and shall give the program of the following sections. We have three main theorems in this paper; however we shall postpone stating Main Theorem 3 till Section 2.

To formulate the problem exactly, we shall fix our notation first. For a natural number d with $d \geq 2$, we denote the family of polynomial maps of degree d by

$$P_d := \left\{ f \in \mathbb{C}[z] \;\middle|\; \deg f = d \right\},$$

and the group of affine transformations on $\mathbb{C}$ by

$$\Gamma := \left\{\gamma(z) = az + b \;\middle|\; a, b \in \mathbb{C},\ a \neq 0\right\},$$

which acts on P_d by $\gamma \cdot f := \gamma \circ f \circ \gamma^{-1}$ for $\gamma \in \Gamma$ and $f \in P_d$; we denote by $\widetilde{P}_d$ the quotient of P_d by Γ, i.e.,

$$\widetilde{P}_d := P_d/\Gamma.$$

Here we say that maps $f, g \in P_d$ are affine conjugate if the equality $g = \gamma \cdot f$ holds for some $\gamma \in \Gamma$, and we call $\widetilde{P}_d$ the moduli space of polynomial maps of degree d hereafter, which is the main object of this paper. Note that if f and g are affine conjugate, then the sets of their fixed-point eigenvalues are the same, i.e., we have $\left\{f'(\zeta) \;\middle|\; f(\zeta) = \zeta\right\} = \left\{g'(\zeta) \;\middle|\; g(\zeta) = \zeta\right\}$, where both sets are considered counted with multiplicity. Concerning fixed points, we put

$$\mathrm{Fix}(f) := \left\{\zeta \in \mathbb{C} \;\middle|\; f(\zeta) = \zeta\right\}$$

for $f \in P_d$, where $\mathrm{Fix}(f)$ is also considered counted with multiplicity; hence we always have $\#\left(\mathrm{Fix}(f)\right) = \deg f$.

Remark 1.1. A fixed point $\zeta \in \mathrm{Fix}(f)$ of a polynomial map $f \in P_d$ is multiple if and only if $f'(\zeta) = 1$.

Proposition 1.2 (Fixed point theorem)**.** *Let d be a natural number with $d \geq 2$ and suppose that a polynomial map $f \in P_d$ has no multiple fixed point. Then we have $\sum_{\zeta \in \mathrm{Fix}(f)} \frac{1}{1 - f'(\zeta)} = 0$.*

Proof. Consider the integration $\frac{1}{2\pi\sqrt{-1}} \oint_{|z|=R} \frac{dz}{z - f(z)}$ for sufficiently large positive real number R. This integral value tends to 0 when R goes to infinity. On the other hand, the integrand has a pole at $\zeta \in \mathbb{C}$ if and only if ζ is a fixed point of f. Moreover its residue at $\zeta \in \mathrm{Fix}(f)$ is $\frac{1}{1 - f'(\zeta)}$. The proposition is thus verified. $\square$

By the fixed point theorem, we find that it is no use to consider the collection of eigenvalues which do not satisfy the equality in Proposition 1.2. Replacing $f'(\zeta)$'s by λ's, we have $\sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 0$, which is also equivalent to $\sum_{i=1}^{d} \prod_{j \neq i} (1 - \lambda_j) = 0$ in the case $\lambda_i \neq 1$ for any $1 \leq i \leq d$; we put

$$\Lambda_d := \left\{ (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \;\middle|\; \sum_{i=1}^{d} \prod_{j \neq i} (1 - \lambda_j) = 0 \right\},$$
$$\widetilde{\Lambda}_d := \Lambda_d/\mathfrak{S}_d \quad \text{and} \quad pr : \Lambda_d \to \widetilde{\Lambda}_d,$$

where $\mathfrak{S}_d$ is the d-th symmetric group and $\mathfrak{S}_d$ acts on Λ_d by the permutation of coordinates. Throughout this paper, we always denote by $\bar{\lambda}$ the equivalent class of $\lambda \in \Lambda_d$ in $\widetilde{\Lambda}_d$, i.e., $\bar{\lambda} = pr(\lambda)$, and never denote the complex conjugate of λ.

To summarize, we have the following:

Proposition 1.3. *we can define the map $\Phi_d : \widetilde{P}_d \to \widetilde{\Lambda}_d$ by $f \mapsto \left(f'(\zeta)\right)_{\zeta \in \mathrm{Fix}(f)}$.*

Proof. If $f \in P_d$ has a multiple fixed point, then at least two elements of $\left\{f'(\zeta) \;\middle|\; \zeta \in \mathrm{Fix}(f)\right\}$ are equal to 1, which implies that $\left(f'(\zeta)\right)_{\zeta \in \mathrm{Fix}(f)}$ belongs to $\widetilde{\Lambda}_d$. The rest is obvious. $\square$

We have thus formulated the problem; the aim of this paper is to analyze the structure of the map Φ_d.

Theorem 1.4. *In the case $d = 2$ or 3, the map Φ_d is bijective.*

This theorem is well-known and easy to prove by a direct calculus. By this theorem, polynomial maps $f \in \widetilde{P}_d$ are completely parameterized by their fixed-point eigenvalues in the case $d = 2$ or 3. Historically, making use of this parameterization, John Milnor [2] started to study complex dynamics in the case of cubic polynomials.

We cannot expect Φ_d to be bijective anymore if $d \geq 4$; yet we can expect Φ_d to be generically finite by the remark below:

Remark 1.5. We have $\widetilde{P}_d \cong \mathbb{C}^{d-1}/(\mathbb{Z}/(d-1)\mathbb{Z})$ and $\widetilde{\Lambda}_d \cong \mathbb{C}^{d-1}$. Especially we have $\dim_{\mathbb{C}} \widetilde{P}_d = \dim_{\mathbb{C}} \widetilde{\Lambda}_d = d-1$.

We are recently informed that Masayo Fujimura [1] also has studied the similar theme independently. She completely studied the map Φ_d in the case $d = 4$, and showed that Φ_d is not surjective for $d \geq 4$. Similar results for rational maps are given in Milnor [3, p.152 Problem 12-d], which is rather an easy exercise.

In the main theorems of this paper, we investigate the map Φ_d for $d \geq 4$ in detail on the domain where polynomial maps have no multiple fixed points; we prepare two more symbols:

$$V_d := \left\{ (\lambda_1, \ldots, \lambda_d) \in \Lambda_d \;\middle|\; \lambda_i \neq 1 \text{ for any } 1 \leq i \leq d \right\},$$
$$\widetilde{V}_d := V_d/\mathfrak{S}_d.$$

Note that $V_d, \widetilde{V}_d$ are Zariski open subsets of $\Lambda_d, \widetilde{\Lambda}_d$ respectively. For any set X, we denote the cardinality of X by $\#(X)$.

Main Theorem 1. *Let d be a natural number with $d \geq 4$ and suppose that $\lambda = (\lambda_1, \ldots, \lambda_d)$ is an element of V_d. Then*

1. *we always have the inequalities $0 \leq \#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) \leq (d-2)!$.*
2. *The cardinality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ is computed in finite steps from the two combinatorial data*

$$\mathcal{I}(\lambda) := \left\{ I \subsetneq \{1, 2, \ldots, d\} \;\middle|\; I \neq \emptyset, \quad \sum_{i \in I} \frac{1}{1-\lambda_i} = 0 \right\},$$
$$\mathcal{K}(\lambda) := \left\{ K \subseteq \{1, 2, \ldots, d\} \;\middle|\; K \neq \emptyset. \text{ If } i, j \in K, \text{ then } \lambda_i = \lambda_j \right\}.$$

3. *If the inclusion relations $\mathcal{I}(\lambda) \subseteq \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda) \subseteq \mathcal{K}(\lambda')$ hold for $\lambda, \lambda' \in V_d$, then the inequality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) \geq \#\left(\Phi_d^{-1}\left(\bar{\lambda}'\right)\right)$ holds.*
4. *The equality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = (d-2)!$ holds if and only if the set $\mathcal{I}(\lambda)$ is empty and the complex numbers $\lambda_1, \ldots, \lambda_d$ are mutually distinct.*
5. *If there exist non-zero integers $c_1, \ldots, c_d$ which satisfy the conditions $\frac{1}{1-\lambda_1} : \cdots : \frac{1}{1-\lambda_d} = c_1 : \cdots : c_d$ and $\sum_{i=1}^{d} |c_i| \leq 2(d-2)$, then the set $\Phi_d^{-1}\left(\bar{\lambda}\right)$ is empty.*
6. *In the case $d \leq 7$, the converse of the assertion (5) holds.*

The exact process of computing the cardinality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ from the data $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ is explained later in Definition 2.1 and Main Theorem 3; it is rather long and complicated.

Conjecture 1.

1. *The converse of the assertion (5) also holds in the case $d \geq 8$.*
2. *If the inclusion relations $\mathcal{I}(\lambda) \subsetneq \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda) \subseteq \mathcal{K}(\lambda')$ hold for $\lambda, \lambda' \in V_d$, then the inequality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) > \#\left(\Phi_d^{-1}\left(\bar{\lambda}'\right)\right)$ holds.*

The conjectures above are completely reduced to the problems on combinatorics by Main Theorem 3.

The local fiber structure of the map Φ_d is also determined by the combinatorial data $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$.

Main Theorem 2.

(1) *For any $\lambda, \lambda' \in V_d$ with $\mathcal{I}(\lambda) = \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda) = \mathcal{K}(\lambda')$, there exist open neighborhoods $\widetilde{U} \ni \bar{\lambda}$, $\widetilde{U}' \ni \bar{\lambda}'$ in $\widetilde{V}_d$ and biholomorphic maps $\mathfrak{L} : \Phi_d^{-1}(\widetilde{U}) \to \Phi_d^{-1}(\widetilde{U}')$, $\widetilde{L} : \widetilde{U} \to \widetilde{U}'$ and $L : U \to U'$ with $L(\lambda) = \lambda'$ such that the following conditions (1a) and (1b) are satisfied, where U, U' are the connected components of $pr^{-1}(\widetilde{U})$, $pr^{-1}(\widetilde{U}')$ containing λ, λ' respectively.*

   (a) *The equalities $\Phi_d \circ \mathfrak{L} = \widetilde{L} \circ \Phi_d|_{\Phi_d^{-1}(\widetilde{U})}$ and $pr \circ L = \widetilde{L} \circ pr|_U$ hold.*

   (b) *For any $\lambda'' \in U$, the equalities $\mathcal{I}(\lambda'') = \mathcal{I}(L(\lambda''))$ and $\mathcal{K}(\lambda'') = \mathcal{K}(L(\lambda''))$ hold.*

(2) *For each $(\mathcal{I}, \mathcal{K}) \in \{(\mathcal{I}(\lambda), \mathcal{K}(\lambda)) \mid \lambda \in V_d\}$, we put*

$$\widetilde{V}(\mathcal{I}, \mathcal{K}) := \left\{ \bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{I}(\lambda) = \mathcal{I} \text{ and } \mathcal{K}(\lambda) = \mathcal{K} \right\},$$

$$\widetilde{V}(\mathcal{I}, *) := \left\{ \bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{I}(\lambda) = \mathcal{I} \right\},$$

$$\widetilde{V}(*, \mathcal{K}) := \left\{ \bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{K}(\lambda) = \mathcal{K} \right\}.$$

Then for any $(\mathcal{I}, \mathcal{K}) \in \{(\mathcal{I}(\lambda), \mathcal{K}(\lambda)) \mid \lambda \in V_d\}$ we have the following:

   (a) *the map $\Phi_d|_{\Phi_d^{-1}(\widetilde{V}(\mathcal{I}, *))} : \Phi_d^{-1}(\widetilde{V}(\mathcal{I}, *)) \to \widetilde{V}(\mathcal{I}, *)$ is proper.*

   (b) *The map $\Phi_d|_{\Phi_d^{-1}(\widetilde{V}(*, \mathcal{K}))} : \Phi_d^{-1}(\widetilde{V}(*, \mathcal{K})) \to \widetilde{V}(*, \mathcal{K})$ is locally homeomorphic.*

   (c) *For each connected component X of $\Phi_d^{-1}(\widetilde{V}(\mathcal{I}, \mathcal{K}))$, the map $\Phi_d|_X : X \to \widetilde{V}(\mathcal{I}, \mathcal{K})$ is an unbranched covering.*

We have nine sections in this paper. In Section 2, we shall give the exact process of computing the cardinality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ from the data $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ in Main Theorem 3, and shall give a problem, a remark and a conjecture concerning the main theorems, one of which is a problem on combinatorics. To state Main Theorem 3 explicitly, we need some more symbols, which will be prepared in Definition 2.1. These symbols are also often referred to in the proof of the main theorems afterward.

From Section 3 to Section 9, we shall give the proofs of Main Theorems 1, 2 and 3. The proofs are self-contained except for the basic knowledge of the intersection theory on the projective space $\mathbb{P}^n$. The most important tool for the proof is the theorem which is an extension of Bezout's theorem on $\mathbb{P}^n$ especially in the case that some components of the common zeros of n homogeneous polynomials are not points or are embedded components, which is stated in Proposition 4.2. The most difficult and most crucial part of the proof is Section 7. Main Theorem 2 is naturally proved in the process of the proofs of Main Theorems 1 and 3. The assertion (5) in Main Theorem 1 is proved in Section 3, and the proofs of the assertions (1), (4) and (6) in Main Theorem 1 are given in Section 5, whereas the proofs of the rest are completed in Section 9.

In Section 3 we shall express the set $\Phi_d^{-1}(\bar{\lambda})$ in another way as follows: for each $\lambda \in V_d$, we shall define the subsets $T_d(\lambda)$, $S_d(\lambda)$ and $B_d(\lambda)$ of $\mathbb{P}^{d-2}$, where $T_d(\lambda)$ is the set of the common zeros of some $(d-2)$ homogeneous polynomials $\varphi_1, \ldots, \varphi_{d-2}$ on $\mathbb{P}^{d-2}$, and $S_d(\lambda), B_d(\lambda)$ are subsets of $T_d(\lambda)$ whose disjoint union is $T_d(\lambda)$, i.e., $T_d(\lambda) = S_d(\lambda) \amalg B_d(\lambda)$. Moreover we shall define the subgroup $\mathfrak{S}(\mathcal{K}(\lambda))$ of $\mathfrak{S}_d$ acting on $S_d(\lambda)$, and shall show the existence of the bijection $\overline{\pi(\lambda)} : S_d(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda)) \cong \Phi_d^{-1}(\bar{\lambda})$ in Proposition 3.3. By Proposition 3.3, we can divide the proof of Main Theorems 1 and 3 into two steps: the first one is to find the cardinality $\#(S_d(\lambda))$, and the second one is to analyze the action of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$.

In Section 4 we shall review the intersection theory on $\mathbb{P}^n$, and shall define the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ for homogeneous polynomials $\varphi_1, \dots, \varphi_m$ on $\mathbb{P}^n$ and a component C of the common zeros of $\varphi_1, \dots, \varphi_m$ with $\operatorname{codim} C = m$ in Definition 4.1. Here, the definition is also well-defined for embedded components C; if C is an irreducible component, then the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ is equal to the usual intersection multiplicity of $\varphi_1, \dots, \varphi_m$ along C. After that, in Proposition 4.2, we shall give the relation among these numbers, which is also reduced to the usual Bezout's theorem if all the components are points. Proposition 4.2 will be utilized crucially for finding the cardinality $\#(S_d(\lambda))$ afterward.

In Sections 5, 6 and 7 we shall find the cardinality $\#(S_d(\lambda))$, based on Section 4. More precisely, in Section 5, we shall give the explicit expression of the set $B_d(\lambda)$ in Lemma 5.5 and shall state Theorems A and B, in which we shall give the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ for each component C of the common zeros of $\varphi_1, \dots, \varphi_{d-2}$ with $\operatorname{codim} C = m$ contained in $B_d(\lambda)$. Some of these components may not be irreducible but embedded, which makes their computation much difficult. Proposition 4.2, Theorems A and B give the exact expression of the cardinality $\#(S_d(\lambda))$. Section 6 is devoted to the proof of Theorem A; Section 7 is devoted to the proof of Theorem B.

In most cases, the action of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$ is free; however in some cases, it is very complicated. In Section 8 we shall analyze the action of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$ in detail, and shall give the exact relation between the cardinalities of $S_d(\lambda)$ and $\Phi_d^{-1}(\bar{\lambda})$ in Theorem E. To summarize, in Section 9 we shall complete the proof of the main theorems.

2. Main Theorem 3

In this section, we shall give the exact process of computing the cardinality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ from the combinatorial data $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ in Definition 2.1 and Main Theorem 3, and shall give a problem, a remark and a conjecture concerning the main theorems. The readers may skip this section if they note that Definition 2.1 is often referred to later in the proof of the main theorems. Reading Sections 3, 4, 5 and 8, the readers will find that Main Theorem 3 is more natural.

Definition 2.1. Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be an element of V_d. Then

- we define the combinatorial data
$$\mathfrak{I}(\lambda) := \left\{ \{I_1, \dots, I_l\} \;\middle|\; \begin{array}{l} I_1 \amalg \cdots \amalg I_l = \{1, \dots, d\}, \quad l \geq 2 \\ I_u \in \mathcal{I}(\lambda) \text{ holds for any } 1 \leq u \leq l \end{array} \right\},$$
where $I_1 \amalg \cdots \amalg I_l$ denotes the disjoint union of $I_1, \dots, I_l$. The partial order $\prec$ in $\mathfrak{I}(\lambda)$ is defined by the refinement of sets. More precisely, for $\mathbb{I}, \mathbb{I}' \in \mathfrak{I}(\lambda)$, the relation $\mathbb{I} \prec \mathbb{I}'$ holds if and only if there exists a map $\chi : \mathbb{I}' \to \mathbb{I}$ such that the equality $I = \coprod_{I' \in \chi^{-1}(I)} I'$ holds for any $I \in \mathbb{I}$. Note that $\mathfrak{I}(\lambda)$ gives the equivalent information as $\mathcal{I}(\lambda)$. (For more detail, see Definition 5.3, Example 1 and Remark 5.4.)
- We denote by $K_1, \dots, K_q$ the collection of maximal elements of $\mathcal{K}(\lambda)$ with respect to the inclusion relations, i.e.,
$$\{K_1, \dots, K_q\} = \left\{ K \in \mathcal{K}(\lambda) \;\middle|\; i \in K,\ j \in \{1, \dots, d\} \setminus K \Longrightarrow \lambda_i \neq \lambda_j \right\}.$$

Note that the equality $K_1 \amalg \cdots \amalg K_q = \{1, \ldots, d\}$ holds. We put $\kappa_w := \#(K_w)$ for $1 \le w \le q$ and denote by g_w the greatest common divisor of $\kappa_1, \ldots, \kappa_{w-1}, \kappa_w - 1, \kappa_{w+1}, \ldots, \kappa_q$ for each $1 \le w \le q$.

- We define the function m by $m(z) := \frac{1}{1-z}$ for $z \in \mathbb{C} \setminus \{1\}$.
- We may assume λ to be in the form

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\kappa_1}, \ldots, \underbrace{\lambda_q, \ldots, \lambda_q}_{\kappa_q}),$$

where $\lambda_1, \ldots, \lambda_q$ are mutually distinct. For each $1 \le w \le q$ and for each divisor t of g_w with $t \ge 2$, we put $d[t] := \frac{d-1}{t} + 1$ and denote by $\lambda[t]$ the element of $V_{d[t]}$ such that

$$\begin{aligned}\lambda[t] := (&\underbrace{m^{-1}(tm(\lambda_1)), \ldots, m^{-1}(tm(\lambda_1))}_{\frac{\kappa_1}{t}}, \ldots, \\ &\underbrace{m^{-1}(tm(\lambda_w)), \ldots, m^{-1}(tm(\lambda_w))}_{\frac{(\kappa_w)-1}{t}}, \ldots, \underbrace{m^{-1}(tm(\lambda_q)), \ldots, m^{-1}(tm(\lambda_q))}_{\frac{\kappa_q}{t}}, \lambda_w).\end{aligned}$$

Note that $\mathcal{I}(\lambda[t])$ is determined by $\mathcal{I}(\lambda), \mathcal{K}(\lambda)$ and t.

Main Theorem 3. *Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be an element of V_d. Then the cardinality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ is computed in the following steps.*

- *For each $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, we define the number $e_{\mathbb{I}}(\lambda)$ by the equality*

$$e_{\mathbb{I}}(\lambda) := \left(\prod_{u=1}^{l} (\#(I_u) - 1)!\right) - \sum_{\substack{\mathbb{I}' \in \mathfrak{I}(\lambda) \\ \mathbb{I}' \succ \mathbb{I},\ \mathbb{I}' \ne \mathbb{I}}} \left(e_{\mathbb{I}'}(\lambda) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k\right)\right),$$

where we put $\chi_u(\mathbb{I}') := \#\left(\left\{I' \in \mathbb{I}' \,\middle|\, I' \subseteq I_u\right\}\right)$ for $\mathbb{I}' \succ \mathbb{I}$.

- *We put*

$$s_d(\lambda) := (d-2)! - \sum_{\mathbb{I} \in \mathfrak{I}(\lambda)} \left(e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#(\mathbb{I})+1}^{d-2} k\right).$$

- *Moreover we define the numbers $c_t(\lambda)$ for $t \in \bigcup_{1 \le w \le q} \left\{t \,\middle|\, t | g_w\right\}$ by the equalities*

$$\sum_{t|b,\ b|g_w} \frac{t}{b} c_b(\lambda) = \frac{s_{d[t]}(\lambda[t])}{\left(\frac{\kappa_1}{t}\right)! \cdots \left(\frac{\kappa_{(w-1)}}{t}\right)! \left(\frac{(\kappa_w)-1}{t}\right)! \left(\frac{\kappa_{(w+1)}}{t}\right)! \cdots \left(\frac{\kappa_q}{t}\right)!} \tag{1}$$

for $(w, t) \in \left\{(w, t) \,\middle|\, 1 \le w \le q,\ t | g_w,\ t \ge 2\right\}$, and

$$c_1(\lambda) + \sum_{w=1}^{q} \left(\sum_{t|g_w,\ t \ge 2} \frac{1}{t} c_t(\lambda)\right) = \frac{s_d(\lambda)}{\kappa_1! \cdots \kappa_q!}, \tag{2}$$

where $t|b$ denotes that t divides b for positive integers t and b.

- *Then the numbers $e_{\mathbb{I}}(\lambda)$, $s_d(\lambda)$ and $c_t(\lambda)$ are non-negative integers. Moreover we have*

$$\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = \sum_t c_t(\lambda) = c_1(\lambda) + \sum_{w=1}^{q} \left(\sum_{t|g_w,\ t \ge 2} c_t(\lambda)\right). \tag{3}$$

Remark 2.2. Verify that all the numbers defined in Main Theorem 3 are determined by the combinatorial data $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$. Especially the number $s_d(\lambda)$ is determined only by $\mathcal{I}(\lambda)$ and corresponds with the cardinality of $S_d(\lambda)$ which will be defined in Definition 3.2.

Problem . *Show in combinatorics that for any $\lambda \in V_d$ and for any t, the number $c_t(\lambda)$ defined above is a non-negative integer. In this paper, its proof is not combinatorial.*

We shall give a remark and a conjecture concerning parameters $\lambda \in \Lambda_d \setminus V_d$.

Remark 2.3. For $\lambda = (\lambda_1, \dots, \lambda_d) \in \Lambda_d \setminus V_d$ with $\#\left\{i \mid \lambda_i = 1\right\} \geq 4$, the inverse image $\Phi_d^{-1}\left(\bar{\lambda}\right)$ may have dimension greater than 1. However, if we put

$$\widetilde{P}''_d := \left\{f \in \widetilde{P}_d \mid f \text{ has at most one multiple fixed point}\right\},$$

then the map $\Phi_d|_{\widetilde{P}''_d} : \widetilde{P}''_d \to \widetilde{\Lambda}_d$ is finite. Moreover the similar results as in the main theorems hold for $\Phi_d|_{\widetilde{P}''_d}$ and for any $\lambda \in \Lambda_d \setminus V_d$, whose proofs are also similar to the proofs of the main theorems; however we shall omit them since it is not interesting to repeat similar discussions.

Conjecture 2. *For any $\zeta \in \mathrm{Fix}(f)$, the holomorphic index of f at ζ is defined to be the complex number $\iota(f, \zeta) := \frac{1}{2\pi\sqrt{-1}} \oint_{|z-\zeta|=\epsilon} \frac{dz}{z - f(z)}$, where ϵ is a sufficiently small positive real number. The index $\iota(f, \zeta)$ is invariant under biholomorphic transformations, and is equal to $\frac{1}{1-f'(\zeta)}$ if ζ is not multiple. We denote by $m(f, \zeta)$ the fixed-point multiplicity of f at $\zeta \in \mathrm{Fix}(f)$. Then we always have $\sum_{\zeta \in \mathrm{Fix}(f)} m(f, \zeta) = \deg f$ and $\sum_{\zeta \in \mathrm{Fix}(f)} \iota(f, \zeta) = 0$. Moreover we have $\iota(f, \zeta) \neq 0$ whenever $m(f, \zeta) = 1$. We define such parameter space and consider the map $\widetilde{\Phi}_d$, in stead of Φ_d, which assigns $f \in \widetilde{P}_d$ to $\widetilde{\Phi}_d(f) = \left([\iota(f, \zeta), m(f, \zeta)]\right)_{\zeta \in \mathrm{Fix}(f)}$. Then it is conjectured that the map $\widetilde{\Phi}_d$ is finite and that the similar results as in the main theorems hold for $\widetilde{\Phi}_d$ and for any parameter value without exception. Note that $\mathrm{Fix}(f)$ is not considered counted with multiplicity only in this conjecture.*

3. Another expression of the set $\Phi_d^{-1}\left(\bar{\lambda}\right)$

In the rest of this paper, we shall always assume that d is a natural number with $d \geq 4$.

An arbitrary polynomial map $f(z) \in \mathbb{C}[z]$ of degree d can be expressed in the form $f(z) = z + \rho(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_d)$, where $\zeta_1, \zeta_2, \dots, \zeta_d$ and ρ are complex numbers with $\rho \neq 0$. In this expression we have $\mathrm{Fix}(f) = \{\zeta_1, \zeta_2, \dots, \zeta_d\}$ and $f'(\zeta_i) = 1 + \rho \prod_{j \neq i}(\zeta_i - \zeta_j)$ for $1 \leq i \leq d$. Hence to show Main Theorems 1 and 3, we only need to count the number of the solutions of the equations $1 + \rho \prod_{j \neq i}(\zeta_i - \zeta_j) = \lambda_i$ for $1 \leq i \leq d$ modulo affine conjugacy. However we do not take this method; we shall prefer to express the equations above in another way. The following lemma is one of the most important key for the proof of the main theorems.

Key Lemma . *Let f be a polynomial map of degree d expressed in the form*

$$f(z) = z + \rho(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_d),$$

where $\zeta_1, \dots, \zeta_d$ and ρ are complex numbers with $\rho \neq 0$. Then for $\lambda = (\lambda_1, \dots, \lambda_d) \in V_d$, the equalities $f'(\zeta_i) = \lambda_i$ hold for $1 \leq i \leq d$ if and only if the equalities

$$\sum_{i=1}^{d} \frac{1}{1 - \lambda_i} \zeta_i^k = \begin{cases} 0 & (1 \leq k \leq d-2) \\ -\frac{1}{\rho} & (k = d-1) \end{cases} \tag{4}$$

hold and $\zeta_1, \dots, \zeta_d$ are mutually distinct.

Proof. Considering the integration $\frac{1}{2\pi\sqrt{-1}}\oint_{|z|=R}\frac{z^k}{z-f(z)}\,dz$ for sufficiently large real number R, we obtain the equalities

$$\sum_{i=1}^{d}\frac{1}{1-f'(\zeta_i)}\,\zeta_i^k=\begin{cases}0 & (0\le k\le d-2)\\ -\frac{1}{\rho} & (k=d-1)\end{cases}\tag{5}$$

in the case that $\zeta_1,\dots,\zeta_d$ are mutually distinct. Since $\lambda_i\neq 1$ for any $1\le i\le d$, the equalities $f'(\zeta_i)=\lambda_i$ for $1\le i\le d$ imply the mutual distinctness of $\zeta_1,\dots,\zeta_d$, which also implies the equalities (5). Thus the necessary condition of the lemma is verified.

On the other hand, suppose the equalities (4) and the mutual distinctness of $\zeta_1,\dots,\zeta_d$. Then the equalities (4) are expressed in the form

$$\begin{pmatrix}1 & 1 & \cdots & 1\\ \zeta_1 & \zeta_2 & \cdots & \zeta_d\\ \zeta_1^2 & \zeta_2^2 & \cdots & \zeta_d^2\\ \vdots & \vdots & \ddots & \vdots\\ \zeta_1^{d-1} & \zeta_2^{d-1} & \cdots & \zeta_d^{d-1}\end{pmatrix}\begin{pmatrix}\frac{1}{1-\lambda_1}\\ \frac{1}{1-\lambda_2}\\ \vdots\\ \frac{1}{1-\lambda_d}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\\ -\frac{1}{\rho}\end{pmatrix}.\tag{6}$$

Moreover the mutual distinctness of $\zeta_1,\dots,\zeta_d$ implies the equalities (5), which are equivalent to the equality obtained from the equality (6) by replacing λ_i by $f'(\zeta_i)$ for each $1\le i\le d$. Therefore since the square matrix in the left hand side of the equality (6) is invertible, we have $\frac{1}{1-f'(\zeta_i)}=\frac{1}{1-\lambda_i}$ for $1\le i\le d$, which completes the proof of Key Lemma. □

On the basis of Key Lemma, we are able to correspond the set $\Phi_d^{-1}\left(\bar{\lambda}\right)$ for $\lambda\in V_d$ to some another one whose cardinality is expected to be easier to count. Recall that $\mathbb{P}^{d-2}$ denotes the complex projective space of dimension $d-2$.

Definition 3.2. For any $\lambda=(\lambda_1,\dots,\lambda_d)\in V_d$, we put

$$\begin{aligned}T_d(\lambda)&:=\left\{(\zeta_1:\cdots:\zeta_{d-1})\in\mathbb{P}^{d-2}\;\middle|\;\sum_{i=1}^{d-1}\frac{1}{1-\lambda_i}\,\zeta_i^k=0\ \text{ for } 1\le k\le d-2\right\},\\ S_d(\lambda)&:=\left\{(\zeta_1:\cdots:\zeta_{d-1})\in T_d(\lambda)\;\middle|\;\zeta_1,\dots,\zeta_{d-1}\text{ and } 0 \text{ are mutually distinct}\right\},\\ B_d(\lambda)&:=T_d(\lambda)\setminus S_d(\lambda)\quad\text{and}\\ \mathfrak{S}\left(\mathcal{K}(\lambda)\right)&:=\left\{\sigma\in\mathfrak{S}_d\;\middle|\;\lambda_{\sigma(i)}=\lambda_i\text{ holds for any } i.\right\}.\end{aligned}$$

Note that $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ is a subgroup of $\mathfrak{S}_d$ determined by $\mathcal{K}(\lambda)$ and is isomorphic to the group $\mathfrak{S}_{\kappa_1}\times\cdots\times\mathfrak{S}_{\kappa_q}$, where $\kappa_1,\dots,\kappa_q$ and $K_1,\dots,K_q$ are those defined in Definition 2.1. The above isomorphism is obtained in the following manner: for each $1\le w\le q$, we can identify $\mathfrak{S}_{\kappa_w}$ with the subgroup of $\mathfrak{S}_d$ consisting the permutations fixing the indices $\{1,\dots,d\}\setminus K_w$. Under this identification, their product $\mathfrak{S}_{\kappa_1}\times\cdots\times\mathfrak{S}_{\kappa_q}$ is also identified with the subgroup of $\mathfrak{S}_d$, which is exactly the subgroup $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$. In the following, we always adopt these identifications.

Proposition 3.3. *Let $\lambda=(\lambda_1,\dots,\lambda_d)$ be an element of V_d. Then we can define the map $\pi(\lambda):S_d(\lambda)\to\Phi_d^{-1}\left(\bar{\lambda}\right)$ by*

$$(\zeta_1:\cdots:\zeta_{d-1})\mapsto f(z)=z+\rho z(z-\zeta_1)\cdots(z-\zeta_{d-1}),$$

where the complex number ρ is determined by the equality $-\frac{1}{\rho} = \sum_{i=1}^{d-1} \frac{1}{1-\lambda_i} \zeta_i^{d-1}$. The map $\pi(\lambda)$ is surjective. The group $\mathfrak{S}(\mathcal{K}(\lambda))$ acts on $S_d(\lambda)$ by the permutation of the coordinates $\zeta_1, \dots, \zeta_{d-1}$ and 0: more precisely, it is defined by

$$\sigma \cdot (\zeta_1 : \cdots : \zeta_{d-1}) := (\zeta_{\sigma^{-1}(1)} - \zeta_{\sigma^{-1}(d)} : \cdots : \zeta_{\sigma^{-1}(d-1)} - \zeta_{\sigma^{-1}(d)})$$

for $\sigma \in \mathfrak{S}_d$ and $(\zeta_1 : \cdots : \zeta_{d-1}) \in S_d(\lambda)$, where we assume $\zeta_d = 0$. Furthermore the map $\pi(\lambda) : S_d(\lambda) \to \Phi_d^{-1}(\bar{\lambda})$ induces the bijection

$$\overline{\pi(\lambda)} : S_d(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda)) \xrightarrow{\cong} \Phi_d^{-1}(\bar{\lambda}).$$

To prove Proposition 3.3, we shall prepare the auxiliary definitions, lemma and proposition.

Definition 3.4. We put

$$Q_d(\lambda) := \left\{ (\zeta_1, \dots, \zeta_d) \in \mathbb{C}^d \;\middle|\; \begin{array}{c} \sum_{i=1}^{d} \frac{1}{1-\lambda_i} \zeta_i^k = 0 \quad \text{for } 1 \le k \le d-2 \\ \zeta_1, \dots, \zeta_d \text{ are mutually distinct} \end{array} \right\},$$

and denote by G the projection map $G : P_d \to \widetilde{P}_d = P_d/\Gamma$.

The groups Γ and $\mathfrak{S}_d$ naturally act on $\mathbb{C}^d$; especially, the subgroup $\mathfrak{S}(\mathcal{K}(\lambda))$ of $\mathfrak{S}_d$ also acts on $\mathbb{C}^d$. The actions of Γ and $\mathfrak{S}_d$ on $\mathbb{C}^d$ commute.

Lemma 3.5. *Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be an element of V_d. Then*

(1) *we can define the map $\varpi(\lambda) : Q_d(\lambda) \to G^{-1} \circ \Phi_d^{-1}(\bar{\lambda})$ by*
$$(\zeta_1, \dots, \zeta_d) \mapsto f(z) := z + \rho(z - \zeta_1) \cdots (z - \zeta_d),$$
where ρ is determined by the equality $-\frac{1}{\rho} = \sum_{i=1}^{d} \frac{1}{1-\lambda_i} \zeta_i^{d-1}$.

(2) *The map $\varpi(\lambda)$ is surjective.*

(3) *The set $Q_d(\lambda)$ is invariant under the action of Γ on $\mathbb{C}^d$.*

(4) *The actions of Γ on $Q_d(\lambda)$ and on $G^{-1} \circ \Phi_d^{-1}(\bar{\lambda})$ commute with the map $\varpi(\lambda)$, i.e., the equality $\varpi(\lambda)(\gamma \cdot \zeta) = \gamma \circ \varpi(\lambda)(\zeta) \circ \gamma^{-1}$ holds for any $\zeta \in Q_d(\lambda)$ and $\gamma \in \Gamma$.*

(5) *The set $Q_d(\lambda)$ is invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $\mathbb{C}^d$.*

(6) *For $\zeta, \zeta' \in Q_d(\lambda)$, the equality $\varpi(\lambda)(\zeta) = \varpi(\lambda)(\zeta')$ holds if and only if the equality $\zeta' = \sigma \cdot \zeta$ holds for some $\sigma \in \mathfrak{S}(\mathcal{K}(\lambda))$.*

Proof. We shall check the existence of the complex number ρ and the necessary condition of the assertion (6); the rests are obvious by Key Lemma.

If we cannot determine $\rho \in \mathbb{C}^*$, then we have $\sum_{i=1}^{d} \frac{1}{1-\lambda_i} \zeta_i^{d-1} = 0$, which implies $\frac{1}{1-\lambda_i} = 0$ for $1 \le i \le d$ by the equality (6); hence the contradiction assures the existence of ρ.

Let $\zeta = (\zeta_1, \dots, \zeta_d), \zeta' = (\zeta_1', \dots, \zeta_d')$ be elements of $Q_d(\lambda)$ with $\varpi(\lambda)(\zeta) = \varpi(\lambda)(\zeta') =: f$. Then by the definition of $\varpi(\lambda)$, there exists a permutation $\sigma \in \mathfrak{S}_d$ with $\zeta' = \sigma \cdot \zeta$. On the other hand, by Key Lemma, we have $f'(\zeta_i) = f'(\zeta_i') = \lambda_i$ for $1 \le i \le d$. Since $\zeta_i' = \zeta_{\sigma^{-1}(i)}$ for $1 \le i \le d$, we have $\lambda_i = \lambda_{\sigma(i)}$ for $1 \le i \le d$, which implies $\sigma \in \mathfrak{S}(\mathcal{K}(\lambda))$. Thus the necessary condition of the assertion (6) is verified. □

Definition 3.6. We put $\widetilde{Q}_d(\lambda) := Q_d(\lambda)/\Gamma$.

Proposition 3.7. *Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be an element of V_d. Then the map $\varpi(\lambda)$ in Lemma 3.5 induces the surjective map $\widetilde{\varpi}(\lambda) : \widetilde{Q}_d(\lambda) \to \Phi_d^{-1}(\bar{\lambda})$. Moreover the group $\mathfrak{S}(\mathcal{K}(\lambda))$ acts on $\widetilde{Q}_d(\lambda)$, which induces the bijection*

$$\overline{\varpi(\lambda)} : \widetilde{Q}_d(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda)) \to \Phi_d^{-1}(\bar{\lambda}).$$

Furthermore $\widetilde{Q}_d(\lambda)$ is canonically identified with $S_d(\lambda)$ by the bijection $\iota(\lambda) : S_d(\lambda) \to \widetilde{Q}_d(\lambda)$ which corresponds $(\zeta_1 : \cdots : \zeta_{d-1}) \in S_d(\lambda)$ to the equivalence class of $(\zeta_1, \ldots, \zeta_{d-1}, 0)$ in $\widetilde{Q}_d(\lambda)$. Under this identification, we have $\widetilde{\varpi}(\lambda) \circ \iota(\lambda) = \pi(\lambda)$. Moreover the actions of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$ and on $\widetilde{Q}_d(\lambda)$ commute with the map $\iota(\lambda)$.

Proof of Propositions 3.7 and 3.3. Proposition 3.7 is a direct consequence of Lemma 3.5, whereas Proposition 3.3 is just a corollary of Proposition 3.7. □

Since $\iota(\lambda) : S_d(\lambda) \cong \widetilde{Q}_d(\lambda)$, we can use both of them in the proof; in the process of finding the cardinality $\#(S_d(\lambda))$ we shall use $S_d(\lambda)$; on the other hand, we shall use $\widetilde{Q}_d(\lambda)$ in the process of analyzing the action of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$.

Proposition 3.8. *The assertion (5) in Main Theorem 1 holds.*

Proof. Since the map $G \circ \varpi(\lambda) : Q_d(\lambda) \to \Phi_d^{-1}(\bar{\lambda})$ is surjective, it suffices to show that the set $Q_d(\lambda)$ is empty. We may assume that the integers $c_1, \ldots, c_j$ are positive and that the integers $c_{j+1}, \ldots, c_d$ are negative. Then the defining equations $\sum_{i=1}^{d} \frac{1}{1-\lambda_i} \zeta_i^k = 0$ for $1 \le k \le d-2$ are equivalent to the equations

$$\underbrace{\zeta_1^k + \cdots + \zeta_1^k}_{c_1} + \cdots + \underbrace{\zeta_j^k + \cdots + \zeta_j^k}_{c_j} = \underbrace{\zeta_{j+1}^k + \cdots + \zeta_{j+1}^k}_{-c_{j+1}} + \cdots + \underbrace{\zeta_d^k + \cdots + \zeta_d^k}_{-c_d}$$

for $1 \le k \le d-2$. Hence the k-th fundamental symmetric expressions of

$$\underbrace{\zeta_1, \ldots, \zeta_1}_{c_1}, \ldots, \underbrace{\zeta_j, \ldots, \zeta_j}_{c_j} \quad \text{and} \quad \underbrace{\zeta_{j+1}, \ldots, \zeta_{j+1}}_{-c_{j+1}}, \ldots, \underbrace{\zeta_d, \ldots, \zeta_d}_{-c_d} \tag{7}$$

coincide for $1 \le k \le d-2$. On the other hand, the condition $\sum_{i=1}^{d} |c_i| \le 2(d-2)$ is equivalent to the condition $\sum_{i=1}^{j} c_i = \sum_{i=j+1}^{d} -c_i \le d-2$. Therefore the left half of (7) should be some permutation of the right half of (7), which contradicts the mutual distinctness of $\zeta_1, \ldots, \zeta_d$. Thus the set $Q_d(\lambda)$ is empty, which completes the proof of the proposition. □

4. Review of the intersection theory on $\mathbb{P}^n$

In this section we shall summarize the fact of the intersection theory on $\mathbb{P}^n$, and shall state Proposition 4.2, which is an extension of Bezout's theorem.

First we shall verify the definition of the degree $\deg C$ of an algebraic variety C in $\mathbb{P}^n$ with $\dim C = k$. Generic $(n-k)$-plane $\mathbb{P}^{n-k} \subset \mathbb{P}^n$ intersects C transversely; we may thus define the degree of C to be the number of intersection points of C with a generic linear subspace $\mathbb{P}^{n-k}$, which does not depend on the choice of $\mathbb{P}^{n-k}$. For example, for any homogeneous polynomial $\varphi(\zeta)$ of degree d on $\mathbb{P}^n$, the degree of the zeros of φ is always d.

Secondly we shall remember the definition of the intersection multiplicity $\mathrm{mult}_{C_\mu}(C, C')$ of varieties C and C' in $\mathbb{P}^n$ along an irreducible component C_μ of $C \cap C'$ with $\dim C_\mu = \dim C + \dim C' - n$. If C_μ is a point, then the intersection multiplicity is defined as follows: in a local coordinate having the origin as C_μ, C meets $C' + \epsilon$ transversely around the origin for generic small $\epsilon \in \mathbb{C}^n$, where $C' + \epsilon$ denotes the translation of C' by ϵ with respect to the given local coordinate; we may thus define the intersection multiplicity $\mathrm{mult}_{C_\mu}(C, C')$ to be the number of intersection points of C and $C' + \epsilon$ around the origin for sufficiently small generic ϵ, which does not depend on the choice of ϵ nor a local coordinate. In the general case with $\dim C_\mu = \dim C + \dim C' - n$, the intersection multiplicity $\mathrm{mult}_{C_\mu}(C, C')$ is defined to be the intersection multiplicity $\mathrm{mult}_p(C \cap H, C' \cap H)$ on H, where p is a generic smooth point of C_μ and H is a submanifold in a neighborhood of p intersecting C_μ transversely at p and with complementary dimension of C_μ. If C' is the zeros of a homogeneous polynomial φ, then we also denote $\mathrm{mult}_{C_\mu}(C, C')$ by $\mathrm{mult}_{C_\mu}(C, \varphi)$.

Next we shall state the relation among the numbers defined above. Let C, C' be algebraic varieties in $\mathbb{P}^n$ with $\dim C = k$ and $\dim C' = k'$, and $C_1, \dots, C_r$ the irreducible components of $C \cap C'$. Moreover suppose that the equality $\dim C_\mu = \dim C + \dim C' - n$ holds for any μ. Then the topological intersection of C and C' is given by $(C \cdot C') = \sum_{\mu=1}^r \mathrm{mult}_{C_\mu}(C, C') \cdot C_\mu$, which implies the equality

$$\deg C \cdot \deg C' = \sum_{\mu=1}^{r} \mathrm{mult}_{C_\mu}(C, C') \cdot \deg C_\mu. \tag{8}$$

On the basis of those mentioned above, we shall state Definition 4.1 and Proposition 4.2.

Definition 4.1. We define the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ for homogeneous polynomials $\varphi_1, \dots, \varphi_m$ on $\mathbb{P}^n$ and an irreducible variety C in $\mathbb{P}^n$ with $\operatorname{codim} C = m$ as follows: if C is not contained in the common zeros of $\varphi_1, \dots, \varphi_m$, then we put $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m) = 0$. Otherwise, we define $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ by the induction of m in the following manner: in the case $m = 1$, the number $\mathrm{mult}_C(\varphi_1)$ is defined to be the usual order of zeros of φ_1 along C; if $m \geq 2$, the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ is defined inductively by the equality

$$\mathrm{mult}_C(\varphi_1, \dots, \varphi_m) = \sum_{\mu=1}^{r} \mathrm{mult}_{C_\mu}(\varphi_1, \dots, \varphi_{m-1}) \cdot \mathrm{mult}_C(C_\mu, \varphi_m), \tag{9}$$

where $C_1, \dots, C_r$ are the components of the common zeros of $\varphi_1, \dots, \varphi_{m-1}$ with codimension $(m-1)$ containing C and not contained in the zeros of φ_m.

Definition 4.1 is valid for an embedded component C of the common zeros of $\varphi_1, \dots, \varphi_m$; if C is an irreducible component, then the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ defined above is equal to the usual intersection multiplicity of $\varphi_1, \dots, \varphi_m$ along C. Note that a component C is said to be irreducible or embedded according as C is maximal or not with respect to the inclusion relation among the family of the components of the common zeros.

Proposition 4.2. *Let $\varphi_1, \dots, \varphi_n$ be homogeneous polynomials on $\mathbb{P}^n$, and $\{C_\mu \mid \mu \in M\}$ the family of the components of the common zeros of $\varphi_1, \dots, \varphi_n$ in $\mathbb{P}^n$. For each $\mu \in M$, we put $\dim C_\mu = l_\mu$. Moreover suppose that for any $1 \leq k \leq n$, if a component C of the common zeros of $\varphi_1, \dots, \varphi_k$ have dimension strictly greater than $n - k$, then C belongs to the family $\{C_\mu \mid \mu \in M\}$. Then C_μ is a component of the common zeros of $\varphi_1, \dots, \varphi_{n-l_\mu}$ for any $\mu \in M$. Moreover we have the equality*

$$\prod_{k=1}^{n} \deg \varphi_k = \sum_{\mu \in M} \left(\deg C_\mu \cdot \mathrm{mult}_{C_\mu}(\varphi_1, \dots, \varphi_{n-l_\mu}) \cdot \prod_{k=n-l_\mu+1}^{n} \deg \varphi_k \right). \tag{10}$$

Proof. First we shall fix our notation. For each $1 \leq l \leq n$, we put $\{C_\mu \mid \mu \in M,\ \dim C_\mu = n - l\} =: \{C_{l,m} \mid 1 \leq m \leq r_l\}$, and denote by $C'_{l,1}, \dots, C'_{l,r'_l}$ the components of the common zeros of $\varphi_1, \dots, \varphi_l$ which do not belong to the family $\{C_\mu \mid \mu \in M\}$. Then by assumption we always have $\dim C'_{l,m} = n - l$ for any l and m. We can also verify that $C_{l,m}$ is a component of the common zeros of $\varphi_1, \dots, \varphi_l$ for any l and m. Moreover we have $r'_n = 0$ by definition.

In the following, we shall show the equality

$$(11)_k \quad \prod_{l=1}^{k} \deg \varphi_l = \sum_{l=1}^{k} \sum_{m=1}^{r_l} \deg C_{l,m} \cdot \mathrm{mult}_{C_{l,m}}(\varphi_1, \ldots, \varphi_l) \cdot \prod_{k'=l+1}^{k} \deg \varphi_{k'} + \sum_{m=1}^{r'_k} \deg C'_{k,m} \cdot \mathrm{mult}_{C'_{k,m}}(\varphi_1, \ldots, \varphi_k)$$

by the induction of k, which will complete the proof since the equality $(11)_n$ is the same as the equality (10). The equality $(11)_1$ is obvious. Multiplying both sides of the equality $(11)_k$ by $\deg \varphi_{k+1}$ and applying the equalities (8) and (9), we obtain the equality $(11)_{k+1}$; thus the proposition is proved. □

Proposition 4.2 is reduced to Bezout's theorem if all the components are points. Proposition 4.2 will be utilized crucially for finding the cardinality $\#(S_d(\lambda))$ in Section 5.

Remark 4.3. The number $\mathrm{mult}_C(\varphi_1, \ldots, \varphi_m)$ may vary if the order of $\varphi_1, \ldots, \varphi_m$ changes. Hence Definition 4.1 may appear to be a little strange in some sense; however this works very well for the computation of the cardinality $\#(S_d(\lambda))$. In the following, we shall give an example in which the number $\mathrm{mult}_{\mathrm{P}_2}(\varphi_1, \varphi_2)$ differs from $\mathrm{mult}_{\mathrm{P}_2}(\varphi_2, \varphi_1)$. Consider $\varphi_1 = x(x-y)$ and $\varphi_2 = x\left(xz - y^2\right)$ on $\mathbb{P}^2 = \{(x : y : z)\}$. We put $\mathrm{P}_1 = \{(1 : 1 : 1)\}$, $\mathrm{P}_2 = \{(0 : 0 : 1)\}$, $C_0 = \{x = 0\}$, $C_1 = \{x = y\}$ and $C_2 = \{xz = y^2\}$. Then the components of the common zeros of φ_1 and φ_2 are C_0, P_1 and P_2; C_0 and P_1 are irreducible, and P_2 is embedded. By Definition 4.1 we have

$$\begin{aligned}
\mathrm{mult}_{\mathrm{P}_2}(\varphi_1, \varphi_2) &= \mathrm{mult}_{C_1}(\varphi_1) \cdot \mathrm{mult}_{\mathrm{P}_2}(C_1, \varphi_2) = 1 \cdot 2 = 2, \\
\mathrm{mult}_{\mathrm{P}_2}(\varphi_2, \varphi_1) &= \mathrm{mult}_{C_2}(\varphi_2) \cdot \mathrm{mult}_{\mathrm{P}_2}(C_2, \varphi_1) = 1 \cdot 3 = 3.
\end{aligned}$$

However Proposition 4.2 holds in any order; we have

$$\begin{aligned}
&\deg C_0 \cdot \mathrm{mult}_{C_0}(\varphi_1) \cdot \deg \varphi_2 + \deg \mathrm{P}_1 \cdot \mathrm{mult}_{\mathrm{P}_1}(\varphi_1, \varphi_2) + \deg \mathrm{P}_2 \cdot \mathrm{mult}_{\mathrm{P}_2}(\varphi_1, \varphi_2) \\
&\qquad = 1 \cdot 1 \cdot 3 + 1 \cdot 1 + 1 \cdot 2 = 6 = \deg \varphi_1 \cdot \deg \varphi_2, \\
&\deg C_0 \cdot \mathrm{mult}_{C_0}(\varphi_2) \cdot \deg \varphi_1 + \deg \mathrm{P}_1 \cdot \mathrm{mult}_{\mathrm{P}_1}(\varphi_2, \varphi_1) + \deg \mathrm{P}_2 \cdot \mathrm{mult}_{\mathrm{P}_2}(\varphi_2, \varphi_1) \\
&\qquad = 1 \cdot 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 3 = 6 = \deg \varphi_2 \cdot \deg \varphi_1.
\end{aligned}$$

5. Outline of finding the cardinality $\#(S_d(\lambda))$

In this section we shall give an outline of finding the cardinality of the set $S_d(\lambda)$ defined in Definition 3.2 for each $\lambda \in V_d$. The proofs of the assertions (1), (4) and (6) in Main Theorem 1 are also given in this section.

For the brevity of notation we put

$$m_i := \frac{1}{1 - \lambda_i} \quad \text{and} \quad \varphi_k(\zeta) := \sum_{i=1}^{d-1} m_i \zeta_i^k$$

for each i and k, and we always assume that $\zeta_d = 0$. Therefore $T_d(\lambda)$ is the set of the common zeros of $\varphi_1, \ldots, \varphi_{d-2}$ in $\mathbb{P}^{d-2}$, and $S_d(\lambda)$ consists of an element $\zeta = (\zeta_1 : \cdots : \zeta_{d-1}) \in T_d(\lambda)$ with mutually distinct $\zeta_1, \ldots, \zeta_{d-1}$ and ζ_d. Moreover we may also consider that $\varphi_k(\zeta) = \sum_{i=1}^{d} m_i \zeta_i^k$.

Lemma 5.1. *Let λ be an element of V_d. Then $S_d(\lambda)$ is discrete in $\mathbb{P}^{d-2}$. Moreover we always have* $\mathrm{mult}_{\zeta_0}(\varphi_1,\dots,\varphi_{d-2}) = 1$ *for any $\zeta_0 \in S_d(\lambda)$.*

Proof. We consider the row vectors

$$\frac{\partial\varphi_k}{\partial\zeta} = \left(\frac{\partial\varphi_k}{\partial\zeta_1},\dots,\frac{\partial\varphi_k}{\partial\zeta_{d-1}}\right) = \left(km_1\zeta_1^{k-1},\dots,km_{d-1}\zeta_{d-1}^{k-1}\right)$$

at $\zeta = \zeta_0 \in S_d(\lambda)$ for $1 \le k \le d-1$. Since $\zeta_1,\dots,\zeta_{d-1}$ are mutually distinct at $\zeta = \zeta_0$ and since $m_i \neq 0$ for any i, the determinant

$$\det {}^t\left({}^t\left(\frac{\partial\varphi_1}{\partial\zeta}\right),\dots,{}^t\left(\frac{\partial\varphi_{d-1}}{\partial\zeta}\right)\right) = (d-1)!\cdot\prod_{i=1}^{d-1} m_i \cdot \det\begin{pmatrix} 1 & \cdots & 1 \\ \zeta_1 & \cdots & \zeta_{d-1} \\ \vdots & \ddots & \vdots \\ \zeta_1^{d-2} & \cdots & \zeta_{d-1}^{d-2}\end{pmatrix}$$

is not equal to zero. Therefore the row vectors $\frac{\partial\varphi_1}{\partial\zeta},\dots,\frac{\partial\varphi_{d-2}}{\partial\zeta}$ are linearly independent at $\zeta = \zeta_0$, which proves the lemma. □

Proposition 5.2. *The assertion (1) in Main Theorem 1 holds*

Proof. Since $S_d(\lambda)$ is discrete and since $\deg\varphi_k = k$ for each k, we always have the inequality $\#\left(S_d(\lambda)\right) \le (d-2)!$ by Proposition 4.2. Hence the surjectivity of the map $\pi(\lambda) : S_d(\lambda) \to \Phi_d^{-1}\left(\bar\lambda\right)$ verifies the proposition. □

Proposition 4.2 and Lemma 5.1 imply that in order to find the cardinality $\#\left(S_d(\lambda)\right)$, we only need to find the degree $\deg C$ and the number $\mathrm{mult}_C(\varphi_1,\dots,\varphi_{d-l})$ for each component C of the common zeros of $\varphi_1,\dots,\varphi_{d-2}$ with $\dim C = l-2$ contained in $B_d(\lambda)$. Before giving the explicit expression of the set $B_d(\lambda)$, we shall make a definition. Recall the definition of $\mathfrak{I}(\lambda)$ for $\lambda \in V_d$ defined in Definition 2.1.

Definition 5.3. Let λ be an element of V_d. For each $\mathbb{I} = \{I_1,\dots,I_l\} \in \mathfrak{I}(\lambda)$, we define the subset $E_d(\mathbb{I})$ of $\mathbb{P}^{d-2}$ by

$$E_d(\mathbb{I}) := \left\{ (\zeta_1 : \cdots : \zeta_{d-1}) \in \mathbb{P}^{d-2} \;\middle|\; \begin{array}{c}\text{If } i,j \in \{1,\dots,d\} \text{ belong to the same } I_u \\ \text{for some } u, \text{ then we have } \zeta_i = \zeta_j.\end{array} \right\}.$$

In the definition of $E_d(\mathbb{I})$, note that we always assume $\zeta_d = 0$. By definition, the relation $\mathbb{I} \prec \mathbb{I}'$ holds for $\mathbb{I},\mathbb{I}' \in \mathfrak{I}(\lambda)$ if and only if the inclusion relation $E_d(\mathbb{I}) \subseteq E_d(\mathbb{I}')$ holds. Moreover if the cardinality of $\mathbb{I} \in \mathfrak{I}(\lambda)$ is l, then $E_d(\mathbb{I})$ is an $(l-2)$-dimensional complex plane in $\mathbb{P}^{d-2}$; hence the degree of $E_d(\mathbb{I})$ is always 1. To help the reader to understand the definition of $\mathfrak{I}(\lambda)$ and Definition 5.3, we shall give an example.

Example 1. Let λ be an element of V_6 such that the equality

$$m_1 : \cdots : m_6 = 1 : 1 : 2 : -1 : -1 : -2$$

holds. Then by definition, we have $\mathfrak{I}(\lambda) = \left\{\mathbb{I}_\omega \;\middle|\; 1 \le \omega \le 8\right\}$, where

$$\mathbb{I}_1 = \big\{\{1,4\},\{2,5\},\{3,6\}\big\},\ \mathbb{I}_2 = \big\{\{1,5\},\{2,4\},\{3,6\}\big\},\ \mathbb{I}_3 = \big\{\{1,4\},\{2,3,5,6\}\big\},$$
$$\mathbb{I}_4 = \big\{\{2,5\},\{1,3,4,6\}\big\},\ \mathbb{I}_5 = \big\{\{3,6\},\{1,2,4,5\}\big\},\ \mathbb{I}_6 = \big\{\{1,5\},\{2,3,4,6\}\big\},$$
$$\mathbb{I}_7 = \big\{\{2,4\},\{1,3,5,6\}\big\} \text{ and } \mathbb{I}_8 = \big\{\{1,2,6\},\{3,4,5\}\big\}.$$

We have $\mathbb{I}_3 \prec \mathbb{I}_1, \mathbb{I}_4 \prec \mathbb{I}_1, \mathbb{I}_5 \prec \mathbb{I}_1, \mathbb{I}_5 \prec \mathbb{I}_2, \mathbb{I}_6 \prec \mathbb{I}_2$ and $\mathbb{I}_7 \prec \mathbb{I}_2$; hence the maximal elements of $\mathfrak{I}(\lambda)$ are $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_8$. Moreover we have

$$\begin{aligned}
&E_6(\mathbb{I}_1) = \left\{(\zeta_1 : \zeta_2 : 0 : \zeta_1 : \zeta_2) \in \mathbb{P}^4 \;\middle|\; (\zeta_1 : \zeta_2) \in \mathbb{P}^1\right\},\\
&E_6(\mathbb{I}_2) = \left\{(\zeta_1 : \zeta_2 : 0 : \zeta_2 : \zeta_1) \in \mathbb{P}^4 \;\middle|\; (\zeta_1 : \zeta_2) \in \mathbb{P}^1\right\},\\
&E_6(\mathbb{I}_3) = \{(1:0:0:1:0)\},\ E_6(\mathbb{I}_4) = \{(0:1:0:0:1)\},\ E_6(\mathbb{I}_5) = \{(1:1:0:1:1)\},\\
&E_6(\mathbb{I}_6) = \{(1:0:0:0:1)\},\ E_6(\mathbb{I}_7) = \{(0:1:0:1:0)\} \text{ and } E_6(\mathbb{I}_8) = \{(0:0:1:1:1)\}.
\end{aligned}$$

$E_6(\mathbb{I}_1)$ and $E_6(\mathbb{I}_2)$ are complex lines in $\mathbb{P}^4$, whereas $E_6(\mathbb{I}_\omega)$ are points for $3 \le \omega \le 8$. We have $E_6(\mathbb{I}_\omega) \subset E_6(\mathbb{I}_1)$ for $\omega = 3, 4$ and 5, while $E_6(\mathbb{I}_\omega) \subset E_6(\mathbb{I}_2)$ for $\omega = 5, 6$ and 7.

Remark 5.4. Since we always have the equality $\sum_{i=1}^d m_i = 0$, we have

$$\mathfrak{I}(\lambda) = \left\{\mathbb{I} \subseteq \mathcal{I}(\lambda) \;\middle|\; \coprod_{I \in \mathbb{I}} I = \{1, \dots, d\}\right\} \quad \text{and} \quad \mathcal{I}(\lambda) = \bigcup_{\mathbb{I} \in \mathfrak{I}(\lambda)} \mathbb{I},$$

which means that the set $\mathfrak{I}(\lambda)$ gives the equivalent information as $\mathcal{I}(\lambda)$.

Now we are in a position to give the explicit expression of the set $B_d(\lambda)$.

Lemma 5.5. *Let λ be an element of V_d. Then we have the equality*

$$B_d(\lambda) = \bigcup_{\mathbb{I} \in \mathfrak{I}(\lambda)} E_d(\mathbb{I}). \tag{12}$$

More strictly, $B_d(\lambda)$ is a union of $E_d(\mathbb{I})$ only for maximal elements $\mathbb{I}$ of $\mathfrak{I}(\lambda)$ as set. However as we shall see later in Example 2, it is better to consider components $E_d(\mathbb{I})$ for $\mathbb{I}$ which are not necessarily maximal in $\mathfrak{I}(\lambda)$. Note that the equality (12) is only an equality as set.

Proof. For any point $\zeta_0 = (\zeta_1 : \cdots : \zeta_{d-1}) \in B_d(\lambda)$, we put

$$\mathbb{I}(\zeta_0) := \left\{ I \subsetneq \{1, 2, \dots, d\} \;\middle|\; \begin{array}{c} I \ne \emptyset. \quad \text{If } i, j \in I, \text{ then } \zeta_i = \zeta_j. \\ \text{If } i \in I \text{ and } j \in \{1, 2, \dots, d\} \setminus I, \text{ then } \zeta_i \ne \zeta_j. \end{array} \right\},$$

$\#(\mathbb{I}(\zeta_0)) =: l$, $\mathbb{I}(\zeta_0) =: \{I_1, \dots, I_l\}$ and $\alpha_u := \zeta_i$ for $i \in I_u$ for each $1 \le u \le l$. Then by definition, $\{1, 2, \dots, d\}$ is a disjoint union of $I_1, \dots, I_l$, and $\alpha_1, \dots, \alpha_l$ are mutually distinct, one of which is equal to zero since $\zeta_d = 0$ and $d \in I_u$ for some $1 \le u \le l$. Moreover since $\zeta_0 \in B_d(\lambda)$, we have $2 \le l \le d - 1$.

Under the notation above, the defining equations $\varphi_k(\zeta_0) = \sum_{u=1}^l \left(\sum_{i \in I_u} m_i\right) \alpha_u^k = 0$ for $1 \le k \le d - 2$ are equivalent to the equality

$$\begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_l \\ \vdots & \ddots & \vdots \\ \alpha_1^{d-2} & \cdots & \alpha_l^{d-2} \end{pmatrix} \begin{pmatrix} \sum_{i \in I_1} m_i \\ \vdots \\ \sum_{i \in I_l} m_i \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

which implies $\sum_{i \in I_u} m_i = 0$ for $1 \le u \le l$ by the inequality $l - 1 \le d - 2$. Therefore we have $\mathbb{I}(\zeta_0) \in \mathfrak{I}(\lambda)$ and $\zeta_0 \in E_d(\mathbb{I}(\zeta_0))$ for any $\zeta_0 \in B_d(\lambda)$, which assures the inclusion relation $B_d(\lambda) \subseteq \bigcup_{\mathbb{I} \in \mathfrak{I}(\lambda)} E_d(\mathbb{I})$. The opposite inclusion relation is clear, which completes the proof of the lemma. □

Proposition 5.6. *The assertion (4) in Main Theorem 1 holds.*

Proof. By Proposition 3.3, Lemma 5.1 and the intersection theory on $\mathbb{P}^{d-2}$, the equality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = (d-2)!$ holds if and only if the set $B_d(\lambda)$ is empty and that the action of $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $S_d(\lambda)$ is trivial. By Lemma 5.5, $B_d(\lambda)$ is empty if and only if $\mathcal{I}(\lambda)$ is empty. On the other hand, if $\lambda_i = \lambda_j$ holds for $i \neq j$, then the action of the permutation $(i,j) \in \mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $S_d(\lambda)$ is not trivial since $d \geq 4$. We have thus completed the proof of the proposition. □

In the rest of this section we shall introduce an example and some theorems that exactly give the number $\mathrm{mult}_C(\varphi_1, \ldots, \varphi_{d-l})$ for each component C of the common zeros of $\varphi_1, \ldots, \varphi_{d-2}$. However their proofs, which are the most crucial and difficult part in the proof of the main theorems, will be given later in Sections 6 and 7.

Theorem A. *Let λ be an element of V_d, and $\mathbb{I} = \{I_1, \ldots, I_l\}$ a maximal element of $\mathfrak{I}(\lambda)$. Then $E_d(\mathbb{I})$ is an irreducible component of the common zeros of $\varphi_1, \ldots, \varphi_{d-l}$ with its intersection multiplicity*

$$\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-l}) = \prod_{u=1}^{l} (\#(I_u) - 1)!.$$

We shall give one more example.

Example 2. We shall consider again $\lambda \in V_6$ with $m_1 : \cdots : m_6 = 1 : 1 : 2 : -1 : -1 : -2$ introduced in Example 1. The notation will follow that in Example 1. In this case, we have $\Phi_6^{-1}\left(\bar{\lambda}\right) = \emptyset$ by the assertion (5) in Main Theorem 1, which also implies $S_6(\lambda) = \emptyset$. Hence in this example, we shall verify $S_6(\lambda) = \emptyset$ by the calculation of intersection multiplicities.

By Example 1 and Lemma 5.5, we have $B_6(\lambda) = E_6(\mathbb{I}_1) \cup E_6(\mathbb{I}_2) \cup E_6(\mathbb{I}_8)$ as set. Moreover by Theorem A, we have $\mathrm{mult}_{E_6(\mathbb{I}_\omega)}(\varphi_1, \varphi_2, \varphi_3) = ((2-1)!)^3 = 1$ for $\omega = 1, 2$, and $\mathrm{mult}_{E_6(\mathbb{I}_8)}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = ((3-1)!)^2 = 4$. Hence the common zeros of φ_1, φ_2 and φ_3 are composed of $E_6(\mathbb{I}_1), E_6(\mathbb{I}_2)$ and some curve C whose degree is $\deg C = 3! - (1+1) = 4$. Moreover since $\deg C \cdot \deg \varphi_4 = 4 \cdot 4 = 16$, we have

$$\#\left(S_6(\lambda)\right) = 16 - \sum_{\zeta \in C \cap \{\varphi_4(\zeta) = 0\} \cap B_6(\lambda)} \mathrm{mult}_\zeta(C, \varphi_4).$$

Here, we have $E_6(\mathbb{I}_8) \subseteq C \cap \{\varphi_4(\zeta) = 0\} \cap B_6(\lambda)$ with $\mathrm{mult}_{E_6(\mathbb{I}_8)}(C, \varphi_4) = 4$.

What occurs in the difference "$16 - 4 = 12$"? It appears to be correct that $\#\left(S_6(\lambda)\right) = 12$; however this is not the case. In practice, the curve C intersects the lines $E_6(\mathbb{I}_1)$ and $E_6(\mathbb{I}_2)$. More precisely, the intersection points of the two curves C and $E_6(\mathbb{I}_1)$ are $E_6(\mathbb{I}_3)$ and $E_6(\mathbb{I}_4)$, while those of C and $E_6(\mathbb{I}_2)$ are $E_6(\mathbb{I}_6)$ and $E_6(\mathbb{I}_7)$; these 4 points do belong to the intersection $C \cap \{\varphi_4(\zeta) = 0\} \cap B_6(\lambda)$. Moreover as we shall see in Theorem B, we have $\mathrm{mult}_{E_6(\mathbb{I}_\omega)}(C, \varphi_4) = \mathrm{mult}_{E_6(\mathbb{I}_\omega)}(\varphi_1, \ldots, \varphi_4) = 3$ for $\omega = 3, 4, 6$ and 7. We thus have the equality $16 - (4+3+3+3+3) = 0$, which assures that $S_6(\lambda)$ is empty and that the intersection points of C and $\{\varphi_4(\zeta) = 0\}$ are $E_6(\mathbb{I}_\omega)$ for $\omega = 3, 4, 6, 7$ and 8, which does not cause any contradiction. To summarize, all the components of the common zeros of φ_1, φ_2, φ_3 and φ_4 contained in $B_6(\lambda)$ are $E_6(\mathbb{I}_\omega)$ for $\omega = 1, 2, 3, 4, 6, 7$ and 8, and the equality

$$4! - (1 \cdot 4 + 1 \cdot 4 + 3 + 3 + 3 + 3 + 4) = 0$$

shows that $S_6(\lambda)$ is an empty set.

As a conclusion of Example 2, we comment about the component $E_6(\mathbb{I}_5)$. The point $E_6(\mathbb{I}_5)$ may also appear as a component of the common zeros. However by Theorem B below, we have $\mathrm{mult}_{E_6(\mathbb{I}_5)}(\varphi_1,\varphi_2,\varphi_3,\varphi_4)=0$, which means that in practice $E_6(\mathbb{I}_5)$ is not a component of the common zeros.

By Example 2, we found that in order to count the number of the set $S_d(\lambda)$, we must also consider the intersection multiplicities of components which are strictly contained in $E_d(\mathbb{I})$ for some maximal $\mathbb{I}\in\mathfrak{I}(\lambda)$.

To state Theorem B, we prepare the notation.

Definition 5.7. For $\lambda=(\lambda_1,\dots,\lambda_d)\in V_d$ and $I\in\mathcal{I}(\lambda)$, we put $\lambda_I:=(\lambda_i)_{i\in I}$.

Note that λ_I always belongs to $V_{\#(I)}$ by definition.

Theorem B. *Let λ be an element of V_d. Then all the possible varieties contained in $B_d(\lambda)$ which may appear as a component of the common zeros of $\varphi_1,\dots,\varphi_{d-2}$ are $E_d(\mathbb{I})$ for some $\mathbb{I}\in\mathfrak{I}(\lambda)$. For any $2\le l\le d-1$ and for any component C of the common zeros of $\varphi_1,\dots,\varphi_{d-l}$ with $\dim C>l-2$, there exists an element $\mathbb{I}\in\mathfrak{I}(\lambda)$ with $C=E_d(\mathbb{I})$. Moreover for any $\mathbb{I}=\{I_1,\dots,I_l\}\in\mathfrak{I}(\lambda)$, we have*

$$\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1,\dots,\varphi_{d-l})=\prod_{u=1}^{l}\left(\Big(\#\left(I_u\right)-1\Big)\cdot\#\Big(S_{\#(I_u)}\left(\lambda_{I_u}\right)\Big)\right), \tag{13}$$

where the cardinality $\#\big(S_{\#(I_u)}\left(\lambda_{I_u}\right)\big)$ is defined to be 1 if $\#(I_u)$ is equal to or smaller than 3.

By definition, the variety $E_d(\mathbb{I})$ is really a component of the common zeros of $\varphi_1,\dots,\varphi_{d-2}$, if and only if the right hand side of the equality (13) is strictly positive.

Remark 5.8. If an element $\mathbb{I}=\{I_1,\dots,I_l\}\in\mathfrak{I}(\lambda)$ is maximal, then $\mathfrak{I}(\lambda_{I_u})$ is empty for any u, which implies $\#\Big(S_{\#(I_u)}\left(\lambda_{I_u}\right)\Big)=\big(\#(I_u)-2\big)!$ by Definition 3.2, Lemmas 5.1 and 5.5. Thus Theorem B includes Theorem A.

By Proposition 4.2 and Theorem B, we have the following:

Proposition C. *Let λ be an element of V_d. Then we have the equality*

$$\#\left(S_d(\lambda)\right)=(d-2)!-\sum_{\mathbb{I}\in\mathfrak{I}(\lambda)}\left(\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1,\dots,\varphi_{d-\#(\mathbb{I})})\cdot\prod_{k=d-\#(\mathbb{I})+1}^{d-2}k\right). \tag{14}$$

As we have seen in Theorem B and Proposition C, the cardinality $\#\left(S_d(\lambda)\right)$ is completely determined by the combinatorial data $\mathfrak{I}(\lambda)$. Moreover it is practically computed only by hand, though the process of its computation is very long and complicated. To relieve the long computation a little, we shall give one more proposition.

Proposition D. *For $\lambda\in V_d$ and $\mathbb{I}=\{I_1,\dots,I_l\}\in\mathfrak{I}(\lambda)$, the number $\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1,\dots,\varphi_{d-l})$ given in the equality (13) is also equal to*

$$\begin{aligned}&\left(\prod_{u=1}^{l}\left(\#\left(I_u\right)-1\right)!\right)\\ &\qquad-\sum_{\substack{\mathbb{I}'\in\mathfrak{I}(\lambda)\\ \mathbb{I}'\succ\mathbb{I},\ \mathbb{I}'\neq\mathbb{I}}}\left(\mathrm{mult}_{E_d(\mathbb{I}')}(\varphi_1,\dots,\varphi_{d-\#(\mathbb{I}')})\cdot\prod_{u=1}^{l}\left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1}k\right)\right),\end{aligned} \tag{15}$$

where $\chi_u\left(\mathbb{I}'\right)$ is the one defined in Main Theorem 3.

Proposition 5.9. *The assertion (6) in Main Theorem 1 holds.*

Proof. The set $\Phi_d^{-1}(\bar{\lambda})$ is empty if and only if the set $S_d(\lambda)$ is empty by Proposition 3.3. On the other hand, the cardinality $\#(S_d(\lambda))$ is completely determined and is computed by the combinatorial data $\mathcal{I}(\lambda)$. Hence to show the assertion (6) in Main Theorem 1, we only need to check all the possible cases of the combinatorial data $\mathcal{I}(\lambda)$. □

Theorem A is just a corollary of Theorem B by Remark 5.8. However the proof of Theorem B is much harder than that of Theorem A. Therefore we shall prove Theorem A first, and based on its proof we shall prove Theorem B. Section 6 is devoted to the proof of Theorem A; Section 7 is devoted to the proofs of Theorem B and Proposition D.

6. Proof of Theorem A

In this section we shall give the proof of Theorem A introduced in Section 5, together with the preparation of the proof of Theorem B.

We shall fix our notation first, which is valid throughout Sections 6 and 7. For a given $\lambda \in V_d$ and $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, we put $\#(I_u) =: r_u + 1$, $(\zeta_i)_{i \in I_u} =: (\zeta_{u,0}, \zeta_{u,1}, \ldots, \zeta_{u,r_u})$, $(\lambda_i)_{i \in I_u} =: (\lambda_{u,0}, \lambda_{u,1}, \ldots, \lambda_{u,r_u})$ and $m_{u,i} := \frac{1}{1-\lambda_{u,i}}$. Here, we assume $\zeta_{l,0} = \zeta_d = 0$. Then we have $\sum_{u=1}^{l}(r_u+1) = d$, $\sum_{i=0}^{r_u} m_{u,i} = 0$, $\varphi_k(\zeta) = \sum_{u=1}^{l}\sum_{i=0}^{r_u} m_{u,i}\zeta_{u,i}^k$ and

$$E_d(\mathbb{I}) = \left\{\zeta \in \mathbb{P}^{d-2} \;\middle|\; \zeta_{u,0} = \zeta_{u,1} = \cdots = \zeta_{u,r_u} \text{ for } 1 \le u \le l\right\} \cong \mathbb{P}^{l-2}.$$

Moreover let $\alpha_1, \alpha_2, \ldots, \alpha_l$ be any mutually distinct complex numbers with $\alpha_l = 0$, and we denote by α the point $\zeta \in E_d(\mathbb{I})$ which satisfies $\zeta_{u,i} = \alpha_u$ for any u and i. In the following, we shall find $\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-l})$ by cutting $E_d(\mathbb{I})$ at α by the plane $\mathcal{H}(\alpha) := \left\{\zeta \in \mathbb{P}^{d-2} \;\middle|\; \zeta_{u,0} = \alpha_u \text{ for } 1 \le u \le l\right\}$. We put $\xi_{u,i} := \zeta_{u,i} - \alpha_u$, $\xi_u := (\xi_{u,1}, \ldots, \xi_{u,r_u}) \in \mathbb{C}^{r_u}$, $\xi := (\xi_1, \ldots, \xi_l) \in \mathbb{C}^{d-l}$ and

$$\psi_k(\xi) := \varphi_k(\alpha + \xi) = \sum_{u=1}^{l}\left(m_{u,0}\alpha_u^k + \sum_{i=1}^{r_u} m_{u,i}\,(\alpha_u + \xi_{u,i})^k\right). \tag{16}$$

Then ξ is a local coordinate system of $\mathcal{H}(\alpha)$ centered at α.

Proposition 6.1. *For any $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$ and for generic $\alpha \in E_d(\mathbb{I})$, we have*

$$\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-l}) = \mathrm{mult}_0(\psi_1, \ldots, \psi_{d-l}). \tag{17}$$

Proof. Obvious by definition. □

In practice, the equality (17) always holds for any α if $\alpha_1, \ldots, \alpha_l$ are mutually distinct, which will be verified by Proposition 7.10.

We shall express the equations $\psi_k(\xi) = 0$ in another way. We put

$$p_{u,k}(\xi_u) = \sum_{i=1}^{r_u} m_{u,i}\xi_{u,i}^k$$

for each u and k. Then we have

$$\begin{aligned}\psi_k(\xi) &= \sum_{u=1}^{l}\left(\left(\sum_{i=0}^{r_u} m_{u,i}\right)\alpha_u^k + \sum_{i=1}^{r_u}\sum_{h=1}^{k} m_{u,i}\binom{k}{h}\alpha_u^{k-h}\xi_{u,i}^h\right)\\ &= \sum_{u=1}^{l}\sum_{h=1}^{k}\binom{k}{h}\alpha_u^{k-h}p_{u,h}(\xi_u),\end{aligned}\tag{18}$$

where $\binom{k}{h} = \frac{k(k-1)\cdots(k-h+1)}{h!}$ denotes the binomial coefficient. Hence $\psi_k(\xi)$ is a linear combination of $p_{u,h}(\xi_u)$ for $1 \le u \le l$ and $1 \le h \le k$.

Proposition 6.2. *The equations $\psi_k(\xi) = 0$ for $1 \le k \le d-l$ are equivalent to the equations*

$$p_{u,k}(\xi_u) = \sum_{v=1}^{l}\sum_{h=r_v+1}^{d-l} a_{u,k,v,h}p_{v,h}(\xi_v)\tag{19}$$

for $1 \le u \le l$ and $1 \le k \le r_u$, where the coefficients $a_{u,k,v,h}$ are some constants which depend only on $r_1, \dots, r_l$ and $\alpha_1, \dots, \alpha_l$.

Proof. It suffices to show the invertibility of the square matrix composed of the coefficients of $p_{u,h}(\xi_u)$ for $1 \le u \le l$ and $1 \le h \le r_u$ in the right hand side of the expressions (18). Proposition 6.2 is therefore reduced to Lemma 6.7, which is given at the end of this section. □

By the aid of Propositions 6.1 and 6.2, we have reduced Theorem A to the following:

Proposition 6.3. *Suppose that the element $\mathbb{I} \in \mathfrak{I}(\lambda)$ is maximal. Then for any complex numbers $a_{u,k,v,h}$, the origin 0 is a discrete solution of the equations (19) for $1 \le u \le l$ and $1 \le k \le r_u$ with its intersection multiplicity $r_1! \cdots r_l!$.*

In the following, we shall prove Proposition 6.3.

Lemma 6.4. *Let $m_1, \dots, m_r$ be complex numbers such that $\sum_{i\in I} m_i \neq 0$ holds for any non-empty $I \subseteq \{1, \dots, r\}$. We put $p_k(\xi) := \sum_{i=1}^{r} m_i\xi_i^k$ for $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r$. Then 0 is the only solution of the equations $p_k(\xi) = 0$ for $1 \le k \le r$ with its intersection multiplicity* $\mathrm{mult}_0(p_1, \dots, p_r) = r!$.

Proof. By the same argument as in the proof of Lemma 5.5, the existence of a solution other than 0 implies the equality $\sum_{i\in I} m_i = 0$ for some non-empty $I \subseteq \{1, \dots, r\}$, which is a contradiction. Thus the uniqueness of the solution is verified.

Moreover by Lemmas 5.1 and 5.5, the set of the common zeros of $p_1, \dots, p_{r-1}$ in $\mathbb{P}^{r-1}$ is discrete and has $(r-1)!$ points, whose intersection multiplicities are all 1. Hence the set of the common zeros of $p_1, \dots, p_{r-1}$ in $\mathbb{C}^r$ consists of $(r-1)!$ lines $\ell_1, \dots, \ell_{(r-1)!}$, all of which pass the origin. Moreover their intersection multiplicities $\mathrm{mult}_{\ell_i}(p_1, \dots, p_{r-1})$ are all 1. Since each line ℓ_i intersects the hypersurface $\{p_r(\xi) = 0\}$ only at the origin, the intersection multiplicity $\mathrm{mult}_0(\ell_i, p_r)$ is equal to r for any i. We thus have the equality $\mathrm{mult}_0(p_1, \dots, p_r) = r \cdot (r-1)! = r!$. □

The most important part of the proof of Proposition 6.3 is to reduce Proposition 6.3 to Lemma 6.4 by replacing all the coefficients $a_{u,k,v,h}$ by 0.

We denote by $A = (a_{u,k,v,h})$ an element of $\mathbb{C}^{(l-1)(d-l)^2}$, where the indices u, k, v, h range in $1 \le u \le l$, $1 \le k \le r_u$, $1 \le v \le l$ and $r_v + 1 \le h \le d-l$. We put

$$D_R := \left\{A = (a_{u,k,v,h}) \in \mathbb{C}^{(l-1)(d-l)^2} \;\middle|\; |a_{u,k,v,h}| < R \text{ for any } u, k, v, h\right\}.$$

Then we can define the map $F : \mathbb{C}^{d-l} \times D_R \to \mathbb{C}^{d-l} \times D_R$ by

$$(\xi, A) \mapsto \left(\left(p_{u,k}(\xi_u) - \sum_{v,h} a_{u,k,v,h} p_{v,h}(\xi_v) \right)_{u,k}, A \right),$$

where the indices u, k range in $1 \le u \le l$ and $1 \le k \le r_u$.

Proposition 6.5. *Suppose that the element $\mathbb{I} \in \mathfrak{I}(\lambda)$ is maximal. Then for any positive real number R and any open neighborhood U_0 of 0 in $\mathbb{C}^{d-l}$, there exist open neighborhoods U, W of 0 in $\mathbb{C}^{d-l}$ with $U \subseteq U_0$ such that the map*

$$(U \times D_R) \cap F^{-1}(W \times D_R) \xrightarrow{F} W \times D_R \tag{20}$$

is proper, finite and surjective, hence a finite branched covering.

In the following, we shall prove Proposition 6.3 first under the assumption of Proposition 6.5; afterward we shall prove Proposition 6.5.

Proof of Proposition 6.3. For any given coefficients $a_{u,k,v,h}$, we take a positive real number R sufficiently large such that the ball D_R contains $A = (a_{u,k,v,h})$. Then the discreteness of the solution 0 is verified by the finiteness of the map (20). Moreover we take an open neighborhood U_0 of 0 in $\mathbb{C}^{d-l}$ sufficiently small such that the only solution of the equations (19) in U_0 is 0. Then the intersection multiplicity of the equations (19) at 0 is equal to the degree of the branched covering map (20), which is also equal to the intersection multiplicity of the equations (19) at 0 with all the coefficients $a_{u,k,v,h}$ equal to 0. Therefore it is $r_1! \cdots r_l!$ by Lemma 6.4, which completes the proof of Proposition 6.3. □

Proof of Proposition 6.5. We put $|\xi_u| := \max_{1 \le i \le r_u} |\xi_{u,i}|$, $Z_u := \left\{ \xi_u \in \mathbb{C}^{r_u} \;\middle|\; |\xi_u| = 1 \right\}$ and $\delta_u := \inf_{\xi_u \in Z_u} \max_{1 \le k \le r_u} |p_{u,k}(\xi_u)|$ for each u. Then by the maximality of $\mathbb{I} \in \mathfrak{I}(\lambda)$ and Lemma 6.4, we have $\delta_u > 0$ for each u, which implies the inequality $\max_{1 \le k \le r_u} |p_{u,k}(\xi_u)| \ge \delta_u |\xi_u|^{r_u}$ for any $\xi_u \in \mathbb{C}^{r_u}$ with $|\xi_u| \le 1$. Hence putting $\delta := \min_{1 \le u \le l} \delta_u$ and $||\xi|| := \max_{1 \le u \le l} |\xi_u|^{r_u}$, we have the inequality

$$\max_{u,k} |p_{u,k}(\xi_u)| \ge \delta \cdot ||\xi|| \tag{21}$$

for $||\xi|| \le 1$.

On the other hand, for any $A = (a_{u,k,v,h}) \in D_R$ and $\xi \in \mathbb{C}^{d-l}$ with $||\xi|| \le 1$, we have

$$\begin{aligned} \max_{u,k} \left| \sum_{v,h} a_{u,k,v,h} p_{v,h}(\xi_v) \right| &\le \sum_{v,h} R \left(\sum_{i=1}^{r_v} |m_{v,i}| \right) |\xi_v|^h \\ &\le L \cdot ||\xi||^{1+\mu}, \end{aligned} \tag{22}$$

where we put $L := R \sum_{v=1}^{l} (d - l - r_v) \left(\sum_{i=1}^{r_v} |m_{v,i}| \right)$ and $\mu := \frac{1}{\max_u r_u}$.

Therefore if we take $\xi \in \mathbb{C}^{d-l}$ with $||\xi|| \leq \left(\frac{\delta}{2L}\right)^{1/\mu}$, then by the inequalities (21) and (22), we have

$$\begin{aligned}
&\max_{u,k}\left|p_{u,k}(\xi_u) - \sum_{v,h} a_{u,k,v,h} p_{v,h}(\xi_v)\right| \\
&\geq \max_{u,k}|p_{u,k}(\xi_u)| - \max_{u,k}\left|\sum_{v,h} a_{u,k,v,h} p_{v,h}(\xi_v)\right| \\
&\geq \delta \cdot ||\xi|| - L \cdot ||\xi||^{1+\mu} \geq \delta \cdot ||\xi|| - L \cdot \frac{\delta}{2L} \cdot ||\xi|| = \frac{\delta}{2} \cdot ||\xi||.
\end{aligned}$$

We define a positive number ϵ sufficiently small such that the inequality $0 < \epsilon < \left(\frac{\delta}{2L}\right)^{1/\mu}$ holds and that the set $U := \left\{\xi \in \mathbb{C}^{d-l} \;\middle|\; ||\xi|| < \epsilon\right\}$ is included by U_0. Moreover we put

$$W := \left\{\eta = (\eta_{u,k}) \in \mathbb{C}^{d-l} \;\middle|\; |\eta| = \max_{u,k}|\eta_{u,k}| < \tfrac{1}{2}\delta\epsilon\right\}.$$

Then we can easily verify that the map (20) is proper. Properness implies the rest of the assertions, which completes the proof of Proposition 6.5. □

The rest of this section is devoted to Lemma 6.7 and its proof.

Definition 6.6. For non-negative integers n, b, k, h with $n > k$ and $b > h$, we denote by $A^{b,h}_{n,k}(\alpha)$ the $(n-k, b-h)$ matrix whose (i,j)-the entry is $\binom{i+k-1}{j+h-1}\alpha^{(i+k)-(j+h)}$ for each i and j. Moreover we put $A^b_{n,k}(\alpha) := A^{b,0}_{n,k}(\alpha)$ and $A^b_n(\alpha) := A^{b,0}_{n,0}(\alpha)$.

By definition, the matrix $A^{b,h}_{n,k}(\alpha)$ is obtained from the (n,b) matrix

$$A^b_n(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \alpha & 1 & 0 & 0 & \cdots & 0 \\ \alpha^2 & 2\alpha & 1 & 0 & \cdots & 0 \\ \alpha^3 & 3\alpha^2 & 3\alpha & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^{n-1} & (n-1)\alpha^{n-2} & \binom{n-1}{2}\alpha^{n-3} & \binom{n-1}{3}\alpha^{n-4} & \cdots & \ddots \end{pmatrix}$$

by cutting off the upper k rows and the left h columns.

Lemma 6.7. *We put $r := r_1 + \cdots + r_l = d - l$, and denote by M the (r,r) square matrix defined by*

$$M = \left(A^{r_1+1,1}_{r+1,1}(\alpha_1), \ldots, A^{r_l+1,1}_{r+1,1}(\alpha_l)\right).$$

Then we have

$$\det M = \frac{r!}{r_1! \cdots r_l!} \cdot \prod_{1 \leq v < u \leq l} (\alpha_u - \alpha_v)^{r_v r_u}.$$

The matrix M defined above is the same as the square matrix composed of the coefficients of $p_{u,h}(\xi_u)$ for $1 \leq u \leq l$ and $1 \leq h \leq r_u$ in the right hand side of the expressions (18); hence Proposition 6.2 is reduced to Lemma 6.7.

To prove Lemma 6.7, we prepare the definition and the lemma.

Definition 6.8. For positive integer b, we denote by X_b the (b,b) diagonal matrix whose (i,i)-th entry is i for $1 \le i \le b$, and by N_b the (b,b) nilpotent matrix whose $(i,i+1)$-th entry is 1 for $1 \le i \le b-1$ and whose other entries are 0, i.e.,

$$X_b = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} \quad \text{and} \quad N_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Lemma 6.9. *For positive integers n and b, we have the equalities*

$$A_{n+1,1}^{b+1,1}(\alpha) = X_n \cdot A_n^b(\alpha) \cdot X_b{}^{-1} \quad \text{and} \quad A_n^n(\beta) \cdot A_n^b(\alpha) = A_n^b(\beta + \alpha).$$

Moreover for positive integers n, b, k with $n > k$ and a non-zero complex number α, we have the equality

$$A_{n,k}^b(\alpha) \cdot \sum_{h=0}^{b-1} \binom{-k}{h} \left(\alpha^{-1} N_b\right)^h = \alpha^k A_{n-k}^b(\alpha),$$

where $\left(\alpha^{-1} N_b\right)^0$ denotes the identity matrix of size (b,b).

Proof. The first equality is easily verified by the equality $\binom{i}{j} = \binom{i-1}{j-1} \cdot \frac{i}{j}$. On the other hand, the second one is proved by the equalities $\binom{i}{h}\binom{h}{j} = \binom{i}{j}\binom{i-j}{h-j}$ and $\sum_{h=0}^{k} \binom{k}{h} \alpha^h \beta^{k-h} = (\alpha+\beta)^k$, while the last one by the equality $\sum_{h=0}^{j} \binom{x}{h}\binom{y}{j-h} = \binom{x+y}{j}$. $\square$

Proof of Lemma 6.7. By Lemma 6.9, we have $A_{r+1,1}^{r_u+1,1}(\alpha_u) = X_r \cdot A_r^{r_u}(\alpha_u) \cdot (X_{r_u})^{-1}$ for each $1 \le u \le l$. Hence putting $M' = \left(A_r^{r_1}(\alpha_1), \dots, A_r^{r_l}(\alpha_l)\right)$, we have the equalities

$$\det M = \det X_r \cdot \det M' \cdot \prod_{u=1}^{l} \det (X_{r_u})^{-1} = \frac{r!}{r_1! \cdots r_l!} \cdot \det M'.$$

Therefore to prove Lemma 6.7, we only need to show the equality

$$\det M' = \prod_{1 \le v < u \le l} (\alpha_u - \alpha_v)^{r_v r_u}. \tag{23}$$

If there exist distinct indices u, v with $\alpha_u = \alpha_v$, then both hand sides of the equality (23) are clearly zero; hence we only need to consider the equality (23) in the case that $\alpha_1, \dots, \alpha_l$ are mutually distinct. Moreover in the case $l = 1$, the equality (23) trivially holds since $\det M' = 1$; we shall show the the equality (23) by the induction of l.

We put $r' = r_2 + \cdots + r_l$ and $\alpha'_u = \alpha_u - \alpha_1$ for $2 \le u \le l$. Then by Lemma 6.9, we have

$$A_r^r(-\alpha_1) \cdot M' = \left(A_r^{r_1}(0), A_r^{r_2}(\alpha'_2), \dots, A_r^{r_l}(\alpha'_l)\right) = \begin{pmatrix} I_{r_1} & * \\ O & \widetilde{M} \end{pmatrix},$$

where we put $\widetilde{M} = \left(A_{r,r_1}^{r_2}(\alpha'_2), \dots, A_{r,r_1}^{r_l}(\alpha'_l)\right)$, and I_{r_1} denotes the identity matrix of size (r_1, r_1). Moreover by Lemma 6.9, we have

$$A_{r,r_1}^{r_u}(\alpha'_u) \cdot \sum_{h=0}^{r_u-1} \binom{-r_1}{h} \left((\alpha'_u)^{-1} N_{r_u}\right)^h = (\alpha'_u)^{r_1} \cdot A_{r'}^{r_u}(\alpha'_u)$$

for each $2 \le u \le l$. Hence putting $M'' = \left(A_{r'}^{r_2}(\alpha_2'), \dots, A_{r'}^{r_l}(\alpha_l')\right)$, we have the equalities

$$\det M' = \det \widetilde{M} = \det M'' \cdot \prod_{u=2}^{l} \left(\alpha_u'\right)^{r_1 r_u},$$

which completes the proof by the induction of l. □

7. Proof of Theorem B

In this section we shall give the proofs of Theorem B and Proposition D introduced in Section 5, which are also highlights of the proof of the main theorems. We shall first give the key proposition to prove Theorem B.

Proposition 7.1. *Let r be a positive integer, and $m_1, \dots, m_r$ non-zero complex numbers with $\sum_{i=1}^r m_i \neq 0$. We put $m = (m_1, \dots, m_r)$,*

$$p_k(\xi) := \sum_{i=1}^{r} m_i \xi_i^k, \quad B(m) := \left\{\xi \in \mathbb{C}^r \;\middle|\; p_k(\xi) = 0 \text{ for } 1 \le k \le r\right\},$$

and $|\xi| := \max_{1 \le i \le r} |\xi_i|$ for $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r$. Then

(1) *for each positive integer h, there exists a positive real number L_h such that the inequality*

$$|p_h(\xi)| \le L_h \cdot \max_{1 \le k \le r} |p_k(\xi)| \tag{24}$$

holds for any $\xi \in \mathbb{C}^r$ with $|\xi| = 1$.

(2) *There exist an open neighborhood O of $B(m) \cap \{\xi \in \mathbb{C}^r \mid |\xi| = 1\}$ in $\mathbb{C}^r$ and a positive real number L' such that the inequality*

$$|p_r(\xi)| \le L' \cdot \max_{1 \le k \le r-1} |p_k(\xi)| \tag{25}$$

holds for any $\xi \in O$.

Proof. We put $m_0 := -\sum_{i=1}^r m_i$,

$$\mathfrak{I}(m) := \left\{ \{I_1, \dots, I_l\} \;\middle|\; \begin{array}{c} I_1 \amalg \cdots \amalg I_l = \{0, \dots, r\}, \quad l \ge 1 \\ I_u \neq \emptyset \text{ and } \sum_{i \in I_u} m_i = 0 \text{ for } 1 \le u \le l \end{array} \right\},$$

$$E(\mathbb{I}) := \left\{(\xi_1, \dots, \xi_r) \in \mathbb{C}^r \;\middle|\; \text{If } i, j \in I \in \mathbb{I}, \text{ then } \xi_i = \xi_j\right\}$$

for each $\mathbb{I} \in \mathfrak{I}(m)$, and

$$\mathbb{I}(\xi) := \left\{ I \subseteq \{0, 1, \dots, r\} \;\middle|\; \begin{array}{c} I \neq \emptyset. \quad \text{If } i, j \in I, \text{ then } \xi_i = \xi_j. \\ \text{If } i \in I \text{ and } j \in \{0, 1, \dots, r\} \setminus I, \text{ then } \xi_i \neq \xi_j. \end{array} \right\}$$

for each $\xi \in B(m)$, where we are assuming $\xi_0 = 0$. Then we have the equality

$$B(m) = \bigcup_{\mathbb{I} \in \mathfrak{I}(m)} E(\mathbb{I}), \tag{26}$$

and we also have $\mathbb{I}(\xi) \in \mathfrak{I}(m)$ and $\xi \in E(\mathbb{I}(\xi))$ for each $\xi \in B(m)$ by the same argument as the proof of Lemma 5.5. Note that in this setting, the set $\mathfrak{I}(m)$ always contains the element $\mathbb{I}_0 := \{\{0, \dots, r\}\}$, and that the equalities $E(\mathbb{I}_0) = \{0\}$ and $\mathbb{I}(0) = \mathbb{I}_0$ hold.

We shall make use of the following auxiliary lemmas:

Lemma 7.2. *There exists an open neighborhood O of $B(m) \cap \{\xi \in \mathbb{C}^r \mid |\xi| = 1\}$ in $\mathbb{C}^r$ such that for each positive integer h, there exists a positive real number L'_h such that the inequality*

$$|p_h(\xi)| \le L'_h \cdot \max_{1 \le k \le r-1} |p_k(\xi)| \tag{27}$$

holds for any $\xi \in O$.

Lemma 7.3. *Let α be a point in $B(m) \setminus \{0\}$. Then there exists an open neighborhood O_α of α in $\mathbb{C}^r$ such that for each positive integer h, there exists a positive real number $L_{\alpha,h}$ such that the inequality*

$$|p_h(\xi)| \le L_{\alpha,h} \cdot \max_{1 \le k \le r+1-\#(\mathbb{I}(\alpha))} |p_k(\xi)| \tag{28}$$

holds for any $\xi \in O_\alpha$.

Note that the implications

$$\text{“Proposition 7.1} \Longrightarrow \text{Lemma 7.2} \Longrightarrow \text{The assertion (2) in Proposition 7.1”}$$

are clear. In the following, we shall prove Lemmas 7.2, 7.3 and the assertion (1) in Proposition 7.1 simultaneously by induction. To make the induction work well, we define the “depth” of a point $\alpha \in B(m)$ by

$$\tau_m(\alpha) := \max \left\{ \nu \;\middle|\; \begin{array}{l} \mathbb{I}(\alpha) =: \mathbb{I}_1 \precneqq \mathbb{I}_2 \precneqq \cdots \precneqq \mathbb{I}_\nu \\ \mathbb{I}_\omega \in \mathfrak{I}(m) \ \text{ for } 1 \le \omega \le \nu \end{array} \right\},$$

where the symbol $\mathbb{I} \precneqq \mathbb{I}'$ for $\mathbb{I}, \mathbb{I}' \in \mathfrak{I}(m)$ denotes that $\mathbb{I}'$ is a refinement of $\mathbb{I}$ with $\mathbb{I} \ne \mathbb{I}'$. Note that the inequality $\tau_m(0) > \tau_m(\alpha)$ holds for any $\alpha \in B(m) \setminus \{0\}$ and that the equality $\tau_m(0) = 1$ holds if and only if $B(m) = \{0\}$.

We shall consider the following assertions for each non-negative integer ν:

$(1)_\nu$ if $\tau_m(0) \le \nu + 1$, then the assertion (1) in Proposition 7.1 holds.
$(2)_\nu$ If $\tau_m(0) \le \nu + 1$, then Lemma 7.2 holds.
$(3)_\nu$ If $\tau_m(\alpha) \le \nu$, then Lemma 7.3 holds.

Note that the assertion $(2)_0$ trivially holds since $\tau_m(0) \le 1$ implies $B(m) = \{0\}$. In the following, we shall show the implications

$$(1)_{\nu-1} \Rightarrow (3)_\nu \Rightarrow (2)_\nu \Rightarrow (1)_\nu$$

for each ν, which will complete the proofs of Lemmas 7.2, 7.3 and Proposition 7.1. We put

$$Z := \{\xi \in \mathbb{C}^r \mid |\xi| = 1\}.$$

Proof of the implication $(3)_\nu \Rightarrow (2)_\nu$. We shall suppose $(3)_\nu$ and prove $(2)_\nu$. When $\tau_m(0) \le \nu + 1$, the inequality $\tau_m(\alpha) \le \nu$ holds for any $\alpha \in Z \cap B(m)$. Hence by the assumption $(3)_\nu$, we can choose, for each $\alpha \in Z \cap B(m)$, an open neighborhood O_α of α and a positive real number $L_{\alpha,h}$ for each $h \in \mathbb{N}$ such that the inequality (28) holds for any $\xi \in O_\alpha$. Since $Z \cap B(m)$ is compact, there exist finite number of open neighborhoods $O_{\alpha_1}, \ldots, O_{\alpha_\mu}$ which cover $Z \cap B(m)$. On the other hand, since $\#(\mathbb{I}(\alpha)) \ge 2$ for any $\alpha \in Z \cap B(m)$, we always have $r + 1 - \#(\mathbb{I}(\alpha)) \le r - 1$. Therefore, putting $O := \bigcup_{1 \le \omega \le \mu} O_{\alpha_\omega}$ and $L'_h := \max_{1 \le \omega \le \mu} L_{\alpha_\omega,h}$ for each h, we have, by the inequality (28), the inequality (27) for any $\xi \in O$. ☐

Proof of the implication $(2)_\nu \Rightarrow (1)_\nu$. We shall suppose $(2)_\nu$ and verify $(1)_\nu$. The set $Z \setminus O$ is compact and does not have common zeros of $p_1, \ldots, p_r$. Hence the infimum $\inf_{\xi \in Z\setminus O} \max_{1\le k\le r} |p_k(\xi)|$ is positive, which assures the existence of a positive real number L_h for each $h \in \mathbb{N}$ satisfying the inequality (24) for any $\xi \in Z \setminus O$. Replacing the maximum of L_h and L'_h by L_h, we obtain the inequality (24) for any $\xi \in Z$. □

In the rest of the proof, we shall suppose $(1)_{\nu-1}$ and prove $(3)_\nu$. We fix $\alpha \in B(m) \setminus \{0\}$ with $\tau_m(\alpha) \le \nu$, put $\mathbb{I}(\alpha) =: \{I_1, \ldots, I_l\}$, and denote by α_u^0 the i-th coordinate of α for $i \in I_u$. Note that $\alpha_1^0, \ldots, \alpha_l^0$ are mutually distinct. Moreover we put $\#(I_u) = r_u + 1$, $(\xi_i)_{i\in I_u} = (\xi_{u,0}, \xi_{u,1}, \ldots, \xi_{u,r_u})$, $(m_i)_{i\in I_u} = (m_{u,0}, m_{u,1}, \ldots, m_{u,r_u})$, $m(I_u) = (m_{u,1}, \ldots, m_{u,r_u})$, $x_{u,i} = \xi_{u,i} - \xi_{u,0}$, $\alpha_u = \xi_{u,0}$, $x_u = (x_{u,1}, \ldots, x_{u,r_u})$, $x = (x_1, \ldots, x_l)$, $|x_u| = \max_{1\le i\le r_u} |x_{u,i}|$ and $|x| = \max_{1\le u\le l} |x_u|$. We may assume $\alpha_l = \alpha_l^0 = \xi_{l,0} = \xi_0 = 0$. We may also consider the coordinates $(\alpha_1, \ldots, \alpha_{l-1}, x)$ as a local coordinate system around α in $\mathbb{C}^r$. Note that the point $(\alpha_1, \ldots, \alpha_{l-1}, x)$ coincides with α if and only if $x = 0$ and $\alpha_u = \alpha_u^0$ for $1 \le u \le l-1$, and that the point $(\alpha_1, \ldots, \alpha_{l-1}, x)$ belongs to $E(\mathbb{I}(\alpha))$ if and only if $x = 0$. Furthermore we put

$$\theta_{u,k}(x_u) = \sum_{i=1}^{r_u} m_{u,i} x_{u,i}^k$$

for $1 \le u \le l$ and $k \in \mathbb{N}$.

Then we have the equality

$$p_k(\xi) = \sum_{u=1}^{l} \sum_{h=1}^{k} \binom{k}{h} \alpha_u^{k-h} \theta_{u,h}(x_u) \tag{29}$$

by the similar computation as in the equalities (18). Moreover by Lemma 6.7, the equalities (29) for $1 \le k \le r+1-l$ are equivalent in some neighborhood of α to the equalities

$$\theta_{u,k}(x_u) = \sum_{h=1}^{r+1-l} b_{u,k,h} p_h(\xi) + \sum_{v=1}^{l} \sum_{h=r_v+1}^{r+1-l} a_{u,k,v,h} \theta_{v,h}(x_v) \tag{30}$$

for $1 \le u \le l$ and $1 \le k \le r_u$, where the coefficients $b_{u,k,h}$ and $a_{u,k,v,h}$ depend only on $r_1, \ldots, r_l$ and $\alpha_1, \ldots, \alpha_{l-1}$. Moreover its dependence is continuous on the domain where $\alpha_1, \ldots, \alpha_{l-1}$ and 0 are mutually distinct. Therefore taking a small open neighborhood Δ of $(\alpha_1^0, \ldots, \alpha_{l-1}^0)$ in $\mathbb{C}^{l-1}$ and a sufficiently large real number R, we may assume that the inequalities

$$|\alpha_u| \le R, \quad |b_{u,k,h}| \le R \quad \text{and} \quad |a_{u,k,v,h}| \le R$$

hold for all u, k, v, h and for any $(\alpha_1, \ldots, \alpha_{l-1}) \in \Delta$.

On the other hand, since $\tau_m(\alpha) \le \nu$, we always have $\tau_{m(I_u)}(0) \le \nu$ for any u. Hence by the assumption $(1)_{\nu-1}$, there exists, for each u and for each positive integer h, a positive real number $L_{u,h}$ such that the inequality

$$|\theta_{u,h}(x_u)| \le L_{u,h} \cdot \max_{1\le k\le r_u} |\theta_{u,k}(x_u)|$$

holds for any $x_u \in \mathbb{C}^{r_u}$ with $|x_u| = 1$. Hence by the homogeneity of $\theta_{u,k}(x_u)$, the inequality

$$|\theta_{u,h}(x_u)| \le L_{u,h} \cdot \max_{1\le k\le r_u} |\theta_{u,k}(x_u)| \cdot |x_u|$$

holds for $h \geq r_u + 1$ and for any $x_u \in \mathbb{C}^{r_u}$ with $|x_u| \leq 1$. Therefore from the equality (30), we have the following for $(\alpha_1, \ldots, \alpha_{l-1}) \in \Delta$ and $|x| \leq 1$:

$$\begin{aligned}\max_{u,k} \left| \sum_{h=1}^{r+1-l} b_{u,k,h} p_h(\xi) \right| &\geq \max_{u,k} |\theta_{u,k}(x_u)| - \max_{u,k} \left| \sum_{v=1}^{l} \sum_{h=r_v+1}^{r+1-l} a_{u,k,v,h} \theta_{v,h}(x_v) \right| \\ &\geq \left(1 - R \sum_{v=1}^{l} \sum_{h=r_v+1}^{r+1-l} L_{v,h} \cdot |x| \right) \max_{u,k} |\theta_{u,k}(x_u)| .\end{aligned}$$

Hence putting

$$J := \max \left\{ 1,\ 2R \sum_{v=1}^{l} \sum_{h=r_v+1}^{r+1-l} L_{v,h} \right\}, \quad L := 2R(r+1-l)$$

$$\text{and} \quad O_\alpha := \left\{ (\alpha_1, \ldots, \alpha_{l-1}, x) \in \mathbb{C}^r \;\middle|\; (\alpha_1, \ldots, \alpha_{l-1}) \in \Delta, |x| < 1/J \right\},$$

we have, for any $\xi = (\alpha_1, \ldots, \alpha_{l-1}, x) \in O_\alpha$, the inequality

$$\max_{u,k} |\theta_{u,k}(x_u)| \leq 2 \max_{u,k} \left| \sum_{h=1}^{r+1-l} b_{u,k,h} p_h(\xi) \right| \leq L \cdot \max_{1 \leq k \leq r+1-l} |p_k(\xi)|. \tag{31}$$

On the other hand, from the equality (29), we have, for each positive integer h, the inequalities

$$|p_h(\xi)| \leq \sum_{u=1}^{l} \sum_{k=1}^{h} \binom{h}{k} R^{h-k} L_{u,k} \cdot \max_{1 \leq k \leq r_u} |\theta_{u,k}(x_u)| \leq L_h \cdot \max_{u,k} |\theta_{u,k}(x_u)| \tag{32}$$

for any $(\alpha_1, \ldots, \alpha_{l-1}, x) \in O_\alpha$, where we put $L_h := \sum_{u=1}^{l} \sum_{k=1}^{h} \binom{h}{k} R^{h-k} L_{u,k}$. Therefore by the inequalities (31) and (32), we obtain

$$|p_h(\xi)| \leq L_h L \cdot \max_{1 \leq k \leq r+1-l} |p_k(\xi)|$$

for any $\xi = (\alpha_1, \ldots, \alpha_{l-1}, x) \in O_\alpha$ and for each h. Thus the assertion $(3)_\nu$ is proved, which completes the proof of Lemmas 7.2, 7.3 and Proposition 7.1. □

In the rest of this section, the notation follows that in Section 6. Therefore λ is an element of V_d, and $\mathbb{I} = \{I_1, \ldots, I_l\}$ an element of $\mathfrak{I}(\lambda)$, which are fixed throughout the rest of this section. Moreover the notation r_u, $\zeta_{u,i}$, $\lambda_{u,i}$, $m_{u,i}$, α_u, α, $\xi_{u,i}$, ξ_u, ξ, $\psi_k(\xi)$, $p_{u,k}(\xi_u)$, $A = (a_{u,k,v,h})$, D_R and the map F is the same as in Section 6.

We shall give the proposition next which is the most important part of the proof of Theorem B, whose proof is essentially based on Proposition 7.1.

Proposition 7.4. *For any positive real numbers R and $1 > \epsilon > 0$, and for any open neighborhood U_0 of 0 in $\mathbb{C}^{d-l}$, there exist open neighborhoods U, W of 0 in $\mathbb{C}^{d-l}$ with $U \subseteq U_0$ such that the map*

$$(U \times D_R) \cap F^{-1} (W_\epsilon \times D_R) \xrightarrow{F} W_\epsilon \times D_R \tag{33}$$

is proper, finite and surjective, hence a finite branched covering, where

$$W_\epsilon := W \cap \Xi_\epsilon \quad \text{and} \quad \Xi_\epsilon := \left\{ \eta = (\eta_{u,k}) \in \mathbb{C}^{d-l} \;\middle|\; \min_{1 \leq u \leq l} |\eta_{u,r_u}| > \epsilon \cdot \max_{u,k} |\eta_{u,k}| \right\}.$$

Proof. Remember that the map $F : \mathbb{C}^{d-l} \times D_R \to \mathbb{C}^{d-l} \times D_R$ is defined by $F(\xi, A) = (\eta, A)$, where $\xi = (\xi_{u,i})$, $\eta = (\eta_{u,k})$, $A = (a_{u,k,v,h})$ and

$$\eta_{u,k} = p_{u,k}(\xi_u) - \sum_{v=1}^{l} \sum_{h=r_v+1}^{d-l} a_{u,k,v,h} p_{v,h}(\xi_v)$$

for $1 \le u \le l$ and $1 \le k \le r_u$. We put

$$|\xi_u| := \max_{1 \le i \le r_u} |\xi_{u,i}|, \quad |\xi| := \max_{1 \le u \le l} |\xi_u|, \quad |\eta| := \max_{u,k} |\eta_{u,k}|,$$

$$\widetilde{B}_u(\lambda_{I_u}) := \left\{ \xi_u \in \mathbb{C}^{r_u} \;\middle|\; p_{u,k}(\xi_u) = 0 \text{ for } 1 \le k \le r_u \right\} \text{ and}$$

$$Z_u := \left\{ \xi_u \in \mathbb{C}^{r_u} \;\middle|\; |\xi_u| = 1 \right\}.$$

By the assertion (1) in Proposition 7.1, there exists a positive real number $L_{u,h}$ for each u and h such that the inequality

$$\left| p_{u,h}(\xi_u) \right| \le L_{u,h} \cdot \max_{1 \le k \le r_u} \left| p_{u,k}(\xi_u) \right|$$

holds for any $\xi_u \in Z_u$. Hence by the homogeneity of $p_{u,k}(\xi_u)$, we have

$$\left| p_{u,h}(\xi_u) \right| \le L_{u,h} \cdot \max_{1 \le k \le r_u} \left| p_{u,k}(\xi_u) \right| \cdot |\xi_u| \tag{34}$$

for any $\xi_u \in \mathbb{C}^{r_u}$ with $|\xi_u| \le 1$ and for each $h \ge r_u + 1$.

On the other hand, by the assertion (2) in Proposition 7.1, there exist an open neighborhood O_u of $\widetilde{B}_u(\lambda_{I_u}) \cap Z_u$ in $\mathbb{C}^{r_u}$ and a positive real number L'_u for each u such that the inequality

$$\left| p_{u,r_u}(\xi_u) \right| \le L'_u \cdot \max_{1 \le k \le r_u - 1} \left| p_{u,k}(\xi_u) \right|$$

holds for any $\xi_u \in O_u$. We put

$$\Omega_u := \left\{ (t\xi_{u,1}, \dots, t\xi_{u,r_u}) \in \mathbb{C}^{r_u} \;\middle|\; t \in \mathbb{R}, \quad t > 0, \quad (\xi_{u,1}, \dots, \xi_{u,r_u}) \in O_u \cap Z_u \right\}$$

for each u and

$$\Omega := \left\{ \xi = (\xi_1, \dots, \xi_l) \in \mathbb{C}^{d-l} \;\middle|\; \xi_u \in \Omega_u \text{ holds for some } 1 \le u \le l \right\}.$$

Then Ω_u is an open neighborhood of $\widetilde{B}_u(\lambda_{I_u}) \setminus \{0\}$ in $\mathbb{C}^{r_u}$, and Ω is an open set in $\mathbb{C}^{d-l}$. Moreover for $\xi_u \in \mathbb{C}^{r_u} \setminus \{0\}$, the point $\xi_u / |\xi_u|$ belongs to the set $O_u \cap Z_u = \Omega_u \cap Z_u$ if and only if $\xi_u \in \Omega_u$. Hence by the homogeneity of $p_{u,k}(\xi_u)$, we have the inequality

$$\left| p_{u,r_u}(\xi_u) \right| \le L'_u \cdot \max_{1 \le k \le r_u - 1} \left| p_{u,k}(\xi_u) \right| \cdot |\xi_u| \tag{35}$$

for any $\xi_u \in \Omega_u$ with $|\xi_u| \le 1$.

For the simplicity of notation, we put

$$L := \max_{1 \le u \le l} \left(\max_{r_u + 1 \le h \le d-l} L_{u,h} \right) \quad \text{and} \quad L' := \max_{1 \le u \le l} L'_u.$$

For any positive real numbers R and $1 > \epsilon > 0$, and for any open neighborhood U_0 of 0 in $\mathbb{C}^{d-l}$, we take a positive real number δ such that the inequality

$$0 < \delta < \min \left\{ 1, \; \frac{\epsilon}{3(l-1)(d-l)RL}, \; \frac{\epsilon}{3L'} \right\}$$

holds and that the set

$$U := \left\{ \xi \in \mathbb{C}^{d-l} \;\middle|\; |\xi| < \delta \right\}$$

is included by U_0.

Then for any $A = (a_{u,k,v,h}) \in D_R$ and $\xi \in U$, we have

$$\begin{aligned}\max_{u,k}\left|\sum_{v=1}^{l}\sum_{h=r_v+1}^{d-l} a_{u,k,v,h}p_{v,h}(\xi_v)\right| &\le \sum_{v=1}^{l}\sum_{h=r_v+1}^{d-l} R\cdot L_{v,h}\cdot|\xi_v|\cdot\max_{1\le k\le r_v}\bigl|p_{v,k}(\xi_v)\bigr| \\ &\le \frac{\epsilon}{3}\cdot\max_{u,k}\bigl|p_{u,k}(\xi_u)\bigr|\end{aligned}$$

by the inequality (34), which implies

$$\begin{aligned}|\eta| = \max_{u,k}|\eta_{u,k}| &\ge \max_{u,k}\bigl|p_{u,k}(\xi_u)\bigr| - \max_{u,k}\left|\sum_{v=1}^{l}\sum_{h=r_v+1}^{d-l} a_{u,k,v,h}p_{v,h}(\xi_v)\right| \\ &\ge \frac{2}{3}\max_{u,k}\bigl|p_{u,k}(\xi_u)\bigr|.\end{aligned} \tag{36}$$

On the other hand, for $A = (a_{u,k,v,h}) \in D_R$ and $\xi \in U \cap \Omega$, we have $\xi_u \in \Omega_u$ for some u, which implies

$$\begin{aligned}|\eta_{u,r_u}| &\le \bigl|p_{u,r_u}(\xi_u)\bigr| + \left|\sum_{v=1}^{l}\sum_{h=r_v+1}^{d-l} a_{u,r_u,v,h}p_{v,h}(\xi_v)\right| \\ &\le L'_u\cdot\max_{1\le k\le r_u-1}\bigl|p_{u,k}(\xi_u)\bigr|\cdot|\xi_u| + \frac{\epsilon}{3}\cdot\max_{u,k}\bigl|p_{u,k}(\xi_u)\bigr| \\ &\le \frac{2\epsilon}{3}\cdot\max_{u,k}\bigl|p_{u,k}(\xi_u)\bigr| \le \epsilon\cdot|\eta|\end{aligned}$$

by the inequality (35). Therefore we have

Lemma 7.5. *For $(\xi, A) \in (U \cap \Omega) \times D_R$, we have $F(\xi, A) \notin \Xi_\epsilon \times D_R$.*

We put

$$\mu_u := \min_{\xi_u\in Z_u\setminus\Omega_u}\ \max_{1\le k\le r_u}\bigl|p_{u,k}(\xi_u)\bigr| \quad\text{and}\quad \mu := \min_{1\le u\le l}\mu_u.$$

Then μ is positive by the compactness of $Z_u \setminus \Omega_u$ for each u. Moreover by the homogeneity of $p_{u,k}(\xi_u)$, we have the inequality

$$\max_{1\le k\le r_u}\bigl|p_{u,k}(\xi_u)\bigr| \ge \mu_u|\xi_u|^{r_u} \tag{37}$$

for any $\xi_u \in \mathbb{C}^{r_u} \setminus \Omega_u$ with $|\xi_u| \le 1$. We put $r := \max_u r_u$.

Lemma 7.6. *For $(\xi, A) \in (U \setminus \Omega) \times D_R$, we have $|\eta| \ge \frac{2}{3}\mu|\xi|^r$.*

Proof. For $\xi \in U \setminus \Omega$, we have $\xi_u \notin \Omega_u$ for any u. Hence for $(\xi, A) \in (U \setminus \Omega) \times D_R$, by the inequalities (36) and (37), we have

$$|\eta| \ge \tfrac{2}{3}\max_{u,k}\bigl|p_{u,k}(\xi_u)\bigr| \ge \tfrac{2}{3}\max_{1\le u\le l}\mu_u|\xi_u|^{r_u} \ge \tfrac{2}{3}\mu|\xi|^r.$$

□

We put

$$W := \left\{\eta = (\eta_{u,k}) \in \mathbb{C}^{d-l}\ \middle|\ |\eta| < \tfrac{2}{3}\mu\cdot\delta^r\right\}.$$

Then Lemma 7.5 implies the inclusion relation

$$(U \times D_R) \cap F^{-1}(W_\epsilon \times D_R) \subseteq (U \setminus \Omega) \times D_R.$$

Therefore for any $(\xi, A) \in (U \times D_R) \cap F^{-1}(W_\epsilon \times D_R)$, we have the inequality $|\eta| \geq \frac{2}{3}\mu|\xi|^r$ by Lemma 7.6, which assures that the map (33) is proper. Properness implies finiteness and surjectivity, which completes the proof of the proposition. □

Proposition 7.7. *The degree of the branched covering map (33) defined in Proposition 7.4 is equal to the right hand side of the equality (13) in Theorem B.*

Proof. To consider the map F on the domain

$$\widetilde{W}' := \left\{(\eta, 0) \in W_\epsilon \times D_R \;\middle|\; \eta_{u,k} = 0 \text{ for } 1 \leq u \leq l \text{ and } 1 \leq k \leq r_u - 1\right\},$$

we define the map $F_u : \mathbb{C}^{r_u} \to \mathbb{C}^{r_u}$ by $F_u(\xi_u) = (p_{u,1}(\xi_u), \ldots, p_{u,r_u}(\xi_u))$, and put

$$X_u := \left\{\xi_u \in \mathbb{C}^{r_u} \;\middle|\; \begin{array}{l} p_{u,k}(\xi_u) = 0 \text{ for } 1 \leq k \leq r_u - 1 \\ p_{u,r_u}(\xi_u) \neq 0 \end{array}\right\}$$

for each u. We prepare the two lemmas:

Lemma 7.8. *The Jacobian of the map F_u is not equal to zero at any point of X_u*

Lemma 7.9. *The degree of the map $p_{u,r_u}|_{X_u} : X_u \to \mathbb{C}^*$ is equal to $r_u \cdot \#(S_{r_u+1}(\lambda_{I_u}))$, where we define $\#(S_{r_u+1}(\lambda_{I_u})) = 1$ if $r_u \leq 2$.*

Lemma 7.8 assures that the branched covering map (33) is unbranched on some neighborhood of $\widetilde{W}'$ in $W_\epsilon \times D_R$, that X_u is a smooth Riemann surface, and that the map $p_{u,r_u}|_{X_u} : X_u \to \mathbb{C}^*$ is unbranched. Hence the degree of the map (33) is equal to that of the map $(U \times D_R) \cap F^{-1}(\widetilde{W}') \xrightarrow{F} \widetilde{W}'$, which is also equal to $\prod_{1 \leq u \leq l} \deg(p_{u,r_u}|_{X_u})$. Therefore Lemmas 7.8 and 7.9 imply the proposition.

We shall show Lemma 7.8 first. Since $p_{u,k}(\xi_u) = \sum_{i=1}^{r_u} m_{u,i}\xi_{u,i}^k$, we have

$$\det(dF_u)(\xi_u) = r_u! \cdot \prod_{i=1}^{r_u} m_{u,i} \cdot \prod_{1 \leq i < j \leq r_u} (\xi_{u,j} - \xi_{u,i})$$

by the similar computation as in the proof of Lemma 5.1. Hence the Jacobian is not equal to zero if and only if $\xi_{u,1}, \ldots, \xi_{u,r_u}$ are mutually distinct. On the other hand, by the similar argument as in the proof of Lemma 5.5, we find that for a common zero $\xi_u = (\xi_{u,1}, \ldots, \xi_{u,r_u})$ of $p_{u,1}, \ldots, p_{u,r_u-1}$, the inequality $p_{u,r_u}(\xi_u) \neq 0$ holds if and only if $0, \xi_{u,1}, \ldots, \xi_{u,r_u}$ are mutually distinct. Hence for any $\xi_u \in X_u$, the Jacobian $\det(dF_u)(\xi_u)$ is not equal to zero, which completes the proof of Lemma 7.8.

Next we shall show Lemma 7.9. Since $p_{u,k}(\xi_u)$ is homogeneous for any u and k, the Riemann surface X_u is invariant under the action of $\mathbb{C}^*$; hence the set

$$\left\{(\xi_{u,1} : \cdots : \xi_{u,r_u}) \in \mathbb{P}^{r_u-1} \;\middle|\; (\xi_{u,1}, \ldots, \xi_{u,r_u}) \in X_u\right\}$$

is well-defined and is equal to $S_{r_u+1}(\lambda_{I_u})$ by definition. Therefore X_u consists of $\#(S_{r_u+1}(\lambda_{I_u}))$ components, each of which is biholomorphic to $\mathbb{C}^*$. Moreover on each component of X_u, the degree of the map p_{u,r_u} is $\deg p_{u,r_u} = r_u$, which completes the proofs of Lemma 7.9 and the proposition. □

On the basis of Propositions 7.4 and 7.7, we shall prove the following:

Proposition 7.10. *Let $\psi_k(\xi)$ be the expression defined in the equality (16). Then the number*

$$\mathrm{mult}_0(\psi_1, \ldots, \psi_{d-l})$$

is equal to the right hand side of the equality (13) in Theorem B.

Proof. We define the map $\Psi : \mathbb{C}^{d-l} \to \mathbb{C}^{d-l}$ by $\Psi(\xi) := (\psi_k(\xi))_{1\le k\le d-l}$, and put

$$Y := \left\{\xi \in \mathbb{C}^{d-l} \mid \psi_1(\xi) = \cdots = \psi_{d-l-1}(\xi) = 0,\ \psi_{d-l}(\xi) \neq 0\right\}.$$

We denote by $M_{(r_1,\ldots,r_l)}$ the square matrix M defined in Lemma 6.7.

Lemma 7.11. *For any open neighborhood $\widetilde{U}'$ of 0 in $\mathbb{C}^{d-l}$, there exist open neighborhoods U', W' of 0 with $U' \subset \widetilde{U}'$ and $W' \subset \mathbb{C}$ such that $Y \cap U'$ is a smooth Riemann surface, that the map*

$$Y \cap U' \cap \psi_{d-l}^{-1}(W' \setminus \{0\}) \overset{\psi_{d-l}}{\to} W' \setminus \{0\} \tag{38}$$

is an unbranched covering, and that the number $\mathrm{mult}_0(\psi_1, \ldots, \psi_{d-l})$ is equal to the degree of the map (38).

Proof. First we shall check that the inequality $\det(d\Psi)(\xi) \neq 0$ holds for any $\xi \in Y \cap U'$, if we take U' sufficiently small. By the similar argument as in the proof of Lemma 5.1, the equality $\det(d\Psi)(\xi) = 0$ holds for $\xi \in U'$ if and only if $\alpha_u + \xi_{u,i} = \alpha_v + \xi_{v,j}$ holds for some u, i, v and j with $(u, i) \neq (v, j)$, which is equivalent to the condition that $\xi_{u,i} = \xi_{u,j}$ holds for some u, i and j with $i \neq j$ if we take U' sufficiently small. Suppose for instance that $\xi_{1,1} = \xi_{1,2}$ holds for some $\xi \in Y \cap U'$. Then putting $\Psi'(\xi) := (\psi_k(\xi))_{1\le k\le d-l-1}$, considering the map $M^{-1}_{(r_1-1,r_2,\ldots,r_l)} \circ \Psi'$, and keeping in mind the inequalities (36), we have $p_{u,k}(\xi) = 0$ for any u and k, which contradicts $\psi_{d-l}(\xi) \neq 0$. Therefore we have $\det(d\Psi)(\xi) \neq 0$ for any $\xi \in Y \cap U'$, which assures that $Y \cap U'$ is a smooth Riemann surface, and that the map (38) is an unbranched covering if we take W' sufficiently small. Moreover since $\det(d\Psi)(\xi) \neq 0$ for any $\xi \in Y \cap U'$, we have $\mathrm{mult}_{Y'}(\psi_1, \ldots, \psi_{d-l-1}) = 1$ for any connected component Y' of $Y \cap U'$; hence we have $\mathrm{mult}_0(\psi_1, \ldots, \psi_{d-l}) = \mathrm{mult}_0(\overline{Y \cap U'}, \psi_{d-l})$ by definition, where $\overline{Y \cap U'}$ is the closure of $Y \cap U'$ in U'. Since $\mathrm{mult}_0(\overline{Y \cap U'}, \psi_{d-l})$ is clearly equal to the degree of the covering map (38), all the assertions in Lemma 7.11 are verified. $\square$

We proceed the proof of the proposition. It is clear that there exists $A = (a_{u,k,v,h}) \in \mathbb{C}^{(l-1)(d-l)^2}$ such that the equality $F(\xi, A) = (M^{-1}_{(r_1,\ldots,r_l)} \circ \Psi(\xi), A)$ holds for any $\xi \in \mathbb{C}^{d-l}$. Let e be the $(d-l, 1)$ column vector whose $(d-l)$-th entry is 1 and whose other entries are 0. Moreover we put $M^{-1}_{(r_1,\ldots,r_l)}e =: \eta = (\eta_{u,k})_{1\le u\le l, 1\le k\le r_u}$. Then the equality $Y \times \{A\} = F^{-1}(\mathbb{C}\eta \setminus \{0\}, A)$ holds, and the map $F|_{Y\times\{A\}}$ is equal to the map $M^{-1}_{(r_1,\ldots,r_l)} \circ \Psi|_Y$. Hence, if we can show $\eta_{u,r_u} \neq 0$ for $1 \le u \le l$, then we have $(\mathbb{C}\eta \setminus \{0\}) \cap W \subseteq W_\epsilon$ for some ϵ, which assures that the degree of the covering map (38) is equal to that of the branched covering map (33); thus the proposition will be verified by Proposition 7.7 and Lemma 7.11.

We shall show $\eta_{u,r_u} \neq 0$ for $1 \le u \le l$. Suppose $\eta_{l,r_l} = 0$ for instance, and put $\eta' = {}^t(\eta_{1,1}, \ldots, \eta_{l,r_l-1}) \in \mathbb{C}^{d-l-1}$ so that the equality $\eta = {}^t({}^t\eta', 0)$ holds. Then by the equality $e = M_{(r_1,\ldots,r_l)}\eta$, we have $0 = M_{(r_1,\ldots,r_{l-1},r_l-1)}\eta'$. Since $M_{(r_1,\ldots,r_{l-1},r_l-1)}$ is invertible, we have $\eta' = 0$, which implies $\eta = 0$ and the contradiction $e = M_{(r_1,\ldots,r_l)}0 = 0$. Therefore $\eta_{u,r_u} \neq 0$ holds for any $1 \le u \le l$, which completes the proof of the proposition. $\square$

We shall complete the proof of Theorem B.

Proof of Theorem B.

Remember the definition of $\mathbb{I}(\alpha) \in \mathfrak{I}(\lambda)$ for $\alpha \in B_d(\lambda)$ in the proof of Lemma 5.5. By Lemma 7.3, we can easily verify that for any $\alpha \in B_d(\lambda)$ there exists an open neighborhood O_α of α in $\mathbb{P}^{d-2}$ such that the equality

$$\left\{\zeta \in O_\alpha \;\middle|\; \varphi_k(\zeta) = 0 \ \text{ for } 1 \le k \le d - \#\left(\mathbb{I}(\alpha)\right)\right\} = B_d(\lambda) \cap O_\alpha$$

holds, which implies the first two assertions in Theorem B. On the other hand, the last assertion in Theorem B is the direct consequence of Propositions 6.1 and 7.10. $\square$

At the end of this section, we shall give the proof of Proposition D.

Proof of Proposition D.

For the brevity of notation, we put

$$\begin{aligned} \mathfrak{I}'(\lambda) &:= \mathfrak{I}(\lambda) \cup \Big\{\{\{1,\dots,d\}\}\Big\} \quad \text{for } \lambda \in V_d, \\ e_{\mathbb{I}}(\lambda) &:= \mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1,\dots,\varphi_{d-\#(\mathbb{I})}) \quad \text{for each } \mathbb{I} \in \mathfrak{I}(\lambda), \text{ and} \\ e_{\{\{1,\dots,d\}\}}(\lambda) &:= (d-1)\cdot \#\left(S_d(\lambda)\right). \end{aligned}$$

Note that $\{\{1,\dots,d\}\}$ is the only minimum element of $\mathfrak{I}'(\lambda)$ with respect to the partial order $\prec$.

Under the notation above, the equality (14) in Proposition C is equivalent to the equality

$$(d-1)! = \sum_{\mathbb{I}\in\mathfrak{I}'(\lambda)} \left(e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#(\mathbb{I})+1}^{d-1} k\right), \tag{39}$$

whereas the equality (13) in Theorem B is rewritten in the form

$$e_{\mathbb{I}}(\lambda) = \prod_{u=1}^{l} e_{\{I_u\}}\left(\lambda_{I_u}\right) = \prod_{I\in\mathbb{I}} e_{\{I\}}\left(\lambda_I\right), \tag{40}$$

where $\mathbb{I} = \{I_1,\dots,I_l\}$ is an element of $\mathfrak{I}(\lambda)$, and $\{I\}$ denotes the minimum element of the set $\mathfrak{I}'\left(\lambda_I\right)$ for each $I \in \mathcal{I}(\lambda)$. Moreover Proposition D is rewritten in the form

$$\prod_{u=1}^{l}\left(\#\left(I_u\right)-1\right)! = \sum_{\mathbb{I}'\in\mathfrak{I}(\lambda),\ \mathbb{I}'\succ\mathbb{I}} \left(e_{\mathbb{I}'}(\lambda)\cdot \prod_{u=1}^{l}\left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k\right)\right) \tag{41}$$

for $\mathbb{I} = \{I_1,\dots,I_l\} \in \mathfrak{I}(\lambda)$, where $\chi_u(\mathbb{I}')$ is the one defined in Main Theorem 3. Note that $\mathbb{I} \succ \mathbb{I}$ holds for any $\mathbb{I} \in \mathfrak{I}'(\lambda)$. To complete the proof of Proposition D, we only need to derive the equality (41) from the equalities (39) and (40).

Note that by definition the equality

$$\left\{\mathbb{I}' \in \mathfrak{I}'(\lambda) \;\middle|\; \mathbb{I}' \succ \mathbb{I}\right\} = \left\{\mathbb{I}_1\cup\cdots\cup\mathbb{I}_l \;\middle|\; \mathbb{I}_u \in \mathfrak{I}'\left(\lambda_{I_u}\right) \ \text{ for } 1 \le u \le l\right\}$$

holds for $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}'(\lambda)$. We have the following equalities for $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$ from the equalities (39) and (40):

$$\begin{aligned}\prod_{u=1}^{l} \left(\#\left(I_u\right) - 1\right)! &= \prod_{u=1}^{l} \left(\sum_{\mathbb{I}_u \in \mathfrak{I}'(\lambda_{I_u})} \left(e_{\mathbb{I}_u}\left(\lambda_{I_u}\right) \cdot \prod_{k=\#(I_u)-\#(\mathbb{I}_u)+1}^{\#(I_u)-1} k \right)\right) \\ &= \sum_{\mathbb{I}_1 \in \mathfrak{I}'\left(\lambda_{I_1}\right)} \cdots \sum_{\mathbb{I}_l \in \mathfrak{I}'\left(\lambda_{I_l}\right)} \prod_{u=1}^{l} \left(\prod_{I'_u \in \mathbb{I}_u} e_{\{I'_u\}}\left(\lambda_{I'_u}\right) \cdot \prod_{k=\#(I_u)-\#(\mathbb{I}_u)+1}^{\#(I_u)-1} k \right) \\ &= \sum_{\mathbb{I}_1 \in \mathfrak{I}'\left(\lambda_{I_1}\right)} \cdots \sum_{\mathbb{I}_l \in \mathfrak{I}'\left(\lambda_{I_l}\right)} \left(e_{\mathbb{I}_1 \cup \cdots \cup \mathbb{I}_l}(\lambda) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\#(\mathbb{I}_u)+1}^{\#(I_u)-1} k \right)\right) \\ &= \sum_{\mathbb{I}' \in \mathfrak{I}(\lambda),\ \mathbb{I}' \succ \mathbb{I}} \left(e_{\mathbb{I}'}(\lambda) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k \right)\right).\end{aligned}$$

The equality (41) is thus obtained, which completes the proof of Proposition D. □

8. Relation between the sets $S_d(\lambda)$ and $\Phi_d^{-1}\left(\bar{\lambda}\right)$

In this section we shall give the explicit relation between the cardinalities $\#\left(S_d(\lambda)\right)$ and $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$. Let λ be an element of V_d, which is fixed throughout this section. Remember the definitions of $K_1, \ldots, K_q,\ \kappa_1, \ldots, \kappa_q,\ g_1, \ldots, g_q$ defined in Definition 2.1, and $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ defined in Definition 3.2. We put

$$\Sigma_d(\lambda) := \left\{ (\zeta_1 : \cdots : \zeta_d) \in \mathbb{P}^{d-1} \;\middle|\; \begin{array}{c} \sum_{i=1}^{d} \zeta_i = 0 \\ \sum_{i=1}^{d} m_i \zeta_i^k = 0 \quad \text{for} \quad 1 \le k \le d-2 \\ \zeta_1, \ldots, \zeta_d \text{ are mutually distinct} \end{array} \right\}.$$

Proposition 8.1. *The bijection $\tilde{\iota} : \Sigma_d(\lambda) \to S_d(\lambda)$ is defined by*

$$(\zeta_1 : \cdots : \zeta_d) \mapsto (\zeta_1 - \zeta_d : \cdots : \zeta_{d-1} - \zeta_d).$$

The group $\mathfrak{S}(\mathcal{K}(\lambda))$ acts on $\Sigma_d(\lambda)$ by the permutation of the homogeneous coordinates. Moreover the actions of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$ and $\Sigma_d(\lambda)$ commute with the map $\tilde{\iota}$; hence we have the bijection $\Sigma_d(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda)) \xrightarrow{\cong} \Phi_d^{-1}\left(\bar{\lambda}\right)$.

Proof. The bijectivity of the map $\iota(\lambda)$ in Proposition 3.7 implies the proposition. □

Lemma 8.2. *Let $\zeta = (\zeta_1 : \cdots : \zeta_d)$ be an element of $\Sigma_d(\lambda)$ and suppose that there exists a non-identity permutation $\sigma \in \mathfrak{S}(\mathcal{K}(\lambda))$ with $\sigma \cdot \zeta = \zeta$. Then there exists a unique suffix i with $\zeta_i = 0$. Moreover if $i \in K_w$, then the fixing subgroup $\left\{\sigma \in \mathfrak{S}\left(\mathcal{K}(\lambda)\right) \;\middle|\; \sigma \cdot \zeta = \zeta\right\}$ of ζ is a cyclic group whose order divides g_w.*

Proof. For any $\sigma \in \mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ with $\sigma \cdot \zeta = \zeta$, there exists a non-zero complex number a satisfying $\zeta_{\sigma^{-1}(i)} = a\zeta_i$ for $1 \le i \le d$, which induces the injective group homomorphism

$$\mathfrak{S}(\zeta) := \left\{\sigma \in \mathfrak{S}\left(\mathcal{K}(\lambda)\right) \;\middle|\; \sigma \cdot \zeta = \zeta\right\} \ni \sigma \stackrel{\mathfrak{a}}{\mapsto} a \in \left\{a \in \mathbb{C}^* \;\middle|\; |a| = 1\right\}.$$

In the following, we shall fix non-identity $\sigma \in \mathfrak{S}(\zeta)$, and denote by t the order of σ. Then $a = \mathfrak{a}(\sigma)$ is a primitive t-th radical root of 1. Moreover the cardinality $\#\left(\left\{\sigma^s(i) \;\middle|\; s \in \mathbb{Z}\right\}\right)$ is equal to 1 or t according as ζ_i is equal to 0 or not.

Suppose that $\zeta_i \neq 0$ holds for any i. Then t is a common divisor of $\kappa_1, \ldots, \kappa_q$. Moreover we may assume

$$m = (\underbrace{m_1, \ldots, m_1}_{t}, \ldots, \underbrace{m_{d/t}, \ldots, m_{d/t}}_{t})$$

and

$$\zeta = (\zeta_1 : a\zeta_1 : \cdots : a^{t-1}\zeta_1 : \cdots : \zeta_{d/t} : a\zeta_{d/t} : \cdots : a^{t-1}\zeta_{d/t}).$$

Under the notation above, the equations $\varphi_k(\zeta) = 0$ for $1 \le k \le d-2$ are equivalent to the equations $\sum_{i=1}^{d/t} m_i \zeta_i^{tk} = 0$ for $1 \le k \le \frac{d}{t} - 1$, which implies $m_i = 0$ for any i by the mutual distinctness of $0, \zeta_1^t, \ldots, \zeta_{d/t}^t$. We thus obtain contradiction, which assures the existence of i with $\zeta_i = 0$.

Next we suppose $\zeta_i = 0$ and $i \in K_w$. Then for any $\sigma \in \mathfrak{S}(\zeta)$, the order t of σ is a common divisor of $\kappa_1, \ldots, \kappa_{w-1}, \kappa_w - 1, \kappa_{w+1}, \ldots, \kappa_q$, i.e., a divisor of g_w. Therefore $\mathfrak{S}(\zeta)$ is isomorphic to a subgroup of $\left\{a \in \mathbb{C}^* \mid a^{g_w} = 1\right\}$ by the map $\mathfrak{a}$, which completes the proof. $\square$

Remember the definitions of $d[t]$ and $\lambda[t]$ in Definition 2.1. In the following, the symbol $a|b$ denotes that a divides b for positive integers a and b.

Theorem E. *If we put $s_d(\lambda) := \#(S_d(\lambda)) = \#(\Sigma_d(\lambda))$ for $\lambda \in V_d$, then the third and fourth steps in Main Theorem 3 hold.*

Proof. For each $t \in \bigcup_{1 \le w \le q} \left\{t \mid t|g_w\right\}$, we put

$$\Theta_t(\lambda) := \left\{ C \in \Sigma_d(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda)) \;\middle|\; \#(C) = \frac{\#(\mathfrak{S}(\mathcal{K}(\lambda)))}{t} \right\}$$

and $c_t(\lambda) := \#(\Theta_t(\lambda))$. Then by Proposition 8.1 and Lemma 8.2, we have

$$\Phi_d^{-1}(\bar{\lambda}) \overset{\cong}{\longleftarrow} \Sigma_d(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda)) = \left(\coprod_{w=1}^{q} \left(\coprod_{t|g_w,\ t \ge 2} \Theta_t(\lambda) \right) \right) \coprod \Theta_1(\lambda),$$

which implies the equalities (2) and (3). Hence to complete the proof, we only need to show the equalities (1) for each t. In the rest of the proof, we shall fix $1 \le w \le q$.

For each t with $t|g_w$ and $t \ge 2$, we define the group $\mathfrak{S}(\mathcal{K}'(\lambda[t]))$ to be isomorphic to $\mathfrak{S}_{\frac{\kappa_1}{t}} \times \cdots \times \mathfrak{S}_{\frac{\kappa_w - 1}{t}} \times \cdots \times \mathfrak{S}_{\frac{\kappa_q}{t}}$. Then $\mathfrak{S}(\mathcal{K}'(\lambda[t]))$ naturally acts on $S_{d[t]}(\lambda[t])$. Note that we always have $\mathfrak{S}(\mathcal{K}'(\lambda[t])) \subseteq \mathfrak{S}(\mathcal{K}(\lambda[t]))$; however in some cases the equality $\mathfrak{S}(\mathcal{K}'(\lambda[t])) = \mathfrak{S}(\mathcal{K}(\lambda[t]))$ does not hold. Moreover for each divisor b of $\frac{g_w}{t}$, we put

$$\Theta'_b(\lambda[t]) := \left\{ C' \in S_{d[t]}(\lambda[t])/\mathfrak{S}\left(\mathcal{K}'(\lambda[t])\right) \;\middle|\; \#(C') = \frac{\#(\mathfrak{S}(\mathcal{K}'(\lambda[t])))}{b} \right\}.$$

Then we have

$$S_{d[t]}(\lambda[t])/\mathfrak{S}\left(\mathcal{K}'(\lambda[t])\right) = \coprod_{b|(g_w/t)} \Theta'_b(\lambda[t]) \tag{42}$$

by the similar argument as in Lemma 8.2.

Let $t,\ b$ be positive integers with $t|b$, $b|g_w$ and $t \ge 2$, and a a primitive b-th radical root of 1. Then a point

$$\left(\zeta_1 : a\zeta_1 : \cdots : a^{b-1}\zeta_1 : \cdots : \zeta_{d[b]-1} : a\zeta_{d[b]-1} : \cdots : a^{b-1}\zeta_{d[b]-1} : 0\right) \in \mathbb{P}^{d-1}$$

represents an element of $\Theta_b(\lambda)$ if and only if

$$\left(\zeta_1^t : a^t z_1^t : \cdots : a^{t((b/t)-1)}\zeta_1^t : \cdots : \zeta_{d[b]-1}^t : a^t z_{d[b]-1}^t : \cdots : a^{t((b/t)-1)}\zeta_{d[b]-1}^t\right) \in \mathbb{P}^{d[t]-2}$$

represents an element of $\Theta'_{b/t}(\lambda[t])$, which gives the bijection between $\Theta_b(\lambda)$ and $\Theta'_{b/t}(\lambda[t])$. The bijection and the equality (42) imply the equalities (1), which completes the proof of the theorem. □

9. Completion of the proof

In Propositions 3.8, 5.2, 5.6 and 5.9, we had already proved the assertions (5), (1), (4) and (6) in Main Theorem 1. In this section we shall complete the rest of the proofs of the main theorems.

Proposition 9.1. *Main Theorem 3 and the assertion (2) in Main Theorem 1 hold.*

Proof. By Theorem B, Propositions C, D and Theorem E, we obtain Main Theorem 3 and the assertion (2) in Main Theorem 1. □

Proposition 9.2. *Main Theorem 2 and the assertion (3) in Main Theorem 1 hold.*

Proof. In the following, we shall always identify V_d with $\left\{(m_1, \ldots, m_d) \in (\mathbb{C}^*)^d \;\middle|\; \sum_{i=1}^d m_i = 0\right\}$ by the correspondence $m_i = \frac{1}{1-\lambda_i}$, and define the following spaces:

$$\begin{aligned}
\widetilde{P}'_d &:= \Phi_d^{-1}(\widetilde{V}_d),\\
\mathcal{X}_d &:= \left\{(\zeta_1, \ldots, \zeta_d, \rho) \in \mathbb{C}^d \times \mathbb{C}^* \;\middle|\; \zeta_1, \ldots, \zeta_d \text{ are mutually distinct}\right\},\\
\widetilde{\mathcal{X}}_d &:= \mathcal{X}_d/\Gamma,\\
\mathcal{Y}_d &:= \left\{((\zeta, \rho), m) \in \widetilde{\mathcal{X}}_d \times V_d \;\middle|\; \sum_{i=1}^d m_i\zeta_i^k = \begin{cases} 0 & (1 \le k \le d-2) \\ -\frac{1}{\rho} & (k = d-1) \end{cases}\right\},\\
(\mathcal{PX})_d &:= \left\{(\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d \;\middle|\; \zeta_1, \ldots, \zeta_d \text{ are mutually distinct}\right\},\\
(\widetilde{\mathcal{PX}})_d &:= (\mathcal{PX})_d/\Gamma,\\
(\mathcal{PV})_d &:= \left\{(m_1 : \cdots : m_d) \in \mathbb{P}^{d-1} \;\middle|\; \sum_{i=1}^d m_i = 0,\ m_i \neq 0 \text{ for } 1 \le i \le d\right\},\\
(\mathcal{PY})_d &:= \left\{(\zeta, m) \in (\widetilde{\mathcal{PX}}) \times (\mathcal{PV})_d \;\middle|\; \sum_{i=1}^d m_i\zeta_i^k = 0 \text{ for } 1 \le k \le d-2\right\},
\end{aligned}$$

where the actions of Γ on $\mathcal{X}_d$ and $(\mathcal{PX})_d$ are defined by

$$\gamma \cdot (\zeta_1, \ldots, \zeta_d, \rho) = \left(\gamma(\zeta_1), \ldots, \gamma(\zeta_d), a^{-d+1}\rho\right) \quad \text{and} \quad \gamma \cdot (\zeta_1, \ldots, \zeta_d) = (\gamma(\zeta_1), \ldots, \gamma(\zeta_d))$$

for $\gamma(z) = az + b \in \Gamma$, $(\zeta_1, \ldots, \zeta_d, \rho) \in \mathcal{X}_d$ and $(\zeta_1, \ldots, \zeta_d) \in (\mathcal{PX})_d$. Then we have the commutative diagram

$$\begin{tikzcd}
(\widetilde{\mathcal{PX}})_d & \widetilde{\mathcal{X}}_d \arrow[l] \arrow[dr] & \\
(\mathcal{PY})_d \arrow[u] \arrow[d, "\widetilde{\Phi}'_d"] & \mathcal{Y}_d \arrow[l] \arrow[r] \arrow[u] \arrow[d, "\Phi'_d"] & \widetilde{P}'_d \arrow[d, "\Phi_d"] \\
(\mathcal{PV})_d & V_d \arrow[l, "P"'] \arrow[r, "pr"] & \widetilde{V}_d,
\end{tikzcd}$$

where each map is defined to be the natural projection except for the maps Φ_d and

$$\widetilde{\mathcal{X}}_d \ni (\zeta_1, \ldots, \zeta_d, \rho) \mapsto z + \rho(z-\zeta_1)\cdots(z-\zeta_d) \in \widetilde{P}'_d.$$

Here, the first projection maps $\mathcal{Y}_d \to \widetilde{\mathcal{X}}_d$ and $(\mathcal{PY})_d \to (\widetilde{\mathcal{PX}})_d$ are isomorphisms. The d-th symmetric group $\mathfrak{S}_d$ acts on $\widetilde{\mathcal{X}}_d$, $\mathcal{Y}_d$ and V_d by the permutation of coordinates; these actions of $\mathfrak{S}_d$ commute with the projection maps $\mathcal{Y}_d \overset{\cong}{\to} \widetilde{\mathcal{X}}_d$ and $\Phi'_d : \mathcal{Y}_d \to V_d$; moreover we have the natural isomorphisms $\mathcal{Y}_d/\mathfrak{S}_d \cong \widetilde{\mathcal{X}}_d/\mathfrak{S}_d \cong \widetilde{P}'_d$ and $V_d/\mathfrak{S}_d \cong \widetilde{V}_d$. The multiplicative group $\mathbb{C}^*$ also acts on $\widetilde{\mathcal{X}}_d$, $\mathcal{Y}_d$ and V_d by $a \cdot (\zeta, \rho) = (\zeta, a^{-1}\rho)$ and $a \cdot (m_1, \ldots, m_d) = (am_1, \ldots, am_d)$ for $a \in \mathbb{C}^*$, $(\zeta, \rho) \in \widetilde{\mathcal{X}}_d$ and $(m_1, \ldots, m_d) \in V_d$; these actions of $\mathbb{C}^*$ are free, commute with the actions of $\mathfrak{S}_d$, and also commute with the projection maps $\mathcal{Y}_d \overset{\cong}{\to} \widetilde{\mathcal{X}}_d$ and $\Phi'_d : \mathcal{Y}_d \to V_d$; moreover we have the natural isomorphisms $\widetilde{\mathcal{X}}_d/\mathbb{C}^* \cong (\widetilde{\mathcal{PX}})_d \cong (\mathcal{PY})_d \cong \mathcal{Y}_d/\mathbb{C}^*$ and $V_d/\mathbb{C}^* \cong (\mathcal{PV})_d$.

Therefore to analyze the fiber structure of the map $\Phi_d|_{\widetilde{P}'_d}$, we only need to consider the second projection map $\widetilde{\Phi}'_d : (\mathcal{PY})_d \to (\mathcal{PV})_d$ and the actions of $\mathfrak{S}_d$ on $\mathcal{Y}_d$ and V_d, most of which had however already been examined since we can make the following identifications as usual:

$$(\widetilde{\mathcal{PX}})_d = \left\{ (\zeta_1 : \cdots : \zeta_{d-1}) \in \mathbb{P}^{d-2} \;\middle|\; \zeta_1, \ldots, \zeta_{d-1}, 0 \text{ are mutually distinct} \right\},$$

$$(\mathcal{PV})_d = \left\{ (m_1 : \cdots : m_{d-1}) \in \mathbb{P}^{d-2} \;\middle|\; \sum_{i=1}^{d-1} m_i \neq 0,\ m_i \neq 0 \text{ for } 1 \leq i \leq d-1 \right\},$$

$$(\mathcal{PY})_d = \left\{ (\zeta, m) \in (\widetilde{\mathcal{PX}}) \times (\mathcal{PV})_d \;\middle|\; \sum_{i=1}^{d-1} m_i \zeta_i^k = 0 \text{ for } 1 \leq k \leq d-2 \right\}.$$

Especially, we have $(\widetilde{\Phi}'_d)^{-1}(P(\lambda)) = S_d(\lambda)$ for any $\lambda \in V_d$.

For each $(\mathcal{I}, \mathcal{K}) \in \{(\mathcal{I}(\lambda), \mathcal{K}(\lambda)) \mid \lambda \in V_d\}$, we put

$$\begin{aligned}
\overline{V(\mathcal{I}, \mathcal{K})} &:= \{\lambda \in V_d \mid \mathcal{I}(\lambda) \supseteq \mathcal{I},\ \mathcal{K}(\lambda) \supseteq \mathcal{K}\}, \\
V(\mathcal{I}, \mathcal{K}) &:= \{\lambda \in V_d \mid \mathcal{I}(\lambda) = \mathcal{I},\ \mathcal{K}(\lambda) = \mathcal{K}\}, \\
V(\mathcal{I}, *) &:= \{\lambda \in V_d \mid \mathcal{I}(\lambda) = \mathcal{I}\}, \\
V(*, \mathcal{K}) &:= \{\lambda \in V_d \mid \mathcal{K}(\lambda) = \mathcal{K}\}
\end{aligned}$$

and $\mathcal{PV}(\mathcal{I}, *) := P(V(\mathcal{I}, *))$. Remember that $\widetilde{V}(\mathcal{I}, \mathcal{K}) = pr(V(\mathcal{I}, \mathcal{K}))$, $\widetilde{V}(\mathcal{I}, *) = pr(V(\mathcal{I}, *))$ and $\widetilde{V}(*, \mathcal{K}) = pr(V(*, \mathcal{K}))$ hold by the definition in Main Theorem 2. Note that $V(\mathcal{I}, \mathcal{K})$ is a Zariski open subset of $\overline{V(\mathcal{I}, \mathcal{K})}$.

First, we shall show the assertion (3) in Main Theorem 1. Let λ_0, λ' be elements of V_d with $\mathcal{I}(\lambda_0) \subseteq \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda_0) \subseteq \mathcal{K}(\lambda')$. Then we have $\lambda' \in \overline{V(\mathcal{I}(\lambda_0), \mathcal{K}(\lambda_0))}$ and $\mathfrak{S}(\mathcal{K}(\lambda_0)) \subseteq \mathfrak{S}(\mathcal{K}(\lambda'))$. By lemma 5.1 and Implicit function theorem, the second projection map $\widetilde{\Phi}'_d$ is locally homeomorphic, which implies that the map Φ'_d is also a local homeomorphism. We put $(\Phi'_d)^{-1}(\lambda') = \{\zeta(1), \ldots, \zeta(s_d(\lambda'))\}$. Then there exist an open neighborhood U of λ' in $\overline{V(\mathcal{I}(\lambda_0), \mathcal{K}(\lambda_0))}$ and holomorphic sections $\tau_j : U \to \mathcal{Y}_d$ for $1 \le j \le s_d(\lambda')$ such that $\Phi'_d \circ \tau_j = id_U$ and $\tau_j(\lambda') = \zeta(j)$. Moreover the action of $\mathfrak{S}(\mathcal{K}(\lambda_0))$ on $(\Phi'_d)^{-1}(\lambda')$ is naturally extended to the action of $\mathfrak{S}(\mathcal{K}(\lambda_0))$ on $\left\{\tau_j(\lambda) \;\middle|\; 1 \le j \le s_d(\lambda')\right\}$ for any $\lambda \in U$. Hence we have $\#\left(\Phi_d^{-1}(\bar{\lambda}_0)\right) \ge \#\left(\Phi_d^{-1}(\bar{\lambda}')\right)$, which completes the proof of the assertion (3) in Main Theorem 1.

Let us prove next the assertion (2) in Main Theorem 2. Since the map Φ'_d is locally homeomorphic and since the map $pr|_{V(*,\mathcal{K})} : V(*,\mathcal{K}) \to \widetilde{V}(*,\mathcal{K})$ is an unbranched covering for each $\mathcal{K} \in \left\{\mathcal{K}(\lambda) \;\middle|\; \lambda \in V_d\right\}$, the map $\Phi_d|_{\Phi_d^{-1}(\widetilde{V}(*,\mathcal{K}))} : \Phi_d^{-1}\big(\widetilde{V}(*,\mathcal{K})\big) \to \widetilde{V}(*,\mathcal{K})$ is a local homeomorphism, which verifies the assertion (2b) in Main Theorem 2. For each $\mathcal{I} \in \left\{\mathcal{I}(\lambda) \;\middle|\; \lambda \in V_d\right\}$, the cardinality of $(\widetilde{\Phi}'_d)^{-1}(m)$ does not depend on the choice of $m \in \mathcal{PV}(\mathcal{I}, *)$, which assures that the map $(\widetilde{\Phi}'_d)^{-1}(\mathcal{PV}(\mathcal{I}, *)) \overset{\widetilde{\Phi}'_d}{\to} \mathcal{PV}(\mathcal{I}, *)$ is an unbranched covering. Hence the map $(\Phi'_d)^{-1}(V(\mathcal{I}, *)) \overset{\Phi'_d}{\to} V(\mathcal{I}, *)$ is also an unbranched covering. Therefore since the map $V(\mathcal{I}, *) \overset{pr}{\to} \widetilde{V}(\mathcal{I}, *)$ is proper, the map $\Phi_d^{-1}\big(\widetilde{V}(\mathcal{I}, *)\big) \overset{\Phi_d}{\to} \widetilde{V}(\mathcal{I}, *)$ is also proper, which verifies the assertion (2a) in Main Theorem 2. The assertions (2a) and (2b) imply the assertion (2c); thus we have completed the proof of the assertion (2) in Main Theorem 2.

Finally, we shall prove the assertion (1) in Main Theorem 2. In the following, we consider V_d as an open dense subset of the vector space $\mathbb{C}^{d-1} = \left\{(m_1, \ldots, m_d) \in \mathbb{C}^d \;\middle|\; \sum_{i=1}^d m_i = 0\right\}$ with the standard inner product. We take $\lambda \in V_d$, and put $\mathcal{I}(\lambda) =: \mathcal{I}$ and $\mathcal{K}(\lambda) =: \mathcal{K}$, which are fixed in the rest of the proof. We denote by $H(\lambda)$ the orthogonal complement of the linear subspace spanned by $V(\mathcal{I}, \mathcal{K})$ in $\mathbb{C}^{d-1}$. Then the space $H(\lambda)$ is invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$. Hence we can take an arbitrarily small open neighborhood $H_\epsilon(\lambda)$ of 0 in $H(\lambda)$ which is invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$. Moreover we denote by $U(\lambda)$ a sufficiently small open neighborhood of λ in $V(\mathcal{I}, \mathcal{K})$. Then the map $H_\epsilon(\lambda) \times U(\lambda) \ni (h, m) \to h + m \in V_d$ defines a local coordinate system around λ in V_d. Hereafter, we identify $(h, m) \in H_\epsilon(\lambda) \times U(\lambda)$ with $h + m \in V_d$.

Since $H_\epsilon(\lambda)$ and $U(\lambda)$ are sufficiently small, we have $\mathcal{I}(h, m) \subseteq \mathcal{I}(\lambda)$ and $\mathcal{K}(h, m) \subseteq \mathcal{K}(\lambda)$ for any $(h, m) \in H_\epsilon(\lambda) \times U(\lambda)$. Moreover $\mathcal{I}(h, m)$ and $\mathcal{K}(h, m)$ do not depend on the choice of $m \in U(\lambda)$. Hence, for each $h \in H_\epsilon(\lambda)$ and for each connected component Y of $(\Phi'_d)^{-1}(\{h\} \times U(\lambda))$, the map $\Phi'_d|_Y : Y \to \{h\} \times U(\lambda)$ is a homeomorphism. Therefore we have the natural isomorphism $(\Phi'_d)^{-1}(H_\epsilon(\lambda) \times U(\lambda)) \to (\Phi'_d)^{-1}(H_\epsilon(\lambda) \times \{\lambda\}) \times U(\lambda)$ which commutes with the projection maps onto $H_\epsilon(\lambda) \times U(\lambda)$.

For each $m \in U(\lambda)$, the space $H_\epsilon(\lambda) \times \{m\}$ is invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$ with a fixed point $(0, m)$. Moreover we have the natural isomorphism $(H_\epsilon(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda))) \times U(\lambda) \cong (H_\epsilon(\lambda) \times U(\lambda))/\mathfrak{S}(\mathcal{K}(\lambda)) \cong pr(H_\epsilon(\lambda) \times U(\lambda))$. Hence $(\Phi'_d)^{-1}(H_\epsilon(\lambda) \times U(\lambda))$ is also invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$, and its action commutes with the isomorphism $(\Phi'_d)^{-1}(H_\epsilon(\lambda) \times U(\lambda)) \to (\Phi'_d)^{-1}(H_\epsilon(\lambda) \times \{\lambda\}) \times U(\lambda)$. Therefore we obtain the isomorphism

$$\Phi_d^{-1}\left(pr(H_\epsilon(\lambda) \times U(\lambda))\right) \cong \Phi_d^{-1}\left(pr\left(H_\epsilon(\lambda) \times \{\bar{\lambda}\}\right)\right) \times U(\lambda)$$

which commutes with the projection maps onto $pr(H_\epsilon(\lambda) \times U(\lambda))$. Hence for each $\lambda \in V(\mathcal{I}, \mathcal{K})$,

$$\left\{ \lambda' \in V(\mathcal{I}, \mathcal{K}) \;\middle|\; \begin{array}{c} \text{the pair } \lambda, \lambda' \text{ satisfies the condition} \\ \text{in the assertion (1) in Main Theorem 2} \end{array} \right\}$$

is an open subset of $V(\mathcal{I}, \mathcal{K})$ containing λ. Since $V(\mathcal{I}, \mathcal{K})$ is connected, we have the assertion (1) in Main Theorem 2. □

To summarize the above mentioned, we have completed the proof of the main theorems.

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